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COMPLETE KÄHLER–EINSTEIN METRICS UNDER CERTAIN HOLOMORPHIC COVERING AND EXAMPLES

by Damin WU & Shing–Tung YAU (*)

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

Abstract. — We establish the unique complete Kähler–Einstein metric with negative scalar curvature on a broad class of complete Kähler manifolds, including those manifolds whose covering space can be biholomorphically embedded into a Kähler manifold with holomorphic sectional curvature bounded above by a negative constant. We further present several new examples of complete noncompact Kähler–Einstein manifolds, generated by the results.

Résumé. — Nous établissons l'unique métrique complète de Kähler–Einstein avec courbure scalaire négative sur une large classe de variétés de Kähler complètes, y compris les variétés dont l'espace de recouvrement peut être biholomorphiquement plongé dans une variété de Kähler à courbure sectionnelle holomorphe limitée au-dessus par une constante négative. Nous présentons en outre plusieurs nouveaux exemples de variétés complètes de Kähler–Einstein non compactes, générés par les résultats.

1. Introduction

In [23] we prove that if a projective manifold M admits a Kähler metric with negative holomorphic sectional curvature, then its canonical bundle K_M is positive. Several special cases have been known before. In particular, the result has also been proven in [8, 9] by assuming the abundance

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conjecture, which has been confirmed for dimension three. We refer to [23] and references therein. The result in [23] has been extended, for example, by [20, 5, 24]. These results can be summarized as below:

Theorem 1.1 ([23, 20, 5, 24]). — *Let (M,ω) be a compact Kähler manifold, and let $H(\omega)$ denote the holomorphic sectional curvature of ω.*

(1) *If $H(\omega) < 0$ everywhere on M, then $K_M > 0$.*
(2) *If $H(\omega) \leqslant 0$ everywhere on M, then K_M is nef.*
(3) *If $H(\omega)$ is quasi-negative, i.e., $H(\omega) \leqslant 0$ everywhere and $H(\omega) < 0$ at one point of M, then $K_M > 0$.*

We remark that, while in the case $H(\omega) < 0$ we can provide a direct proof by using the complex Monge–Ampère type equation combining the Schwarz lemma, in the case of $H(\omega)$ quasi-negative one seems to have to use $\int_M c_1(K_M)^n > 0$ to conclude the bigness of K_M. This relies on Demailly's fundamental work on the Morse inequality (see [4] for example).

More recently in [25] we extend the result to complete noncompact Kähler manifolds. Note that on a compact Kähler manifold M, the ampleness of K_M is equivalent to the existence of Kähler–Einstein metric on M with negative scalar curvature. We can therefore characterize the positivity of K_M on a complete noncompact Kähler manifolds by the existence of a complete Kähler–Einstein metric with negative scalar curvature.

Theorem 1.2 ([25, Theorem 3]). — *Let (M,ω) be a complete Kähler manifold such that the holomorphic sectional curvature $H(\omega)$ of ω satisfies $-B \leqslant H(\omega) \leqslant -A$ for two positive constants A, B. Then, M admits a unique complete Kähler–Einstein metric ω_{KE} with Ricci curvature equal to -1, satisfying $C^{-1}\omega \leqslant \omega_{\mathrm{KE}} \leqslant C\omega$ for some constant $C > 0$. Furthermore, the curvature tensor of ω_{KE} and all its covariant derivatives are bounded.*

Through the proof of Theorem 1.2, we develop the effective version of quasi-bounded geometry. By using the quasi-bounded geometry, we further show that the three classical invariant metrics, the Bergman metric, the Kobayashi–Royden metric, and the Kähler–Einstein metric of negative scalar curvature are quasi-isometric to each other on a simply-connected complete Kähler manifold of negatively pinched sectional curvature ([25]).

In this note we give a further extension, which connects to the study of the fourth classical invariant metric, the Carathéodory–Reiffen metric.

Theorem 1.3. — *Let (M,ω) be a complete Kähler manifold. Assume the holomorphic sectional curvature $H(\omega)$ of ω satisfies $-B \leqslant H(\omega) \leqslant A$ for two positive constants A and B. Suppose that M has a holomorphic* *covering space* $\pi : \widetilde{M} \to M$ *such that for each point* $\widetilde{x} \in \widetilde{M}$*, there exists a holomorphic map* F *from* $\widetilde{M}$ *to a Kähler manifold* (N, ω_N) *such that* $H(\omega_N) \leqslant -1$ *and*

$$F^*\omega_N \geqslant C_1\widetilde{\omega} \quad \text{at } \widetilde{x} \tag{1.1}$$

where $\widetilde{\omega} = \pi^*\omega$ *is the covering metric, and* $C_1 > 0$ *is a constant independent of* $\widetilde{x}$*. Then,* M *admits a unique complete Kähler–Einstein metric* ω_{KE} *satisfying*

$$C^{-1}\omega \leqslant \omega_{\mathrm{KE}} \leqslant C\omega \quad \text{on } M,$$

and the curvature tensor of ω_{KE} *and all its covariant derivatives are bounded on* M*.*

Theorem 1.3 includes Theorem 1.2, since one can simply take $\widetilde{M} = M = N$ or let $\widetilde{M}$ to be the universal covering of M, $N = M$, and F be the canonical projection. On the other hand, Theorem 1.3 can be viewed as a complete noncompact extension of a result of H. Wu [26, Theorem 2] (see also [10, Corollary 1.2]) concerning the Carathéodory hyperbolicity.

Corollary 1.4. — *Let* (M, ω) *be a compact Kähler manifold.*

(1) [26, 10] *If* M *is Carathéodory hyperbolic, then* K_M *is ample.*
(2) *If* K_M *is nef and* M *is Carathéodory hyperbolic at one point, then* K_M *is big.*

Corollary 1.4 (1) is contained in H. Wu [26] and S. Kikuta [10], which do not require the Kähler condition. Their approaches, however, do not achieve the second part (2), since they require M to be Carathéodory hyperbolic at least on a dense open subset. Corollary 1.4 (2) may be compared with Theorem 1.1 (3). Our proof of (2) again combines the Monge–Ampère type equation with Demailly's Morse inequality.

One goal of this paper is to construct complete Kähler–Einstein metrics on a broader class of manifolds, which may not have negative holomorphic curvature over the whole manifold. The following theorem is useful in some situation.

Theorem 1.5. — *Let* (M, ω) *be a complete Kähler manifold with bounded sectional curvature, and let* $\pi : \widetilde{M} \to M$ *has a holomorphic covering space. Assume* $\widetilde{E} \subset \widetilde{M}$ *is either compact or* $\widetilde{M} \setminus \widetilde{E}$ *is a bounded domain with respect to* $\widetilde{\omega} = \pi^*\omega$*, such that*

(1) $\mathrm{dd}^c \log \widetilde{\omega}^n \geqslant C_1\widetilde{\omega}$ *on* $\widetilde{M} \setminus \widetilde{E}$*, where* $C_1 > 0$ *is a constant;*
(2) *for each* $\widetilde{x} \in \widetilde{E}$*, there exists a holomorphic map* F *from* $\widetilde{M}$ *to a Kähler manifold* (N, ω_N) *with* $H(\omega_N) \leqslant -1$ *such that* $F^*\omega_N \geqslant C_2\,\widetilde{\omega}$ *at* $\widetilde{x}$*, where* C_2 *is a constant independent of* $\widetilde{x}$*.*

In the case (M,ω) has the quasi-bounded geometry, assume only that $\widetilde{E}$ is closed in $\widetilde{M}$ and satisfy (1) *and* (2)*. Then, M admits a unique complete Kähler–Einstein metric ω_{KE} which is uniformly equivalent to ω, and the curvature tensor of ω_{KE} has bounded covariant derivatives of arbitrary order.*

A motivational example for Theorem 1.3 and Theorem 1.5 is the moduli space of Riemann surfaces, whose covering space is the Teichmüller space (see [14] for example). The Bers embedding theorem exemplifies the map F from the covering space to a large ball in $\mathbb{C}^n$ so that the pullback metric under F is nondegenerate. Another example for Theorem 1.5 is the quasi-projective surface $M = \overline{M} \setminus D$ with positive logarithmic canonical bundle, where D is a Riemann surface of genus greater than two. Then, there exists a small tubular neighborhood of D in $\overline{M}$ such that the closure of $U \setminus D$ in $\widetilde{M} = M$ satisfies the requirement of Theorem 1.5 (cf. [17]).

