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GLOBAL STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH NON ZERO BOUNDARY CONDITIONS AND APPLICATIONS

by Corentin AUDIARD (*)

Abstract. — We consider the Schrödinger equation on a half space in any dimension with a class of nonhomogeneous boundary conditions including Dirichlet, Neuman and the so-called transparent boundary conditions. Building upon recent local in time Strichartz estimates (for Dirichlet boundary conditions), we obtain global Strichartz estimates for initial data in $H^s, 0 \leqslant s \leqslant 2$ and boundary data in a natural space $\mathcal{H}^s$. For $s \geqslant 1/2$, the issue of compatibility conditions requires a thorough analysis of the $\mathcal{H}^s$ space. As an application we solve nonlinear Schrödinger equations and construct global asymptotically linear solutions for small data. A discussion is included on the appropriate notion of scattering in this framework, and the optimality of the $\mathcal{H}^s$ space.

Résumé. — On considère l'équation de Schrödinger sur le demi espace en dimension arbitraire pour une classe de conditions au bord non homogènes, incluant les conditions de Dirichlet, Neumann, et « transparentes ». Le principal résultat consiste en des estimations de Strichartz globales pour des données initiales H^s, $0 \leqslant s \leqslant 2$ et des données au bord dans un espace naturel $\mathcal{H}^s$, il améliore les estimées de Strichartz locales en temps obtenues récemment par d'autres auteurs dans le cas des conditions de Dirichlet. Pour $s \geqslant 1/2$, la définition des conditions de compatibilité requiert une étude précise des espaces $\mathcal{H}^s$. En application, on résout des équations de Schrödinger non linéaires, et on construit des solutions dispersives globales si les données sont petites. On discute également le sens précis donné à « solution dispersive », ainsi que la question de l'optimalité de l'espace $\mathcal{H}^s$.

Keywords: Schrödinger equation, dispersive estimates, boundary conditions, Kreiss–Lopatinskii, compatibility condition.
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1. Introduction

We consider the initial boundary value problem (IBVP) for the Schrödinger equation on a half space

$$\begin{cases} i\partial_t u + \Delta u = f, \\ u|_{t=0} = u_0, \\ B(u|_{y=0}, \partial_y u|_{y=0}) = g, \end{cases} \qquad (x,y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}_t^+, \tag{1.1}$$

where the notation $\mathbb{R}_t$ emphasizes the time variable. B is defined as follows: we denote $\mathcal{L}$ the Fourier–Laplace transform on $\mathbb{R}^{d-1} \times \mathbb{R}_t^+$

$$g \to \mathcal{L}g(\xi,\tau) := \int_0^\infty \int_{\mathbb{R}^{d-1}} e^{-\tau t - ix\xi} g(x,t) \mathrm{d}x\mathrm{d}t, \qquad (\xi,\tau) \in \mathbb{R}^{d-1} \times \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant 0\},$$

and B satisfies

$$\mathcal{L}(B(a,b)) = b_1(\xi,\tau)\mathcal{L}(a) + b_2(\xi,\tau)\mathcal{L}(b),$$
$$\text{with } b_1, b_2 \text{ smooth on } \mathrm{Re}(\tau) > 0 \text{ and}$$
$$\forall\ \lambda > 0,\ \ b_1(\lambda\xi, \lambda^2\tau) = b_1(\xi,\tau),\ \ b_2(\lambda\xi,\lambda^2\tau) = \lambda^{-1} b_2(\xi,\tau).$$

This kind of boundary conditions was considered by the author [3] for a large class of dispersive equations on the half space. They are natural considering the homogeneity of the equation, they include Dirichlet ($b_1 = 1$, $b_2 = 0$) and Neuman boundary conditions ($b_1 = 0$, $b_2 = (|\xi|^2 - i\tau)^{-1/2}$, see Section 3 for the choice of the square root), but also the important case of *transparent* boundary conditions ($b_1 = 1$, $b_2 = -(|\xi|^2 - i\tau)^{-1/2}$). The label transparent comes from the fact that the solution of the homogeneous IBVP with transparent boundary conditions coincides on $y \geqslant 0$ with the solution of the Cauchy problem that has for initial value the function u_0 extended by 0 for $y \leqslant 0$ (for motivation and more details see [1]).

Our aim here is to prove the well-posedness of the IBVP under natural assumptions on B detailed in Section 3, and prove that the solutions satisfy Strichartz estimates.

Let us recall that the linear, pure Cauchy problem on $\mathbb{R}^d$ can be solved by elementary semi-group arguments, and its fundamental solution is explicitly given by $\frac{e^{-|x|^2/(4it)}}{(4i\pi t)^{d/2}}$, an immediate consequence being the dispersion estimate $\|e^{it\Delta}u_0\|_{L^\infty} \lesssim \|u_0\|_{L^1}/t^{d/2}$. A more delicate, but essential consequence are Strichartz estimates:

$$\text{for } p > 2,\ \ \frac{2}{p} + \frac{d}{q} = \frac{d}{2},\ \ \|e^{it\Delta}u_0\|_{L^p(\mathbb{R}_t, L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.2}$$

Such estimates are a key tool for the analysis of nonlinear Schrödinger equations (NLS) (see the reference book [13]). Any pair (p,q) that satisfies the identity above is called admissible. In the limit case $p^* = 2, q^* = 2d/(d-2)$, in view of the critical Sobolev embedding $H^1 \hookrightarrow L^{q*}$ such estimates correspond (scaling wise) to a gain of one derivative. It is easily seen that (1.2) remains true if $\mathbb{R}_t$ is replaced by $[0,T]$, and by Hölder's inequality, the estimate is true on $[0,T]$ for $q \geqslant 2, 2/p + d/q \geqslant d/2$. For such indices it is usually called a Strichartz estimate with "loss of derivatives".

The study of the IBVP is significantly more difficult even for homogeneous Dirichlet boundary conditions: the existence of dispersion estimates remained essentially open until very recently (see the announcement [19]), and it is now well understood that Strichartz estimates strongly depend on the geometry of the domain. One of the first breakthroughs on the analysis of Strichartz estimates for the homogeneous BVP was due to Burq, Gérard and Tzvetkov [11], who proved that if the domain is *non trapping*(1) and Δ_D is the Dirichlet Laplacian

$$\text{for } p \geqslant 2,\quad \frac{1}{p} + \frac{d}{q} = \frac{d}{2},\quad \|e^{it\Delta_D} u_0\|_{L^p L^q} \lesssim \|u_0\|_{L^2},$$

this corresponds to Strichartz estimates with loss of $1/2$ derivative. Numerous improvements have been obtained since [2, 7], up to Strichartz estimates without loss of derivatives [7, 18], and their usual consequences for semilinear problems. Very recently, Killip, Visan and Zhang [21] shrinked even more the gap between the IVP and the IBVP by proving the global well-posedness of the quintic defocusing Schrödinger equation posed on the exterior of a convex compact set, while the same result for the Cauchy problem (see [14]) was a major achievement.

Less results are available for nonhomogeneous boundary value problems, although the theory in dimension 1 made very significant progresses. Actually, even in the simplest settings of a half space the two following fundamental questions have not received completely satisfying answers yet

(1) Given smooth boundary data, what algebraic condition should satisfy B for the BVP to be well-posed ?
(2) For such B, given $s \geqslant 0$ what is the optimal regularity of the boundary data to ensure $u \in C_t H^s$?

In dimension one, with Dirichlet boundary conditions, question 2 is now well understood (see [17]): for a solution $u \in C_t H^s(\mathbb{R}^+)$, the natural space for the boundary data is $H^{s/2+1/4}(\mathbb{R}_t^+)$. An easy way to understand this

(1) A typical example is the exterior of a compact star shaped domain.

regularity assumption is that it is precisely the regularity of the trace of solutions of the Cauchy problem, as can be seen from the celebrated sharp Kato smoothing. Let us recall here the classical argument of [20]

$$\begin{aligned}
e^{it\Delta}u_0 &= \int_{\mathbb{R}} e^{-it|\xi|^2} e^{ix\xi}\widehat{u_0}\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^+} e^{-it\eta}\big(e^{ix\sqrt{\eta}}\widehat{u_0} + e^{i-x\sqrt{\eta}}\widehat{u_0}\big)\mathrm{d}\xi \\
&\quad \Rightarrow \|e^{it\Delta}u_0|_{x=0}\|_{\dot{H}^{s/2+1/4}} \\
&\qquad \sim \int_{\mathbb{R}^+}(|\widehat{u_0}(\sqrt{\eta})|^2 + |\widehat{u_0}(-\sqrt{\eta})|^2)|\eta|^{s+1/2}\mathrm{d}\eta \\
&\qquad \sim \int_{\mathbb{R}} |\widehat{u_0}(\xi)|^2|\xi|^{2s}\mathrm{d}\xi \\
&\qquad\qquad \leqslant \|u_0\|_{H^s}^2.
\end{aligned}$$

Sharp Strichartz estimates without loss of derivatives were also derived, so that local well-posedness can be deduced for various nonlinear problems. The Cauchy theory has been recently significantly improved by Bona, Sun and Zhang [9], where the authors study the IBVPs with spatial domain $\mathbb{R}^+$ and $[0, L]$. An interesting feature is that (contrary to the IBVP for the KdV equation) the natural space for the boundary data must be replaced by $H^{s/2+1/2}(\mathbb{R}_t^+)$ when the domain is $[0, L]$, and this space is optimal. The dispersive estimates on $[0, L]$ are obtained by technics of harmonic analysis, in the spirit of the fundamental results of Bourgain [10] for the Schrödinger equation on the torus.

Moreover the authors obtain the *global* well-posednes in H^1 under various assumptions on the nonlinearity. The global well-posedness is based on intricate energy estimates. Finally let us mention that A. S. Fokkas developed the so-called unified transform method (in the spirit of inverse scattering), a method for computing explicitly solutions to boundary value problems in dimension 1. Since the seminal paper [15], the theory received numerous improvements, with the most recent contribution [16] dealing also with the nonlinear Schrödinger equation on the half-line. To our knowledge, Strichartz estimates have not yet been obtained through this approach.

The BVP in dimension $\geqslant 2$ poses new difficulties, because the geometry can be more complex, and waves propagating along the boundary are harder to control (this issue appears even with the trivial geometry of the half space). We expect that the answer to question 2 strongly depends on the domain. Due to its role for control problems, the Schrödinger equation in bounded domain has received significant attention, see [12, 27, 29] and references therein. In unbounded domains with non trivial geometry, the regularity of the boundary data is different and Strichartz estimates with loss can be derived (see the author's contribution [4]).

In this article we only consider the case where the domain is the half space. The Schrödinger equation shares some (limited) similarities with hyperbolic equations, for which question 1 has been clarified in the seminal work of Kreiss [22]: there is a purely algebraic condition, the so-called Kreiss–Lopatinskii condition, which leads to Hadamard type instability if it is violated (see the book [5, Section 4 and references therein]). This condition was extended by the author in [3] for a class of linear dispersive equations posed on the half space. A consequence of the main result was that if this condition is satisfied then (1.1) is well posed in C_tH^s for boundary data in $L^2(\mathbb{R}_t, H^{s+1/2}(\mathbb{R}^{d-1})) \cap H^{s/2+1/4}(\mathbb{R}_t, L^2)$, a space that, scaling wise, is a natural higher dimensional version of $H^{s/2+1/4}(\mathbb{R}_t)$. We point out however that the Kreiss–Lopatinskii condition derived in [3] was quite restrictive, and in particular forbid the Neuman boundary condition, a limitation which is lifted here.

On the issue of Strichartz estimates, Y. Ran, S. M. Sun and B. Y. Zhang considered in [28] the IBVP (1.1) on a half space with nonhomogeneous Dirichlet boundary conditions. They derived explicit solution formulas in the spirit of their work on the Korteweg de Vries equation with J. Bona [8], and managed to use them to obtain local in time Strichartz estimates without loss of derivatives. A very interesting feature was that the existence of solutions in C_TH^s only required boundary data in some space $\mathcal{H}^s$ which has the same scaling as $L^2_tH^{s+1/2}\cap H_t^{s/2+1/4}L^2$ but is slightly weaker. We refer to Section 2.3 for a precise definition of $\mathcal{H}^s$. The space $\mathcal{H}^s$ is in some way optimal, as it is exactly the space where traces of solutions of the Cauchy problem belong, see Proposition 3.9. Note however that in the appendix we provide a construction showing that it is less accurate for evanescent waves (solutions that exist only for BVPs and remain localized near the boundary).

Although not stated explicitly in [28], we might roughly summarize their linear results as follows:

Theorem 1.1 ([28]). — *For $s \geqslant 0$, $s \not\equiv 1/2\ [2\mathbb{Z}]$, $(u_0, f, g) \in H^s(\mathbb{R}^{d-1} \times \mathbb{R}^+) \times L^1([0,T], H^s) \times \mathcal{H}^s([0,T])$. If (u_0, f, g) satisfy appropriate compatiblity conditions, the IBVP (1.1) with Dirichlet boundary conditions has a unique solution $u \in C([0,T], H^s)$, moreover for any (p, q) such that $p > 2$, $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$ and $T > 0$ it satisfies the a priori estimate*

$$\|u\|_{L^p([0,T],W^{s,q})} \lesssim \|u_0\|_{H^s} + \|f\|_{L^1([0,T],H^s)} + \|g\|_{\mathcal{H}^s([0,T])}.$$

In Theorems 1.3,1.4, we provide two improvements to this result: we allow more general boundary conditions, and our Strichartz estimates are *global* in time with a larger range of integrability indices for f (any dual admissible pair). Some consequences for nonlinear problems are then drawn in Section 4.

For the full IBVP the smoothness of solutions does not only depend on the smoothness of the data, but also on some *compatibility conditions*, the simplest one being $u_0|_{y=0} = g|_{t=0}$ in the case of Dirichlet boundary conditions. This compatibility condition is trivially satisfied if $u_0|_{y=0} = g|_{t=0} = 0$ (that is, $u_0 \in H_0^1$), but the non trivial case is mathematically relevant and important for nonlinear problems. It is delicate to describe compatibility conditions for a general boundary operator B, therefore we shall split the analysis in the following two simpler problems:

- General boundary conditions, "trivial" compatibility conditions in Theorem 1.3,
- Dirichlet boundary conditions, general compatibility conditions in Theorem 1.4.

As $\mathcal{H}^s$ is not embedded into continuous functions, $g|_{t=0}$ does not have an immediate meaning. Therefore we thoroughly study the functional spaces $\mathcal{H}^s$ in Section 2.3, including trace properties which allow us to rigorously define the compatibility conditions, including the intricate case $s = 1/2$ where $g|_{t=0}$ has no sense, but a new global compatiblity condition is required. The main new consequence for nonlinear problems is a scattering result in H^1 for (u_0, g) small in $H^1 \times \mathcal{H}^1$. To our knowledge, all previous global well-posedness results required more smoothness on g.

1.1. Statement of the main results

Let us begin with a word on the first order compatiblity condition: if $u_0 \in H^s(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, $s > 1/2$, $u_0|_{y=0}$ is well defined and belongs to $H^{s-1/2}(\mathbb{R}^{d-1})$. We will prove in Proposition 2.1 the embedding $\mathcal{H}^s \subset C_t H^{s-1/2}(\mathbb{R}^{d-1})$, therefore if $u \in C_t \mathcal{H}^s$ solves (1.1), necessarily

$$\text{for } s > 1/2, g|_{t=0} = u_0|_{y=0}. \tag{1.3}$$

(1.3) is the first order compatibility condition. If $s = 1/2$, (1.3) does not makes sense, but a subtler condition is required: let Δ' the laplacian on $\mathbb{R}^{d-1}$, then

$$\text{if } s = 1/2, \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{|e^{-it^2\Delta'} g(x, t^2) - u_0(x, t)|^2}{t} \mathrm{d}t\mathrm{d}x < \infty. \tag{1.4}$$

This is reminiscent of the famous Lions–Magenes global compatibility condition for traces on domains with corners, with a twist due to the Schrödinger evolution, see Definition (2.5) and Section 3.3 for more details. When we say "the compatibility condition is satisfied", we implicitly mean the strongest compatiblity condition that makes sense, so that for $s < 1/2$ nothing is required. It is not difficult to define recursively higher order compatibility conditions (see e.g. [4, Section 2]). Note however that higher order compatibility conditions involve also the trace $f|_{y=t=0}$, which makes sense only if f has some time regularity. We do not treat this issue in the paper.

For nonlinear applications we are only interested by the H^1 regularity, so we choose to consider indices of regularity $s \in [0,2]$. Our main result requires a few notions: see Section 2 for the definition of the functional spaces $\mathcal{H}^s$, $\mathcal{H}^s_0$ and $\mathcal{H}^{1/2}_{00}$ and Section 3 for the definition of the Kreiss–Lopatinskii condition.

We use the following definition of solution:

Definition 1.2. — *A function $u \in C(\mathbb{R}^+_t, L^2)$ is a solution of* (1.1) *if there exists a sequence $(u_0^n, f^n, g^n) \in H^2(\mathbb{R}^{d-1} \times \mathbb{R}^+) \times L^p(\mathbb{R}^+_t, W^{2,q}) \times (L^2(\mathbb{R}^+_t, H^2) \cap H^1(\mathbb{R}^+_t, L^2))$, with*

$$\|(u_0, f, g) - (u_0^n, f^n, g^n)\|_{L^2 \times L_t^{p_1'} L^{q_1'} \times \mathcal{H}^0} \longrightarrow_n 0,$$

such that there exists a solution $u^n \in C_t H^2 \cap C_t^1 L^2$ to the corresponding IBVP and u_n converges to u in $C_t L^2$. A $C_t H^s$ solution is a solution in the $C_t L^2$ sense with additional regularity.

In our statements we shall use the following convention for any $(p,q) \in [1,\infty]^2$

$$B^0_{q,2}(\mathbb{R}^{d-1} \times \mathbb{R}^+) := L^q, \quad B^2_{q,2}(\mathbb{R}^{d-1} \times \mathbb{R}^+) := W^{2,q}, \qquad B^0_{p,2}(\mathbb{R}^+_t) := L^p, \qquad B^1_{p,2}(\mathbb{R}^+_t) := W^{1,p}. \tag{1.5}$$

These equalities are *not true* for the usual definition of Besov spaces, but they allow us to give shorter statements for a regularity parameter $s \in [0,2]$.

Theorem 1.3. — *If B satisfies the Kreiss–Lopatinskii condition* (3.4)*, for $s \in [0,2]$, (p_1, q_1) an admissible pair,*

$$(u_0, f, g) \in H^s_0(\mathbb{R}^{d-1} \times \mathbb{R}^+) \times \left(L^{p_1'}(\mathbb{R}^+_t, B^s_{q_1',2}) \cap B^s_{p_1',2}(\mathbb{R}^+_t, L^{q_1'})\right) \times \mathcal{H}^s_0(\mathbb{R}^+),$$

(if $s = 1/2$, $(u_0, g) \in H^{1/2}_{00} \times \mathcal{H}^{1/2}_{00}$), then the IBVP (1.1) *has a unique solution $u \in C(\mathbb{R}^+, H^s)$, and for any (p,q) such that $p > 2$, $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, it*

satisfies the a priori estimate

$$\|u\|_{L^p(\mathbb{R}_t^+,B^s_{q,2})\cap B^{s/2}_{p,2}(\mathbb{R}_t^+,L^q)} \lesssim \|u_0\|_{H^s}+\|f\|_{L^{p_1'}(\mathbb{R}_t^+,B^s_{q_1',2})\cap B^s_{p_1',2}(\mathbb{R}_t^+,L^{q_1'})}+\|g\|_{\mathcal{H}^s(\mathbb{R}_t^+)}.$$

Moreover, solutions are causal, in the sense that if $(u_i)_{i=1,2}$ *are solutions corresponding to initial data* $(u_{0,i},f_i,g_i)$*, such that* $u_{0,1}=u_{0,2}$*,* $f_1|_{[0,T]}=f_2|_{[0,T]}$*,* $g_1|_{[0,T]}=g_2|_{[0,T]}$*, then* $u_1|_{[0,T]}=u_2|_{[0,T]}$*.*

For the Dirichlet BVP, well-posedness with non trivial compatibility conditions holds:

Theorem 1.4. — *In the case of Dirichlet boundary conditions, for* $s\in[0,2]$*,* (p_1,q_1) *an admissible pair,*

$$(u_0,f,g)\in H^s(\mathbb{R}^{d-1}\times\mathbb{R}^+)\times\left(L^{p_1'}(\mathbb{R}_t^+,B^s_{q_1',2})\cap B^s_{p_1',2}(\mathbb{R}_t^+,L^{q_1'})\right)\times\mathcal{H}^s(\mathbb{R}_t^+),$$

that satisfy the compatiblity condition, then (1.1) *has a unique solution* $u\in C(\mathbb{R}_t^+,H^s)$*, moreover for any* (p,q) *such that* $p>2,\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$ *it satisfies the a priori estimate*

$$\|u\|_{L^p(\mathbb{R}_t^+,B^s_{q,2})\cap B^{s/2}_{p,2}(\mathbb{R}_t^+,L^q)} \lesssim \|u_0\|_{H^s}+\|f\|_{L^{p_1'}(\mathbb{R}_t^+,B^s_{q_1',2})\cap B^s_{p_1',2}(\mathbb{R}_t^+,L^{q_1'})}+\|g\|_{\mathcal{H}^s(\mathbb{R}_t^+)}.$$

Note that we have the usual range of indices for the integrability of f but some time regularity is required. Such requirements are common for hyperbolic BVP (e.g. [26, Proposition 4.3.1]), and the regularity required here is sharp in term of scaling, so that we are able to deduce the usual nonlinear well-posedness results from our linear estimates in Section 4.

