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Tome 68, nº 4 (2018), p. 1715-1830.

<http://aif.cedram.org/item?id=AIF_2018__68_4_1715_0>

Ann. Inst. Fourier, Grenoble
68, 4 (2018) 1715-1830

DOUBLY-RESONANT SADDLE-NODES IN $(\mathbb{C}^3, 0)$ AND THE FIXED SINGULARITY AT INFINITY IN PAINLEVÉ EQUATIONS: ANALYTIC CLASSIFICATION

by **Amaury BITTMANN**

Abstract. — In this work, we consider germs of analytic singular vector fields in $\mathbb{C}^3$ with an isolated and doubly-resonant singularity of saddle-node type at the origin. Such vector fields come from irregular two-dimensional differential systems with two opposite non-zero eigenvalues, and appear for instance when studying the irregular singularity at infinity in Painlevé equations $(P_j)_{j=I,\dots,V}$ for generic values of the parameters. Under suitable assumptions, we prove a theorem of analytic normalization over sectorial domains, analogous to the classical one due to Hukuhara–Kimura–Matuda for saddle-nodes in $\mathbb{C}^2$. We also prove that these sectorial normalizing maps are in fact the Gevrey-1 sums of the formal normalizing map, the existence of which has been proved in a previous paper. Finally we provide an analytic classification under the action of fibered diffeomorphisms, based on the study of the so-called *Stokes diffeomorphisms* obtained by comparing consecutive sectorial normalizing maps *à la* Martinet–Ramis / Stolovitch for 1-resonant vector fields.

Résumé. — Dans ce travail, nous considérons des germes de champs de vecteurs singuliers dans $(\mathbb{C}^3, 0)$ ayant une singularité isolée doublement résonante de type noeud-col à l'origine. Ces champs de vecteurs proviennent de systèmes différentiels irréguliers en dimension deux, avec deux valeurs propres opposées non-nulles, et apparaissent par exemple dans l'étude des singularités irrégulières à l'infini des équations de Painlevé $(P_j)_{j=I,\dots,V}$ pour des valeurs génériques des paramètres. Sous des conditions adéquates, nous démontrons un théorème de normalisation analytique sur des domaines sectoriels, analogue à un résultat de Hukuhara, Kimura et Matuda pour les noeud-cols dans $\mathbb{C}^2$. Nous prouvons également que ces normalisations sectorielles sont en fait les sommes 1-Gevrey de la normalisation formelle, dont l'existence a été prouvée dans un précédent papier. Nous terminons en fournissant une classification analytique sous l'action de difféomorphismes fibrés, basée sur l'étude des *difféomorphismes de Stokes* obtenus en comparant les normalisations sectorielles consécutives *à la* Martinet–Ramis / Stolovitch pour des champs de vecteurs 1-résonants.

Keywords: Painlevé equations, singular vector field, irregular singularity, resonant singularity, analytic classification, Stokes diffeomorphisms.
2010 *Mathematics Subject Classification:* 34M30, 34M35, 34M40, 34M55.

1. Introduction

As in [3], we consider (germs of) singular vector fields Y in $\mathbb{C}^3$ which can be written in appropriate coordinates $(x, \mathbf{y}) := (x, y_1, y_2)$ as

$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + F_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + F_2(x, \mathbf{y})) \frac{\partial}{\partial y_2}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ and F_1, F_2 are germs of holomorphic functions in $(\mathbb{C}^3, 0)$ of homogeneous valuation (order) at least two. They represent irregular two-dimensional differential systems having two opposite non-zero eigenvalues:

$$\begin{cases} x^2 \dfrac{\mathrm{d}y_1(x)}{\mathrm{d}x} = -\lambda y_1(x) + F_1(x, \mathbf{y}(x)) \\ x^2 \dfrac{\mathrm{d}y_2(x)}{\mathrm{d}x} = \lambda y_2(x) + F_2(x, \mathbf{y}(x)). \end{cases}$$

These we call doubly-resonant vector fields of saddle-node type (or simply *doubly-resonant saddle-nodes*). We will impose more (non-generic) conditions in the sequel. The motivation for studying such vector fields is at least of two types.

(1) There are two independent resonance relations between the eigenvalues (here 0, $-\lambda$ and λ): we generalize then the study in [17, 18]. More generally, this work is aimed at understanding singularities of vector fields in $\mathbb{C}^3$. According to a theorem of resolution of singularities in dimension less than three in [20], there exists a list of "final models" for singularities (*log-canonical*) obtained after a finite procedure of *weighted blow-ups* for three dimensional singular analytic vector fields. In this list, we find in particular doubly-resonant saddles-nodes, as those we are interested in. In dimension 2, these final models have been intensively studied (for instance by Martinet, Ramis, Ecalle, Ilyashenko, Teyssier, . . .) from the view point of both formal and analytic classification (some important questions remain unsolved, though). In dimension 3, the problems of formal and analytic classification are still open questions, although Stolovitch has performed such a classification for 1-resonant vector fields in any dimension [28]. The presence of two kinds of resonance relations brings new difficulties.

(2) Our second main motivation is the study of the irregular singularity at infinity in Painlevé equations$(P_j)_{j=I,\ldots,V}$, for generic values of the parameters (cf. [33]). These equations were discovered by Paul Painlevé [23] because the only movable singularities of the solutions are poles (the so-called *Painlevé property*). Their study

> has become a rich domain of research since the important work of Okamoto [21]. The fixed singularities of the Painlevé equations, and more particularly those at infinity, where notably investigated by Boutroux with his famous *tritronquées* solutions [5]. Recently, several authors provided more complete information about such singularities, studying "quasi-linear Stokes phenomena" and also giving connection formulas; we refer to the following (non-exhaustive) sources [6, 7, 11, 12, 13, 14]. Stokes coefficients are invariant under local changes of analytic coordinates, but do not form a complete invariant of the vector field. To the best of our knowledge there currently does not exist a general analytic classification for doubly-resonant saddle-nodes. Such a classification would provide a new framework allowing to analyse Stokes phenomena in that class of singularities.

In this paper we provide an analytic classification under the action of fibered diffeomorphisms for a specific (to be defined later on) class of doubly-resonant saddle-nodes which contains the Painlevé case. For this purpose, the main tool is a theorem of analytic normalization over sectorial domain (*à la* Hukuhara–Kimura–Matuda [9] for saddle-nodes in $(\mathbb{C}^2, 0)$) for a specific class (to be defined later on) of doubly-resonant saddle-nodes which contains the Painlevé case. The analytic classification for this class of vector fields, inspired by the important works [17, 18, 28] for 1-resonant vector fields, is based on the study of so-called *Stokes diffeomorphisms*, which are the transition maps between different sectorial domains for the normalization.

In [32, 33] Yoshida shows that doubly-resonant saddle-nodes arising from the compactification of Painlevé equations $(P_j)_{j=I,\dots,V}$ (for generic values for the parameters) are conjugate to vector fields of the form:

$$Z = x^2 \frac{\partial}{\partial x} + (-(1+\gamma y_1 y_2) + a_1 x) y_1 \frac{\partial}{\partial y_1} + (1+\gamma y_1 y_2 + a_2 x) y_2 \frac{\partial}{\partial y_2}, \tag{1.2}$$

with $\gamma \in \mathbb{C}$ and $(a_1, a_2) \in \mathbb{C}^2$ such that $a_1 + a_2 = 1$. One should notice straight away that this "conjugacy" does not agree with what is traditionally (in particular in this paper) meant by conjugacy, for Yoshida's transform $\Psi(x, \mathbf{y}) = (x, \psi_1(x, \mathbf{y}), \psi_2(x, \mathbf{y}))$ takes the form

$$\psi_i(x, \mathbf{y}) = y_i \left(1 + \sum_{\substack{(k_0, k_1, k_2) \in \mathbb{N}^3 \\ k_1 + k_2 > 1}} \frac{q_{i,\mathbf{k}}(x)}{x^{k_0}} y_1^{k_1+k_0} y_2^{k_1+k_0}\right), \tag{1.3}$$

where each $q_{i,\mathbf{k}}$ is formal power series although x *appears with negative exponents*. This expansion may not even be a formal Laurent series. It is, though, the asymptotic expansion along $\{x = 0\}$ of a function analytic in a domain

$$\{(x,\mathbf{z}) \quad S \times \mathbf{D}(0,\mathbf{r}) \mid |z_1 z_2| < \nu |x|\} \tag{1.4}$$

for some small $\nu > 0$, where S is a sector of opening greater than π with vertex at the origin and $\mathbf{D}(0,\mathbf{r})$ is a polydisc of small poly-radius $\mathbf{r} = (r_1, r_2) \quad (\mathbb{R}_{>0})^2$. Moreover the $(q_{i,\mathbf{k}}(x))_{i,\mathbf{k}}$ are actually Gevrey-1 power series. The drawback here is that the transforms are convergent on regions so small that taken together they cannot cover an entire neighborhood of the origin in $\mathbb{C}^3$ (which seems to be problematic to obtain an analytic classification *à la* Martinet–Ramis).

Several authors studied the problem of convergence of formal transformations putting vector fields as in (1.1) into "normal forms". Shimomura, improving on a result of Iwano [10], shows in [27] that analytic doubly-resonant saddle-nodes satisfying more restrictive conditions are conjugate (formally and over sectors) to vector fields of the form

$$x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x) y_2 \frac{\partial}{\partial y_2} \tag{1.5}$$

via a diffeomorphism whose coefficients have asymptotic expansions as $x \to 0$ in sectors of opening greater than π. Stolovitch then generalized this result to any dimension in [28]. More precisely, Stolovitch's work offers an analytic classification of vector fields in $\mathbb{C}^{n+1}$ with an irregular singular point, without further hypothesis on eventual additional resonance relations between eigenvalues of the linear part. However, as Iwano and Shimomura did, he needed to impose other assumptions, among which the condition that the restriction of the vector field to the invariant hypersurface $\{x = 0\}$ is a linear vector field. In [4], the authors obtain a *Gevrey-1 summable* "normal form", though not as simple as Stolovitch's one and not unique a priori, but for more general kind of vector field with one zero eigenvalue. However, the same assumption on hypersurface $\{x = 0\}$ is required (the restriction is a linear vector field). Yet from [33] (and later [3]) stems the fact that this condition is not met in the case of Painlevé equations $(P_j)_{j=I,\ldots,V}$. In comparison, we merely ask here that the restricted vector field be orbitally linearizable (see Definition 1.7), i.e. the foliation induced by Y on $\{x = 0\}$ (and not the vector field $Y_{|\{x=0\}}$ itself) be linearizable. The fact that this condition is fulfilled by the singularities of

Painlevé equations formerly described is well-known. As discussed in Remark 1.17, the more general context also introduces new phenomena and technical difficulties as compared to prior classification results.

1.1. Scope of the paper

The action of local analytic / formal diffeomorphisms Ψ fixing the origin on local holomorphic vector fields Y of type (1.1) by change of coordinates is given by

$$\Psi_* Y := \mathrm{d}\Psi(Y) \circ \Psi^{-1}. \tag{1.6}$$

In [3] we performed the formal classification of such vector fields by exhibiting an explicit universal family of vector fields for the action of formal changes of coordinates at 0 (called a family of normal forms). Such a result seems currently out of reach in the analytic category: it is unlikely that an explicit universal family for the action of local analytic changes of coordinates be described anytime soon. If we want to describe the space of equivalent classes (of germs of a doubly-resonant saddle-node under local analytic changes of coordinates) with same formal normal form, we therefore need to find a complete set of invariants which is of a different nature. We call *moduli space* this quotient space and would like to give it a (non-trivial) presentation based on functional invariants *à la* Martinet–Ramis [17, 18]. We only deal here with x-fibered local analytic conjugacies acting on vector fields of the form (1.1) with some additional assumptions detailed further down (see Definitions 1.1, 1.3 and 1.7). Importantly, these hypothesis are met in the case of Painlevé equations mentioned above. The classification under the action of general (not necessarily fibered) diffeomorphisms can be found in [2]).

First we prove a theorem of analytic sectorial normalizing map (over a pair of opposite "wide" sectors of opening greater than π whose union covers a full punctured neighborhood of $\{x = 0\}$). Then we attach to each vector field a complete set of invariants given as transition maps (over "narrow" sectors of opening less than π) between the sectorial normalizing maps. Although this viewpoint has become classical since the work of Martinet and Ramis, and has latter been generalized by Stolovitch as already mentioned, our approach has some geometric flavor. For instance, we avoid the use of fixed-point methods altogether to establish the existence of the normalizing maps, and generalize instead the approach of Teyssier [31, 30] relying on path-integration of well-chosen 1-forms (following Arnold's method of characteristics [1]). As a by-product of this normalization we deduce that the normalizing sectorial diffeomorphisms are Gevrey-1 asymptotic to the normalizing formal power series of [3], retrospectively proving their 1-summability (with respect to the $x-$coordinate). When the vector field additionally supports a symplectic transverse structure (which is again the case of Painlevé equations) we prove that the (essentially unique) sectorial normalizing map is performed by a transversally symplectic diffeomorphism. We deduce from this a theorem of analytic classification under the action of *transversally symplectic* diffeomorphisms.

1.2. Definitions and main results

To state our main results we need to introduce some notations and nomenclature.

- For $n \quad \mathbb{N}_{>0}$, we denote by $(\mathbb{C}^n, 0)$ an (arbitrary small) open neighborhood of the origin in $\mathbb{C}^n$.
- We denote by $\mathbb{C}\{x, \mathbf{y}\}$, with $\mathbf{y} = (y_1, y_2)$, the $\mathbb{C}$-algebra of germs of holomorphic functions at the origin of $\mathbb{C}^3$, and by $\mathbb{C}\{x, \mathbf{y}\}^\times$ the group of invertible elements for the multiplication (also called units), i.e. elements U such that $U(0) = 0$.
- $\chi(\mathbb{C}^3, 0)$ is the Lie algebra of germs of singular holomorphic vector fields at the origin $\mathbb{C}^3$. Any vector field in $\chi(\mathbb{C}^3, 0)$ can be written as
$$Y = b(x, y_1, y_2)\frac{\partial}{\partial x} + b_1(x, y_1, y_2)\frac{\partial}{\partial y_1} + b_2(x, y_1, y_2)\frac{\partial}{\partial y_2}$$
with $b, b_1, b_2 \quad \mathbb{C}\{x, y_1, y_2\}$ vanishing at the origin.
- $\mathrm{Diff}(\mathbb{C}^3, 0)$ is the group of germs of a holomorphic diffeomorphism fixing the origin of $\mathbb{C}^3$. It acts on $\chi(\mathbb{C}^3, 0)$ by conjugacy: for all
$$(\Phi, Y) \quad \mathrm{Diff}(\mathbb{C}^3, 0) \times \chi(\mathbb{C}^3, 0)$$
we define the push-forward of Y by Φ by
$$\Phi \ (Y) := (\mathrm{d}\Phi \cdot Y) \circ \Phi^{-1}, \tag{1.7}$$
where $\mathrm{d}\Phi$ is the Jacobian matrix of Φ.
- $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0)$ is the subgroup of $\mathrm{Diff}(\mathbb{C}^3, 0)$ of fibered diffeomorphisms preserving the x-coordinate, i.e. of the form $(x, \mathbf{y}) \to (x, \phi(x, \mathbf{y}))$.
- We denote by $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ the subgroup of $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0)$ formed by diffeomorphisms tangent to the identity.

All these concepts have *formal* analogues, where we only suppose that the objects are defined with formal power series, not necessarily convergent near the origin.

Definition 1.1. — *A* diagonal doubly-resonant saddle-node *is a vector field* $Y \in \chi(\mathbb{C}^3, 0)$ *of the form*

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2}, \tag{1.8}$$

with $\lambda \in \mathbb{C}^*$ *and* $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ *of order at least two. We denote by* $\mathcal{SN}_{\mathrm{diag}}$ *the set of such vector fields.*

Remark 1.2. — One can also define the foliation associate to a diagonal doubly-resonant saddle-node in a geometric way. A vector field $Y \in \chi(\mathbb{C}^3, 0)$ is orbitally equivalent to a diagonal doubly-resonant saddle-node (i.e. Y is conjugate to some VX, where $V \in \mathbb{C}\{x, \mathbf{y}\}^{\times}$ and $X \in \mathcal{SN}_{\mathrm{fib}}$) if and only if the following conditions hold:

1. (1) $\mathrm{Spec}(\mathrm{D}_0 Y) = \{0, -\lambda, \lambda\}$ with $\lambda \neq 0$;
2. (2) there exists a germ of irreducible analytic hypersurface $\mathcal{H}_0 = \{S = 0\}$ which is transverse to the eigenspace E_0 (corresponding to the zero eigenvalue) at the origin, and which is stable under the flow of Y;
3. (3) $\mathcal{L}_Y(S) = U.S^2$, where $\mathcal{L}_Y$ is the Lie derivative of Y and $U \in \mathbb{C}\{x, \mathbf{y}\}^{\times}$.

By Taylor expansion up to order 1 with respect to $\mathbf{y}$, given a vector field $Y \in \mathcal{SN}_{\mathrm{diag}}$ written as in (1.1) we can consider the associate 2-dimensional system:

$$x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \alpha(x) + \mathbf{A}(x)\mathbf{y}(x) + \mathbf{F}(x, \mathbf{y}(x)), \tag{1.9}$$

with $\mathbf{y} = (y_1, y_2)$, such that the following conditions hold:

- $\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix}$, with $\alpha_1, \alpha_2 \in \mathbb{C}\{x\}$ and $\alpha_1, \alpha_2 \in \mathrm{O}(x^2)$
- $\mathbf{A}(x) \in \mathrm{Mat}_{2,2}(\mathbb{C}\{x\})$ with $\mathbf{A}(0) = \mathrm{diag}(-\lambda, \lambda)$, $\lambda \in \mathbb{C}^*$
- $\mathbf{F}(x, \mathbf{y}) = \begin{pmatrix} F_1(x, \mathbf{y}) \\ F_2(x, \mathbf{y}) \end{pmatrix}$, with $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ and $F_1, F_2 \in \mathrm{O}(\|\mathbf{y}\|^2)$.

Based on this expression, we state:

Definition 1.3. — *The* residue *of* $Y \in \mathcal{SN}_{\mathrm{diag}}$ *is the complex number*

$$\mathrm{res}(Y) := \left(\frac{\mathrm{Tr}(\mathbf{A}(x))}{x} \right)_{|x=0}.$$

We say that Y *is* non-degenerate *(resp.* strictly non-degenerate) *if* $\mathrm{res}(Y) \notin \mathbb{Q}_{\leq 0}$ *(resp.* $\Re(\mathrm{res}(Y)) > 0$*).*

Remark 1.4. — It is obvious that there is an action of $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$ on $\mathcal{SN}_{\mathrm{diag}}$. The residue is an invariant of each orbit of $\mathcal{SN}_{\mathrm{fib}}$ under the action of $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$ by conjugacy (see [3]).

The main result of [3] can now be stated as follows:

Theorem 1.5 ([3]). — *Let $Y \in \mathcal{SN}_{\mathrm{diag}}$ be non-degenerate. Then there exists a unique formal fibered diffeomorphism $\hat{\Phi}$ tangent to the identity such that:*

$$\hat{\Phi}_*(Y) = x^2\frac{\partial}{\partial x} + (-\lambda + a_1x + c_1(y_1y_2))y_1\frac{\partial}{\partial y_1} + (\lambda + a_2x + c_2(y_1y_2))y_2\frac{\partial}{\partial y_2}, \tag{1.10}$$

where $\lambda \in \mathbb{C}^*$, $c_1, c_2 \in \mathbb{C}[[v]]$ are formal power series in $v = y_1y_2$ without constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \mathrm{res}(Y) \in \mathbb{C}\setminus\mathbb{Q}_{\leq 0}$.

Definition 1.6. — *The vector field obtained in* (1.10) *is called the* formal normal form *of Y. The formal fibered diffeomorphism $\hat{\Phi}$ is called the* formal normalizing map *of Y.*

The above result is valid for formal objects, without considering problems of convergence. The first main result in this work states that this formal normalizing map is analytic in sectorial domains, under some additional assumptions that we are now going to precise.

Definition 1.7.

- *We say that a germ of a vector field X in $(\mathbb{C}^2,0)$ is* orbitally linear *if*
$$X = U(\mathbf{y})\left(\lambda_1y_1\frac{\partial}{\partial y_1} + \lambda_2y_2\frac{\partial}{\partial y_2}\right),$$
for some $U(\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}^\times$ and $(\lambda_1,\lambda_2) \in \mathbb{C}^2$.
- *We say that a germ of vector field X in $(\mathbb{C}^2,0)$ is analytically (resp.* formally*)* orbitally linearizable *if X is analytically (resp. formally) conjugate to an orbitally linear vector field.*
- *We say that a diagonal doubly-resonant saddle-node $Y \in \mathcal{SN}_{\mathrm{diag}}$ is* div-integrable *if $Y_{|\{x=0\}} \in \chi(\mathbb{C}^2,0)$ is (analytically) orbitally linearizable.*

Remark 1.8. — Alternatively we could say that the foliation associated to $Y_{|\{x=0\}}$ is linearizable. Since $Y_{|\{x=0\}}$ is analytic at the origin of $\mathbb{C}^2$ and has two opposite eigenvalues, it follows from a classical result of Brjuno (see [16]), that $Y_{|\{x=0\}}$ is analytically orbitally linearizable if and only if it is formally orbitally linearizable.

Definition 1.9. — *We denote by $SN_{\text{diag},0}$ the set of strictly non-degenerate diagonal doubly-resonant saddle-nodes which are div-integrable.*

The vector field corresponding to the irregular singularity at infinity in the Painlevé equations $(P_j)_{j=I,\dots,V}$ is orbitally equivalent to an element of $SN_{\text{fib},0}$, for generic values of the parameters (see [33]). We can now state the first main result of our paper (we refer to section 2. for more details on *1-summability*).

Theorem 1.10. — *Let $Y \in SN_{\text{diag},0}$ and let $\hat{\Phi}$ (given by Theorem 1.5) be the unique formal fibered diffeomorphism tangent to the identity such that*

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2\frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1y_2))y_1\frac{\partial}{\partial y_1} \\ &\qquad + (\lambda + a_2 x + c_2(y_1y_2))y_2\frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}}, \end{aligned}$$

where $\lambda \neq 0$ and $c_1(v), c_2(v) \in v\mathbb{C}[[v]]$ are formal power series without constant term. Then:

1. *the normal form Y_{norm} is analytic* (i.e. $c_1, c_2 \in \mathbb{C}\{v\}$), *and it also is div-integrable, i.e. $c_1 + c_2 = 0$;*
2. *the formal normalizing map $\hat{\Phi}$ is 1-summable (with respect to x) in every direction $\theta \neq \arg(\pm\lambda)$.*
3. *there exist analytic sectorial fibered diffeomorphisms Φ_+ and Φ_-, (asymptotically) tangent to the identity, defined in sectorial domains of the form $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, where*

$$S_+ := \left\{ x \in \mathbb{C} \,\middle|\, 0 < |x| < r \text{ and } \left|\arg\left(\frac{x}{i\lambda}\right)\right| < \frac{\pi}{2} + \epsilon \right\}$$
$$S_- := \left\{ x \in \mathbb{C} \,\middle|\, 0 < |x| < r \text{ and } \left|\arg\left(\frac{-x}{i\lambda}\right)\right| < \frac{\pi}{2} + \epsilon \right\}$$

   (for any $\epsilon \in \left]0, \frac{\pi}{2}\right[$ and some $r > 0$ small enough), which admit $\hat{\Phi}$ as weak Gevrey-1 asymptotic expansion in these respective domains, and which conjugate Y to Y_{norm}. Moreover Φ_+ and Φ_- are the unique such germs of analytic functions in sectorial domains (see Definition 2.2).

Remark 1.11. — Although item (3) above is a straightforward consequence of the *1-summability* of $\hat{\Phi}$ (item (2) above), we will in fact start by proving item (3) in Corollary 4.2, and establish the 1-summability of item (2) in a second step (see Proposition 5.6). What we will obtain at first directly is only the *weak* 1-summability (see Subsection 2.3) of $\hat{\Phi}$

(see Proposition 4.18), and not immediately the 1-summability. To obtain the “true” 1-summability, we will need to prove that the transition maps between Φ_+ and Φ_- are exponentially close to the identity (see Proposition 5.2), and then to use a fundamental theorem of Martinet and Ramis (see Theorem 2.22).

Definition 1.12. — *We call Φ_+ and Φ_- the* sectorial normalizing maps *of $Y \in \mathcal{SN}_{\text{diag},0}$.*

They are the 1-sums of $\hat{\Phi}$ along the respective directions $\arg(i\lambda)$ and $\arg(-i\lambda)$. Notice that Φ_+ and Φ_- are *germs of analytic sectorial fibered diffeomorphisms*, i.e. they are of the form

$$\begin{aligned}\Phi_+ : S_+ \times (\mathbb{C}^2,0) &\longrightarrow S_+ \times (\mathbb{C}^2,0)\\ (x,\mathbf{y}) &\longmapsto (x, \Phi_{+,1}(x,\mathbf{y}), \Phi_{+,2}(x,\mathbf{y}))\end{aligned}$$

and

$$\begin{aligned}\Phi_- : S_- \times (\mathbb{C}^2,0) &\longrightarrow S_- \times (\mathbb{C}^2,0)\\ (x,\mathbf{y}) &\longmapsto (x, \Phi_{-,1}(x,\mathbf{y}), \Phi_{-,2}(x,\mathbf{y}))\end{aligned}$$

(see Section 2. for a precise definition of *germ of analytic sectorial fibered diffeomorphism*). The fact that they are also *(asymptotically) tangent to the identity* means that we have:

$$\Phi_\pm(x,\mathbf{y}) = \mathrm{Id}(x,\mathbf{y}) + \mathrm{O}(\|(x,\mathbf{y})\|^2)\,.$$

In fact, we can prove the uniqueness of the sectorial normalizing maps under weaker assumptions.

Proposition 1.13. — *Let φ_+ and φ_- be two germs of sectorial fibered diffeomorphisms in $S_+ \times (\mathbb{C}^2,0)$ and $S_- \times (\mathbb{C}^2,0)$ respectively, where S_+ and S_- are as in Theorem 1.10, which are (asymptotically) tangent to the identity and such that*

$$(\varphi_\pm)_*(Y) = Y_{\text{norm}}\,.$$

Then, they necessarily coincide with the sectorial normalizing maps Φ_+ and Φ_- defined above.

Since two analytically conjugate vector fields are also formally conjugate, we fix now a normal form

$$Y_{\text{norm}} = x^2\frac{\partial}{\partial x} + (-\lambda + a_1 x - c(v))y_1\frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(v))y_2\frac{\partial}{\partial y_2}\,,$$

with $\lambda \in \mathbb{C}^*$, $\Re(a_1+a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ vanishing at the origin.

Definition 1.14. — *We denote by* $[Y_{\rm norm}]$ *the set of germs of holomorphic doubly-resonant saddle-nodes in* $(\mathbb{C}^3, 0)$ *which are formally conjugate to* $Y_{\rm norm}$ *by formal fibered diffeomorphisms tangent to the identity, and denote by* ${}^{[Y_{\rm norm}]}/_{\mathrm{Diff}_{\rm fib}(\mathbb{C}^3,0;\mathrm{Id})}$ *the set of orbits of the elements in this set under the action of* $\mathrm{Diff}_{\rm fib}(\mathbb{C}^3, 0; \mathrm{Id})$.

According to Theorem 1.10, to any $Y \in [Y_{\rm norm}]$ we can associate two sectorial normalizing maps Φ_+, Φ_-, which can in fact extend analytically in domains $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$, where $S_\pm$ is an asymptotic sector in the direction $\arg(\pm i\lambda)$ with opening 2π (see Definition 2.3):

$$(S_+, S_-) \in \mathcal{AS}_{\arg(i\lambda),2\pi} \times \mathcal{AS}_{\arg(-i\lambda),2\pi}\,.$$

Then, we consider two germs of sectorial fibered diffeomorphisms $\Phi_\lambda, \Phi_{-\lambda}$ analytic in $S_\lambda, S_{-\lambda}$, with

$$\text{(1.11)}\qquad \begin{aligned} S_\lambda &:= S_+ \cap S_- \cap \left\{ \Re\left(\frac{x}{\lambda}\right) > 0 \right\} \in \mathcal{AS}_{\arg(\lambda),\pi} \\ S_{-\lambda} &:= S_+ \cap S_- \cap \left\{ \Re\left(\frac{x}{\lambda}\right) < 0 \right\} \in \mathcal{AS}_{\arg(-\lambda),\pi} \end{aligned}$$

defined by:

$$\begin{cases} \Phi_\lambda := \left(\Phi_+ \circ \Phi_-^{-1}\right)_{|S_\lambda \times (\mathbb{C}^2,0)} \in \mathrm{Diff}_{\rm fib}(\mathcal{S}_{\arg(\lambda),\ }; \mathrm{Id}), & \forall \epsilon \in [0, \pi[\\ \Phi_{-\lambda} := \left(\Phi_- \circ \Phi_+^{-1}\right)_{|S_{-\lambda} \times (\mathbb{C}^2,0)} \in \mathrm{Diff}_{\rm fib}(\mathcal{S}_{\arg(-\lambda),\ }; \mathrm{Id}), & \forall \epsilon \in [0, \pi[\,. \end{cases}$$

Notice that $\Phi_\lambda, \Phi_{-\lambda}$ are *isotropies* of $Y_{\rm norm}$, i.e. they satisfy:

$$\text{(1.12)}\qquad (\Phi_{\pm\lambda})_* (Y_{\rm norm}) = Y_{\rm norm}\,.$$

Definition 1.15. — *With the above notations, we define* $\Lambda_\lambda(Y_{\rm norm})$ *(resp.* $\Lambda_{-\lambda}(Y_{\rm norm})$*) as the group of germs of sectorial fibered isotropies of* $Y_{\rm norm}$*, tangent to the identity, and admitting the identity as Gevrey-1 asymptotic expansion (see Definition 2.4) in sectorial domains of the form* $S_\lambda \times (\mathbb{C}^2, 0)$ *(resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$*), with* $S_{\pm\lambda} \in \mathcal{AS}_{\arg(\pm\lambda),\pi}$*. The two sectorial isotropies* Φ_λ *and* $\Phi_{-\lambda}$ *defined above are called the* Stokes diffeomorphisms *associate to* $Y \in [Y_{\rm norm}]$.

Our second main result gives the moduli space for the analytic classification that we are looking for.

Theorem 1.16. — *The map*

$$\begin{aligned} {}^{[Y_{\rm norm}]}/_{\mathrm{Diff}_{\rm fib}(\mathbb{C}^3,0;\mathrm{Id})} &\longrightarrow \Lambda_\lambda(Y_{\rm norm}) \times \Lambda_{-\lambda}(Y_{\rm norm}) \\ Y &\longrightarrow (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned}$$

is well-defined and bijective.

In particular, the result states that Stokes diffeomorphisms only depend on the class of $Y \quad [Y_{\mathrm{norm}}]$ in the quotient $[Y_{\mathrm{norm}}]/\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$. We will give a description of this set of invariants in terms of power series in the *space of leaves* in Section 5.

Remark 1.17. — In this paper we start by proving a theorem of sectorial normalizing map analogous to the classical one due to Hukuhara–Kimura–Matuda for saddle-nodes in $(\mathbb{C}^2, 0)$ [9], generalized later by Stolovitch in any dimension in [28]. Unlike the method based on a fixed point theorem used by these authors, we have used a more geometric approach (following the works of Teyssier [30, 31]) based on the resolution of an homological equation, by integrating a well chosen 1-form along asymptotic paths. This latter approach turned out to be more efficient to deal with the fact that $Y_{|\{x=0\}}$ is not necessarily linearizable. Indeed, if we look at [28] in details, one of the first problem is that in the irregular systems that needs to be solved by a fixed point method (for instance equation (2.7) in the cited paper), the non-linear terms would not be divisible by the "time" variable t in our situation. This would complicate the different estimations that are done later in the cited work. This is the first main new phenomena we have met. Then we will see that the sectorial normalizing maps Φ_+, Φ_- in the corollary above admit in fact the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5 as "true" Gevrey-1 asymptotic expansion in $S_+ \quad S_{\arg(\lambda),\eta}$ and $S_- \quad S_{\arg(-\lambda),\eta}$ respectively. This will be proved by studying $\Phi_+ \circ (\Phi_-)^{-1}$ in $S_+ \cap S_-$ (and more generally any germ of sectorial fibered isotropy of Y_{norm} in "narrow" sectorial neighborhoods $S_{\pm\lambda} \quad S_+ \cap S_-$ which admits the identity as weak Gevrey-1 asymptotic expansion). The main difficulty is to prove that such a sectorial isotropy of Y_{norm} over the "narrow" sectors described above is necessarily exponentially close to the identity (see Lemma 5.20). This will be done *via* a detailed analysis of these maps in the space of leaves (see Definition 5.10). In fact, this is the second main new difficulty we have met, which is due to the presence of the "resonant" term

$$\frac{c_m (y_1 y_2)^m \log(x)}{x}$$

in the exponential expression of the first integrals of the vector field (see (5.3)). In [28], similar computations are done in Subsection 3.4.1. In this part of the paper, infinitely many irregular differential equations appear when identifying terms of same homogeneous degree. The fact that $Y_{|\{x=0\}}$ is linear implies that these differential equations are all linear and independent of each others (i.e. they are not mixed together). In our situation, this is not the case and then more complicated.

1.3. Painlevé equations: the transversally Hamiltonian case

In [33] Yoshida shows that a vector field in the class $\mathcal{SN}_{\text{fib},0}$ naturally appears after a suitable compactification (given by the so-called Boutroux coordinates [5]) of the phase space of Painlevé equations $(P_j)_{j=I,\dots,V}$, for generic values of the parameters. In these cases the vector field presents an additional transverse Hamiltonian structure. Let us illustrate these computations in the case of the first Painlevé equation:

$$(P_I) \qquad \frac{\mathrm{d}^2 z_1}{\mathrm{d}t^2} = 6z_1^2 + t\,.$$

As is well known since Okamoto [22], (P_I) can be seen as a non-autonomous Hamiltonian system

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = -\dfrac{\partial H}{\partial z_2} \\ \dfrac{\partial z_2}{\partial t} = \dfrac{\partial H}{\partial z_1} \end{cases}$$

with Hamiltonian

$$H(t, z_1, z_2) := 2z_1^3 + tz_1 - \frac{z_2^2}{2}. \tag{1.13}$$

More precisely, if we consider the standard symplectic form $\omega := dz_1 \wedge dz_2$ and the vector field

$$Z := \frac{\partial}{\partial t} - \frac{\partial H}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial H}{\partial z_1}\frac{\partial}{\partial z_2} \tag{1.14}$$

induced by (P_I), then the Lie derivative

$$\mathcal{L}_Z(\omega) = \left(\frac{\partial^2 H}{\partial t \partial z_1}\mathrm{d}z_1 + \frac{\partial^2 H}{\partial t \partial z_2}\mathrm{d}z_2\right) \wedge \mathrm{d}t = \mathrm{d}z_1 \wedge \mathrm{d}t$$

belongs to the ideal $\langle \mathrm{d}t \rangle$ generated by $\mathrm{d}t$ in the exterior algebra $\Omega^\bullet(\mathbb{C}^3)$ of differential forms in variables (t, z_1, z_2). Equivalently, for any $t_1, t_2 \in \mathbb{C}$ the flow of Z at time $(t_2 - t_1)$ acts as a *symplectomorphism* between fibers $\{t = t_1\}$ and $\{t = t_2\}$. The weighted compactification given by the Boutroux coordinates [5] defines a chart near $\{t = \infty\}$ as follows:

$$\begin{cases} z_2 = y_2 x^{-\frac{3}{5}} \\ z_1 = y_1 x^{-\frac{2}{5}} \\ t = x^{-\frac{4}{5}}. \end{cases}$$

In the coordinates (x, y_1, y_2), the vector field Z is transformed, up to a translation $y_1 \to y_1 + \zeta$ with $\zeta = \frac{i}{\sqrt{6}}$, to the vector field

$$\tilde{Z} = -\frac{5}{4x^{\frac{1}{5}}} Y \tag{1.15}$$

where

$$Y = x^2\frac{\partial}{\partial x} + \left(-\frac{4}{5}y_2 + \frac{2}{5}xy_1 + \frac{2\zeta}{5}x\right)\frac{\partial}{\partial y_1} + \left(-\frac{24}{5}y_1^2 - \frac{48\zeta}{5}y_1 + \frac{3}{5}xy_2\right)\frac{\partial}{\partial y_2}\,. \tag{1.16}$$

We observe that Y is a strictly non-degenerate doubly-resonant saddle-node as in Definitions 1.1 and 1.3 with residue $\mathrm{res}(Y) = 1$. Furthermore we have:

$$\begin{cases} \mathrm{d}t = -\frac{4}{5}5^{\frac{4}{5}}x^{-\frac{9}{5}}\mathrm{d}x \\ \mathrm{d}z_1 \wedge \mathrm{d}z_2 = \frac{1}{x}(\mathrm{d}y_1 \wedge \mathrm{d}y_2) + \frac{1}{5x^2}(2y_1\mathrm{d}y_2 - 3y_2\mathrm{d}y_1) \wedge \mathrm{d}x \\ \qquad\qquad\quad \in \frac{1}{x}(\mathrm{d}y_1 \wedge \mathrm{d}y_2) + \langle \mathrm{d}x \rangle\,, \end{cases}$$

where $\langle \mathrm{d}x \rangle$ denotes the ideal generated by $\mathrm{d}x$ in the algebra of holomorphic forms in $\mathbb{C}^* \times \mathbb{C}^2$. We finally obtain

$$\begin{cases} L_Y\left(\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x}\right) = \frac{1}{5x}(3y_2\mathrm{d}y_1 - (2\zeta + 2y_1)\mathrm{d}y_2) \wedge \mathrm{d}x \\ L_Y(\mathrm{d}x) = 2x\mathrm{d}x\,. \end{cases}$$

Therefore, both $L_Y(\omega)$ and $L_Y(\mathrm{d}x)$ are differential forms who lie in the ideal $\langle \mathrm{d}x \rangle$, in the algebra of germs of meromorphic 1-forms in $(\mathbb{C}^3, 0)$ with poles only in $\{x = 0\}$. This motivates the following:

Definition 1.18. — *Consider the rational 1-form*

$$\omega := \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x}\,. \tag{1.17}$$

We say that vector field Y is transversally Hamiltonian *(with respect to ω and $\mathrm{d}x$) if*

$$L_Y(\mathrm{d}x) \in \langle \mathrm{d}x \rangle \text{ and } L_Y(\omega) \in \langle \mathrm{d}x \rangle\,. \tag{1.18}$$

For any open sector $S \subset \mathbb{C}$, we say that a germ of sectorial fibered diffeomorphism Φ in $S \times (\mathbb{C}^2, 0)$ is transversally symplectic *(with respect to ω and $\mathrm{d}x$) if*

$$\Phi^*(\omega) \in \omega + \langle \mathrm{d}x \rangle$$

(Here $\Phi^*(\omega)$ denotes the pull-back of ω by Φ). We denote by $\mathrm{Diff}_\omega(\mathbb{C}^3, 0; \mathrm{Id})$ the group of transversally symplectic diffeomorphisms which are tangent to the identity.

Remark 1.19.