In Section 5, we present several new examples of the complete noncompact Kähler–Einstein manifold of negative Ricci curvature, generated by Theorem 1.2, Theorem 1.3, and Theorem 1.5. In particular, Theorem 1.5 allows us to construct a new class of complete noncompact Kähler–Einstein surfaces modeled on $\mathbb{D} \times \mathbb{D}^*$; see Example 5.5.

2. Preliminary

Let us recall some standard results concerning the covering metric. Let $\pi : \widetilde{M} \to M$ be a covering space of a Riemannian manifold M. One can pullback the metric g on M to a Riemannian metric $\widetilde{g} \equiv \pi^* g$, called the *covering metric*, on $\widetilde{M}$. Then, $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry. The following result on completeness is standard; see, for example, [13, p. 176, Theorem 4.6].

Proposition 2.1. — *The Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is complete if and only if (M, g) is complete.*

We shall use an elementary topological lemma, whose proof is omitted.

Lemma 2.2. — *Let M be a compact smooth manifold and $\pi : \widetilde{M} \to M$ be a covering. Then, there exists a compact subset $\widetilde{\Sigma}$ of $\widetilde{M}$ such that $\pi(\widetilde{\Sigma}) = M$.*

Next, we recall the definition of Carathéodory hyperbolicity (cf. [12, p. 367] and [26, p. 646]). The *Carathéodory–Reiffen pseudometric* $\mathfrak{C}_M$ on a complex manifold M is defined as below. For any point $x \in M$ and any holomorphic tangent vector $v \in T'_x M$, let

$$\begin{aligned}\mathfrak{C}_M(x,v) &\equiv \sup\{|V|_{\mathcal{P}}; \psi \in \mathrm{Hol}(M,\mathbb{D}), \psi_*(v) = V\}\\ &= \sup\{|V|_{\mathcal{P}}; \psi \in \mathrm{Hol}(M,\mathbb{D}), \psi(x) = 0, \psi_*(v) = V\}.\end{aligned}$$

where $\mathrm{Hol}(M,N)$ denotes the set of all holomorphic maps from M to N, $\mathcal{P}$ denotes the Poincaré metric $\mathrm{d}z \otimes \mathrm{d}\bar{z}/(1-|z|^2)^2$ on the unit disk $\mathbb{D}$, and $\psi_*(v) = \mathrm{d}\psi_x(v) = \mathrm{d}\psi(v)$.

A complex manifold M is said to be *Carathéodory hyperbolic*, or *$\mathfrak{C}$-hyperbolic*, at a point $x_0 \in M$ if there exists a holomorphic covering $\pi : \widetilde{M} \to M$ and a point $\widetilde{x}_0 \in \widetilde{M}$ such that $\pi(\widetilde{x}_0) = x_0$ and

$$\mathfrak{C}_{\widetilde{M}}(\widetilde{x}_0, v) > 0 \quad \text{for all } v \in T'_{\widetilde{x}_0}\widetilde{M}, v \neq 0.$$

The manifold M is said to be *Carathéodory hyperbolic* or *$\mathfrak{C}$-hyperbolic* if there exists a holomorphic covering $\pi : \widetilde{M} \to M$ such that $\mathfrak{C}_M(\widetilde{x}, v) > 0$ for every $\widetilde{x} \in \widetilde{M}$ and $v \in T'_{\widetilde{x}}\widetilde{M}$. The reason we pass to the holomorphic covering is due to the fact that $\mathfrak{C}_M \equiv 0$ for any compact complex manifold M.

The following lemma is pointed out in [26, p. 647, Lemma 1].

Lemma 2.3. — *An n-dimensional complex manifold M is $\mathfrak{C}$-hyperbolic at a point if and only if there exists a holomorphic covering $\pi : \widetilde{M} \to M$ and a point $\widetilde{x}_0 \in \widetilde{M}$ such that there exists a holomorphic map $\Psi : \widetilde{M} \to \mathbb{B}^n$ satisfying $\Psi(x_0) = 0$ and $\mathrm{d}\Psi : T'_{\widetilde{x}_0}\widetilde{M} \to T'_0\mathbb{B}^n$ is nonsingular, where $\mathbb{B}^n$ denotes the unit open ball in $\mathbb{C}^n$.*

One ingredient in our proofs is the following Schwarz type lemma, whose proof follows immediately from adapting the argument in [27] and [16, Proposition 4] to [23, Proposition 9].

Proposition 2.4. — *Let $F : M \to N$ be a non-constant holomorphic map between two Kähler manifolds (M^m, ω_M) and (N^n, ω_N). Assume that the Ricci curvature $\mathrm{Ric}(\omega_M)$ of ω_M satisfies*

$$\mathrm{Ric}(\omega_M) \geqslant \lambda\omega_M + \mu\omega_N \quad \textit{at a point } p;$$

and that the holomorphic sectional curvature $H(\omega_N)$ of ω_N satisfies

$$H(\omega_N) \leqslant -\kappa \quad \textit{at the point } F(p),$$

where λ, μ, κ are constants with $\mu \geqslant 0$ and $\kappa \geqslant 0$. Then,

$$\Delta_{\omega_M} \log S \geqslant \left(\frac{l+1}{2l}\kappa + \frac{\mu}{m}\right) S + \lambda \quad \textit{at } p,$$

where $S \equiv tr_{\omega_M}(F^*\omega_N)$, and l is the rank of dF_p.

Another ingredient is the quasi-bounded geometry established in [25, Theorem 9], using W. X. Shi's derivative estimates of curvature [18] (compare [25, Lemma 13]).

Lemma 2.5. — *Let (M, ω) be an n-dimensional complete Kähler manifold with holomorphic curvature $-B \leqslant H(\omega) \leqslant A$ for two positive constants A and B. Then,*

(1) *the manifold M admits a complete Kähler metric ω_1 such that ω_1 is quasi-isometric to ω, and the curvature tensor of ω_1 and its kth covariant derivatives are bounded by constants depending only on n, A, B, and k.*

(2) *Furthermore, (M, ω_1) has the quasi-bounded geometry in the following sense: There exists a constant $r > 0$ depending only in n, A and B such that for every point $x \in M$, there is a nonsingular map $\psi_x : B(r) \to M$, $\psi_x(0) = x$, such that*

$$C^{-1}\omega_{\mathbb{C}^n} \leqslant \psi_x^*\omega_1 \leqslant C\omega_{\mathbb{C}^n} \quad \text{on } B(r),$$

with constant $C > 0$ depending only on n, A, B, and that the kth derivatives of metric components of $\psi_x^*\omega_1$ on $B(r)$ with respect to the natural coordinates in $\mathbb{C}^n$ are bounded by constants depending only on n, A, B, and k. Here $B(r)$ denotes an open ball in $\mathbb{C}^n$ of radius r centered at the origin, and $\omega_{\mathbb{C}^n}$ is the standard Kähler form on $\mathbb{C}^n$.

