# ANNALES

## DE

# L'INSTITUT FOURIER

# LOG-CANONICAL THRESHOLDS IN REAL AND COMPLEX DIMENSION 2

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*Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday*

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ABSTRACT. — We study the set of log-canonical thresholds (or critical integrability indices) of holomorphic (resp. real analytic) function germs in $\mathbb{C}^2$ (resp. $\mathbb{R}^2$). In particular, we prove that the ascending chain condition holds, and that the positive accumulation points of decreasing sequences are precisely the integrability indices of holomorphic (resp. real analytic) functions in dimension 1. This gives a new proof of a theorem of Phong–Sturm.

RÉSUMÉ. — Nous étudions l'ensemble des seuils log-canoniques (ou indices d'intégrabilité critiques) des germes de fonctions holomorphes (resp. réel analytiques) dans $\mathbb{C}^2$ (resp. $\mathbb{R}^2$). En particulier, nous prouvons que la condition de la chaîne ascendante est vraie et que les points d'accumulation positifs des séquences décroissantes sont précisément les indices d'intégrabilité des fonctions holomorphes (resp. réel analytiques) en dimension 1. Cela donne une nouvelle preuve d'un théorème de Phong-Sturm.

## 1. Introduction

Let $f$ be a holomorphic or real analytic function defined in a neighbourhood of the origin in $\mathbb{C}^n$ or $\mathbb{R}^n$. We define the critical integrability index, or log-canonical threshold, of $f$ at the origin to be

$$c_0(f) := \sup \left\{ c > 0 \ : \exists \, \varepsilon > 0 \text{ such that } \int_{B_\varepsilon(0)} |f|^{-c} < +\infty \right\}$$

The number $c_0(f)$ is a measure of the order of vanishing of $f$ at the origin; indeed in dimension 1 it is easy to see that $c_0(f)$ is inverse proportional

to the order of vanishing, while in higher dimension $c_0(f)$ is an important and subtle invariant of the set $\{f = 0\}$. In real analysis the critical integrability index plays a prominent role in the asymptotic analysis of oscillatory integral operators [12, 25, 36], and has recently appeared as an important invariant in statistics (see [21] and the references therein). In complex differential and algebraic geometry, the critical integrability index (and its generalizations to ideal sheaves) appear in geometric applications including existence of Kähler–Einstein metrics on Fano manifolds (where $c_0(f)$ is called the $\alpha$-invariant) [3, 7, 31, 32, 33, 34, 35], and in the minimal model program (where $c_0(f)$ is called the log-canonical threshold) [16, 17, 29, 30]. The reader should note that our definition of the critical integrability index differs by a factor of 2 from the usual definition in the complex setting.

In their study of complex singularity exponents and applications to Kähler–Einstein metrics on Fano orbifolds, Demailly–Kollár [7] proposed a series of remarkable conjectures regarding the numbers $c_0(f)$, including the following

CONJECTURE 1.1. — *The set*

$$\mathcal{C}(n) = \{c_0(f) : f \in \mathcal{O}_{\mathbb{C}^n,0}\}$$

*satisfies the ascending chain condition: precisely, any ascending sequence eventually stabilizes.*

A closely related, and well-known conjecture, called the Ascending Chain Condition (ACC) conjecture, appears in the algebraic geometry literature dating back to Shokurov [30] (see also [16, 19]). Shokurov himself established the 2-dimensional case of the ACC conjecture using Mori's minimal model program [29]. There has been remarkable recent progress on the ACC conjecture, and it is now established in dimension 3 by work of Alexeev [1], and for smooth varieties in arbitrary dimension by de Fernex–Ein–Mustaţă [10]. Finally, the ACC was established in full generality by Hacon–McKernan–Xu [13]. We refer the reader also to the earlier works [11, 19] where important partial results were obtained. The proof of the smooth case of the Shokurov's ACC conjecture implies

THEOREM 1.2 (de Fernex–Ein–Mustaţă [10, Theorem 1.1]). — *Conjecture 1.1 holds.*

In complex dimension 2 there have been several contributions to Conjecture 1.1. Igusa [15] computed the log-canoncial thresholds of irreducible plane curves with singularities at the origin. This was subsequently improved by Kuwata [20] who computed all of the critical integrability indices

in dimension 2 using algebraic techniques. Phong–Sturm, using analytic techniques they developed in [27], described all the critical integrability indices in dimension 2 [28], giving an analytic proof of Conjecture 1.1, and characterizing all the accumulation points of $\mathcal{C}(2)$. Furthermore, their results hold more generally in the setting of real analytic functions in $\mathbb{R}^2$ [28, Remark 1]. Favre–Jonsson proved Conjecture 1.1 in complex dimension 2 [9] as a by product of their robust algebraic techniques based on valuations. More recently, Hai–Hiep–Hung [14] gave another analytic proof of Conjecture 1.1 in dimension 2. Summarizing;

THEOREM 1.3. — *Let $\mathcal{C}(2)$ denote the set of integrability indices for holomorphic (resp. real analytic) germs defined near the origin in $\mathbb{C}^2$ (resp. $\mathbb{R}^2$). Then $\mathcal{C}(2)$ satisfies the ascending chain condition. Furthermore, the accumulation points of $\mathcal{C}(2)$ are precisely $\{0\} \cup (\mathcal{C}(1)\backslash 2)$.*

In this generality, Theorem 1.3 is due to Phong–Sturm [28, Theorem A, Remark 1]. The goal of this note is to give a new "elementary" proof of Theorem 1.3. The main idea in our approach is to use the connection between integrability indices and convex bodies. Recall that the Newton polyhedron of $f$ is the convex polyhedron $NP(f) \subset \mathbb{R}^2_{\geqslant 0}$ constructed as the convex hull of the set $(p, q) + \mathbb{R}^2_{\geqslant 0}$, where $x^p y^q$ appears in the Taylor series of $f$ with non-zero coefficient (see Section 2 for more details). The Newton polyhedron is not coordinate invariant, but in dimension two one can find a holomorphic (resp. real analytic) change of coordinates so that $NP(f)$ computes $c_0(f)$.