(1) The flow of a transversally Hamiltonian vector field X defines a map between fibers $\{x = x_1\}$ and $\{x = x_2\}$ which sends $\omega_{|x=x_1}$ onto $\omega_{|x=x_2}$, since

$$(\exp(X))\ (\omega) \quad \omega + \ \mathrm{d}x\ .$$

(2) A fibered sectorial diffeomorphism Φ is transversally symplectic if and only if $\det(\mathrm{d}\Phi) = 1$.

Definition 1.20. — *A* transversally Hamiltonian doubly-resonant saddle-node *is a transversally Hamiltonian vector field which is conjugate,* via *a transversally symplectic di eomorphism, to one of the form*

$$Y = x^2\frac{\partial}{\partial x} + (-\lambda y_1 + F_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + (\lambda y_2 + F_2(x,\mathbf{y}))\frac{\partial}{\partial y_2}\,, \tag{1.19}$$

with $\lambda \quad \mathbb{C}$ *and* F_1, F_2 *analytic in* $(\mathbb{C}^3, 0)$ *and of order at least* 2*.*

Notice that a transversally Hamiltonian doubly-resonant saddle-node is necessarily strictly non-degenerate (since its residue is always equal to 1), and also div-integrable (see Section 3). It follows from Yoshida's work [33] that the doubly-resonant saddle-nodes at infinity in Painlevé equations $(P_j)_{j=I,\dots,V}$ (for generic values of the parameters) all are transversally Hamiltonian. We recall the second main result from [3].

Theorem 1.21 ([3]). — *Let* $Y \quad SN_{\text{diag}}$ *be a diagonal doubly-resonant saddle-node which is supposed to be transversally Hamiltonian. Then, there exists a unique formal fibered transversally symplectic di eomorphism* $\hat{\Phi}$*, tangent to identity, such that:*

$$\begin{aligned}\hat{\Phi}\ (Y) = x^2\frac{\partial}{\partial x} &+ (-\lambda + a_1x - c(y_1y_2))y_1\frac{\partial}{\partial y_1}\\ &+ (\lambda + a_2x + c(y_1y_2))y_2\frac{\partial}{\partial y_2}\\ =: Y_{\text{norm}}\,,&\end{aligned} \tag{1.20}$$

where $\lambda \quad \mathbb{C}$, $c(v) \quad v\mathbb{C}\llbracket v\rrbracket$ *a formal power series in* $v = y_1y_2$ *without constant term and* $a_1, a_2 \quad \mathbb{C}$ *are such that* $a_1 + a_2 = 1$*.*

As a consequence of Theorem 1.21, Theorem 1.10 we have the following:

Theorem 1.22. — *Let* Y *be a transversally Hamiltonian doubly-resonant saddle-node and let* $\hat{\Phi}$ *be the unique formal normalizing map given by Theorem 1.21. Then the associate sectorial normalizing maps* Φ_+ *and* Φ_- *are also transversally symplectic.*

Proof. — Since $\hat{\Phi}$ is 1-summable in $S_{\pm} \times (\mathbb{C}^2, 0)$, the formal power series $\det(\mathrm{d}\hat{\Phi})$ is also 1-summable in $S_{\pm} \times (\mathbb{C}^2, 0)$, and its asymptotic expansion has to be the constant 1. By uniqueness of the 1-sum, we thus have $\det(\mathrm{d}\Phi_{\pm}) = 1$.

Let us fix a normal form Y_{norm} as in Theorem 1.22, and consider two sectorial domains $S_{\lambda} \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ as in (1.11). Then, the Stokes diffeomorphisms $(\Phi_{\lambda}, \Phi_{-\lambda})$ defined in the previous subsection as
□

$$\begin{aligned} \Phi_{\lambda} &:= (\Phi_{+} \circ \Phi_{-}^{-1})_{|S_{\lambda} \times (\mathbb{C}^2,0)} \\ \Phi_{-\lambda} &:= (\Phi_{-} \circ \Phi_{+}^{-1})_{|S_{-\lambda} \times (\mathbb{C}^2,0)}, \end{aligned}$$

are also transversally symplectic.

Definition 1.23. — *We denote by* $\Lambda_{\lambda}^{\omega}(Y_{\mathrm{norm}})$ *(resp.* $\Lambda_{-\lambda}^{\omega}(Y_{\mathrm{norm}})$*) the group of germs of sectorial fibered isotropies of* Y_{norm}*, admitting the identity as Gevrey-1 asymptotic expansion in sectorial domains of the form* $S_{\lambda} \times (\mathbb{C}^2, 0)$ *(resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$*), and which are transversally symplectic.*

Let us denote by $[Y_{\mathrm{norm}}]_{\omega}$ the set of germs of vector fields which are formally conjugate to Y_{norm} *via* (formal) transversally symplectic diffeomorphisms tangent to the identity. As a consequence of Theorems (1.16) and (1.22), we can now state the following result.

Theorem 1.24. — *The map*

$$\begin{aligned} [Y_{\mathrm{norm}}]_{\omega} / \mathrm{Diff}_{\omega}(\mathbb{C}^3, 0; \mathrm{Id}) &\longrightarrow \Lambda_{\lambda}^{\omega}(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}^{\omega}(Y_{\mathrm{norm}}) \\ Y &\longrightarrow (\Phi_{\lambda}, \Phi_{-\lambda}) \end{aligned}$$

is a well-defined bijection.

1.4. Outline of the paper

In Section 2, we introduce the different tools we need concerning asymptotic expansion, Gevrey-1 series and 1-summability. We will in particular introduce a notion of "*weak*" 1-summability.

In Section 3, we prove Proposition 3.1, which states that we can always formally conjugate a non-degenerate doubly-resonant saddle-node which is also div-integrable to its normal form up to remaining terms of order $\mathrm{O}(x^N)$, for all N $\mathbb{N}_{>0}$, and the conjugacy is actually 1-summable.

In Section 4, we prove that for all Y $\mathcal{SN}_{\mathrm{fib},0}$, there exists a unique pair of sectorial normalizing maps (Φ_{+}, Φ_{-}) tangent to the identity which conjugates Y to its normal form Y_{norm} over sectors with opening greater than π and arbitrarily close to 2π. The existence is given by Corollary 4.2, while the uniqueness clause stated in Proposition 1.13 is proved thanks to Proposition 4.16. Moreover, we will see that Φ_+ and Φ_- both admit the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5 as weak Gevrey-1 asymptotic expansion (see Proposition 4.18).

In Section 5, we show that the Stokes diffeomorphisms Φ_λ and $\Phi_{-\lambda}$, which admit a priori the identity only as weak Gevrey-1 asymptotic expansion, admit in fact the identity as "true" Gevrey-1 asymptotic expansion. This will be done by studying more generally the germs of sectorial isotropies of the normal form in sectorial domains with "narrow" opening (see Corollary 5.2). Using a theorem by Martinet and Ramis [17] reformulated in Theorem 2.22, which is a "non-abelian" version of the Ramis–Sibuya theorem, we will obtain the fact that $\hat{\Phi}$ is 1-summable in every direction $\theta = \arg(\pm\lambda)$, of 1-sums Φ_+ and Φ_- respectively in the corresponding domains (see Corollary 5.6). We then give a short proof of Theorem 1.10, just by using the different lemmas and propositions needed and proved earlier in this paper. After that, we will once again use Theorem 2.22 in order to obtain both Theorems 1.16 and 1.24. We give in Proposition 5.24 a description of the moduli space of analytic classification in terms of some spaces of power series in the space of leaves.

Acknowledgments. The author thanks his PhD advisors, namely Daniel Panazzolo and Loïc Teyssier: their many advices were absolutely necessary to do this work. He also thanks Laurent Stolovitch for his interest and his advices.

2. Background

We refer the reader to [4, 15, 17, 25] for a detailed introduction to the theory of asymptotic expansion, Gevrey series and summability (see also [28] for a useful discussion of these concepts), where one can find the proofs of the classical results we recall (but we do not prove here). We call $x \in \mathbb{C}$ the *independent* variable and $\mathbf{y} := (y_1, \dots, y_n) \in \mathbb{C}^n$, $n \in \mathbb{N}$, the *dependent* variables. As usual we define $\mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}$ for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, and $|\mathbf{k}| = k_1 + \dots + k_n$. The notions of asymptotic expansion, Gevrey-1 power series and 1-summability presented here are always considered with respect to the independent variable x living in (open) sectors, the dependent variable $\mathbf{y}$ belonging to poly-discs

$$\mathbf{D}(\mathbf{0}, \mathbf{r}) := \{\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_1| < r_1, \dots |y_n| < r_n\},$$

of poly-radius $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{R}_{>0})^n$. Given an open subset

$$U \subset \mathbb{C}^{n+1} = \{(x, \mathbf{y}) \in \mathbb{C} \times \mathbb{C}^n\}$$

we denote by $\mathcal{O}(U)$ the algebra of holomorphic function in U. The algebra of germs of analytic functions of m variables $\mathbf{x} := (x_1, \ldots, x_m)$ at the origin is denoted by $\mathbb{C}\{\mathbf{x}\}$. The results recalled in this section are valid when $n = 0$. Some statements for which we do not give a proof can be proved exactly as in the classical case $n = 0$, uniformly in the dependent multi-variable $\mathbf{y}$. For convenience and homogeneity reasons we will present some classical results not in their original (and more general) form, but rather in more specific cases which we will need. Finally, we will introduce a notion of *weak* Gevrey-1 summability, which we will compare to the classical notion of 1-summability.

2.1. Sectorial germs

Given $r > 0$, and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we denote by $S(r, \alpha, \beta)$ the following open sector:

$$S(r, \alpha, \beta) := \{x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \alpha < \arg(x) < \beta\}.$$

Let $\theta \in \mathbb{R}$, $\eta \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$.

Definition 2.1.

(1) *An* $x-$sectorial neighborhood *(or simply* sectorial neighborhood*)* of the origin *(in* $\mathbb{C}^{n+1}$*)* in the direction θ with opening η *is an open set* $S \subset \mathbb{C}^{n+1}$ *such that*

$$S \supset S\left(r, \theta - \frac{\eta}{2} - \epsilon, \theta + \frac{\eta}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$$

for some $r > 0$, $\mathbf{r} \in (\mathbb{R}_{>0})^n$ *and* $\epsilon > 0$. *We denote by* $(\mathcal{S}_{\theta,\eta}, \leqslant)$ *the directed set formed by all such neighborhoods, equipped with the order relation*

$$S_1 \leqslant S_2 \iff S_1 \supset S_2\,. \tag{2.1}$$

(2) *The algebra of* germs of holomorphic functions in a sectorial neighborhood of the origin in the direction θ with opening η *is the direct limit*

$$\mathcal{O}(\mathcal{S}_{\theta,\eta}) := \varinjlim \mathcal{O}(S)$$

with respect to the directed system defined by $\{\mathcal{O}(S) : S \in \mathcal{S}_{\theta,\eta}\}$.

We now give the definition of *a (germ of a) sectorial diffeomorphism*.

Definition 2.2.

(1) *Given an element* $\mathcal{S} \in \mathcal{S}_{\theta,\eta}$, *we denote by* $\mathrm{Diff}_{\mathrm{fib}}(\mathcal{S};\mathrm{Id})$ *the set of holomorphic fibered diffeomorphisms of the form*

$$\begin{aligned} \Phi : \quad \mathcal{S} &\longrightarrow \Phi(\mathcal{S}) \\ (x,\mathbf{y}) &\longmapsto (x, \phi_1(x,\mathbf{y}), \phi_2(x,\mathbf{y})), \end{aligned}$$

such that $\Phi(x,\mathbf{y}) - \mathrm{Id}(x,\mathbf{y}) = \mathrm{O}(\|x,\mathbf{y}\|^2)$, *as* $(x,\mathbf{y}) \to (0,\mathbf{0})$ *in* $\mathcal{S}$.[1]

(2) *The set of* germs of (fibered) sectorial diffeomorphisms in the direction θ with opening η, tangent to the identity, *is the direct limit*

$$\mathrm{Diff}_{\mathrm{fib}}(\mathcal{S}_{\theta,\eta};\mathrm{Id}) := \varinjlim \mathrm{Diff}_{\mathrm{fib}}(\mathcal{S};\mathrm{Id})$$

with respect to the directed system defined by $\{\mathrm{Diff}_{\mathrm{fib}}(\mathcal{S};\mathrm{Id}) : \mathcal{S} \in \mathcal{S}_{\theta,\eta}\}$. *We equip* $\mathrm{Diff}_{\mathrm{fib}}(\mathcal{S}_{\theta,\eta};\mathrm{Id})$ *with a group structure as follows: given two germs* $\Phi, \Psi \in \mathrm{Diff}_{\mathrm{fib}}(\mathcal{S}_{\theta,\eta};\mathrm{Id})$ *we chose corresponding representatives* $\Phi_0 \in \mathrm{Diff}_{\mathrm{fib}}(\mathcal{S};\mathrm{Id})$ *and* $\Psi_0 \in \mathrm{Diff}_{\mathrm{fib}}(\mathcal{T};\mathrm{Id})$ *with* $\mathcal{S}, \mathcal{T} \in \mathcal{S}_{\theta,\eta}$ *such that* $\mathcal{T} \subset \Phi_0(\mathcal{S})$ *and let* $\Psi \circ \Phi$ *be the germ defined by* $\Psi_0 \circ \Phi_0$.[2]

We will also need the notion of *asymptotic sectors.*

Definition 2.3. — *An* (open) asymptotic sector of the origin in the direction θ and with opening η *is an open set* $S \subset \mathbb{C}$ *such that*

$$S \supset \bigcup_{0 \leqslant \eta' < \eta} \mathcal{S}_{\theta,\eta'}.$$

We denote by $\mathcal{AS}_{\theta,\eta}$ *the set of all such (open) asymptotic sectors.*

2.2. Gevrey-1 power series and 1-summability

2.2.1. Gevrey-1 asymptotic expansions

In this subsection we fix a formal power series which we write under two forms:

$$\hat{f}(x,\mathbf{y}) = \sum_{k \geqslant 0} f_k(\mathbf{y}) x^k = \sum_{(j_0,\mathbf{j}) \in \mathbb{N}^{n+1}} f_{j_0,\mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x,\mathbf{y}]],$$

using the canonical identification $\mathbb{C}[[x,\mathbf{y}]] = \mathbb{C}[[x]][[\mathbf{y}]] = \mathbb{C}[[\mathbf{y}]][[x]]$. We also fix a norm $\|\cdot\|$ in $\mathbb{C}^{n+1}$.

[1] This condition implies in particular that $\Phi(\mathcal{S}) \in \mathcal{S}_{\theta,\eta}$.

[2] One can prove that this definition is independent of the choice of the representatives.

Definition 2.4.

- *A function* f *analytic in a domain* $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ *admits* $\hat f$ *as asymptotic expansion* in the sense of Gérard–Sibuya *in this domain if for all closed sub-sector* $S' \subset S(r,\alpha,\beta)$ *and compact* $\mathbf{K}\subset\mathbf{D}(\mathbf{0},\mathbf{r})$*, for all* $N\in\mathbb{N}$*, there exists a constant* $C_{S',K,N}>0$ *such that:*
$$\Big\| f(x,\mathbf{y}) - \sum_{j_0+j_1+\ldots j_n \leqslant N} f_{j_0,\mathbf{j}} x^{j_0}\mathbf{y}^{\mathbf{j}} \Big\| \leqslant C_{S',K,N} \,\|(x,\mathbf{y})\|^{N+1}$$
for all $(x,\mathbf{y})\in S'\times K$.
- *A function* f *analytic in a domain* $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ *admits* $\hat f$ *as* asymptotic expansion (with respect to x) *if for all* $k\in\mathbb{N}$*,* $f_k(\mathbf{y})$ *is analytic in* $\mathbf{D}(\mathbf{0},\mathbf{r})$*, and if for all closed sub-sector* $S'\subset S(r,\alpha,\beta)$*, compact subset* $\mathbf{K}\subset\mathbf{D}(\mathbf{0},\mathbf{r})$ *and* $N\in\mathbb{N}$*, there exists* $A_{S',K,N}>0$ *such that:*
$$\Big\| f(x,\mathbf{y}) - \sum_{k\geqslant 0}^{N} f_k(\mathbf{y})x^k \Big\| \leqslant A_{S',K,N}|x^{N+1}|$$
for all $(x,\mathbf{y})\in S'\times K$.
- *An analytic function* f *in a sectorial domain* $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ *admits* $\hat f$ *as* Gevrey-1 asymptotic expansion *in this domain, if for all* $k\in\mathbb{N}$*,* $f_k(\mathbf{y})$ *is analytic in* $\mathbf{D}(\mathbf{0},\mathbf{r})$*, and if for all closed sub-sector* $S'\subset S(r,\alpha,\beta)$*, there exists* $A,C>0$ *such that:*
$$\Big\| f(x,\mathbf{y}) - \sum_{k=0}^{N-1} f_k(\mathbf{y})x^k \Big\| \leqslant AC^N(N!)|x^N|$$
for all $N\in\mathbb{N}$ *and* $(x,\mathbf{y})\in S'\times\mathbf{D}(\mathbf{0},\mathbf{r})$.

Remark 2.5.

(1) If a function admits $\hat f$ as Gevrey-1 asymptotic expansion in $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$, then it also admits $\hat f$ as asymptotic expansion.
(2) If a function admits $\hat f$ as asymptotic expansion in $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$, then it also admits $\hat f$ as asymptotic expansion in the sense of Gérard–Sibuya.
(3) An asymptotic expansion (in any of the different senses described above) is unique.

As a consequence of Stirling formula, we have the following characterization for functions admitting 0 as Gevrey-1 asymptotic expansion.

Proposition 2.6. — *The set of analytic functions admitting* 0 *as Gevrey-1 asymptotic expansion at the origin in a sectorial domain* $S(r,\alpha,\beta) \times \mathbf{D}(\mathbf{0},\mathbf{r})$ *is exactly the set of analytic functions* f *in* $S(r,\alpha,\beta) \times \mathbf{D}(\mathbf{0},\mathbf{r})$ *such that for all closed sub-sector* $S' \subset S(r,\alpha,\beta)$ *and all compact* $\mathbf{K} \subset \mathbf{D}(\mathbf{0},\mathbf{r})$*, there exist* $A_{S',K}, B_{S',K} > 0$ *such that:*

$$|f(x,\mathbf{y})| \leqslant A_{S',K} \exp\left(-\frac{B_{S',K}}{|x|}\right).$$

We say that such a function is exponentially flat at the origin in the corresponding domain.

2.2.2. Borel transform and Gevrey-1 power series

Definition 2.7.

- *We define the Borel transform* $\mathcal{B}(\hat{f})$ *of* $\hat{f}$ *as:*

$$\mathcal{B}(\hat{f})(t,\mathbf{y}) := \sum_{k \geqslant 0} \frac{f_k(\mathbf{y})}{k!} t^k .$$

- *We say that* $\hat{f}$ *is Gevrey-1 if* $\mathcal{B}(\hat{f})$ *is convergent in a neighborhood of the origin in* $\mathbb{C} \times \mathbb{C}^n$*. Notice that in this case the* $f_k(\mathbf{y})$, $k \geqslant 0$*, are all analytic in a same polydisc* $\mathbf{D}(\mathbf{0},\mathbf{r})$*, of poly-radius* $\mathbf{r} = (r_n \ldots, r_n) \in (\mathbb{R}_{>0})^n$*, so that* $\mathcal{B}(\hat{f})$ *is analytic in* $\mathrm{D}(0,\rho) \times \mathbf{D}(\mathbf{0},\mathbf{r})$*, for some* $\rho > 0$*. Possibly by reducing* $\rho, r_1, \ldots, r_n > 0$*, we can assume that* $\mathcal{B}(\hat{f})$ *is bounded in* $\mathrm{D}(0,\rho) \times \mathbf{D}(\mathbf{0},\mathbf{r})$*.*

Remark 2.8.

(1) If a sectorial function f admits $\hat{f}$ for Gevrey-1 asymptotic expansion as in Definition 2.4 then $\hat{f}$ is a Gevrey-1 formal power series.

(2) The set of all Gevrey-1 formal power series is an algebra closed under partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$.

Remark 2.9. — For technical reasons we will sometimes need to use another definition of the Borel transform, that is:

$$\widetilde{\mathcal{B}}(\hat{f})(t,\mathbf{y}) := \sum_{k \geqslant 0} f_{k+1}(\mathbf{y}) \frac{t^k}{k!} .$$

The first definition we gave has the advantage of being “directly” invertible (*via* the Laplace transform) for all 1-summable formal power series (see next subsection), but behaves not so good with respect to the product. On the contrary, the second definition will be “directly” invertible only for 1-summable formal power series with null constant term (otherwise a translation is needed). However, the advantage of the second Borel transform is that it changes a product into a convolution product:

$$\widetilde{B}(\hat{f}\hat{g}) = (\widetilde{B}(\hat{f}) * \widetilde{B}(\hat{g}))\,,$$

where the convolution product of two analytic functions $h_1 h_2$ is defined by

$$(h_1 * h_2)(t,\mathbf{y}) := \int_0^t h_1(s,\mathbf{y})h_2(s-t,\mathbf{y})\mathrm{d}s\,.$$

The property of being Gevrey-1 or not does not depend on the choice of the definition we take for the Borel transform.

2.2.3. Directional 1-summability and Borel–Laplace summation

Definition 2.10. — *Given $\theta \in \mathbb{R}$ and $\delta > 0$, we define the infinite sector in the direction θ with opening δ as the set*

$$A^{\infty}_{\theta,\delta} := \left\{ t \in \mathbb{C} \;\middle|\; |\arg(t) - \theta| < \frac{\delta}{2} \right\}.$$

We say that $\hat{f}$ is 1-summable in the direction $\theta \in \mathbb{R}$, *if the following three conditions holds:*

- *$\hat{f}$ is a Gevrey-1 formal power series;*
- *$B(\hat{f})$ can be analytically continued to a domain of the form $A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0},\mathbf{r})$;*
- *there exists $\lambda > 0, M > 0$ such that:*

$$\forall (t,\mathbf{y}) \in A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0},\mathbf{r}),\ |B(\hat{f})(t,\mathbf{y})| \leqslant M\exp(\lambda|t|)\,.$$

In this case we set $\Delta_{\theta,\delta,\rho} := A^{\infty}_{\theta,\delta} \cup \mathrm{D}(0,\rho)$ and

$$\|\hat{f}\|_{\lambda,\theta,\delta,\rho,\mathbf{r}} := \sup_{(t,\mathbf{y}) \in \Delta_{\theta,\delta,\rho} \times \mathbf{D}(\mathbf{0},\mathbf{r})} |B(\hat{f})(t,\mathbf{y})\exp(-\lambda|t|)|\,.$$

If the domain is clear from the context we will simply write $\|\hat{f}\|_{\lambda}$.

Remark 2.11.

(1) For fixed $(\lambda,\theta,\delta,\rho,\mathbf{r})$ as above, the set $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$ of formal power series $\hat{f}$ 1-summable in the direction θ and such that $\|\hat{f}\|_{\lambda,\theta,\delta,\rho,\mathbf{r}} < +\infty$ is a Banach vector space for the norm $\|\cdot\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}$. We simply write $(\mathfrak{B}_{\lambda}, \|\cdot\|_{\lambda})$ when there is no ambiguity.

(2) We will also need a norm well-adapted to the second Borel transform $\widetilde{B}$ (cf. Remark 2.9), that is:

$$\|\hat{f}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}} := \sup_{(t,\mathbf{y}) \in \Delta_{\theta,\delta,\rho} \times \mathbf{D}(\mathbf{0},\mathbf{r})} |B(\hat{f})(t,\mathbf{y})(1+\lambda^2|t|^2)\exp(-\lambda|t|)|\,.$$

We write then $\mathfrak{B}^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$ the set space of formal power series $\hat{f}$ which are 1-summable in the direction θ and such that $\|\hat{f}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}} < +\infty$.

(3) If $\lambda' > \lambda$, then $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}} \subset \mathfrak{B}_{\lambda',\theta,\delta,\rho,\mathbf{r}}$ and $\mathfrak{B}^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}} \subset \mathfrak{B}^{\mathrm{bis}}_{\lambda',\theta,\delta,\rho,\mathbf{r}}$.

Proposition 2.12 ([4, Proposition 4.]). — *If $\hat{f},\hat{g} \in \mathfrak{B}^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$, then $\hat{f}\hat{g} \in \mathfrak{B}^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$ and:*

$$\|\hat{f}\hat{g}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}} \leqslant \frac{4\pi}{\lambda}\|\hat{f}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}\|\hat{g}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}.$$

Remark 2.13. — If $\lambda \geqslant 4\pi$, then $\|\cdot\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$ is a sub-multiplicative norm, i.e.

$$\|\hat{f}\hat{g}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}} \leqslant \|\hat{f}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}\|\hat{g}\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho,\mathbf{r}}.$$

Definition 2.14. — *Let g be analytic in a domain and $A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0},\mathbf{r})$ and let $\lambda > 0, M > 0$ such that*

$$\forall (t,\mathbf{y}) \in A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0},\mathbf{r}),\ |g(t,\mathbf{y})| \leqslant M\exp(\lambda|t|).$$

We define the Laplace transform *of g in the direction θ as:*

$$\mathcal{L}_\theta(g)(x,\mathbf{y}) := \int_{e^{i\theta}\mathbb{R}_{>0}} g(t,\mathbf{y})\exp\left(-\frac{t}{x}\right)\frac{\mathrm{d}t}{x},$$

which is absolutely convergent for all $x \in \mathbb{C}$ with $\Re\left(\frac{e^{i\theta}}{x}\right) > \lambda$ and for all $\mathbf{y} \in \mathbf{D}(\mathbf{0},\mathbf{r})$, and analytic with respect to $(x,\mathbf{y})$ in this domain.

Remark 2.15. — As for the Borel transform, there also exists another definition of the Laplace transform, that is:

$$\widetilde{\mathcal{L}}_\theta(g)(x,\mathbf{y}) := \int_{e^{i\theta}\mathbb{R}_{>0}} g(t,\mathbf{y})\exp\left(-\frac{t}{x}\right)\mathrm{d}t.$$

Proposition 2.16. — *A formal power series $\hat{f} \in \mathbb{C}[[x,\mathbf{y}]]$ is 1-summable in the direction θ if and only if there exists a germ of a sectorial holomorphic function $f_\theta \in \mathcal{O}(S_{\theta,\pi})$ which admits $\hat{f}$ as Gevrey-1 asymptotic expansion in some $S \in S_{\theta,\pi}$. Moreover, f_θ is unique (as a germ in $\mathcal{O}(S_{\theta,\pi})$) and in particular*

$$f_\theta = \mathcal{L}_\theta(\mathcal{B}(\hat{f})).$$

The function (germ) f_θ is called the 1-sum of $\hat{f}$ in the direction θ.

Remark 2.17. — With the second definitions of Borel and Laplace transforms given above, we have a similar result for formal power series of the form $\hat{f}(x,\mathbf{y}) = \sum_k f_k(\mathbf{y})x^k$ with:

$$f_\theta = \widetilde{\mathcal{L}}_\theta(\widetilde{\mathcal{B}}(\hat{f})) + \hat{f}(0,\mathbf{y}).$$

We recall the following well-known result.

Lemma 2.18. — *The set* $\Sigma_\theta \subset \mathbb{C}[[x,\mathbf{y}]]$ *of* 1*-summable power series in the direction* θ *is an algebra closed under partial derivatives. Moreover the map*

$$\begin{aligned} \Sigma_\theta &\longrightarrow \mathcal{O}(S_{\theta,\pi}) \\ \hat{f} &\longmapsto f_\theta \end{aligned}$$

is an injective morphism of differential algebras.

Definition 2.19. — *A formal power series* $\hat{f} \in \mathbb{C}[[x,\mathbf{y}]]$ *is called* 1-summable *if it is 1-summable in all but a finite number of directions, called* Stokes directions. *In this case, if* $\theta_1,\ldots,\theta_k \in \mathbb{R}/2\pi\mathbb{Z}$ *are the possible Stokes directions, we say that* $\hat{f}$ *is 1-summable except for* $\theta_1,\ldots,\theta_k$*. More generally, we say that an* $m-$*uple* $(f_1,\ldots,f_m) \in \mathbb{C}[[x,\mathbf{y}]]^m$ *is Gevrey-1 (resp. 1-summable in direction* θ*) if this property holds for each component* $f_j, j=1,\ldots,m$*. Similarly, a formal vector field (or diffeomorphism) is said to be Gevrey-1 (resp. 1-summable in direction* θ*) if each one of its components has this property.*

The following classical result deals with composition of 1-summable power series (an elegant way to prove it is to use an important theorem of Ramis–Sibuya).

Proposition 2.20. — *Let* $\hat{\Phi}(x,\mathbf{y}) \in \mathbb{C}[[x,\mathbf{y}]]$ *be 1-summable in directions* θ *and* $\theta-\pi$*, and let* $\Phi_+(x,\mathbf{y})$ *and* $\Phi_-(x,\mathbf{y})$ *be its 1-sums directions* θ *and* $\theta-\pi$ *respectively. Let also* $\hat{f}_1(x,\mathbf{z}),\ldots,\hat{f}_n(x,\mathbf{z})$ *be 1-summable in directions* θ*,* $\theta-\pi$*, and* $f_{1,+},\ldots,f_{n,+}$*, and* $f_{1,-},\ldots,f_{n,-}$ *be their 1-sums in directions* θ *and* $\theta-\pi$ *respectively. Assume that*

$$\hat{f}_j(0,\mathbf{0}) = 0, \text{ for all } j=1,\ldots,n\,. \tag{2.2}$$

Then

$$\hat{\Psi}(x,\mathbf{z}) := \hat{\Phi}(x,\hat{f}_1(x,\mathbf{z}),\ldots,\hat{f}_n(x,\mathbf{z}))$$

is 1-summable in directions $\theta,\theta-\pi$*, and its 1-sum in the corresponding direction is*

$$\Psi_\pm(x,\mathbf{z}) := \Phi_\pm(x,f_{1,\pm}(x,\mathbf{z}),\ldots,f_{n,\pm}(x,\mathbf{z}))\,,$$

which is a germ of a sectorial holomorphic function in this direction.

Consider $\hat{Y}$ a formal singular vector field at the origin and a formal fibered diffeomorphism $\hat{\varphi}:(x,\mathbf{y}) \mapsto (x,\hat{\phi}(x,\mathbf{y}))$. Assume that both $\hat{Y}$ and $\hat{\varphi}$ are 1-summable in directions θ and $\theta-\pi$, for some $\theta \in \mathbb{R}$, and denote by Y_+,Y_- (resp. φ_+,φ_-) their 1-sums in directions θ and $\theta-\pi$ respectively.

As a consequence of Proposition 2.20 and Lemma 2.18, we can state the following:

Corollary 2.21. — *Under the assumptions above, $\hat{\varphi}\ (\hat{Y})$ is 1-summable in both directions θ and $\theta - \pi$, and its 1-sums in these directions are $\varphi_+(Y_+)$ and $\varphi_-(Y_-)$ respectively.*

2.2.4. An important result by Martinet and Ramis

We are going to make an essential use of an isomorphism theorem proved in [17]. This result is of paramount importance in the present paper since it will be a key tool in the proofs of both Theorems 1.10 and 1.16 (see Section 5). Let us consider two open asymptotic sectors S and S at the origin in directions θ and $\theta - \pi$ respectively, both of opening π:

$$S \quad AS_{\theta,\pi}$$
$$S \quad AS_{\theta-\pi,\pi}$$

(see Definition 2.3). In this particular setting, the cited theorem can be stated as follows.

Theorem 2.22 ([17, Théorème 5.2.1]). — *Consider a pair of germs of sectorial di eomorphisms*

$$(\varphi, \varphi\ ) \quad \mathrm{Diff}_{\mathrm{fib}}(S_{\theta,0}; \mathrm{Id}) \times \mathrm{Diff}_{\mathrm{fib}}(S_{\theta-\pi,0}; \mathrm{Id})$$

such that φ and φ extend analytically and admit the identity as Gevrey-1 asymptotic expansion in $S \times (\mathbb{C}^2, 0)$ and $S \times (\mathbb{C}^2, 0)$ respectively. Then, there exists a pair (ϕ_+, ϕ_-) of germs of sectorial fibered di eomorphisms

$$(\phi_+, \phi_-) \quad \mathrm{Diff}_{\mathrm{fib}}(S_{\theta+\frac{\pi}{2},\eta}; \mathrm{Id}) \times \mathrm{Diff}_{\mathrm{fib}}(S_{\theta-\frac{\pi}{2},\eta}; \mathrm{Id})$$

with $\eta \quad]\pi, 2\pi[$, which extend analytically in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, for some $S_+ \quad AS_{\theta+\frac{\pi}{2},2\pi}$ and $S_- \quad AS_{\theta-\frac{\pi}{2},2\pi}$, such that:

$$\phi_+ \circ (\phi_-)^{-1}_{|S \times (\mathbb{C}^2,0)} = \varphi$$
$$\phi_+ \circ (\phi_-)^{-1}_{|S \ \times (\mathbb{C}^2,0)} = \varphi\ .$$

There also exists a formal di eomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_+ and ϕ_- both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively.

In particular, in the theorem above $\hat{\phi}$ is 1−summable in every direction except θ and $\theta - \pi$, and its 1-sums in directions $\theta + \frac{\pi}{2}$ and $\theta - \frac{\pi}{2}$ respectively are ϕ_+ and ϕ_-. For future use, we are going to prove a *"transversally symplectic"* version of the above theorem.

Corollary 2.23. — *With the assumptions and notations of Theorem 2.22, if φ and φ both are transversally symplectic (see Definition 1.18), then there exists a germ of an analytic fibered di eomorphism* ψ $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ *(tangent to the identity), such that*

$$\sigma_+ := \phi_+ \circ \psi \quad \textit{and} \quad \sigma_- := \phi_- \circ \psi$$

both are transversally symplectic. Moreover we also have:

$$\sigma_+ \circ (\sigma_-)^{-1}_{|S\times(\mathbb{C}^2,0)} = \varphi$$
$$\sigma_+ \circ (\sigma_-)^{-1}_{|S\ \times(\mathbb{C}^2,0)} = \varphi\ .$$

Proof. — We recall that for any germ φ of a sectorial fibered diffeomorphism which is tangent to the identity, φ is transversally symplectic if and only if $\det(\mathrm{D}\varphi) = 1$. First of all, let us show that

$$\det(\mathrm{D}\phi_+) = \det(\mathrm{D}\phi_-) \text{ in } (S_+ \cap S_-) \times (\mathbb{C}^2, 0)\,.$$

Since ϕ_+ and ϕ_- both are sectorial fibered diffeomorphism which are tangent to the identity and transversally symplectic, then

$$\det(\phi_+ \circ (\phi_-)^{-1}_{|(S_+\cap S_-)\times(\mathbb{C}^2,0)}) = 1\,.$$

The *chain rule* implies immediately that

$$\det(\mathrm{D}\phi_+) = \det(\mathrm{D}\phi_-) \text{ in } (S_+ \cap S_-) \times (\mathbb{C}^2, 0)\,.$$

Thus, this equality allows us to define (thanks to the Riemann's Theorem of removable singularities) a germ of analytic function f $\mathcal{O}(\mathbb{C}^3, 0)$. Notice that $f(0,0,0) = 1$ because ϕ_+ and ϕ_- are tangent to the identity. Now, let us look for an element ψ $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ of the form

$$\psi : (x, y_1, y_2) \to (x, \psi_1(x, \mathbf{y}), y_2) \tag{2.3}$$

such that

$$\sigma_+ := \phi_+ \circ \psi \text{ and } \sigma_- := \phi_- \circ \psi$$

both be transversally symplectic. An easy computation gives:

$$\det(\sigma_\pm) = (\det(\mathrm{D}\phi_\pm) \circ \psi) \det(\mathrm{D}\psi) = 1$$

i.e.

$$(f \circ \psi) \det(\mathrm{d}\psi) = 1\,.$$

According to (2.3), we must have:

$$(f \circ \psi)\frac{\partial \psi_1}{\partial y_1} = 1\,. \tag{2.4}$$

Let us define

$$F(x, y_1, y_2) := \int_0^{y_1} f(x, z, y_2)\mathrm{d}z\,,$$

so that (2.4) can be integrated as

$$F \circ \psi = y_1 + h(x, y_2)\,,$$

for some $h \in \mathbb{C}\{x, y_2\}$. Notice that

$$\frac{\partial F}{\partial y_1}(0,0,0) = 1$$

since $f(0,0,0) = 1$. Let us chose $h = 0$. Then, we have to solve

$$F \circ \psi = y_1\,,$$

with unknown $\psi \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ as in (2.4). If we define

$$\Phi : (x, \mathbf{y}) \mapsto (x, F(x, \mathbf{y}), y_2)\,,$$

the latter problem is equivalent to find ψ as above such that:

$$\Phi \circ \psi = \mathrm{Id}\,.$$

Since $\mathrm{D}\Phi_0 = \mathrm{Id}$, the inverse function theorem gives us the existence of the germ $\psi = \Phi^{-1} \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$.

2.3. Weak Gevrey-1 power series and weak 1-summability

We present here a weaker notion of 1-summability that we will also need. Any function $f(x, \mathbf{y})$ analytic in a domain $U \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, where $U \subset \mathbb{C}$ is open, and bounded in any domain $U \times \overline{\mathbf{D}}(\mathbf{0}, \mathbf{r}')$ with $r_1' < r_1, \ldots, r_n' < r_n$, can be written

$$f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}}\,, \tag{2.5}$$

where for all $\mathbf{j} \in \mathbb{N}^n$, $F_{\mathbf{j}}$ is analytic and bounded on U, and defined *via* the Cauchy formula:

$$F_{\mathbf{j}}(x) = \frac{1}{(2i\pi)^n} \int_{|z_1| = r_1'} \cdots \int_{|z_n| = r_n'} \frac{f(x, \mathbf{z})}{(z_1)^{j_1+1} \ldots (z_n)^{j_n+1}} \mathrm{d}z_n \ldots \mathrm{d}z_1\,.$$

Notice that the convergence of the function series above is uniform in every compact with respect to x and $\mathbf{y}$. In the same way, any formal power series $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ can be written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}}\,.$$

Definition 2.24.

- *The formal power series $\hat{f}$ is said to be* weakly Gevrey-1 *if for all $\mathbf{j} \in \mathbb{N}^n$, $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$ is a Gevrey-1 formal power series.*

- *A function*
$$f(x,\mathbf{y}) = \sum_{\mathbf{j}\in\mathbb{N}^n} F_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}}$$
analytic and bounded in a domain $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$*, admits* $\hat{f}$ *as* weak Gevrey-1 asymptotic expansion *in* $x\in S(r,\alpha,\beta)$*, if for all* $\mathbf{j}\in\mathbb{N}^n$*,* $F_{\mathbf{j}}$ *admits* $\hat{F}_{\mathbf{j}}$ *as Gevrey-1 asymptotic expansion in* $S(r,\alpha,\beta)$*.*
- *The formal power series* $\hat{f}$ *is said to be* weakly 1-summable in the direction $\theta\in\mathbb{R}$*, if the following conditions hold:*
  - *for all* $\mathbf{j}\in\mathbb{N}^n$*,* $\hat{F}_{\mathbf{j}}(x)\in\mathbb{C}[\![x]\!]$ *is 1-summable in the direction* θ*, whose 1-sum in the direction* θ *is denoted by* $F_{\mathbf{j},\theta}$*;*
  - *the series* $f_\theta(x,\mathbf{y}) := \sum_{\mathbf{j}\in\mathbb{N}^n} F_{\mathbf{j},\theta}(x)\mathbf{y}^{\mathbf{j}}$ *defines a germ of a sectorial holomorphic function in a sectorial neighborhood attached to the origin in the direction* θ *with opening greater than* π*.*

In this case, $f_\theta(x,\mathbf{y})$ *is called* the weak 1-sum of $\hat{f}$ in the direction θ*.*

As a consequence to the classical theory of summability and Gevrey asymptotic expansions, we immediately have the following:

Lemma 2.25.