(3) *Assume $E \subset M$ is either a compact set or satisfies that $M \setminus E$ is a bounded domain. If $\mathrm{dd}^c \log \omega^n \geqslant A_1\omega$ on $M \setminus E$ for some constant $A_1 > 0$, then*

$$\mathrm{dd}^c \log \omega_1^n \geqslant \frac{A_1}{4}\omega_1 \quad \text{on } M \setminus E.$$

Proof. — Statement (1) follows from Shi's derivative estimates, and statement (2) follows from statement (1) and [25, Theorem 9(1)]. Only statement (3) requires a proof.

Let us recall the approach of Ricci flow and set up the notation. Consider

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\bar\beta}(x,t) = -4R_{\alpha\bar\beta}(x,t), & x \in M, t \geqslant 0, \\ g_{\alpha\bar\beta}(x,0) = g_{\alpha\bar\beta}(x), & x \in M, \end{cases} \tag{2.1}$$

where $g_{\alpha\bar\beta}(x)$ is the metric component of ω at x. By Shi's derivative estimates, there exists a constant $\theta_0(n) > 0$ depending only on n such that (2.1) admits a smooth family of Kähler metrics $\{g_{\alpha\bar\beta}(x,t)\}$ for $0 \leqslant$

$t \leqslant \theta_0(n)/(A+B)$, satisfying for each $l \in \mathbb{Z}_{\geqslant 0}$,

$$\sup_{x\in M} |\nabla^l R_{\alpha\bar\beta\gamma\bar\sigma}(x,t)|^2_{g_{\alpha\bar\beta}(x,t)} \leqslant \frac{C(l,n)(A+B)^2}{t^l}, \qquad \text{for all } 0 < t \leqslant \frac{\theta_0(n)}{A+B}. \tag{2.2}$$

It follows that

$$e^{-C(n)} g_{\alpha\bar\beta}(x) \leqslant g_{\alpha\bar\beta}(x,t) \leqslant e^{C(n)} g_{\alpha\bar\beta}(x).$$

for all $0 < t \leqslant \theta_0(n)/(A+B)$. Here $C(n) > 0$ denotes a generic constant depending only on n. Thus, for an arbitrary $t \in (0, \theta_0(n)/(A+B)]$,

$$\omega_1(x,t) = \frac{\sqrt{-1}}{2} g_{\alpha\bar\beta}(x,t) \mathrm{d}z^\alpha \wedge \mathrm{d}\bar z^\beta$$

satisfies statement (1) and (2).

For statement (3), if $M \setminus E$ is a bounded domain then its closure $\overline{M \setminus E}$ is compact in M. Then, $(\overline{M \setminus E}) \times [0, \theta_0(n)/(A+B)]$ is compact in $M \times [0, \theta_0(n)/(A+B)]$. Note that $\mathrm{dd}^c \log \omega^n \geqslant A_1 \omega$ is the same as $-R_{\alpha\bar\beta}(x) \geqslant A_1 g_{\alpha\bar\beta}(x)$. By the uniform continuity, there exists a small $0 < t_0 < \theta_0(n)/(A+B)$, depending on E, such that

$$R_{\alpha\bar\beta}(x,t) + \frac{A_1}{2} g_{\alpha\bar\beta}(x,t) \leqslant 0, \quad x \in M \setminus E, 0 < t \leqslant t_0.$$

Thus, then $\omega_1 = \{g_{\alpha\bar\beta}(x,t_0)\}$ satisfies statements (1), (2), and (3).

To show (3) when E is a compact subset of M, we first apply the uniform continuity on $\partial E \times [0, \theta_0(n)/(A+B)]$ to obtain a small $0 < t_1 < \theta_0(n)/(A+B)$ such that

$$R_{\alpha\bar\beta}(x,t) + \frac{A_1}{2} g_{\alpha\bar\beta}(x,t) \leqslant 0, \quad \text{for all } (x,t) \in \partial E \times [0, t_1].$$

Note that

$$\frac{\partial}{\partial t} R_{\alpha\bar\beta} = 4\Delta R_{\alpha\bar\beta} + 4 g^{\mu\bar\nu} g^{\gamma\bar\sigma} R_{\alpha\bar\beta\mu\bar\sigma} R_{\gamma\bar\nu} - 4 g^{\mu\bar\nu} R_{\alpha\bar\nu} R_{\mu\bar\beta}.$$

It follows that

$$\left(\frac{\partial}{\partial t} R_{\alpha\bar\beta}\right) \eta^\alpha \bar\eta^\beta \leqslant 4(\Delta R_{\alpha\bar\beta}) \eta^\alpha \bar\eta^\beta + C(n)(A+B)^2 |\eta|^2,$$

for all $x \in M$ and $0 \leqslant t \leqslant t_1$. Applying Proposition 2.6 below with $W_{\alpha\bar\beta} = R_{\alpha\bar\beta}$, $\kappa = A_1/2$, $k_0 = C(n)(A+B)^2$, $C_0 = C(n)(A+B)$, and $C_1 = C(n)(A+B)^2$ yields

$$R_{\alpha\bar\beta}(x,t) \eta^\alpha \bar\eta^\beta \leqslant C(n)(A+B)^2 t - \frac{A_1}{2}$$

for all $x \in M \setminus E$, $\eta \in T'_x M$, $|\eta| = 1$, $0 \leqslant t \leqslant t_1$. Let

$$t_2 = \min\left\{t_1, \frac{A_1}{C(n)(A+B)^2}\right\} > 0.$$

Then,

$$R_{\alpha\bar{\beta}}(x, t_2) \leqslant -\frac{A_1}{4} g_{\alpha\bar{\beta}}(x, t_2), \quad x \in M \setminus E.$$

Hence, the metric $\omega_1 = \{g_{\alpha\bar{\beta}}(x, t_2)\}$ satisfies requirements (1), (2), (3). □

Proposition 2.6. — *Assume on a complete noncompact Kähler manifold M the Ricci flow equation*

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -4R_{\alpha\bar{\beta}}(x,t), \quad \text{on } M \times [0,T], \tag{2.3}$$

admits a smooth solution $g_{\alpha\bar{\beta}}(x,t) > 0$ for all $x \in M$ and $0 \leqslant t \leqslant T$, whose curvature tensor satisfying

$$\sup_{M \times [0,T]} |R_{\alpha\bar{\beta}\gamma\bar{\sigma}}(x,t)|^2 \leqslant k_0 \tag{2.4}$$

for some constant $k_0 > 0$. Suppose a smooth tensor $\{W_{\alpha\bar{\beta}}(x,t)\}$ on M with complex conjugation $\overline{W}_{\alpha\bar{\beta}}(x,t) = W_{\beta\bar{\alpha}}(x,t)$ satisfies

$$\left(\frac{\partial}{\partial t} W_{\alpha\bar{\beta}}\right) \eta^\alpha \bar{\eta}^\beta \leqslant \left(\Delta W_{\alpha\bar{\beta}}\right) \eta^\alpha \bar{\eta}^\beta + C_1 |\eta|^2_{\omega(x,t)}, \tag{2.5}$$

for all $x \in M$, $\eta \in T'_x M$, $0 \leqslant t \leqslant T$, where $\Delta \equiv 2g^{\alpha\bar{\beta}}(x,t)(\nabla_{\bar{\beta}}\nabla_\alpha + \nabla_\alpha \nabla_{\bar{\beta}})$ and C_1 is a constant. Let

$$h(x,t) = \max\left\{W_{\alpha\bar{\beta}}\eta^\alpha\bar{\eta}^\beta; \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\right\},$$

for all $x \in M$ and $0 \leqslant t \leqslant T$. Suppose that

$$\sup_{x \in M,\, 0 \leqslant t \leqslant T} |h(x,t)| \leqslant C_0, \tag{2.6}$$

$$\sup_{x \in M \setminus E} h(x,0) \leqslant -\kappa, \tag{2.7}$$

$$\sup_{x \in \partial E,\, 0 \leqslant t \leqslant T} h(x,t) \leqslant -\kappa, \tag{2.8}$$

for some constants $C_0 > 0$ and κ, where $E \subset M$ is compact. Then,

$$h(x,t) \leqslant \left(4C_0\sqrt{nk_0} + C_1\right) t - \kappa.$$

for all $x \in M \setminus E$, $0 \leqslant t \leqslant T$.