THEOREM 1.4. — *In $\mathbb{C}^2$ (resp. $\mathbb{R}^2$), there exists a holomorphic (resp. real analytic) change of coordinates of the form*

$$(\tilde{x}, \tilde{y}) = (x - Q(y), y) \quad \text{or} \quad (\tilde{x}, \tilde{y}) = (x, y - Q(x))$$

*so that*

$$c_0(f) = 2\delta_{NP} \qquad (\text{resp. } \delta_{NP})$$

*where $\delta_{NP}(f)^{-1}$ is the Newton distance of $f$ in the coordinates $(\tilde{x}, \tilde{y})$ defined by $(\delta_{NP}^{-1}, \delta_{NP}^{-1}) \in \partial NP(f)$.*

Assuming Theorem 1.4, our proof of Theorem 1.3 is based on the observation that the Newton polyhedron is characterized by "discrete" data (namely the integral vertices), and so one should expect some rigidity for the possible values obtained by intersecting with the diagonal. By Theorem 1.4 this implies rigidity for integrability indices. The referee kindly pointed out to us that ACC part of Theorem 1.3 can also be deduced from Theorem 1.4 using an algebraic result of Maclagan [22, Corollary 5.2].

Theorem 1.4 was proved by Varčhenko [36]. The description of the coordinate change in the real setting is implicit in [36, Lemma 3.6], and these techniques generalize to the complex case. In the real case, Theorem 1.4 appears explicitly in the work of Phong–Stein–Sturm [26] as an application of their techniques for studying stability of certain oscillatory integrals. In the complex case, a version of Theorem 1.4 is proved in [2], omitting only the description of the change of variables. The techniques of [2] are algebraic, making use of the local topological zeta function of Denef–Loeser [8]. We remark that the explicit description of the coordinate change will be important for our argument. In $\mathbb{R}^n, n \geqslant 3$, an analogue of Theorem 1.4 was proved by the author with Greenleaf and Pramanik [6] though crucially, one must allow coordinate changes involving fractional power series. Since the result we need is not explicitly stated in the literature we shall give a self-contained proof of Theorem 1.4 in the spirit of the methods developed by Phong–Stein–Sturm [26], and later work of the author with Greenleaf and Pramanik [6]. These techniques are based on an analytic resolution of singularities algorithm (building on work of Bierstone–Milman [4, 5] and Parusiński [23, 24]) and sharp estimates. Furthermore, we view this as an opportunity to illustrate the techniques of [6] in an example.

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## 2. Background

Suppose $f(x,y)$ is a holomorphic (resp. real analytic) function defined in a neighbourhood of $0 \in \mathbb{C}^2$ (resp. $\mathbb{R}^2$) and with $f(0,0) = 0$. We expand $f$ as a power series

$$(2.1) \qquad f(x,y) = \sum_{(p,q)\in\mathbb{N}^2} a_{p,q} x^p y^q.$$

DEFINITION 2.1. — *The Newton polyhedron of $f$, denoted $NP(f)$ is the convex set defined as*

$$NP(f) = \mathrm{ConvexHull}(\{(p,q) + \mathbb{R}^2_{\geqslant 0} : a_{p,q} \neq 0\}).$$

In words, for every monomial $x^p y^q$ appearing in the Taylor series of $f$, one attaches a copy of the positive orthant at $(p,q)$, and then takes the convex hull. The result is a non-compact convex body with polyhedral boundary.

*Remark 2.2.* — It is important to note that the Newton polyhedron is *not* independent of the choice of coordinates. For example, the polynomial $f(x, y) = (y - x)^N$ has Newton polyhedron $\{p + q \geqslant N\}$, while $g(x, y) = y^N$ has Newton polyhedron $\{q \geqslant N\}$. But clearly $f = g(x, y - x)$.

DEFINITION 2.3. — *The Newton distance, denoted $\delta_{NP}$ is defined by*

$$\delta_{NP}^{-1}(f) = \inf\{t \in \mathbb{R}_{\geqslant 0} : (t, t) \in NP(f)\}$$

That is $(\delta_{NP}^{-1}, \delta_{NP}^{-1})$ is the point where the diagonal line $\{p = q\}$ intersects $\partial NP(f)$. The Newton polyhedron plays a fundamental role in the study of the zeroes of $f$; we will expand upon some of these connections, but refer the reader to [18] for a more thorough discussion.

We define the order of $f$ at 0 to be $N$, denoted $\mathrm{ord}_0\, f = N$, if $N$ is the minimal homogeneous degree of a non-zero monomial appearing in (2.1). Equivalently, $N$ is the order of vanishing of $f$ restricted to a generic line in $\mathbb{C}^2$ or $\mathbb{R}^2$. Therefore, up to making a linear change of coordinates, we can assume that

$$f(0, y) = cy^N + O(y^{N+1}), \quad c \neq 0.$$

By the Weierstrass preparation theorem we can write

$$f(x, y) = (unit) \cdot P(x, y)$$

where *unit* denotes a non-vanishing holomorphic (resp. real analytic) function, and $P(x, y)$ is a Weierstrass polynomial of degree $N$ in $y$;

$$P(x, y) = y^N + \sum_{\ell=0}^{N-1} b_\ell(x) y^\ell$$

with $b_\ell(x)$ holomorphic (resp. real analytic) functions vanishing at $x = 0$.