1.2. Plan of the article

In Section 2 we recall a number of standard results on Sobolev spaces, and describe the $\mathcal{H}^s$ spaces (completeness, duality, density properties ...). Section 3 starts with the definition of the Kreiss–Lopatinskii condition, and is then devoted to the proof of Theorems 1.3 and 1.4. In Section 4, under classical restrictions on the nonlinearity we prove the local well-posedness in H^1 of the Dirichlet IBVP, and global well-posedness for small data. Finally Section 5 is devoted to the description of the long time behaviour of the global small solutions: we prove that in some sense they behave as the restriction to $y\geqslant 0$ of solutions of the linear Cauchy problem. The appendix A is a small discussion on the optimality of the space $\mathcal{H}^s$.

2. Notations and functional background

2.1. Notations

The Fourier transform of a function u is denoted $\widehat{u}$. As we will use Fourier transform in the (x,y) variable, x variable or (x,t) variable, we use when necessary the less ambiguous notation $\mathcal{F}_{x,y}u, \mathcal{F}_x u, \mathcal{F}_{x,t}u$, for example

$$\widehat{u} = \mathcal{F}_{x,t}u := \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} u(x,t)e^{-ix\cdot\xi - i\delta t}\mathrm{d}x\mathrm{d}t.$$

The notation $\mathbb{R}_t$ emphasizes the time variable.

Lebesgue spaces on a set Ω are denoted $L^p(\Omega)$. For X a Banach space $L^p_t X := L^p(\mathbb{R}_t, X)$ or depending on the context $L^p(\mathbb{R}_t^+, X)$, similarly $L^p_T X := L^p([0,T], X)$. Similarly, L^p_x refers to functions defined on $\mathbb{R}^{d-1}$. When dealing with nonlinear problems, we shall use the convenient but unusual notation $L^p = L^{1/p}$.

We write $a \lesssim b$ if $a \leqslant Cb$ with C a positive constant. Similarly, $a \sim b$ if there exists $C_1, C_2 > 0$ such that $C_1 a \leqslant b \leqslant C_2 b$.

2.2. Functional spaces

$\mathcal{S}'(\mathbb{R}^d)$ is the set of tempered distributions, dual of $\mathcal{S}(\mathbb{R}^d)$. $L^p(\Omega)$ is the Lebesgue space, we follow the usual notation $p' := p/(p-1)$. For $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}|^2 \mathrm{d}\xi < \infty \right\}.$$

$\dot{H}^s$ is the homogeneous Sobolev space. For Ω open, $H^s(\Omega)$ is defined as the set of restrictions to Ω of distributions in $H^s(\mathbb{R}^n)$, with the restriction norm

$$\|u\|_{H^s(\Omega)} = \inf_{v \text{ extension of } u} \|v\|_{H^s(\mathbb{R}^d)}.$$

Similarly, for X a Banach space, $H^s(\Omega, X)$ denotes the Sobolev space of X valued distributions. We recall a few facts (see e.g. [24, 25]):

(1) For n integer, Ω smooth simply connected, $H^n(\Omega, X)$ coincides topologically with $\{u : \int_\Omega \sum_{|\alpha|\leqslant n} |\partial^\alpha u|^2 \mathrm{d}x\}$, that is $\|u\|_{H^n(\Omega)} \sim (\int_\Omega \sum_{|\alpha|\leqslant n} |\partial^\alpha v|^2 \mathrm{d}x)^{1/2}$, with constants that depend on Ω, s. If $\Omega = I$ is an interval the constants only depend on $1/|I|$ and s, in particular if I is unbounded they only depend on s. The same is true if Ω is a half space.

(2) For any $s \geqslant 0$, there exists a continuous extension operator $T_s : H^t(\Omega, X) \to H^t(\mathbb{R}^d, X)$ for $t \leqslant s$, moreover T_s can be chosen such that it is valued into functions supported in $\{x : d(x, \Omega) \leqslant 1\}$. If $s < 1/2$, the zero extension is such an operator and in this case the operator's norm does not depend on Ω.

(3) $H^s_0(\Omega)$ is the closure in Ω of C^∞_c. The extension by zero outside Ω is continuous $H^s_0(\Omega) \to H^s(\mathbb{R}^d)$ if $s \not\equiv 1/2$ $[\mathbb{Z}]$, but not if $s \equiv 1/2$ $[\mathbb{Z}]$. However it is continuous on the Lions–Magenes space $H^{1/2}_{00}$ with norm

$$\|u\|_{H^{1/2}_{00}} = \|u\|_{H^{1/2}} + \left(\int_\Omega \frac{u^2(x)}{d(x, \Omega^c)} \mathrm{d}x \right)^{1/2}, \tag{2.1}$$

and $H^{1/2}_{00} = [L^2, H^1_0]_{1/2}$ (see [32, Section 33]).

For $n \in \mathbb{N}$, $W^{n,p}(\mathbb{R}^d)$ is the Sobolev space with norm $(\sum_{|\alpha| \leqslant n} \int |\partial^\alpha u|^p \mathrm{d}x)^{1/p}$. The Besov spaces on $\mathbb{R}^d$ are denoted $B^s_{p,q}(\mathbb{R}^d)$, they are defined by real interpolation [6]

$$\forall\ 0 \leqslant s \leqslant 2,\ \ B^s_{p,q}(\mathbb{R}^d) = [L^p(\mathbb{R}^d), W^{2,p}(\mathbb{R}^d)]_{s/2,q}.$$

As for Sobolev spaces $B^s_{p,q}(\Omega)$ is defined by restriction. Due to the existence of extension operators, it is equivalent to define $B^s_{p,q}(\Omega) = {}^p(\Omega), W^{2,p}(\Omega)]_{s/2,q}$, the norm equivalence depends on Ω. For $n \in \mathbb{N}$, the following inclusions stand ([6, Theorem 6.4.4])

$$\forall\ p \geqslant 2,\ \ B^n_{p,2}(\Omega) \subset W^{n,p}(\Omega), W^{n,p'}(\Omega) \subset B^n_{p',2}(\Omega).$$

The extension by zero outside some set (which depend on the context) is generically denoted P_0, the restriction operator is denoted R.

2.3. The $\mathcal{H}^s$ spaces

2.3.1. Structure and traces

Proposition 2.1. — *For $s \geqslant 0$, we define the space $\mathcal{H}^s(\mathbb{R}^{d-1} \times \mathbb{R}_t)$ as the set of tempered distributions g such that $\widehat{g} \in L^1_{loc}$ and*

$$\|g\|^2_{\mathcal{H}^s(\mathbb{R}^{d-1}_x \times \mathbb{R}_t)} := \iint_{\mathbb{R}^{d-1} \times \mathbb{R}} (1 + |\xi|^2 + |\delta|)^s \sqrt{||\xi|^2 + \delta|} |\widehat{g}|^2 \mathrm{d}\delta \mathrm{d}\xi < \infty.$$

When d is unambiguous, we write for conciseness $\mathcal{H}^s(\mathbb{R}_t)$.

It is a complete Hilbert space, in which $C_c^\infty(\mathbb{R}_x^{d-1}\times\mathbb{R}_t)$ is dense, and has equivalent norm

$$\begin{aligned}\|g\|_{\mathcal{H}^s} &:= \left(\iint_{\mathbb{R}^{d-1}\times\mathbb{R}}(1+|\xi|^2+|\delta|)^s\sqrt{||\xi|^2+\delta|}|\widehat{g}|^2\mathrm{d}\delta\mathrm{d}\xi\right)^{1/2}\\ &\sim\left(\iint_{\mathbb{R}^{d-1}\times\mathbb{R}}(1+|\xi|^{2s}+||\xi|^2+\delta|^s)\sqrt{||\xi|^2+\delta|}|\widehat{g}|^2\mathrm{d}\delta\mathrm{d}\xi\right)^{1/2}.\end{aligned}$$

The space $\mathcal{H}^0$ is denoted $\mathcal{H}$. The map $u\to\nabla_x u$ is continuous $\mathcal{H}^s\to\mathcal{H}^{s-1}$ for $s\geqslant 1$, and $u\to\partial_t u$ is continuous $\mathcal{H}^s\to\mathcal{H}^{s-2}$ for $s\geqslant 2$.

For $s>1/2$, $\mathcal{H}^s\hookrightarrow C\big(\mathbb{R}_t,H^{s-1/2}(\mathbb{R}_x^{d-1})\big)$, in particular for any $t\in\mathbb{R}$, the trace operator $g\mapsto g(\,\cdot\,,t)$ is continuous $\mathcal{H}^s\to H^{s-1/2}$.

Proof. — Obviously, $\mathcal{H}^s\subset\mathcal{H}^{s'}$ for $s>s'$. Let $g\in\mathcal{H}$, from Cauchy–Schwarz's inequality

$$\begin{aligned}&\iint_{\mathbb{R}^{d-1}\times\mathbb{R}}|\widehat{g}(\xi,\delta)|(1+|\xi|+|\delta|)^{-d}\mathrm{d}\xi\mathrm{d}\delta\\ &\qquad\leqslant\|g\|_{\mathcal{H}}\left(\iint\frac{1}{(1+|\xi|+|\delta|)^{2d}\sqrt{||\xi|^2+\delta|}}\mathrm{d}\xi\mathrm{d}\delta\right)^{1/2}\\ &\qquad\lesssim\|g\|_{\mathcal{H}}\left(\int_{\mathbb{R}^{d-1}}\frac{1}{(1+|\xi|)^{d+1}}\mathrm{d}\xi\mathrm{d}\delta\right)^{1/2}\\ &\qquad\lesssim\|g\|_{\mathcal{H}},\end{aligned}$$

thus the embedding $\mathcal{H}\hookrightarrow\mathcal{S}'$ is continuous. We define the measure μ by $\mathrm{d}\mu=(1+|\xi|^2+|\delta|)^s\sqrt{||\xi|^2+\delta|}\mathrm{d}\delta\mathrm{d}\xi$. If g_n is a Cauchy sequence in $\mathcal{H}^s$, $\widehat{g_n}$ is a Cauchy sequence in $L^2(\mathrm{d}\mu)$. By completeness of Lebesgue spaces, there exists $v\in L^2(\mathrm{d}\mu)$ such that $\|\widehat{g_n}-v\|\longrightarrow 0$. From the previous computations, $\mathcal{F}_{x,t}^{-1}(v)\in\mathcal{S}'$ and $\lim_{\mathcal{S}'}g_n=\mathcal{F}^{-1}v\in\mathcal{H}^s$.

The density of C_c^∞ in $\mathcal{H}^s$ is obtained through the usual procedure. The equivalence of norms is a consequence of the elementary inequality $|a+b|^s\geqslant(1-2^{-1/s})^s(|a|^s-2|b|^s)$.

Let us now consider the trace problem. We start with the existence of a trace at $t=0$:

$$g(x,0)=\int_{\mathbb{R}^{d-1}\times\mathbb{R}} e^{ix\cdot\xi}\widehat{g}(\xi,\delta)\mathrm{d}\delta\mathrm{d}\xi,$$

$$\begin{aligned}\Rightarrow \|g(\,\cdot\,,0)\|_{H^{s-1/2}}^2 &= \int_{\mathbb{R}^{d-1}} |(1+\xi|)^{2s-1}\left|\int_{\mathbb{R}}\widehat{g}\mathrm{d}\delta\right|^2\mathrm{d}\xi\\ &\leqslant \int_{\mathbb{R}^{d-1}}\bigg(\int_{\mathbb{R}}|\widehat{g}|^2\sqrt{||\xi|^2+\delta|}(1+|\xi|^2+|\delta|)^s\mathrm{d}\delta\\ &\qquad\times(1+|\xi|)^{2s-1}\int_{\mathbb{R}}\frac{1}{\sqrt{||\xi|^2+\delta|}(1+|\xi|^2+|\delta|)^s}\mathrm{d}\delta\bigg)\mathrm{d}\xi.\end{aligned}$$

Now clearly $\int_{\mathbb{R}}\frac{1}{\sqrt{||\xi|^2+\delta|}(1+|\xi|^2+|\delta|)^s}\mathrm{d}\delta$ is bounded for $|\xi|\leqslant 1$, and for $|\xi|\geqslant 1$ setting $\delta=|\xi|^2\mu$

$$|\xi|^{2s-1}\int_{\mathbb{R}}\frac{1}{\sqrt{||\xi|^2+\delta|}(|\xi|^2+|\delta|)^s}\mathrm{d}\tau\leqslant\int_{\mathbb{R}}\frac{1}{\sqrt{|1+\mu|}(1+|\mu|)^s}\mathrm{d}\mu<\infty.$$

Therefore the trace at $t=0$ maps continuously $\mathcal{H}^s(\mathbb{R}_t)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$. It is easily checked that the map $T_r: g\to g(\,\cdot\,,\cdot+r)$ is an isometry $\mathcal{H}^s\to\mathcal{H}^s$ and for any $g\in\mathcal{H}^s$, $\lim_0\|T_rg-g\|_{\mathcal{H}^s}=0$. Combining this observation with the existence of the trace at $t=0$ implies the embedding $\mathcal{H}^s\hookrightarrow C_tH^{s-1/2}$. □

Finally, we identify $(\mathcal{H}^s)'$ in a natural way:

Proposition 2.2 (Duality of $\mathcal{H}^s$ spaces). — *For $s>0$, the topological dual $(\mathcal{H}^s)'$ is the set of tempered distributions g' such that $\widehat{g'}\in L^1_{loc}$ and*

$$\|g'\|_{(\mathcal{H}^s)'}^2=\iint_{\mathbb{R}^{d-1}\times\mathbb{R}_t}\frac{(1+|\xi|^2+|\delta|)^{-s}}{\sqrt{||\xi|^2+\delta|}}|\widehat{g'}|^2\mathrm{d}\delta\mathrm{d}\xi<\infty,$$

$\mathcal{S}(\mathbb{R}^n)$ is dense in $(\mathcal{H}^s)'$, and $(\mathcal{H}^s)'$ acts on $\mathcal{H}^s$ with the L^2 duality bracket

$$\langle g,\,g'\rangle_{\mathcal{H}^s,(\mathcal{H}^s)'}=\iint\widehat{g}\overline{\widehat{g'}}\mathrm{d}\delta\mathrm{d}\xi.$$

2.3.2. Restrictions, extensions

Definition 2.3. — *For $s\geqslant 0$, I an interval the space $\mathcal{H}^s(I)$ is the set of restrictions to $\mathbb{R}^{d-1}\times I$ of distributions in $\mathcal{H}^s(\mathbb{R}_t)$, with norm $\|g\|_{\mathcal{H}^s(I)}:=\inf_{\widetilde{g}\text{ extension}}\|\widetilde{g}\|_{\mathcal{H}^s}$.*

For $s \not\equiv 1/2\ [\mathbb{Z}]$, we define $\mathcal{H}_0^s = \mathcal{H}^s$ if $s < 1/2$, and for $s > 1/2$

$$\mathcal{H}_0^s((a,b)) = \{g \in \mathcal{H}^s((a,b)) : \\ \forall\ 0 \leqslant 2k \leqslant [s-1/2], \lim_{a,b} \|\partial_t^k g(\,\cdot\,,t)\|_{H^{s-2k-1/2}} = 0\}.$$

Obviously, if a (or b) is finite, the definition above simply amounts to $\partial_t^k g(\,\cdot\,,a) = 0$.

A very convenient observation is that $\mathcal{H}^s$ is a kind of Bourgain space: let Δ' be the laplacian on $\mathbb{R}^{d-1}$, we have using the change of variable $\delta - \xi^2 = \mu$

$$\begin{aligned}\|e^{-it\Delta'} g\|^2_{\dot{H}_t^{(1+2s)/4} L_x^2 \cap \dot{H}_t^{1/4} H^s} \\ &= \iint |\delta|^{1/2}\big(1+|\delta|^s+|\xi|^{2s}\big)\big|\mathcal{F}_{x,t} e^{-it\Delta} g\big|^2 \mathrm{d}\delta \mathrm{d}\xi \\ &= \iint |\delta|^{1/2}\big(1+|\delta|^s+|\xi|^{2s}\big)|\widehat{g}(\xi,\delta-\xi^2)|^2 \mathrm{d}\delta \mathrm{d}\xi \\ &\sim \iint |\xi^2+\mu|^{1/2}\big(1+|\mu|^s+|\xi|^{2s}\big)|\widehat{g}(\xi,\mu)|^2 \mathrm{d}\mu \mathrm{d}\xi.\end{aligned}$$

so that $\|g\|_{\mathcal{H}^s} \sim \|e^{-it\Delta'} g\|_{\dot{H}^{(1+2s)/4} L_x^2 \cap \dot{H}^{1/4} H^s}$. The following results are elementary consequences of this remark and the classical theory of Sobolev spaces.

Corollary 2.4. — *Let I an interval, $g \in \mathcal{H}^s(I)$. We define the zero extension $P_0 : g \mapsto P_0 g$*

$$P_0 g(\,\cdot\,,t) = \begin{cases} g(\,\cdot\,,t) & \text{if } t \in I, \\ 0 & \text{else.} \end{cases}$$

We have the following assertions:

(1) *With constants only depending on s*

$$\|g\|_{\mathcal{H}^s(I)} \sim \|e^{-it\Delta'} g\|_{\dot{H}^{(2s+1)/4}(I,L^2) \cap \dot{H}^{1/4}(I,H^s)}.$$

(2) *For any $s \geqslant 0$, there exists an extension operator $\mathcal{T}_s$ such that for $k \leqslant s$, $\mathcal{T}_s : \mathcal{H}^k(I) \to \mathcal{H}^k(\mathbb{R})$ is continuous and for any $g \in \mathcal{H}^s(I)$, $\mathcal{T}_s g(t) = 0$ for $t \notin (\inf I - 1, \sup I + 1)$. If $s < 1/2$, P_0 is such an operator.*

(3) *For $s \geqslant 0$, $g \in \mathcal{H}^s(\mathbb{R})$, then $\lim_{T\to\infty} \|g\|_{\mathcal{H}^s([T,\infty[)} = 0$.*

(4) *For $s \geqslant 0$, $\mathcal{H}_0^s(\mathbb{R}) = \mathcal{H}^s$, moreover if $s \not\equiv 1/2\ [\mathbb{Z}]$ P_0 is continuous $\mathcal{H}_0^s(I) \to \mathcal{H}^s(\mathbb{R})$.*

(5) *The restriction operator $(\mathcal{H}(\mathbb{R}))' \to (\mathcal{H}(I))', g \mapsto P_0^*(g)$ is a continuous surjection.*

Proof. — (1) is a direct consequence of the definition of Sobolev spaces by restriction.