Proof. — The argument is similar to that of [25, Lemma 15]. We only point out the differences. Suppose there exist $(x_1, t_1) \in (M \setminus E) \times [0, T]$ such that

$$h(x_1, t_1) - Ct_1 + \kappa > 0, \quad C \equiv 4C_0\sqrt{nk_0} + C_1.$$

By (2.7), we have $t_1 > 0$. Invoke the function $\theta(x,t) \in C^\infty(M \times [0,T])$ with

$$0 < \theta(x,t) \leqslant 1,$$
$$\frac{\partial \theta}{\partial t} - \Delta\theta + 2\frac{|\nabla\theta|^2}{\theta} \leqslant -\theta$$
$$\frac{C_2^{-1}}{1 + d_0(x, x_0)} \leqslant \theta(x,t) \leqslant \frac{C_2}{d_0(x, x_0)}$$

on $M \times [0,T]$, where $x_0 \in M$ is a fixed point and d_0 is the distance with respect to $g_{\alpha\bar{\beta}}(x, 0)$ (cf. [18, p. 124, Lemma 4.6]). Then,

$$0 < m_0 \equiv \sup_{x \in M\setminus E, 0 \leqslant t \leqslant T} [(h(x,t) - Ct + \kappa)\theta(x,t)] < C_0 + |\kappa|.$$

Now pick a geodesic ball $B(x_0; \Lambda)$ with respect to d_0 centered at x_0 of radius

$$\Lambda = \max\left\{\frac{2C_2(C_0 + |\kappa|)}{m_0},\ \max_{y \in E} d_0(x_0, y) + 1\right\} > 0.$$

Then, $B(x_0, \Lambda) \supset E$, and for all $x \in M$ with $d_0(x, x_0) \geqslant \Lambda$,

$$(h(x,t) - Ct + \kappa)\theta(x,t) \leqslant \frac{C_2(C_0 + |\kappa|)}{1 + d_0(x, x_0)} \leqslant \frac{m_0}{2}.$$

Thus, $(h(x,t) - Ct + \kappa)\theta$ must attain its supremum m_0 at a point

$$(x_*, t_*) \in \big(\big(B(x_0; \Lambda) \setminus E\big) \sqcup \partial E\big) \times [0, T].$$

By (2.7), we have $t_* > 0$. Notice that $x_* \notin \partial E$, by virtue of (2.8). Hence,

$$x_* \in B(x_0; \Lambda) \setminus E,$$

that is, x_* is an interior point. This allows us to apply the maximum principle to get a contradiction.

The rest of the proof follows entire the same as that of [25, Lemma 15], with the factor 4 instead of 8 in the estimate, which is due to

$$\frac{\partial}{\partial t}\left(\frac{W_{\alpha\bar{\beta}}\eta^\alpha\bar{\eta}^\beta}{|\eta|^2}\right) = \frac{1}{|\eta|^2}\left(\frac{\partial}{\partial t}W_{\alpha\bar{\beta}}\right)\eta^\alpha\bar{\eta}^\beta + \frac{4}{|\eta|^4}(R_{\gamma\bar{\sigma}}\eta^\gamma\bar{\eta}^\sigma)(W_{\alpha\bar{\beta}}\eta^\alpha\bar{\eta}^\beta),$$

if $\partial\eta/\partial t = 0$. □

3. Theorem 1.3 and hyperbolicity

We first prove Theorem 1.3. Then, we indicate its application to the Carathéodory hyperbolicity.

Proof of Theorem 1.3. — By Lemma 2.5, we can assume, without loss of generality, that (M,ω) has quasi-bounded geometry. Consider the Monge–Ampère equation

$$(\mathrm{MA})_t \qquad \begin{cases} (t\omega + \mathrm{dd}^c \log\omega^n + \mathrm{dd}^c u)^n = e^u \omega^n, \\ c_t^{-1}\omega \leqslant \omega_t \equiv t\omega + \mathrm{dd}^c \log\omega^n + \mathrm{dd}^c u \leqslant c_t\,\omega \quad \text{on } M, \end{cases}$$

where $\mathrm{dd}^c \log\omega^n = -\operatorname{Ric}(\omega)$, $n = \dim M$, and the constant $c_t > 1$ may depend on t. We use the continuity method to produce a solution for $t = 0$. The nonemptyness, openness, and the estimate $\sup_M u \leqslant C$ follow from the same arguments in [25, Lemma 31]. We denote by C the generic constant depending only on n and ω.

To get the second order estimate we pass to the covering space $\widetilde{M}$. Let $\widetilde{\omega} = \pi^*\omega$, $\widetilde{u} = \pi^* u$, and $\widetilde{\omega_t} = \pi^*\omega_t$. Note that both $\widetilde{\omega}$ and $\widetilde{\omega_t}$ are Kähler, since d commutes with π^*. By Proposition 2.1, the metrics $\widetilde{\omega}$ and $\widetilde{\omega_t}$ are both complete, satisfying

$$\widetilde{\omega_t}^n = e^{\widetilde{u}}\widetilde{\omega}^n \quad \text{on } \widetilde{M}.$$

This implies

$$\operatorname{Ric}(\widetilde{\omega_t}) = -\widetilde{\omega_t} + t\widetilde{\omega} \geqslant -\widetilde{\omega_t}.$$

For each $x \in \widetilde{M}$, applying Proposition 2.4 to $F : (\widetilde{M}, \widetilde{\omega_t}) \to (N, \omega_N)$ yields

$$\Delta_{\widetilde{\omega}_t} \log \widetilde{S} \geqslant \left(\frac{l+1}{2l} + \frac{t}{n}\right)\widetilde{S} - 1,$$

where $\widetilde{S} = \operatorname{tr}_{\widetilde{\omega}_t}(F^*\omega_N)$, and l is the rank of $\mathrm{d}F$ at a point in $\widetilde{M}$, and $l \geqslant n$ by the assumption (1.1). Since $\widetilde{\omega_t}$ is complete with Ricci curvature bounded below by -1, we apply the second author's upper bound lemma (see, for example, [1, p. 353, Theorem 8]) to obtain

$$\widetilde{S} \leqslant \frac{2n}{n+1}.$$

Then,

$$C_1\widetilde{\omega} \leqslant F^*\omega_N \leqslant \frac{2n}{n+1}\widetilde{\omega_t} \quad \text{at } \widetilde{x}.$$

It follows that

$$C_1\omega \leqslant \frac{2n}{n+1}\omega_t \quad \text{at } x = \pi(\widetilde{x}).$$

This together with $\sup_M u \leqslant C$ imply the desired second order estimate

$$\frac{(n+1)C_1}{2n}\omega \leqslant \omega_t \leqslant C\omega,$$

as in [25, Lemma 29]. The closedness of T then follows from the same process as in [25, Lemma 29]. Hence, $t = 0 \in T$. This establishes the existence of the desired Kähler–Einstein metric.