(1) *The weak Gevrey-1 asymptotic expansion of an analytic function in a domain* $S(r,\alpha,\beta)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ *is unique.*

(2) *The weak 1-sum of a weak 1-summable formal power series in the direction* θ*, is unique as a germ in* $\mathcal{O}(S_{\theta,\pi})$*.*

(3) *The set* $\Sigma_\theta^{(\mathrm{weak})}\subset\mathbb{C}[\![x,\mathbf{y}]\!]$ *of weakly* 1*-summable power series in the direction* θ *is an algebra closed under partial derivatives. Moreover the map*
$$\begin{aligned}\Sigma_\theta^{(\mathrm{weak})} &\longrightarrow \mathcal{O}(S_{\theta,\pi})\\ \hat{f} &\longmapsto f_\theta\end{aligned}$$
is an injective morphism of differential algebras.

The following proposition is an analogue of Proposition 2.20 for weak 1-summable formal power series, with the a stronger condition instead of (2.2).

Proposition 2.26. — *Let*
$$\hat{\Phi}(x,\mathbf{y}) = \sum_{\mathbf{j}\in\mathbb{N}^n} \hat{\Phi}_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}}\in\mathbb{C}[\![x,\mathbf{y}]\!]$$
and
$$\hat{f}^{(k)}(x,\mathbf{z}) = \sum_{\mathbf{j}\in\mathbb{N}^n} \hat{F}_{\mathbf{j}}^{(k)}(x)\mathbf{z}^{\mathbf{j}}\in\mathbb{C}[\![x,\mathbf{z}]\!],$$

for $k = 1, \dots, n$, be n+1 formal power series which are weakly 1-summable in directions θ and $\theta - \pi$. Let us denote by $\Phi_+, f_+^{(1)}, \dots, f_+^{(n)}$ (resp. Φ_-, $f_-^{(1)}, \dots, f_-^{(n)}$) their respective weak 1-sums in the direction θ (resp. $\theta - \pi$). Assume that $\hat{F}_{\mathbf{0}}^{(k)} = 0$ for all $k = 1, \dots, n$. Then,

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}(x, \hat{f}^{(1)}(x, \mathbf{z}), \dots, \hat{f}^{(n)}(x, \mathbf{z}))$$

is weakly 1-summable directions θ and $\theta - \pi$, and its 1-sum in the corresponding direction is

$$\Psi_{\pm}(x, \mathbf{z}) = \Phi_{\pm}(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})),$$

which is a germ of a sectorial holomorphic function in this direction with opening π.

Proof. — First of all,

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}(x, \hat{f}^{(1)}(x, \mathbf{z}), \dots, \hat{f}^{(n)}(x, \mathbf{z}))$$

is well defined as formal power series since for all $k = 1, \dots, n$, $\hat{F}_{\mathbf{0}}^{(k)} = 0$. It is also clear that

$$\Psi_{\pm}(x, \mathbf{z}) := \Phi_{\pm}(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z}))$$

is an analytic in a domain $S_+ \subset S_{\theta,\pi}$ (resp. $S_- \subset S_{\theta-\pi,\pi}$), because $f_{\pm}^{(k)}(x, \mathbf{0}) = 0$ for all $k = 1, \dots, n$. Finally, we check that $\Psi_{\pm}$ admits $\hat{\Psi}$ as weak Gevrey-1 asymptotic expansion in $S_{\pm}$. Indeed:

$$\begin{aligned}
\Psi_{\pm}(x, \mathbf{z}) &= \Phi_{\pm}(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})) \\
&= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) (f_{\pm}^{(1)}(x, \mathbf{z}))^{j_1} \dots (f_{\pm}^{(1)}(x, \mathbf{z}))^{j_n} \\
&= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) \left(\sum_{|\mathbf{l}| \geqslant 1} (F_{\mathbf{l}}^{(1)})_{\pm}(x) \mathbf{z}^{\mathbf{l}} \right)^{j_1} \dots \\
&\qquad \dots \left(\sum_{|\mathbf{l}| \geqslant 1} (F_{\mathbf{l}}^{(n)})_{\pm}(x) \mathbf{z}^{\mathbf{l}} \right)^{j_n} \\
&= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Psi_{\mathbf{j}})_{\pm}(x) \mathbf{y}^{\mathbf{j}}
\end{aligned}$$

where for all $\mathbf{j} \in \mathbb{N}^n$, $(\Psi_{\mathbf{j}})_{\pm}(x)$ is obtained as a finite number of additions and products of the $(\Phi_{\mathbf{k}})_{\pm}$,$(F_{\mathbf{k}}^{(1)})_{\pm}$,...,$(F_{\mathbf{k}}^{(n)})_{\pm}$, $|\mathbf{k}| \leqslant |\mathbf{l}|$. The same computation is valid for the associated formal power series, and allows us to define the $\hat{\Psi}_{\mathbf{j}}(x)$, for all $\mathbf{j} \in \mathbb{N}^n$. Then, each $(\Psi_{\mathbf{j}})_{\pm}$ has $\hat{\Psi}_{\mathbf{j}}$ as Gevrey-1 asymptotic expansion in $S_{\pm}$.

As a consequence of Proposition 2.26 and Lemma 2.25, we have an analogue version of Corollary (2.21) in the weak 1-summable case. Consider $\hat{Y}$ a formal singular vector field at the origin and a formal fibered diffeomorphism $\hat{\varphi} : (x, \mathbf{y}) \to (x, \hat{\phi}(x, \mathbf{y}))$ such that $\hat{\phi}(x, \mathbf{0}) = \mathbf{0}$. Assume that both $\hat{Y}$ and $\hat{\varphi}$ are weakly 1-summable in directions θ and $\theta - \pi$, for some $\theta \in \mathbb{R}$, and denote by Y_+, Y_- (resp. φ_+, φ_-) their weak 1-sums in directions θ and $\theta - \pi$ respectively.

Corollary 2.27. — *Under the assumptions above, $\hat{\varphi}_*(\hat{Y})$ is weakly 1-summable in both directions θ and $\theta - \pi$, and its 1-sums in these directions are $\varphi_{+*}(Y_+)$ and $\varphi_{-*}(Y_-)$ respectively.*

2.4. Weak 1-summability *versus* 1-summability

As in the previous subsection, let a formal power series $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[\![x, \mathbf{y}]\!]$ which is written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}},$$

so that its Borel transform is

$$\mathcal{B}(\hat{f})(t, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \mathcal{B}(\hat{F}_{\mathbf{j}})(t)\mathbf{y}^{\mathbf{j}}.$$

The next lemma is immediate.

Lemma 2.28.

(1) *The power series $\mathcal{B}(\hat{f})(t, \mathbf{y})$ is convergent in a neighborhood of the origin in $\mathbb{C}^{n+1}$ if and only if the $\mathcal{B}(\hat{F}_{\mathbf{j}})$, $\mathbf{j} \in \mathbb{N}^n$, are all analytic and bounded in a same disc $\mathrm{d}(0, \rho)$, $\rho > 0$, and if there exists $B, L > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\sup_{t \in \mathrm{d}(0,\rho)} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leqslant L.B^{|\mathbf{j}|}$.*

(2) *If (1) is satisfied, then $\mathcal{B}(\hat{f})$ can be analytically continued to a domain $A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ if and only if for all $\mathbf{j} \in \mathbb{N}^n$, $\mathcal{B}(\hat{F}_{\mathbf{j}})$ can be analytically continued to $A^{\infty}_{\theta,\delta}$ and if for all compact $K \subset A^{\infty}_{\theta,\delta}$, there exists $B, L > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\sup_{t \in K} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leqslant L.B^{|\mathbf{j}|}$.*

(3) *If (1) and (2) are satisfied, then there exists $\lambda, M > 0$ such that:*

$$\forall (t, \mathbf{y}) \in A^{\infty}_{\theta,\delta} \times \mathbf{D}(\mathbf{0}, \mathbf{r}),\ |\mathcal{B}(\hat{f})(t, \mathbf{y})| \leqslant M. \exp(\lambda|t|)$$

if and only if there exists $\lambda, L, B > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$,

$$\forall t \in A^{\infty}_{\theta,\delta},\ |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leqslant L.B^{|\mathbf{j}|} \exp(\lambda|t|).$$

Remark 2.29.

(1) Condition (1) above states that the formal power series $\hat{f}$ is Gevrey-1.
(2) As usual, there exists an equivalent lemma for the second definitions of the Borel transform (see Remark 2.9).

The following corollary gives a link between 1-summability and weak 1-summability.

Corollary 2.30. — *Let*

$$\hat{f}(x,\mathbf{y}) = \sum_{\mathbf{j}\in\mathbb{N}^n} \hat{F}_{\mathbf{j}}(x)\mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x,\mathbf{y}]] \tag{2.6}$$

be a formal power series. Then, $\hat{f}$ is 1-summable in the direction $\theta \in \mathbb{R}$, of 1-sum $f \in \mathcal{O}(S_{\theta,\pi})$, if and only if the following two conditions hold:

- *$\hat{f}$ is weakly 1-summable in the direction θ;*
- *there exists λ, δ, ρ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\|\hat{F}_{\mathbf{j}}\|_{\lambda,\theta,\delta,\rho} < \infty$ and the power series $\sum_{\mathbf{j}\in\mathbb{N}^n} \|\hat{F}_{\mathbf{j}}\|_{\lambda,\theta,\delta,\rho}\mathbf{y}^{\mathbf{j}}$ is convergent near the origin of $\mathbb{C}^n$.*

Proof. — This is an immediate consequence of Lemma 2.28.

Remark 2.31. — We can replace the norm $\|\cdot\|_{\lambda,\theta,\delta,\rho}$ by $\|\cdot\|^{\mathrm{bis}}_{\lambda,\theta,\delta,\rho}$ in the second point of the above corollary.

Notice that there exists formal power series which are weakly 1-summable in some direction but which are not Gevrey-1: for instance, the series

$$\hat{f} := \sum_{j} \hat{F}_j(x)y^j\,,$$

where for all $j \in \mathbb{N}$, $\hat{F}_j(x)$ is such that $\mathcal{B}(\hat{F}_j)(t) = \frac{1}{t+\frac{1}{j}}$, is weakly 1-summable in the direction $0 \in \mathbb{R}$, but is *not* Gevrey-1, since $\mathcal{B}(\hat{F}_j)$ has a pole in every $-\frac{1}{j} \underset{j\to+\infty}{\longrightarrow} 0$.

2.5. Some useful tools on 1-summability of solutions of singular linear differential equations

For future reuse, we give here two results on the 1-summability of formal solutions to some singular linear differential equations with 1-summable right hand side, which generalize (and precise) a similar result proved in [17] (Proposition p. 126). The authors use a norm $\|\cdot\|_{\beta}$, but we will need to use a norm $\|\cdot\|^{\mathrm{bis}}_{\beta}$ later (in the proof of Proposition 3.15).

Proposition 2.32. — *Let $\hat{b}$ be a formal power series 1-summable in the direction θ; consider a domain $\Delta_{\theta,\delta,\rho}$ as in Definition 2.10. Let use denote by b_θ its 1-sum in this direction θ. Let us also fix $\alpha, k \in \mathbb{C}$.*

(1) *Assume $\|\hat{b}\|_\beta^{\mathrm{bis}} < +\infty$ and that $k \in \mathbb{C}\setminus\{0\}$ is such that $d_k := \operatorname{dist}(-k, \Delta_{\theta,\delta,\rho}) > 0$ and $\beta d_k > C|\alpha k|$, where $C > 0$ is a constant large enough, independent from parameters $k, \beta, \theta, \delta, \rho$ (for instance, one can take $C = \frac{2\exp(2)}{5} + 5$). Then, the irregular singular differential equation*

$$x^2 \frac{\mathrm{d}a}{\mathrm{d}x}(x) + (1 + \alpha x) k a(x) = \hat{b}(x) \tag{2.7}$$

has a unique formal solution $\hat{a}$ such that $\hat{a}(0) = \frac{1}{k}\hat{b}(0)$. Moreover, $\hat{a}$ is 1-summable in the direction θ, and

$$\|\hat{a}\|_\beta \leqslant \frac{\beta}{\beta d_k - C|\alpha k|} \|\hat{b}\|_\beta\,. \tag{2.8}$$

Finally, the 1-sum a_θ of $\hat{a}$ in the direction θ is the only solution to

$$x^2 \frac{\mathrm{d}a_\theta}{\mathrm{d}x}(x) + (1 + \alpha x) k a_\theta(x) = b_\theta(x)$$

which is bounded in some $S_{\theta,\pi}$ $\mathsf{S}_{\theta,\pi}$.

(2) *Assume $\|\hat{b}\|_\beta < +\infty$ and that $\Re(k) > 0$. Then the regular singular differential equation*

$$x \frac{\mathrm{d}a}{\mathrm{d}x}(x) + k a(x) = \hat{b}(x) \tag{2.9}$$

admits a unique formal solution $\hat{a}$ which is also 1-summable in the direction θ, of 1-sum a_θ. Moreover, a_θ is the only germ of solution to the differential equation

$$x \frac{\mathrm{d}a}{\mathrm{d}x}(x) + k a(x) = b_\theta(x)$$

which is bounded in some $S_{\theta,\pi}$ $\mathsf{S}_{\theta,\pi}$.

Proof. — (1) Since $\hat{b}$ is 1-summable in the direction θ, we can choose $\rho > 0$ and $\delta > 0$ such that $\mathcal{B}(\hat{b})$ can be analytically continued to (and is bounded in) any domain of the form $\Delta_{\theta,\delta,\rho} \cap \overline{\mathrm{D}}(0,R)$, $R > 0$.

Let us apply the Borel transform $\mathcal{B}$ to equation (2.7): we obtain

$$(t + k)\mathcal{B}(\hat{a})(t) + \alpha k \int_0^t \mathcal{B}(\hat{a})(s)\mathrm{d}s = \mathcal{B}(\hat{b})(t)\,. \tag{2.10}$$

The derivative with respect to t of this equation shows that $\mathcal{B}(\hat{a})$ is solution of a linear differential equation, with only one (regular) singularity at $t =$

$-k$ (but this singularity is not in $\Delta_{\theta,\delta,\rho}$ by assumption):

$$(t+k)\frac{\mathrm{d}\mathcal{B}(\hat{a})}{\mathrm{d}t}(t) + (1+\alpha k)\mathcal{B}(\hat{a})(t) = \frac{\mathrm{d}\mathcal{B}(\hat{b})}{\mathrm{d}t}(t)\,.$$

Since $\mathcal{B}(\hat{b})$ can be analytically continued to $\Delta_{\theta,\delta,\rho}$, the same goes for $\frac{\mathrm{d}\mathcal{B}(\hat{b})}{\mathrm{d}r}(t)$ and then for $\mathcal{B}(\hat{a})$. Since $\mathcal{B}(\hat{a})(0) = \frac{\mathcal{B}(\hat{b})(0)}{k} = \frac{\hat{b}'(0)}{k}$, we can write:

$$\begin{aligned}
\mathcal{B}(\hat{a})(t) &= (t+k)^{-1-\alpha k}\left(\hat{b}'(0).k^{\alpha k} + \int_0^t \frac{\mathrm{d}\mathcal{B}(\hat{b})}{\mathrm{d}s}(s).(s+k)^{\alpha k}\mathrm{d}s\right)\\
&= (t+k)^{-1-\alpha k}\Big(\hat{b}'(0).k^{\alpha k} + \mathcal{B}(\hat{b})(t).(t+k)^{\alpha k} - \mathcal{B}(\hat{b})(0).k^{\alpha k}\\
&\qquad - \alpha k\int_0^t \mathcal{B}(\hat{b})(s).(s+k)^{\alpha k-1}\mathrm{d}s\Big)\\
&= (t+k)^{-1-\alpha k}\Big(\mathcal{B}(\hat{b})(t).(t+k)^{\alpha k}\\
&\qquad - \alpha k\int_0^t \mathcal{B}(\hat{b})(s).(s+k)^{\alpha k-1}\mathrm{d}s\Big)\\
\mathcal{B}(\hat{a}) &= \frac{\mathcal{B}(\hat{b})(t)}{(t+k)} - \alpha k.(t+k)^{-1-\alpha k}\int_0^t \mathcal{B}(\hat{b})(s).(s+k)^{\alpha k-1}\mathrm{d}s\,.
\end{aligned}$$

The fact that $\mathcal{B}(\hat{b})$ is bounded in any domain of the form $\Delta_{\theta,\delta,\rho}\cap\overline{\mathrm{D}}(0,R)$, $R>0$, implies that the same goes for $\mathcal{B}(\hat{a})$. Let us prove inequality (2.8). For all $R>0$, for all Gevrey-1 series $\hat{f}\in\mathbb{C}[\![x,\mathbf{y}]\!]$ such that $\mathcal{B}(\hat{f})$ can be analytically continued to $\Delta_{\theta,\delta,r}$, we set:

$$\left\|\hat{f}\right\|_{\beta,R}^{\mathrm{bis}} := \sup_{t\in\Delta_{\theta,\delta,\rho}\cap\overline{\mathrm{D}}(0,R)}\left\{\left|\mathcal{B}(\hat{f})(t)(1+\beta^2|t|^2)\exp(-\beta|t|)\right|\right\}\in\mathbb{R}\cup\{\infty\}\,.$$

Notice that $\left\|\hat{f}\right\|_{\beta}^{\mathrm{bis}} = \sup_{R>0}\left\{\left\|\hat{f}\right\|_{\beta,R}^{\mathrm{bis}}\right\}$ for all $\hat{f}$ as above, and that for all $R>0$, $\left\|\hat{a}\right\|_{\beta,R}^{\mathrm{bis}} < +\infty$, since $\mathcal{B}(\hat{a})$ is bounded in any domain of the form $\Delta_{\theta,\delta,\rho}\cap\overline{\mathrm{D}}(0,R)$. Fix some $R>0$, and let $t\in\Delta_{\theta,\delta,\rho}\cap\overline{\mathrm{D}}(0,R)$. From equation (2.10) we obtain

$$\mathcal{B}(\hat{a})(t) = \frac{1}{(t+k)}\left(\mathcal{B}(\hat{b})(t) - \alpha k\int_0^t \mathcal{B}(\hat{a})(s)\mathrm{d}s\right) \tag{2.11}$$

an then

$$\begin{aligned}|\tilde{B}(\hat{a})(t)| &\leqslant \frac{1}{|t+k|}\left(\|\hat{b}\|^{\mathrm{bis}}_{\beta}\frac{\exp(\beta|t|)}{1+\beta^2|t|^2}+|\alpha k|.\|\hat{a}\|^{\mathrm{bis}}_{\beta,R}\int_0^{|t|}\frac{\exp(\beta u)}{1+\beta^2u^2}\mathrm{d}u\right)\\ &\leqslant \frac{1}{d_k}\frac{\exp(\beta|t|)}{1+\beta^2|t|^2}\left(\|\hat{b}\|^{\mathrm{bis}}_{\beta}+|\alpha k|\,\|\hat{a}\|^{\mathrm{bis}}_{\beta,R}\frac{C}{\beta}\right),\end{aligned}$$

with $C = \frac{2\exp(2)}{5}+5$. Here we use the following:

Lemma 2.33. — *There exists a constant $C > 0$ (e.g. $C = \frac{2\exp(2)}{5}+5$), such that for all $\beta > 0$, we have:*

$$\forall t \geqslant 0,\ \int_0^t \frac{\exp(\beta u)}{1+\beta^2u^2}\mathrm{d}u \leqslant \frac{C}{\beta}\frac{\exp(\beta t)}{1+\beta^2t^2}.$$

Proof. — Let $F : u \mapsto \frac{\exp(\beta u)}{1+\beta^2u^2}$, for $u \geqslant 0$. For $t \in [0, \frac{2}{\beta}]$, we have:

$$\int_0^t F(u)\mathrm{d}u \leqslant \frac{\exp(2)}{5}.\frac{2}{\beta},$$

since F is an increasing function over $\mathbb{R}_+$:

$$F'(u) = \beta F(u).\frac{(1-\beta u)^2}{1+\beta^2u^2} \geqslant 0.$$

Moreover for all $t \geqslant 0$, we have $F(t) \geqslant F(0) = 1$. Hence for all $t \in [0, \frac{2}{\beta}]$:

$$\int_0^t F(u)\mathrm{d}u \leqslant \frac{\exp(2)}{5}.\frac{2}{\beta}.F(t).$$

For $t \geqslant \frac{2}{\beta}$, the following inequality holds:

$$\int_0^t F(u)\mathrm{d}u \leqslant \frac{\exp(2)}{5}.\frac{2}{\beta}F(t) + \int_{\frac{2}{\beta}}^t F(u)\mathrm{d}u. \tag{2.12}$$

In addition, if $u \geqslant \frac{\beta}{2}$, then:

$$\frac{(1-\beta u)^2}{1+\beta^2u^2} \geqslant \frac{1}{5}, \tag{2.13}$$

Therefore, for all $u \geqslant \frac{\beta}{2}$:

$$F'(u) = \beta F(u).\frac{(1-\beta u)^2}{1+\beta^2u^2} \geqslant \frac{\beta}{5}F(u).$$

Hence:

$$\begin{aligned}\int_0^t F(u)\mathrm{d}u &\leqslant \int_0^{\frac{2}{\beta}} F(u)\mathrm{d}u + \int_{\frac{2}{\beta}}^t F(u)\mathrm{d}u\,.\\ &\leqslant \frac{\exp(2)}{5}.\frac{2}{\beta}F(t) + \frac{5}{\beta}\int_{\frac{2}{\beta}}^t F'(u)\mathrm{d}u\\ &\leqslant \frac{F(t)}{5\beta}.(2\exp(2)+25)\,.\end{aligned}$$

Let us go back to the proof of the original lemma. Finally, we have:

$$\left\|\hat{a}\right\|^{\mathrm{bis}}_{\beta,R} \leqslant \frac{1}{d_k}\left(\left\|\hat{b}\right\|^{\mathrm{bis}}_{\beta} + \frac{C.|\alpha k|.\left\|\hat{a}\right\|^{\mathrm{bis}}_{\beta,R}}{\beta}\right), \tag{2.14}$$

and consequently:

$$\left\|\hat{a}\right\|^{\mathrm{bis}}_{\beta,R} \leqslant \frac{\beta}{\beta d_k - C|\alpha k|}\left\|\hat{b}\right\|^{\mathrm{bis}}_{\beta}.$$

As a conclusion:

$$\left\|\hat{a}\right\|^{\mathrm{bis}}_{\beta} \leqslant \frac{\beta}{\beta d_k - C|\alpha k|}\left\|\hat{b}\right\|^{\mathrm{bis}}_{\beta},$$

and a_θ is the 1-sum of $\hat{a}$ in the direction θ.

(2) Let us write $\hat{b}(x) = \sum_{j\geqslant 0} b_j x^j$. A direct computation shows that the only formal solution to equation (2.9) is $\hat{a}(x) = \sum_{j\geqslant 0} a_j x^j$ where for all $j \in \mathbb{N}$, $a_j = \frac{b_j}{j+k}$: it exists since $k \notin \mathbb{Z}_{\leqslant 0}$, and then $k+j \neq 0$. In particular, we see immediately that $\hat{a}$ is Gevrey-1, because the same goes for $\hat{b}$. In other words, the Borel transform $\mathcal{B}(\hat{a})$ is analytic in some disc $D(0,\rho)$, $\rho > 0$. In $D(0,\rho)$, $\mathcal{B}(\hat{a})$ satisfies:

$$t\frac{\mathrm{d}\mathcal{B}(\hat{a})}{\mathrm{d}t}(t) + k\mathcal{B}(\hat{a})(t) = \mathcal{B}(\hat{b})(t)\,. \tag{2.15}$$

The general solution near the origin to this equation is

$$y(t) = \frac{c}{t^k} + \frac{1}{t^k}\int_0^t \mathcal{B}(\hat{b})(s)s^{k-1}\mathrm{d}s\,,\ c \in \mathbb{C}.$$

In particular, the only solution analytic in $D(0,\rho)$ is the one with $c=0$, i.e.

$$\mathcal{B}(\hat{a})(t) = \frac{1}{t^k}\int_0^t \mathcal{B}(\hat{b})(s)s^{k-1}\mathrm{d}s\,.$$

Since $\mathcal{B}(\hat{b})$ can be analytically continued to an infinite domain that have denoted by $\Delta_{\theta,\delta,\rho}$ bisected by $\mathbb{R}_+e^{i\theta}$ (because $\hat{b}$ is 1-summable in the direction θ), $\mathcal{B}(\hat{a})$ can also be analytically continued to the same domain.

Moreover, there exists $\beta > 0$ such that $\left\|\hat{b}\right\|_{\beta} < +\infty$, i.e. $\forall t \in \Delta_{\theta,\delta,\rho}$:

$$|B(\hat{b})(t)| \leqslant \left\|\hat{b}\right\|_{\beta} \exp(\beta|t|)\,.$$

Thus, for all $t \in \Delta_{\theta,\delta,\rho}$, we have:

$$\begin{aligned}
&|\exp(-\beta|t|)B(\hat{a})(t)| \\
&\qquad \leqslant \frac{1}{|t^k|}\int_0^{|t|} |\exp(-\beta|t|)| \left|B(\hat{b})\left(s\,\mathrm{e}^{i\arg(t)}\right)\right| \left|s^{k-1}\,\mathrm{e}^{i(k-1)\arg(t)}\right| \mathrm{d}s \\
&\qquad \leqslant \frac{1}{|t|^{\Re(k)}}\int_0^{|t|} |\exp(-\beta s)| \left|B(\hat{b})\left(s\,\mathrm{e}^{i\arg(t)}\right)\right| s^{\Re(k)-1}\mathrm{d}s \\
&\qquad \leqslant \frac{\left\|\hat{b}\right\|_{\beta}}{|t|^{\Re(k)}}\int_0^{|t|} s^{\Re(k)-1}\mathrm{d}s \\
&\qquad = \frac{\left\|\hat{b}\right\|_{\beta}}{\Re(k)}\,.
\end{aligned}$$

Thus, $\hat{a}$ is 1-summable in the direction θ.

3. 1-summable preparation up to any order N

The aim of this section is to prove that we can always formally conjugate a non-degenerate doubly-resonant saddle-node, which is also div-integrable, to its normal form up to a remainder of order $\mathrm{O}(x^N)$ for every $N \in \mathbb{N}_{>0}$. Moreover, we prove that this conjugacy is in fact 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, hence analytic over sectorial domains of opening at least π.

Proposition 3.1. — *Let $Y \in \mathcal{SN}_{\mathrm{diag}}$ be a non-degenerate diagonal doubly-resonant saddle-node which is div-integrable, such that $\mathrm{d}_0 Y = \mathrm{diag}(0,-\lambda,\lambda)$, $\lambda \neq 0$. Then, for all $N \in \mathbb{N}_{>0}$, there exists a formal fibered diffeomorphism $\Psi^{(N)} \in \widehat{\mathrm{Diff}}_{\mathrm{fib}}(\mathbb{C}^3;\mathrm{Id})$ tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$ such that:*

$$\begin{aligned}
\left(\Psi^{(N)}\right)_*(Y) = x^2\frac{\partial}{\partial x} &+ \left(\left(-\left(\lambda + d^{(N)}(y_1y_2)\right) + a_1 x\right) + x^N F_1^{(N)}(x,\mathbf{y})\right) y_1\frac{\partial}{\partial y_1} \\
&+ \left(\left(\lambda + d^{(N)}(y_1y_2) + a_2 x\right) + x^N F_2^{(N)}(x,\mathbf{y})\right) y_2\frac{\partial}{\partial y_2} \\
&=: Y^{(N)}\,,
\end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $(a_1+a_2) = \mathrm{res}(Y) \in \mathbb{C}\setminus\mathbb{Q}_{\leqslant 0}$, $d^{(N)}(v) \in v\mathbb{C}\{v\}$ is an analytic germ vanishing at the origin, and $F_1^{(N)}, F_2^{(N)} \in \mathbb{C}[[x,\mathbf{y}]]$ are 1-summable in *the direction θ, and of order at least one with respect to $\mathbf{y}$. Moreover, one can choose $d^{(2)} = \cdots = d^{(N)}$ for all $N > 2$.*

Definition 3.2. — *A vector field $Y^{(N)}$ as is the proposition above is said to be* normalized up to order N.

Remark 3.3.

(1) Observe that this result does not require the more restrictive assumption of being "strictly non-degenerate" (i.e. $\Re(a_1 + a_2) > 0$).
(2) As a consequence of Corollary 2.21, the 1-sum $\Psi_\theta^{(N)}$ of $\Psi^{(N)}$ in the direction θ is a germ of sectorial fibered diffeomorphism tangent to the identity, i.e. $\Psi_\theta^{(N)} \in \mathit{Diff}_{\text{fib}}(S_{\theta,\pi}; \mathrm{Id})$, which conjugates Y to the 1-sum $Y_\theta^{(N)}$ of $Y^{(N)}$ in the direction θ.

In order to prove this result we will proceed in several steps and use after each step Proposition 2.20 and Corollary 2.21 in order to prove the 1-summability in every direction $\theta = \arg(\pm\lambda)$ of the different objects. First, we will normalize analytically the vector field restricted to $\{x = 0\}$. Then, we will straighten the formal separatrix to $\{y_1 = y_2 = 0\}$ in suitable coordinates. Next, we will simplify the linear terms with respect to $\mathbf{y}$. After that, we will straighten two invariant hypersurfaces to $\{y_1 = 0\}$ and $\{y_2 = 0\}$. Finally, we will conjugate the vector field to its final normal form up to remaining terms of order $\mathrm{O}(x^N)$.

3.1. Analytic normalization on the hyperplane $\{x = 0\}$

3.1.1. Transversally Hamiltonian *versus* div-integrable

We start by proving that an element of $\mathcal{SN}_{\text{diag}}$ which is transversally Hamiltonian is necessarily div-integrable.

Proposition 3.4. — *If $Y \in \mathcal{SN}_{\text{diag}}$ is transversally Hamiltonian, then Y is div-integrable.*

Proof. — Let us consider more generally a diagonal doubly-resonant saddle-node $Y \in \mathcal{SN}_{\text{diag}}$ such that $Y_{|\{x=0\}}$ is a Hamiltonian vector field with respect to $\mathrm{d}y_1 \wedge \mathrm{d}y_2$ (this is the case if Y is transversally Hamiltonian): there exists a Hamiltonian $H(\mathbf{y}) = \lambda y_1 y_2 + O(\|\mathbf{y}\|^3) \in \mathbb{C}\{\mathbf{y}\}$, such that

$$Y = x^2 \frac{\partial}{\partial x} + \left(\left(-\frac{\partial H}{\partial y_2} + x F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\frac{\partial H}{\partial y_1} + x F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \right),$$

where $F_1, F_2 \in \mathbb{C}\{x,\mathbf{y}\}$ are vanishing at the origin. If we define $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ and $\nabla H := {}^t(\mathrm{d}H)$, then $Y_{|\{x=0\}} = J\nabla H$. According to the Morse lemma for holomorphic functions, there exists a germ of an analytic change of coordinates $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ given by

$$\mathbf{y} = (y_1, y_2) \mapsto \varphi(\mathbf{y}) = (y_1 + O(\|\mathbf{y}\|^2), y_2 + O(\|\mathbf{y}\|^2)), \tag{3.1}$$

such that $\hat{H}(\mathbf{y}) := H(\varphi^{-1}(\mathbf{y})) = y_1 y_2$. Let us now recall a trivial result from linear algebra.

Lemma 3.5. — *Let $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, and $P \in M_2(\mathbb{C})$. Then, $PJ\,{}^tP = \det(P)J$.*

As a consequence we have:

Corollary 3.6. — *Let $H \in \mathbb{C}\{\mathbf{y}\}$ be a germ of an analytic function at 0, $Y_0 := J\nabla H$ the associated Hamiltonian vector field in $\mathbb{C}^2$ (for the usual symplectic form $dy_1 \wedge dy_2$), and an analytic diffeomorphism near the origin denoted by φ. Then:*

$$\varphi_*(Y_0) := (\mathrm{D}\varphi \circ \varphi^{-1}) \cdot (Y_0 \circ \varphi^{-1}) = \det(\mathrm{D}\varphi \circ \varphi^{-1}) J \nabla \hat{H}, \tag{3.2}$$

where $\hat{H} := H \circ \varphi^{-1}$.

As a conclusion we have proved that Y is div-integrable.

3.1.2. General case

Now we prove how to normalize the restriction to $\{x = 0\}$ of a div-integrable element of SN_{diag}.

Proposition 3.7. — *Let $Y \in SN_{\mathrm{diag}}$ be div-integrable. Then, there exists $\psi \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ of the form*

$$\psi : (x, \mathbf{y}) \mapsto (x, y_1 + O(\|\mathbf{y}\|^2), y_2 + O(\|\mathbf{y}\|^2))$$

such that

$$\psi_*(Y) = x^2 \frac{\partial}{\partial x} + (-(\lambda + d(v))y_1 + xT_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + ((\lambda + d(v))y_2 + xT_2(x,\mathbf{y}))\frac{\partial}{\partial y_2},$$

with $v := y_1 y_2$, $d(v) \in v\mathbb{C}\{v\}$ and $T_1, T_2 \in \mathbb{C}\{x, \mathbf{y}\}$ vanishing at the origin.

Proof. — By assumption, and according to a theorem due to Brjuno (cf. [16]), up to a first transformation analytic at the origin in $\mathbb{C}^2$, we can suppose that

$$Y_{|\{x=0\}} = (\lambda + h(\mathbf{y}))\left(-y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right).$$

Then, it remains to apply the following lemma to $Y_{|\{x=0\}}$.

Lemma 3.8. — *Let Y_0 be a germ of analytic vector field in $(\mathbb{C}^2, 0)$ of the form*

$$Y_0 = (\lambda + h(\mathbf{y}))\left(-y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right), \tag{3.3}$$

with $h \in \mathbb{C}\{\mathbf{y}\}$ vanishing at the origin. Then there exists $\phi \in \mathrm{Diff}(\mathbb{C}^2, 0; \mathrm{Id})$ such that

$$\phi_*(Y_0) = (\lambda + d(v))\left(-y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right), \tag{3.4}$$

with $v := y_1y_2$ and $d \in v\mathbb{C}\{v\}$.

Remark 3.9. — In other words, we have removed every non-resonant term in $h(\mathbf{y})$. In fact, we re-obtain here a particular case (with one vector field in dimension 2) of the principal result in [29] (which is itself inspired of Vey's works).

Proof. — We claim that ϕ can be chosen of the form

$$\phi(\mathbf{y}) = \left(y_1 e^{-\gamma(\mathbf{y})}, y_2 e^{\gamma(\mathbf{y})}\right),$$

for a conveniently chosen $\gamma \in \mathbb{C}\{\mathbf{y}\}$. Indeed, let us study how such a diffeomorphism acts on Y_0. Let us consider $U := (\lambda + h(v))$ and $L := (-y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2})$, such that $Y_0 = UL$. An easy computation shows:

$$\begin{aligned}\phi_*(Y_0) &= \phi_*(U.L)\\ &= ([U\cdot(1 - \mathcal{L}_L(\gamma))]\circ\phi^{-1})L,\end{aligned}$$

where $\mathcal{L}_L$ is the Lie derivative of associate to L. We want to find γ such that the unit

$$D := [U(1 - \mathcal{L}_L(\gamma))]\circ\phi^{-1}$$

is free from *non-resonant* terms, i.e. is of the form

$$D = \lambda + d(y_1y_2) = \lambda + \sum_{k>1} d_k (y_1y_2)^k.$$

Notice that if a unit $W = \sum_{k>0} W_k (y_1 y_2)^k \in \mathbb{C}\{\mathbf{y}\}^\times$ is free from non-resonant terms, then:

$$\begin{cases} W \circ \phi^{-1} = W \\ \mathcal{L}_L(W) = 0\,. \end{cases}$$

Thus, let us split both U and γ in a "resonant" and a "non-resonant" part:

$$\begin{cases} U = U_{\text{res}} + U_{\text{n-res}} \\ \gamma = \gamma_{\text{res}} + \gamma_{\text{n-res}} \end{cases}$$

where

$$\begin{cases} U_{\text{n-res}} = \sum_{k_1 \neq k_2} U_{k_1,k_2} y_1^{k_1} y_2^{k_2} \\ U_{\text{res}} = \sum_{k} U_{k,k} (y_1 y_2)^k \\ \gamma_{\text{n-res}} = \sum_{k_1 \neq k_2} \gamma_{k_1,k_2} y_1^{k_1} y_2^{k_2} \\ \gamma_{\text{res}} = \sum_{k} \gamma_{k,k} (y_1 y_2)^k. \end{cases}$$

Then the non-resonant terms of $U(1 - \mathcal{L}_L(\gamma))$ are

$$(U_{\text{n-res}} - (U_{\text{n-res}} + U_{\text{res}}) \mathcal{L}_L(\gamma_{\text{n-res}})) \circ \phi^{-1}\,.$$

Hence, the partial differential equation we want to solve is:

$$\mathcal{L}_L(\gamma) = \frac{U_{\text{n-res}}}{U_{\text{res}} + U_{\text{n-res}}}\,.$$

One sees immediately that this equation admit an analytic solution (and even infinitely many solutions) $\gamma \in \mathbb{C}\{\mathbf{y}\}$, since the unit $U \in \mathbb{C}\{\mathbf{y}\}$ is analytic.

3.2. 1-summable simplification of the "dependent" affine part

We are concerned by studying vector fields of the form

$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x,\mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x,\mathbf{y})) \frac{\partial}{\partial y_2}\,, \tag{3.5}$$

with

$$\begin{cases} f_1(x,\mathbf{y}) = -d(y_1 y_2) y_1 + x T_1(x,\mathbf{y}) \\ f_2(x,\mathbf{y}) = d(y_1 y_2) y_2 + x T_2(x,\mathbf{y}), \end{cases}$$

where $d(v) \in v\mathbb{C}\{v\}$ and $T_1, T_2 \in \mathbb{C}\{x,\mathbf{y}\}$ are of order at least one.

Proposition 3.10. — *Let $Y \in SN_{\mathrm{diag}}$ be a doubly-resonant saddle-node of the form*

$$Y = x^2\frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x,\mathbf{y}))\frac{\partial}{\partial y_2}\,,$$

where $f_1, f_2 \in \mathbb{C}\{x,\mathbf{y}\}$ are such that $f_1(x,\mathbf{y}), f_2(x,\mathbf{y}) = \mathrm{O}(\|(x,\mathbf{y})\|^2)$. Then there exist formal power series $\hat{y}_1$, $\hat{y}_2$, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2 \in x\mathbb{C}[\![x]\!]$ which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal fibered diffeomorphism

$$\hat{\Phi} : (x,y_1,y_2) \longmapsto \Big(x, \hat{y}_1(x) + (1+\hat{\alpha}_1(x))y_1 + \hat{\beta}_1(x)y_2, \\ \hat{y}_2(x) + \hat{\alpha}_2(x)y_1 + (1+\hat{\beta}_1(x))y_2\Big)\,,$$

(which is tangent to the identity and 1- summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates Y to

$$\hat{\Phi}_*(Y) = x^2\frac{\partial}{\partial x} + ((-\lambda + a_1x)y_1 + \hat{F}_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} \\ + ((\lambda + a_2x)y_2 + \hat{F}_2(x,\mathbf{y}))\frac{\partial}{\partial y_2}\,,$$

where $a_1, a_2 \in \mathbb{C}$ and $\hat{F}_1, \hat{F}_2 \in \mathbb{C}[\![x,\mathbf{y}]\!]$ are of order at least 2 with respect to $\mathbf{y}$, and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

Remark 3.11. — Notice that $\hat{\Phi}_{|\{x=0\}} = \mathrm{Id}$, so that $\hat{F}_i(0,\mathbf{y}) = f_i(0,\mathbf{y})$ for $i = 1,2$. Moreover, the residue of $\hat{\Phi}_*(Y)$ is $a_1 + a_2$.