*Remark 2.4.* — We point out that, for a generic choice of coordinates we can also arrange that $f(x, 0) = c'x^N + O(x^{N+1})$. In particular, we can ensure that $f(x, y)$ can be written simultaneously as a Weierstrass polynomial of order $N$ in $x$ and a Weierstrass polynomial of order $N$ in $y$ (up to multiplication by a unit). This will be a convenient choice of coordinates to make in Section 4.

For completeness, and since we will later have to allow changes of coordinates, we will compute the critical integrability index of a general holomorphic/real analytic function $f$. Up to discarding a non-vanishing function, we can take $f$ to be of the form

$$f(x, y) := x^\alpha y^\beta P(x, y)$$

where $P(x,y)$ is a Weierstrass polynomial in $y$ of degree $N$ with $P(x,0) \not\equiv 0$. It is a classical fact that after possibly shrinking the neighbourhood of the origin we can factor

$$(2.2) \qquad P(x,y) = \prod_{\nu=1}^{N} (y - \varphi_\nu(x))$$

where the $\varphi_\nu$ are convergent Puiseux series solutions of the form

$$\varphi_\nu(x) = b_\nu x^{a_\nu} + O(x^{A_\nu})$$

with $b_\nu \in \mathbb{C}^*$ and $a_\nu \in \mathbb{Q}_{>0}$, $A_\nu > a_\nu$. More precisely, each root $\varphi_\nu$ is obtained from a complex analytic function $g_\nu(t)$ by a subsitution $t = x^{1/N}$ for some positive integer $N$. In the real case this can be deduced from the Newton–Puiseux theorem by first extending $f(x,y)$ to a complex analytic function $f(z,w)$ defined in a neighbourhood of zero, and then restricting the Puiseux series solutions to the set $\mathrm{Im}(z) = \mathrm{Im}(w) = 0$. We refer the reader to [25, Section 3], and [6, Section 2] and the references therein for a complete discussion of the convergence properties of Puiseux series, as well as their properties in the real analytic case. Let

$$0 < a_1 < \cdots < a_k$$

be the distinct leading exponents of the $\varphi_\nu$, and let $F_1, \ldots, F_k$ be the non-vertical/horizontal faces of the Newton polyhedron of $f$, numbered from left to right. The leading exponents are determined by

$$\frac{1}{a_i} = - \text{ the slope of } F_i.$$

Furthermore, the number of solutions $\varphi_\nu$ with a given leading order $a_i$ is nothing but the length of the line obtained by projecting $F_i$ onto the $q$-axis. This discussion is somewhat backwards, since the Puiseux series solutions are in general constructed from the Newton polyhedron inductively [18].

Let us introduce some notation. Set

$$S_i = \{\varphi_\nu(x) : \varphi_\nu(x) = bx^{a_i} + \cdots \text{ for some } b \neq 0\}, \qquad m_i := \#S_i$$

to be the collection of Puiseux series roots with leading order $a_i$. For $0 \leqslant i \leqslant k$, let

$$(2.3) \quad A_i = \sum_{j \leqslant i} m_j a_j + \alpha, \qquad B_i = \sum_{j > i} m_j + \beta, \qquad \delta_i^{-1} := \frac{A_i + a_i B_i}{1 + a_i}.$$

A short computation shows that $(A_i, B_i)$ are the vertices of the Newton polyhedron, and $(\delta_i^{-1}, \delta_i^{-1})$ is the point of intersection of the diagonal with the prolongation of the face $F_i$. We define the *main face* of $\partial NP(f)$ to be

the face or vertex of $\partial NP(f)$ which meets the diagonal; in particular, the main face may not be a face at all.

For each pair of roots $\varphi_\mu, \varphi_\nu$ we can write

$$\varphi_\mu = b_\mu x^{a_\mu} + \cdots, \qquad \varphi_\mu - \varphi_\nu = b_{\mu\nu} x^{r_{\mu\nu}} + \cdots,$$

with $b_\mu, b_{\mu\nu} \neq 0$, and we define the *order of contact* between $\varphi_\mu$ and $\varphi_\nu$ to be $r_{\mu\nu}$. We define $r_{\mu\nu} = +\infty$ if $\varphi_\mu - \varphi_\nu \equiv 0$. In what follows it will be convenient to fix two constants $\varepsilon, D$, which we define as

$$\varepsilon = \frac{3}{4} \min_\mu \min_{\nu:\varphi_\mu \not\equiv \varphi_\nu} \{|b_\mu|, |b_{\mu\nu}|\}, \qquad D = 2 \max_{\mu,\nu} \{|b_\mu|, |b_{\mu\nu}|\}.$$

*Remark 2.5.* — The precise values of $\varepsilon, D$ will not matter. What does matter is an upper bound for $\varepsilon$ and a lower bound for $D$, which we have fixed with the above definition. That is, if $0 < \tilde{\varepsilon} < \varepsilon < D < \tilde{D} < +\infty$, then our arguments work just as well with $\tilde{\varepsilon}, \tilde{D}$.

Roots with order of contact larger than their leading order play an important role in the estimates to follow, and so we will introduce the sets

$$S_{i,\ell} = \{\varphi_\nu(x) : \varphi_\nu(x) = b_{i\ell} x^{a_i} + \cdots\}, \qquad m_{i\ell} := \#S_{i,\ell}.$$

Let

(2.4)                    $$a_i < r_1 < r_2 < \cdots < r_{m-1} < r_m = +\infty$$

be the list of contact orders between $\varphi_\mu, \varphi_\nu \in S_{i,\ell}$ (so that $r_1 > a_i$); note that we do not impose that $\varphi_\mu, \varphi_\nu$ are distinct.

## 3. Proof of Theorem 1.4

We will give the proof of Theorem 1.4 in the complex case, though the reader can check that the argument works just as well in the real case; see [26] and especially [6] for the necessary modifications. Our goal in this section is to estimate the integral

(3.1)                    $$\int_U |f|^{-2c}$$

where $U \ni 0$ is an open set that we can shrink as we please. Note we have inserted a factor of 2 for convenience (this is customary in the complex setting). We first establish that $c_0(f) \leqslant 2\delta_{NP}$, by estimating the integral in a region which is "far" from the roots of $P$.