The uniqueness of a complete Kähler–Einstein metric with negative scalar curvature is a standard result, as a consequence of the second author's Schwarz lemma (see [2, Proposition 5.5] for example). The proof is therefore completed. □

Proof of Corollary 1.4. — For statement (1), it suffices to verify condition (1.1) in Theorem 1.3. This follows immediately from the compactness of M and the Carathéodory hyperbolicity. More precisely, by Lemma 2.2, there exists a compact subset $\widetilde{\Sigma} \subset \widetilde{M}$ such that $\pi(\widetilde{\Sigma}) \supset M$. It is then sufficient to verify (1.1) on $\widetilde{\Sigma}$.

By Lemma 2.3, for each $\widetilde{x} \in \widetilde{\Sigma}$, there is a nonsingular holomorphic map $\Psi_{\widetilde{x}} : \widetilde{M} \to \mathbb{B}^n$. By the inverse function theorem, $\Psi_{\widetilde{x}}$ is a biholomorphism on a neighborhood $U_{\widetilde{x}}$ of $\widetilde{x}$ in $\widetilde{M}$, and satisfies

$$c_{\widetilde{x}}^{-1}\widetilde{\omega} \leqslant \Psi_{\widetilde{x}}^*\omega_{\mathfrak{B}} \leqslant c_{\widetilde{x}}\widetilde{\omega} \quad \text{on } U_{\widetilde{x}},$$

where $\omega_{\mathfrak{B}} \equiv -\mathrm{dd}^c \log(1-|z|^2)$ is the Bergman metric on $\mathbb{B}^n$ with holomorphic sectional curvature identically equal to -2, and $c_{\widetilde{x}} > 0$ is a constant depends on $\widetilde{x}$.

Fix a finite cover $\{U_{\widetilde{x}_j}\}_{j=1}^N$ for $\widetilde{\Sigma}$. Let

$$C_M = \max\{c_{\widetilde{x}_1}, \ldots, c_{\widetilde{x}_N}\} > 0.$$

Then, the constant $C_M > 0$ depending only on M such that

$$C_M^{-1}\widetilde{\omega} \leqslant \Psi_{\widetilde{x}}^*\omega_{\mathfrak{B}} \quad \text{for all } \widetilde{x} \in \widetilde{\Sigma}.$$

This completes the proof of statement (1).

For statement (2), one cannot expect to have (1.1). However, we can still use the approach of Theorem 1.3. By [23, Proposition 8], for each small $t > 0$, there exists a smooth function u on M satisfies the Monge–Ampère type equation

$$(\mathrm{MA})_t \qquad \begin{cases} (t\omega + \mathrm{dd}^c \log\omega^n + \mathrm{dd}^c u)^n = e^u\omega^n, \\ \omega_t \equiv t\omega + \mathrm{dd}^c \log\omega^n + \mathrm{dd}^c u > 0 \quad \text{on } M. \end{cases}$$

It follows that the Ricci curvature

$$\mathrm{Ric}(\omega_t) = -\omega_t + t\omega.$$

Let $\pi : \widetilde{M} \to M$ be the holomorphic covering space of M satisfying that there exists a point $\widetilde{x}_0 \in \widetilde{M}$ such that $\mathfrak{C}_{\widetilde{M}}(\widetilde{x}_0, v) > 0$ for every nonzero $v \in T'_{\widetilde{x}_0}\widetilde{M}$. By Lemma 2.3, there exists a holomorphic map $\Psi : \widetilde{M} \to \mathbb{B}^n$ such that $\Psi(\widetilde{x}_0) = 0$ and $\Psi_* : T'_{\widetilde{x}_0}\widetilde{M} \to T'_0\mathbb{B}^n$ is nonsingular.

Let $\widetilde{\omega} = \pi^*\omega$ be the pullback metric on $\widetilde{M}$. Then,

$$\widetilde{\omega}_t \equiv \pi^*\omega_t = t\widetilde{\omega} + \mathrm{dd}^c \log \widetilde{\omega}^n + \mathrm{dd}^c\widetilde{u}, \quad \widetilde{u} = \pi^*u.$$

Both $\widetilde{\omega}$ and $\widetilde{\omega}_t$ are complete Kähler metrics. Equation $(\mathrm{MA})_t$ induces an equation on $\widetilde{M}$:

$$\widetilde{\omega}_t^n = (t\widetilde{\omega} + \mathrm{dd}^c \log \widetilde{\omega}^n + \mathrm{dd}^c\widetilde{u})^n = e^{\widetilde{u}}\widetilde{\omega}^n. \tag{3.1}$$

Applying the maximum principle to $(\mathrm{MA})_t$ yields $\sup_M u \leqslant C$, which is the same as

$$\sup_{\widetilde{M}} \widetilde{u} \leqslant C.$$

To get the C^2 estimate of $(\mathrm{MA})_t$, we shall make use of (3.1). It follows that

$$\mathrm{Ric}(\widetilde{\omega}_t) = -\widetilde{\omega}_t + t\omega \geqslant -\widetilde{\omega}_t.$$

Apply Proposition 2.4 to $\Psi : (\widetilde{M}, \widetilde{\omega}_t) \to (\mathbb{B}^n, \omega_{\mathfrak{B}})$ with $\widetilde{S} = \mathrm{tr}_{\widetilde{\omega}_t}(\Psi^*\omega_{\mathfrak{B}})$ to obtain that

$$\Delta_{\widetilde{\omega}_t} \log \widetilde{S} \geqslant \left(\frac{l+1}{l} + \frac{t}{n}\right)\widetilde{S} - 1,$$

where l is the rank of $\mathrm{d}\Psi$ at a point. Applying the maximum principle yields

$$\widetilde{S} \leqslant \left(\frac{l+1}{l} + \frac{t}{n}\right)^{-1} \leqslant \frac{l}{l+1}.$$

By the assumption and Lemma 2.3, $\mathrm{d}\Psi$ at $\widetilde{x}_0$ has rank n. It follows that

$$C_0^{-1}\widetilde{\omega} \leqslant \Psi^*\omega_{\mathfrak{B}} \leqslant C_0\widetilde{\omega} \quad \text{at } \widetilde{x}_0.$$

Here $C_0 > 0$ is a constant which depends on the fixed point $\widetilde{x}_0$ but not on t. Hence,

$$\mathrm{tr}_{\omega_t}\,\omega(x_0) = \mathrm{tr}_{\widetilde{\omega}_t}\,\widetilde{\omega}(\widetilde{x}_0) \leqslant C_0\widetilde{S}(\widetilde{x}_0) \leqslant \frac{nC_0}{1+n},$$

where $x_0 = \pi(\widetilde{x}_0)$. It follows that

$$e^{-\frac{1}{n}\max_M u} \leqslant \left(\frac{\omega^n}{\omega_t^n}\right)^{1/n}(x_0) \leqslant \frac{\mathrm{tr}_{\omega_t}\,\omega}{n}(x_0) \leqslant \frac{C_0}{1+n}.$$

Hence, $-\max_M u \leqslant n\log(C_0/(n+1))$. It then follows from the process [24, after (8), p. 908] that

$$\int_M c_1(K_M)^n = \lim_{t\to 0}\int_M \omega_t^n > 0.$$

This implies the bigness of K_M, as a consequence of Demailly's Morse inequality. □

4. Proof of Theorem 1.5

Proof of Theorem 1.5. — It is sufficient to show the existence of the Kähler–Einstein metric. As in the proof of Theorem 1.3, we use the continuity method to solve the Monge–Ampère type equation

$$\begin{cases} (t\omega + \mathrm{dd}^c \log \omega^n + \mathrm{dd}^c u)^n = e^u \omega^n, \\ c_t^{-1}\omega \equiv t\omega + \mathrm{dd}^c \log \omega^n + \mathrm{dd}^c u < c_t \omega \quad \text{on } M \end{cases}$$

for $t = 0$. The nonemptyness and openness follow from the same process as in the proof of Theorem 1.3. Furthermore, we have

$$\sup_M u \leqslant C. \tag{4.1}$$

To get the closedness, we lift the equation to $\widetilde{\omega}_t^n = e^{\widetilde{u}}\widetilde{\omega}^n$ on $\widetilde{M}$, where $\widetilde{\omega} = \pi^*\omega$ and $\widetilde{\omega}_t = \pi^*\omega_t$ are both complete metrics on $\widetilde{M}$.