(2) According to Section 2.2, there exists an extension operator T such that

$$\|T(e^{-it\Delta'}g)\|_{\dot H^{(1+2s)/4}(\mathbb{R},L^2_x)\cap \dot H^{1/4}(\mathbb{R},H^s)} \lesssim \|e^{-it\Delta'}g\|_{\dot H^{(1+2s)/4}(I,L^2_x)\cap \dot H^{1/4}(I,H^s)} \lesssim \|g\|_{\mathcal{H}^s(I)}.$$

It is then clear that $\mathcal{T} = e^{it\Delta'}T(e^{-it\Delta'})$ defines a continuous extension operator.

(3) If r is an integer, $\lim_{T\to\infty}\|f\|_{H^r([T,\infty[)} = 0$ is clear, then we can conclude by a density argument and the inequality

$$\|e^{-it\Delta}g\|_{\dot H^{(1+2s)/4}(I,L^2_x)\cap \dot H^{1/4}(I,H^s)} \leqslant \|e^{-it\Delta}g\|_{H^k(I,L^2)\cap H^1(I,H^s)}, \quad k\geqslant (1+2s)/4.$$

(4) Let $g\in \mathcal{H}^s(\mathbb{R})$. By continuity of the trace and point (3)

$$\lim_{\infty}\|\partial_t^k g(\,\cdot\,,t)\|_{H^{s-2k-1/2}} \lesssim \lim_{\infty}\|g\|_{\mathcal{H}^s([T,\infty))},$$

the limit at $-\infty$ follows from a symmetry argument.

Now fix $a\in\mathbb{R}$. If for $0\leqslant 2k\leqslant s-1/2$, $\partial_t^k g(\,\cdot\,,a)=0$, this implies clearly $\partial_t^k(e^{-it\Delta}g)(\,\cdot\,,a)=0$, so that we can apply the continuity of the extension by 0 for $e^{-it\Delta'}g$ in the usual Sobolev spaces.

(5) Continuity follows from point (4), the surjectivity from the definition of $\mathcal{H}(I)$. □

Similarly to the Sobolev space $H^{1/2}(\mathbb{R}^+)$, the zero extension is *not* continuous $\mathcal{H}^{1/2}(\mathbb{R}^+)\to\mathcal{H}^{1/2}(\mathbb{R})$. Nevertheless, we observe that $P_0g\in\mathcal{H}^{1/2}(\mathbb{R})$ if $e^{-it\Delta'}P_0g = P_0e^{-it\Delta'}g\in \dot H^{1/2}L^2\cap \dot H^{1/4}H^{1/2}$, which is true if $e^{-it\Delta'}g\in \dot H^{1/2}(\mathbb{R}^+,L^2)\cap \dot H^{1/4}(\mathbb{R}^+,H^{1/2})$ and (according to (2.1))

$$I(g) := \int_{\mathbb{R}^+\times\mathbb{R}^{d-1}} \frac{|e^{-it\Delta'}g(x,t)|^2}{t}\mathrm{d}t\mathrm{d}x < \infty. \tag{2.2}$$

Or more compactly $e^{-it\Delta'}g\in \dot H^{1/2}_{00}(\mathbb{R}^+,L^2)\cap \dot H^{1/4}(\mathbb{R}^+,H^{1/2})$, endowed with the norm

$$\|e^{-it\Delta'}g\|_{\dot H^{1/2}_{00}L^2\cap \dot H^{1/4}H^{1/2}} := \|e^{-it\Delta'}g\|_{\dot H^{1/2}L^2\cap \dot H^{1/4}H^{1/2}} + I(g)^{1/2}.$$

These observations lead to the following definition:

Definition 2.5. — *We denote* $\mathcal{H}^{1/2}_{00}(\mathbb{R}^+) := \{g \in \mathcal{H}^{1/2}(\mathbb{R}^+) : P_0 g \in \mathcal{H}^{1/2}(\mathbb{R})\}$, *it coincides with* $\{g : e^{-it\Delta} g \in \dot{H}^{1/2}_{00} \cap \dot{H}^{1/4} H^{1/2}\}$, *and is a Banach space for the norm*

$$\|g\|_{\mathcal{H}^{1/2}_{00}} = \|e^{-it\Delta'} g\|_{\dot{H}^{1/2}L^2 \cap \dot{H}^{1/4}H^{1/2}} + I(g)^{1/2}. \tag{2.3}$$

Remark 2.6. — Of course we could also define $\mathcal{H}^{1/2}_{00}(I)$, but it is not useful for this paper.

2.3.3. Interpolation

For basic definitions of interpolation, we refer to [6, Sections 3.1 and 4.1]. We denote $[\cdot,\cdot]_\theta$ the complex interpolation functor and $[\cdot,\cdot]_{\theta,2}$ the real interpolation functor with parameter 2.

Proposition 2.7. — *For* $s_0, s_1 \geqslant 0$, $0 < \theta < 1$ *we have*

$$[\mathcal{H}^{s_0}, \mathcal{H}^{s_1}]_\theta = \mathcal{H}^{(1-\theta)s_0 + \theta s_1} \text{ (complex interpolation)},$$
$$[\mathcal{H}^{s_0}, \mathcal{H}^{s_1}]_{\theta,2} = \mathcal{H}^{(1-\theta)s_0 + \theta s_1} \text{ (real interpolation)}.$$

Proof. — By Fourier transform we are reduced to the interpolation of weighted L^2 spaces. For real interpolation, this is Theorem 5.4.1 of [6], for complex interpolation this is Theorem 5.5.3. □

The interpolation of $\mathcal{H}^s_0$ spaces is a bit more delicate.

Proposition 2.8. — *For* $0 < \theta < 1$, $\theta \neq 1/4$, I *an interval we have*

$$[\mathcal{H}_0(I), \mathcal{H}^2_0(I)]_\theta = \mathcal{H}^{2\theta}_0(I) \text{ (complex interpolation)},$$
$$[\mathcal{H}_0(I), \mathcal{H}^2_0(I)]_{\theta,2} = \mathcal{H}^{2\theta}_0(I) \text{ (real interpolation)}.$$

If $s_0 = 0, s_1 = 2, \theta = 1/4$, *then*

$$[\mathcal{H}_0(\mathbb{R}^+), \mathcal{H}^2_0(\mathbb{R}^+)]_{1/4} = \mathcal{H}^{1/2}_{00}(\mathbb{R}^+) \text{ (complex interpolation)},$$
$$[\mathcal{H}(\mathbb{R}^+), \mathcal{H}^2_0(\mathbb{R}^+)]_{1/4,2} = \mathcal{H}^{1/2}_{00}(\mathbb{R}^+) \text{ (real interpolation)}.$$

Proof. — We only detail the case $I = \mathbb{R}^+$, the case of a general interval is similar. According to Corollary 2.4, for $s \in [0,2] \setminus \{1/2\}$ the zero extension P_0, resp. the restriction $\mathcal{R}$ to $\mathbb{R}^+$, is a continuous operators $\mathcal{H}^s_0(\mathbb{R}^+) \to \mathcal{H}^s(\mathbb{R})$, resp. $\mathcal{H}^s(\mathbb{R}) \to \mathcal{H}^s(\mathbb{R}^+)$, with $\mathcal{R} \circ P_0 = \mathrm{Id}$. Therefore by interpolation

$$P_0\big([\mathcal{H}(\mathbb{R}^+), \mathcal{H}^2_0(\mathbb{R}^+)]_{s,2}\big) \subset \mathcal{H}^{2s}(\mathbb{R}),$$

and from the existence of traces, if $s > 1/4$, for $g \in [\mathcal{H}, \mathcal{H}_0^2]_{s,2}$, $g(0) = \lim_{0^-} P_0 g(t) = 0$, thus $[\mathcal{H}, \mathcal{H}_0^2]_{s,2} \subset \mathcal{H}_0^{2s}(\mathbb{R}^+)$. Conversely, for $g \in \mathcal{H}^s(\mathbb{R})$, we define

$$Sg : t \in (0,\infty) \to g(t) - 3g(-t) + 2g(-2t).$$

Clearly, it is continuous $\mathcal{H}^s(\mathbb{R}) \to \mathcal{H}^s(\mathbb{R}^+)$ for $0 \leqslant s \leqslant 2$, and when it makes sense $Sg(0) = 0$, $\partial_t Sg(0) = 0$ thus it is $\mathcal{H}_0^s(\mathbb{R}^+)$ valued. By interpolation S is continuous $\mathcal{H}^{2s}(\mathbb{R}) \to [\mathcal{H}(\mathbb{R}^+), \mathcal{H}_0^2(\mathbb{R}^+)]_{s,2}$. Now for $s \neq 1/2$ we can observe that $S \circ P_0 = \mathrm{Id}$ on $\mathcal{H}_0^s(\mathbb{R}^+)$, therefore $\mathcal{H}_0^{2s}(\mathbb{R}^+) \subset [\mathcal{H}, \mathcal{H}_0^2]_{s,2}$ and the identification is complete.

If $s = 1/2$, we observe that the same argument can be applied provided P_0 acts continuously $\mathcal{H}_{00}^{1/2}(\mathbb{R}^+) \to \mathcal{H}^{1/2}(\mathbb{R})$, but this is true according to Definition 2.5. □

2.4. Interpolation spaces and composition estimates

In order to treat nonlinear problems, estimates in $B^s_{p,2}L^q$ require some composition estimates.

Proposition 2.9. — *Let A be a Banach space. For $0 < \theta < 1$, $[L^p(\mathbb{R}, A), W^{1,p}(\mathbb{R}, A)]_{\theta,2} = B^\theta_{p,2}(\mathbb{R}, A)$ the fractional Besov space endowed with the norm*

$$\begin{aligned}\|u\|^2_{B^\theta_{p,2}A} &:= \int_0^\infty \left(\frac{\|u(\,\cdot\,+h) - u(\,\cdot\,)\|_A}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h} + \|u\|^2_{L^pA} \\ &:= \|u\|^2_{\dot{B}^\theta_{p,2}A} + \|u\|^2_{L^pA}.\end{aligned}$$

For completeness we include a short proof in the spirit of [32] of this well-known result.

Proof. — We use the K-method for interpolation. Let $K(h) = \inf_{u=u_0+u_1} \|u_0\|_{L^pA} + h\|u_1\|_{W^{1,p}A}$. If $u \in [L^p(\mathbb{R}, A), W^{1,p}(\mathbb{R}, A)]_{\theta,2}$, then for any $h \geqslant 0$ there exists (u_0, u_1) with $u = u_0 + u_1$, $\|u_0\|_{L^pA} + h\|u_1\|_{W^{1,p}A} \leqslant 2K(h)$ and $\|u\|_{[L^pA, W^{1,p}A]_{\theta,2}} := (\int_0^\infty (K(h)/h^\theta)^2 \mathrm{d}h/h)^{1/2} < \infty$. The standard estimate $\|u_1(\,\cdot+h) - u_1(\,\cdot\,)\|_{L^p} \leqslant h\|u_1\|_{W^{1,p}}$ implies

$$\int_0^\infty \left(\frac{\|u(\,\cdot+h) - u(\,\cdot\,)\|_{L^pA}}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h} \leqslant 4\int_0^\infty \left(\frac{K(h)}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h}.$$

Conversely, assume the left hand side of the equation above is finite and $u \in L^pA$. For $h > 0$, $\rho_h = \rho(\,\cdot/h)/h$ with $\rho \in C_c^\infty, \rho \geqslant 0, \int \rho = 1$,

$\operatorname{supp}(\rho) \subset [-1,1]$, we set $u_0 = u - \rho_h * u$, $u_1 = \rho_h * u$. Minkowski's inequality gives

$$\begin{aligned}\|u-\rho_h*u\|_{L^pA} &\leqslant \int_{-h}^{h} \rho_h(s)\|u(\,\cdot\,)-u(\,\cdot-s)\|_{L^pA}\mathrm{d}s\\ &\lesssim \frac{1}{h}\int_0^h \|u(\,\cdot+s)-u(\,\cdot\,)\|_{L^pA}\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\|(\rho_h)'*u\|_{L^pA} &\leqslant \int_{-h}^{h} \|\rho_h'(s)\big(u(\,\cdot-s)-u(\,\cdot\,)\big)\|_{L^pA}\mathrm{d}s\\ &\lesssim \frac{1}{h^2}\int_0^h \|u(\,\cdot+s)-u(\,\cdot\,)\|_{L^pA}\mathrm{d}s,\end{aligned}$$

therefore

$$\begin{aligned}K(h) &\leqslant \|u-\rho_h*u\|_{L^pA} + h\|\rho_h*u\|_{W^{1,p}A}\\ &\lesssim h\|u\|_{L^pA} + \frac{1}{h}\int_0^h \|u(\,\cdot+s)-u(\,\cdot\,)\|_{L^pA}\mathrm{d}s.\end{aligned}$$

Also, it is obvious that for $h \geqslant 1$, $K(h) \leqslant \|u\|_{L^p}$. By integration

$$\int_0^\infty \left(\frac{K(h)}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h} \lesssim \|u\|_{L^pA}^2 + \int_0^1 \left(\int_0^h \|u(\,\cdot+s)-u(\,\cdot\,)\|_{L^pA}\mathrm{d}s\right)^2 \frac{\mathrm{d}h}{h^{3+2\theta}}.$$

We set $f(h) = \|u(\,\cdot+h)-u(\,\cdot\,)\|_{L^pA}$, $F(h) = \int_0^h f\mathrm{d}s$. An integration by parts and Cauchy–Schwarz's inequality gives

$$\begin{aligned}\int_0^\infty \big(F(h)\big)^2 \frac{\mathrm{d}h}{h^{3+2\theta}} &= \frac{2}{2+2\theta}\int_0^\infty f(h)F(h)\frac{\mathrm{d}h}{h^{2+2\theta}}\\ &\leqslant \frac{2}{2+\theta}\left(\int_0^\infty \left(\frac{f(h)}{h^\theta}\right)^2\frac{\mathrm{d}h}{h}\right)^{1/2}\left(\int_0^\infty F(h)^2\frac{\mathrm{d}h}{h^{3+2\theta}}\right)^{1/2},\end{aligned}$$

from which we deduce

$$\int_0^\infty \left(\frac{K(h)}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h} \lesssim \|u\|_{L^pA}^2 + \int_0^\infty \left(\frac{\|u(\,\cdot+h)-u(\,\cdot\,)\|_{L^pA}}{h^\theta}\right)^2 \frac{\mathrm{d}h}{h}. \qquad \square$$

Proposition 2.10. — *Let $F : \mathbb{C} \to \mathbb{C}$ such that $|F(u)| \lesssim |u|^a$, $|F'(u)| \lesssim |u|^{a-1}$, $a > 1$. Then*

$$\text{for } 0 < s < 1, \quad \|F(u)\|_{B^s_{p,2}(\mathbb{R}_t,L^q)} \lesssim \|u\|^{a-1}_{L^{p_1}(\mathbb{R}_t,L^{q_1})}\|u\|_{B^s_{p_2,2}(\mathbb{R}_t,L^{q_2})},$$

with

$$p_1,q_1,p_2,q_2 \geqslant 1,\quad \frac{1}{q} = \frac{a-1}{q_1} + \frac{1}{q_2},\quad \frac{1}{p} = \frac{a-1}{p_1} + \frac{1}{p_2}.$$

Proof. — The $L^p_tL^q$ part of the norm is simply estimated with Hölder's inequality on $|u|^{a-1}\times|u|$. For the $\dot B^s_{p,2}$ part, let $1/p=1/p_3+1/p_2$, $1/q=1/q_3+1/q_2$:

$$\begin{aligned}
\int_0^\infty &\left(\frac{\|F(u)(\cdot+h)-F(u)(\cdot)\|_{L^p_tL^q}}{h^s}\right)^2\frac{\mathrm{d}h}{h}\\
&\lesssim \int_0^\infty \left(\frac{\|(|u(\cdot+h)|^{a-1}+|u(\cdot)|^{a-1})|u(\cdot+h)-u(\cdot)|\|_{L^p_tL^q}}{h^s}\right)^2\frac{\mathrm{d}h}{h}\\
&\lesssim \int_0^\infty \left(\frac{\|(u^{a-1}\|_{L^{p_3}_tL^{q_3}}\|u(\cdot+h)-u(\cdot)\|_{L^{p_2}_tL^{q_2}}}{h^s}\right)^2\frac{\mathrm{d}h}{h}\\
&= \|u\|^{2(a-1)}_{L^{p_1}_tL^{q_1}}\int_0^\infty \left(\frac{\|u(\cdot+h)-u(\cdot)\|_{L^{p_2}_tL^{q_2}}}{h^s}\right)^2\frac{\mathrm{d}h}{h}\\
&\leqslant \|u\|^{2(a-1)}_{L^{p_1}_tL^{q_1}}\|u\|^2_{B^s_{p_2,2}L^{q_2}}. \qquad\square
\end{aligned}$$

Finally, as the nonlinear problems require to construct local solutions, we shall use the following extension lemma.

Lemma 2.11. — *Let $p\geqslant 1, 0<s<1$ with $sp>1$, A a Banach space. For any $0<T\leqslant 1$, there exists an extension operator $P_T: B^s_{p,2}([0,T],A)\to B^s_{p,2}(\mathbb{R}_t,A)$ such that $P_Tu(\cdot,t)=0$ if $t\notin[-T,2T]$ and (with constants unbounded as $sp\to 1$)*

$$\begin{cases}\|P_Tu\|_{L^p(\mathbb{R}_t,A)}\lesssim \|u\|_{L^p([0,T],A)},\\ \|P_Tu\|_{B^s_{p,2}(\mathbb{R}_t,A)}\lesssim T^{1/p-s}\|u\|_{B^s_{p,2}([0,T],A)}.\end{cases}\tag{2.4}$$

Proof. — We fix $\chi\in C^\infty_c([0,1[)$, $\chi(0)=1$, and define the operator

$$P_1: B^s_{p,2}([0,1],A)\to B^s_{p,2}(\mathbb{R},A)u\mapsto\begin{cases}u(t), & 0\leqslant t\leqslant 1,\\ u(2-t)\chi(t-1), & 0\leqslant t\leqslant 2,\\ u(-t)\chi(-t), & -1\leqslant t\leqslant 0,\\ 0, & \text{else.}\end{cases}$$

It is not difficult to check that P_1 is bounded $L^p([0,1],A)\to L^p_tA$, $W^{1,p}([0,1],L^q)\to W^{1,p}_tL^q$, with bounds independent of p, thus it is also bounded $B^s_{p,2}([0,1],A)\to B^s_{p,2}(\mathbb{R},A)$. Let D_λ be the dilation operator $D_\lambda: u\mapsto u(\cdot,\lambda\cdot)$, we set

$$P_T=D_{1/T}\circ P_1\circ D_T.$$

From a direct computation, $\|P_Tu\|_{L^p_tA}\leqslant 3\|\chi\|_\infty\|u\|_{L^p_TA}$, thus we are left to prove the second inequality in (2.4).

As $sp>1$, by Sobolev's embedding $\|D_Tu\|^2_{L^\infty([0,1],A)}\lesssim \|u\|_{B^s_{p,2}([0,T],A)}$ thus

$$\|P_1D_Tu\|_{L^p_tL^q}\lesssim \|P_1D_Tu\|_{L^\infty A}\lesssim \|D_Tu\|_{L^\infty([0,1],A)}\lesssim \|u\|_{B^s_{p,2}([0,T],L^q)}.$$

On the other hand for $v\in B^s_{p,2}$ an extension of u, basic computations give

$$\|D_Tv\|^2_{\dot{B}^s_{p,2}L^q}=\int_0^\infty \frac{\|(D_Tv)(t+h)-(D_Tv)(t)\|^2_{L^p_tL^q}}{h^{1+2s}}\mathrm{d}h\leqslant T^{2s-2/p}\|v\|^2_{B^s_{p,2}},$$

thus for $T\leqslant 1$, $\|P_1D_Tu\|_{B^s_{p,2}(\mathbb{R}_t,L^q)}\lesssim \|u\|_{B^s_{p,2}([0,1],L^q)}$, from which we get with the same scaling argument

$$\|D_{1/T}P_1D_Tu\|_{B^s_{p,2}(\mathbb{R}_t,L^q)}\lesssim T^{1/s-1/p}\|u\|_{B^s_{p,2}([0,1])}. \qquad \square$$

3. Linear estimates

The plan to solve (1.1) is based on a superposition principle: let us denote abusively u_0 an extension of u_0 to $\mathbb{R}^d$. If we can solve the Cauchy problem

$$\begin{cases} i\partial_tv+\Delta v=f, \\ v|_{t=0}=u_0, \end{cases}\qquad (x,y,t)\in\mathbb{R}^d\times\mathbb{R}.$$

and the boundary value problem

$$\begin{cases} i\partial_tw+\Delta w=0, \\ w|_{t=0}=0, \\ B(w|_{y=0},\partial_yw|_{y=0})=g-B(v|_{y=0},\partial_yv|_{y=0}), \end{cases} \tag{3.1}$$
$$(x,y,t)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}^+.$$

then $v|_{y\geqslant 0}+w$ is the solution to (1.1). For this strategy to be fruitful we need a number of results: Strichartz estimates for v, trace estimates for $v|_{y=0},\partial_yv|_{y=0}$, existence and Strichartz estimates for w. This is the program that we follow through Section 3.