*Notation 3.1.* — We will use the symbol $\preceq$ to denote "less than or equal, up to multiplication by a positive constant", so $a \preceq b$ means $a \leqslant Cb$ for some $C > 0$.

Fix $0 < \eta \ll 1$ and for $1 \leqslant i \leqslant k - 1$, consider the regions

$$R_{0,1} = \{D|x|^{a_1} < |y| < \varepsilon\} \times \{0 \leqslant |x| \leqslant \eta\}$$

(3.2)     $$R_{i,i+1} = \{D|x|^{a_{i+1}} < |y| < \varepsilon|x|^{a_i}\} \times \{0 \leqslant |x| \leqslant \eta\}$$

$$R_{k,\infty} = \{0 < |y| < \varepsilon|x|^{a_k}\} \times \{0 \leqslant |x| \leqslant \eta\}.$$

We will explain how to estimate the integral over $R_{i,i+1}$ in detail, but let us first state the results for $R_{0,1}$ and $R_{k,\infty}$ first. The integral over $R_{0,1}$ is finite if and only if

$$c\alpha < 1 \quad \text{and} \quad c < \frac{1 + a_1}{(A_1 + a_1 B_1)}$$

while the integral over $R_{k,\infty}$ is finite if and only if

$$c\beta < 1 \quad \text{and} \quad c < \frac{1 + a_k}{(A_k + a_k B_k)}.$$

We impose these conditions from now on. To estimate the integral on $R_{i,i+1}$ it suffices to sharply estimate the quantity $|y - \varphi_\nu(x)|$. There are two cases. First, if $\varphi_\nu \in S_j$ for $j > i$ then we get the estimate

$$|y - \varphi_\nu(x)| \leqslant |y| + |\varphi_\nu(x)| \leqslant \varepsilon|x|^{a_i} + \frac{D}{2}|x|^{a_j} \leqslant 2\varepsilon|x|^{a_i}$$

$$|y - \varphi_\nu(x)| \geqslant |y| - |\varphi_\nu(x)| \geqslant |y| - \frac{D}{2}|x|^{a_j} \geqslant \frac{1}{2}|y|.$$

On the other hand, if $\varphi_\nu \in S_j$ for $j \leqslant i$, then the definition of $\varepsilon$ gives

$$|y - \varphi_\nu(x)| \leqslant |y| + |\varphi_\nu(x)| \leqslant \varepsilon|x|^{a_i} + \frac{D}{2}|x|^{a_j} \leqslant D|x|^{a_j}$$

$$|y - \varphi_\nu(x)| \geqslant |\varphi_\nu(x)| - |y| \geqslant \frac{4\varepsilon}{3}|x|^{a_j} - \varepsilon|x|^{a_i} \geqslant \frac{1}{3}\varepsilon|x|^{a_j}.$$

Thus, in $R_{i,i+1}$ we have

(3.3)                    $$|x|^{A_i}|y|^{B_i} \preceq |f| \preceq |y|^\beta |x|^{A_i + a_i(B_i - \beta)}.$$

We estimate, using $1 > c\beta$

$$\int_{R_{i,i+1}} |f|^{-2c}$$

$$\succeq \int_{0 < |x| < \eta} \int_{C|x|^{a_{i+1}} < |y| < \varepsilon|x|^{a_i}} |y|^{1 - 2c\beta} |x|^{1 - 2c(A_i + a_i(B_i - \beta))} \mathrm{d}|y|\mathrm{d}|x|$$

$$\succeq \int_{0 < |x| < \eta} |x|^{1 + 2a_i - 2c(A_i + a_i B_i)} \mathrm{d}|x|$$

from which we see that the integral over $R_{i,i+1}$ is infinite if

$$c \geqslant \delta_i = \frac{1 + a_i}{A_i + a_i B_i}.$$

Summarizing, we have proved that the integral (3.1) diverges if

$$c \geqslant \min_i \left\{ \frac{1}{\alpha}, \frac{1}{\beta}, \delta_i \right\} = \delta_{NP}.$$

Next we show that this bound is sharp on the regions (3.2). So suppose $c < \delta_{NP}$. Using the estimate (3.3)

$$\int_{R_{i,i+1}} |f|^{-2c} \preceq \int_{\{0<|x|<\eta\}} \int_{C|x|^{a_{i+1}}<|y|<\varepsilon|x|^{a_i}} |y|^{1-2cB_i}|x|^{1-2cA_i} \mathrm{d}|y|\mathrm{d}|x|$$

$$\preceq \int_{\{0<|x|<\eta\}} |x|^{1+2a_i-2c(A_i+a_iB_i)}\mathrm{d}|x|$$

provided $1 > cB_i$. If instead $1 < cB_i$, then we get

$$\int_{R_{i,i+1}} |f|^{-2c} \preceq \int_{\{0<|x|<\eta\}} |x|^{1+2a_{i+1}-2c(A_i+a_{i+1}B_i)}\mathrm{d}|x|$$

Since $A_i + a_{i+1}B_i = A_{i+1} + a_{i+1}B_{i+1}$, we see that the integral over $R_{i,i+1}$ is finite if $c < \min\{\delta_i, \delta_{i+1}\} \leqslant \delta_{NP}$. The regions $R_{0,1}, R_{k,\infty}$ are estimated similarly and we conclude:

LEMMA 3.2. — *The integral of* $|f|^{-2c}$ *over the regions in* (3.2) *converges if and only if*

$$c < \min_i \left\{ \frac{1}{\alpha}, \frac{1}{\beta}, \delta_i \right\} = \delta_{NP}.$$