If $\widetilde{E} \subset \widetilde{M}$ is either compact or $\widetilde{M} \setminus \widetilde{E}$ is a bounded domain with respect to $\widetilde{\omega}$, then by Lemma 2.5 we can assume that ω and $\widetilde{\omega}$ have the quasi-bounded geometry. In the following, we consider the general case that $\widetilde{E}$ is a closed subset of $\widetilde{M}$.

For each $\widetilde{x} \in \widetilde{E}$, we apply Proposition 2.4 to $F : (\widetilde{M}, \widetilde{\omega}_t) \to (N, \omega_N)$ to obtain

$$\Delta_{\widetilde{\omega}_t} \log \widetilde{S} \geqslant \left(\frac{l+1}{l} + \frac{t}{n}\right)\widetilde{S} - 1,$$

where

$$\widetilde{S} = \mathrm{tr}_{\widetilde{\omega}_t}(F^*\omega_N).$$

By assumption (2), we have $l \geqslant n$. Applying the second author's upper bound lemma (see, for example, [1, p. 353, Theorem 8] or [22, p. 407, Lemma 3.2]) yields

$$\widetilde{S} \leqslant \frac{n}{n+1}, \quad \text{i.e.,} \quad F^*\omega_N \leqslant \frac{n}{n+1}\widetilde{\omega}_t.$$

This together with assumption (2) imply that

$$C_2\widetilde{\omega} \leqslant \frac{2n}{n+1}\widetilde{\omega}_t, \quad \text{i.e.,} \quad \mathrm{tr}_{\widetilde{\omega}_t}\widetilde{\omega} \leqslant \frac{2n}{(n+1)C_2} \qquad \text{on } \widetilde{E}. \tag{4.2}$$

It remains to estimate $\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega}$ on $\widetilde{M}\setminus\widetilde{E}$. For this we need $\inf_{\widetilde{M}}\widetilde{u}$. Note that (4.2) implies that

$$e^{-\widetilde{u}} = \frac{\widetilde{\omega}^n}{\widetilde{\omega}_t^n} \leqslant \left(\frac{\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega}}{n}\right)^n \leqslant \left(\frac{2}{(n+1)C_2}\right)^n \quad \text{on } \widetilde{E}.$$

Hence,

$$\inf_{\widetilde{E}}\widetilde{u} \geqslant n\log\frac{C_2(n+1)}{2}. \tag{4.3}$$

On the other hand, by assumption (1) we can apply the second author's upper bound lemma to

$$(t\widetilde{\omega} + \mathrm{dd}^c\log\widetilde{\omega}^n + \mathrm{dd}^c\widetilde{u})^n = e^{\widetilde{u}}\widetilde{\omega}^n \tag{4.4}$$

on $\widetilde{M}\setminus\widetilde{E}$ to obtain

$$\inf_{\widetilde{M}\setminus\widetilde{E}}\widetilde{u} \geqslant \log\frac{(\mathrm{dd}^c\log\widetilde{\omega}^n)^n}{\widetilde{\omega}^n} \geqslant -n\log C_1. \tag{4.5}$$

Indeed, if $\widetilde{u}$ attains its infimum in $\widetilde{E}$, then it is already estimated by (4.3). If $\widetilde{u}$ attains its infimum at an interior point in $\widetilde{M}\setminus\widetilde{E}$, then (4.5) follows from applying the usual maximum principle to (4.4). The difficulty lies in the case that $\widetilde{u}$ tends to its infimum at infinity. By assumption (1) and (4.4), we have

$$e^{-\widetilde{u}} = \frac{\widetilde{\omega}^n}{\widetilde{\omega}_t^n} \leqslant C_1^{-n}\frac{(\widetilde{\omega}_t - t\widetilde{\omega} - \mathrm{dd}^c\widetilde{u})^n}{\widetilde{\omega}_t^n} \leqslant C_1^{-n}\left[1 + \frac{1}{n}\Delta_{\widetilde{\omega}_t}(-\widetilde{u})\right]^n.$$

We then apply the second author's upper bound lemma or generalized maximum principle to the following inequality

$$\Delta_{\widetilde{\omega}_t}(-\widetilde{u}) \geqslant nC_1e^{-\widetilde{u}/n} - n$$

to obtain (4.5). Thus, combining (4.3) and (4.5) yields

$$\inf_{\widetilde{M}}\widetilde{u} \geqslant -C(n, C_1, C_2).$$

Applying Proposition 2.4 with $N = M$, $F =$ identity map yields

$$\begin{aligned}\Delta_{\widetilde{\omega}_t}\left(\log(\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega}) - A\widetilde{u}\right) &\geqslant \left(\frac{t}{n} - \frac{(n+1)\kappa_1}{2n}\right)\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega} - A\Delta_{\widetilde{\omega}_t}\widetilde{u} - 1\\ &\geqslant \left(AC_1 - \frac{(n+1)\kappa_1}{2n}\right)\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega} - An - 1 \quad \text{on } \widetilde{M}\setminus\widetilde{E},\end{aligned}$$

where $\kappa_1 = \sup H(\widetilde{\omega})$ and we use (1) and the fact that $\operatorname{Ric}(\widetilde{\omega}_t) = -\widetilde{\omega}_t + t\widetilde{\omega} \geqslant -\widetilde{\omega}_t$. Fix a constant A such that

$$AC_1 - \frac{(n+1)\kappa_1}{2n} \geqslant 1.$$

It follows that

$$\Delta_{\widetilde{\omega}_t}\left(\log(\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega}) - A\widetilde{u}\right) \geqslant C(\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega})e^{-A\widetilde{u}} - An - 1 \tag{4.6}$$

on $\widetilde{M}\setminus\widetilde{E}$. Applying the second author's upper bound lemma yields

$$\sup_{\widetilde{M}\setminus\widetilde{E}}(\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega}) \leqslant C = C(C_1, n, \kappa_1, C_2).$$

This together with (4.2) implies the desired second order estimate

$$\operatorname{tr}_{\widetilde{\omega}_t}\widetilde{\omega} \leqslant C \quad \text{on } \widetilde{M}.$$

Hence,

$$\operatorname{tr}_{\omega_t}\omega \leqslant C \quad \text{on } M.$$

This together with (4.1) yields the desired closedness. The proof is therefore completed. □

5. Examples

In the following examples, when we say *a manifold possesses a complete Kähler–Einstein metric*, we mean the manifold possesses a unique complete Kähler–Einstein metric with Ricci curvature equal to -1, and the curvature tensor of Kähler–Einstein metric and all its covariant derivatives are bounded, unless otherwise indicated.

Example 5.1. — Let Ω be a bounded strictly pseudoconvex domain with smooth boundary in a Stein manifold (for instance, an open ball in $\mathbb{C}^n$). Then, every closed complex submanifold Σ of Ω possesses a complete Kähler–Einstein metric.