3.1. The pure boundary value problem

Consider the linear boundary value problem

$$\begin{cases} i\partial_tu+\Delta u=0, \\ B(u|_{y=0},\partial_yu|_{y=0})=g, \\ u(\,\cdot\,,0)=0. \end{cases} \tag{3.2}$$

We use the following notion of solution (slightly stronger than Definition 1.2):

Definition 3.1. — *Let $g \in \mathcal{H}_0^s(\mathbb{R}^+)$. We say that u is a solution of the BVP* (3.2) *if $u \in C(\mathbb{R}^+, H^s)$, there exists a sequence $g_n \in \cap_{k\geqslant 0} H_0^k(\mathbb{R}^{d-1} \times \mathbb{R}_t^+)$ with $\|g - g_n\|_{\mathcal{H}^s} \to_n 0$ and smooth solutions $u_n \in C^\infty(\mathbb{R}^+, \cap_{k\geqslant 0} H^k)$ of* (3.2) *with boundary data g_n such that $\|u - u_n\|_{L^\infty H^s} \to_n 0$.*

3.1.1. The Kreiss–Lopatinskii condition

We recall the notation of the introduction

$$\mathcal{L}(B(a,b))(\xi,\tau) = b_1 \mathcal{L}a(\xi,\tau) + b_2 \mathcal{L}b(\xi,\tau),$$

with b_1, b_2 anisotropically homogeneous: $b_1(\lambda\xi, \lambda^2\tau) = b_1(\xi,\tau)$, $b_2(\lambda\xi, \lambda^2\tau) = \lambda^{-1} b_2(\xi,\tau)$. Of course, the operator B must satisfy some conditions. First of all, it should be defined independently of $\mathrm{Re}(\tau) := \gamma > 0$, so according to Paley–Wiener's theorem we assume that b_1, b_2 are holomorphic in τ on $\{(\tau,\xi) \in \mathbb{C} \times \mathbb{R}^{d-1}, \mathrm{Re}(\tau) > 0\}$. Moreover we assume that b_1 extends continuously on $\{(i\delta,\xi) \in (\mathbb{R} \times \mathbb{R}^{d-1}) \setminus \{0\}\}$, and a.e. in (δ,ξ), $\lim_{\gamma\to 0} b_2(\xi, \gamma + i\delta)$ exists.

The Kreiss–Lopatinskii condition is an algebraic condition that we introduce with the following heuristic: assume that (3.2) has a solution $u \in C_b(\mathbb{R}_t^+, \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}^+))$, and consider its Fourier–Laplace transform $\mathcal{L}u(\xi, y, \tau) = \iint e^{-\tau t + i\xi x} u(x,y,t) \mathrm{d}x\, \mathrm{d}t$. Then $\mathcal{L}u$ satisfies

$$\partial_y^2 \mathcal{L}u = (|\xi|^2 - i\tau)\mathcal{L}u.$$

The condition $\lim_{y\to\infty} \mathcal{L}u(y) = 0$ requires

$$\mathcal{L}u = e^{-\sqrt{|\xi|^2 - i\tau}\, y} \mathcal{L}u(y=0). \tag{3.3}$$

Here, $\sqrt{\cdot}$ is the square root defined on $\mathbb{C} \setminus i\mathbb{R}^+$ such that $\sqrt{-1} = -i$. From (3.3), the condition $B(u|_{y=0}, \partial_y u|_{y=0}) = g$ rewrites

$$(b_1 - \sqrt{|\xi|^2 - i\tau}\, b_2)\mathcal{L}u(0) = \mathcal{L}g,$$

so that $\mathcal{L}u(0)$ is uniquely determined from $\mathcal{L}g$ *with uniform bounds* if

$$\exists\ \alpha, \beta > 0 : \forall\ (\gamma, \delta, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{d-1}, \qquad \alpha \leqslant \left| \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cdot \left(1 - \sqrt{|\xi|^2 - i\tau}\right) \right| \leqslant \beta. \tag{3.4}$$

Definition 3.2. — *B satisfies the (generalized) Kreiss–Lopatinskii condition if* (3.4) *is true.*

By homogeneity b_1 is uniformly bounded, thus (3.4) implies that $b_2\sqrt{|\xi|^2-i\tau}$ is uniformly bounded for $\mathrm{Re}(\tau)\geqslant 0$, although b_2 may be infinite at some points $(\xi,i\delta)$. The vector $V_-:=\big(1-\sqrt{|\xi|^2-i\tau}\big)$ is the so-called stable eigenvector, and algebraically (3.4) means that the symbol of B, as a linear operator $\mathbb{C}^2\to\mathbb{C}$, defines an isomorphism $\mathrm{span}(V_-)\to\mathbb{C}$.

Obviously, the Dirichlet boundary condition $b_D:=(1,0)$ satisfies the uniform Kreiss Lopatinskii condition. It is also possible to include the Neuman boundary condition as well as the transparent boundary condition into this framework by setting

$$\mathcal{L}B_N(a,b)=\frac{\mathcal{L}b(\xi,\tau)}{\sqrt{|\xi|^2-i\tau}}\text{ (Neuman)},\tag{3.5}$$

$$\mathcal{L}B_T(a,b)=\mathcal{L}a(\xi,\tau)-\frac{\mathcal{L}b(\xi,\tau)}{\sqrt{|\xi|^2-i\tau}}\text{ (Transparent)}.\tag{3.6}$$

With this convention, $b_N\cdot V_-=-1$ and $b_T\cdot V_-=2$, so that both satisfy the Kreiss–Lopatinskii condition. Let us point out that in the case of Neuman boundary conditions, $B_N(u,\partial_y u)\in\mathcal{H}$ is equivalent to $\partial_y u|_{y=0}\in\mathcal{H}'$, indeed

$$\|P_0 g\|^2_{\mathcal{H}(\mathbb{R})}=\int_{\mathbb{R}^d}\frac{|\mathcal{L}(\partial_y u|_{y=0})|^2}{||\xi|^2+\delta|}\sqrt{||\xi|^2+\delta|}\mathrm{d}\xi\mathrm{d}\delta=\|P_0\partial_y u|_{y=0}\|^2_{\mathcal{H}'}.\tag{3.7}$$

3.1.2. The Kreiss–Lopatinskii condition and the backward BVP

For general boundary conditions, the boundary value problem is not always reversible. Indeed if we solve (3.2) for $t\leqslant 0$, g supported in $\mathbb{R}_t^-$, the parameter γ in the Laplace transform is negative therefore the appropriate square root in formula (3.3) is defined on $\mathbb{C}\setminus i\mathbb{R}^-$, and maps -1 to i. Let us denote it sq. Even if we dismiss analyticity issues, there is no reason that "backward (3.4)" stands

$$\exists\ \alpha,\beta>0:\forall\ (\gamma,\delta,\xi)\in\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}^{d-1},\qquad \alpha\leqslant\Big|\big(b_1b_2\big)\cdot\big(1-\mathrm{sq}(|\xi|^2-i\tau)\big)\Big|\leqslant\beta.\tag{3.8}$$

For example, take the forward transparent boundary condition $(b_1,b_2)=(1,\frac{-1}{\sqrt{|\xi|^2+\delta}})$, then

$$\forall\ (\xi,\delta)\text{ such that }|\xi|^2+\delta<0,\ \begin{pmatrix}1\\ \frac{-i}{\sqrt{||\xi|^2+\delta|}}\end{pmatrix}\cdot\begin{pmatrix}1\\ -i\sqrt{||\xi|^2+\delta|}\end{pmatrix}=0,$$

and therefore the backward Kreiss–Lopatinskii condition fails in the region $\{|\xi|^2+\delta<0\}$. Note however that the Kreiss–Lopatinskii condition is true for the backward Dirichlet boundary value problem. It is also true for the Neuman boundary value problem provided we choose $(b_1, b_2) = (0, 1/\mathrm{sq}(|\xi|^2 - i\tau))$ instead of $(b_1, b_2) = (0, 1/\sqrt{|\xi|^2 - i\tau})$. The fact that the BVP with transparent boundary condition is not reversible is rather natural: the dissipation due to waves going out of the domain prevents to go back in time.

3.1.3. Well-posedness

The main result of this section states that Theorem 1.3 is true in the case of the pure BVP.

Proposition 3.3. — *If B satisfies the Kreiss–Lopatinskii condition* (3.4)*, and $g \in \mathcal{H}_0^s(\mathbb{R}^+)$, $0 \leqslant s \leqslant 2$ ($\mathcal{H}_{00}^{1/2}$ if $s = 1/2$), the problem* (3.2) *has a unique solution. Moreover it satisfies*[2]

$$\text{(3.9)}\quad \text{for } 0 \leqslant s \leqslant 2,\ \frac{2}{p} + \frac{d}{q} = \frac{d}{2},\ p > 2,$$
$$\|u\|_{L^p(\mathbb{R}^+, B^s_{q,2}) \cap B^{s/2}_{p,2}(\mathbb{R}^+, L^q)} \lesssim \|g\|_{\mathcal{H}_0^s(\mathbb{R}^+)}.$$

Proof.

Existence. — We first justify the existence of g_n as in Definition 3.1. For any $M > 0$, according to Corollary 2.4 there exists $g_M \in \mathcal{H}^s(\mathbb{R})$ that coincides with g for $t \in [0, M]$, and vanishes if $t \leqslant 0$ or $t \geqslant M + 1$. Next we shift $g_M^\delta(x, t) = g_M(x, t - \delta)$, and recall $\lim_0 \|g_M^\delta - g_M\|_{\mathcal{H}^s} = 0$. Let $\rho \in C_c^\infty(\mathbb{R}^{d-1} \times \mathbb{R}_t)$ with $\mathrm{supp}(\rho) \subset \{|t| \leqslant 1\}$, $\int \rho\, \mathrm{d}x\mathrm{d}t = 1$. Then setting $\rho_\varepsilon = \rho(\,\cdot\,/\varepsilon)/\varepsilon^d$,

$$\begin{aligned}&\|\rho_\varepsilon * g_M^\delta - g_M^\delta\|^2_{\mathcal{H}^s}\\ &= \iint_{\mathbb{R}^{d-1}\times\mathbb{R}} |1 - \widehat{\rho}(\varepsilon\xi, \varepsilon\delta)|^2 |\widehat{g_M^\delta}|^2 (1 + |\xi|^2 + |\delta|)^s \sqrt{||\xi|^2 + \delta|}\mathrm{d}\delta\mathrm{d}\xi \to_\varepsilon 0,\\ &\qquad \mathrm{supp}(\rho_\varepsilon * g_M^\delta) \subset \{(x, t) : \delta - \varepsilon \leqslant t \leqslant M + 1 + \delta + \varepsilon\}.\end{aligned}$$

Now we remark $g_M^\delta \in \mathcal{H}^s \Rightarrow e^{-it\Delta} g \in \dot{H}^{1/4}(\mathbb{R}, L^2) \subset L^4(\mathbb{R}, L^2)$, thus $\rho_\varepsilon * g_M^\delta \in \cap_{k \geqslant 0} H^k$. Moreover $\lim_{M\to\infty} \|g\|_{\mathcal{H}^s([M,\infty[)} = 0$ thus if $P_0 g$ is the extension by zero for $t \leqslant 0$

$$\lim_{M\to\infty} \|g_M - P_0 g\|_{\mathcal{H}^s(\mathbb{R}^{d-1}\times\mathbb{R}_t)} = 0.$$

(2) We recall our unusual notation $B^1_{p,2} := W^{1,p}$, $B^2_{q,2} := W^{2,q}$

We also remark that for $\varepsilon \leqslant \delta, \operatorname{supp}(\rho_\varepsilon * g_M^\delta) \subset \{t \geqslant 0\}$, so that an appropriate choice of $\varepsilon_n \leqslant \delta_n, M_n$, provides a smooth sequence (g_n) as in Definition 3.1.

For such g_n, we postpone the existence of a smooth solution u_n and a priori estimate (3.9) to the next paragraphs. Now if (3.9) is true *for smooth solutions*, the case $(p,q) = (\infty, 2)$ implies that (u_n) converges to a solution u in $L^\infty H^s$, and the estimate on u_n for general (p,q) provides the estimate on u.

Uniqueness. — It is again a consequence of the a priori estimate applied to the smooth solutions.

The main issue is thus to prove estimate (3.9): it was obtained very recently in [28] with $\|u\|_{L^pW^{s,q}}$ in the left hand side for *bounded* time intervals. While the core of the $L_t^pL^q$ estimate does not require significant modifications we include a full proof for comfort of the reader.

Proof of estimate (3.9).

The case $s = 0$. — We assume that the Kreiss–Lopatinskii condition (3.4) is satisfied, and that $g \in \cap_{k\geqslant 0} H_0^k(\mathbb{R}^{d-1} \times \mathbb{R}_t^+)$. Let us look back at the formal computation leading to (3.3), which makes sense for smooth functions and reads

$$\mathcal{L}u = e^{-\sqrt{|\xi|^2 - i\tau}\, y} \frac{\mathcal{L}g}{b \cdot V_-(\xi, \tau)}.$$

According to the Kreiss–Lopatinskii condition (3.4), $|\mathcal{L}g/(b \cdot V_-)| \sim |\mathcal{L}g|$, uniformly in (τ, ξ), so that using Paley–Wiener's theorem $\mathcal{L}g/(b \cdot V_-)$ is the Fourier–Laplace transform of some g_1 supported in $t \geqslant 0$.

Now we let $\gamma \to 0$: since $g \in \cap_{k\geqslant 0} H_0^k(\mathbb{R}^{d-1} \times \mathbb{R}_t^+)$, its zero extension belongs to $\cap_{k\geqslant 0} H^k(\mathbb{R}^{d-1} \times \mathbb{R}_t)$, and we can (abusively) identify $\mathcal{L}g(\xi, i\delta) = \widehat{P_0 g}(\xi, \delta)$, with

$$\forall\ k \geqslant 0, \quad \int_{\mathbb{R}^{d-1}\times\mathbb{R}} (1 + |\xi|^2 + |\delta|^2)^k |\widehat{P_0 g}|^2 \mathrm{d}\delta \mathrm{d}\xi < \infty.$$

Since $|\widehat{g_1}| \sim |\widehat{P_0 g}|$, $g_1 \in \cap_{k\geqslant 0} H^k(\mathbb{R}^{d-1} \times \mathbb{R}_t)$ and for any $s \geqslant 0$, $\|g_1\|_{\mathcal{H}^s} \sim \|g\|_{\mathcal{H}^s}$, moreover it is supported in $t \geqslant 0$ thus its restriction belongs to $\cap_{k\geqslant 0} \mathcal{H}_0^s(\mathbb{R}_t^+)$. Since by construction $u|_{y=0} = g_1$, we are reduced to solve the IBVP (3.2) with smooth Dirichlet boundary condition g_1. We abusively denote $\widehat{u}(\xi, \delta)$ for $\mathcal{L}u(\xi, i\delta)$, drop the index 1 of g_1 and simply assume

$$\widehat{u}(\xi, \delta) = e^{-\sqrt{|\xi|^2 + \delta}\, y} \widehat{g}(\xi, \delta), g \in \cap_{k\geqslant 0} H_0^k.$$

If $\delta + |\xi|^2 \geqslant 0$, $\sqrt{\delta + |\xi|^2} \in \mathbb{R}^+$is the usual square root, else $\sqrt{\delta + |\xi|^2} = -i\sqrt{|\delta + |\xi|^2|}$. The solution $u(x, y, t)$ is then obtained by inverse Fourier transform. We split the integral depending on the sign of $\delta + |\xi|^2$, the change of variables $\delta + |\xi|^2 = \pm\eta^2$ gives

$$\begin{aligned}
u(x,y,t) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_{\delta\leqslant -|\xi|^2} e^{iy\sqrt{|\delta+|\xi|^2|}}e^{i(\delta t+x\cdot\xi)}\widehat{g}(\xi,\delta)\mathrm{d}\delta\mathrm{d}\xi\\
&\quad+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_{\delta>-|\xi|^2} e^{-y\sqrt{\delta+|\xi|^2}}e^{i(\delta t+x\cdot\xi)}\widehat{g}(\xi,\delta)\mathrm{d}\delta\mathrm{d}\xi\\
&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_0^\infty e^{i(y\eta+x\cdot\xi)}e^{-it(|\xi|^2+\eta^2)}2\eta\widehat{g}(\xi,-\eta^2-|\xi|^2)\mathrm{d}\eta\,\mathrm{d}\xi\\
&\quad+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_0^\infty e^{-y\eta+ix\cdot\xi}e^{it(-|\xi|^2+\eta^2)}2\eta\widehat{g}(\xi,-|\xi|^2+\eta^2)\mathrm{d}\eta\,\mathrm{d}\xi\\
&:= u_1+u_2. \qquad (3.10)
\end{aligned}$$

From the smoothness of g the integrals are absolutely convergent, infinitely differentiable in x, y, t, and give a solution to (3.2), so that the formal computation is justified for smooth solutions. Moreover, the formula is well defined for $t \in \mathbb{R}$ (and actually cancels for $t < 0$ by Paley–Wiener's theorem), therefore we will focus on proving the seemingly stronger, but more natural estimate

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|g\|_{\mathcal{H}(\mathbb{R}^+)}. \qquad (3.11)$$

Control of u_1. — Let $\widehat{\phi}(\xi,\eta) := 2\eta\widehat{g}(\xi,-\eta^2-|\xi|^2)1_{\eta\geqslant 0}$, we observe $u_1(x,y,t) = e^{it\Delta}\phi$, so that the classical Strichartz estimate (1.2) gives

$$\begin{aligned}
\|u_1\|_{L^p(\mathbb{R}^+,L^q(\mathbb{R}^{d-1}\times\mathbb{R}^+)} &\leqslant \|u_1\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))}\\
&\lesssim \|\phi\|_{L^2}\\
&\sim \|\widehat{\phi}\|_{L^2} \qquad (3.12)\\
&\sim \iint \eta^2|\widehat{g}(\xi,-|\xi|^2-\eta^2)|^2\mathrm{d}\eta\mathrm{d}\xi \qquad (3.13)\\
&\sim \int_{\mathbb{R}^d}\int_{-\infty}^{-|\xi|^2}\sqrt{||\xi|^2+\delta|}|\widehat{g}(\xi,\delta)|^2\mathrm{d}\delta\mathrm{d}\xi \qquad (3.14)\\
&\leqslant \|g\|_{\mathcal{H}}^2. \qquad (3.15)
\end{aligned}$$

Control of u_2. — As mentioned before, it is more convenient to let t vary in $\mathbb{R}$ rather than $\mathbb{R}^+$, obviously bounds in $L^p(\mathbb{R}, L^q(\mathbb{R}^{d-1}\times\mathbb{R}^+))$ imply bounds in $L^p(\mathbb{R}^+, L^q(\mathbb{R}^{d-1}\times\mathbb{R}^+))$.