*In particular,* $c_0(f) \leqslant 2\delta_{NP}$ *for any coordinate system.*

Remark 3.3. — Note that if $\mathrm{ord}_0 f = N$ then we always have $c_0(f) \leqslant 2\delta_{NP} \leqslant 4N^{-1}$. This bound can be derived in a more elementary way than this by converting to spherical coordinates; see [27, Lemma 5.1]

It remains to show that the integral (3.1) is finite for $c < \delta_{NP}$. This will be achieved by estimating the integral on the parts of $U$ not covered by the regions described in (3.2). For $1 \leqslant i \leqslant k$ we write

(3.4) $$V_i = \{\varepsilon|x|^{a_i} < |y| < D|x|^{a_i}\} \times \{0 \leqslant |x| \leqslant \eta\}.$$

The $V_i$ together with the regions in (3.2) cover a neighbourhood of 0. The estimates on $V_i$ are somewhat more involved, owing to the fact that $V_i$ contains the roots in $S_i$. As our estimates need to be essentially sharp, we must take care in how we decompose $V_i$, particularly with respect to isolating the roots of $f$. The correct way to do this is by organizing the

roots of $f$ according to their complexity, measured by the order of contact. Let us focus only on the roots in $S_{i,\ell}$.

DEFINITION 3.4. — *A cluster at level $r$ is a set $C_r \subset S_{i,\ell}$ such that, for all $\varphi_\mu, \varphi_\nu \in C_r$ we have*

$$\varphi_\mu - \varphi_\nu = b_{\mu\nu} x^{r_{\mu\nu}} + \cdots$$

*for $b_{\mu\nu} \neq 0$ and $r_{\mu\nu} \geqslant r$. Furthermore, $C_r$ is maximal with this property in the sense that if $\varphi_\mu \in C_r$, and $\varphi_\nu \notin C_r$ then*

$$\varphi_\mu - \varphi_\nu = b_{\mu\nu} x^{r_{\mu\nu}} + \cdots$$

*for $b_{\mu\nu} \neq 0$ and $r_{\mu\nu} < r$.*

Note that if $\varphi_\mu \in C_r$ and $\varphi_\nu \notin C_r$, then the order of contact between $\varphi_\mu$ and $\varphi_\nu$ depends only on $C_r$ and not on the choice of $\varphi_\mu$.

DEFINITION 3.5. — *Let $C_r$ be a cluster at level $r$, and fix $\varphi_\mu \in C_r$. The maximal order of contact of $C_r$ is*

$$\gamma(C_r) := \max_\nu \{ r_{\mu\nu} : \varphi_\mu - \varphi_\nu = b_{\mu\nu} x^{r_{\mu\nu}} + \cdots \quad \text{where} \ \ \varphi_\nu \notin C_r \}.$$

*Observe that $\gamma(C_r) < r$.*

An important point is that each root $\varphi_\nu \in S_i$ appears in a list of distinct clusters with decreasing order

$$\{\varphi_\nu\} = C_\infty(\varphi_\nu) \subsetneq C_{r_1} \subsetneq C_{r_2} \subsetneq \cdots \subsetneq S_{i,\ell} \subset S_i$$

where $r_k = \gamma(C_{r_{k-1}})$. Decomposing $V_i$ will involve the use of sets adapted to clusters; we call these sets "horns".

DEFINITION 3.6. — *A hollow horn about a cluster $C_r$ at level $r$ is*
$$HH(C_r) := \{ |x|^r \preceq |y - C_r| \preceq |x|^r \}$$
$$= \bigcap_{\varphi_j \in C_r} \{ \varepsilon |x|^r < |y - \varphi_j| \leqslant D|x|^r \}.$$

*A solid horn about a cluster $C_r$ at level $r$ is*
$$SH(C_r) := \{ |y - C_r| \preceq |x|^{\gamma(C_r)} \}$$
$$= \bigcup_{\varphi_j \in C_r} \{ |y - \varphi_j| \leqslant \varepsilon |x|^{\gamma(C_r)} \}.$$

PROPOSITION 3.7. — *If $C_r \subset S_{i,\ell} \subset S_i$ is a cluster at level $r$, then*

$$\int_{SH(C_r)} |f|^{-2c} < +\infty$$

*provided*

$$(3.5) \qquad c < \min \left\{ m_{i\ell}^{-1}, \frac{1 + a_i + (r_{m-1} - a_i)}{A_i + a_i B_i + m_{i\ell}(r_{m-1} - a_i)}, \delta_{NP} \right\}$$

*where $r_{m-1}$ is the maximal, non-infinite order of contact between roots in $S_{i,\ell}$ (see (2.4))*

*Proof.* — Suppose $c$ satisfies the bound in (3.5). The proof goes by induction on the level of the cluster, beginning with clusters at level $\infty$. By definition, a cluster at level $\infty$ is a set consisting of one of the distinct roots of $P$, counted with multiplicity. We will estimate the integral over a solid horn about a cluster at level $\infty$. To this end, fix $\varphi_\nu \in S_{i,\ell}$, which we assume has maximal order of contact $r_\nu$. A solid horn about the cluster $\{\varphi_\nu\}$ consists of the region

$$(3.6) \qquad \{ |y - \varphi_\nu(x)| < \varepsilon |x|^{r_\nu} \}.$$

On this region we can estimate $|f|$ from below in the following way. First, if $\varphi_\mu \in S_j \cap S_{i,\ell}^c$, or if $\varphi_\mu \equiv 0$, then

$$(3.7) \qquad |y - \varphi_\mu(x)| \geqslant |\varphi_\mu - \varphi_\nu| - |y - \varphi_\nu| \geqslant \frac{\varepsilon}{4} |x|^{a_j},$$

where we used the definition of $\varepsilon$. On the other hand, if $\varphi_\mu \in S_{i,\ell}$ but $\varphi_\mu \neq \varphi_\nu$ then we have the (possibly wasteful) estimate

$$|y - \varphi_\mu(x)| \geqslant |\varphi_\mu - \varphi_\nu| - |y - \varphi_\nu| \geqslant \frac{\varepsilon}{4} |x|^{r_\nu},$$

again, using the definition of $\varepsilon$. Letting $d_\nu$ denote the multiplicity of $\varphi_\nu$, or equivalently the cardinality of the cluster, we get the estimate

$$|f| \geqslant |y - \varphi_\nu|^{d_\nu} |x|^{A_i + a_i B_i + m_{i\ell}(r_\nu - a_i) - d_\nu r_\nu}.$$