To see this, recall that Ω possesses a complete Kähler metric ω of the bounded geometry and negatively pinched holomorphic curvature (cf. [2] and [11, p. 281, Theorem 2]). Then, the restriction $\omega|_\Sigma$ of ω defines a complete Kähler metric on Σ with negatively pinched holomorphic curvature, in view of the decreasing property of holomorphic curvature on submanifolds and the bounded geometry. It then follows from Theorem 1.2 that Σ possesses a complete Kähler–Einstein metric, which is uniformly equivalent to ω.

Example 5.2. — Let $\mathbb{D}$ be the unit open disk in $\mathbb{C}$ associated with the Poincaré metric $\mathcal{P} = \mathrm{d}z \otimes \mathrm{d}\bar{z}/(1-|z|^2)^2$. Denote by $\mathbb{D}^n$ be the n-polydisk $\mathbb{D}\times\cdots\times\mathbb{D}$ with the product metric of Poincaré metrics. Then, similar to Example 5.1, every closed complex submanifold of $\mathbb{D}^n$ possesses a complete Kähler–Einstein metric. □

Example 5.3. — Let M be a smooth noncompact quotient of the unit open ball in $\mathbb{C}^n$ (a model case is $M = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ when $n = 1$). Then, every closed complex submanifold of X possesses a complete Kähler–Einstein metric.

This assertion is parallel to the compact case of smooth ball quotient ([23, p. 597]). Note that the Bergman metric $\omega_{\mathfrak{B}}$ on the unit ball descends to M a complete Kähler–Einstein metric of the quasi-bounded geometry and constant negative holomorphic curvature. The assertion then follows from either Theorem 1.2 or Theorem 1.3, together with the decreasing property of holomorphic curvature on complex submanifolds.

Example 5.4. — Let V be an irreducible, smooth, complex (quasi-)projective variety. Then, for every point $x \in V$, there exists a Zariski neighborhood $U = V \setminus Z$ of x such that U possesses a complete Kähler–Einstein metric of finite volume, where Z is an algebraic subvariety of V.

Indeed, by Lemma 2.3 in [7, p. 25], there is a quasi-projective variety $U = X \setminus Z$ containing x such that U can be smoothly embedded into

$$S_1 \times \cdots \times S_N$$

as a closed algebraic submanifold, in which $S_j = \mathbb{CP}^1 \setminus \{z_1, \ldots, z_{N_j}\}$ and $N_j \geqslant 3$. Note that each S_j admits a complete Kähler–Einstein metric ω_j with Poincaré growth near z_i, $i = 1, \ldots, N_j$. The metric ω_j has Gauss curvature -1 and quasi-bounded geometry. Then, the product $S_1 \times \cdots \times S_N$ has negatively pinched holomorphic sectional curvature, quasi-bounded geometry, and finite volume; so does the submanifold U. The assertion then follows from Theorem 1.2 or Theorem 1.3.

Example 5.5. — Let Σ be a closed complex surface in $\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D}$, and let $D \subset \Sigma$ be a smooth divisor, i.e., a Riemann surface, in Σ. For simplicity, we assume that

$$\overline{D} \cap \partial\mathbb{D}^3 \subset \{(z^1, z^2, z^3) \in \overline{\mathbb{D}^3} : |z^1| = 1, |z^2| < 1, z^3 = 0\}, \tag{5.1}$$

and $\overline{D}$ intersect $\partial\mathbb{D}^3$ transversally at each intersection point. Here by the closure $\overline{G}$ and boundary ∂G of a set G we mean the topological closure and topological boundary of G in $\mathbb{C}^3$. We *claim* that the complex surface $\Sigma \setminus D$ possesses a complete Kähler–Einstein metric.

Let us make some remarks before proving the claim. First, a special case of (Σ, D) is that $\Sigma = \mathbb{D}^2 \times \{0\}$ and D is a Riemann surface in Σ such that $\overline{D}$ only intersects $\partial\mathbb{D}^2$ at some points in $\{|z^1| = 1, |z^2| < 1\}$, and the intersection is transversal. The model case is $\mathbb{D} \times \mathbb{D}^*$ where $\mathbb{D}^* \equiv \mathbb{D} \setminus \{0\}$.

Second, the complex surface $\Sigma \setminus D$ is not covered by the previous examples, neither by the known complete Kähler–Einstein manifolds constructed in, for example, [28, 3, 19, 22], since $\Sigma \setminus D$ does not have finite volume, nor by [15, Theorem, p. 49] since $\Sigma \setminus D$ is not compactly contained in the Stein manifold Σ.

Third, the hypersurface D of Σ cannot be replaced by a submanifold of Σ with codimension $\geqslant 2$, by virtue of Theorem 1.3 in [6, p. 1272]. The closure of D necessary intersects $\partial\mathbb{D}^3$; for, otherwise D would be a compact Stein manifold.

Proof of the claim. — We shall apply Theorem 1.5 with $M = \widetilde{M} = \Sigma \setminus D$. We need to construct a complete Kähler metric ω of quasi-bounded geometry satisfying conditions (1) and (2) in Theorem 1.5.

Denote by $\omega_{\mathcal{P}}$ the product of the Poincaré metrics on $\mathbb{D}^3$. Then, $\omega_{\mathcal{P}}|_\Sigma$ defines a complete Kähler metric on Σ of negatively pinched holomorphic curvature and the quasi-bounded geometry, and so does $\omega_{\mathcal{P}}|_D$ on D. As in Example 5.2, the submanifolds Σ and D possess complete Kähler–Einstein metrics, denoted by ζ and η, respectively, which are uniformly equivalent to $\omega_{\mathcal{P}}|_\Sigma$ and $\omega_{\mathcal{P}}|_D$ and have the quasi-bounded geometry.

Choose a smooth metric h_1 on $[D]$ over Σ such that h_1 can be smoothly extended over $\overline{\Sigma}$ and that h_1 is identically constant outside a tubular neighborhood U of D in Σ. It follows that

$$\mathrm{dd}^c \log \frac{\det \zeta_{i\bar{j}}}{h_1} = \mathrm{dd}^c \log \zeta^2 - \mathrm{dd}^c \log h_1 = \zeta - \mathrm{dd}^c \log h_1$$

which is positive outside U, where $\det \xi_{i\bar{j}}$ is the coefficients of ξ^2 in terms of the local coordinates on Σ. Let

$$h = h_1[(1-|z^1|^2)(1-|z^2|^2)(1-|z^3|^2)]^N,$$

where (z^1, z^2, z^3) are the global coordinates on $\mathbb{D}^3$, and $N > 0$ is a fixed a large number such that

$$\omega_K \equiv \mathrm{dd}^c \log \frac{\det \zeta_{i\bar{j}}}{h} = \zeta - \mathrm{dd}^c \log h_1 + N\omega_{\mathcal{P}}|_\Sigma > 0$$

everywhere on Σ.