The idea in [28] is to use a TT^* argument similar to the classical one for the Schrödinger equation, namely if we set $\widehat{\psi} = 2\eta\widehat{g}(\xi, -|\xi|^2+\eta^2)1_{\eta\geqslant 0}$ then (3.10) reads

$$u_2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} e^{-y|\eta|+ix\cdot\xi}e^{it(-|\xi|^2+\eta^2)}\widehat{\psi}(\xi,\eta)\mathrm{d}\eta\,\mathrm{d}\xi := T(\psi),$$

with $\|\psi|\|_{L^2} \lesssim \|g\|_{\mathcal{H}}$. Consider T as an operator

$$L^2(\mathbb{R}^{d-1}\times\mathbb{R}) \to L^p(\mathbb{R}_t, L^q(\mathbb{R}^{d-1}\times\mathbb{R}^+)),$$

the TT^* argument consists in proving

$$\|TT^*\|_{L^{p'}L^{q'}\to L^pL^q} < \infty.$$

If such a bound holds true, then $\|T^*f\|_{L^2}^2 = \langle TT^*f, f\rangle \lesssim \|f\|^2_{L^{p'}L^{q'}}$, thus T^* is continuous $L^{p'}L^{q'} \to L^2$, and by duality $T : L^2 \to L^pL^q$ is continuous, which gives the expected bound $\|u_2\|_{L^pL^q} \lesssim \|g\|_{\mathcal{H}}$. Now let us write

$$\begin{aligned} u_2(x,y,t) &= \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^d}\iint_{\mathbb{R}^d} e^{-y|\eta|-it(|\xi|^2-\eta^2)+ix\cdot\xi}e^{-ix_1\cdot\xi-iy_1\eta} \\ &\qquad\qquad\times\psi(x_1,y_1)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}\eta\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^d}\left(\iint_{\mathbb{R}^d} e^{-y|\eta|-it(|\xi|^2-\eta^2)+ix\cdot\xi}e^{-ix_1\cdot\xi-iy_1\eta}\mathrm{d}\xi\mathrm{d}\eta\right) \\ &\qquad\qquad\times\psi(x_1,y_1)\mathrm{d}x_1\mathrm{d}y_1. \end{aligned}$$

We denote(3) $X = (x,y) \in \mathbb{R}^{d-1}\times\mathbb{R}^+$, $X_1 = (x_1,y_1) \in \mathbb{R}^d$, observe that $T\psi$ can be seen as the action of a kernel with parameter $K_t(X,X_1)$ on $\psi(X_1)$:

$$u_2(x,y,t) = \frac{1}{(2\pi)^d}Op(K_t)\cdot\psi.$$

According to the TT^* argument, it suffices to bound $Op(K_t)\circ Op(K_t)^* : L^{p'}(\mathbb{R}, L^{q'}(\mathbb{R}^{d-1}\times\mathbb{R}^+)) \to L^p(\mathbb{R}, L^q(\mathbb{R}^{d-1}\times\mathbb{R}^+))$. After a few computations one may check

$$\begin{aligned} Op(K_t)\circ Op(K_t)^* f &= \int_{\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}_s}\left(\int_{\mathbb{R}^d}K_t(X,X_1)\overline{K_s}(X_2,X_1)\mathrm{d}X_1\right) \\ &\qquad\qquad f(X_2,s)\mathrm{d}X_2\mathrm{d}s \end{aligned}$$

$$:= \int_{\mathbb{R}_s}\left(\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}N_{t,s}(X,X_2)f(X_2,s)\mathrm{d}X_2\right)\mathrm{d}s \tag{3.16}$$

$$= \int_{\mathbb{R}_s}\big(\mathrm{Op}(N_{t,s})\cdot f(\,\cdot\,,s)\big)(X)\mathrm{d}s \tag{3.17}$$

(3) While X corresponds to the space variable (x,y) that we use throughout the paper, the variable X_1 is purely artificial.

Lemma 3.4. — *We have for* $(X, X_2) \in (\mathbb{R}^{d-1}\times\mathbb{R}^+)^2$

$$N_{t,s}(X,X_2) = (2\pi)^d \int_{\mathbb{R}^d} e^{i(|\xi|^2-\eta^2)(s-t)} e^{i\xi\cdot(x-x_2)} e^{-(y+y_2)|\eta|}\mathrm{d}\eta\mathrm{d}\xi.$$

Proof. — According to identity (3.16)

$$\begin{aligned}
N_{t,s} &= \int_{\mathbb{R}^d} K_t(X,X_1)\overline{K_s}(X_2,X_1)dX_1\\
&= \int_{\mathbb{R}^{3d}} e^{-it(|\xi|^2-\eta^2)+ix\cdot\xi-y|\eta|-i(\xi\cdot x_1+\eta y_1)}\\
&\qquad\qquad \times e^{is(|\xi_1|^2-\eta_1^2)-ix_2\cdot\xi_1-y_2|\eta_1|+i(\xi_1\cdot x_1+\eta_1 y_1)}\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\eta_1\mathrm{d}\xi_1\mathrm{d}X_1\\
&= (2\pi)^d\int_{\mathbb{R}^d} e^{-it(|\xi|^2-\eta^2)+ix\cdot\xi-y|\eta|}\mathcal{F}_{X_1\to\xi,\eta}\\
&\qquad\qquad \times \mathcal{F}^{-1}_{(\xi_1,\eta_1)\to X_1}\Big(e^{-ix_2\cdot\xi_1-y_2|\eta_1|+is(|\xi_1|^2-\eta_1^2)}\Big)\mathrm{d}\xi\mathrm{d}\eta\\
&= (2\pi)^d\int_{\mathbb{R}^d} e^{-it(|\xi|^2-\eta^2)+ix\cdot\xi-y|\eta|}\, e^{-ix_2\xi-y_2|\eta|+is(|\xi|^2-\eta^2)}\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}$$

which is the expected result. □

The estimate of $N_{t,s}$ requires a (classical) substitute to Plancherel's formula:

Lemma 3.5. — *The map* $\mathcal{L}: f\to\int_0^\infty e^{-\lambda y}f(y)\mathrm{d}y$ *is continuous* $L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)$.

Proof. — We have

$$\begin{aligned}
\|\mathcal{L}f\|_2^2 = \langle\mathcal{L}f,\mathcal{L}f\rangle &= \int_{(\mathbb{R}^+)^3} e^{-\lambda(y_1+y_2)}f(y_1)\overline{f}(y_2)\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}\lambda\\
&= \int_{(\mathbb{R}^+)^2}\frac{f(y_1)\overline{f}(y_2)}{y_1+y_2}\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}$$

Splitting $(\mathbb{R}^+)^2 = \{y_2\leqslant y_1\}\cup\{y_1\leqslant y_2\}$, we remark

$$\begin{aligned}
\|\mathcal{L}f\|_2^2 = \int_0^\infty f(y_1)\frac{1}{y_1}\int_0^{y_1}\frac{\overline{f}(y_2)}{1+y_2/y_1}\mathrm{d}y_2\mathrm{d}y_1&\\
+\int_0^\infty\overline{f}(y_2)\frac{1}{y_2}\int_0^{y_2}\frac{f(y_1)}{1+y_1/y_2}\mathrm{d}y_1\mathrm{d}y_2.&
\end{aligned}$$

One easily concludes using $|f(y_2)/(1+y_2/y_1)|\leqslant|f(y_2)|$ and Hardy's inequality. □

Proposition 3.6. — *The operator* $Op(N_{t,s})$ *satisfies for* $2\leqslant p\leqslant\infty$

$$\|Op(N_{t,s})v\|_{L^p(\mathbb{R}^{d-1}\times\mathbb{R}^+)}\lesssim\frac{\|v\|_{L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+)}}{|t-s|^{d(1/2-1/p)}}. \tag{3.18}$$

Proof. — The case $p=\infty$: according to Proposition 3.4

$$\begin{aligned}N_{t,s}(X,X_2) &= \int_{\mathbb{R}^d} e^{i(|\xi|^2-\eta^2)(s-t)} e^{i\xi\cdot(x-x_2)} e^{-(y+y_2)|\eta|} \mathrm{d}\eta \mathrm{d}\xi \\ &= \int_{\mathbb{R}^{d-1}} e^{i|\xi|^2(s-t)} e^{i\xi\cdot(x-x_2)} \mathrm{d}\xi \int_{\mathbb{R}} e^{i\eta^2(t-s)} e^{-(y+y_2)|\eta|} \mathrm{d}\eta \\ &= \frac{e^{i\frac{|x-x_2|^2}{4(t-s)}}}{(4i\pi(t-s))^{(d-1)/2}} \int_{\mathbb{R}} e^{i\eta^2(t-s)} e^{-(y+y_2)|\eta|} \mathrm{d}\eta\end{aligned}$$

The Van Der Corput lemma implies

$$\begin{aligned}\left|\int_{\mathbb{R}} e^{i\eta^2(t-s)} e^{-(y+y_2)|\eta|} \mathrm{d}\eta\right| &\lesssim \frac{\|e^{-(y+y_2)|\eta|}\|_{L^\infty_\eta} + \|(e^{-(y+y_2)|\eta|})'\|_{L^1_\eta}}{\sqrt{|t-s|}} \\ &\lesssim \frac{1}{|t-s|^{1/2}}.\end{aligned}$$

Therefore $|N_{t,s}| \lesssim 1/|t-s|^{d/2}$ uniformly in X,X_2, this implies the case $p=\infty$.

For the case $p=2$ we use Plancherel's formula and Lemma 3.5:

$$\begin{aligned}&\|\mathrm{Op}(N_{t,s})v\|_{L^2} \\ &= \left\| \int_{\mathbb{R}^{d-1}} e^{i|\xi|^2(s-t)} e^{i\xi\cdot x} \int_{\mathbb{R}\times\mathbb{R}^+\mathbb{R}^{d-1}} e^{-i\xi\cdot x_2 - i\eta^2(s-t)-(y+y_2)\eta} v(X_2) \mathrm{d}\eta dX_2 \mathrm{d}\xi \right\|_{L^2_{xy}} \\ &\sim \left\|\left\| \int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot x_2} \int_{\mathbb{R}\times\mathbb{R}^+} e^{-i\eta^2(s-t)} e^{-(y+y_2)|\eta|} v(X_2) \mathrm{d}\eta \mathrm{d}y_2 \mathrm{d}x_2 \right\|_{L^2_\xi}\right\|_{L^2_y} \\ &\sim \left\|\left\| \int_{\mathbb{R}} e^{-|\eta|y} \int_{\mathbb{R}^+} e^{-i\eta^2(s-t)} e^{-y_2|\eta|} v(X_2) \mathrm{d}y_2 \mathrm{d}\eta \right\|_{L^2_y}\right\|_{L^2_{x_2}} \\ &\lesssim \left\|\left\| e^{-i\eta^2(s-t)} \int_{\mathbb{R}^+} e^{-y_2|\eta|} v(X_2) \mathrm{d}y_2 \right\|_{L^2_\eta}\right\|_{L^2_{x_2}} \\ &\lesssim \|v(X_2)\|_{L^2_{X_2}}.\end{aligned}$$

The general case follows from an interpolation argument. □

The estimate on $\mathrm{Op}(K_t)\circ\mathrm{Op}(K_t)^*$ now follows from the Hardy–Littlewood–Sobolev lemma (e.g. [23, Theorem 2.6]): for $p>2$, $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$, we have $1+\frac{1}{p}=\frac{1}{p'}+d(\frac{1}{2}-\frac{1}{q})$, thus

$$\begin{aligned}\|\mathrm{Op}(K_t)\circ\mathrm{Op}(K_t)^*f\|_{L^p_tL^q_X}&=\left\|\int_{\mathbb{R}}\mathrm{Op}(N_{t,s})f(\,\cdot\,,s)\mathrm{d}s\right\|_{L^p_tL^q_X}\\&\lesssim\left\|\int_{\mathbb{R}}\frac{\|f(\,\cdot\,,s)\|_{L^{q'}_X}}{|t-s|^{d(1/2-1/q)}}\mathrm{d}s\right\|_{L^p_t}\\&\lesssim\|f\|_{L^{p'}_tL^{q'}_X}.\end{aligned}$$

Using the TT^* argument this ends estimate (3.9) for the case $s=0$.

The case $s=2$. — By differentiation of formula (3.10), for $|\alpha|+\beta+2\gamma\leqslant 2$, and using the case $s=0$

$$\begin{aligned}\|\partial_x^\alpha\partial_y^\beta\partial_t^\gamma u\|_{L^p_tL^q}&\lesssim\left(\iint||\xi|^2+\delta|^{(1+\beta)/2}|\xi|^\alpha|\delta|^\gamma|\widehat{g}|^2\mathrm{d}\delta\mathrm{d}\xi\right)^{1/2}\\&\lesssim\|g\|^2_{\mathcal{H}^2_0(\mathbb{R}^+_t)}.\end{aligned}$$

Remark 3.7. — We recall that in the inequality above, $\widehat{g}$ abusively denotes $\widehat{P_0g}$. Since P_0g must belong to $\mathcal{H}^2$ we can not simply take $g\in\mathcal{H}^2(\mathbb{R}^+)$.

Obviously, the same argument applies as soon as s is an even integer, but since the non-integer case is slightly more delicate, we chose to consider only $s\leqslant 2$ for simplicity.

The case $0<s<2$. — This is an interpolation argument. For $p>2$, $2/p+d/q=d/2$, the solution map is continuous

$$\mathcal{H}(\mathbb{R}^+)\to L^p(\mathbb{R}_t,L^q),\mathcal{H}^2_0(\mathbb{R}^+)\to L^p(\mathbb{R}_t,W^{2,q})\cap W^{1,p}(\mathbb{R}_t,L^q),$$

thus by interpolation it is continuous

$$[\mathcal{H},\mathcal{H}^2_0]_{s,2}\to L^p(\mathbb{R}_t,B^{2s}_{q,2})\cap B^s_{p,2}(\mathbb{R}_t,L^q),$$

this gives the result by using the interpolation identities of Proposition 2.8 and by restriction on $t\geqslant 0$. □

3.1.4. The boundary value problems on $[-T,\infty[$ and $\mathbb{R}_t$

A natural question (and actually useful in the rest of the paper) is the solvability of the BVP on other time intervals than $[0,\infty[$. As we mentioned before, the backward BVP can be ill-posed. However translations have a better behaviour: first, we extend the operator $(a,b)\to B(a,b)$ to distributions in $\mathcal{H}(\mathbb{R})\times\mathcal{H}'(\mathbb{R})$ with the formula

$$B(a,b)=\mathcal{F}^{-1}_{x,t}\big(b_1(\xi,i\delta)\widehat{a}(\xi,\delta)+b_2(\xi,i\delta)\widehat{b}(\xi,\delta)\big).$$

Under the Kreiss–Lopatinskii condition, this extension maps $\mathcal{H}\times\mathcal{H}'\to\mathcal{H}(\mathbb{R})$. For $g\in\mathcal{H}(\mathbb{R})$ smooth, supported in $t\geqslant 0$ and u a smooth solution to the pure BVP (3.2), we define $u_T=u(t+T)$ for some $T\in\mathbb{R}$. Then from the explicit formula (3.3), u_T satisfies

$$\mathcal{F}B(u_T|_{y=0},\partial_y u_T|_{y=0})=e^{-iT\delta}\mathcal{L}g(\xi,i\delta)=\mathcal{F}(g(\,\cdot+T)),$$

so that u_T is a solution of the BVP

$$\begin{cases} i\partial_t v+\Delta v=0, (x,y,t)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times[-T,\infty[,\\ B(v|_{y=0},\partial_y v|_{y=0})=g(\,\cdot+T),\\ v(\,\cdot\,,-T)=0.\end{cases}$$

Therefore up to the appropriate translation of g, to solve a BVP on $[-T,\infty[$ is equivalent to solve a BVP on $[0,\infty[$. A useful consequence of this remark is the well-posedness of the BVP posed on $\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}_t$.

Corollary 3.8. — *Consider the boundary value problem*

$$\begin{cases} i\partial_t u+\Delta u=0,\\ B(u|_{y=0},\partial_y u|_{y=0})=g, \quad (x,y,t)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}_t.\\ \lim_{t\to-\infty}u(\,\cdot\,,t)=0.\end{cases}\tag{3.19}$$

If B satisfies the Kreiss–Lopatinskii condition (3.4) *and $g\in\mathcal{H}^s(\mathbb{R})$, $0\leqslant s\leqslant 2$, there exists a unique solution $u\in C(\mathbb{R},H^s)$, moreover it satisfies estimate* (3.9) *with $\mathbb{R}_t^+$ replaced by $\mathbb{R}_t$.*

If g vanishes on $\mathbb{R}^{d-1}\times]-\infty,T]$, then so does u on $(\mathbb{R}^{d-1}\times\mathbb{R}^+)\times]-\infty,T]$.

Proof. — Fix $g\in\mathcal{H}^s(\mathbb{R})$. By density there exists $g_n\in C_c^\infty(\mathbb{R}^{d-1}\times\mathbb{R}_t)$ such that

$$\|g-g_n\|_{\mathcal{H}^s(\mathbb{R}_t)}\longrightarrow_n 0.$$

We can assume that g_n is supported in $[-T_n,\infty[$, and T_n is increasing. By translation invariance in time, there exists a smooth solution u_n to

$$\begin{cases} i\partial_t u_n+\Delta u_n=0,\\ B(u_n|_{y=0},\partial_y u_n|_{y=0})=g_n, \quad (x,y,t)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times[-T_n,\infty[.\\ u_n(\,\cdot\,,-T_n)=0.\end{cases}\tag{3.20}$$

As was pointed out in the proof of estimate (3.9), setting $u_n|_{]-\infty,-T_n[}=0$ defines a smooth extension of u_n, which solves the boundary value problem with $g_n|_{]-\infty,-T_n]}=0$.

Let $n\geqslant p$, then $\operatorname{supp}(u_n-u_p)\subset[-T_n,\infty[$ and a priori estimate (3.9) implies

$$\|u_n-u_p\|_{L^\infty(\mathbb{R},H^s(\mathbb{R}))}\lesssim\|g_n-g_p\|_{\mathcal{H}^s([-T_n,\infty[)}\lesssim\|g_n-g_p\|_{\mathcal{H}^s(\mathbb{R})}.$$

This implies that (u_n) converges to some $u \in C_tH^s$. Moreover

$$\forall\ n \in \mathbb{N}, \lim_{-\infty} \|u_n(t)\|_{H^s} = 0 \Rightarrow \lim_{-\infty} \|u(t)\|_{H^s} = 0.$$

The other estimates can be obtained as for Proposition 3.3.

In the case where g is supported in $\mathbb{R}^{d-1} \times [T, \infty[$, it suffices to observe that we can assume that g_n is supported in $\mathbb{R}^{d-1} \times [T + 1/n, \infty[$, and use the previous observation on the support of smooth solutions. □

3.2. Estimates for the Cauchy problem

3.2.1. Pure Cauchy problem

We recall (see (3.4)) that the Kreiss–Lopatinskii condition reads $\alpha \leqslant |b_1 - \sqrt{|\xi|^2 + \delta} b_2| \leqslant \beta$, therefore we define Λ the Fourier multiplier of symbol $\sqrt{||\xi|^2 + \delta|}$ that acts on functions defined on $\mathbb{R}^{d-1} \times \mathbb{R}_t$. In order to control $\|B(u|_{y=0}, \partial_y u|_{y=0})\|_{\mathcal{H}^s}$ it suffices to control $\|u|_{y=0}\|_{\mathcal{H}^s}$ and $\|\Lambda^{-1}\partial_y u|_{y=0}\|_{\mathcal{H}^s}$.