The proof of Proposition 3.10 is postponed to Subsection 3.2.2.

3.2.1. Technical lemmas on irregular differential systems

Lemma 3.12. — *There exists a pair of formal power series*

$$(\hat{y}_1(x), \hat{y}_2(x)) \in (x\mathbb{C}[\![x]\!])^2$$

which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal diffeomorphism given by

$$\hat{\Phi}_1(x,y_1,y_2) = (x, y_1 - \hat{y}_1(x), y_2 - \hat{y}_2(x)),$$

(which is 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates Y in (3.5) to:

$$\hat{Y}_1(x,\mathbf{y}) = x^2\frac{\partial}{\partial x} + (-\lambda y_1 + \hat{g}_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + (\lambda y_2 + \hat{g}_2(x,\mathbf{y}))\frac{\partial}{\partial y_2}, \tag{3.6}$$

where $\hat{g}_1, \hat{g}_2$ are formal power series of order at least 2 such that $\hat{g}_1(x,\mathbf{0}) = \hat{g}_2(x,\mathbf{0}) = 0$, and are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

In other words, in the new coordinates, the curve given by $(y_1, y_2) = (0, 0)$ is invariant by the flow of the vector field, and contains the origin in its closure: it is usually called the (formal, 1-summable) *center manifold.*

Proof. — This is an immediate consequence of an important theorem by Ramis and Sibuya on the summability of formal solutions to irregular differential systems [24]. This theorem proves the existence and the 1-summability in every direction $\theta = \arg(\pm\lambda)$, of $\hat{y}_1$ and $\hat{y}_2$: $(\hat{y}_1(x), \hat{y}_2(x))$ is defined as the unique formal solution to

$$\begin{cases} x^2 \dfrac{\mathrm{d}y_1}{\mathrm{d}x} = -\lambda y_1(x) + f_1(x, y_1(x), y_2(x)) \\ x^2 \dfrac{\mathrm{d}y_2}{\mathrm{d}x} = \lambda y_2(x) + f_2(x, y_1(x), y_2(x)), \end{cases}$$

such that $(\hat{y}_1(0), \hat{y}_2(0)) = (0, 0)$. The 1-summability of $\hat{g}_1$ and $\hat{g}_2$ comes from Proposition 2.20.

The next step is aimed at changing to linear terms with respect to $\mathbf{y}$ in "diagonal" form.

Lemma 3.13. — *There exists a pair of formal power series* $(\hat{p}_1, \hat{p}_2)$ $(\mathbb{C}[[x]])^2$ *which are 1-summable in every direction* $\theta = \arg(\pm\lambda)$, *such that the formal fibered diffeomorphism given by*

$$\hat{\Phi}_2(x, y_1, y_2) = (x, y_1 + x\hat{p}_2(x)y_2, y_2 + x\hat{p}_1(x)y_1),$$

(which is tangent to the identity and 1-summable in every direction $\theta = \arg(\pm\lambda)$*) conjugates* $\hat{Y}_1$ *in* (3.6), *to*

$$\hat{Y}_2(x, \mathbf{y}) = x^2 \frac{\partial}{\partial x} + ((-\lambda + x\hat{a}_1(x))y_1 + \hat{H}_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + ((\lambda + x\hat{a}_2(x))y_2 + \hat{H}_2(x, \mathbf{y})) \frac{\partial}{\partial y_2}, \tag{3.7}$$

where $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$ *are formal power series which are 1-summable in every direction* $\theta = \arg(\pm\lambda)$ *and* $\hat{H}_1, \hat{H}_2$ *are of order at least* 2 *with respect to* $\mathbf{y}$.

Proof. — Let us write

$$\begin{cases} \hat{g}_1(x, \mathbf{y}) = x\hat{b}_1(x)y_1 + x\hat{c}_1(x)y_2 + \hat{G}_1(x, \mathbf{y}) \\ \hat{g}_2(x, \mathbf{y}) = x\hat{c}_2(x)y_1 + x\hat{b}_2(x)y_2 + \hat{G}_2(x, \mathbf{y}), \end{cases}$$

where $\hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{G}_1, \hat{G}_2$ are formal power series 1-summable in the direction $\theta = \arg(\pm\lambda)$, such that $\hat{G}_1$ and $\hat{G}_2$ are of order at least 2 with respect to $\mathbf{y}$. Let us consider the following irregular differential system naturally associated to $\hat{Y}_1$:

$$x^2 \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x}(x) = \hat{\mathbf{B}}(x)\mathbf{z}(x) + \hat{\mathbf{G}}(x, \mathbf{z}(x)) , \tag{3.8}$$

where

$$\hat{\mathbf{B}}(x) = \begin{pmatrix} -\lambda + x\hat{b}_1(x) & x\hat{c}_1(x) \\ x\hat{c}_2(x) & \lambda + x\hat{b}_2(x) \end{pmatrix}, \quad \hat{\mathbf{G}}(x, \mathbf{z}(x)) = \begin{pmatrix} \hat{G}_1(x, \mathbf{z}(x)) \\ \hat{G}_2(x, \mathbf{z}(x)) \end{pmatrix}.$$

We are looking for

$$\hat{\mathbf{P}}(x) = \begin{pmatrix} 1 & x\hat{p}_2(x) \\ x\hat{p}_1(x) & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[x]]) ,$$

where $\hat{p}_1, \hat{p}_2$ are 1-summable formal power series in x every direction $\theta = \arg(\pm\lambda)$, such that the linear transformation given by $\mathbf{z}(x) = \hat{\mathbf{P}}(x)\mathbf{y}(x)$ changes equation (3.8) to

$$x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) = \hat{\mathbf{A}}(x)\mathbf{y}(x) + \hat{\mathbf{H}}(x, \mathbf{y}(x)) ,$$

with

$$\hat{\mathbf{A}}(x) = \begin{pmatrix} -\lambda + x\hat{a}_1(x) & 0 \\ 0 & \lambda + x\hat{a}_2(x) \end{pmatrix}, \quad \hat{\mathbf{H}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{H}_1(x, \mathbf{y}(x)) \\ \hat{H}_2(x, \mathbf{y}(x)) \end{pmatrix},$$

where $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$ are 1-summable formal power series in x every direction $\theta = \arg(\pm\lambda)$. We have:

$$x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) = \hat{\mathbf{P}}(x)^{-1}(\hat{\mathbf{B}}(x)\hat{\mathbf{P}}(x) - x^2 \frac{\mathrm{d}\hat{\mathbf{P}}}{\mathrm{d}x}(x))\mathbf{y}(x) + \hat{\mathbf{P}}(x)^{-1}\hat{\mathbf{G}}(x, \hat{\mathbf{P}}(x)\mathbf{y}(x))$$

and we want to determine $\hat{\mathbf{A}}(x)$ and $\hat{\mathbf{P}}(x)$ as above so that

$$\hat{\mathbf{B}}(x)\hat{\mathbf{P}}(x) - x^2 \frac{\mathrm{d}\hat{\mathbf{P}}}{\mathrm{d}x}(x) = \hat{\mathbf{A}}(x) .$$

This gives four equations:

$$\begin{cases} \hat{a}_1(x) = \hat{b}_1(x) + x\hat{c}_1(x)\hat{p}_1(x) \\ \hat{a}_2(x) = \hat{b}_2(x) + x\hat{c}_2(x)\hat{p}_2(x) \\ x^2 \frac{\mathrm{d}\hat{p}_1}{\mathrm{d}x}(x) = (2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x))\hat{p}_1(x) + \hat{c}_2(x) - x^2\hat{c}_1(x)\hat{p}_1(x)^2 \\ x^2 \frac{\mathrm{d}\hat{p}_2}{\mathrm{d}x}(x) = (-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x))\hat{p}_2(x) + \hat{c}_1(x) - x^2\hat{c}_2(x)\hat{p}_2(x)^2 . \end{cases} \tag{3.9}$$

Thanks to the theorem by Ramis and Sibuya on the summability of formal solutions to irregular systems [24], we have the existence and the 1-summability in every direction $\theta = \arg(\pm\lambda)$, of $\hat{p}_1$ and $\hat{p}_2$: $(\hat{p}_1(x), \hat{p}_2(x))$ is defined as the unique formal solution to

$$\begin{cases} x^2\dfrac{\mathrm{d}\hat{p}_1}{\mathrm{d}x}(x) = (2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x))\hat{p}_1(x) + \hat{c}_2(x) - x^2\hat{c}_1(x)\hat{p}_1(x)^2 \\ x^2\dfrac{\mathrm{d}\hat{p}_2}{\mathrm{d}x}(x) = (-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x))\hat{p}_2(x) + \hat{c}_1(x) - x^2\hat{c}_2(x)\hat{p}_2(x)^2 \end{cases}$$

such that

$$(\hat{p}_1(0), \hat{p}_2(0)) = \left(\frac{-\hat{c}_2(0)}{2\lambda}, \frac{\hat{c}_1(0)}{2\lambda}\right).$$

Notice that $\hat{a}_1$ and $\hat{a}_2$ are defined by the first two equations in (3.9). Finally, $\hat{\mathbf{H}}$ is defined by

$$\hat{\mathbf{H}}(x,\mathbf{y}) := \hat{\mathbf{P}}(x)^{-1}\hat{\mathbf{G}}(x, \hat{\mathbf{P}}(x)\mathbf{y}),$$

and, by Proposition 2.20, it is 1-summable in every direction $\theta = \arg(\pm\lambda)$.

The goal of the last following lemma is to transform $\hat{a}_1(x)$ and $\hat{a}_2(x)$ in (3.7) to constant terms.

Lemma 3.14. — *There exists a pair of formal power series* $(\hat{q}_1, \hat{q}_2) \in (\mathbb{C}[[x]])^2$ *with* $\hat{q}_1(0) = \hat{q}_2(0) = 1$*, which are 1-summable in every direction* $\theta = \arg(\pm\lambda)$*, such that the formal fibered diffeomorphism of the form*

$$\hat{\Phi}_3(x, y_1, y_2) = (x, \hat{q}_1(x)y_1, \hat{q}_2(x)y_2),$$

(which is tangent to the identity and 1-summable in every direction $\theta = \arg(\pm\lambda)$*) conjugates* $\hat{Y}_2$ *in* (3.7)*, to*

$$\hat{Y}_3(x,\mathbf{y}) = x^2\frac{\partial}{\partial x} + ((-\lambda + a_1x)y_1 + \hat{F}_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + ((\lambda + a_2x)y_2 + \hat{F}_2(x,\mathbf{y}))\frac{\partial}{\partial y_2},$$

where $\hat{F}_1, \hat{F}_2$ *are formal power series of order at least* 2 *with respect to* $\mathbf{y}$ *which are 1-summable in every direction* $\theta = \arg(\pm\lambda)$ *and* $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$*.*

Proof. — We can associate to $\hat{Y}_2$ the following irregular differential system:

$$x^2\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x}(x) = \hat{\mathbf{A}}(x)\mathbf{z}(x) + \hat{\mathbf{H}}(x, \mathbf{z}(x)),$$

and we are looking for a change of coordinates of the form $\mathbf{z}(x) = \hat{\mathbf{Q}}(x)\mathbf{y}(x)$, where

$$\hat{\mathbf{Q}}(x) = \begin{pmatrix} \hat{q}_1(x) & 0 \\ 0 & \hat{q}_2(x) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[x]])$$

with $\hat{q}_1(0) = \hat{q}_2(0) = 1$, such that the new system is

$$x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) = \mathbf{A}(x)\mathbf{y}(x) + \hat{\mathbf{F}}(x, \mathbf{y}(x)),$$

with

$$\mathbf{A}(x) = \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}, \quad \hat{\mathbf{F}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{F}_1(x, \mathbf{y}(x)) \\ \hat{F}_2(x, \mathbf{y}(x)) \end{pmatrix},$$

and $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$. We have

$$\begin{aligned} x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) &= \underbrace{\hat{\mathbf{Q}}(x)^{-1}\left(\hat{\mathbf{A}}(x)\hat{\mathbf{Q}}(x) - x^2 \frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}x}(x)\right)}_{= \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}} \mathbf{y}(x) + \hat{\mathbf{Q}}(x)^{-1}\hat{\mathbf{H}}(x, \hat{\mathbf{Q}}(x)\mathbf{y}(x)) \end{aligned}$$

so that

$$x^2 \frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}x}(x) = \hat{\mathbf{A}}(x)\hat{\mathbf{Q}}(x) - \hat{\mathbf{Q}}(x) \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}$$

and we obtain:

$$\begin{cases} x^2 \frac{\mathrm{d}\hat{q}_1}{\mathrm{d}x}(x) = x\hat{q}_1(x)(\hat{a}_1(x) - a_1) \\ x^2 \frac{\mathrm{d}\hat{q}_2}{\mathrm{d}x}(x) = x\hat{q}_2(x)(\hat{a}_2(x) - a_2) \end{cases} \quad \begin{cases} \frac{\mathrm{d}\hat{q}_1}{\mathrm{d}x}(x) = \hat{q}_1(x)\left(\frac{\hat{a}_1(x) - a_1}{x}\right) \\ \frac{\mathrm{d}\hat{q}_2}{\mathrm{d}x}(x) = \hat{q}_2(x)\left(\frac{\hat{a}_2(x) - a_2}{x}\right) \end{cases}$$

$$\begin{cases} \hat{q}_1(x) = \exp\left(\int_0^x \frac{\hat{a}_1(s) - a_1}{s}\mathrm{d}s\right) \\ \hat{q}_2(x) = \exp\left(\int_0^x \frac{\hat{a}_2(s) - a_2}{s}\mathrm{d}s\right) \end{cases} \text{if we set } \hat{q}_1(0) = \hat{q}_2(0) = 1,$$

and the expression $\int_0^x \frac{\hat{a}_j(s) - a_j}{s}\mathrm{d}s$, for $j = 1, 2$, means the only anti-derivative of $\frac{\hat{a}_j(s) - a_j}{s}$ without constant term. Since $\hat{a}_1$ and $\hat{a}_2$ are 1-summable in every direction $\theta = \arg(\pm\lambda)$, the same goes for $\hat{q}_1$ and $\hat{q}_2$, and then for $\hat{F}_1$ and $\hat{F}_2$ by Proposition 2.20.

3.2.2. Proof of Proposition 3.10.

We are now able to prove Proposition 3.10.

Proof of Proposition 3.10. — We have to use successively Lemma 3.8 (with $Y_0 := Y_{|\{x=0\}}$), followed by Proposition 3.10, then Proposition 3.15 and finally Proposition 3.19, using at each step Corollary 2.21 to obtain the 1-summability.

3.3. 1-summable straightening of two invariant hypersurfaces

For any $\theta \in \mathbb{R}$, we recall that we denote by F_θ the 1-sum of a 1-summable series $\hat{F}$ in the direction θ. Let $\theta \in \mathbb{R}$ with $\theta \neq \arg(\pm\lambda)$ and consider a formal vector field $\hat{Y}$, 1-summable in the direction θ of 1-sum Y_θ, of the form

$$\hat{Y} = x^2\frac{\partial}{\partial x} + (\lambda_1(x)y_1 + \hat{F}_1(x,\mathbf{y}))\frac{\partial}{\partial y_1} + (\lambda_2(x)y_2 + \hat{F}_2(x,\mathbf{y}))\frac{\partial}{\partial y_2}, \tag{3.10}$$

where:

- $\lambda_1(x) = -\lambda + a_1 x$
- $\lambda_2(x) = \lambda + a_2 x$
- $\lambda \neq 0$
- $a_1, a_2 \in \mathbb{C}$
- for $j = 1, 2$,
$$\hat{F}_j(x,\mathbf{y}) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^2\\|\mathbf{n}|\geq 2}} \hat{F}_{\mathbf{n}}^{(j)}(x)\mathbf{y}^{\mathbf{n}} \in \mathbb{C}[\![x,\mathbf{y}]\!]$$
is 1-summable in the direction θ of 1-sum
$$F_{j,\theta}(x,\mathbf{y}) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^2\\|\mathbf{n}|\geq 2}} F_{j,\mathbf{n},\theta}(x)\mathbf{y}^{\mathbf{n}}\,.$$
In particular, there exists $A, B, \mu > 0$ such that for all $\mathbf{n} \in \mathbb{N}^2$, $|n| \geq 2$, for $j = 1, 2$:
$$\forall t \in \Delta_{\theta,\epsilon,\rho},\ |\tilde{\mathcal{B}}(\hat{F}_{j,\mathbf{n}})(t)| \leq A.B^{|\mathbf{n}|}\frac{\exp(\mu)}{1+\mu^2|t|^2}\,,$$
for some $\rho > 0$ and $\epsilon > 0$ such that $(\mathbb{R}.\lambda) \cap A_{\theta,\epsilon} = \emptyset$ (see Definition 2.10 and Remark 2.11 for the notations). Notice that $F_{j,\theta}$ is analytic and bounded in some sectorial neighborhood $S \in \mathcal{S}_{\theta,\pi}$ of the origin. For technical reasons, we use in this subsection the alternative definition of the Borel transform $\tilde{\mathcal{B}}$, with its associate norm $\|\cdot\|_\mu^{\mathrm{bis}}$ (see Remarks 2.9 and 2.11 and Proposition 2.12).

Proposition 3.15. — *Under the assumptions above, there exists a pair of formal power series* $(\hat{\phi}_1, \hat{\phi}_2) \in (\mathbb{C}[\![x, \mathbf{y}]\!])^2$ *of order at least two with respect to* $\mathbf{y}$ *which are 1-summable in every direction* $\theta = \arg(\pm\lambda)$*, such that the formal fibered diffeomorphism*

$$\hat{\Phi}(x, \mathbf{y}) = (x, y_1 + \hat{\phi}_1(x, \mathbf{y}), y_2 + \hat{\phi}_2(x, \mathbf{y}))\,,$$

(which is tangent to the identity and 1-summable in every direction $\theta = \arg(\pm\lambda)$*) conjugates* $\hat{Y}$ *in* (3.10) *to*

$$\hat{Y}_{\mathrm{prep}} = x^2\frac{\partial}{\partial x} + ((-\lambda + a_1 x) + y_2\hat{R}_1(x, \mathbf{y}))y_1\frac{\partial}{\partial y_1} + ((\lambda + a_2 x) + y_1\hat{R}_2(x, \mathbf{y}))y_2\frac{\partial}{\partial y_2},$$

where $\hat{R}_1, \hat{R}_2 \in \mathbb{C}[\![x, \mathbf{y}]\!]$ *are 1-summable in every direction* $\theta = \arg(\pm\lambda)$*.*

Proof. — We follow and adapt the proof of analytic straightening of invariant curves for resonant saddles in two dimensions in [19]. We are looking for

$$\hat{\Psi}(x, \mathbf{y}) = (x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y}))\,,$$

with $\hat{\psi}_1, \hat{\psi}_2$ of order at least 2, and $\hat{R}_1, \hat{R}_2$ as above such that:

$$\hat{\Psi}_*(\hat{Y}_{\mathrm{prep}}) = \hat{Y}\,,$$

i.e.

$$\mathrm{d}\hat{\Psi} \cdot \hat{Y}_{\mathrm{prep}} = \hat{Y} \circ \hat{\Psi}\,. \tag{3.11}$$

Then, we will set $\Phi := \Psi^{-1}$. Let us write

$$\begin{aligned}
\hat{T}_1 &:= y_1 y_2 \hat{R}_1 = \sum_{|\mathbf{n}| \geqslant 2} \hat{T}_{1,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\\
\hat{T}_2 &:= y_1 y_2 \hat{R}_2 = \sum_{|\mathbf{n}| \geqslant 2} \hat{T}_{2,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\\
\hat{\psi}_1 &= \sum_{|\mathbf{n}| \geqslant 2} \hat{\psi}_{1,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\\
\hat{\psi}_2 &= \sum_{|\mathbf{n}| \geqslant 2} \hat{\psi}_{2,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\,,
\end{aligned}$$

so that equation (3.11) becomes:

$$x^2\frac{\partial\hat{\psi}_1}{\partial x^2} + \left(1 + \frac{\partial\hat{\psi}_1}{\partial y_1}\right)(\lambda_1(x)y_1 + \hat{T}_1) + \frac{\partial\hat{\psi}_1}{\partial y_2}(\lambda_2(x)y_2 + \hat{T}_2) = \lambda_1(x)(y_1 + \hat{\psi}_1) + \hat{F}_1(x, y_1 + \hat{\psi}_1, y_2 + \hat{\psi}_2)$$

and

$$x^2\frac{\partial\hat\psi_2}{\partial x^2}+\frac{\partial\hat\psi_2}{\partial y_1}(\lambda_1(x)y_1+\hat T_1)+\left(1+\frac{\partial\hat\psi_2}{\partial y_2}\right)(\lambda_2(x)y_2+\hat T_2)$$
$$=\lambda_2(x)(y_2+\hat\psi_2)+\hat F_2(x,y_1+\hat\psi_1,y_2+\hat\psi_2)\,.$$

These equations can be written as:

(3.12)
$$\begin{cases}\displaystyle\sum_{|\mathbf{n}|>2}\left(\delta_{1,\mathbf{n}}(x)\hat\psi_{1,\mathbf{n}}(x)+x^2\frac{\mathrm{d}\hat\psi_{1,\mathbf{n}}}{\mathrm{d}x}(x)+\hat T_{1,\mathbf{n}}(x)\right)\mathbf{y}^{\mathbf{n}}\\ \displaystyle =\hat F_1(x,y_1+\hat\psi_1(x,\mathbf{y}),y_2+\hat\psi_2(x,\mathbf{y}))-\hat T_1(x)\frac{\partial\hat\psi_1}{\partial y_1}(x,\mathbf{y})-\hat T_2(x)\frac{\partial\hat\psi_1}{\partial y_2}(x,\mathbf{y})\\ \displaystyle =:\sum_{|\mathbf{n}|\geq 2}\zeta_{1,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\\ \displaystyle\sum_{|\mathbf{n}|>2}\left(\delta_{2,\mathbf{n}}(x)\hat\psi_{2,\mathbf{n}}(x)+x^2\frac{\mathrm{d}\hat\psi_{2,\mathbf{n}}}{\mathrm{d}x}(x)+\hat T_{2,\mathbf{n}}(x)\right)\mathbf{y}^{\mathbf{n}}\\ \displaystyle =\hat F_2(x,y_1+\hat\psi_1(x,\mathbf{y}),y_2+\hat\psi_2(x,\mathbf{y}))-\hat T_1(x)\frac{\partial\hat\psi_2}{\partial y_1}(x,\mathbf{y})-\hat T_2(x)\frac{\partial\hat\psi_2}{\partial y_2}(x,\mathbf{y})\\ \displaystyle =:\sum_{|\mathbf{n}|>2}\zeta_{2,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}\end{cases}$$

where $\delta_{j,\mathbf{n}}(x)=\lambda_1(x)n_1+\lambda_2(x)n_2-\lambda_j(x)$, $j=1,2$. We are looking for $\hat T_1,\hat T_2$ such that

$$\begin{cases}\hat T_{1,\mathbf{n}}=0, & \text{if } n_1=0 \text{ or } n_2=0\\ \hat T_{2,\mathbf{n}}=0, & \text{if } n_1=0 \text{ or } n_2=0\,.\end{cases}$$

Notice that $\zeta_{j,\mathbf{n}}$, for $j=1,2$ and $|\mathbf{n}|>2$, depends only on the $\hat\psi_{i,\mathbf{k}}$'s and the $\hat F_{i,\mathbf{k}}$'s, for $i=1,2$, $|\mathbf{k}|<\mathbf{n}$. We can then determine the coefficients $\hat\psi_{j,\mathbf{n}}$ and $\hat T_{j,\mathbf{n}}$, $j=1,2$, $|\mathbf{n}|>2$, by induction on $|\mathbf{n}|$, setting

$$\begin{cases}\hat T_{1,\mathbf{n}}=0, & \text{if } n_1=0 \text{ or } n_2=0\\ \hat T_{2,\mathbf{n}}=0, & \text{if } n_1=0 \text{ or } n_2=0\\ \hat\psi_{1,\mathbf{n}}=0, & \text{if } n_1>1 \text{ and } n_2>1\\ \hat\psi_{2,\mathbf{n}}=0, & \text{if } n_1>1 \text{ and } n_2>1\,,\end{cases}$$

and solving for each $\mathbf{n}=(n_1,n_2)\in\mathbb{N}^2$ with $|\mathbf{n}|>2$, the equations

$$\begin{cases}\displaystyle\delta_{1,\mathbf{n}}(x)\hat\psi_{1,\mathbf{n}}(x)+x^2\frac{\mathrm{d}\hat\psi_{1,\mathbf{n}}}{\mathrm{d}x}(x)=\zeta_{1,\mathbf{n}}(x), & \text{if } n_1=0 \text{ or } n_2=0\\ \displaystyle\delta_{2,\mathbf{n}}(x)\hat\psi_{2,\mathbf{n}}(x)+x^2\frac{\mathrm{d}\hat\psi_{2,\mathbf{n}}}{\mathrm{d}x}(x)=\zeta_{2,\mathbf{n}}(x), & \text{if } n_1=0 \text{ or } n_2=0\,.\end{cases}$$

Lemma 3.16. — *There exists $\beta > 4\pi, M > 0$ such that for all $\mathbf{n} \in \mathbb{N}^2$ with $|\mathbf{n}| \geqslant 2$, and for $j = 1, 2$, $\left\|\zeta_{j,\mathbf{n}}\right\|_{\beta}^{\mathrm{bis}} < +\infty$ and:*

$$\left\|\hat{\psi}_{j,\mathbf{n}}\right\|_{\beta}^{\mathrm{bis}} \leqslant M. \left\|\zeta_{j,\mathbf{n}}\right\|_{\beta}^{\mathrm{bis}},$$

where the norm corresponds to the domain $\Delta_{\theta,\epsilon,\rho}$ (see Definition 2.10).

Proof. — For $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ with $n_1 + n_2 \geqslant 2$ we have :

$$\delta_{1,\mathbf{n}}(x) = \lambda_1(x)(n_1 - 1) + \lambda_2(x)n_2$$
$$= \begin{cases} \lambda(n_2 + 1) + x(-a_1 + a_2 n_2), & \text{if } n_1 = 0 \\ -\lambda(n_1 - 1) + a_1 x(n_1 - 1), & \text{if } n_2 = 0 \end{cases}$$

and

$$\delta_{2,\mathbf{n}}(x) = \lambda_2(x)(n_2 - 1) + \lambda_1(x)n_1$$
$$= \begin{cases} \lambda(n_2 - 1) + a_2 x(n_2 - 1), & \text{if } n_1 = 0 \\ -\lambda(n_1 + 1) + x(-a_2 + a_1 n_1), & \text{if } n_2 = 0. \end{cases}$$

We will only deal with $\delta_{1,\mathbf{n}}(x)$ (the case of $\delta_{2,\mathbf{n}}(x)$ being similar). Notice that we are exactly in the situation of Proposition 2.32. In particular, using notation in this definition, we respectively have on the one hand:

$$k = \lambda(n_2 + 1), \alpha = \frac{(-a_1 + a_2 n_2)}{\lambda(n_2 + 1)}, \tag{3.13}$$

and

$$d_k = \min\{|\lambda(n_2 + 1)| - \rho, |\lambda(n_2 + 1)||\sin(\theta + \epsilon)|, |\lambda(n_2 + 1)||\sin(\theta - \epsilon)|\}$$

$n_1 = 0$, and on the other hand

$$k = -\lambda(n_1 + 1), \alpha = \frac{(-a_2 + a_1 n_1)}{-\lambda(n_1 + 1)}, \tag{3.14}$$

and

$$d_k = \min\{|\lambda(n_1 + 1)| - \rho, |\lambda(n_1 + 1)||\sin(\theta + \epsilon)|, |\lambda(n_1 + 1)||\sin(\theta - \epsilon)|\} \tag{3.15}$$

when $n_2 = 0$. We can chose the domain $\Delta_{\theta,\epsilon,\rho}$ corresponding to the 1-summability of $\hat{F}_1$ and $\hat{F}_2$ with $0 < \rho < |\lambda|$, so that $d_k > 0$, since $\epsilon > 0$ is such that $(\mathbb{R}.\lambda) \cap A_{\theta,\epsilon} = \emptyset$. Finally, we chose

$$\beta > \frac{C(|a_1| + |a_2|)}{\min\{|\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)|\}} > 0$$

(with $C = \frac{2\exp(2)}{5} + 5$), so that $\left\|\hat{F}_1\right\|_{\beta}^{\mathrm{bis}} < +\infty$. This choice of β implies $\beta d_k > C|\alpha k|$ as needed in Proposition 2.32, in both considered situations, namely $n_1 = 0$ and $n_2 = 0$ respectively. Since for $j = 1, 2$ and $|\mathbf{n}| \geq 2$, $\zeta_{j,\mathbf{n}}$ depends only on the $\hat{\psi}_{i,\mathbf{k}}$'s and the $\hat{F}_{i,\mathbf{k}}$'s, for $i = 1, 2$, $|\mathbf{k}| < \mathbf{n}$, we deduce by induction that

$$\begin{cases} \|\zeta_{1,\mathbf{n}}\|^{\text{bis}}_{\beta} < +\infty, & \text{if } n_1 = 0 \text{ or } n_2 = 0 \\ \|\zeta_{2,\mathbf{n}}\|^{\text{bis}}_{\beta} < +\infty, & \text{if } n_1 = 0 \text{ or } n_2 = 0 \end{cases}$$

and then, thanks to Proposition 2.32:

$$\|\hat{\psi}_{j,\mathbf{n}}\|^{\text{bis}}_{\beta} \leq \left(\frac{\beta}{\beta(|\lambda| - \rho) - C(|a_1| + |a_2|)}\right) . \|\zeta_{j,\mathbf{n}}\|^{\text{bis}}_{\beta} \quad \text{for } j = 1, 2.$$

The lemma is proved, with

$$M = \max\left(\frac{\beta}{\beta \min\{|\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)|\} - C(|a_1| + |a_2|)}\right) .$$

In order to finish the proof of Proposition 3.15, we have to prove that for $j = 1, 2$, the series $\overline{\hat{\psi}_j} := \sum_{\mathbf{n} \in \mathbb{N}^2} \|\hat{\psi}_{j,\mathbf{n}}\|^{\text{bis}}_{\beta} \mathbf{y}^{\mathbf{n}}$ is convergent in a poly-disc $\mathbf{D}(\mathbf{0}, \mathbf{r})$, with $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$ (then, Corollary 2.30 gives 1-summability). We will prove this by using a method of dominant series. Let us introduce some useful notations. If $(\mathfrak{B}, \|\cdot\|)$ is a Banach algebra, for any formal power series $f(\mathbf{y}) = \sum_{\mathbf{n}} f_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[\![\mathbf{y}]\!]$, we define $\overline{f} := \sum_{\mathbf{n}} \|f_{\mathbf{n}}\| \mathbf{y}^{\mathbf{n}}$, and $\overline{\overline{f}}(y) := \overline{f}(y, y)$. If $g = \sum_{\mathbf{n}} g_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[\![\mathbf{y}]\!]$ is another formal power series, we write $\overline{f} \preccurlyeq \overline{g}$ if for all $\mathbf{n} \in \mathbb{N}^2$, we have $\|f_{\mathbf{n}}\| \leq \|g_{\mathbf{n}}\|$. We remind the following classical result (the proof is performed in [26] when $(\mathfrak{B}, \|\cdot\|) = (\mathbb{C}, |\cdot|)$, but the same proof works for any Banach algebra).

Lemma 3.17 ([26, Theorem 2.2, p. 48]). — *For $j = 1, 2$, let*

$$f_j = \sum_{|\mathbf{n}| \geq 2} f_{j,\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[\![\mathbf{y}]\!]$$

be formal power series with coefficients in a Banach algebra $(\mathfrak{B}, \|\cdot\|)$, and of order at least two. Consider also two other series

$$g_j = \sum_{|\mathbf{n}| \geq 2} g_{j,\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}\{\mathbf{y}\}, \, j = 1, 2,$$

of order at least two, which have a non-zero radius of convergence at the origin. Assume that there exists $\sigma > 0$ such that for $j = 1, 2$:

$$\sigma \overline{f_j} \preccurlyeq \overline{g_j}(y_1 + \overline{f_1}, y_2 + \overline{f_2}) .$$

Then, f_1 and f_2 have a non-zero radius of convergence.

Taking $\beta > 4\pi$, according to Proposition 2.12, for all $\hat{f}, \hat{g} \in \mathfrak{B}^{\text{bis}}_{\beta}$, we have:

$$\|\hat{f}\hat{g}\|^{\text{bis}}_{\beta} \leq \|\hat{f}\|^{\text{bis}}_{\beta} \|\hat{g}\|^{\text{bis}}_{\beta} .$$

This implies that $(\mathfrak{B}_\beta^{\mathrm{bis}}, \|\cdot\|_\beta^{\mathrm{bis}})$ is a Banach algebra as needed in the above lemma. It remains to prove that there exists $\sigma > 0$ such that for $j = 1, 2$:

$$\sigma\overline{\hat{\psi}_j} \quad \overline{\hat{F}_j}\left(y_1 + \overline{\hat{\psi}_1}, y_2 + \overline{\hat{\psi}_2}\right).$$

Remember that there exists $M > 0$ such that for $j = 1, 2$:

$$\left\|\hat{\psi}_{j,\mathbf{n}}\right\|_\beta^{\mathrm{bis}} \leqslant M.\left\|\zeta_{j,\mathbf{n}}\right\|_\beta^{\mathrm{bis}}$$

where

$$\begin{cases} \zeta_1 := \displaystyle\sum_{|\mathbf{n}|\geqslant 2} \zeta_{1,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}} \\ \quad = \hat{F}_1(x, y_1 + \hat{\psi}_1(x,\mathbf{y}), y_2 + \hat{\psi}_2(x,\mathbf{y})) - \hat{T}_1(x)\dfrac{\partial\hat{\psi}_1}{\partial y_1}(x,\mathbf{y}) - \hat{T}_2(x)\dfrac{\partial\hat{\psi}_1}{\partial y_2}(x,\mathbf{y}) \\ \zeta_2 := \displaystyle\sum_{|\mathbf{n}|\geqslant 2} \zeta_{2,\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}} \\ \quad = \hat{F}_2(x, y_1 + \hat{\psi}_1(x,\mathbf{y}), y_2 + \hat{\psi}_2(x,\mathbf{y})) - \hat{T}_1(x)\dfrac{\partial\hat{\psi}_2}{\partial y_1}(x,\mathbf{y}) - \hat{T}_2(x)\dfrac{\partial\hat{\psi}_2}{\partial y_2}(x,\mathbf{y}). \end{cases}$$

If we set $\sigma := \frac{1}{M}$, then we have

$$\begin{cases} \sigma\overline{\hat{\psi}_1} \quad \overline{\zeta_1} \quad \overline{\hat{F}}_1(x, y_1 + \overline{\hat{\psi}_1}(x,\mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x,\mathbf{y})) \\ \qquad + \overline{\hat{T}_1}(x)\dfrac{\partial\overline{\hat{\psi}_1}}{\partial y_1}(x,\mathbf{y}) + \overline{\hat{T}_2}(x)\dfrac{\partial\overline{\hat{\psi}_1}}{\partial y_2}(x,\mathbf{y}) \\ \sigma\overline{\hat{\psi}_2} \quad \overline{\zeta_2} \quad \overline{\hat{F}}_2(x, y_1 + \overline{\hat{\psi}_1}(x,\mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x,\mathbf{y})) \\ \qquad + \overline{\hat{T}_1}(x)\dfrac{\partial\overline{\hat{\psi}_2}}{\partial y_1}(x,\mathbf{y}) + \overline{\hat{T}_2}(x)\dfrac{\partial\overline{\hat{\psi}_2}}{\partial y_2}(x,\mathbf{y}). \end{cases}$$

Moreover, we recall that

$$\begin{cases} \hat{T}_{1,\mathbf{n}} = 0, & \text{if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0, & \text{if } n_1 = 0 \text{ or } n_2 = 0 \end{cases} \qquad \begin{cases} \hat{\psi}_{1,\mathbf{n}} = 0, & \text{if } n_1 \geqslant 1 \text{ and } n_2 \geqslant 1 \\ \hat{\psi}_{2,\mathbf{n}} = 0, & \text{if } n_1 \geqslant 1 \text{ and } n_2 \geqslant 1, \end{cases}$$

so that we have in fact more precise dominant relations:

$$\begin{cases} \sigma\overline{\hat{\psi}_1} \quad \overline{\zeta_1} \quad \overline{\hat{F}}_1\left(x, y_1 + \overline{\hat{\psi}_1}(x,\mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x,\mathbf{y})\right) \\ \sigma\overline{\hat{\psi}_2} \quad \overline{\zeta_2} \quad \overline{\hat{F}}_2\left(x, y_1 + \overline{\hat{\psi}_1}(x,\mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x,\mathbf{y})\right). \end{cases}$$

It remains the apply the lemma above to conclude.

Remark 3.18. — In the previous proposition, assume that for $j=1,2$,

$$\hat{F}_j(x,\mathbf{y}) = \sum_{\mathbf{n}\in\mathbb{N}^2,\, |\mathbf{n}|\geqslant 2} \hat{F}^{(j)}_{\mathbf{n}}(x)\mathbf{y}^{\mathbf{n}}$$

in the expression of $\hat{Y}$ satisfies

$$\begin{cases} \hat{F}^{(1)}_{\mathbf{n}}(0)=0, & \forall\, \mathbf{n}=(n_1,n_2) \mid n_1+n_2\geqslant 2 \text{ and } \left(n_1=0 \text{ or } n_2=0\right) \\ \hat{F}^{(2)}_{\mathbf{n}}(0)=0, & \forall\, \mathbf{n}=(n_1,n_2) \mid n_1+n_2\geqslant 2 \text{ and } \left(n_1=0 \text{ or } n_2=0\right). \end{cases}$$

Then, the diffeomorphism $\hat{\Phi}$ in the proposition can be chosen to be the identity on $\{x=0\}$, so that

$$\begin{cases} y_1y_2\hat{R}_1(x,\mathbf{y}) = \hat{F}_1(0,\mathbf{y}) + x\hat{S}_1(x,\mathbf{y}) \\ y_1y_2\hat{R}_2(x,\mathbf{y}) = \hat{F}_2(0,\mathbf{y}) + x\hat{S}_2(x,\mathbf{y}), \end{cases}$$

where $\hat{S}_1,\hat{S}_2$ are 1-summable in the direction $\theta = \arg(\pm\lambda)$ and both $\hat{F}_1(0,\mathbf{y}),\hat{F}_2(0,\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}$ are convergent in neighborhood of the origin in $\mathbb{C}^2$. Indeed, we easily see by induction on $|\mathbf{n}| = n_1+n_2\geqslant 2$ that $\hat{\psi}_1$ and $\hat{\psi}_2$ can be chosen "divisible" by x, and that ζ_1,ζ_2 are such that $\zeta_{j,\mathbf{n}}(x)$ is also "divisible" by x if $n_1=0$ or $n_2=0$.