Thanks to the bound $c < m_{i\ell}^{-1} \leqslant d_\nu^{-1}$, integrating the estimate over the region in (3.6) we get

$$\int |f|^{-2c} \preceq \int_0^\varepsilon |x|^{1 + 2r_\nu - 2c(A_i + a_i B_i + m_{i\ell}(r_\nu - a_i))} \mathrm{d}|x|$$

and so we see that the integral converges if

$$(3.8) \qquad c < \frac{1 + a_i + (r_\nu - a_i)}{A_i + a_i B_i + m_{i\ell}(r_\nu - a_i)}.$$

We state the following trivial lemma, for convenience

LEMMA 3.8. — *Let*

$$c(x) = \frac{1 + a_i + x}{A_i + a_i B_i + m_{i\ell} x}.$$

*Then $c(x) \geqslant c(0)$ for $x > 0$ if and only if $m_{i\ell} \leqslant c(0)^{-1}$. If $m_{i\ell} > c(0)^{-1}$, then $c(x)$ is an increasing function of $x > 0$.*

By the lemma, we see that the inequality in (3.8) is implied by the inequality in Proposition 3.7, and hence establishes the proposition for clusters at level $\infty$.

Now, suppose we have proved Proposition 3.7 for solid horns about clusters at level $r \geqslant r_{s+1}$. Let $C_{r_s}$ be a cluster at level $r_s$. To ease notation, let us denote $r = r_s$. Our goal is to estimate the integral over the solid horn $SH(C_r)$. The key point is that $C_r$ can be written as a disjoint union of clusters $C_{\hat{r}} \subsetneq C_r$ each having $\gamma(C_{\hat{r}}) = r$. This allows us to decompose

$$SH(C_r) = E_1 \sqcup E_2 \sqcup E_3$$

where,

$$E_1 = \bigcup_{C_{\hat{r}} \subsetneq C_r} SH(C_{\hat{r}})$$

$$E_2 = HH(C_r)$$

$$E_3 = \bigcup_{\varphi_j \in C_r} \{ D|x|^r \leqslant |y - \varphi_j| < \varepsilon|x|^{\gamma(C_r)} \}.$$

the first union being taking over the $C_{\hat{r}}$ described above. Since we have already established the estimate on the solid horns $SH(C_{\hat{r}})$ for $\hat{r} > r$, it suffices to estimate the integral on $E_2$ and $E_3$. Let us consider the estimate on $E_2$. First, if $\varphi_\mu \in S_j \cap S_{i,\ell}^c$, or $\varphi_\mu \equiv 0$, then (3.7) still holds, so it suffices to estimate $|y - \varphi_\mu|$ when $\varphi_\mu \in S_{i,\ell}$. There are then two cases, depending on whether $\varphi_\mu \in C_r$ or $\varphi_\mu \in S_{i,\ell} \cap C_r^c$. If $\varphi_\mu \in C_r$, then from the definition of $HH(C_r)$ we have

$$|y - \varphi_\mu| \succeq |x|^r.$$

If instead $\varphi_\mu \in S_{i,\ell} \cap C_r^c$ then for any $\varphi_j \in C_r$ we have

$$|y - \varphi_\mu| \geqslant |\varphi_j - \varphi_\mu| - |\varphi_j - y| \geqslant \varepsilon|x|^{\gamma(C_r)} - D|x|^r \succeq |x|^r,$$

and so we obtain the estimate

$$|f| \succeq |x|^{A_i + a_i B_i + m_{ik}(r - a_i)}.$$

Integrating this estimate over the hollow horn $HH(C_r)$ yields the bound

$$c < \frac{1 + a_i + (r - a_i)}{A_i + a_i B_i + m_{i\ell}(r - a_i)}.$$

Since $r \leqslant r_{m-1}$, Lemma 3.8 shows that this bound is implied by the assumptions of Proposition 3.7. It remains to estimate the integral over the

region $E_3$. On this region the estimate (3.7) still holds, so it suffices to estimate $|y - \varphi_\mu|$ when $\varphi_\mu \in S_{i,\ell}$. We consider separately each of the regions

$$U_j := \{D|x|^r \leqslant |y - \varphi_j| < \varepsilon|x|^{\gamma(C_r)}\}.$$

For any $\varphi_\mu \in C_r$, different from $\varphi_j$ we have the estimate

$$|y - \varphi_\mu| \geqslant |y - \varphi_j| - |\varphi_j - \varphi_\mu| \geqslant \frac{1}{2}|y - \varphi_j|$$

using the definitions of $D, U_j$. On the other hand, if $\varphi_\mu \in S_{i,\ell} \cap C_r^c$ then we have the estimate

$$|y - \varphi_\mu| \geqslant |\varphi_j - \varphi_\mu| - |y - \varphi_j| \geqslant \frac{\varepsilon}{4}|x|^{\gamma(C_r)}.$$

Combining these estimates gives

(3.9) $\qquad |f| \geqslant |y - \varphi_j|^{\#C_r} |x|^{A_i + a_i B_i + m_{i,\ell}(\gamma(C_r) - a_i) - \#C_r \cdot \gamma(C_r)}.$