By the adjunction formula $(K_\Sigma + [D])|_D = K_D$, the metric $(\det \zeta_{i\bar{j}})/h$ restricted to D defines a volume form Φ_D on D. Let $e^{u_D} = \eta/\Phi_D$. Then u_D and all its covariant derivatives are bounded with respect to $\omega_{\mathcal{P}}|_D$. Observe that

$$\omega_K|_D = \mathrm{dd}^c \log(\eta e^{-u_D}) = \mathrm{dd}^c \log \eta - \mathrm{dd}^c u_D = \eta - \mathrm{dd}^c u_D.$$

We can extend u_D to a smooth function u on Σ satisfying that $u|_D = u_D$ and that $\omega_K + \mathrm{dd}^c u$ is uniformly equivalent to $\omega_{\mathcal{P}}|_\Sigma$ (see for example [21, p. 815]); that is,

$$\begin{cases} (\omega_K + \mathrm{dd}^c u)|_D = \eta, \\ C^{-1}\omega_{\mathcal{P}}|_\Sigma \leqslant \omega_K + \mathrm{dd}^c u \leqslant C\,\omega_{\mathcal{P}}|_\Sigma, \quad \text{for some constant } C > 0. \end{cases} \tag{5.2}$$

Define

$$\omega = A(\omega_K + \mathrm{dd}^c u) - \mathrm{dd}^c \log(\log|s|^2)^2 \quad \text{on } \Sigma \setminus D,$$

where $A > 0$ is a large constant, s is a holomorphic defining section of D on Σ, and the norm $|s|^2$ is with respect to metric h_1 on $[D]$ such that $|s|^2 < 1/e$. We would like to show that ω is a desired complete Kähler metric on $\Sigma \setminus D$. Note that

$$-\mathrm{dd}^c \log(\log|s|^2)^2 = \frac{2\mathrm{d}\log|s|^2 \wedge \mathrm{d}^c \log|s|^2}{(\log|s|^2)^2} + \frac{2\mathrm{dd}^c \log h_1}{-\log|s|^2}. \tag{5.3}$$

Fix a large constant A so that

$$A(\omega_K + \mathrm{dd}^c u) + \frac{2\mathrm{dd}^c \log h_1}{-\log|s|^2} \geqslant \omega_K + \mathrm{dd}^c u \geqslant C^{-1}\omega_{\mathcal{P}}|_\Sigma.$$

Thus, $\omega > 0$ on $\Sigma \setminus D$.

To see the completeness of ω, we only need to consider ω near D. Note that, near D but away from $\partial\mathbb{D}^3$, the normal direction to D is dominated by the second term on the right of (5.3), i.e.,

$$\frac{2\mathrm{d}\log|s|^2 \wedge \mathrm{d}^c \log|s|^2}{(\log|s|^2)^2},$$

which has Poincaré growth. For any $x \in \overline{D} \cap \partial\mathbb{D}^3$, by (5.1) we can assume that near x the divisor D is given by $\{z^2 = 0\} \cap \Sigma$, and hence,

$$|s|^2 = h_1|z^2|^2 \quad \text{in a neighborhood of } x \text{ in } \Sigma.$$

Then, the metric

$$\begin{aligned} \omega \geqslant C^{-1}\omega_{\mathcal{P}}|_\Sigma &+ \frac{2\mathrm{d}\log|s|^2 \wedge \mathrm{d}^c \log|s|^2}{(\log|s|^2)^2} \\ &\sim C^{-1}\frac{(\sqrt{-1}/2)\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1}{(1-|z^1|^2)^2} + \frac{\sqrt{-1}\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{|z^2|^2(\log|z^2|^2)^2} \end{aligned}$$

on $V \setminus D$ as $|z^1| \to 1$ and $z^2 \to 0$. Here the symbol $\sim$ gives the leading order term of an asymptotic expansion. This shows the completeness of ω. It is clear that ω has the quasi-bounded geometry.

Next to compute $\mathrm{dd}^c \log \omega^2$. Note that

$$\begin{aligned}\omega^2 &= 2\left[A(\omega_K + \mathrm{dd}^c u) + \frac{2\mathrm{dd}^c \log h_1}{-\log|s|^2}\right] \wedge \frac{2\mathrm{d}\log|s|^2 \wedge \mathrm{d}^c \log|s|^2}{(\log|s|^2)^2} \\ &\quad + \left[A(\omega_K + \mathrm{dd}^c u) + \frac{2\mathrm{dd}^c \log h_1}{-\log|s|^2}\right]^2 \\ &= 4\frac{A(\omega_K + \mathrm{dd}^c u) \wedge \mathrm{d}\log|s|^2 \wedge \mathrm{d}^c \log|s|^2}{(\log|s|^2)^2}\left[1 + \frac{f}{\log|s|^2}\right],\end{aligned}$$

where f is a smooth function on $\Sigma \setminus D$ and all covariant derivatives of f are bounded with respect to ω. By (5.2), we have

$$\mathrm{dd}^c \log \omega^2 \sim \eta - \mathrm{dd}^c \log(\log|s|^2)^2$$

on a deleted neighborhood of D in Σ away from $\partial\mathbb{D}^3$, as $|s| \to 0$. Hence, we obtain

$$\mathrm{dd}^c \log \omega^2 \geqslant \frac{1}{2A}\omega$$

on a smaller deleted neighborhood of D in Σ away from $\partial\mathbb{D}^3$.

For any $x \in \overline{D} \cap \partial\mathbb{D}^3$, there is a neighborhood V of x in $\mathbb{C}^3$ such that $\Sigma \cap V = \{z^3 = 0\}$ and $D \cap V = \{z^2 = z^3 = 0\}$. Then, $|s|^2 = h_1|z^2|^2$. Note that

$$\begin{aligned}\omega_K + \mathrm{dd}^c u &= \zeta - \mathrm{dd}^c \log h_1 + \mathrm{dd}^c u + N\omega_{\mathcal{P}}|_\Sigma \\ &\sim a_{1\bar{1}}\frac{(\sqrt{-1}/2)\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1}{(1-|z^1|^2)^2} + \eta_{1\bar{2}}\frac{\sqrt{-1}}{2}\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^2 \\ &\qquad + \eta_{2\bar{1}}\frac{\sqrt{-1}}{2}\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^1 + \eta_{2\bar{2}}\frac{\sqrt{-1}}{2}\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2,\end{aligned}$$

on $(V \cap \Sigma) \setminus D$ as $|z^1| \to 1$ and $z^2 \to 0$, where $a_{1\bar{1}}$ and $\eta_{2\bar{2}}$ are positive bounded smooth functions on $\overline{V}$ and $|\eta_{1\bar{2}}| = |\eta_{2\bar{1}}| = O\big((1-|z^1|^2)^{-1}\big)$. It follows that

$$\omega^2 \sim Aa_{1\bar{1}}\frac{(\sqrt{-1})^2\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{(1-|z^1|^2)^2|z^2|^2(\log|z^2|^2)^2} \quad \text{on } (V \cap \Sigma) \setminus D,$$

as $|z^1| \to 1$ and $z^2 \to 0$. Hence,

$$\mathrm{dd}^c \log \omega^2 \geqslant \frac{1}{2A}\omega$$

on a smaller deleted neighborhood of D in $V \cap \Sigma$. Therefore, there exists a tubular neighborhood $D_{2\delta} = \{|s| < 2\delta\}$ of D in Σ such that

$$\mathrm{dd}^c \log \omega^2 \geqslant \frac{1}{2A}\omega \quad \text{on } D_{2\delta} \setminus D.$$

Let $E = \Sigma \setminus D_\delta$. Then, E is a closed subset of $\Sigma \setminus D$. Notice that $\log(-\log|s|^2)$ is a bounded smooth function on the closure $\overline{E}$ in $\mathbb{C}^3$. Then,

$$-C_1\omega_{\mathcal{P}}|_\Sigma \leqslant -2\mathrm{dd}^c\log(-\log|s|^2) \leqslant C_1\omega_{\mathcal{P}}|_\Sigma \quad \text{on } E,$$

for some constant $C_1 > 0$. It follows that

$$\omega \leqslant (AC + C_1)\,\omega_{\mathcal{P}}|_\Sigma \quad \text{on } E.$$

Thus, the complete Kähler metric ω of the quasi-bounded geometry satisfies conditions (1) and (2) of Theorem 1.5. The claim then follows from Theorem 1.5. □

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