3.4. 1-summable normal form up to arbitrary order N

We consider now a (formal) non-degenerate diagonal doubly-resonant saddle node, which is supposed to be div-integrable and 1-summable in every direction $\theta=\arg(\pm\lambda)$, of the form

$$\hat{Y}_{\text{prep}} = x^2\frac{\partial}{\partial x} + \left(-\lambda + a_1x - d(y_1y_2) + x\hat{S}_1(x,\mathbf{y})\right) y_1\frac{\partial}{\partial y_1} \\ + \left(\lambda + a_2x + d(y_1y_2) + x\hat{S}_2(x,\mathbf{y})\right) y_2\frac{\partial}{\partial y_2},$$

where:

- $\lambda \in \mathbb{C}\setminus\{0\}$;
- $\hat{S}_1,\hat{S}_2 \in \mathbb{C}[\![x,\mathbf{y}]\!]$ are of order at least one with respect to $\mathbf{y}$ and 1-summable in every direction $\theta\in\mathbb{R}$ with $\theta=\arg(\pm\lambda)$;
- $a := \operatorname{res}(\hat{Y}_{\text{prep}}) = a_1+a_2 \notin \mathbb{Q}_{\leqslant 0}$;
- $d(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1y_2$ vanishing at the origin.

As usual, we denote by $Y_{\text{prep},\theta}, S_{1,\theta}, S_{2,\theta}$ the respective 1-sums of $\hat{Y}, \hat{S}_1, \hat{S}_2$ in the direction θ. Let us introduce some useful notations:

$$\hat{Y}_{\text{prep}} = Y_0 + D\overrightarrow{C} + R\overrightarrow{R}\,, \tag{3.16}$$

where

- $\overrightarrow{C} := -y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$
- $\overrightarrow{R} := y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$
- $Y_0 := \lambda\overrightarrow{C} + x\left(x\frac{\partial}{\partial x} + a_1y_1\frac{\partial}{\partial y_1} + a_2y_2\frac{\partial}{\partial y_2}\right)$
- $D(x,\mathbf{y}) = d(y_1y_2) + xD^{(1)}(x,\mathbf{y}) = d(y_1y_2) + x\left(\frac{\hat{S}_2-\hat{S}_1}{2}\right)$ is 1-summable in the direction θ of 1-sum D_θ : it is called the "*tangential*" part. D_θ is also dominated by $\|\mathbf{y}\| = \max(|y_1|,|y_2|)$ (D is of order at least one with respect to $\mathbf{y}$).
- $R(x,\mathbf{y}) = xR^{(1)}(x,\mathbf{y}) = x\left(\frac{\hat{S}_2+\hat{S}_1}{2}\right)$ is 1-summable in the direction θ of 1-sum R_θ: it is called the "*radial*" part. R_θ is also dominated by $\|\mathbf{y}\|_\infty = \max(|y_1|,|y_2|)$ (R is of order at least one with respect to $\mathbf{y}$).

The following proposition gives the existence of a 1-summable normalizing map, up to any order $N \in \mathbb{N}_{>0}$, with respect to x.

Proposition 3.19. — *Let*

$$\hat{Y}_{\text{prep}} = Y_0 + D\overrightarrow{C} + R\overrightarrow{R} \tag{3.17}$$

be as above. Then for all $N \in \mathbb{N}_{>0}$ there exist $d^{(N)}(v) \in \mathbb{C}\{v\}$ of order at least one and $\Phi^{(N)} \in \mathrm{Diff}_{\text{fib}}(\mathbb{C}^3,0;\mathrm{Id})$ which conjugates $\hat{Y}_{\text{prep}}$ (resp. its 1-sums $Y_{\text{prep},\theta}$ in the direction θ) to

$$Y^{(N)} = Y_0 + \left(d^{(N)}(y_1y_2) + x^N D^{(N)}(x,\mathbf{y})\right)\overrightarrow{C} + x^N R^{(N)}(x,\mathbf{y})\overrightarrow{R}$$
$$\left(\text{resp. } Y_\theta^{(N)} = Y_0 + \left(d^{(N)}(y_1y_2) + x^N D_\theta^{(N)}(x,\mathbf{y})\right)\overrightarrow{C} + x^N R_\theta^{(N)}(x,\mathbf{y})\overrightarrow{R}\right)$$

where $D^{(N)}, R^{(N)}$ are 1-summable in the direction θ, of order at least one with respect to $\mathbf{y}$, of 1-sums $D_\theta^{(N)}, R_\theta^{(N)}$ in the direction θ. Moreover, one can choose $d^{(2)} = \cdots = d^{(N)}$ for all $N > 2$, and $d^{(1)} = d$.

Proof. — The proof is performed by induction on N.

- The case $N = 1$ is the initial situation here, and is already proved with $\hat{Y}_{\text{prep}} = Y^{(1)}$.
- Assume that the result holds for $N \in \mathbb{N}_{>0}$. We will proceed in three steps.

First step. — Let us write

$$R^{(N)}(x,\mathbf{y}) = \sum_{n_1+n_2>1} R^{(N)}_{n_1,n_2}(x) y_1^{n_1} y_2^{n_2}\,.$$

We are looking for an analytic solution τ to the equations:

$$\begin{aligned} L_{Y^{(N)}}(\tau) &= -x^N R^{(N)} + \left(x^{N+1}\tilde{R}^{(N+1)}\right)\circ\Lambda_\tau \\ L_{Y_\theta^{(N)}}(\tau) &= -x^N R_\theta^{(N)} + \left(x^{N+1}\tilde{R}_\theta^{(N+1)}\right)\circ\Lambda_\tau\,, \end{aligned} \tag{3.18}$$

for a convenient choice of $\tilde{R}^{(N+1)}, \tilde{R}_\theta^{(N+1)}$, with

$$\Lambda_\tau(x,\mathbf{y}) := \left(x, y_1\exp(\tau(x,\mathbf{y})), y_2\exp(\tau(x,\mathbf{y}))\right),$$

and

$$\tau(x,\mathbf{y}) = x^{N-1}\tau_0(y_1y_2) + x^N\tau_1(\mathbf{y})\,,$$

where $\tau_1(\mathbf{y}) = \sum_{j_1 = j_2} \tau_{1,j_1j_2} y_1^{j_1} y_2^{j_2}$. More concretely, Λ_τ is the formal flow of $\vec{\mathcal{R}}$ at "time" $\tau(x,\mathbf{y})$.

If we admit for a moment that such an analytic solution τ exists, then $\Lambda_\tau \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$ and therefore $\Lambda_\tau^{-1} \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$. Consider $d^{(N)}$ and $\tilde{D}^{(N-1)}$ such that

$$\begin{aligned} d^{(N+1)}(z_1z_2) + x^{N-1}\tilde{D}^{(N-1)}(x,\mathbf{z}) &\\ := \left(d^{(N)}(y_1y_2) + x^N D^{(N)}(x,\mathbf{y})\right)&\circ\Lambda_\tau^{-1}(x,\mathbf{z}) \\ d^{(N+1)}(z_1z_2) + x^{N-1}\tilde{D}_\theta^{(N-1)}(x,\mathbf{z}) &\\ := \left(d^{(N)}(y_1y_2) + x^N D_\theta^{(N)}(x,\mathbf{y})\right)&\circ\Lambda_\tau^{-1}(x,\mathbf{z}), \end{aligned}$$

with $\tilde{D}^{(N-1)} = 0$ if $N=1$. Consequently, the two equations given in (3.18) imply that

$$\begin{aligned} (\Lambda_\tau)_*\left(Y^{(N)}\right) = Y_0 + &\left(d^{(N+1)}(z_1z_2) + x^{N-1}\tilde{D}^{(N-1)}(x,\mathbf{z})\right)\vec{\mathcal{C}} \\ &+ x^{N+1}\tilde{R}^{(N+1)}(x,\mathbf{z})\vec{\mathcal{R}} \\ (\Lambda_\tau)_*\left(Y_\theta^{(N)}\right) = Y_0 + &\left(d^{(N+1)}(z_1z_2) + x^{N-1}\tilde{D}_\theta^{(N-1)}(x,\mathbf{z})\right)\vec{\mathcal{C}} \\ &+ x^{N+1}\tilde{R}_\theta^{(N+1)}(x,\mathbf{z})\vec{\mathcal{R}}\,. \end{aligned}$$

Indeed:

$$\begin{aligned}
\mathrm{d}\Lambda_\tau \cdot Y^{(N)} &= \begin{pmatrix} \mathcal{L}_{Y^{(N)}}(x) \\ \mathcal{L}_{Y^{(N)}}(y_1 \exp(\tau(x,\mathbf{y}))) \\ \mathcal{L}_{Y^{(N)}}(y_2 \exp(\tau(x,\mathbf{y}))) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (\mathcal{L}_{Y^{(N)}}(y_1) + y_1(\mathcal{L}_{Y^{(N)}}(\tau)))\exp(\tau(x,\mathbf{y})) \\ (\mathcal{L}_{Y^{(N)}}(y_2) + y_2(\mathcal{L}_{Y^{(N)}}(\tau)))\exp(\tau(x,\mathbf{y})) \end{pmatrix} \\
&= \Big(Y_0 + \Big(d^{(N+1)} + x^{N-1}\tilde{D}^{(N-1)} \Big) \vec{\mathcal{C}} \\
&\qquad + x^{N+1}\tilde{R}^{(N+1)} \vec{\mathcal{R}} \Big) \circ \Lambda_\tau(x,\mathbf{y}) .
\end{aligned}$$

These computations are also true with the corresponding 1-sums of formal objects considered here, i.e. with $Y_\theta^{(N)}$, $D_\theta^{(N)}$, $\tilde{D}_\theta^{(N-1)}$, $\tilde{R}_\theta^{(N+1)}$ instead of $Y^{(N)}$, $D^{(N)}$, $\tilde{D}^{(N-1)}$, $\tilde{R}^{(N+1)}$ respectively. We use Proposition 2.20 to obtain the 1-summability of the objects defined by compositions.

Let us prove that there exists a germ of analytic function of the form

$$\tau(x,\mathbf{y}) = x^{N-1}\tau_0(y_1y_2) + x^N\tau_1(\mathbf{y}) ,$$

of order ate least one with respect to $\mathbf{y}$ in the origin, with

$$\tau_1(\mathbf{y}) = \sum_{j_1 = j_2} \tau_{1,j_1j_2} y_1^{j_1} y_2^{j_2}$$

satisfying equation (3.18). This equation can be written

$$\begin{aligned}
x^2\frac{\partial\tau}{\partial x} &+ \Big(-\lambda + a_1x - d^{(N)}(y_1y_2) - x^N D^{(N)}(x,\mathbf{y}) + x^N R^{(N)}(x,\mathbf{y}) \Big) y_1\frac{\partial\tau}{\partial y_1} \\
&+ \Big(\lambda + a_2x + d^{(N)}(y_1y_2) + x^N D^{(N)}(x,\mathbf{y}) + x^N R^{(N)}(x,\mathbf{y}) \Big) y_2\frac{\partial\tau}{\partial y_2} \\
&\qquad = -x^N R^{(N)} + \Big(x^{N+1}\tilde{R}^{(N+1)} \Big) \circ \Lambda_\tau ,
\end{aligned}$$

or equivalently

$$\begin{aligned}
x^2\frac{\partial\tau}{\partial x} + a_1xy_1\frac{\partial\tau}{\partial y_1} + a_2xy_2\frac{\partial\tau}{\partial y_2} &+ \Big(\lambda + d^{(N)}(y_1y_2) + x^N D^{(N)}(x,\mathbf{y}) \Big) \mathcal{L}_{\vec{\mathcal{C}}}(\tau) \\
&+ \Big(x^N R^{(N)}(x,\mathbf{y}) \Big) \mathcal{L}_{\vec{\mathcal{R}}}(\tau) = -x^N R^{(N)} + \Big(x^{N+1}\tilde{R}^{(N+1)} \Big) \circ \Lambda_\tau .
\end{aligned}$$

Let us consider terms of degree N with respect to x:

$$(3.19)\quad (N-1)\tau_0(y_1y_2) + \left(a_1 + a_2 + 2\delta_{N,1}R^{(N)}(0,\mathbf{y})\right) y_1y_2\frac{\partial\tau_0}{\partial v}(y_1y_2) + \left(\lambda + d^{(N)}(y_1y_2)\right) L_{\vec{C}}(\tau_1) = -R^{(N)}(0,\mathbf{y})$$

(here $\delta_{N,1}$ is the Kronecker notation), and let us define

$$R^{(N)}_{\mathrm{res}}(0,v) := \sum_{k\geqslant 1} R^{(N)}_{k,k}(0)v^k\,.$$

We use now the fact that $\mathrm{Im}(L_C)\oplus\mathrm{Ker}(L_C)$ is a direct sum, and that $\mathrm{Ker}(L_C)$ is the set of formal power series in the resonant monomial $v = y_1y_2$. Isolating the term $L_C(\tau_1)$ on the one hand, and the others on the other hand, the direct sum above gives us:

$$\begin{cases} v\left(a_1 + a_2 + 2\delta_{N,1}R^{(N)}_{\mathrm{res}}(0,v)\right)\dfrac{\mathrm{d}\tau_0}{\mathrm{d}v}(v) + (N-1)\tau_0(v) = -R^{(N)}_{\mathrm{res}}(0,v)\\ \tau_0(0) = 0 \end{cases}$$

and

$$\begin{cases} L_{\vec{C}}(\tau_1) = \dfrac{-1}{\lambda + d^{(N)}(y_1y_2)}\Big(2\delta_{N,1}\left(R^{(N)}(0,\mathbf{y}) - R^{(N)}_{\mathrm{res}}(0,v)\right) y_1y_2\frac{\mathrm{d}\tau_0}{\mathrm{d}v}(y_1y_2)\\ \qquad\qquad + R^{(N)}(0,\mathbf{y}) - R^{(N)}_{\mathrm{res}}(0,v)\Big)\\ \tau_1(0) = 0\,. \end{cases}$$

Since $R^{(N)}$ is analytic with respect to $\mathbf{y}$, $R^{(N)}_{\mathrm{res}}(0,v)$ is analytic near $v = 0$. Furthermore, as $R^{(N)}_{\mathrm{res}}(0,0) = 0$ and $a_1+a_2 \notin \mathbb{Q}_{\leqslant 0}$, the first of the two equation above has a unique formal solution τ_0 with $\tau_0(0)$, and this solution is convergent in a neighborhood of the origin. Once τ_0 is determined, there exists a unique formal solution τ_1 to the second equation satisfying $\tau_1(\mathbf{y}) = \sum_{j_1\neq j_2} \tau_{1,j_1j_2}y_1^{j_1}y_2^{j_2}$, which is moreover convergent in a neighborhood of the origin of $\mathbb{C}^2$.

Therefore Λ_τ is a germ of analytic diffeomorphism fixing the origin, fibered, tangent to the identity and conjugates $Y^{(N)}$ (resp. $Y^{(N)}_\theta$) to $\tilde{Y}^{(N)} := (\Lambda_\tau)_*(Y^{(N)})$ (resp. to $\tilde{Y}^{(N)}_\theta := (\Lambda_\tau)_*(Y^{(N)}_\theta)$).

Equation (3.19) implies that both $(L_{Y^{(N)}}(\tau) + x^N R^{(N)})$ and $(L_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)})$ are divisible by x^{N+1}, so that we can define:

$$\tilde{R}^{(N+1)}(x, \mathbf{z}) := \left(\frac{L_{Y^{(N)}}(\tau) + x^N R^{(N)}}{x^{N+1}} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z})$$
$$\tilde{R}_\theta^{(N+1)}(x, \mathbf{z}) := \left(\frac{L_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)}}{x^{N+1}} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) .$$

By Proposition 2.20, $\tilde{R}^{(N+1)}$ (resp. $\tilde{D}^{(N-1)}$) is 1-summable in the direction θ, of 1-sum $\tilde{R}_\theta^{(N+1)}$ (resp. $\tilde{D}_\theta^{(N-1)}$).

Finally, notice that $d^{(N+1)} \circ \Lambda_\tau(0, \mathbf{y}) = d^{(N)}(y_1, y_2)$, $\tau(0, \mathbf{y}) = 0$ and then $\Lambda_\tau(0, \mathbf{y}) = (0, y_1, y_2)$ if $N > 1$, so that $d^{(N+1)} = d^{(N)}$ when $N > 1$.

Second step. — Exactly as in the previous step which dealt with the "radial part" (in fact the computations are even easier here), we can prove the existence of a germ of an analytic function σ, solution to the equation:

$$\begin{aligned} L_{\tilde{Y}^{(N)}}(\sigma) &= -x^{N-1}\tilde{D}^{(N-1)} + (x^N \tilde{\tilde{D}}^{(N)}) \circ \Gamma_\sigma \\ L_{\tilde{Y}_\theta^{(N)}}(\sigma) &= -x^{N-1}\tilde{D}_\theta^{(N-1)} + (x^N \tilde{\tilde{D}}_\theta^{(N)}) \circ \Gamma_\sigma , \end{aligned} \tag{3.20}$$

for a good choice of $\tilde{\tilde{D}}^{(N)}, \tilde{\tilde{D}}_\theta^{(N)}$, with

$$\Gamma_\sigma(x, \mathbf{z}) := (x, y_1 \exp(-\sigma(x, \mathbf{z})), y_2 \exp(\sigma(x, \mathbf{z})))$$

and

$$\sigma(x, \mathbf{z}) = x^{N-2}\sigma_0(z_1 z_2) + x^{N-1}\sigma_1(\mathbf{z}) ,$$

where $\sigma_1(\mathbf{z}) = \sum_{j_1 = j_2} \sigma_{1, j_1, j_2} z_1^{j_1} z_2^{j_2}$. Here, we take $\sigma_0 = 0$ if $N = 1$. Notice that Γ_σ is the formal flow of $\overrightarrow{\mathcal{C}}$ at "time" $\sigma(x, \mathbf{z})$.

Again, as in the first step with the "radial part", we have on a $\Gamma_\sigma \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ and then also $\Gamma_\sigma^{-1} \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$. If we consider $\tilde{\tilde{R}}^{(N+1)}$ and $\tilde{\tilde{R}}_\theta^{(N+1)}$ such that

$$\tilde{\tilde{R}}^{(N+1)}(x, \mathbf{y}) := \tilde{R}^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y})$$
$$\tilde{\tilde{R}}_\theta^{(N+1)}(x, \mathbf{z}) := \tilde{R}_\theta^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) ,$$

then it follows from (3.20) that

$$\begin{aligned}(\Gamma_\sigma)\ (\tilde{Y}^{(N)}) &= Y_0 + \left(d^{(N+1)}(y_1y_2) + x^N \tilde{\tilde{D}}^{(N)}(x,\mathbf{y}) \right) \vec{C} \\ &\quad + x^{N+1}\tilde{\tilde{R}}^{(N+1)}(x,\mathbf{y})\vec{R} \\ (\Gamma_\sigma)\ (\tilde{Y}_\theta^{(N)}) &= Y_0 + \left(d^{(N+1)}(y_1y_2) + x^N \tilde{\tilde{D}}_\theta^{(N)}(x,\mathbf{y}) \right) \vec{C} \\ &\quad + x^{N+1}\tilde{\tilde{R}}_\theta^{(N+1)}(x,\mathbf{y})\vec{R}\ .\end{aligned}$$

Notice that the degree of the monomial x^{N+1} in front of $\tilde{\tilde{R}}^{(N+1)}$ is indeed $N+1$ (and not N): this essentially comes form the fact that Γ_σ (and Γ_σ^{-1}) preserves the resonant monomial $v = y_1y_2$. We choose:

$$\begin{aligned}\tilde{\tilde{D}}^{(N)}(x,\mathbf{y}) &:= \left(\frac{L_{\tilde{Y}^{(N)}}(\sigma) + x^{N-1}\tilde{D}^{(N-1)}}{x^N} \right) \circ \Gamma_\sigma^{-1}(x,\mathbf{y}) \\ \tilde{\tilde{D}}_\theta^{(N)}(x,\mathbf{y}) &:= \left(\frac{L_{\tilde{Y}_\theta^{(N)}}(\sigma) + x^{N-1}\tilde{D}_\theta^{(N-1)}}{x^N} \right) \circ \Gamma_\sigma^{-1}(x,\mathbf{y})\,.\end{aligned}$$

By Proposition 2.20, $\tilde{\tilde{D}}^{(N)}$ (resp. $\tilde{\tilde{R}}^{(N+1)}$) is 1-summable in the direction θ, of 1-sum $\tilde{\tilde{D}}_\theta^{(N)}$ (resp. $\tilde{\tilde{R}}_\theta^{(N+1)}$). We finally define $\tilde{\tilde{Y}}^{(N)} := (\Gamma_\sigma)\ (\tilde{Y}^{(N)})$, and $\tilde{\tilde{Y}}_\theta^{(N)} := (\Gamma_\sigma)\ (\tilde{Y}_\theta^{(N)})$.

Third (and last) step. — As in both previous steps, we can prove the existence of a germ of an analytic function φ, solution to the equation:

$$\begin{aligned}L_{\tilde{\tilde{Y}}^{(N)}}(\varphi) &= -x^N\tilde{\tilde{D}}^{(N)} + \left(x^{N+1}D^{(N+1)} \right) \circ \Gamma_\phi\,, \\ L_{\tilde{\tilde{Y}}_\theta^{(N)}}(\varphi) &= -x^N\tilde{\tilde{D}}_\theta^{(N)} + \left(x^{N+1}D_\theta^{(N+1)} \right) \circ \Gamma_\phi\,,\end{aligned}$$

for a good choice of $D^{(N+1)}, D_\theta^{(N+1)}$, with

$$\Gamma_\phi(x,\mathbf{y}) := (x, y_1\exp(-\varphi(x,\mathbf{y})), y_2\exp(\varphi(x,\mathbf{y})))$$

and

$$\varphi(x,\mathbf{y}) = x^{N-1}\varphi_0(y_1y_2) + x^N\varphi_1(\mathbf{y})\,,$$

where $\varphi_1(\mathbf{y}) = \sum_{j_1=j_2} \varphi_{1,j_1,j_2} y_1^{j_1}y_2^{j_2}$.

Again, we have on a $\Gamma_\phi \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$ and then also $\Gamma_\phi^{-1} \in \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$. If we consider $R^{(N+1)}$ and $R_\theta^{(N+1)}$ such that

$$\begin{aligned}R^{(N+1)}(x,\mathbf{y}) &:= \tilde{\tilde{R}}^{(N+1)} \circ \Gamma_\phi^{-1}(x,\mathbf{y}) \\ R_\theta^{(N+1)}(x,\mathbf{z}) &:= \tilde{\tilde{R}}_\theta^{(N+1)} \circ \Gamma_\phi^{-1}(x,\mathbf{y})\,,\end{aligned}$$

then we have:

$$
\begin{aligned}
(\Gamma_\phi)\ (\tilde{\tilde{Y}}^{(N)}) &= Y_0 + \left(d^{(N+1)}(y_1y_2) + x^{N+1}D^{(N+1)}(x,\mathbf{y})\right)\overrightarrow{\overrightarrow{C}} \\
&\quad + x^{N+1}R^{(N+1)}(x,\mathbf{y})\overrightarrow{\overrightarrow{R}} \\
(\Gamma_\phi)\ (\tilde{\tilde{Y}}_\theta^{(N)}) &= Y_0 + \left(d^{(N+1)}(y_1y_2) + x^{N+1}D_\theta^{(N+1)}(x,\mathbf{y})\right)\overrightarrow{\overrightarrow{C}} \\
&\quad + x^{N+1}R_\theta^{(N+1)}(x,\mathbf{y})\overrightarrow{\overrightarrow{R}}\,.
\end{aligned}
$$

As above, notice that the degree of the monomial x^{N+1} in front of $R^{(N+1)}$ is indeed $N+1$. We choose:

$$
\begin{aligned}
D^{(N+1)}(x,\mathbf{y}) &:= \left(\frac{L_{\tilde{\tilde{Y}}^{(N)}}(\varphi) + x^N\tilde{\tilde{D}}^{(N)}}{x^{N+1}}\right)\circ\Gamma_\phi^{-1}(x,\mathbf{y}) \\
D_\theta^{(N+1)}(x,\mathbf{y}) &:= \left(\frac{L_{\tilde{\tilde{Y}}_\theta^{(N)}}(\varphi) + x^N\tilde{\tilde{D}}_\theta^{(N)}}{x^{N+1}}\right)\circ\Gamma_\phi^{-1}(x,\mathbf{y})\,.
\end{aligned}
$$

By Proposition 2.20, $D^{(N+1)}$ (resp. $R^{(N+1)}$) is 1-summable in the direction θ, of 1-sum $D_\theta^{(N)}$ (resp. $R_\theta^{(N+1)}$). We finally define $Y^{(N+1)} := (\Gamma_\phi)\ (\tilde{\tilde{Y}}^{(N)})$, and $Y_\theta^{(N+1)} := (\Gamma_\phi)\ (\tilde{\tilde{Y}}_\theta^{(N)})$.

3.5. Proof of Proposition 3.1

We now give a short proof of Proposition 3.1, using the different results proved in this section.

Proof of Proposition 3.1. — We just have to use consecutively Proposition 3.7 (applied to $Y_0 := Y_{|\{x=0\}}$), Proposition 3.10, Proposition 3.15 and finally Proposition 3.19, using at each time Corollary 2.21 in order to obtain the directional 1-summability.

4. Sectorial analytic normalization

The aim of this section is to prove that for any $Y \in SN_{\mathrm{diag},0}$ and for any $\eta \in [\pi, 2\pi[$, there exists a unique pair

$$(\Phi_+, \Phi_-) \in \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(i\lambda),\eta}; \mathrm{Id}) \times \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(-i\lambda),\eta}; \mathrm{Id})$$

whose elements analytically conjugate Y to its normal form Y_{norm} (given by Theorem 1.5) in sectorial neighborhoods of the origin with wide opening.

The existence of sectorial normalizing maps Φ_+ and Φ_- in domains of the form $S_+ \in \mathcal{S}_{\arg(i\lambda),\eta}$ and $S_- \in \mathcal{S}_{\arg(-i\lambda),\eta}$ for all $\eta \in [\pi, 2\pi[$, is equivalent to the existence of a sectorial normalizing map Φ_θ in domains $S \in \mathcal{S}_{\theta,\pi}$, for all $\theta \in \mathbb{R}$ such that $\theta = \arg(\pm\lambda)$. At the end of this section we will also prove that Φ_+ and Φ_- both admit the unique formal normalizing map $\hat{\Phi}$ (given by Theorem 1.5) as weak Gevrey-1 asymptotic expansion in domains $S_+ \in \mathcal{S}_{\arg(i\lambda),\eta}$ and $S_- \in \mathcal{S}_{\arg(-i\lambda),\eta}$ respectively. In particular, this will prove that $\hat{\Phi}$ is weakly 1-summable in every direction $\theta = \arg(\pm\lambda)$. We start with a vector field $Y^{(N)}$ normalized up to order $N > 2$ as in Proposition 3.1. First of all, we prove the existence of germs of sectorial analytic functions $\alpha_+ \in \mathcal{O}(S_+), \alpha_- \in \mathcal{O}(S_-)$, which are solutions to homological equations of the form:

$$\mathcal{L}_{Y^{(N)}}(\alpha_\pm) = x^{M+1}A_\pm(x, \mathbf{y}),$$

where $M \in \mathbb{N}_{>0}$ and $A_\pm \in \mathcal{O}(S_\pm)$ is analytic in $S_\pm$ (see Lemma 4.6). In order to construct such solutions, we will integrate some appropriate meromorphic 1-form on asymptotic paths (see Subsection 4.4). Once we have these solutions α_+, α_-, we will construct the desired germs of sectorial diffeomorphisms as the flows of some elementary linear vector fields at "time" $\alpha_\pm(x, \mathbf{y})$. After that, we will prove in Subsection 4.5 that there exist unique germs of sectorial fibered diffeomorphisms tangent to the identity which conjugate $Y \in \mathcal{SN}_{\text{fib},0}$ to its normal form, by studying the sectorial isotropies in sectorial domains with wide opening. We go on using the notations introduced in Subsection 3.4, i.e.

- $\lambda \in \mathbb{C}$
- $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$
- $\overrightarrow{C} := -y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$
- $\overrightarrow{R} := y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$
- $Y_0 := \lambda\overrightarrow{C} + x\left(x\frac{\partial}{\partial x} + a_1y_1\frac{\partial}{\partial y_1} + a_2y_2\frac{\partial}{\partial y_2}\right)$.

For $\epsilon \in]0, \frac{\pi}{2}[$ and $r > 0$, we will consider two sectors, namely

$$S_+(r,\epsilon) := S\left(r, \arg(i\lambda) - \frac{\pi}{2} - \epsilon, \arg(i\lambda) + \frac{\pi}{2} + \epsilon\right)$$

and

$$S_-(r,\epsilon) = S\left(r, \arg(-i\lambda) - \frac{\pi}{2} - \epsilon, \arg(-i\lambda) + \frac{\pi}{2} + \epsilon\right).$$

Let us consider a (weakly) 1-summable non-degenerate div-integrable doubly-resonant saddle-node normalized up to an order $N+2$, with $N > 0$:

$$\begin{aligned} Y^{(N+2)} &= Y_0 + \left(c(y_1y_2) + x^{N+2}D^{(N+2)}(x,\mathbf{y})\right)\vec{\mathcal{C}} + x^{N+2}R^{(N+2)}(x,\mathbf{y})\vec{\mathcal{R}} \\ &\quad \text{(formal)} \\ Y_{\pm}^{(N+2)} &= Y_0 + \left(c(y_1y_2) + x^{N+2}D_{\pm}^{(N+2)}(x,\mathbf{y})\right)\vec{\mathcal{C}} + x^{N+2}R_{\pm}^{(N+2)}(x,\mathbf{y})\vec{\mathcal{R}} \\ &\quad \text{(analytic in } S_{\pm}(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r})) \end{aligned}$$

where $D^{(N+2)}, R^{(N+2)}$ are of order at least one with respect to $\mathbf{y}$, and (weak) 1-summable in every direction $\theta \in \mathbb{R}$ with $\theta = \arg(\pm\lambda)$: their respective (weak) 1-sums in the direction $\arg(\pm i\lambda)$ are $D_{\pm}^{(N+2)}, R_{\pm}^{(N+2)}$, which can be analytically extended in $S_{\pm}(r,\epsilon) \times (\mathbb{C}^2, 0)$. In order to have the complete sectorial normalizing map, we have to assume now that our vector field is *strictly non-degenerate*, i.e.

$$\boxed{(a_1 + a_2) > 0\,.}$$

Proposition 4.1. — *Under the assumptions above, for all $\eta \in\,]\pi, 2\pi[$, there exist two germs of sectorial fibered diffeomorphisms*

$$\begin{aligned} \Psi_{+} &\in \mathrm{Diff}_{\mathrm{fib}}(\mathcal{S}_{\arg(i\lambda),\eta}; \mathrm{Id}) \\ \Psi_{-} &\in \mathrm{Diff}_{\mathrm{fib}}(\mathcal{S}_{\arg(-i\lambda),\eta}; \mathrm{Id}) \end{aligned}$$

of the form

$$\Psi_{\pm} : (x,\mathbf{y}) \mapsto \left(x, \mathbf{y} + \mathrm{O}(\|\mathbf{y}\|^2)\right), \qquad (4.1)$$

which conjugate $Y_{\pm}^{(N+2)}$ to its formal normal form

$$Y_{\mathrm{norm}} = x^2\frac{\partial}{\partial x} + \left(-\lambda + a_1x - c(y_1y_2)\right)y_1\frac{\partial}{\partial y_1} + \left(\lambda + a_2x + c(y_1y_2)\right)y_2\frac{\partial}{\partial y_2},$$

where $c(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1y_2$ vanishing at the origin. Moreover, we can choose $\Psi_{\pm}$ above such that

$$\Psi_{\pm}(x,\mathbf{y}) = \mathrm{Id}(x,\mathbf{y}) + x^N\mathbf{P}_{\pm}^{(N)}(x,\mathbf{y}),$$

where $\mathbf{P}_{\pm}^{(N)} = (0, P_{1,\pm}, P_{2,\pm})$ is analytic in $S_{\pm}(r,\epsilon) \times (\mathbb{C}^2, 0)$ (for some $r > 0$ and $\epsilon > \frac{\eta}{2}$) and of order at least two with respect to $\mathbf{y}$.

By combining Propositions 3.1 and 4.1 we immediately obtain the following result.

Corollary 4.2. — *Let $Y \in \mathcal{SN}_{\mathrm{fib},0}$ be a strictly non-degenerate diagonal doubly-resonant saddle-node which is div-integrable. Then, for all*

$\eta \in]\pi, 2\pi[$, there exist two germs of sectorial fibered diffeomorphisms

$$\begin{cases} \Phi_+ \in \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(i\lambda),\eta}; \mathrm{Id}) \\ \Phi_- \in \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(-i\lambda),\eta}; \mathrm{Id}) \end{cases}$$

tangent to the identity such that:

$$\begin{aligned} (\Phi_\pm)_* (Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} \\ &\quad + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\mathrm{norm}} , \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$, and $c(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1 y_2$ vanishing at the origin.

As already mentioned, we will also prove at the end of this section that Φ_+ and Φ_- are unique as germs (see Proposition 1.13), and that they are the weak 1-sums of the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5.

4.1. Proof of Proposition 4.1

We give here two consecutive propositions which allow to prove Proposition 4.1 as an immediate consequence. When we say that a function $f : U \to \mathbb{C}$ is *dominated* by another $g : U \to \mathbb{R}_+$ in U, it means that there exists $L > 0$ such that for all $u \in U$, we have $|f(u)| \leqslant L.g(u)$.

Proposition 4.3. — *Let $Y_\pm^{(N+2)} = Y_0 + D_\pm \overrightarrow{\mathcal{C}} + R_\pm \overrightarrow{\mathcal{R}}$, where*

$$\begin{cases} D_\pm(x, \mathbf{y}) = c(y_1 y_2) + x^{N+2} D_\pm^{(N+2)}(x, \mathbf{y}) \\ R_\pm(x, \mathbf{y}) = x^{N+2} R_\pm^{(N+2)}(x, \mathbf{y}) , \end{cases}$$

with $N \in \mathbb{N}_{>0}$, $c(v) \in v\mathbb{C}\{v\}$ of order at least one, and $D_\pm^{(N+2)}, R_\pm^{(N+2)}$ analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_\infty$.

Assume $\Re(a_1 + a_2) > 0$. Then, possibly by reducing $r > 0$ and the neighborhood $(\mathbb{C}^2, 0)$, there exist two germs of sectorial fibered diffeomorphisms φ_+ and φ_- in $S_+(r, \epsilon) \times (\mathbb{C}^2, 0)$ and $S_-(r, \epsilon) \times (\mathbb{C}^2, 0)$ respectively, which conjugate $Y_\pm^{(N+2)}$ to

$$Y_{\overrightarrow{\mathcal{C}},\pm} := Y_0 + C_\pm \overrightarrow{\mathcal{C}} , \tag{4.2}$$

where $C_\pm(x,\mathbf{y}) = D_\pm \circ \varphi_\pm^{-1}(x,\mathbf{z})$*. Moreover we can chose* $\varphi_\pm$ *to be of the form*

$$\varphi_\pm(x,\mathbf{y}) = (x, y_1 \exp(\rho_\pm(x,\mathbf{y})), y_2 \exp(\rho_\pm(x,\mathbf{y}))) \,, \tag{4.3}$$

where $\rho_\pm(x,\mathbf{y}) = x^{N+1}\tilde{\rho}_\pm(x,\mathbf{y})$ *and* $\tilde{\rho}_\pm$ *is analytic in* $S_\pm(r,\epsilon) \times (\mathbb{C}^2,0)$ *and dominated by* $\|\mathbf{y}\|_\infty$*.*

Remark 4.4. — Notice that $\varphi_\pm^{-1}$ is of the form

$$\varphi_\pm^{-1}(x,\mathbf{z}) = \left(x, z_1\left(1 + x^{N+1}\vartheta(x,\mathbf{z})\right), z_2\left(1 + x^{N+1}\vartheta(x,\mathbf{z})\right)\right),$$

where ϑ is analytic in $S_\pm(r,\epsilon) \times (\mathbb{C}^2,0)$ and dominated by $\|\mathbf{z}\|_\infty$. Consequently:

$$C_\pm(x,\mathbf{z}) = c(z_1 z_2) + x^{N+1} C_\pm^{(N+1)}(x,\mathbf{z}) \,,$$

where c is the same as above and $C_\pm$ is analytic in $S_\pm(r,\epsilon) \times (\mathbb{C}^2,0)$ and dominated by $\|\mathbf{z}\|_\infty$.

Proposition 4.5. — *Let* $Y_{C,\pm} := Y_0 + C_\pm \overrightarrow{\mathcal{C}}$*, where*

$$C_\pm(x,\mathbf{z}) = c(z_1 z_2) + x^{N+1} C_\pm^{(N+1)}(x,\mathbf{z}) \,,$$

with $N \in \mathbb{N}_{>0}$*,* $c(v) \in v\mathbb{C}\{v\}$ *of order at least one, and* $C_\pm^{(N+1)}$ *analytic in* $S_\pm(r,\epsilon) \times (\mathbb{C}^2,0)$ *and dominated by* $\|\mathbf{z}\|_\infty$*. Assume* $\Re(a_1 + a_2) > 0$*. Then, possibly by reducing* $r > 0$ *and the neighborhood* $(\mathbb{C}^2,0)$*, there exist two germs of sectorial fibered diffeomorphisms* ψ_+ *and* ψ_- *in* $S_+(r,\epsilon) \times (\mathbb{C}^2,0)$ *and* $S_-(r,\epsilon) \times (\mathbb{C}^2,0)$ *respectively, which conjugate* $Y_{C,\pm}$ *to*

$$Y_{\mathrm{norm}} := Y_0 + c(v)\overrightarrow{\mathcal{C}} \,. \tag{4.4}$$

Moreover, we can chose $\psi_\pm$ *to be of the form*

$$\psi_\pm(x,\mathbf{z}) = (x, z_1 \exp(-\chi_\pm(x,\mathbf{z})), z_2 \exp(\chi_\pm(x,\mathbf{z}))) \,, \tag{4.5}$$

where $\chi_\pm(x,\mathbf{z}) = x^N \tilde{\chi}_\pm(x,\mathbf{z})$ *and* $\tilde{\chi}$ *is analytic in* $S_\pm(r,\epsilon) \times (\mathbb{C}^2,0)$ *and dominated by* $\|\mathbf{z}\|_\infty$*.*

If we assume for a moment the two propositions above, the proof of Proposition becomes obvious.

Proof of Proposition 4.1. — It is an immediate consequence of the consecutive application of the previous two propositions, just by taking $\Psi_\pm = \psi_\pm \circ \varphi_\pm$ with the notations above.

4.2. Proof of Propositions 4.3 and 4.5

In order to prove Propositions 4.3 and 4.5, we will need the following lemmas. The first one gives the existence of analytic solutions (in sectorial domains) to a homological equations we need to solve.