The reader may note the similarity between this estimate and the one obtained in the initial step of the induction (where $r = +\infty$). Again, thanks to the bound $\#C_r \leqslant m_{i\ell} < c^{-1}$, integrating (3.9) over the region $U_j$ gives the bound

$$c < \frac{1 + a_i + (\gamma(C_r) - a_i)}{A_i + a_i B_i + m_{i\ell}(\gamma(C_r) - a_i)}.$$

Appealing again to Lemma 3.8 establishes Proposition 3.7. $\qquad \square$

Let's assume now that $m_{i\ell} \leqslant \delta_{NP}^{-1}$ for all $i, \ell$ and finish the proof of Theorem 1.4. In this case, Proposition 3.7, combined with Lemma 3.8 implies that the integral of $|f|^{-2c}$ over solid horns about clusters $C_r \subset S_{i,\ell}$ is finite provided $c < \delta_{NP}$. Up to decreasing $\varepsilon$ and increasing $D$ (see Remark 2.5), it is easy to see that the region $V_i$, defined in (3.4), can be decomposed into a union of solid horns and hollow horns about the $S_{i,\ell}$. Since the $S_{i,\ell}$ are themselves clusters, the estimates on solid horns about $S_{i,\ell}$ follow from Proposition 3.7. The estimates on the hollow horns about the $S_{i,\ell}$ are identical to those already obtained. Alternatively, one can start from the very beginning, considering the $S_i$ themselves as clusters and running the same induction. We have

PROPOSITION 3.9. — In $\mathbb{C}^2$ (resp. $\mathbb{R}^2$), if $m_{i\ell} \leqslant \delta_{NP}^{-1}$ for all $i, \ell$, then we have

$$c_0(f) = 2\delta_{NP} \qquad (\text{resp. } \delta_{NP}.)$$

Theorem 1.4 now follows from Proposition 3.10 below. Before embarking on the proof, the reader may consider the examples mentioned in Remark 2.2. In that case the function $f = (y - x)^N$ does not satisfy the assumptions of Proposition 3.9. However, $f$ can be transformed to the

function $g = y^N$, to which Proposition 3.9 does apply, by a holomorphic change of variables.

PROPOSITION 3.10. — *In the above notation, we have*

$$m_{i\ell} \leqslant \delta_{NP}^{-1}$$

*unless $m_{i\ell}$ corresponds to the main face of $\partial NP(f)$. If $m_{i\ell} > \delta_{NP}^{-1}$ then there exists a holomorphic function $Q(\,\cdot\,)$ so that after making the change of coordinates*

$$(\tilde{x}, \tilde{y}) = (x - Q(y), y), \quad \text{or} \quad (\tilde{x}, \tilde{y}) = (x, y - Q(x))$$

*the critical integrability index $c_0(f)$ is computed by the Newton distance of $f(\tilde{x}, \tilde{y})$.*

*Proof.* — We sketch a proof, following Phong–Stein–Sturm [26, Theorem 5]. Let $\pi_p, \pi_q$ be the projections to the $p$ and $q$-axes respectively. For every face $F_i$ of $NP(f)$ let

$$A_i = \text{ Length of } \pi_q(F_i), \qquad B_i = \text{ Length of } \pi_p(F_i)$$

Recall that $A_i$ is the cardinality of the set of roots with leading order determined by the slope of $F_i$, so if $F_i$ is right of the main face, then $m_{i\ell} \leqslant A_i \leqslant \delta_{NP}$. Now suppose we have a solution

$$\varphi_\nu = bx^{p_\nu/q_\nu} + \cdots$$

with $p_\nu, q_\nu$ relatively prime, and corresponding to a face $F_i$. If $\zeta$ is a $q_\nu$-th root of unity then it is easy to see that

$$\tilde{\varphi}_\nu = \zeta bx^{p_\nu/q_\nu} + \cdots$$

is also a root of $f$, and therefore $m_{i\ell} \leqslant A_i/q_\nu$. Since $A_i/B_i = q_\nu/p_\nu$, we can further conclude

$$A_i \geqslant q_\nu m_{i\ell} \qquad B_i \geqslant p_\nu m_{i\ell}.$$

Now if $F_i$ is left of the main face, then we fhave $B_i \leqslant \delta_{NP}$, and so $m_{i\ell} \leqslant \delta_{NP}$. If the main face is a vertex we're done, so we can assume that is not the case. Let $F_i$ denote the main face. $F_i$ is cutout by the line

$$L(p, q) := \frac{qp_\nu + q_\nu p}{p_\nu + q_\nu} - \delta_{NP}^{-1} = 0,$$

and the point $(0, A_i)$ lies in the region $\{L \leqslant 0\}$. Writing this out gives

$$\delta_{NP}^{-1} \geqslant \frac{A_i p_\nu}{p_\nu + q_\nu} \geqslant m_{i\ell} \frac{p_\nu q_\nu}{p_\nu + q_\nu}.$$

If $m_{i\ell} > \delta_{NP}^{-1}$, then $p_\nu + q_\nu > p_\nu q_\nu$ so either $p_\nu$ or $q_\nu$ must be 1. It remains only to construct the change of variables. Let us assume $q_\nu = 1$ for simplicity, otherwise we write the roots as functions of $y$ and the argument is the same. By assumption we have $m_{i\ell} > \delta_{NP}^{-1}$ roots of the form

$$\varphi_\nu = bx^{p_\nu} + \cdots.$$

We take $Q(x) = bx^{p_\nu} + \cdots$ to be the powerseries, or polynomial of maximal degree so that there are more than $\delta_{NP}^{-1}$ roots $\varphi_\nu$ with

$$\varphi_\nu = Q(x) + \text{ higher order terms.}$$

It is then not hard to check that in the new coordinates $(\tilde{x}, \tilde{y}) = (x, y - Q(x))$ we have $\tilde{\delta}_{NP} < \delta_{NP}$ and $m_{i\ell} \leqslant \delta_{NP}^{-1}$ for every $i, \ell$. An equivalent approach is to construct $Q(x)$ inductively, beginning with the coordinate transformation $\varphi(x, y) = (x, y - bx^{p_\nu})$, and examining the change in the Newton polygon. We leave the details to the reader. □