Proposition 3.9. — *The solution $e^{it\Delta}u_0$ of the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (x, y, t) \in \mathbb{R}^{d+1},$$

satisfies the following estimates for $0 \leqslant s \leqslant 2$:

$$\forall\ p > 2, \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad \|u\|_{L^p(\mathbb{R}_t, B^s_{q,2}) \cap B^{s/2}_{p,2}(\mathbb{R}_t, L^q)} \lesssim \|u_0\|_{H^s}, \tag{3.21}$$

$$\|u|_{y=0}\|_{\mathcal{H}^s(\mathbb{R}_t)} + \|\Lambda^{-1}(\partial_y u|_{y=0})\|_{\mathcal{H}^s(\mathbb{R}_t)} \lesssim \|u_0\|_{H^s}. \tag{3.22}$$

Proof. — The $L^pB^s_{q,2}$ estimate in (3.21) is the classical Strichartz estimate, see e.g. [13, Corollary 2.3.9]. Since $\partial_t u = i\Delta u$, $\|u\|_{W^{1,p}L^q} \lesssim \|u_0\|_{H^2}$, and the $B^{s/2}_{p,2}L^q$ bound follows by interpolation. For the trace estimate, we observe that the solution of the Cauchy problem satisfies

$$\forall\ (x, y, t) \in \mathbb{R}^{d+1},$$
$$(e^{it\Delta}u_0)(x, y) = \frac{1}{(2\pi)^d} \iint e^{-i(|\xi|^2+\eta^2)t} e^{ix\cdot\xi + iy\eta} \widehat{u_0}(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta,$$
$$\Rightarrow (e^{it\Delta}u_0)(x, 0) = \frac{1}{(2\pi)^d} \iint e^{-i(|\xi|^2+\eta^2)t} e^{ix\cdot\xi} \widehat{u_0}(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta.$$

We consider the integral over $\eta \geqslant 0$, and use the change of variables $\delta = -(\eta^2 + |\xi|^2)$

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^+} & e^{-i(|\xi|^2+\eta^2)t} e^{ix\cdot\xi} \widehat{u_0}(\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{-|\xi|^2} e^{i\delta t} e^{ix\cdot\xi} \frac{\widehat{u_0}\big(\xi, \sqrt{||\xi|^2+\delta|}\big)}{\sqrt{||\xi|^2+\delta|}} \mathrm{d}\delta \mathrm{d}\xi \\
&:= (2\pi)^d \mathcal{F}_{x,t}^{-1}(\psi).
\end{aligned}$$

Then for $s \geqslant 0$, reversing the change of variable

$$\begin{aligned}
&\|\mathcal{F}_{x,t}^{-1}(\psi)\|_{\mathcal{H}^s(\mathbb{R}_t)}^2 \\
&\quad = \int_{\mathbb{R}^d} \sqrt{|\delta+|\xi|^2|} (1+|\delta|+|\xi|^2)^s |\psi(\xi,\delta)|^2 \mathrm{d}\delta \mathrm{d}\xi \\
&\quad = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{-|\xi|^2} \sqrt{|\delta+|\xi|^2|} (1+|\delta|+|\xi|^2)^s \left| \frac{\widehat{u_0}\big(\xi, \sqrt{||\xi|^2+\delta|}\big)}{\sqrt{||\xi|^2+\delta|}} \right|^2 \mathrm{d}\delta \mathrm{d}\xi \\
&\quad \lesssim \iint_{\mathbb{R}^{d-1}\times\mathbb{R}^+} (1+|\xi|^2+|\eta|^2)^s |\widehat{u_0}(\xi,\eta)|^2 \mathrm{d}\eta \mathrm{d}\xi \sim \|u_0\|_{H^s}^2.
\end{aligned}$$

Symmetric computations can be carried for $\eta \in \mathbb{R}^-$, we conclude

$$\|e^{it\Delta} u_0|_{y=0}\|_{\mathcal{H}^s(\mathbb{R}_t)} \lesssim \|u_0\|_{H^s}.$$

The estimate for $\|\Lambda^{-1}(\partial_y u|_{y=0})\|_{\mathcal{H}^s}$ is done similarly by writing

$$(\partial_y u|_{y=0}) = \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d} e^{-i(|\xi|^2+\eta^2)t} e^{ix\cdot\xi} i\eta \widehat{u_0}(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta,$$

and using the fact that after the change of variable, the η factor becomes $\sqrt{||\xi|^2+\delta|}$, so that it balances precisely the symbol of Λ^{-1}. □

Remark 3.10. — Inequality (3.22) is a multi-dimensional variant (not new) of the sharp Kato-smoothing property that we already mentioned in the introduction. It is clear that the argument also works for $s \geqslant 2$.

3.2.2. Pure forcing problem

We consider $u = \int_0^t e^{i(t-s)\Delta} f(s) \mathrm{d}s$ solution of

$$\begin{cases} i\partial_t u + \Delta u = if, \\ u(\,\cdot\,, 0) = 0. \end{cases} \quad (x,t) \in \mathbb{R}^{d+1}.$$

Our aim is to obtain an estimate of the kind $\|u|_{y=0}\|_{\mathcal{H}^s(\mathbb{R}_t)} \lesssim \|f\|_{L^1_t H^s}$. If the integral $\int_0^t$ was replaced by $\int_0^\infty$, we might simply apply Proposition 3.9

$$
\begin{aligned}
u|_{y=0} = e^{it\Delta}\Big(\int_0^\infty e^{-is\Delta} f(s)\mathrm{d}s\Big)\Big|_{y=0}&,\\
\Rightarrow \Big\| e^{it\Delta}\Big(\int_0^\infty e^{-is\Delta} f(s)\mathrm{d}s\Big)\Big|_{y=0}\Big\|_{\mathcal{H}^s}&\\
\leqslant \Big\|\int_0^\infty e^{-is\Delta} f(s)\mathrm{d}s\Big\|_{H^s} &\lesssim \|f\|_{L^1_t H^s}.
\end{aligned}
$$

Combined with Proposition 3.9, this implies $\|\int_0^\infty e^{i(t-s)\Delta} f(s)\mathrm{d}s|_{y=0}\|_{\mathcal{H}^s} \lesssim \|f\|_{L^1_t H^s}$. Unfortunately, due to the intricate nature of $\mathcal{H}^s$, which measures both time and space regularity, we can not apply the celebrated Christ–Kiselev lemma to deduce bounds for $\int_0^t e^{i(t-s)\Delta} f(s)\mathrm{d}s|_{y=0}$ (see also Remark 3.12 for a discussion on this issue). Nevertheless, we have the following proposition.

Proposition 3.11. — *For $0 < s < 2$, (p,q), and (p_1,q_1) admissible pairs, we have*

$$
\Big\|\int_0^t e^{i(t-s)\Delta} f(s)\mathrm{d}s\Big\|_{L^{p_1}(\mathbb{R}_t, B^s_{q_1,2})\cap B^s_{p_1,2}(\mathbb{R}_t, L^{q_1})} \lesssim \|f\|_{L^{p'}(\mathbb{R}_t, B^s_{q',2})\cap B^{s/2}_{p',2}(\mathbb{R}_t, L^{q'})}, \tag{3.23}
$$

$$
\Big\|\int_0^t e^{i(t-\tau)\Delta} f(\tau)\mathrm{d}\tau\Big|_{y=0}\Big\|_{\mathcal{H}^s(\mathbb{R}_t)} \lesssim \|f\|_{L^{p'}(\mathbb{R}_t, B^s_{q',2})\cap B^{s/2}_{p',2}(\mathbb{R}_t, L^{q'})}, \tag{3.24}
$$

$$
\Big\|\Lambda^{-1}\Big(\partial_y \int_0^t e^{i(t-\tau)\Delta} f(\tau)\mathrm{d}\tau\Big)\Big|_{y=0}\Big\|_{\mathcal{H}^s(\mathbb{R}_t)} \lesssim \|f\|_{L^{p'}(\mathbb{R}_t, B^s_{q',2})\cap B^{s/2}_{p',2}(\mathbb{R}_t, L^{q'})}. \tag{3.25}
$$

Proof. — We start with (3.23) and (3.24). As a first reduction, we point out that according to the usual Strichartz estimates (see [13, Theorem 2.3.3

to Corollary 2.3.9]) and Proposition 3.9

$$\begin{aligned}
\left\| e^{it\Delta} \int_{-\infty}^{0} e^{-is\Delta} f(s)\mathrm{d}s|_{y=0} \right\|_{\mathcal{H}^s(\mathbb{R}_t)} &\lesssim \left\| \int_{-\infty}^{0} e^{-is\Delta} f(s)\mathrm{d}s \right\|_{H^s(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \\
&\lesssim \|f\|_{L_t^{p'} B^s_{q',2}}, \\
\left\| e^{it\Delta} \int_{-\infty}^{0} e^{-is\Delta} f(s)\mathrm{d}s \right\|_{L_t^p B^s_{q,2}} &\lesssim \left\| \int_{-\infty}^{0} e^{-is\Delta} f(s)\mathrm{d}s \right\|_{H^s} \\
&\lesssim \|f\|_{L_t^{p'} B^s_{q',2}}, \\
\text{and} \quad \left\| \partial_t \int_{-\infty}^{0} e^{i(t-s)\Delta} f(s)\mathrm{d}s \right\|_{L^pL^q} &= \left\| \int_{-\infty}^{0} e^{i(t-s)\Delta} \Delta f(s)\mathrm{d}s \right\|_{L_t^pL^q} \\
&\lesssim \|f\|_{L_t^{p'} W^{2,q'}}.
\end{aligned}$$

So, by interpolation

$$\left\| e^{it\Delta} \int_{-\infty}^{0} e^{-is\Delta} f(s)\mathrm{d}s \right\|_{B^{s/2}_{p,2} L^q} \lesssim \|f\|_{L^{p'} B^s_{q',2}}.$$

Therefore, it suffices to estimate $\int_{-\infty}^{t} e^{i(t-s)\Delta} f(s)\mathrm{d}s$, which is the solution of $i\partial_t u + \Delta u = if$, $\lim_{-\infty} u = 0$. In this case, the analog of (3.23) is also a consequence of the classical results in [13], and the analog of (3.24) relies on the following duality argument.

The case $s = 0$. — We fix $g \in \mathcal{H}'(\mathbb{R})$ and denote v the solution of the backward Neuman boundary value problem

$$\begin{cases} i\partial_t v + \Delta v = 0, \\ \lim_{+\infty} v(t) = 0 & (x,y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}_t. \\ \partial_y v|_{y=0} = g \end{cases}$$

According to the discussion in Section 3.1.2 and Corollary 3.8, this problem is well-posed and the solution is in $\cap_{(\rho,\gamma)\text{ admissible}} L_t^{\rho} L^{\gamma}$. We extend v on $\mathbb{R}^d \times \mathbb{R}_t$ by reflection

$$v(x,y,t) = \begin{cases} v(x,y,t), & y \geqslant 0, \\ v(x,-y,t), & y < 0. \end{cases}$$

In particular, $v|_{y=0^-}=v|_{y=0^+}$ and $\partial_y v|_{y=0^-}=-\partial_y v|_{y=0^+}=-g$. Using a density argument, the following integration by part is justified:

$$\begin{aligned}\int_{\mathbb{R}_t}\int_{\mathbb{R}^d} if\overline{v}\mathrm{d}x\mathrm{d}y\mathrm{d}t &= \int_{\mathbb{R}_t}\int_{\mathbb{R}^d} u\overline{i\partial_t v+\Delta v}\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\quad + \int_{\mathbb{R}_t}\int_{\mathbb{R}^{d-1}} -u|_{y=0}\overline{\partial_y v|_{y=0^-}}+u|_{y=0}\overline{\partial_y v|_{y=0^+}}\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\mathbb{R}_t}\int_{\mathbb{R}^{d-1}} \partial_y u|_{y=0}\overline{v|_{y=0^-}}-\partial_y u|_{y=0}\overline{v|_{y=0^+}}\mathrm{d}x\mathrm{d}t \\ &= 2\int_{\mathbb{R}_t}\int_{\mathbb{R}^{d-1}} u|_{y=0}\overline{g}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Taking the supremum over $\|g\|_{\mathcal{H}'}=1$, by duality we deduce

$$\|u|_{y=0}\|_{\mathcal{H}(\mathbb{R}_t)}\leqslant \frac{1}{2}\|f\|_{L^{p'}(\mathbb{R}_t,L^{q'})}\sup_{\|g\|_{\mathcal{H}'}=1}\|v\|_{L_t^pL^q}\lesssim \|f\|_{L_t^{p'}L^{q'}}. \tag{3.26}$$

Higher order estimates. — We recall that Δ' is the Laplacian in the x variable. If $f\in L_t^{p'}W^{2,q'}$, then $\Delta' u$ is the solution of

$$i\partial_t\Delta' u+\Delta\Delta' u=\Delta' f,\quad \lim_{-\infty}\Delta' u(t)=0,$$

therefore the estimate for $s=0$ implies $\|\Delta' u|_{y=0}\|_{\mathcal{H}}\lesssim\|\Delta' f\|_{L_t^{p'}L^{q'}}\lesssim \|f\|_{L_t^{p'}W^{2,q'}}$. By interpolation we get for $0<s<2$

$$\int_{\mathbb{R}^d}\sqrt{||\xi|^2+\delta|}(1+|\xi|^{2s})|\widehat{u|_{y=0}}|^2\mathrm{d}\delta\mathrm{d}\xi\lesssim\|f\|^2_{L_t^{p'}B^s_{q',2}}. \tag{3.27}$$

Similarly, if $f\in W^{1,p'}L^{q'}$, then $\partial_t u$ satisfies

$$i\partial_t\partial_t u+\Delta\partial_t u=\partial_t f,\quad \lim_{-\infty}\partial_t u(t)=0,$$

the estimate for $s=0$ gives $\|\partial_t u|_{y=0}\|_{\mathcal{H}}\lesssim\|\partial_t f\|_{L_t^{p'}L^{q'}}$ and by interpolation again

$$\int_{\mathbb{R}^d}\sqrt{||\xi|^2+\delta||}(1+|\delta|^s)|\widehat{u|_{y=0}}|^2\mathrm{d}\delta\mathrm{d}\xi\lesssim\|f\|^2_{B^s_{p',2}L^{q'}}. \tag{3.28}$$

Combining (3.27) and (3.28) implies for $0<s<2$

$$\|u|_{y=0}\|_{\mathcal{H}^s}\lesssim\|f\|_{B^{s/2}_{p',2}L^{q'}\cap L_t^{p'}B^s_{q',2}}.$$

Estimate (3.25). — For $s=0$, we only sketch the similar duality argument: consider v solution of the backward BVP with Dirichlet boundary condition g, and extend it on $\mathbb{R}^d \times \mathbb{R}_t$ as an odd function in the y variable. The same computations as for (3.24) lead to

$$\sup_{g\in\mathcal{H}(\mathbb{R}_t)} \int_{\mathbb{R}_t\times\mathbb{R}^{d-1}} \partial_y u|_{y=0}\overline{g}\mathrm{d}x\mathrm{d}t \lesssim \|f\|_{L^{p'}L^{q'}}\|g\|_{\mathcal{H}(\mathbb{R}_t)},$$
$$\Rightarrow \|\partial_y u|_{y=0}\|_{\mathcal{H}'(\mathbb{R}_t)} \lesssim \|f\|_{L^{p'}(L^{q'})},$$

according to (3.7), this estimate is precisely (3.25) for $s=0$. The case $0 < s \leqslant 2$ follows from the same differentiation/interpolation argument. □

Remark 3.12. — The space $L^{p'}B^s_{q',2}\cap B^{s/2}_{p',2}L^{q'}$ seems natural at least scaling wise. In the case of dimension 1, Holmer [17] managed to prove (3.24) with only $\|f\|_{L^{p'}W^{s,q'}}$ in the right hand side under the condition $s<1/2$. For $s\geqslant 1/2$, it is convenient to add some time regularity.

A (very formal) argument is as follows: suppose that u is a smooth solution of $i\partial_t u+\Delta u = f, u|_{t=0}=0$. If $u|_{y=0}\in\mathcal{H}^2$, then $f|_{y=0} = i\partial_t(u|_{y=0}) + (\Delta u)|_{y=0}$, where $i\partial_t g\in\mathcal{H}$ and $w=\Delta u$ satisfies $i\partial_t w+\Delta w=\Delta f, w|_{t=0}=0$, so that the a priori estimate for $s=0$ gives $(\Delta u)|_{y=0}\in\mathcal{H}$. Therefore $f|_{t=0}$ should belong to $\mathcal{H}$, which can not be deduced from $f\in L^1_tH^2$.

Now if $f\in W^{1,1}_tL^2\cap L^1_tH^2$, from the numerology of Sobolev embeddings one expects

$$f\in W_t^{3/4,1}H^{1/2} \Rightarrow \text{ “almost” } f|_{y=0}\in W_t^{3/4,1}L^2 \hookrightarrow H_t^{1/4}L^2,$$
$$f\in W_t^{1/2,1}H^{1} \Rightarrow \text{ “almost” } f|_{y=0}\in W_t^{1/2,1}H^{1/2} \hookrightarrow L_t^{2}H^{1/2},$$

in particular, $f|_{y=0}$ almost belongs to $H^{1/4}L^2\cap L^2H^{1/2}\hookrightarrow\mathcal{H}$.

3.3. Proof of Theorems 1.3 and 1.4

Up to using regularized data $u_0^n\in H^2_0, f_n\in W_0^{1,p'}L^{q'}\cap L^{p'}W_0^{2,q'}, g_n\in\mathcal{H}^2_0$ all quantities are well-defined, so we mainly focus on the issue of a priori estimates in this paragraph.

Proof of Theorem 1.3. — First we point out a confusion to avoid for the operator B: if $B_{\mathbb{R}}$ is the Fourier multiplier with same symbol as B, P_0 the zero extension to $t\leqslant 0$, and R the restriction to $t\geqslant 0$, we have

$$B = R\circ B_{\mathbb{R}}\circ P_0.$$

We recall that P_0 (resp. R) is continuous $\mathcal{H}^s_0(\mathbb{R}^+)\to\mathcal{H}^s(\mathbb{R})$, $s\neq 1/2$ (resp. $\mathcal{H}^s(\mathbb{R})\to\mathcal{H}^s_0(\mathbb{R}^+)$), and by duality $P_0:\mathcal{H}'(\mathbb{R}^+)\to\mathcal{H}'(\mathbb{R}), R:\mathcal{H}'(\mathbb{R})\to\mathcal{H}'(\mathbb{R}^+)$ are continuous.

The case $s=0$. — We follow the method and notations from the beginning of Section 3: let v the solution of the Cauchy problem, w the solution of (3.1), that is

$$\begin{cases} i\partial_t v+\Delta v=f, \\ v|_{t=0}=u_0, \end{cases} \quad \begin{cases} i\partial_t w+\Delta w=0, \\ w|_{t=0}=0, \\ B(w|_{y=0},\partial_y w|_{y=0})=g-B(v|_{y=0},\partial_y v|_{y=0}). \end{cases}$$

Since $\|v\|_{L^pL^q}\lesssim \|u_0\|_{L^2}+\|f\|_{L_t^{p_1'}L^{q_1'}}$ (Propositions 3.9 and 3.11), it suffices to check that w exists and $\|w\|_{L_t^pL^q}\lesssim \|u_0\|_{L^2}+\|f\|_{L_t^{p_1'}L^{q_1'}}+\|g\|_{\mathcal{H}}$. Let us write $B_{\mathbb{R}}(a,b)=B_{1,\mathbb{R}}(a)+B_{2,\mathbb{R}}(b)$. According to the Kreiss–Lopatinskii condition the symbols b_1 and $\sqrt{|\delta+|\xi|^2|}b_2$ are bounded uniformly in (δ,ξ). From the estimates of Section 3.2, $\|v|_{y=0}\|_{\mathcal{H}(\mathbb{R}_t)}+\|\partial_y v|_{y=0}\|_{\mathcal{H}'(\mathbb{R}_t)}\lesssim \|u_0\|_{L^2}$, this implies

$$\|B_{1,\mathbb{R}}\circ P_0\circ R(u|_{y=0})\|_{\mathcal{H}(\mathbb{R}_t)}\lesssim \|P_0\circ Ru|_{y=0}\|_{\mathcal{H}(\mathbb{R}_t)}\lesssim \|u_0\|_{L^2}.$$

and

$$\begin{aligned} &\|B_{2,\mathbb{R}}\circ P_0\circ R(\partial_y u|_{y=0})\|^2_{\mathcal{H}(\mathbb{R}_t)} \\ &\qquad = \iint |b_2(\xi,\delta)|^2\sqrt{||\xi|^2+\delta|}\,|\mathcal{F}_{x,t}\big(P_0\circ R(\partial_y u|_{y=0})\big)|^2\mathrm{d}\xi\mathrm{d}\delta \\ &\qquad \lesssim \iint (||\xi|^2+\delta|)^{-1/2}\,|\mathcal{F}_{x,t}\big(P_0\circ R(\partial_y u|_{y=0})\big)|^2\mathrm{d}\xi\mathrm{d}\delta \\ &\qquad = \|P_0\circ R(\partial_y u|_{y=0})\|^2_{\mathcal{H}'(\mathbb{R}_t)}\lesssim \|u_0\|_{L^2}, \end{aligned}$$

We can now apply Proposition 3.3 which gives the existence of w with the expected Strichartz estimate, then $v+w$ solves (1.1).