Lemma 4.6. — *Let $Z_\pm := Y_0 + C_\pm(x,\mathbf{y})\vec{\mathcal{C}} + xR_\pm^{(1)}(x,\mathbf{y})\vec{\mathcal{R}}$, with $C_\pm$, $R_\pm^{(1)}$ analytic in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$ and dominated by $\|\mathbf{y}\|_\infty$ and let also $A_\pm(x,\mathbf{y})$ be analytic in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$ and dominated by $\|\mathbf{y}\|_\infty$. Then for all $M\in\mathbb{N}_{>0}$, possibly by reducing $r>0$ and the neighborhood $(\mathbb{C}^2,0)$, there exists a solution $\alpha_\pm$ to the homological equation*

$$\mathcal{L}_{Z_\pm}(\alpha_\pm) = x^{M+1}A_\pm(x,\mathbf{y})\,, \tag{4.6}$$

such that $\alpha_\pm(x,\mathbf{y}) = x^M\tilde{\alpha}_\pm(x,\mathbf{y})$, where $\tilde{\alpha}_\pm$ is a germ of analytic function in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$ and dominated by $\|\mathbf{y}\|_\infty$.

We will prove this lemma in Subsection 4.4. The following lemma proves that $\varphi_\pm$ and $\psi_\pm$ constructed in Propositions 4.3 and 4.5 are indeed germs of sectorial fibered diffeomorphisms in domains of the form $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$.

Lemma 4.7. — *Let $f_\pm, g_\pm$ be two germs of analytic and bounded functions in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$, which tend to 0 as $(x,\mathbf{y})\to(0,\mathbf{0})$ in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$. Then*

$$\phi_\pm : (x,\mathbf{y}) \mapsto (x, y_1\exp(f_\pm(x,\mathbf{y})), y_2\exp(g_\pm(x,\mathbf{y})))$$

defines a germ of sectorial fibered diffeomorphism analytic in $S_\pm(r,\epsilon)\times(\mathbb{C}^2,0)$ (possibly by reducing $r>0$ and the neighborhood $(\mathbb{C}^2,0)$).

Let us explain why these lemmas imply Propositions 4.3 and 4.5.

Proof of both Propositions 4.3 and 4.5. — It is sufficient to apply Lemma 4.6 with $M = N+1$, $A_\pm = -R_\pm^{(N+2)}$, $\alpha_\pm = \rho_\pm$ and $Z_\pm = Y_\pm^{(N+2)}$ for Proposition 4.3, and with $M = N$, $A_\pm = -C_\pm^{(N+1)}$, $\alpha_\pm = \chi_\pm$ and $Z_\pm = Y_{\vec{\mathcal{C}},\pm}$ for Proposition 4.5. Then we use Lemma 4.7 to see that $\varphi_\pm$ and $\psi_\pm$ are germs of sectorial fibered diffeomorphisms on the considered domains, and we finally check that they do the conjugacy we want. With the notations above:

$$
\begin{aligned}
\mathrm{d}\varphi_\pm \cdot Y_\pm^{(N+2)} &= \begin{pmatrix} L_{Y_\pm^{(N+2)}}(x) \\ L_{Y_\pm^{(N+2)}}(y_1 \exp(\rho_\pm(x,\mathbf{y}))) \\ L_{Y_\pm^{(N+2)}}(y_2 \exp(\rho_\pm(x,\mathbf{y}))) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (L_{Y_\pm^{(N+2)}}(y_1) + y_1(L_{Y_\pm^{(N+2)}}(\rho_\pm))) \exp(\rho_\pm(x,\mathbf{y})) \\ (L_{Y_\pm^{(N+2)}}(y_2) + y_2(L_{Y_\pm^{(N+2)}}(\rho_\pm))) \exp(\rho_\pm(x,\mathbf{y})) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - D_\pm(x,\mathbf{y})) y_1 \exp(\rho_\pm(x,\mathbf{y})) \\ (\lambda + a_2 x + D_\pm(x,\mathbf{y})) y_2 \exp(\rho_\pm(x,\mathbf{y})) \end{pmatrix} \\
&\qquad \left(\text{we have used } L_{Y_\pm^{(N+2)}}(\rho_\pm) = -x^{N+2} R_\pm^{(N+2)}\right) \\
&= (Y_0 + C_\pm \vec{\vec{C}}) \circ \varphi_\pm(x,\mathbf{y}) \\
&= Y_{\vec{\vec{C}},\pm} \circ \varphi_\pm(x,\mathbf{y}),
\end{aligned}
$$

so that $(\varphi_\pm)_*(Y_\pm^{(N+2)}) = Y_{\vec{\vec{C}},\pm}$ and then

$$
\begin{aligned}
\mathrm{d}\psi_\pm \cdot Y_{\vec{\vec{C}},\pm} &= \begin{pmatrix} L_{Y_{\vec{\vec{C}},\pm}}(x) \\ L_{Y_{\vec{\vec{C}},\pm}}(z_1 \exp(-\chi(x,\mathbf{z}))) \\ L_{Y_{\vec{\vec{C}},\pm}}(z_2 \exp(\chi(x,\mathbf{z}))) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (L_{Y_{\vec{\vec{C}},\pm}}(z_1) + z_1(L_{Y_{\vec{\vec{C}},\pm}}(\chi))) \exp(-\chi(x,\mathbf{z})) \\ (L_{Y_{\vec{\vec{C}},\pm}}(z_2) + z_2(L_{Y_{\vec{\vec{C}},\pm}}(\chi))) \exp(\chi(x,\mathbf{z})) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - c(z_1 z_2)) z_1 \exp(-\chi(x,\mathbf{z})) \\ (\lambda + a_2 x + c(z_1 z_2)) z_2 \exp(\chi(x,\mathbf{y})) \end{pmatrix} \\
&\qquad \left(\text{we have used } L_{Y_{\vec{\vec{C}},\pm}}(\chi_\pm) = -x^{N+1} C_\pm^{(N+1)}\right) \\
&= (Y_0 + c(u)\vec{\vec{C}}) \circ \psi_\pm(x,\mathbf{z}) \\
&= Y_{\mathrm{norm}} \circ \psi_\pm(x,\mathbf{z}),
\end{aligned}
$$

so that $(\psi_\pm)_*(Y_{\vec{\vec{C}},\pm}) = Y_{\mathrm{norm}}$.

4.3. Proof of Lemma 4.7

Proof of Lemma 4.7. — We consider two germs of analytic functions $f_\pm, g_\pm$ in $S_\pm(r,\epsilon) \times (\mathbb{C}^2, 0)$ which tend to 0 as $(x,\mathbf{y}) \to (0,\mathbf{0})$ in $S_\pm(r,\epsilon) \times$

$(\mathbb{C}^2, 0)$, and we define

$$\phi_\pm : (x, \mathbf{y}) \to (x, y_1 \exp(f_\pm(x, \mathbf{y})), y_2 \exp(g_\pm(x, \mathbf{y}))) .$$

Let us first prove that $\phi_\pm$ is into. Let $\mathbf{x} = (x, y_1, y_2)$ and $\mathbf{x}' = (x', y_1', y_2')$ in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ such that $\phi_\pm(\mathbf{x}) = \phi_\pm(\mathbf{x}')$. Since $\phi_\pm$ is fibered, necessarily $x = x'$. Then assume that $(y_1, y_2) \neq (y_1', y_2')$, such that

$$\|(y_1 - y_1', y_2 - y_2')\|_\infty > 0$$

and for instance $\|(y_1 - y_1', y_2 - y_2')\|_\infty = |y_1 - y_1'| > 0$ (the other case can be done similarly). We denote by $D_{\mathbf{y}}$ the derivative with respect to variables (y_1, y_2). According to the mean value theorem:

$$\left|\frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y_1'}\right| \leqslant \sup_{\mathbf{z} \in [\mathbf{y}, \mathbf{y}']} \left\| D_{\mathbf{y}}(e^{f_\pm})(x, \mathbf{z}) \right\|_\infty , \tag{4.7}$$

where $\mathbf{z} = (z_1, z_2)$, $\mathbf{y} = (y_1, y_2)$ and $\mathbf{y}' = (y_1', y_2')$. Consequently we have:

$$\begin{aligned}
0 &= \left| y_1 e^{f_\pm(\mathbf{x})} - y_1' e^{f_\pm(\mathbf{x}')} \right| \\
&= \left| e^{f_\pm(\mathbf{x})} \right| |y_1 - y_1'| \left| 1 + \frac{y_1'}{e^{f_\pm(\mathbf{x})}} . \frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y_1'} \right| \\
&\geqslant \left| e^{f_\pm(\mathbf{x})} \right| |y_1 - y_1'| \left(1 - \left| \frac{y_1'}{e^{f_\pm(\mathbf{x})}} \right| \left| \frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y_1'} \right| \right) \\
&\geqslant \left| e^{f_\pm(\mathbf{x})} \right| |y_1 - y_1'| \left(1 - \left| \frac{y_1'}{e^{f_\pm(\mathbf{x})}} \right| \left| \sup_{\mathbf{z} \in [\mathbf{y}, \mathbf{y}']} \left\| D_{\mathbf{y}}(e^{f_\pm})(x, \mathbf{z}) \right\|_\infty \right| \right) .
\end{aligned}$$

Assume that we chose $(\mathbb{C}^2, 0) = \mathbf{D}(\mathbf{0}, \mathbf{r})$ small enough such that $f_\pm$ is analytic in

$$S_\pm(r, \epsilon) \times D(0, 3r_1 + \delta) \times D(0, 3r_2 + \delta)$$

with $\delta > 0$ small. Without lost of generality we can take $r_1 = r_2$. We apply Cauchy's integral formula to $z_1 \mapsto e^{f_\pm(x, z_1, z_2)}$, for all fixed z_2, integrating on the circle of center 0 and radius $3r_1 = 3r_2$. Similarly we also apply Cauchy's integral formula to $z_2 \mapsto e^{f_\pm(x, z_1, z_2)}$, for all fixed z_1, integrating on the circle of center 0 and radius $3r_2 = 3r_1$. Then we obtain

$$\sup_{\mathbf{z} \in [\mathbf{y}, \mathbf{y}']} \left\| D_{\mathbf{y}}(e^{f_\pm})(x, \mathbf{z}) \right\|_\infty \leqslant \frac{3}{4r_1} . \exp\left(\sup_{\mathbf{x} \in S_\pm(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (|f_\pm(\mathbf{x})|) \right) ,$$

such that:

$$\begin{aligned} 0 &= \left| y_1 e^{f_\pm(\mathbf{x})} - y_1 e^{f_\pm(\mathbf{x})} \right| \\ &\geqslant \left| e^{f_\pm(\mathbf{x})} \right| |y_1 - y_1| \left(1 - \frac{3}{4} \exp\left(\sup_{\mathbf{x} \quad S_\pm(r,\ ) \times \mathbf{D}(\mathbf{0},\mathbf{r})} (2|f_\pm(\mathbf{x})|) \right)\right) . \end{aligned}$$

Since $f_\pm(\mathbf{x}) \underset{\mathbf{x}\to\mathbf{0}}{\longrightarrow} 0$, we can choose r, r_1 and r_2 small enough such that:

$$\exp\left(\sup_{\mathbf{x} \quad S_\pm(r,\ ) \times \mathbf{D}(\mathbf{0},\mathbf{r})} (2|f_\pm(\mathbf{x})|) \right) \leqslant \frac{5}{4} < \frac{4}{3} .$$

Finally we obtain:

$$\begin{aligned} 0 &= \left| y_1 e^{f_\pm(\mathbf{x})} - y_1 e^{f_\pm(\mathbf{x})} \right| \\ &\geqslant \left| e^{f_\pm(\mathbf{x})} \right| \frac{|y_1 - y_1|}{16} > 0. \end{aligned}$$

and so, if $y_1 = y_1$, we have $0 = |y_1 e^{f_\pm(\mathbf{x})} - y_1 e^{f_\pm(\mathbf{x})}| > 0$, which is a contradiction. Conclusion: $(y_1, y_2) = (y_1, y_2)$ and then $\phi_\pm$ is into in $S_\pm(r,\epsilon) \times (\mathbb{C}^2, 0)$. Since $\phi_\pm$ is into and analytic in $S_\pm(r,\epsilon) \times (\mathbb{C}^2, 0)$, it is a biholomorphism between $S_\pm(r,\epsilon) \times (\mathbb{C}^2, 0)$ and its image which is necessarily open (an analytic function is open), and of the same form.

4.4. Resolution of the homological equation: proof of Lemma 4.6

The goal of this subsection is to prove Lemma 4.6 by studying the existence of paths asymptotic to the singularity and tangent to the foliation, and then to use them to construct the solution to the homological equation (4.6). *For convenience and without lost of generality we assume* $\lambda = 1$ *during this subsection* (otherwise we can divide our vector field by $\lambda = 0$, make $x \to \lambda x$ and finally consider $\exp(-i\arg(\lambda)).S_\pm(r,\epsilon)$ instead of $S_\pm(r,\epsilon)$: these modifications do not change a_1 and a_2, .

4.4.1. Domain of stability and asymptotic paths

We consider

$$\begin{aligned} Z_\pm &= Y_0 + C_\pm(x,\mathbf{y}) \vec{C} + x R_\pm^{(1)}(x,\mathbf{y}) \vec{R} \\ &= \begin{pmatrix} x^2 \\ y_1(-(1 + C_\pm(x,\mathbf{y})) + a_1 x + x R_\pm^{(1)}(x,\mathbf{y})) \\ y_2(1 + C_\pm(x,\mathbf{y}) + a_2 x + x R_\pm^{(1)}(x,\mathbf{y})) \end{pmatrix} \end{aligned}$$

with $(a_1+a_2)>0$, and $C_\pm, R_\pm^{(1)}$ analytic in $S_\pm(r,\epsilon)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ and dominated by $\mathbf{y}_\infty$. More precisely, we consider the Cauchy problem of unknown $\mathbf{x}(t) := (x(t), y_1(t), y_2(t))$, with real and increasing time $t>0$, associated to

$$X_\pm := \frac{\pm i}{1+(\frac{a_2-a_1}{2})x + C_\pm} Z_\pm\,,$$

i.e.

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\pm i x^2}{1+(\frac{a_2-a_1}{2})x + C_\pm} \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = \dfrac{\pm i y_1}{1+(\frac{a_2-a_1}{2})x + C_\pm}\left(-(1+C_\pm(x,\mathbf{y})) + a_1 x + xR_\pm^{(1)}(x,\mathbf{y})\right) \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = \dfrac{\pm i y_2}{1+(\frac{a_2-a_1}{2})x + C_\pm}\left(1+C_\pm(x,\mathbf{y}) + a_2 x + xR_\pm^{(1)}(x,\mathbf{y})\right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \quad S_\pm(r,\epsilon)\times\mathbf{D}(\mathbf{0},\mathbf{r}). \end{cases} \tag{4.8}$$

We denote by $(t,\mathbf{x}_0)\to\Phi^t_{X_\pm}(\mathbf{x}_0)$ the flow of $X_\pm$ with increasing time $t>0$ and with initial point $\mathbf{x}_0$: $\Phi^0_{X_\pm}(\mathbf{x}_0)=\mathbf{x}_0$. We will prove the following:

Proposition 4.8. — *For all $\epsilon\ \ ]0,\frac{\pi}{2}[$, there exists finite sectors $S_\pm(r,\epsilon)$, $S_\pm(r\ ,\epsilon)$ with $r, r\ >0$ and an open domain $\Omega_\pm$ stable by the flow of (4.8) with increasing time $t>0$ such that*

$$S_\pm(r\ ,\epsilon)\times\mathbf{D}(\mathbf{0},\mathbf{r}\ )\quad\Omega_\pm\quad S_\pm(r,\epsilon)\times\mathbf{D}(\mathbf{0},\mathbf{r})\,,$$

(cf. Figure 4.1). Moreover, if $\mathbf{x}_0\quad\Omega_\pm$ then the corresponding solution of (4.8), namely $\mathbf{x}(t) := \Phi^t_{X_\pm}(\mathbf{x}_0)$ exists for all $t>0$ and $\mathbf{x}(t)\to\mathbf{0}$ as $t\to+\infty$.

Remark 4.9. — This will prove that the solution $\mathbf{x}(t)$ to (4.8) exists for all $t>0$ and tends to the origin: it defines a path tangent to the foliation and asymptotic to the origin. Moreover, notice that the domain $\Omega_\pm$ depends on the choice of r and $r\ >0$.

Definition 4.10. — *We define the* asymptotic path *with base point $\mathbf{x}_0\quad\Omega_\pm$ associated to $X_\pm$ the path $\gamma_{\pm,\mathbf{x}_0} := \{\Phi^t_{X_\pm}(\mathbf{x}_0),\ t>0\}$.*

For convenience and without lost of generality we only detail the case where "$\pm=+$" (the case where "$\pm=-$" is totally similar). If we write

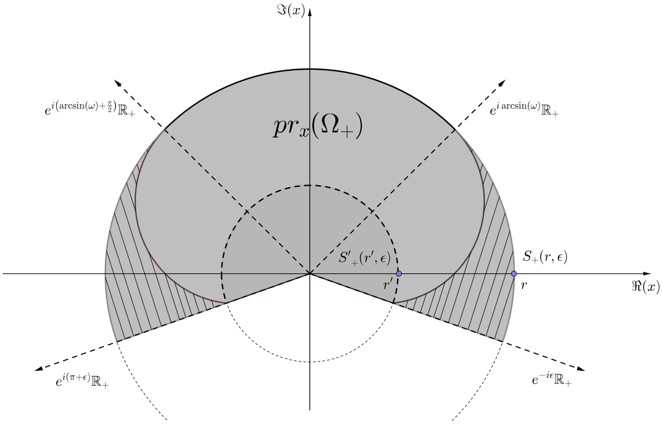

Figure 4.1. Representation of the projection $pr_x(\Omega_+)$ of the stable domain Ω_+ in the x-space.

$a := a_1 + a_2$ and $b := \frac{a_2 - a_1}{2}$, in the case "$\pm = +$" we have:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{ix^2}{1+bx+C_+(x,\mathbf{y})} \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = iy_1\left(-1+\left(\dfrac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)x\right) \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = iy_2\left(1+\left(\dfrac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)x\right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \in S_+(r,\epsilon)\times\mathbf{D}(\mathbf{0},\mathbf{r}). \end{cases}$$

We also consider the differential equations satisfied by $|x(t)|$, $|y_1|(t)$, $|y_2|(t)$ and $\theta(t) := \arg(x(t))$:

$$\begin{cases} \dfrac{\mathrm{d}|x(t)|}{\mathrm{d}t} = |x(t)|\,\Re\left(\dfrac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))}\right) \\ \dfrac{\mathrm{d}|y_1|(t)}{\mathrm{d}t} = |y_1|(t)\,\Re\left(ix(t)\left(\dfrac{\frac{a}{2}+R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))}\right)\right) \\ \dfrac{\mathrm{d}|y_2|(t)}{\mathrm{d}t} = |y_2|(t)\,\Re\left(ix(t)\left(\dfrac{\frac{a}{2}+R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))}\right)\right) \\ \dfrac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \Im\left(\dfrac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))}\right). \end{cases}$$

For any non-zero complex number ζ and positive numbers $R, B > 0$, we denote by $\Sigma_+(\zeta, R, B)$ the sector of radius R bisected by $i\overline{\zeta}\mathbb{R}_+$ and of opening $\pi - 2\arcsin(B) = 2\arccos(B)$:

$$\begin{aligned}\Sigma_+(\zeta, R, B) &:= \{x \quad D(0,R) \mid \quad (\zeta x) > B|\zeta x|\} \\ &= \{x \quad D(0,R) \mid -\arccos(B) < \arg(x) - \arg(i\bar{\zeta}) < \arccos(B)\}.\end{aligned}$$

For $T, R > 0$, we denote by $\Theta_+(R,T)$ (resp. $\Theta_-(R,T)$) the sector of radius R bisected by $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) and of opening $2\arccos(T)$:

$$\begin{aligned}\Theta_+(R,T) &:= \{x \quad D(0,R) \mid \quad (x) > T|x|\} \\ &= \{x \quad D(0,R) \mid -\arccos(T) < \arg(x) < \arccos(T)\} \\ \Theta_-(R,T) &:= \{x \quad D(0,R) \mid \quad (x) < -T|x|\} \\ &= \{x \quad D(0,R) \mid -\arccos(T) < \arg(x) - \pi < \arccos(T)\}\end{aligned}$$

Since $\quad (a) > 0$ by assumption, we can choose $\omega \quad]0, \frac{(a)}{|a|}[$, such that $\Sigma_+(a, r, \omega\ )$ contains $i\mathbb{R}_{>0}$. Indeed, we have

$$|\arg(i) - \arg(i\bar{a})| = |\arg(a)| < \arccos(\omega\ )\,.$$

In particular, we have:

$$0 < \arccos(\omega\ ) - |\arg(a)| < \frac{\pi}{2}$$

so that

$$0 < \cos(\arccos(\omega\ ) - |\arg(a)|) < 1\,.$$

Hence we take $\omega > 0$ such that

$$\omega \quad]\cos(\arccos(\omega\ ) - |\arg(a)|), 1[, \tag{4.9}$$

and then $\Sigma_+(1, r, \omega) \quad \Sigma_+(a, r, \omega\ )$. Indeed, if $x \quad \Sigma_+(1, r, \omega)$, then:

$$-\arccos(\omega) < \arg(x) - \frac{\pi}{2} < \arccos(\omega)\,, \tag{4.10}$$

and therefore

$$\begin{aligned}|\arg(x) - \arg(i.\bar{a})| &< \arccos(\omega) + |\arg(a)| && \text{(by (4.10))} \\ &< \arccos(\omega\ ) && \text{(by (4.9))}.\end{aligned}$$

Finally, we fix $\mu \quad]0, \sqrt{1-\omega^2}[$ small enough such that

$$\begin{aligned}\Theta_+(r,\mu) \cap \Sigma_+(1, r, \omega) &= \\ \Theta_-(r,\mu) \cap \Sigma_+(1, r, \omega) &=\end{aligned}$$

and

$$S_+(r,\epsilon) \quad \Sigma_+(1, r, \omega) \quad \Theta_+(r,\mu) \quad \Theta_-(r,\mu)\,.$$

More precisely, we must have $0 < \epsilon < \arccos(\mu)$. The idea is now to study the behavior of $t \to x(t)$ (where $t \to \mathbf{x}(t) = (x(t), y_1(t), y_2(t))$ is the solution of (4.8)) over each domains $\Sigma_+(1, r, \omega), \Theta_+(r,\mu), \Theta_-(r,\mu)$ (cf. Figure 4.2).

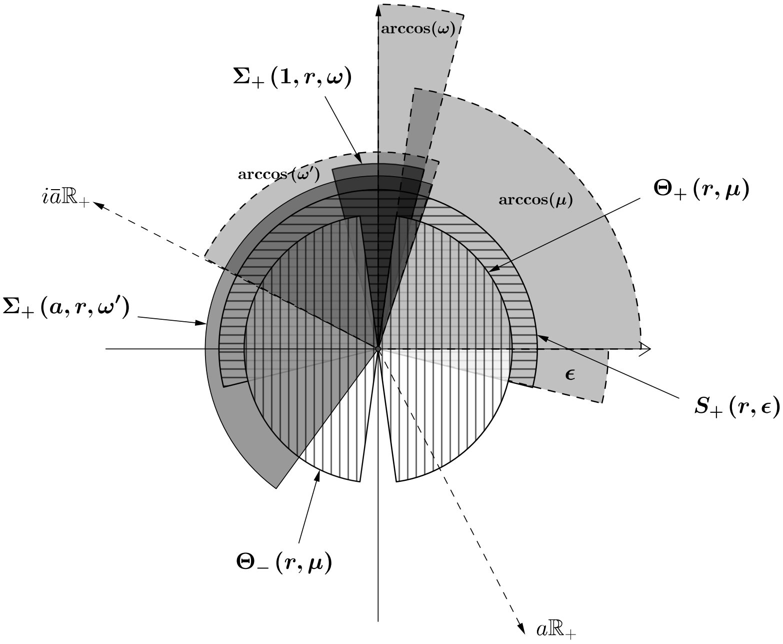

Figure 4.2. Representation of domains $\Sigma_+(1,r,\omega)$, $\Sigma_+(a,r,\omega\ )$, $\Theta_+(r,\mu)$, $\Theta_-(r,\mu)$, $S_+(r,\epsilon)$ *(with modified radii for more clarity).*

We can now prove the following result, which is a precision of Proposition 4.8.

Lemma 4.11.

(1) *There exists* $r, r_1, r_2 > 0$ *such that* $\Sigma_+(1,r,\omega) \times \mathbf{D}(\mathbf{0},\mathbf{r})$ *is stable by the flow of* (4.8) *with increasing time* $t > 0$*. Moreover in this region* $|x(t)|$, $|y_1|(t)$ *and* $|y_2|(t)$ *decrease and go to* 0 *as* $t \to +\infty$*.*

(2) *There exists* $0 < r\ < r$, $0 < r_1\ < r_1$, $0 < r_2\ < r_2$ *and an open domain* Ω_+ *stable under the action flow of* (4.8) *with increasing time* $t > 0$ *such that*

$$S_+(r\ ,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r}\ ) \quad \Omega_+ \quad S_+(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r})\,.$$

Moreover, if $x_0 \quad \Theta_+(r\ ,\mu)$ *(resp.* $x_0 \quad \Theta_-(r\ ,\mu)$*), then* $\theta(t) = \arg(x(t)), t > 0$ *is increasing (resp. decreasing) as long as* $x(t)$

remains in $\Theta_+(r',\mu)$ (resp. $\Theta_-(r',\mu)$). Finally, there exists $t_0 > 0$ such that for all $t > t_0$, $x(t) \in \Sigma_+(1,r,\omega)$.

Proof. — We fix $\delta \in]0,\min(\omega,\mu)[$, $\delta' \in]0,\omega'[$ and we take $r > 0$ small enough such that for all $\mathbf{x} = (x,\mathbf{y}) \in S_+(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r})$, we have

$$\begin{cases} \left|\dfrac{1}{1+bx+C_+(\mathbf{x})} - 1\right| < \delta \\ \left|\dfrac{\frac{a}{2}+R_+^{(1)}(\mathbf{x})}{1+bx+C_+(\mathbf{x})} - \frac{a}{2}\right| < \delta' . \end{cases}$$

Consequently for all $\mathbf{x} \in S_+ \times \mathbf{D}(\mathbf{0},\mathbf{r})$ we have the following estimations:

$$\begin{cases} -|x|(1+\delta) < \Re\left(\dfrac{ix}{1+bx+C_+(\mathbf{x})}\right) < |x|(1+\delta) \\ -|x|\left(\left|\dfrac{a}{2}\right|+\delta'\right) < \Re\left(ix\left(\dfrac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right) < |x|\left(\left|\dfrac{a}{2}\right|+\delta'\right) . \end{cases}$$

Moreover:

- if $x \in \Sigma_+(1,r,\omega)$ then

$$\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < -|x|(\omega-\delta)\,; \tag{4.11}$$

- if $x \in \Sigma_+(a,r,\omega')$ (in particular if $x \in \Sigma_+(1,r,\omega)$) then

$$\Re\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right) < -|x|(\omega'-\delta')\,; \tag{4.12}$$

- if $x \in \Theta_-(r,\mu)$ (resp. $\Theta_+(r,\mu)$) then

$$\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < -|x|(\mu-\delta)$$
$$\left(\text{resp. } \Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) > |x|(\mu-\delta)\right) .$$

Hence:

- for all $t > 0$

$$-(1+\delta)|x(t)|^2 < \frac{\mathrm{d}|x(t)|}{\mathrm{d}t} < -(1+\delta)|x(t)|^2 \tag{4.13}$$

and then, as long as $\mathbf{x}(t) \in S_+(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r})$, we have

$$|x(t)| > \frac{|x_0|}{1+(1+\delta)|x_0|t}\,;$$

- for all $t > 0$, if $x(t) \in \Sigma_+(1, r, \omega)$, then

$$\frac{\mathrm{d}|x(t)|}{\mathrm{d}t} < -(\omega - \delta)|x(t)|^2 \tag{4.14}$$

  and

$$\begin{cases} \dfrac{\mathrm{d}|y_1|(t)}{\mathrm{d}t} < -(\omega - \delta)|y_1|(t)|x(t)| \\ \dfrac{\mathrm{d}|y_2|(t)}{\mathrm{d}t} < -(\omega - \delta)|y_2|(t)|x(t)| \end{cases} \tag{4.15}$$

  so that $|x(t)|, |y_1|(t)$ and $|y_2|(t)$ are decreasing as long as $x(t) \in \Sigma_+(1, r, \omega)$;
- for all $t > 0$, if $x(t) \in \Theta_-(r, \mu)$ (resp. $\Theta_+(r, \mu)$) then

$$\left(\begin{array}{l} \dfrac{\mathrm{d}\theta}{\mathrm{d}t}(t) < -(\mu - \delta)|x(t)| < \dfrac{-(\mu - \delta)|x_0|}{1 + (1 + \delta)|x_0|t} \\ \text{resp. } \dfrac{\mathrm{d}\theta}{\mathrm{d}t}(t) > (\mu - \delta)|x(t)| > \dfrac{(\mu - \delta)|x_0|}{1 + (1 + \delta)|x_0|t} \end{array} \right)$$

  so that $t \mapsto \theta(t)$ is strictly decreasing (resp. increasing) as long as $x(t) \in \Theta_-(r, \mu)$ (resp. $\Theta_+(r, \mu)$). Moreover, if $\theta_0 = \theta(0)$ is such that $x_0 = x(0) \in \Theta_-(t, \mu) \setminus \Sigma_+(1, r, \omega)$ (resp. $\Theta_+(r, \mu) \setminus \Sigma_+(1, r, \omega)$), then as long as $x(t) \in \Theta_-(r, \mu)$ (resp. $\Theta_+(r, \mu)$) we have:

$$\left(\begin{array}{l} \theta(t) < \theta_0 - (\dfrac{\mu - \delta}{1 + \delta}) \ln(1 + (1 + \delta)|x_0|t) \\ \text{resp. } \theta(t) > \theta_0 + (\dfrac{\mu - \delta}{1 + \delta}) \ln(1 + (1 + \delta)|x_0|t) \end{array} \right).$$

  We see that $x(t) \in \Sigma_+(1, r, \omega)$ for all

$$\left(\begin{array}{l} t > t_0 := \dfrac{(\exp(\frac{1+\delta}{\mu-\delta}(\theta_0 - \frac{\pi}{2} - \arccos(\omega))) - 1)}{(1 + \delta)|x_0|} \\ \text{resp. } t_0 := \dfrac{(\exp(\frac{1+\delta}{\mu-\delta}(\frac{\pi}{2} - \arccos(\omega) - \theta_0)) - 1)}{(1 + \delta)|x_0|} \end{array} \right).$$

  Indeed, if $t > t_0$, with t_0 as above, and if $x(t) \in \Theta_+(r, \mu)$, then we have:

$$\begin{aligned} \theta(t) &> \theta_0 + \left(\frac{\mu - \delta}{1 + \delta} \right) \ln(1 + (1 + \delta)|x_0|t) \\ &> \theta_0 + \left(\frac{\mu - \delta}{1 + \delta} \right) \ln\left(\exp\left(\left(\frac{1 + \delta}{\mu - \delta} \right) \left(\theta_0 - \frac{\pi}{2} - \arccos(\omega) \right) \right) \right) \\ &= \theta_0 + \frac{\pi}{2} - \arccos(\omega) - \theta_0 = \frac{\pi}{2} - \arccos(\omega) \end{aligned}$$

and therefore

$$-\arccos(\omega) < \arg(x(t)) - \frac{\pi}{2} < 0\,.$$

Hence, we have $x(t) \quad \Sigma_+(1,r,\omega)$. Moreover, notice that

$$t_0 \leqslant \frac{\exp((\frac{1+\delta}{\mu-\delta})(\epsilon+\arcsin(\omega)))}{(1+\delta)|x_0|}\,. \tag{4.16}$$

On the one hand $\Sigma_+(1,r,\omega)\times\mathbf{D}(\mathbf{0},\mathbf{r})$ is stable by the flow of (4.8) with increasing time $t \geqslant 0$. Indeed in this region $|x(t)|, |y_1|(t)$ and $|y_2|(t)$ are decreasing, and as soon as $x(t)$ goes in $\Sigma_+(1,r,\omega)\cap\Theta_-(r,\mu)$ (resp. $\Sigma_+(1,r,\omega)\cap\Theta_+(r,\mu)$), which is non-empty and contains a part of the boundary of $\Sigma_+(1,r,\omega)$ with constant argument, $\theta(t)$ is decreasing (resp. increasing). Then, $x(t)$ remains in $\Sigma_+(1,r,\omega)$. On the other hand, as long as we are $x(t)$ belongs to $\Theta_-(r,\mu)$ (resp. $\Theta_+(r,\mu)$) we can re-parametrized the solutions by $(-\theta)$ (*resp* θ) (we are now going to make an abuse of notation, writing when needed $x(\theta)$ or $x(t)$):

$$\begin{cases}
\dfrac{\mathrm{d}|x|}{\mathrm{d}(-\theta)} = -|x|\dfrac{(\frac{ix}{1+bx+C_+(\mathbf{x})})}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |x|.\dfrac{1+\delta}{\mu-\delta} \\
\left(\text{resp. } \dfrac{\mathrm{d}|x|}{\mathrm{d}\theta} = |x|\dfrac{(\frac{ix}{1+bx+C_+(\mathbf{x})})}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |x|.\dfrac{1+\delta}{\mu-\delta}\right) \\
\dfrac{\mathrm{d}|y_1|}{\mathrm{d}(-\theta)} = -|y_1|\dfrac{\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right)}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |y_1|.\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta} \\
\left(\text{resp. } \dfrac{\mathrm{d}|y_1|}{\mathrm{d}\theta} = |y_1|\dfrac{\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right)}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |y_1|.\frac{|\frac{a}{2}|+\delta}{\mu-\delta}\right) \\
\dfrac{\mathrm{d}|y_2|}{\mathrm{d}(-\theta)} = -|y_2|\dfrac{\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right)}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |y_2|.\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta} \\
\left(\text{resp. } \dfrac{\mathrm{d}|y_2|}{\mathrm{d}\theta} = |y_2|\dfrac{\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right)}{(\frac{ix}{1+bx+C_+(\mathbf{x})})} \leqslant |y_2|.\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}\right)
\end{cases}.$$

Hence, if $\theta_0 := \theta(0)$ is such that $x_0 := x(0) \in \Theta_-(r,\mu)$ (resp. $\Theta_+(r,\mu)$), for $t \leqslant t_0$ we have:

$$
\begin{cases}
|x(t)| \leqslant |x_0| \exp\left(\dfrac{1+\delta}{\mu-\delta}(\theta_0 - \theta(t))\right) \\
\quad \left(\text{resp. } |x(t)| \leqslant |x_0| \exp\left(\dfrac{1+\delta}{\mu-\delta}(\theta(t) - \theta_0)\right)\right) \\
|y_1|(t) \leqslant |y_{1,0}| \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\theta_0 - \theta(t))\right) \\
\quad \left(\text{resp. } |y_1|(t) \leqslant |y_{1,0}| \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\theta(t) - \theta_0)\right)\right) \\
|y_2|(t) \leqslant |y_{2,0}| \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\theta_0 - \theta(t))\right) \\
\quad \left(\text{resp. } |y_1|(t) \leqslant |y_{1,0}| \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\theta(t) - \theta_0)\right)\right)
\end{cases}. \tag{4.17}
$$

Definition 4.12. — *We define the domain Ω_+ as the set of all*

$$\mathbf{x} = (x, y_1, y_2) \in S_+(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r})$$

such that:

- if $\Im(x) \geqslant \omega|x|$ then:

$$
\begin{cases}
|x| \leqslant r \exp\left(\dfrac{1+\delta}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\
|y_1| \leqslant r_1 \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\
|y_2| \leqslant r_2 \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right)
\end{cases};
$$

- if $\Im(x) \leqslant -\omega|x|$ then:

$$
\begin{cases}
|x| \leqslant r \exp\left(\dfrac{1+\delta}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\
|y_1| \leqslant r_1 \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\
|y_2| \leqslant r_2 \exp\left(\dfrac{|\frac{a}{2}|+\delta}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right)
\end{cases}.
$$

We see that Ω_+ is stable by the flow of (4.8) with increasing time $t \geqslant 0$. We have seen that for any initial condition in Ω_+, the solution exists for any $t \geqslant 0$, stays in Ω_+, and after a finite time $t_0 \geqslant 0$ enters and remains in $\Sigma_+(1, r, \omega)$. Finally, we have:

$$S_+(r', \epsilon') \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_+ \subset S_+(r,\epsilon) \times \mathbf{D}(\mathbf{0},\mathbf{r}),$$

where

$$\begin{cases} r = r\exp\left(-\left(\frac{1+\delta}{\mu-\delta}\right)(\epsilon+\arcsin(\omega))\right) < r \\ r_1 = r_1\exp\left(-\left(\frac{\left|\frac{a}{2}\right|+\delta}{\mu-\delta}\right)(\epsilon+\arcsin(\omega))\right) < r_1 \\ r_2 = r_2\exp\left(-\left(\frac{\left|\frac{a}{2}\right|+\delta}{\mu-\delta}\right)(\epsilon+\arcsin(\omega))\right) < r_2\,. \end{cases}$$

Let $\mathbf{x}_0 = (x_0, \mathbf{y}_0) \in \Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. From (4.14) and (4.15) we have for all $t \geqslant 0$:

$$\begin{cases} |x(t)| \leqslant \frac{|x_0|}{1+(\omega-\delta)|x_0|t} \\ |y_1|(t) \leqslant \frac{|y_{1,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\omega-\delta}{1+\delta}}} \\ |y_1|(t) \leqslant \frac{|y_{2,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\omega-\delta}{1+\delta}}}\,, \end{cases} \tag{4.18}$$

which proves that the solutions goes to $\mathbf{0}$ as $t \to +\infty$.

Remark 4.13. — Another stable domain Ω_- is defined similarly when dealing with the case "$\pm = -$"

4.4.2. Construction of a sectorial analytic solution to the homological equation

We consider the meromorphic 1-form $\tau := \frac{\mathrm{d}x}{x^2}$, which satisfies $\tau\cdot(Z_\pm) = 1$. Let also $A_\pm(x, \mathbf{y})$ be analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_\infty$, and $M \in \mathbb{N}_{>0}$. The following proposition is a precision of Lemma 4.6.

Proposition 4.14. — *For all* $\mathbf{x}_0 \in \Omega_\pm$ *(see Definition 4.12), the integral defined by*

$$\alpha_\pm(\mathbf{x}_0) := -\int_{\gamma_{\pm,\mathbf{x}_0}} x^{M+1} A_\pm(\mathbf{x})\,\tau$$

is absolutely convergent (the integration path $\gamma_{\pm,\mathbf{x}_0}$ *is the one of Definition 4.10). Moreover, the function* $\mathbf{x}_0 \mapsto \alpha_\pm(\mathbf{x}_0)$ *is analytic in* $\Omega_\pm$*, satisfies*

$$L_{Z_\pm}(\alpha_\pm) = x^{M+1} A_\pm(\mathbf{x})$$

and $\alpha_\pm(x, \mathbf{y}) = x^M \tilde{\alpha}_\pm(x, \mathbf{y})$*, where* $\tilde{\alpha}_\pm$ *is analytic on* $\Omega_\pm$ *and dominated by* $\|\mathbf{y}\|_\infty$*.*

Proof. — We are going to use the estimations obtained in the previous paragraph.