## 4. Proof of Theorem 1.3

Let $\{f_n\}$ be a sequence of holomorphic (or real analytic) functions with $c_0(f_n)$ converging to $c_\infty > 0$. By Remark 3.3, up to passing to a subsequence, we can assume that the $\mathrm{ord}_0 f_n = N$ for all $n$. By Remark 2.4, after making a generic linear change of coordinates we can assume that each $f_n$ has $\mathrm{ord}_0 f_n(x, 0) = \mathrm{ord}_0 f_n(0, y) = N$, and that $f_n$ can be written simultaneously as a Weierstrass polynomial of order $N$ in $x$ and $y$. By Theorem 1.4, for each $n \in \mathbb{N}$, we can make a change of coordinates of the type described in Proposition 3.10 so that $c_0(f_n)$ is computed by the Newton distance $\delta_{NP}(f_n)$. Suppose that the $n$-th change of variables is of the form

$$(\tilde{x}, \tilde{y}) = (x, y - Q_n(x))$$

Then we can write

(4.1) $$f_n(\tilde{x}, \tilde{y}) = (unit) \cdot \left( \tilde{y}^N + \sum_{i=0}^{N-1} \tilde{b}_\ell(\tilde{x}) \tilde{y}^\ell \right)$$

with $\tilde{b}_\ell(\tilde{x})$ holomorphic (resp. real analytic) and vanishing at the origin. The same thing holds in the case that $(\tilde{x}, \tilde{y}) = (x - Q_n(y), y)$, using the representation of $f_n$ as a Weierstrass polynomial of degree $N$ in $x$. By renaming the variables we can always assume that there are coordinates where $\delta_{NP}(f_n)$ computes $c_0(f_n)$ and in these coordinates $f_n$ has the form (4.1). In particular, each Newton polygon $NP(f_n)$ has a vertex at the point $(0, N)$, and

so contains the set $(0, N) + \mathbb{R}^2_{\geqslant 0}$. Every vertex of $\partial NP(f_n)$ lying to the left of the diagonal must be an element of the set

$$\mathcal{L} := \{(p, q) \in \mathbb{N}^2 : q \geqslant p, \ q \leqslant N\}.$$

Since $\mathcal{L}$ is finite, by passing to a subsequence, we can assume that $NP(f_n) \cap \mathcal{L}$ is independent of $n$. Let $(p^*, q^*)$ be the vertex of $\partial NP(f_n)$ lying in $\mathcal{L}$ and with $p^*$ maximal– that is, $(p^*, q^*)$ is the first vertex of $\partial NP(f_n)$ lying on, or left of, the diagonal. There are then two cases (we assume the complex case now, to fix constants).

If $p^* = q^*$, then $c_0(f_n) = 2\delta_{NP}(f_n) = 2/p^*$ and we're finished, so we can assume $p^* < q^*$. For simplicity, we now consider the increasing and decreasing cases separately, though the principle is the same.

*Increasing case.* — Let $(p_n, q_n)$ be the left most vertex of $NP(f_n)$ lying in

$$\mathcal{R} = \{(p, q) \in \mathbb{N}^2 : q < p\}.$$

That is $(p_n, q_n)$ is the first vertex lying strictly to the right of the diagonal. Since $p^* < q^*$, we have that $2c_0(f_n)^{-1} = \delta_{NP}(f_n)^{-1}$ lies in the interior of the line connecting $(p^*, q^*)$ to $(p_n, q_n)$. Moreover, thanks to the fact that $c_0(f_n)$ is increasing, we know that $(p_n, q_n)$ must lie in the region of $\mathcal{R}$ lying below the line connecting $(p^*, q^*)$ to $(p_1, q_1)$. But this region is compact and hence there are only finitely many possible choice of $(p_n, q_n)$. In particular, $\delta_{NP}(f_n)$ must stabilize.

*Decreasing case.* — Essentially the same argument works in the decreasing case. Assume that

$$c_0(f_n) \searrow c_\infty.$$

By Proposition 3.7 we have $c_0(f_1) \leqslant 2$, and so $c_\infty < 2$. If $c_\infty = \frac{2}{q^*}$ then we're done, since $q* > 1$, and so $2/q^* \in \mathcal{C}(1)\backslash 2$. Thus, we can assume that $c_\infty > 2/q^*$. We now consider the line $L$ through $(p^*, q^*)$ and $(c_\infty^{-1}, c_\infty^{-1})$. Since $2c_\infty^{-1} < q^*$, $L$ must intersect the $p$-axis at some point. Since $c_0(f_n)$ is decreasing, the points $(p_n, q_n)$ lie in $\mathcal{R}$, but below $L$. This set is bounded, and hence contains only finitely many lattice points. It follows that $c_0(f_n)$ must stabilize. It remains only to demonstrate that points in $\mathcal{C}(1)\backslash 2$ are actually accumulation points. This can be deduced by applying Proposition 3.9 to the functions $f_n = y^n - x^m$, which gives that

$$c_0(f_n) = \frac{2(m + n)}{mn}$$

provided $1 \leqslant \frac{mn}{m+n}$; if $m > 1$, this is always satisfied for $n$ sufficiently large. Letting $n \to \infty$ we find that $2/m \in \mathcal{C}(1)\backslash 2$ is an accumulation point of $\mathcal{C}(2)$, finishing the proof.                                                                    $\square$

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