The causality follows by taking the difference of two solutions and using the property on support of solutions in Corollary 3.8.

The case $s=2$. — Here we assume $f\in L_t^{p'}W^{2,q'}\cap W_t^{1,p'}L^{q'}$, $u_0\in H_0^2(\mathbb{R}^{d-1}\times\mathbb{R}^+), g\in \mathcal{H}_0^2(\mathbb{R}^+)$. According to Proposition 3.3, we can use again a superposition principle provided

$$\begin{aligned} &B(v|_{y=0},\partial_y v|_{y=0})\in \mathcal{H}_0^2(\mathbb{R}^+) \\ &\qquad\qquad \text{or equivalently } B_{\mathbb{R}}\circ P_0\circ R(v|_{y=0},\partial_y v|_{y=0})\in \mathcal{H}^2(\mathbb{R}), \end{aligned}$$

since for any g, $B_{\mathbb{R}}\circ P_0 g$ is supported in $t\geqslant 0$. Now $u_0\in H_0^2$, thus $v|_{y=t=0}=u_0|_{y=0}=0$, therefore estimate (3.22) and Corollary 2.4 imply

$$\|B_{1,\mathbb{R}}\circ P_0\circ R(v|_{y=0})\|_{\mathcal{H}^2(\mathbb{R})}\lesssim \|u_0\|_{H^2}+\|f\|_{L^{p'}W^{2,q'}\cap W^{1,p'}L^{q'}}.$$

Moreover, estimate (3.22) also implies

$$\begin{aligned}\|\Delta'\partial_y v|_{y=0}\|^2_{\mathcal{H}'(\mathbb{R}_t)} + \|\partial_t\partial_y v|_{y=0})\|^2_{\mathcal{H}'(\mathbb{R}_t)}&\\ \sim \iint_{\mathbb{R}^d} \frac{|\widehat{\partial_y v|_{y=0}}|^2}{\sqrt{||\xi|^2+\delta|}}(|\xi|^2+|\delta|)^2\mathrm{d}\xi\mathrm{d}\delta&\\ \lesssim \|u_0\|^2_{H^2}6 + \|f\|^2_{L^{p'}W^{2,q'}\cap W^{1,p'}L^{q'}}.&\end{aligned}$$

But since $u_0 \in H^2_0(\mathbb{R}^{d-1}\times\mathbb{R}^+)$, $\partial_y v|_{y=t=0} = \partial_y u_0|_{y=0} = 0$ thus

$$\begin{aligned}\partial_t P_0\circ R(\partial_y v|_{y=0}) &= P_0\circ R(\partial_t\partial_y v|_{y=0}),\\ \Delta' P_0\circ R(\partial_y v|_{y=0}) &= P_0\circ R(\Delta'\partial_y v|_{y=0}).\end{aligned}$$

By continuity of $P_0\circ R:\mathcal{H}'\to\mathcal{H}'$, $(P_0\circ R(\partial_t\partial_y v|_{y=0}), P_0\circ R(\Delta'\partial_y v|_{y=0}))\in(\mathcal{H}')^2$. Finally, using the boundedness of $b_2\sqrt{||\xi|^2+\delta|}$ we get

$$\begin{aligned}\|B_{2,\mathbb{R}}\circ P_0\circ R(\partial_y v|_{y=0})\|_{\mathcal{H}^2}&\\ \lesssim \|\partial_t P_0\circ R(\partial_y v|_{y=0})\|_{\mathcal{H}'} + \|\Delta' P_0\circ R(\partial_y v|_{y=0})\|_{\mathcal{H}'}&\\ \lesssim \|u_0\|_{H^2} + \|f\|_{L^{p'}W^{2,q'}\cap W^{1,p'}L^{q'}}&\end{aligned}$$

which implies as expected $B_{2,\mathbb{R}}\circ P_0\circ R(\partial_y v|_{y=0})\in\mathcal{H}^2(\mathbb{R})$.

The case $0<s<2$. — After fixing an extension operator, since $(u_0,f)\to B(v|_{y=0},\partial_y v|_{y=0})$ is continuous $L^2\times L^{p'}_t L^{q'}\to\mathcal{H}(\mathbb{R}^+)$ and $H^2_0\times\big(W^{1,p'}_t L^{q'}\cap L^{p'}_t W^{2,q'}\big)\to\mathcal{H}^2_0(\mathbb{R}^+)$, the general case follows by interpolation. ☐

Proof of Theorem 1.4. — Let $s\in[0,2]$. We fix extensions of u_0, f to $y\leqslant 0$ and solve

$$\begin{cases} i\partial_t v+\Delta v = f,\\ v|_{t=0}=u_0,\end{cases}\qquad (x,y,t)\in\mathbb{R}^d\times\mathbb{R}.$$

From the estimates for the Cauchy problem, $v|_{y=0}\in\mathcal{H}^s(\mathbb{R})$. Consider the BVP

$$\begin{cases} i\partial_t w+\Delta w=0,\\ w|_{t=0}=0,\\ w|_{y=0}=g-v|_{y=0},\end{cases}\qquad (x,y,t)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}^+. \tag{3.29}$$

If $s>1/2$, the trace $v|_{y=t=0}=u_0|_{y=0}$ is well defined and belong to $H^{s-1/2}$. Moreover the compatibility condition gives $(g-v|_{y=0})|_{t=0}=g|_{t=0}-u_0|_{y=0}=0$ so that for $s\in[0,2]\setminus\{1/2\}$, $g-v|_{y=0}\in\mathcal{H}^s_0(\mathbb{R}^+)$. From Proposition 3.3 there exists a unique solution $w\in C(\mathbb{R}^+_t,H^s)$ to (3.29). Now $u:=v|_{y\geqslant 0}+w$ is a solution of (1.1), it satisfies the expected estimate because according to Propositions 3.3, 3.9 and 3.11, v and w do.

In the case $s=1/2$, we first note that

$$\forall\ t\geqslant 0,\int_0^t e^{i(t-s)\Delta}f(s)\mathrm{d}s=\int_0^t e^{i(t-s)\Delta}P_0\circ R(f(s))\mathrm{d}s,$$

and since $p_1'\leqslant 2$, $P_0\circ R(f)\in B^{1/4}_{p_1',2}(\mathbb{R}_t,L^{q_1'})\cap L^{p_1'}(\mathbb{R}_t,B^{1/2}_{q_1',2})$. From Proposition 3.11 $\int_0^t e^{i(t-s)\Delta}P_0\circ Rf(s)\mathrm{d}s|_{y=0}\in \mathcal{H}^{1/2}(\mathbb{R})$, and clearly vanishes for $t\leqslant 0$, therefore $R\big(\int_0^t e^{i(t-s)\Delta}f(s)\mathrm{d}s|_{y=0}\big)\in \mathcal{H}^{1/2}_{00}(\mathbb{R}^+)$ (by Definition 2.5 of $\mathcal{H}^{1/2}_{00}$). In order to solve (3.29), we are left to prove that if the compatibility condition is satisfied, then $(e^{it\Delta}u_0)|_{y=0}-g\in\mathcal{H}^{1/2}_{00}(\mathbb{R}^+)$. From the previous estimates, we know $(e^{it\Delta}u_0)|_{y=0}-g\in\mathcal{H}^{1/2}(\mathbb{R}^+)$, and we must check condition (2.2), that is:

$$\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}\frac{\big|(e^{it\partial_y^2}u_0)|_{y=0}-e^{-it\Delta'}g(x,t)\big|^2}{t}\mathrm{d}t\mathrm{d}x<\infty.$$

Using the change of variable $t\to\sqrt{t}$, the compatibility condition (1.4) ensures

$$\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}\frac{\big|u_0(x,\sqrt{t})-e^{-it\Delta'}g(x,t)\big|^2}{t}\mathrm{d}t\mathrm{d}x<\infty.$$

Therefore we only need to estimate $u_0(x,\sqrt{t})-(e^{it\partial_y^2}u_0)|_{y=0}$. We use the following interpolation argument: if $u_0\in H^1(\mathbb{R}^d)$, the identity $u_0(x,\sqrt{t})-(e^{it\partial_y^2}u_0)|_{y=0}=u_0(x,\sqrt{t})-u_0(x,0)+u_0(x,0)-(e^{it\partial_y^2}u_0)|_{y=0}$ makes sense, and thanks to Hardy's inequality

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}&\frac{|u_0(x,\sqrt{t})-u_0(x,0)|^2}{t^{3/2}}\mathrm{d}t\mathrm{d}x\\ &=2\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}\frac{|u_0(x,y)-u_0(x,0)|^2}{y^2}\mathrm{d}y\mathrm{d}x\lesssim\|\partial_yu_0\|_{L^2}^2.\end{aligned}$$

Similarly, the sharp Kato smoothing (3.22) implies $\|(e^{it\partial_y^2}u_0)|_{y=0}\|_{\dot{H}^{3/4}_tL^2}\lesssim\|u_0\|_{H^1}$ so that the (fractional) Hardy's inequality gives

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}&\frac{|(e^{it\partial_y^2}u_0)|_{y=0}-u_0(x,0)|^2}{t^{3/2}}\mathrm{d}t\mathrm{d}x\\ &\lesssim\|(e^{it\partial_y^2}u_0)|_{y=0}\|^2_{\dot{H}^{3/4}_tL^2}\lesssim\|u_0\|^2_{H^1}.\end{aligned}$$

On the other hand, we have by a similar simpler argument

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}}&\frac{|u_0(x,\sqrt{t})|^2+|(e^{it\partial_y^2}u_0)|_{y=0}|^2}{t^{1/2}}\\ &\lesssim\|u_0\|^2_{L^2}+\|(e^{it\partial_y^2}u_0)|_{y=0}\|^2_{\dot{H}^{1/4}L^2}\lesssim\|u_0\|^2_{L^2}.\end{aligned}$$

We deduce by interpolation

$$\iint_{\mathbb{R}^+\times\mathbb{R}^{d-1}} \frac{|u_0(x,\sqrt{t}) - (e^{it\partial_y^2}u_0)|_{y=0}|^2}{t}\mathrm{d}t \lesssim \|u_0\|_{H^{1/2}}^2.$$

This implies $(e^{it\Delta}u_0)|_{y=0} - g \in \mathcal{H}_{00}^{1/2}$. We can then end the proof[(4)] as for the case $s \neq 1/2$. □

4. Local and global existence

For simplicity, we only consider nonlinearities of the type $\varepsilon|u|^{a-1}u$, $a > 1, \varepsilon \in \{-1,1\}$ Dirichlet boundary conditions, $u_0 \in H^1$. More general nonlinearities and indices of regularity can be treated with similar methods, see from [13, Chapter 4].

Since so far we have always considered global solution, some clarifications for local solutions of nonlinear problems are required. For P_T an extension operator as in Lemma 2.11, consider the map $\Phi : v \in L^\infty(\mathbb{R}_t^+, H^1) \mapsto \Phi(v)$ the solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon|P_T v|^{a-1}P_T v, \\ u|_{t=0} = u_0, \\ u|_{y=0} = g. \end{cases} \tag{4.1}$$

If $1 < a < 1+4/(d-2)$, $2 < a+1 < 2d/(d-2)$ thus by Sobolev's embedding $v \in L_t^\infty L^{a+1}$. If $(p, a+1)$ is admissible, we deduce $|P_T v|^{a-1}P_T v \in L_t^{p'} \cap L^{(a+1)'}$, and according to Theorem 1.4 Φ is well-defined $L_t^\infty H^1 \to C_t L^2$.

We say that u is a local solution on $[0,T]$ of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon|u|^{a-1}u, \\ u|_{t=0} = u_0, \\ u|_{y=0} = g. \end{cases} \tag{4.2}$$

if u is the restriction on $[0,T]$ of a fixed point of Φ.

Theorem 4.1. — *Let $(u_0, g) \in H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+) \times \mathcal{H}^1(\mathbb{R}_t^+)$ such that $u_0|_{y=0} = g|_{t=0}$, $1 < a < 1 + 4/(d-2)$. The IBVP (4.2) has a unique maximal solution in $C([0,T_{\max}), H^1)$. If $T_{\max} < \infty$, $\overline{\lim}_{T_{\max}}\|u(t)\|_{H^1} = \infty$. For any T such that u exists on $[0,T]$ and (p,q) an admissible pair, then $u \in L^p([0,T], W^{1,q}) \cap B_{p,2}^{1/2}([0,T], L^q)$.*

(4) Note that u_0 was extended to $\mathbb{R}^n$, but the argument clearly independent of the choice of the extension operator.

If moreover $1+4/d \leqslant a$, *there exists* $\varepsilon > 0$ *such that if* $\|u_0\|_{H^1}+\|g\|_{\mathcal{H}^1} < \varepsilon$ *then the solution is global and for* (p,q) *admissible,* $u \in L^p(\mathbb{R}_t^+, W^{1,q}) \cap B^{1/2}_{p,2}(\mathbb{R}_t^+, L^q)$.

Proof. — We use the convenient notation $L^{1/p} = L^p$. Let us recall shortly the classical Kato's argument, with some modifications to handle time regularity.

Local existence. — For M to fix later, we set S the closed ball of radius M in $L^\infty(\mathbb{R}^+, H^1) \cap L^p(\mathbb{R}^+, W^{1,q}) \cap B^{1/2}_{p,2}(\mathbb{R}^+, L^q)$, $q = a+1$, (p,q) admissible. We use on S the following distance

$$d(u,v) = \|u-v\|_{L^\infty(\mathbb{R}^+,L^2)\cap L^q(\mathbb{R}^+,L^r)}.$$

(S,d) is a complete set (see e.g. [13, Section 4.4]). We fix an extension operator P_T as in Lemma 2.11: such that for any $v \in S$,

$$\begin{aligned}(4.3)\quad & \operatorname{supp}(P_Tv) \subset \mathbb{R}^{d-1}\times\mathbb{R}^+\times[-T,2T],\\ & \|P_Tv\|_{B^{1/2}_{p,2}(\mathbb{R}_t,L^q)} \lesssim T^{1/p-1/2}\|v\|_{B^{1/2}_{p,2}(\mathbb{R}_t,L^q)},\end{aligned}$$

and we construct a fixed point to Φ, with Φ defined at (4.1).

Combining the inclusions $B^1_{q,2} \subset W^{1,q}$, $B^1_{q',2} \supset W^{1,q'}$ (see [6, Theorem 6.4.4]), with the linear estimates of Theorem 1.4 we get

$$\begin{aligned}&\|\Phi(v)\|_{L^\infty_t H^1\cap L^p_t W^{1,q}\cap B^{1/2}_{p,2}(\mathbb{R}^+,L^q)}\\ &\qquad\lesssim \|u_0\|_{H^1}+\|g\|_{\mathcal{H}^1}+\||P_Tv|^{a-1}P_Tv\|_{L^{p'}W^{1,q'}\cap B^{1/2}_{p',2}L^{q'}}.\end{aligned}$$

Using $aq'=q$, $\frac{1-2/q}{a-1} = 1/q$, the embedding $H^1 \hookrightarrow L^q$ and assumption (4.3), we have

$$\begin{aligned}(4.4)\quad &\||P_Tv|^{a-1}P_Tv\|_{L^{p'}(\mathbb{R},W^{1,q'})}\\ &\lesssim \|P_Tv\|^a_{L_t^{ap'}L^q}+\|P_Tv\|^{a-1}_{L_t^\infty L^q}\|\nabla P_Tv\|_{L_t^pL^q}\|1\|_{L^{1-2/p}([-T,2T],L^\infty)}\\ &\lesssim T^{a-1/(p')}\|v\|^a_{L_t^\infty H^1}+T^{1-2/p}\|P_Tv\|^{a-1}_{L^\infty H^1}\|\nabla P_Tv\|_{L^p_TL^q}\\ &\lesssim (T^{a-1/p'}+T^{1-2/p})M^a.\end{aligned}$$

Similarly for the time regularity, we have using Proposition 2.10 and Lemma 2.11

$$\begin{aligned}\||P_Tv|^{a-1}P_Tv\|_{B^{1/2}_{p',2}(\mathbb{R},L^{q'})} &\lesssim \|(P_Tv)^{a-1}\|_{L_T^{1-2/p}L^{1-2/q}}\|P_Tv\|_{B^{1/2}_{p,2}L^q}\\ &\lesssim T^{1-2/p}\|v\|^{a-1}_{L_t^\infty L^q}T^{1/p-1/2}\|v\|_{B^{1/2}_{p,2}L^q}\\ &\lesssim T^{1/2-1/p}M^a.\end{aligned}$$

Therefore for $0 \leqslant T \leqslant 1$,

$$\|\Phi(v)\|_{L^\infty H^1\cap L^pW^{1,q}\cap B^{1/2}_{p,2}L^q} \lesssim \|u_0\|_{H^1} + \|g\|_{\mathcal{H}^1} + (T^{a-1/p'} + T^{1/2-1/p})M^a.$$

Choosing $M > \|u_0\|_{H^1} + \|g\|_{\mathcal{H}^1}$, T small enough, Φ maps S into S. Then from similar computations

$$\begin{aligned}(4.5)\quad \|\Phi(u) - \Phi(v)\|_{L^\infty_t L^2\cap L^p_t L^q} \\ \lesssim T^{1-2/p}(\|u\|_{L^\infty_t H^1} + \|v\|_{L^\infty_t H^1})^{a-1}\|u-v\|_{L^p_t L^q}.\end{aligned}$$

Up to decreasing T, the usual fixed point argument gives the existence of a unique fixed point in S for T small enough. Estimate (4.5) also implies uniqueness in $L^\infty H^1$, and by causality the solution does not depend on the choice of the extension operator.

Thanks to the local well-posedness in H^1, and the fact that the compatibility condition is clearly propagated by the flow, the existence and uniqueness of a maximal solution follows.

Global existence. — Let us go back to (4.4), assuming $a \geqslant 1+4/d$. Then $\frac{1}{p} = \frac{d(a-1)}{4(a+1)}$ and

$$\frac{1}{a-1}\left(1-\frac{2}{p}\right) - \frac{1}{p} = \frac{1}{a-1}\left(1-\frac{a+1}{p}\right) = \frac{1}{a-1}\left(1-\frac{d(a-1)}{4}\right) \leqslant 0,$$
$$\frac{1}{a}\left(1-\frac{1}{p}\right) - \frac{1}{p} = \frac{1}{a}\left(1-\frac{a+1}{p}\right) = \frac{1}{a}\left(1-\frac{d(a-1)}{4}\right) \leqslant 0.$$

Therefore $L^{ap'} \cap L^{\frac{1}{a-1}\left(1-\frac{2}{p}\right)} \subset L^\infty \cap L^p$. As we work with small data, we can assume that the solution exists on $[0, T_0]$, $T_0 \geqslant 1$, and for any $T \geqslant T_0$, using $H^1 \hookrightarrow L^q$

$$\begin{aligned}\||u|^{a-1}u\|_{L^{p'}_T W^{1,q'}} &\lesssim \|u\|^a_{L^{ap'}_T L^q} + \|u\|^{a-1}_{L_T^{\frac{1}{a-1}\left(1-\frac{2}{p}\right)}L^q}\|\nabla u\|_{L^p_T L^q} \\ &\lesssim \|u\|^a_{L^\infty_T H^1\cap L^pW^{1,q}}.\end{aligned}$$

The same computations can be applied to estimate time regularity, so that setting $m(T) = \|u\|_{L^\infty_T H^1\cap L^p_T W^{1,q}\cap B^{1/2}_{p,2}([0,T],L^q)}$, we have with C independent of $T \geqslant T_0$

$$m(T) \leqslant C(\|u_0\|_{H^1} + \|g\|_{\mathcal{H}^1} + m(T)^a).$$

If $\|u_0\|_{H^1} + \|g\|_{\mathcal{H}^1} \leqslant \varepsilon$ small enough, then from the fixed point argument $m(1) \leqslant A\varepsilon$ for some $A > 0$. Choosing $B > \max(A, C)$ and ε small enough such that $C + CB^a\varepsilon^{a-1} < B$, for any $T \in [0, T_{\max}[$, $m(T) \leqslant B\varepsilon$ thus $T_{\max} = \infty$. Since $u \in L^\infty H^1 \cap L^p(\mathbb{R}^+_t, W^{1,q}) \cap B^{1/2}_{p,2}(\mathbb{R}^+_t, L^q)$ for some (p,q)

admissible, it is also true for arbitrary admissible (p, q) by using the same computations. □

Remark 4.2. — For the Schrödinger equation on $\mathbb{R}^d$, global well-posedness for small data is known provided $a_S < a$, where $a_S = (\sqrt{d^2+12d+4}+d+2)/(2d) < 1+4/d$ is the so-called Strauss exponent, see [30]. Strichartz estimates for “non admissible pairs” ([13, Section 2.4]) are the missing tool for reaching this range.