- Let us start by proving that the integral above is convergent. We begin with:

$$\alpha_\pm(\mathbf{x}_0) = -\int_0^{+\infty} \frac{x(t)^{M+1} A_\pm(\mathbf{x}(t))}{x(t)^2} \frac{ix(t)^2}{1 + bx(t) + C_+(\mathbf{x}(t))} \mathrm{d}t$$
$$= -i \int_0^{+\infty} \frac{x(t)^{M+1} A_\pm(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \mathrm{d}t \,.$$

Since $\mathbf{x}(t) \in \Omega_\pm$ for all $t \geqslant 0$ and $A_\pm(x, \mathbf{y})$ is dominated by $\|\mathbf{y}\|_\infty$, we have then:

$$\left| \frac{x(t)^{M+1} A_\pm(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| \leqslant C |x(t)|^{M+1} \|\mathbf{y}(t)\|_\infty \tag{4.19}$$

where $C > 0$ is some constant, independent of $\mathbf{x}_0$ and t. For $t \geqslant 0$ big enough, we deduce from paragraph 4.4.1 that:

$$\left| \frac{x(t)^{M+1} A_\pm(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right|$$
$$\leqslant C \|\mathbf{y}_0\| \left(\frac{|x_0|}{1 + (\omega - \delta)|x_0| t} \right)^{M+1} \frac{1}{(1 + (1 + \delta)|x_0| t)^{\frac{\omega - \delta}{1 + \delta}}}$$
$$= \underset{t \to +\infty}{\mathrm{O}} \left(\frac{1}{t^{M+1}} \right)$$

and then the integral is absolutely convergent.

- Let us prove the analyticity of $\alpha_\pm$ in $\Omega_\pm$: it is sufficient to prove that it is analytic in every compact $K \subset \Omega_\pm$. Let K be such a compact subset. Let $L > 0$ such that for all $\mathbf{x} \in K$, we have:

$$\left| \frac{A_\pm(\mathbf{x})}{1 + bx + C_+(\mathbf{x})} \right| \leqslant L.$$

Since K in a compact subset of $\Omega_\pm \subset S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and $S_\pm(r, \epsilon)$ is open ($0 \notin S_\pm(r, \epsilon)$), there exists $\delta > 0$ such that for all $\mathbf{x} = (x, y_1, y_2) \in K$, we have $\delta < |x| < r$. Finally, according to the several estimates in paragraph 4.4.1, there exists $B > 0$ such that for all $\mathbf{x}_0 \in K$ and $t \geqslant 0$, we have:

$$|x(t)| \leqslant B \frac{|x_0|}{1 + (\omega - \delta)|x_0| t} \,.$$

Hence:

$$\left| \frac{x(t)^{M+1} A_\pm(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| \leqslant L B^{M+1} \frac{|x_0|^{M+1}}{(1 + (\omega - \delta)|x_0| t)^{M+1}}$$
$$\leqslant \frac{L B^{M+1} r^{M+1}}{(1 + (\omega - \delta)\delta t)^{M+1}},$$

and the classical theorem concerning the analyticity of integral with parameters proves that $\alpha_\pm$ is analytic in any compact $K \subset \Omega_\pm$, and consequently in $\Omega_\pm$.

- Let us write $F(\mathbf{x}) := \frac{\pm i x^{M+1} A_\pm(\mathbf{x})}{1+bx+C_+(\mathbf{x})}$, so that

$$\alpha_\pm(\mathbf{x}_0) = -\int_0^{+\infty} F(\Phi^t_{X_\pm}(\mathbf{x}_0))\mathrm{d}t\,.$$

For all $\mathbf{x}_0 \in \Omega_\pm$, the function $t \to \mathbf{x}(t) = \Phi^t_{X_\pm}(\mathbf{x}_0)$ satisfies:

$$\frac{\partial}{\partial t}(\Phi^t_{X_\pm}(\mathbf{x}_0)) = \frac{\pm i}{1 + b|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))| + C_+(\Phi^t_{X_\pm}(\mathbf{x}_0))} Z_\pm(\Phi^t_\pm(\mathbf{x}))\,.$$

The classical theorem about the analyticity of integral with parameters tells us that we can compute the derivatives inside the integral symbol:

$$\begin{aligned}
&(L_{Z_\pm}\alpha_\pm)(\mathbf{x}_0)\\
&= -\int_0^{+\infty} L_{Z_\pm}(F\circ\Phi^s)(\mathbf{x}_0)\mathrm{d}s\\
&= -\int_0^{+\infty} \mathrm{d}F(\Phi^s_{X_\pm}(\mathbf{x}_0)).\mathrm{d}\Phi^s_{X_\pm}(\mathbf{x}_0).Z_\pm(\mathbf{x}_0)\mathrm{d}s\\
&= -\int_0^{+\infty} \mathrm{d}F(\Phi^s_{X_\pm}(\mathbf{x})).\frac{\partial}{\partial t}(\Phi^{s+t}_{X_\pm}(\mathbf{x}_0))_{|t=0}\left(\pm\frac{1+bx_0+C_\pm(\mathbf{x}_0)}{i}\right)\mathrm{d}s\\
&= -\left(\pm\frac{1+bx_0+C_\pm(\mathbf{x}_0)}{i}\right).\int_0^{+\infty} \mathrm{d}F(\Phi^s_{X_\pm}(\mathbf{x}_0)).\frac{\partial}{\partial t}(\Phi^t_{X_\pm}(\mathbf{x}_0))_{|t=s}\mathrm{d}s\\
&= -\left(\pm\frac{1+bx_0+C_\pm(\mathbf{x}_0)}{i}\right).\int_0^{+\infty} \frac{\partial}{\partial s}(F\circ\Phi^s_{X_\pm}(\mathbf{x}_0))ds\\
&= -\left(\pm\frac{1+bx_0+C_\pm(\mathbf{x}_0)}{i}\right).[F\circ\Phi^s_{X_\pm}(\mathbf{x}_0)]^{s=+\infty}_{s=0}\\
&= -\left(\pm\frac{1+bx_0+C_\pm(\mathbf{x}_0)}{i}\right).(-F(\mathbf{x}_0))\\
&= x_0^{M+1}A_\pm(\mathbf{x}_0).
\end{aligned}$$

- Let us prove that $\tilde{\alpha}_\pm(x,\mathbf{y}) := \frac{\alpha_\pm(x,\mathbf{y})}{x^M}$ is bounded and dominated by $\|\mathbf{y}\|_\infty$ in $\Omega_\pm$. The fact that it is analytic in $\Omega_\pm$ is clear because $\alpha_\pm$ is analytic there and $0 \notin \Omega_\pm$. As above, there exists there exists $C > 0$ such that for all $\mathbf{x}_0 := (x_0, \mathbf{y}_0) \in \Omega_\pm$ and for all $t > 0$:

$$\left|\frac{x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))^{M+1}A_\pm(\Phi^t_{X_\pm}(\mathbf{x}_0))}{(1+bx(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))+C_+(\Phi^t_{X_\pm}(\mathbf{x}_0)))}\right| \leqslant C\|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))\|^{M+1}\|\mathbf{y}(\Phi^t_{X_\pm}(\mathbf{x}_0))\|_\infty.$$

We will only deal with the case where $x_0 \in \Theta_\pm(r,\mu)$ (the case where $\Sigma_\pm(1,r,\omega)$ is easier and can be deduced from that case). On the one hand from (4.17) we have for all $t \leq t_0$:

$$\begin{cases} \left|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))\right| \leq D|x_0|, & \text{where } D := \exp\left(\frac{1+\delta}{\mu-\delta}(\arccos(\mu)+\epsilon)\right) \\ \left\|\mathbf{y}(\Phi^{t}_{X_\pm}(\mathbf{x}_0))\right\|_\infty \leq D'\left\|\mathbf{y}_0\right\|_\infty, & \text{where } D' := \exp\left(\frac{\left|\frac{a}{2}\right|+\delta}{\mu-\delta}(\arccos(\mu)+\epsilon)\right). \end{cases}$$

On the other hand we have seen in (4.18) that for all $t \geq t_0$:

$$\begin{cases} \left|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))\right| \leq \dfrac{\left|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))\right|}{1+(\omega-\delta)\left|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))\right|(t-t_0)} \\ \left\|\mathbf{y}(\Phi^{t}_{X_\pm}(\mathbf{x}_0))\right\|_\infty \leq \left\|\mathbf{y}_0\right\|_\infty. \end{cases}$$

Hence, we use the Chasles relation and the estimations above to obtain:

$$\begin{aligned} |\tilde{\alpha}_\pm(x_0,\mathbf{y}_0)| &\leq \frac{|\alpha_\pm(x_0,\mathbf{y}_0)|}{|x_0|^M} \\ &\leq \frac{CD^{M+1}D'\left\|\mathbf{y}_0\right\|_\infty |x_0|^{M+1}|t_0|}{|x_0|^M} \\ &\quad + \frac{C\left\|\mathbf{y}_0\right\|_\infty}{|x_0|^M}\int_{t_0}^{+\infty}\frac{\mathrm{d}t}{(1+(\omega-\delta)|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))|(t-t_0))} \\ &\leq CD^{M+1}D'\left\|\mathbf{y}_0\right\|_\infty |x_0||t_0| \\ &\quad + \frac{C\left\|\mathbf{y}_0\right\|_\infty |x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))|^{M+1}}{M(\omega-\delta)|x_0|^M\,|x(\Phi^{t_0}_{X_\pm}(\mathbf{x}_0))|}; \end{aligned}$$

and according to (4.16) we have

$$|\tilde{\alpha}_\pm(x_0,\mathbf{y}_0)| \leq \left(\frac{D^2D'}{(1+\delta)} + \frac{1}{M(\omega-\delta)}\right)CD^M\left\|\mathbf{y}_0\right\|_\infty. \tag{4.20}$$

4.5. Sectorial isotropies in "wide" sectors and uniqueness of the normalizations: proof of Proposition 1.13.

We consider a normal form Y_{norm} as given by Corollary 4.2. We study here the germs of sectorial isotropies of the normal form Y_{norm} in $S_\pm \times (\mathbb{C}^2,0)$, where $S_\pm \in \mathcal{S}_{\arg(\pm i\lambda),\eta}$ is a sectorial neighborhood of the origin with opening $\eta \in \,]\pi,2\pi[$ in the direction $\arg(\pm i\lambda)$. Proposition 1.13 states that the normalizing maps (Φ_+,Φ_-) are unique as sectorial germs. It is a straightforward consequence of Proposition 4.16 below, which show that the only sectorial fibered isotropy (tangent to the identity) of the normal form in over "wide" sector (i.e. of opening $> \pi$) is the identity itself.

Definition 4.15. — *A germ of sectorial fibered diffeomorphism $\Phi_{\theta,\eta}$ in the direction $\theta \in \mathbb{R}$ with opening $\eta > 0$ and tangent to the identity, is a germ of fibered sectorial* isotropy *of Y_{norm} (in the direction $\theta \in \mathbb{R}$ with opening $\eta > 0$ and tangent to the identity) if $(\Phi_{\theta,\eta})_* (Y_{\text{norm}}) = Y_{\text{norm}}$ in $S \subset S_{\theta,\eta}$. We denote by $\text{Isot}_{\text{fib}}(Y, S_{\theta,\eta}; \text{Id}) \subset \text{Diff}_{\text{fib}}(S_{\theta,\eta}; \text{Id})$ the subset formed composed of these elements.*

Proposition 1.13 is an immediate consequence of the following one.

Proposition 4.16. — *For all $\eta \in]\pi, 2\pi[$:*

$$\text{Isot}_{\text{fib}}(Y_{\text{norm}}, S_{\arg(\pm i\lambda),\eta}; \text{Id}) = \{\text{Id}\}\,.$$

Proof. — Let

$$\phi : (x, \mathbf{y}) \mapsto (x, \phi_1(x,\mathbf{y}), \phi_2(x,\mathbf{y})) \in \text{Isot}_{\text{fib}}(Y_{\text{norm}}, S_{\arg(\pm i\lambda),\eta}; \text{Id})$$

be a germ of a sectorial fibered isotropy (tangent to the identity) of Y_{norm} in $S_\pm \subset S_{\arg(\pm i\lambda),\eta}$ with $\eta \in]\pi, 2\pi[$. Possibly by reducing our domain, we can assume that $S_\pm$ is bounded and of the form $S_\pm \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ (where, as usual, $S_\pm$ is an adapted sector and $\mathbf{D}(\mathbf{0}, \mathbf{r})$ a polydisc), and that ϕ is bounded in this domain. We have

$$\phi_* (Y_{\text{norm}}) = Y_{\text{norm}}$$

i.e.

$$\mathrm{d}\phi \cdot Y_{\text{norm}} = Y_{\text{norm}} \circ \phi$$

which is also equivalent to:

$$\tag{4.21} \begin{cases} x^2 \dfrac{\partial \phi_1}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \dfrac{\partial \phi_1}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \dfrac{\partial \phi_1}{\partial y_2} \\ \qquad = \phi_1 (-1 - c(\phi_1 \phi_2) + a_1 x) \\ x^2 \dfrac{\partial \phi_2}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \dfrac{\partial \phi_2}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \dfrac{\partial \phi_2}{\partial y_2} \\ \qquad = \phi_2 (1 + c(\phi_1 \phi_2) + a_2 x)\,. \end{cases}$$

Let us consider $\psi := \phi_1 \phi_2$. Then

$$x^2 \frac{\partial \psi}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \psi}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \psi}{\partial y_2} = (a_1 + a_2) x \psi.$$

By assumption we can write

$$\psi(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 2} \psi_{j_1, j_2}(x) y_1^{j_1} y_2^{j_2}\,,$$

where $\psi_{j_1,j_2}(x)$ is analytic and bounded in $S_\pm$ for all $j_1, j_2 \geqslant 0$ and such that

$$\sum_{j_1+j_2\geqslant 1} \Big(\sup_{x\in S_\pm} (|\psi_{j_1,j_2}(x)|)\Big) y_1^{j_1} y_2^{j_2}$$

is convergent near the origin of $\mathbb{C}^2$ (e.g. in $\mathbf{D}(\mathbf{0},\mathbf{r})$). Consequently, with an argument of uniform convergence in every compact subset, we have for all $j_1, j_2 \geqslant 0$:

$$x^2 \frac{\mathrm{d}\psi_{j_1;j_2}}{\mathrm{d}x}(x) + (j_2 - j_1 + (a_1(j_1-1) + a_2(j_2-1))x)\psi_{j_1,j_2}(x) = (j_1 - j_2) \sum_{l=1}^{\min(j_1,j_2)} \psi_{j_1-l,j_2-l}(x) c_l\,.$$

For $j_1 = j_2 = j \geqslant 1$, we have

$$\psi_{j,j}(x) = b_{j,j} x^{-(j-1)(a_1+a_2)}, \qquad b_{j,j} \in \mathbb{C}.$$

Since $\Re(a_1 + a_2) > 0$, the function $x \mapsto \psi_{j,j}(x)$ is bounded near the origin if and only if $b_{j,j} = 0$ or $j = 1$. For $j_1 > j_2$, we see recursively that $\psi_{j_1,j_2}(x) = 0$. Indeed, we obtain by induction that

$$\psi_{j_1,j_2}(x) = b_{j_1,j_2} \exp\left(\frac{j_2 - j_1}{x}\right) x^{-(a_1(j_1-1)+a_2(j_2-1))},$$

and since it has to be bounded on $S_\pm$, we necessarily have $b_{j_1,j_2} = 0$. Similarly, for $j_1 < j_2$, we see recursively that $\psi_{j_1,j_2}(x) = 0$. As a conclusion, $\psi(x, \mathbf{y}) = b_{1,1}y_1y_2 = y_1y_2$ (we must have $b_{1,1} = 1$ since ϕ is tangent to the identity). We can now solve separately each equation in (4.21):

$$\begin{cases} x^2 \dfrac{\partial\phi_1}{\partial x} + (-1 - c(y_1y_2) + a_1x)y_1 \dfrac{\partial\phi_1}{\partial y_1} + (1 + c(y_1y_2) + a_2x)y_2 \dfrac{\partial\phi_1}{\partial y_2} \\ \qquad = \phi_1(-1 - c(y_1y_2) + a_1x) \\ x^2 \dfrac{\partial\phi_2}{\partial x} + (-1 - c(y_1y_2) + a_1x)y_1 \dfrac{\partial\phi_2}{\partial y_1} + (1 + c(y_1y_2) + a_2x)y_2 \dfrac{\partial\phi_2}{\partial y_2} \\ \qquad = \phi_2(1 + c(y_1y_2) + a_2x)\,. \end{cases}$$

As above for $i = 1, 2$ we can write

$$\phi_i(x, \mathbf{y}) = \sum_{j_1+j_2\geqslant 1} \phi_{i,j_1,j_2}(x) y_1^{j_1} y_2^{j_2}\,,$$

where $\phi_{i,j_1,j_2}(x)$ is analytic and bounded in $S_\pm$ for all $j_1, j_2 \geqslant 0$ and such that

$$\sum_{j_1+j_2\geqslant 1} \left(\sup_{x\in S_\pm} (|\phi_{i,j_1,j_2}(x)|)\right) y_1^{j_1} y_2^{j_2}$$

is a convergent entire series near the origin of $\mathbb{C}^2$ (e.g. in $\mathbf{D}(\mathbf{0},\mathbf{r})$). As above, using the uniform convergence in every compact subset and identifying terms of same homogeneous degree (j_1, j_2), we obtain:

$$\left\{\begin{aligned} & x^2\frac{\mathrm{d}\phi_{1,j_1;j_2}}{\mathrm{d}x}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2 j_2)x)\phi_{1,j_1,j_2}(x) \\ & \quad = \sum_{l=1}^{\min(j_1,j_2)} \phi_{1,j_1-l,j_2-l}(x)(j_1 - j_2 - 1)c_l \\ & x^2\frac{\mathrm{d}\phi_{2,j_1;j_2}}{\mathrm{d}x}(x) + (j_2 - j_1 - 1 + (a_1 j_1 + a_2(j_2 - 1))x)\phi_{2,j_1,j_2}(x) \\ & \quad = \sum_{l=1}^{\min(j_1,j_2)} \phi_{2,j_1-l,j_2-l}(x)(j_1 - j_2 + 1)c_l\,. \end{aligned}\right.$$

From this we deduce:

$$\left\{\begin{aligned} \phi_{1,1,0}(x) &= p_{1,0} \quad \mathbb{C}\setminus\{0\} \\ \phi_{2,0,1}(x) &= q_{0,1} \quad \mathbb{C}\setminus\{0\} \end{aligned}\right.$$

with $p_{1,0}q_{0,1} = 1$. Then, using the assumption that $\phi_{i,j_1,j_2}(x)$ is analytic and bounded in $S_\pm$ for all $j_1, j_2 > 0$, we see (by induction on $j > 1$) that

$$j > 1 \quad \left\{\begin{aligned} \phi_{1,j+1,j} &= 0 \\ \phi_{2,j,j+1} &= 0\,. \end{aligned}\right.$$

Indeed, we show recursively that for all $j > 1$, we have:

$$x^2\frac{\mathrm{d}\phi_{1,j+2,j+1}}{\mathrm{d}x}(x) + (j+1)(a_1 + a_2)x\phi_{1,j+2,j+1}(x) = 0\,,$$

and the general solution to this equation is:

$$\phi_{1,j+2,j+1}(x) = p_{j+2,j+1}x^{-(j+1)(a_1+a_2)}\,, \text{ with } p_{j+2j+1} \quad \mathbb{C}\,.$$

The quantity $\phi_{1,j+2,j+1}(x)$ is bounded near the origin if and only if $p_{j+2,j+1} = 0$, since $(a_1 + a_2) > 0$. The same arguments work for $\phi_{2,j,j+1}$,

$j > 1$. Consequently:

$$\begin{cases} x^2 \dfrac{\mathrm{d}\phi_{1,j_1;j_2}}{\mathrm{d}x}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2 j_2)x)\phi_{1,j_1,j_2}(x) \\ \qquad = (j_1 - j_2 - 1) \displaystyle\sum_{l=1}^{\min(j_1,j_2)} \phi_{1,j_1-l,j_2-l}(x) c_l \\ x^2 \dfrac{\mathrm{d}\phi_{2,j_1;j_2}}{\mathrm{d}x}(x) + (j_2 - j_1 - 1 + (a_1 j_1 + a_2(j_2 - 1))x)\phi_{2,j_1,j_2}(x) \\ \qquad = (j_1 - j_2 + 1) \displaystyle\sum_{l=1}^{\min(j_1,j_2)} \phi_{2,j_1-l,j_2-l}(x) c_l \,. \end{cases}$$

Once again, we see recursively that for $j_1 > j_2 + 1$, $\phi_{1,j_1,j_2}(x) = 0$. Indeed, we obtain by induction that

$$\phi_{1,j_1,j_2}(x) = p_{j_1,j_2} \exp\left(\frac{j_2 - j_1 + 1}{x}\right) x^{-(a_1(j_1-1)+a_2 j_2)},$$

and since this has to be bounded on $S_\pm$, we necessarily have $p_{j_1,j_2} = 0$, and therefore $\phi_{1,j_1,j_2}(x) = 0$. Similarly, for $j_1 < j_2 + 1$, we prove that $\phi_{j_1,j_2}(x) = 0$. As a conclusion, $\phi_1(x, \mathbf{y}) = y_1$. By exactly the same kind of arguments we have $\phi_2(x, \mathbf{y}) = y_2$.

4.6. Weak 1-summability of the normalizing map

Let us consider the same data as in Lemma 4.6. The following lemma states that an analytic solution to the considered homological equation in $S_\pm \in S_{\arg(\pm i\lambda),\eta}$ with $\eta \in [\pi, 2\pi[$, admits a weak Gevrey-1 asymptotic expansion in this sector. In other words, it is the weak 1-sum of a formal solution the homological equation. Let us re-use the notations introduced at the beginning of the latter section.

Lemma 4.17. — *Let*

$$Z := Y_0 + C(x, \mathbf{y})\overrightarrow{\mathcal{C}} + xR^{(1)}(x, \mathbf{y})\overrightarrow{\mathcal{R}}$$

be a formal vector field weakly 1-summable in $S_\pm \in S_{\arg(\pm i\lambda),\eta}$, *with* $\eta \in [\pi, 2\pi[$ *and* $C, R^{(1)}$ *of order at least one with respect to* $\mathbf{y}$. *We denote by*

$$Z_\pm := Y_0 + C_\pm(x, \mathbf{y})\overrightarrow{\mathcal{C}} + xR^{(1)}_\pm(x, \mathbf{y})\overrightarrow{\mathcal{R}}$$

the associate weak 1-sum in $S_\pm$. *Let also* $A \in \mathbb{C}[\![x, \mathbf{y}]\!]$ *be weakly 1-summable in* $S_\pm$, *of 1-sum* $A_\pm$ *and of order at least one with respect to* $\mathbf{y}$. *Then, any sectorial germ of an analytic function of the form* $\alpha_\pm(x, \mathbf{y}) = x^M \tilde{\alpha}_\pm(x, \mathbf{y})$, *with $M \in \mathbb{N}_{>0}$ and $\tilde{\alpha}_\pm$ analytic in $S_\pm$, which is dominated by $\|\mathbf{y}\|_\infty$ and satisfies*

$$L_{Z_\pm}(\alpha_\pm) = x^{M+1} A_\pm(x, \mathbf{y}),$$

has a Gevrey-1 asymptotic expansion in $S_\pm$, denoted by α. Moreover, α is a formal solution to

$$L_Z(\alpha) = x^{M+1} A(x, \mathbf{y}).$$

Proof. — Let us write Z as follow:

$$Z = x^2 \frac{\partial}{\partial x} + (-(\lambda + d(y_1 y_2)) + a_1 x + F_1(x, \mathbf{y})) y_1 \frac{\partial}{\partial y_1} + (\lambda + d(y_1 y_2) + a_2 x + F_2(x, \mathbf{y})) y_2 \frac{\partial}{\partial y_2},$$

with F_1, F_2 weakly 1-summable in $S_\pm \in \mathcal{S}_{\arg(\pm i\lambda),\eta}$, with $\eta \in \left]\pi, 2\pi\right[$, of weak 1-sums $F_{1,\pm}, F_{2,\pm}$ respectively, which are dominated by $\|\mathbf{y}\|$, and with $d(v) \in v\mathbb{C}\{v\}$ without constant term. Consider the Taylor expansion with respect to $\mathbf{y}$ of d,F_1,F_2,A and α:

$$\begin{cases} d(y_1 y_2) = \sum_{k>1} d_k y_1^k y_2^k \\ F_1(x, \mathbf{y}) = \sum_{j_1+j_2>1} F_{1,\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ F_2(x, \mathbf{y}) = \sum_{j_1+j_2>1} F_{2,\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ A(x, \mathbf{y}) = \sum_{j_1+j_2>1} A_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ \alpha(x, \mathbf{y}) = \sum_{j_1+j_2>1} \alpha_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \end{cases}$$

(same expansions are valid in $S_\pm$ for the corresponding weak 1-sums). As usual, possibly by reducing $S_\pm$, we can assume that $S_\pm = S_\pm \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ (where $S_\pm$ is an adapted sector and $\mathbf{D}(\mathbf{0}, \mathbf{r})$ a polydisc). The homological equation

$$L_Z(\alpha) = x^{M+1} A_\pm(x, \mathbf{y})$$

can be re-written:

$$x^2 \frac{\partial \alpha}{\partial x} + (-(\lambda + d(y_1 y_2)) + a_1 x + F_{1,\pm}(x, \mathbf{y})) y_1 \frac{\partial \alpha}{\partial y_1} + (\lambda + d(y_1 y_2) + a_2 x + F_{2,\pm}(x, \mathbf{y})) y_2 \frac{\partial \alpha}{\partial y_2} = x^{M+1} A_\pm(x, \mathbf{y}).$$

Using normal convergence in any compact subset of $S_\pm$, we can compute the partial derivatives of

$$\alpha(x, \mathbf{y}) = \sum_{j_1+j_2>1} \alpha_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$$

with respect to x, y_1 or y_2 term by term, in order to obtain after identification: $\quad \mathbf{j} = (j_1, j_2) \quad \mathbb{N}^2$,

$$x^2 \frac{\mathrm{d}\alpha_{\mathbf{j},\pm}}{\mathrm{d}x}(x) + (\lambda(j_2 - j_1) + (a_1 j_1 + a_2 j_2)x)\alpha_{\mathbf{j},\pm}(x) = G_{\mathbf{j},\pm}(x)\,, \tag{4.22}$$

where $G_{\mathbf{j},\pm}(x)$ depends only on $d_k, F_{1,\mathbf{k},\pm}, F_{2,\mathbf{k},\pm}, \alpha_{\mathbf{k},\pm}$ and $A_{\mathbf{l},\pm}$, for $k \leqslant \min(j_1, j_2)$, $|\mathbf{k}| \leqslant |\mathbf{j}| - 1$ and $|\mathbf{l}| \leqslant |\mathbf{j}|$. We obtain a similar differential equation for the associated formal power series. Let us prove by induction on $|\mathbf{j}| \geqslant 0$ that:

(1) $G_{\mathbf{j},\pm}$ is the 1-sum of $G_{\mathbf{j}}$ in $S_\pm$,
(2) $G_{j,j}(0) = 0$ if $\mathbf{j} = (j, j)$
(3) $\alpha_{\mathbf{j},\pm}$ is the 1-sum $\alpha_{\mathbf{j}}$ in $S_\pm$.

It is paramount to use the fact that for all $\mathbf{j} \quad \mathbb{N}^2$, $\alpha_{\mathbf{j},\pm}$ is bounded in $S_\pm$.

- For $\mathbf{j} = (0, 0)$, we have $G_{(0,0)} = 0$ and then $\alpha_{(0,0)} = 0$.
- Let $\mathbf{j} = (j_1, j_2) \quad \mathbb{N}^2$ with $|\mathbf{j}| = j_1 + j_2 \geqslant 1$. Assume the property holds for all $\mathbf{k} \quad \mathbb{N}^2$ with $|\mathbf{k}| \leqslant |\mathbf{j}| - 1$.
  (1) Since $G_{\mathbf{j}}(x)$ depends only on $d_k, F_{1,\mathbf{k}}, F_{2,\mathbf{k}}, \alpha_{\mathbf{k}}$ and $A_{\mathbf{l}}$, for $k \leqslant \min(j_1, j_2)$, $|\mathbf{k}| \leqslant |\mathbf{j}| - 1$ and $|\mathbf{l}| \leqslant |\mathbf{j}|$, then $G_{\mathbf{j}}$ is 1-summable in $S_\pm$, of 1-sum $G_{\mathbf{j},\pm}$.
  (2) We also see that $G_{j,j}(0) = 0$, if $\mathbf{j} = (j, j)$.
  (3) If $j_1 = j_2$, then point 1. in Proposition 2.32 tells us that there exists a unique formal solution $\alpha_{\mathbf{j}}(x)$ to the irregular differential equation we are looking at, and such that $\alpha_{\mathbf{j}}(0) = \frac{1}{\lambda(j_2 - j_1)} G_{\mathbf{j}}(0)$. Moreover, this solution is 1-summable in $S_\pm$ since the same goes for $G_{\mathbf{j}}$.
  (4) If however $j_1 = j_2 = j \geqslant 1$, since $G_{(j,j)}(0) = 0$ we can write $G_{(j,j)}(x) = x\tilde{G}_{(j,j)}(x)$ with $\tilde{G}_{(j,j)}(x)$ 1-summable in $S_\pm$, and then the differential equation becomes regular:

  $$x \frac{\mathrm{d}\alpha_{(j,j),\pm}}{\mathrm{d}x}(x) + (a_1 + a_2) j \alpha_{(j,j),\pm}(x) = \tilde{G}_{(j,j),\pm}(x)\,.$$

  Since $\quad (a_1 + a_2) > 0$, according to point 2. in Proposition 2.32, the latter equation has a unique formal solution $\alpha_{(j,j)}(x)$ such that $\alpha_{(j,j)}(0) = \frac{\tilde{G}_{(j,j)}(0)}{(a_1 + a_2) j}$, and this solution is moreover 1-summable in $S_\pm$, and its 1-sum is the only solution to this equation bounded in $S_\pm$. Thus, it is necessarily $\alpha_{(j,j),\pm}$.

We are now able to prove the weak 1-summability of the formal normalizing map.

Proposition 4.18. — *The sectorial normalizing maps* (Φ_+, Φ_-) *in Corollary 4.2 are the weak 1-sums in* $S_\pm \setminus S_{\arg(\pm\lambda),\eta}$ *of the formal normalizing map* $\hat{\Phi}$ *given by Theorem 1.5, for all* $\eta \in [\pi, 2\pi[$*. In particular,* $\hat{\Phi}$ *is weakly 1-summable, except for* $\arg(\pm\lambda)$*.*

Proof. — The normalizing map $\Phi_\pm$ from Corollary 4.2 is constructed as the composition of two germs of sectorial diffeomorphisms, using successively Propositions 3.1 and 4.1. The sectorial map obtained in Proposition 3.1 is 1-summable except in directions $\arg(\pm\lambda)$. The sectorial transformation in Proposition 4.1 is constructed as the composition of two germs of sectorial diffeomorphisms, using successively Proposition 4.3 and 4.5. Both of these two sectorial maps are built thanks to Lemma 4.6. Lemma 4.17 above justifies that each of these maps admits in fact a weak Gevrey-1 asymptotic expansion in a domain of the form $S_\pm \setminus S_{\arg(\pm\lambda),\eta}$, for all $\eta \in [\pi, 2\pi[$. Consequently, the same goes for the sectorial diffeomorphisms of Proposition 4.1, and then for those of Corollary 4.2 (we used here Proposition 2.26 for the composition). Using item 3 in Lemma 2.25, we deduce that the weak Gevrey-1 asymptotic expansion of the sectorial normalizing maps of Corollary 4.2 is therefore a formal normalizing map, such as the one given by Theorem 1.5. By uniqueness of such a normalizing map, it is $\hat{\Phi}$.

5. Analytic classification

In this section, we end up the proofs of both Theorems 1.10 and 1.16. In order to do this, we prove that the Stokes diffeomorphisms Φ_λ and $\Phi_{-\lambda}$ obtained from the germs of sectorial normalizing maps Φ_+ and Φ_-, which a priori admit identity as *weak* Gevrey-1 asymptotic expansion, admit in fact identity as "true" Gevrey-1 asymptotic expansion. This will be done by studying more generally germs of sectorial isotropies of the normal form Y_{norm} in sectorial domains with "narrow" opening, and by considering theses isotropies in the of space of leaves. Using Theorem 2.22, which is a "non-abelian" version of the Ramis–Sibuya theorem due to Martinet and Ramis [17], this will has as consequence the fact that the sectorial normalizing maps Φ_+ and Φ_- both admit the formal normalizing map $\hat{\Phi}$ as Gevrey-1 asymptotic expansion in the corresponding sectorial domains. This proves Theorem 1.10. Moreover, another consequence will be Theorem 1.16. Finally, we will describe the moduli space of analytic classification in terms of some spaces of power series.

From now on, we fix a normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2))y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2))y_2 \frac{\partial}{\partial y_2} \,,$$

with $\lambda \quad \mathbb{C}$, $\quad (a_1 + a_2) > 0$ and $c \quad v\mathbb{C}\{v\}$ vanishing at the origin. We denote by $[Y_{\text{norm}}]$ the set of germs of holomorphic doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$, formally conjugate to Y_{norm} by formal fibered diffeomorphisms tangent to the identity. We refer the reader to Definition 2.1 for notions relating to sectors.

Definition 5.1. — *We define* $\Lambda_{\lambda}^{(\text{weak})}(Y_{\text{norm}})$ *(resp.* $\Lambda_{-\lambda}^{(\text{weak})}(Y_{\text{norm}})$*) as the group of germs of sectorial fibered isotropies of* Y_{norm}*, admitting the identity as* weak *Gevrey-1 asymptotic expansion in sectorial domains of the form* $S_\lambda \times (\mathbb{C}^2, 0)$ *(resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$*), where:*

$$\begin{aligned} S_\lambda &\quad AS_{\arg(\lambda),\pi} \\ S_{-\lambda} &\quad AS_{\arg(-\lambda),\pi} \end{aligned}$$

(see Definition 2.3).

We recall the notations given in the introduction: we have defined $\Lambda_\lambda(Y_{\text{norm}})$ (resp. $\Lambda_{-\lambda}(Y_{\text{norm}})$) as the group of germs of sectorial fibered isotropies of Y_{norm}, admitting the identity as Gevrey-1 asymptotic expansion in sectorial domains of the form $S_\lambda \times (\mathbb{C}^2, 0)$ (resp. $S_{-\lambda} \times (\mathbb{C}^2, 0)$). It is clear that we have:

$$\Lambda_{\pm\lambda}(Y_{\text{norm}}) \quad \Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}}) \quad \text{Isot}_{\text{fib}}(Y, S_{\arg(\pm\lambda),\eta}; \text{Id}), \quad \eta \quad]0, \pi[\,.$$

The main result of this section is the following.

Proposition 5.2. — *Any* $\psi \quad \Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}})$ *admits the identity as Gevrey-1 asymptotic expansion in* $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$*. In other words:*

$$\Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}}) = \Lambda_{\pm\lambda}(Y_{\text{norm}}) \,.$$

5.1. Proofs of the main results (assuming Proposition 5.2)

In this subsection, we prove the main results of this paper, assuming Proposition 5.2 above holds.

5.1.1. Analytic invariants: Stokes diffeomorphisms

According to Corollary 4.2, to any $Y \quad [Y_{\mathrm{norm}}]$, we can associate a pair of germs of sectorial fibered isotropies in $S_\lambda \times (\mathbb{C}^2,0)$ and $S_{-\lambda} \times (\mathbb{C}^2,0)$ respectively, denoted by $(\Phi_\lambda, \Phi_{-\lambda})$:

□

$$\begin{aligned} \Phi_\lambda &:= (\Phi_+ \circ \Phi_-^{-1})_{|S_\lambda \times (\mathbb{C}^2,0)} \quad \mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(\lambda),\eta}; \mathrm{Id})\,, \qquad \eta \quad]0,\pi[\\ \Phi_{-\lambda} &:= (\Phi_- \circ \Phi_+^{-1})_{|S_{-\lambda} \times (\mathbb{C}^2,0)} \quad \mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(-\lambda),\eta}; \mathrm{Id})\,, \qquad \eta \quad]0,\pi[\,, \end{aligned}$$

where (Φ_+, Φ_-) is the pair of the sectorial normalizing maps given by Corollary 4.2.

Proposition 5.3. — *For any given* $\eta \quad]0,\pi[$ *the map*

$$\begin{aligned} [Y_{\mathrm{norm}}] &\longrightarrow \mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(\lambda),\eta}; \mathrm{Id}) \times \mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(-\lambda),\eta}; \mathrm{Id}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda})\,, \end{aligned}$$

actually ranges in $\Lambda_\lambda^{(\mathrm{weak})}(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}^{(\mathrm{weak})}(Y_{\mathrm{norm}})$.

Proof. — The fact that the sectorial normalizing maps Φ_+, Φ_- given by Corollary 4.2 both conjugate $Y \quad [Y_{\mathrm{norm}}]$ to Y_{norm} in the corresponding sectorial domains proves that the arrow above is well-defined, with values in $\mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(\lambda),\eta}; \mathrm{Id}) \times \mathrm{Isot}_{\mathrm{fib}}(Y, S_{\arg(-\lambda),\eta}; \mathrm{Id})$, for all $\eta \quad]0,\pi[$. The fact that $\Phi_{\pm\lambda}$ admits the identity as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times (\mathbb{C}^2,0)$ comes from Proposition 4.18 (Φ_+ and Φ_- admits the same weak Gevrey-1 asymptotic expansion in $S_\lambda \times (\mathbb{C}^2,0)$ and $S_{-\lambda} \times (\mathbb{C}^2,0)$) and from Proposition 2.26.

The subgroup $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id}) \quad \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0)$ formed by fibered diffeomorphisms tangent to the identity acts naturally on $[Y_{\mathrm{norm}}]$ by conjugacy. Now we show that the uniqueness of germs of sectorial normalizing maps (Φ_+, Φ_-) implies that the Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$ of a vector field $Y \quad [Y_{\mathrm{norm}}]$ is invariant under the action of $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$. Furthermore, this map is one-to-one.

Proposition 5.4. — *The map*

$$\begin{aligned} [Y_{\mathrm{norm}}] &\longrightarrow \Lambda_\lambda^{(\mathrm{weak})}(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}^{(\mathrm{weak})}(Y_{\mathrm{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned}$$

factorizes through a one-to-one map

$$\begin{aligned} [Y_{\mathrm{norm}}]/\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id}) &\longrightarrow \Lambda_\lambda^{(\mathrm{weak})}(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}^{(\mathrm{weak})}(Y_{\mathrm{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda})\,. \end{aligned}$$

Remark 5.5. — This very result means that the Stokes diffeomorphisms encode completely the class of Y in the quotient ${}^{[Y_{\mathrm{norm}}]}/_{\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})}$ as they separate conjugacy classes.