5. Asymptotic behaviour

The aim of this section is to show that the global small solution constructed in Section 4 scatters in the sense that it is asymptotically linear. For the Cauchy problem, the classical definition(5) is

$$\exists\ \varphi \in H^1 : \lim_{t\to+\infty} \|e^{-it\Delta}u(t) - \varphi\|_{H^1} = 0. \tag{5.1}$$

We propose a natural extension for the Dirichlet boundary value problem: we define the resolvent operator $\Phi(g, s, t, u_0) = v(t, \cdot\,)$ where v is the solution of

$$\begin{cases} i\partial_r v + \Delta v = 0, \\ v|_{r=s} = u_0, \\ v|_{x_d=0} = g. \end{cases}$$

Note that by reversibility of the boundary value problem with Dirichlet boundary conditions, $\Phi(g, s, t, u_0)$ is well defined if g is defined on $[s, t]$, in particular we do not require $s \leqslant t$, and we have the usual formulas

$$\begin{aligned} \Phi(g, s, t, \Phi(g, s_1, s, u_0)) &= \Phi(g, s_1, t, u_0), \\ \Phi(g+h, s, t, u_0+v_0) &= \Phi(g, s, t, u_0) + \Phi(h, s, t, v_0), \end{aligned}$$

and we will freely use the fact that linear estimates directly give estimates on Φ.

In view of (5.1), the natural definition for scattering is then:

Definition 5.1. — *If u is a global solution to* (4.2), *we say that it scatters in H^1 if*

$$\exists\ \varphi \in H^1 : \lim_{t\to+\infty} \|\Phi(g, t, 0, u(t)) - \varphi\|_{H^1} = 0.$$

(5) up to some flexibility for the functional settings.

Remark 5.2. — Since the flow acts continuously on H^1, this is equivalent to the more "forward" definition

$$\exists\ \varphi \in H^1 : \lim_t \|\Phi_g(0,t,\varphi) - u(t)\|_{H^1} = 0,$$

which has the advantage of making sense for non reversible BVP (but is not as easily checked).

PROPOSITION 5.3. — *The global solution constructed in Section 4 scatters in H^1.*

Proof. — It suffices to check that $\Phi(g,t,0,u(t))$ is a Cauchy sequence. We keep the same notation as in the previous section. For $t > s$, we have

$$\Phi(g,t,0,u(t)) - \Phi(g,s,0,u(s)) = \Phi(0,t,0,u(t) - \Phi(g,s,t,u(s)))$$

On the other hand, $u(t) - \Phi(g,s,t,u(s))$ is the value at time t of the solution of

$$\begin{cases} i\partial_r z + \Delta z = |u|^{a-1}u 1_{r\geqslant s}, \\ z|_{r=s} = u(s) - u(s) = 0, \\ z|_{y=0} = 0. \end{cases}$$

We deduce

$$\|\Phi(g,t,0,u(t)) - \Phi(g,s,0,u(s))\|_{H^1} \\ \lesssim \||u|^{a-1}u\|_{L^{p'}([s,\infty[,W^{1,q'})\cap B^{1/2}_{p',2}([s,\infty[,L^{q'})} \to_s 0,$$

therefore by Cauchy's criterion $\Phi(g,t,0,u(t))$ converges in H^1. □

Due to the presence of boundary conditions, there is some "room" for other definitions of scattering. The purpose of the next proposition is to show that the asymptotic behaviour is actually trivial, in the sense that the solution converges to the restriction on $y \geqslant 0$ of $e^{it\Delta}\varphi$ for some $\varphi \in H^1(\mathbb{R}^d)$. We denote Δ_D the Dirichlet laplacian.

PROPOSITION 5.4. — *There exists $\varphi \in H^1_0$ such that*

$$\|u(t) - e^{it\Delta_D}\varphi\|_{H^1} \to_t 0.$$

Equivalently, u converges as $t \to \infty$ to the restriction on $y \geqslant 0$ of the solution of

$$\begin{cases} i\partial_t v + \Delta v = 0, \\ v|_{t=0} = A(\varphi), \end{cases} \quad x \in \mathbb{R}^d$$

where $A(\varphi)$ is the antisymetric extension on $y \leqslant 0$ of φ.

Proof. — Let us fix $\mathcal{R}$ a lifting operator $H^{1/2}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, $\mathcal{P}$ an extension operator $\mathcal{H}^1(\mathbb{R}_t^+) \to \mathcal{H}^1(\mathbb{R}_t)$. We define

$$\mathcal{P}_t : \mathcal{H}^1([t,\infty[\times\mathbb{R}^{d-1}) \to \mathcal{H}^1(\mathbb{R}\times\mathbb{R}^d), g \to \mathcal{P}(g(\,\cdot+t))(\,\cdot-t),$$

so that for $r \geqslant t, \mathcal{P}_t g(r) = g(r)$. We now consider the backward operator:

$$\Phi(\mathcal{P}_t g, t, 0, u).$$

For $t > s$, $\Phi\big(\mathcal{P}_t g, t, 0, u(t)) - \Phi(\mathcal{P}_s g, s, 0, u(s)\big) = \Phi\big(\mathcal{P}_t g - \mathcal{P}_s g, t, 0, u(t) - \Phi(g, s, t, u(s))\big)$. We already know (see the previous proof) that

$$\|u(t) - \Phi(g, s, t, u(s))\|_{H^1} \to_s 0,$$

moreover $\lim_{t\to\infty} \|g\|_{\mathcal{H}^1([t,\infty[)} = 0$ (Corollary 2.4(3)), thus $\|\mathcal{P}_t(g)\|_{\mathcal{H}^1} \to_\infty 0$. We deduce $\|\Phi\big(\mathcal{P}_t g, t, 0, u(t)) - \Phi(\mathcal{P}_s g, s, 0, u(s)\big)\| \to_{s,t} 0$, thus from Cauchy's criterion

$$\exists\ \varphi \in H^1 : \lim_t \Phi(\mathcal{P}_t g, t, 0, u(t)) = \varphi.$$

We remark now that $\Phi(\mathcal{P}_t g, t, \tau, u(t)) - e^{i(\tau-t)\Delta_D}(u(t) - \mathcal{R}g(t))$ is the solution of

$$\begin{cases} i\partial_\tau w + \Delta w = 0, \\ w|_{\tau=t} = \mathcal{R}g(t), \\ w|_{y=0} = \mathcal{P}_t g. \end{cases}$$

Since $\|g\|_{\mathcal{H}^1([t,\infty[)} \to_\infty 0$, we have $\|\mathcal{P}_t g\|_{\mathcal{H}^1} \to_\infty 0$ and from the embedding $\mathcal{H}^1 \hookrightarrow C([t,\infty|, H^{1/2})$, we have $\|\mathcal{R}g(t)\|_{H^1} \to_\infty 0$, this implies

$$\lim_{t\to\infty} \|\Phi(\mathcal{P}_t g, t, \tau, u(t)) - e^{i(\tau-t)\Delta_D}(u(t) - \mathcal{R}g(t))\|_{L^\infty_\tau H^1} \to 0,$$

in particular for $\tau = 0$,

$$\lim_{t\to\infty} \|\Phi(\mathcal{P}_t g, t, 0, u(t)) - e^{-it\Delta_D}(u(t) - \mathcal{R}g(t))\|_{H^1} = 0.$$

As $\Phi(\mathcal{P}_t g, t, 0, u(t)) \to_t \varphi \in H^1$, we deduce $e^{-it\Delta_D}(u(t) - \mathcal{R}g(t))) \to_t \varphi$ too. Furthermore for any t, $e^{-it\Delta_D}(u(t) - \mathcal{R}g(t)) \in H_0^1$ which is closed so $\varphi \in H_0^1$. Finally from $\|\mathcal{R}g(t)\|_{H^1} \to_t 0$ we conclude $\|u(t) - e^{it\Delta_D}\varphi\|_{H^1} \to 0$.

The equivalent statement simply comes from the fact that $A(e^{it\Delta_D}\varphi) = e^{it\Delta}A\varphi$. □

Appendix A. Remarks on the optimality of $\mathcal{H}$

A natural question is wether $\mathcal{H}$ is the weakest space for which the solution to (1.1) is C_tL^2. We consider the BVP

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ \lim_{-\infty} u(t) = 0, \\ u|_{y=0} = g. \end{cases}$$

We formulate our problem as follows

(A.1) Is there a weight $p > 0$ such that

$$\|u\|_{C_tL^2} \lesssim \left(\int |\widehat{g}|^2 p(\xi,\delta)\mathrm{d}\delta\mathrm{d}\xi\right)^{1/2} \quad \text{and} \quad \inf \frac{p(\xi,\delta)}{\sqrt{||\xi|^2+\delta|}} = 0?$$

The aim of this section is to show that the answer to this question is positive, even under the stronger assumptions that $p \leqslant \sqrt{||\xi|^2+\delta|}$ and for any $\lambda > 0$, $p(\lambda\xi, \lambda^2\delta) = \lambda p(\xi,\delta)$. However we will see that region where the inf is realized is a bit peculiar.

We recall that the solution is given by $\mathcal{L}u = e^{-y\sqrt{|\xi|^2+\delta}}\widehat{g}$, and that we can split u as

$$\begin{aligned} u(x,y,t) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_0^\infty e^{i(y\eta+x\cdot\xi)}e^{-it(|\xi|^2+\eta^2)}2\eta\widehat{g}(\xi,-\eta^2-|\xi|^2)\mathrm{d}\eta\,\mathrm{d}\xi \\ &\quad + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_0^\infty e^{-y\eta+ix\cdot\xi}e^{it(-|\xi|^2+\eta^2)}2\eta\widehat{g}(\xi,-|\xi|^2+\eta^2)\mathrm{d}\eta\,\mathrm{d}\xi \\ &= u_1+u_2. \end{aligned}$$

This splits the frequencies in two regions $\{\delta < -|\xi|^2\} := \mathcal{R}_h$ and $\{\delta > -|\xi|^2\} := \mathcal{R}_e$. In the usual terminology of boundary value problems these are the hyperbolic and elliptic regions (see [31] in the context of the Schrödinger equation). According to Plancherel's formula,

$$\|u_1(t=0)\|_{L^2} \sim \|\eta\widehat{g}(\xi,-\eta^2-\xi^2)\|_{L^2_{\xi,\eta}} \sim \|\widehat{g}(\xi,\delta)\,||\xi|^2+\delta|^{1/4}\|_{L^2},$$

therefore the weight $\sqrt{||\xi|^2+\delta|}$ can not be modified in $\mathcal{R}_h$.

In $\mathcal{R}_e$, we set $J(\xi,\eta) = \sqrt{\eta/p(\xi,-|\xi|^2+\eta^2)}$, $\varphi(\xi,\eta) = 2\widehat{g}(\xi,-|\xi|^2+|\eta|^2)\sqrt{\eta p}$. We remark that (A.1) is equivalent to $\sup J = +\infty$, and

$$\int |\varphi|^2(\xi,\eta)\mathrm{d}\eta = 2\int_{\mathbb{R}^{d-1}}\int_{-|\xi|^2}^\infty |\widehat{g}(\xi,\delta)|^2 p(\xi,\delta)\mathrm{d}\delta,$$

Now without loss of generality we can assume that for any $(\xi,\eta),\varphi(\xi,\eta)\in\mathbb{R}^+$, and we bound

$$\begin{aligned}
\text{(A.2)}\quad &\|u_2(\,\cdot\,,t)\|_{L^2_{x,y}}\\
&\sim\left\|\int_0^\infty e^{-y\eta}2\eta\widehat{g}(\xi,-|\xi|^2+\eta^2)\mathrm{d}\eta\right\|_{L^2_{\xi,y}}\\
&=\left\|\int_0^\infty e^{-y\eta}\varphi(\xi,\eta)J(\xi,\eta)\mathrm{d}\eta\right\|_{L^2_{\xi,y}}\\
&=\left\|\left(\int_{[0,\infty[^3}e^{-y(\eta_1+\eta_2)}\varphi(\xi,\eta_1)\varphi(\xi,\eta_2)J(\xi,\eta_1)J(\xi,\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}y\right)^{1/2}\right\|_{L^2_\xi}\\
&=\left(\int_{\mathbb{R}^{d-1}}\int_{[0,\infty[^2}\frac{J(\xi,\eta_1)J(\xi,\eta_2)}{\eta_1+\eta_2}\varphi(\xi,\eta_1)\varphi(\xi,\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\xi\right)^{1/2}
\end{aligned}$$

Using the decomposition $(\mathbb{R}^+)^2=\{\eta_1<\eta_2\}\cup\{\eta_2<\eta_1\}$, we see that (A.2) is bounded by $\|\varphi\|^2_{L^2}$ if

$$\text{(A.3)}\qquad T:\varphi\mapsto\frac{J(\xi,\eta_1)}{\eta_1}\int_0^{\eta_1}J(\xi,\eta_2)\varphi(\xi,\eta_2)\mathrm{d}\eta_2\ \text{is bounded}\ L^2\to L^2.$$

Due to scaling invariances, it seems natural to add some homogeneity assumptions: if u is a solution of the BVP with boundary data g, then for any $\lambda>0$, $\lambda^{d/2}u(\lambda x,\lambda y,\lambda^2 t)$ is a solution with boundary data $g(\lambda x,\lambda^2 t)$ and same C_tL^2 norm. The norm of the boundary data is scale invariant if

$$\int|\widehat{g}(\xi,\delta)|^2\frac{p(\lambda\xi,\lambda^2\delta)}{\lambda}\mathrm{d}\xi\mathrm{d}\delta=\int|\widehat{g}(\xi,\delta)|^2p(\xi,\delta)\mathrm{d}\xi\mathrm{d}\delta,$$

which is true provided p is anisotropically homogeneous: $p(\lambda\xi,\lambda^2\delta)=\lambda p(\xi,\delta)$. This is equivalent to the $J(\lambda\xi,\lambda\eta)=J(\xi,\eta)$. Somewhat surprisingly, even with these strong assumptions it is possible to construct J satisfying (A.1).

Proposition A.1. — *There exists $p(\xi,\delta)$ such that* (A.1) *is true, moreover we can choose p such that*

$$\forall\ (\lambda,\xi,\delta)\in\mathbb{R}^{+*}\times\mathbb{R}^{d-1}\times\mathbb{R},\quad p(\lambda\xi,\lambda^2\delta)=\lambda p(\xi,\delta),$$
$$\text{and}\quad p(\xi,\delta)\leqslant\sqrt{||\xi|^2+\delta|}.$$

Proof. — We keep the notations of the discussion above. For simplicity, we assume $d=2$, and define:

$$r(\xi,\eta)=\begin{cases}j & \text{if } 2^j-2^{-j}\leqslant\frac{\eta}{\xi}\leqslant 2^j,\ j\in\mathbb{N},\\ 0 & \text{else.}\end{cases}$$

Obviously, $J := 1 + r \geqslant 1$ is 0-homogeneous and unbounded, thus

$$p = \sqrt{\delta+\xi^2}/J^2(\xi, \sqrt{\delta+\xi^2}) \leqslant \sqrt{\delta+\xi^2}, \quad \inf p = 0 \quad \text{and} \quad p(\lambda\xi, \lambda^2\delta) = \lambda p(\xi, \delta).$$

Developping in (A.3) $J(\xi,\eta_1)J(\xi,\eta_2) = 1+r(\xi,\eta_1)+r(\xi,\eta_2)+r(\xi,\eta_1)r(\xi,\eta_2)$ it suffices to estimate each term separately. By symmetry, we can simply consider the integral over $\eta_1 \geqslant \eta_2$. The term with 1 is bounded thanks to Hardy's inequality, for the term with $r(\xi,\eta_2)$ we write

$$\begin{aligned}
\int_0^\infty \frac{1}{\eta_1^2}\left(\int_0^{\eta_1} r(\xi,\eta_2)\varphi(\xi,\eta_2)\mathrm{d}\eta_2\right)^2 \mathrm{d}\eta_1
&\lesssim \sum_{k=0}^\infty \int_{\xi 2^{k-1}}^{\xi 2^k} \frac{1}{\eta_1^2}\left(\int_0^{2^k\xi} r\varphi(\xi,\eta_2)\mathrm{d}\eta_2\right)^2 \mathrm{d}\eta_1 \\
&\lesssim \sum_{k=0}^\infty \frac{2^{-k}}{\xi}\left(\sum_{j=0}^k \int_{\xi(2^j-2^{-j})}^{\xi 2^j} j\varphi \mathrm{d}\eta_2\right)^2 \\
&\lesssim \sum_{k=0}^\infty \frac{2^{-k}}{\xi}\left(\sum_{j=0}^k \|\varphi(\xi,\cdot)\|_{L^2([\xi(2^j-2^{-j}),\xi 2^j])} j\sqrt{2^{-j}\xi}\right)^2 \\
&\lesssim \big\|\|\varphi(\xi,\cdot)\|_{L^2([\xi(2^j-2^{-j}),\xi 2^j])}\big\|_{l^2_j} \leqslant \|\varphi(\xi,\cdot)\|^2_{L^2_\eta}.
\end{aligned}$$

Similarly for the term with $r(\xi,\eta_1)$

$$\begin{aligned}
\int_0^\infty \frac{r^2(\xi,\eta_1)}{\eta_1^2}\left(\int_0^{\eta_1} \varphi(\xi,\eta_2)\mathrm{d}\eta_2\right)^2 \mathrm{d}\eta_1
&\lesssim \sum_{k=0}^\infty \int_{\xi(2^k-2^{-k})}^{\xi 2^k} \frac{k^2}{\eta_1^2}\left(\int_0^{2^k\xi} \varphi(\xi,\eta_2)\mathrm{d}\eta_2\right)^2 \mathrm{d}\eta_1 \\
&\lesssim \sum_{k=0}^\infty \frac{k^2 2^{-3k}}{\xi}\|\varphi(\xi,\cdot)\|^2_{L^2} 2^k\xi \lesssim \|\varphi(\xi,\cdot)\|^2_{L^2_\eta}.
\end{aligned}$$

The last term $r(\xi,\eta_1)r(\xi,\eta_2)$ is easier to estimate, we conclude by integration in ξ

$$\int_{\mathbb{R}}\int_{(\mathbb{R}^+)^2} \frac{J(\xi,\eta_1)J(\xi,\eta_2)}{\eta_1+\eta_2}\varphi(\xi,\eta_1)\varphi(\xi,\eta_2)\mathrm{d}\eta_2\mathrm{d}\eta_1\mathrm{d}\xi \lesssim \|\varphi\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \sim \|\widehat{g}\|_{L^2(p\mathrm{d}\delta\mathrm{d}\xi)},$$

despite the fact that J is larger than 1 and unbounded. □

Remark A.2. — Let us point out that the contribution of the elliptic region $\mathcal{R}_e$ to the solution corresponds to a superposition of so-called evanescent waves, that do not propagate like solutions of the Cauchy problem: for (δ, ξ) such that $\delta + |\xi|^2 > 0$, the wave $e^{-y\sqrt{||\xi|^2+\delta|}}e^{i(\delta t + x\cdot\xi)}$ is a solution of the Schrödinger equation on $\mathbb{R}^{d-1} \times \mathbb{R}^+$ remaining localized near the boundary.

As mentionned before, for frequencies that correspond to propagating waves, the weight $\sqrt{\delta + |\xi|^2}$ is optimal.

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