Proof. — First of all, let us prove that the latter map is well-defined. Let $Y, \tilde{Y}\quad [Y_{\mathrm{norm}}]$ and $\Theta\quad \mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ be such that $\Theta\ (Y) = \tilde{Y}$. We denote by $\Phi_\pm$ (resp. $\tilde{\Phi}_\pm$) the sectorial normalizing maps of Y (resp. $\tilde{Y}$), and $(\Phi_\lambda, \Phi_{-\lambda})$ (resp. $(\tilde{\Phi}_\lambda, \tilde{\Phi}_{-\lambda})$) the Stokes diffeomorphisms of Y (resp. $\tilde{Y}$). By assumption, $\tilde{\Phi}_\pm \circ \Theta$ is also a germ of a sectorial fibered normalization of Y in $S_\pm \times (\mathbb{C}^2, 0)$, which is tangent to the identity. Thus, according to the uniqueness statement in Theorem 1.10:

$$\Phi_\pm = \tilde{\Phi}_\pm \circ \Theta\,.$$

Consequently, in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$ we have

$$\begin{aligned}\Phi_\lambda &= (\Phi_+ \circ \Phi_-^{-1})_{|S_\lambda \times (\mathbb{C}^2,0)}\\ &= \tilde{\Phi}_+ \circ \Theta \circ \Theta^{-1} \circ \tilde{\Phi}_-\\ &= \tilde{\Phi}_\lambda\,,\end{aligned}$$

and similarly

$$\begin{aligned}\Phi_{-\lambda} &= (\Phi_- \circ \Phi_+^{-1})_{|S_{-\lambda} \times (\mathbb{C}^2,0)}\\ &= \tilde{\Phi}_- \circ \Theta \circ \Theta^{-1} \circ (\tilde{\Phi}_+)^{-1}\\ &= \tilde{\Phi}_{-\lambda}\,.\end{aligned}$$

Let us prove that the map is one-to-one. Let $Y, \tilde{Y}\quad [Y_{\mathrm{norm}}]$ share the same Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$. We denote by $\Phi_\pm$ (resp. $\tilde{\Phi}_\pm$) the germ of a sectorial fibered normalizing map of Y (resp. $\tilde{Y}$) $S_\pm \times (\mathbb{C}^2, 0)$. We have:

$$\begin{aligned}&\Phi_+ \circ (\Phi_-)^{-1} = \Phi_\lambda = \tilde{\Phi}_+ \circ (\tilde{\Phi}_-)^{-1} &&\text{in } S_\lambda \times (\mathbb{C}^2, 0)\\ &\Phi_- \circ (\Phi_+)^{-1} = \Phi_{-\lambda} = \tilde{\Phi}_- \circ (\tilde{\Phi}_+)^{-1} &&\text{in } S_{-\lambda} \times (\mathbb{C}^2, 0)\,.\end{aligned}$$

Thus:

$$\begin{aligned}&(\tilde{\Phi}_+)^{-1} \circ \Phi_+ = (\tilde{\Phi}_-)^{-1} \circ \Phi_- &&\text{in } S_\lambda \times (\mathbb{C}^2, 0)\\ &(\tilde{\Phi}_+)^{-1} \circ \Phi_+ = (\tilde{\Phi}_-)^{-1} \circ \Phi_- &&\text{in } S_{-\lambda} \times (\mathbb{C}^2, 0)\,.\end{aligned}$$

We can then define a map φ analytic in a domain of the form $(D(0, r) \setminus \{0\}) \times \mathbf{D(0, r)}$ by setting:

$$\begin{aligned}&\varphi_{|S_+} = (\tilde{\Phi}_+)^{-1} \circ \Phi_+ &&\text{in } S_+\\ &\varphi_{|S_-} = (\tilde{\Phi}_-)^{-1} \circ \Phi_- &&\text{in } S_-\,.\end{aligned}$$

This map is analytic and bounded in $(D(0,r)\setminus\{0\}) \times \mathbf{D}(\mathbf{0},\mathbf{r})$, and the Riemann singularity theorem tells us that this map can be analytically extended to the entire poly-disc $D(0,r) \times \mathbf{D}(\mathbf{0},\mathbf{r})$. As a conclusion, φ $\mathrm{Diff}_{\mathrm{fib}}(\mathbb{C}^3,0;\mathrm{Id})$, $\Phi_\pm = \tilde{\Phi}_\pm \circ \varphi$ and $\varphi\ \ (Y) = \tilde{Y}$.

5.1.2. Proof of Theorem 1.10: 1-summability of the formal normalization

As a first consequence of Proposition 5.2, we obtain Proposition 5.6, which states that the formal normalizing map from Theorem 1.5 [3] is in fact 1-summable.

Proposition 5.6. — *The unique formal normalizing map $\hat{\Phi}$ given in 1.5 is the Gevrey-1 asymptotic expansion of the unique germs of sectorial normalizing maps Φ_+ and Φ_- in $S_+ \times (\mathbb{C}^2,0)$ and $S_- \times (\mathbb{C}^2,0)$ respectively. In particular, $\hat{\Phi}$ is 1-summable in every direction $\theta = \arg(\pm\lambda)$, and (Φ_+,Φ_-) is its Borel–Laplace 1-sum.*

Proof. — Let us consider the unique germs of a sectorial normalizing map Φ_+ and Φ_- in $S_+ \times (\mathbb{C}^2,0)$ and $S_- \times (\mathbb{C}^2,0)$ respectively, and their associated Stokes diffeomorphisms:

$$\Phi_\lambda = (\Phi_+ \circ \Phi_-^{-1})_{|S_\lambda \times (\mathbb{C}^2,0)} \quad \Lambda_\lambda^{(\mathrm{weak})}(Y_{\mathrm{norm}})$$
$$\Phi_{-\lambda} = (\Phi_- \circ \Phi_+^{-1})_{|S_{-\lambda} \times (\mathbb{C}^2,0)} \quad \Lambda_{-\lambda}^{(\mathrm{weak})}(Y_{\mathrm{norm}})\,.$$

According to Proposition 5.2,

$$\Lambda_{\pm\lambda}^{(\mathrm{weak})}(Y_{\mathrm{norm}}) = \Lambda_{\pm\lambda}(Y_{\mathrm{norm}})\,,$$

so that Φ_λ and $\Phi_{-\lambda}$ both admit the identity as Gevrey-1 asymptotic expansion, in $S_\lambda \times (\mathbb{C}^2,0)$ and $S_{-\lambda} \times (\mathbb{C}^2,0)$ respectively. Then, Theorem 2.22 gives the existence of

$$(\phi_+,\phi_-) \quad \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(i\lambda),\eta};\mathrm{Id}) \times \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(-i\lambda),\eta};\mathrm{Id})$$

for all η $]\pi,2\pi[$, such that:

$$\phi_+ \circ (\phi_-)^{-1}_{|S_\lambda \times (\mathbb{C}^2,0)} = \Phi_\lambda$$
$$\phi_- \circ (\phi_+)^{-1}_{|S_{-\lambda} \times (\mathbb{C}^2,0)} = \Phi_{-\lambda}\,,$$

and the existence of a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_+ and ϕ_- both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_+ \times (\mathbb{C}^2,0)$ and $S_- \times (\mathbb{C}^2,0)$ respectively. In particular, we have:

$$((\Phi_+)^{-1} \circ \phi_+)_{|(S_\lambda \quad S_{-\lambda}) \times (\mathbb{C}^2,0)} = ((\Phi_-)^{-1} \circ \phi_-)_{|(S_\lambda \quad S_{-\lambda}) \times (\mathbb{C}^2,0)}\,.$$

This proves that the function Φ defined by $(\Phi_+)^{-1} \circ \phi_+$ in $S_+ \times (\mathbb{C}^2, 0)$ and by $(\Phi_-)^{-1} \circ \phi_-$ in $S_- \times (\mathbb{C}^2, 0)$ is well-defined and analytic in $D(0,r) \setminus \{0\} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. Since it is also bounded, it can be extended to an analytic map Φ in $D(0,r) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ by Riemann's theorem. Hence:

□

$$\begin{aligned} \phi_+ &= \Phi_+ \circ \Phi \\ \phi_- &= \Phi_- \circ \Phi\,. \end{aligned}$$

In particular, by composition, Φ_+ and Φ_- both admit $\hat{\phi} \circ \Phi^{-1}$ as Gevrey-1 asymptotic expansion in $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ respectively. Since Φ_+ and Φ_- conjugates Y to Y_{norm} and since the notion of asymptotic expansion commutes with the partial derivative operators, the formal diffeomorphism $\hat{\phi} \circ \Phi^{-1}$ formally conjugates Y to Y_{norm}. Finally, notice that $\hat{\phi} \circ \Phi^{-1}$ is necessarily tangent to the identity. Hence, by uniqueness of the formal normalizing map given by Theorem 1.5, we deduce that $\hat{\phi} \circ \Phi^{-1} = \hat{\Phi}$, the unique formal normalizing map tangent to the identity.

We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. It is a straightforward consequence of Proposition 5.6 above.

5.1.3. Proof of Theorem 1.16

Proof of Theorem 1.16. — Propositions 5.4, together with Proposition 5.2, tell us that the considered map is well-defined and one-to-one. It remains to prove that this map is onto. Let

□

$$\begin{aligned} &\Phi_\lambda \quad \Lambda_\lambda(Y_{\mathrm{norm}}) \\ &\Phi_{-\lambda} \quad \Lambda_{-\lambda}(Y_{\mathrm{norm}})\,. \end{aligned}$$

According to Theorem 2.22, there exists

$$(\phi_+, \phi_-) \quad \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(i\lambda),\eta}; \mathrm{Id}) \times \mathrm{Diff}_{\mathrm{fib}}(S_{\arg(-i\lambda),\eta}; \mathrm{Id})$$

with $\eta \quad]\pi, 2\pi[$, which extend analytically to $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, such that:

$$\phi_\pm \circ (\phi \quad)^{-1}_{|S_{\pm\lambda} \times (\mathbb{C}^2,0)} = \Phi_{\pm\lambda}$$

and there also exists a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that $\phi_\pm$ both admit $\hat{\phi}$ as asymptotic expansion in $S_\pm \times (\mathbb{C}^2, 0)$. Let us consider the two germs of sectorial vector fields obtained as

$$Y_\pm := (\phi_\pm^{-1}) \ (Y_{\mathrm{norm}})$$

In particular, since $\hat{\phi}$ is the Gevrey-1 asymptotic expansion of $\phi_\pm$, the vector fields $Y_\pm$ both admit $(\hat{\phi})\ (Y_{\text{norm}})$ as Gevrey-1 asymptotic expansion. The fact that $\phi_+ \circ (\phi_-)^{-1}$ is an isotropy of Y_{norm} implies immediately that $Y_+ = Y_-$ on

$$(S_+ \cap S_-) \times (\mathbb{C}^2, 0) = (S_\lambda \quad S_{-\lambda}) \times (\mathbb{C}^2, 0)\,. \tag{5.1}$$

Then, the vector field Y, which coincides with $Y_\pm$ in $S_\pm \times (\mathbb{C}^2, 0)$, defines a germ of analytic vector field in $(\mathbb{C}^3, 0)$ by Riemann's theorem. By construction, $Y \quad \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})\ (Y_{\text{norm}})$ and admits $(\Phi_\lambda, \Phi_{-\lambda})$ as Stokes diffeomorphisms.

5.1.4. Proof of Theorem 1.24

In a similar way, we prove now Theorem 1.24.

Proof of Theorem 1.24. — Let $Y_{\text{norm}} \quad \mathcal{SN}_{\text{diag},0}$ be a normal form which is also transversally symplectic. We refer to Subsection 1.3 for the notations. It is clear from Theorems 1.16 and 1.22 that the mapping is well-defined and one-to-one. It remains to prove that it is also onto. Let

□

$$\begin{array}{l} \Phi_\lambda \quad \Lambda_\lambda^\omega(Y_{\text{norm}}) \\ \Phi_{-\lambda} \quad \Lambda_{-\lambda}^\omega(Y_{\text{norm}})\,. \end{array}$$

Since $\Lambda_\lambda^\omega(Y_{\text{norm}}) \quad \Lambda_\lambda(Y_{\text{norm}})$ and $\Lambda_{-\lambda}^\omega(Y_{\text{norm}}) \quad \Lambda_{-\lambda}(Y_{\text{norm}})$, according to Theorem 2.22 there exists

$$(\phi_+, \phi_-) \quad \text{Diff}_{\text{fib}}(S_{\arg(i\lambda),\eta}; \text{Id}) \times \text{Diff}_{\text{fib}}(S_{\arg(-i\lambda),\eta}; \text{Id})$$

with $\eta \quad]\pi, 2\pi[$, which extend analytically in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, such that:

$$\phi_\pm \circ (\phi \quad)^{-1}_{|S_{\pm\lambda} \times (\mathbb{C}^2,0)} = \Phi_{\pm\lambda}$$

and there also exists a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that $\phi_\pm$ both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_\pm \times (\mathbb{C}^2, 0)$. According to Corollary 2.23, there exists a germ of an analytic fibered diffeomorphism $\psi \quad \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ (tangent to the identity), such that

$$\sigma_\pm := \phi_\pm \circ \psi$$

both are transversally symplectic. Then, we have:

$$\sigma_\pm \circ (\Psi \quad)^{-1}_{|S_{\pm\lambda} \times (\mathbb{C}^2,0)} = \Phi_{\pm\lambda}\,.$$

The end of the proof goes exactly as at the end of the proof of the previous theorem.

5.2. Sectorial isotropies in narrow sectors and space of leaves: proof of Proposition 5.2.

A normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2}$$

is fixed for some $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ (vanishing at the origin). The aim of this subsection is to prove Proposition 5.2 stated at the beginning of this section. Let us denote $a := \text{res}(Y_{\text{norm}}) = a_1 + a_2$, $m := \frac{1}{a}$ and

$$c(v) = \sum_{k=1}^{+\infty} c_k v^k .$$

If $m \notin \mathbb{N}_{>0}$, we set $c_m := 0$. We also define the following power series

$$\tilde{c}(v) = m \sum_{k \neq m} \frac{c_k}{k - m} v^k , \tag{5.2}$$

and we notice that $\tilde{c}(v) \in v\mathbb{C}\{v\}$.

5.2.1. Sectorial first integrals and the space of leaves

In a sectorial neighborhood of the origin of the form $S_\lambda \times (\mathbb{C}^2, 0)$ (resp. $S_{-\lambda} \times (\mathbb{C}^2, 0)$), with $S_{\pm\lambda} \in \mathcal{S}_{\arg(\pm\lambda), \epsilon}$ and $\epsilon \in]0, \pi[$, we can give three first integrals of Y_{norm} which are analytic in the considered domain. Let us start with the following proposition.

Proposition 5.7. — *The following quantities are first integrals of Y_{norm}, analytic in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$:*

$$\begin{cases} w_{\pm\lambda} := \dfrac{y_1 y_2}{x^a} \\ h_{1,\pm\lambda}(x, \mathbf{y}) := y_1 \exp\left(\dfrac{-\lambda}{x} + \dfrac{c_m (y_1 y_2)^m \log(x)}{x} + \dfrac{\tilde{c}(y_1 y_2)}{x} \right) x^{-a_1} \\ h_{2,\pm\lambda}(x, \mathbf{y}) := y_2 \exp\left(\dfrac{\lambda}{x} - \dfrac{c_m (y_1 y_2)^m \log(x)}{x} - \dfrac{\tilde{c}(y_1 y_2)}{x} \right) x^{-a_2} \end{cases} \tag{5.3}$$

(we fix here a branch of the logarithm analytic in $S_{\pm\lambda}$, and we write simply h_j and w instead of $h_{j,\pm\lambda}$ and $w_{\pm\lambda}$ respectively, if there is no ambiguity on the sector $S_{\pm\lambda}$). Moreover, we have the relation:

$$h_1 h_2 = w .$$

Proof. — It is an elementary computation.

Remark 5.8. — In other words, in a sectorial domain, we can parametrize a leaf (which is not in $\{x = 0\}$) of the foliation associated to Y_{norm} by:

$$\left\{\begin{array}{l} y_1(x) = h_1 \exp\left(\frac{\lambda}{x} - c_m (h_1h_2)^m \log(x) - \frac{\tilde{c}(h_1h_2x^a)}{x}\right) x^{a_1} \\ y_2(x) = h_2 \exp\left(-\frac{\lambda}{x} + c_m (h_1h_2)^m \log(x) + \frac{\tilde{c}(h_1h_2x^a)}{x}\right) x^{a_2} \end{array}\right. \tag{5.4}$$

$(h_1, h_2) \quad \mathbb{C}^2$.

Corollary 5.9. — *The map*

$$\begin{aligned} H_{\pm\lambda} : S_{\pm\lambda} \times (\mathbb{C}^2, 0) &\longrightarrow S_{\pm\lambda} \times \mathbb{C}^2 \\ (x, \mathbf{y}) &\longmapsto (x, h_{1,\pm\lambda}(x, \mathbf{y}), h_{2,\pm\lambda}(x, \mathbf{y})), \end{aligned}$$

(where $h_{1,\pm\lambda}, h_{2,\pm\lambda}$ are defined in (5.3)) is a sectorial germ of a fibered analytic map in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$, which is into. Moreover, there exists an open neighborhood of the origin in $\mathbb{C}^2$, denoted by $\Gamma_{\pm\lambda} \quad \mathbb{C}^2$, such that

$$H_{\pm\lambda}(S_{\pm\lambda} \times (\mathbb{C}^2, 0)) = S_{\pm\lambda} \times \Gamma_{\pm\lambda}\,.$$

In particular, $H_\pm$ induces a fibered biholomorphism

$$S_{\pm\lambda} \times (\mathbb{C}^2, 0) \xrightarrow{H_{\pm\lambda}} S_{\pm\lambda} \times \Gamma_{\pm\lambda}$$

which conjugates Y_{norm} to $x^2 \frac{\partial}{\partial x}$, i.e.

$$(H_{\pm\lambda}) \ (Y_{\text{norm}}) = x^2 \frac{\partial}{\partial x}\,.$$

Definition 5.10. — *We call* $\Gamma_{\pm\lambda}$ the space of leaves of Y_{norm} *in* $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$.

Remark 5.11. — The set $\Gamma_{\pm\lambda}$ depends on the choice of the neighborhood $(\mathbb{C}^2, 0)$, but also on the choice of the sectorial neighborhood $S_{\pm\lambda}$ $S_{\arg(\pm\lambda),}$.

5.2.2. Sectorial isotropies in the space of leaves

Now, we consider a germ of a sectorial isotropy $\psi_{\pm\lambda} \quad \Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}})$ and we denote by $\Gamma_{\pm\lambda}$ the (germ of an) open subset of $\mathbb{C}^2$ such that:

$$H_{\pm\lambda} \circ \psi_\pm(S_{\pm\lambda} \times (\mathbb{C}^2, 0)) = S_{\pm\lambda} \times \Gamma_{\pm\lambda}\,.$$

Proposition 5.12. — *With the notations and assumptions above, the map*

$$\Psi_{\pm\lambda} := H_{\pm\lambda} \circ \psi_\pm \circ H_{\pm\lambda}^{-1} : S_{\pm\lambda} \times \Gamma_{\pm\lambda} \longrightarrow S_{\pm\lambda} \times \Gamma_{\pm\lambda} \tag{5.5}$$

is a sectorial germ of a fibered biholomorphism from $S_{\pm\lambda} \times \Gamma_{\pm\lambda}$ *to* $S_{\pm\lambda} \times \Gamma_{\pm\lambda}$*, which is of the form:*

$$\Psi_{\pm\lambda}(x, h_1, h_2) = (x, \Psi_{1,\pm\lambda}(h_1, h_2), \Psi_{2,\pm\lambda}(h_1, h_2)) .$$

In particular, $\Psi_{1,\pm\lambda}$ *and* $\Psi_{2,\pm\lambda}$ *are analytic and depend only on* $(h_1, h_2) \in \Gamma_{\pm\lambda}$*, while* $\Psi_{\pm\lambda}$ *induces a biholomorphism (still written* $\Psi_{\pm\lambda}$*):*

$$\begin{aligned} \Psi_{\pm\lambda} : \Gamma_{\pm\lambda} &\longrightarrow \Gamma_{\pm\lambda} \\ (h_1, h_2) &\longmapsto (\Psi_{1,\pm\lambda}(h_1, h_2), \Psi_{2,\pm\lambda}(h_1, h_2)) . \end{aligned}$$

Proof. — We only have to prove that $\Psi_{1,\pm\lambda}$ and $\Psi_{2,\pm\lambda}$ depend only on $(h_1, h_2) \in \Gamma_{\pm\lambda}$. By assumption, $\Psi_{\pm\lambda}$ is an isotropy of $x^2\frac{\partial}{\partial x}$:

$$(\Psi_{\pm\lambda})_* \left(x^2\frac{\partial}{\partial x}\right) = x^2\frac{\partial}{\partial x} . \tag{5.6}$$

We immediately obtain:

$$\frac{\partial \Psi_{1,\pm\lambda}}{\partial x} = \frac{\partial \Psi_{2,\pm\lambda}}{\partial x} = 0 .$$

In the space of leaves $\Gamma_{\pm\lambda}$ equipped with coordinates (h_1, h_2), we denote by w the product of h_1 and h_2:

$$w(h_1, h_2) := h_1 h_2 .$$

We define the two following quantities:

$$\begin{cases} f_1(x, w) := \exp\left(\dfrac{\lambda}{x} - c_m w^m \log(x) - \dfrac{\tilde{c}(wx^a)}{x}\right) x^{a_1} \\ f_2(x, w) := \exp\left(-\dfrac{\lambda}{x} + c_m w^m \log(x) + \dfrac{\tilde{c}(wx^a)}{x}\right) x^{a_2} , \end{cases} \tag{5.7}$$

such that the leaves of the foliations are parametrized by:

$$\begin{cases} y_1(x) = h_1 f_1(x, h_1 h_2) \\ y_2(x) = h_2 f_2(x, h_1 h_2) \end{cases} , \quad (h_1, h_2) \in \mathbb{C}^2 .$$

Notice that:

$$f_1(x, w) f_2(x, w) = x^a . \tag{5.8}$$

Moreover, one checks immediately the following statement.

Lemma 5.13. — *For all* $w \in \mathbb{C}$*:*

$$\begin{cases} \lim\limits_{\substack{x \to 0 \\ x \in S_\lambda}} |f_1(x, w)| = \lim\limits_{\substack{x \to 0 \\ x \in S_{-\lambda}}} |f_2(x, w)| = +\infty \\ \lim\limits_{\substack{x \to 0 \\ x \in S_{-\lambda}}} |f_1(x, w)| = \lim\limits_{\substack{x \to 0 \\ x \in S_\lambda}} |f_2(x, w)| = 0 . \end{cases}$$

Using notations of Proposition 5.12, we also assume from now on that $(\mathbb{C}^2,0)=\mathbf{D}(\mathbf{0},\mathbf{r})$, with $\mathbf{r}=(r_1,r_2)\in(\mathbb{R}_{>0})^2$ and $r_1,r_2>0$ small enough so that

$$\psi_{\pm\lambda}(S_{\pm\lambda}\times\mathbf{D}(\mathbf{0},\mathbf{r}))\subset S_{\pm\lambda}\times\mathbf{D}(\mathbf{0},\mathbf{r}')$$

for some $\mathbf{r}'=(r'_1,r'_2)\in(\mathbb{R}_{>0})^2$. Let us now define in a general way the following set associated to the sector $S_{\pm\lambda}$ and to a polydisc $\mathbf{D}(\mathbf{0},\tilde{\mathbf{r}})$, with $\tilde{\mathbf{r}}:=(\tilde{r}_1,\tilde{r}_2)$.

Definition 5.14. — *For all $x\in S_{\pm\lambda}$ et $\tilde{\mathbf{r}}:=(\tilde{r}_1,\tilde{r}_2)\in(\mathbb{R}_{>0})^2$, we define*

$$\Gamma_{\pm\lambda}(x,\tilde{\mathbf{r}}):=\left\{(h_1,h_2)\in\mathbb{C}^2 \;\middle|\; |h_j|\leqslant\frac{\tilde{r}_j}{|f_j(x,h_1h_2)|},\ \text{for } j\in\{1,2\}\right\}$$

We also consider the:

$$\begin{aligned}\Gamma_{\pm\lambda}(\tilde{\mathbf{r}})&:=\bigcup_{x\in S_{\pm\lambda}}\Gamma_{\pm\lambda}(x,\tilde{\mathbf{r}})\\ &=\left\{(h_1,h_2)\in\mathbb{C}^2 \;\middle|\; \exists x\in S_{\pm\lambda}\text{ s.t. } |h_j|\leqslant\frac{\tilde{r}_j}{|f_j(x,h_1h_2)|},\ \text{for } j\in\{1,2\}\right\}\end{aligned}$$

(cf. Figure 5.1).

Since we assume now that $(\mathbb{C}^2,0)=\mathbf{D}(\mathbf{0},\mathbf{r})$, then we have:

$$\Gamma_{\pm\lambda}=\Gamma_{\pm\lambda}(\mathbf{r}),$$

and

$$\Gamma_{\pm\lambda}\subset\Gamma_{\pm\lambda}(\mathbf{r}').$$

Remark 5.15.

(1) It is important to notice that the particular form of $\Psi_{\pm\lambda}$ implies that the image of any fiber

$$\{x=x_0\}\times\Gamma_{\pm\lambda}(x_0,\mathbf{r})$$

by $\Psi_{\pm\lambda}$ is included in a fiber of the form

$$\{x=x_0\}\times\Gamma_{\pm\lambda}(x_0,\mathbf{r}').$$

(2) If $(h_1,h_2)\in\Gamma_{\pm\lambda}(x,\mathbf{r})$, then

$$|h_1h_2|<\frac{r_1r_2}{|x^a|}. \tag{5.9}$$

(3) As $(h_1,h_2)\in\Gamma_{\pm\lambda}$ varies the values of $w=h_1h_2$ cover the whole $\mathbb{C}$.

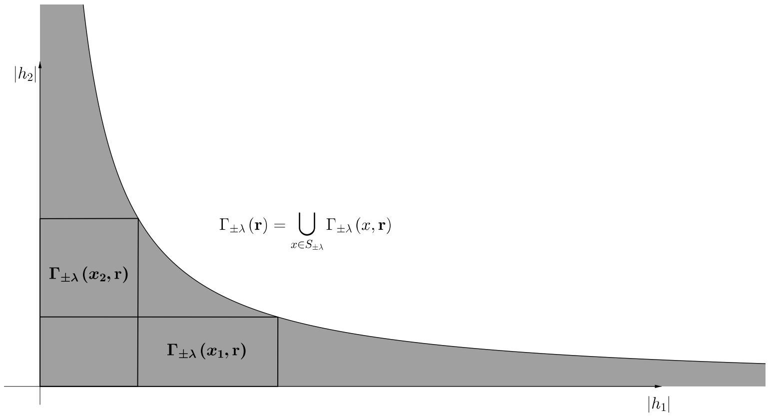

Figure 5.1. Representation of the space of leaves in terms of $|h_1|$ *and* $|h_2|$ *when* $c = 0$*: in this case, it is a* Reinhardt domain *(cf.* [8]*).*

5.2.3. Action on the resonant monomial in the space of leaves

Let us study the action of $\Psi_{\pm\lambda}$ on the resonant monomial $w = h_1 h_2$ in the space of leaves.

Lemma 5.16. — *We consider a biholomorphism*

$$\begin{aligned} \Psi_{\pm\lambda} : \Gamma_{\pm\lambda} &\longrightarrow \Gamma_{\pm\lambda} \\ (h_1, h_2) &\longmapsto \left(\Psi_{1,\pm}(h_1, h_2), \Psi_{2,\pm}(h_1, h_2)\right), \end{aligned}$$

such that for all $x \in S_{\pm\lambda}$*, we have*

$$\Psi_{\pm\lambda}(\Gamma_{\pm\lambda}(x_0, \mathbf{r})) \subset \Gamma_{\pm\lambda}(x_0, \mathbf{r}) .$$

We also define $\Psi_{w,\pm\lambda} := \Psi_{1,\pm\lambda}\Psi_{2,\pm\lambda}$*. Then, for all* $n \in \mathbb{N}$*, there exists entire (i.e. analytic over* $\mathbb{C}$*) functions* $\Psi_{w,\lambda,n}$ *and* $\Psi_{w,-\lambda,n}$ *such that*

$$\begin{cases} \Psi_{w,\lambda}(h_1, h_2) = \sum_{n \geq 0} \Psi_{w,\lambda,n}(h_1 h_2) h_1^n \\ \Psi_{w,-\lambda}(h_1, h_2) = \sum_{n > 0} \Psi_{w,-\lambda,n}(h_1 h_2) h_2^n . \end{cases}$$

Moreover, the series above uniformly converge (for the sup*-norm) in every subset of* $\Gamma_{\pm\lambda}$ *of the form* $\Gamma_{\pm\lambda}(\tilde{\mathbf{r}})$*, with* $\tilde{\mathbf{r}} := (\tilde{r}_1 \tilde{r}_2)$ *and*

$$0 < \tilde{r}_j < r_j , \quad \forall j \in \{1, 2\}$$

(cf. Definition 5.14). More precisely, for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ *such that*

$$0 < \tilde{r}_j + \delta < r_j\,, \quad \forall j \in \{1,2\}$$

for all $x \in S_{\pm\lambda}$ *and* $w \in \mathbb{C}$ *we have*

$$|wx^{a}| \leqslant \tilde{r}_1\tilde{r}_2 \Longrightarrow \begin{cases} |\Psi_{w,\lambda,n}(w)| \leqslant \dfrac{r_1 r_2}{|x^{a}|}\left|\dfrac{f_1(x,w)}{\tilde{r}_1+\delta}\right|^{n}, \\ |\Psi_{w,-\lambda,n}(w)| \leqslant \dfrac{r_1 r_2}{|x^{a}|}\left|\dfrac{f_2(x,w)}{\tilde{r}_2+\delta}\right|^{n}, \end{cases} \quad n \geqslant 0\,.$$

Proof. — Let us give the proof for $\Psi_{w,\lambda}, \Psi_{1,\lambda}$ and $\Psi_{2,\lambda}$ in Γ_λ (the same proof applies also for $\Psi_{w,-\lambda}$ in $\Gamma_{-\lambda}$ by exchanging the role played by h_1 and h_2). We fix some $0 < \tilde{r}_j < r_j$, $j \in \{1,2\}$, and $\delta > 0$ such that

$$0 < \tilde{r}_j + \delta < r_j\,, \quad \forall j \in \{1,2\}.$$

For a fixed value $w \in \mathbb{C}$, we consider the restriction of $\Psi_{w,\lambda}$ to the hypersurface $M_w := \{h_1 h_2 = w\} \cap \Gamma_\lambda$: this restriction is analytic in M_w. The map

$$\varphi_w : h_1 \mapsto \Psi_{w,\lambda}(h_1, \frac{w}{h_1})$$

is analytic in

$$M_{w,1} := \bigcup_{\substack{x \in S_\lambda \\ |wx^{a}| < r_1 r_2}} \Omega_{x,w}\,,$$

where for all $x \in S_\lambda$ with $|wx^{a}| < r_1 r_2$, the set $\Omega_{x,w}$ is the following annulus:

$$\Omega_{x,w} := \left\{ h_1 \in \mathbb{C} \;\middle|\; \left|\frac{w f_2(x,w)}{r_2}\right| < |h_1| < \left|\frac{r_1}{f_1(x,w)}\right| \right\}.$$

In particular, φ_w admits a Laurent expansion

$$\varphi_w(h_1) = \Psi_{w,+}\left(h_1, \frac{w}{h_1}\right) = \sum_{n \geqslant -L} \Psi_{w,+,n}(w) h_1^{n}$$

in every annulus $\Omega_{x,w}$, with $x \in S_\lambda$ such that $|wx^{a}| < r_1 r_2$. Moreover for all $x \in S_\lambda$ such that $|wx^{a}| < r_1 r_2$, Cauchy's formula gives

$$\Psi_{w,\lambda,n}(w) = \frac{1}{2i\pi}\oint_{\gamma(x,w)} \frac{\Psi_{w,\lambda}(h_1, \frac{w}{h_1})}{h_1^{n+1}} \mathrm{d}h_1\,, \text{ for all } n \in \mathbb{N}, \tag{5.10}$$

where $\gamma(x,w)$ is any circle (oriented positively) centered at the origin with a radius $\rho(x,w)$ satisfying

$$\left|\frac{w f_2(x,w)}{r_2}\right| < \rho(x,w) < \left|\frac{r_1}{f_1(x,w)}\right|.$$

If $|wx^{\mathbf{a}}| < (\tilde{r}_1 + \delta)(\tilde{r}_2 + \delta)$, we can take for instance

$$\rho(x, w) = \left|\frac{\tilde{r}_1 + \delta}{f_1(x, w)}\right| .$$

Therefore, for all $x \in S_\lambda$ and all $w \in \mathbb{C}$ such that $|wx^{\mathbf{a}}| \leqslant \tilde{r}_1\tilde{r}_2$, for all $\xi \in \mathbb{C}$ with $|\xi| < \delta$, we also have:

$$\Psi_{w,\lambda,n}(w + \xi) = \frac{1}{2i\pi} \oint_{\gamma(x,w)} \frac{\Psi_{w,\lambda}(h_1, \frac{w+\xi}{h_1})}{h_1^{n+1}} \mathrm{d}h_1 \, , \text{ for all } n \in \mathbb{Z}, \tag{5.11}$$

where $\gamma(x, w)$ is the same circle (of radius $\rho(x, w) = |\frac{\tilde{r}_1+\delta}{f_1(x,w)}|$) for all $|\xi| < \delta$. Moreover, since for all $x \in S_\lambda$, we have

$$\Psi_\lambda(\Gamma_\lambda(x, \mathbf{r})) \subset \Gamma_\lambda(x, \mathbf{r}') ,$$

and since for all $(h_1, h_2) \in \Gamma_\lambda(x, \mathbf{r}')$ we have

$$|h_1 h_2| \leqslant \frac{r_1' r_2'}{|x^{\mathbf{a}}|} ,$$

then for all $x \in S_\lambda$ and $w \in \mathbb{C}$ such that $|wx^{\mathbf{a}}| \leqslant \tilde{r}_1\tilde{r}_2$, the following inequality holds for all h_1 with $|h_1| < \frac{r_1}{f_1(x,w)}$:

$$|\Psi_{w,\lambda}(h_1, \frac{w}{h_1})| < \frac{r_1' r_2'}{|x^{\mathbf{a}}|} . \tag{5.12}$$

The well-known theorem regarding integrals depending analytically on a parameter asserts that for all $n \in \mathbb{Z}$ the mapping $\Psi_{w,\lambda,n}$ is analytic near any point $w \in \mathbb{C}$. Hence, it is an entire function (i.e. analytic over $\mathbb{C}$). Moreover, the inequality above and the Cauchy's formula together imply that for all $n \in \mathbb{Z}$ and for all $(x, w) \in S_\lambda \times \mathbb{C}$ such that $|wx^{\mathbf{a}}| \leqslant \tilde{r}_1\tilde{r}_2$, we have:

$$|\Psi_{w,\lambda,n}(w)| < \frac{r_1' r_2'}{|x^{\mathbf{a}}|\rho(x, w)^n} = \frac{r_1' r_2'}{|x^{\mathbf{a}}|} \left|\frac{f_1(x, w)}{\tilde{r}_1 + \delta}\right|^n . \tag{5.13}$$

According to Lemma 5.13, for a fixed value $w \in \mathbb{C}$, if $n < 0$, the right hand-side tends to 0 as x tends to 0 in S_λ. This implies in particular that $\Psi_{w,\lambda,n} = 0$ for all $n < 0$. Consequently:

$$\Psi_{w,\lambda}\left(h_1, \frac{w}{h_1}\right) = \sum_{n \geqslant 0} \Psi_{w,\lambda,n}(w) h_1^n . \tag{5.14}$$

Moreover, for all $w \in \mathbb{C}$ the series converges normally in every domain of the form

$$\Omega_{x,w} := \left\{ h_1 \in \mathbb{C} \,\middle|\, |h_1| \leqslant \left|\frac{\tilde{r}_1}{f_1(x, w)}\right| \right\} , \text{ for all } x \in S_\lambda \, , 0 < \tilde{r}_1 < r_1,$$

since the Laurent expansion's range is $n > 0$. This actually means that the series converges normally in an entire neighborhood of the origin in $\mathbb{C}$. In particular, for all fixed $w \in \mathbb{C}$, the map

$$h_1 \to \Psi_{w,\lambda}\left(h_1, \frac{w}{h_1}\right) = \sum_{n>0} \Psi_{w,\lambda,n}(w)h_1^n$$

is analytic in a neighborhood of the origin. Finally, the series

$$\Psi_{w,\lambda}(h_1,h_2) = \sum_{n>0} \Psi_{w,\lambda,n}(h_1h_2)h_1^n \tag{5.15}$$

converges normally, and hence its sum is analytic in every domain of the form $\Gamma_\lambda(\tilde{\mathbf{r}})$, with $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$.

5.2.4. Action on the resonant monomial

Since $\psi_{\pm\lambda} \in \Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}})$, the mapping $\psi_{\pm\lambda}$ is of the form

$$\psi_{\pm\lambda}(x,\mathbf{y}) = (x, \psi_{1,\pm\lambda}(x,\mathbf{y}), \psi_{2,\pm\lambda}(x,\mathbf{y}))\,, \tag{5.16}$$

with $\psi_{1,\pm\lambda}, \psi_{2,\pm\lambda}$ analytic and bounded in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$. Moreover, by assumption $\psi_{\pm\lambda}$ admits the identity as weak Gevrey-1 asymptotic expansion, i.e. we have a normally convergent expansion:

$$\psi_{i,\pm\lambda}(x,\mathbf{y}) = y_i + \sum_{\mathbf{k}\in\mathbb{N}^2} \psi_{i,\pm\lambda,\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}}\,, \tag{5.17}$$

where $\psi_{i,\pm\lambda,\mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion, for $i = 1,2$ and all $\mathbf{k} = (k_1,k_2) \in \mathbb{N}^2$.

Lemma 5.17. — *With the notations and assumptions above, let us define $\psi_{v,\pm\lambda} := \psi_{1,\pm\lambda}\psi_{2,\pm\lambda}$. Then $\psi_{v,\lambda}$ and $\psi_{v,-\lambda}$ can be expanded as the series*

$$\begin{cases} \psi_{v,\lambda}(x,\mathbf{y}) = y_1y_2 + x^a \displaystyle\sum_{n>1} \Psi_{w,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n \\ \psi_{v,-\lambda}(x,\mathbf{y}) = y_1y_2 + x^a \displaystyle\sum_{n>1} \Psi_{w,-\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right)^n \end{cases}$$

which are normally convergent in every subset of $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$ of the form $S_{\pm\lambda} \times \overline{\mathbf{D}}(\mathbf{0},\tilde{\mathbf{r}})$, where $\overline{\mathbf{D}}(\mathbf{0},\tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1,\tilde{r}_2)$ such that

$$0 < \tilde{r}_j < r_j\,, \quad \forall j \in \{1,2\}.$$

Here $\Psi_{w,\lambda,n}$ and $\Psi_{w,-\lambda,n}$, for $n \in \mathbb{N}$, are the ones appearing in Lemma 5.16. Moreover, for all closed sub-sector $S \subset S_{\pm\lambda}$ and for all closed poly-disc $\overline{\mathbf{D}} \subset \mathbf{D}(\mathbf{0},\mathbf{r})$, there exists $A, B > 0$ such that:

$$|\psi_{v,\pm\lambda}(x,y_1,y_2) - y_1y_2| \leqslant A\exp\left(-\frac{B}{|x|}\right), \quad \forall (x,\mathbf{y}) \in S \times \overline{\mathbf{D}}. \tag{5.18}$$

In particular, $\psi_{v,\pm\lambda}$ admits y_1y_2 as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$.

Proof. — By definition, we have

$$\Psi_{\pm\lambda} \circ H_{\pm\lambda} = H_{\pm\lambda} \circ \psi_{\pm\lambda}.$$

In particular, for all $(x,\mathbf{y}) \in S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$:

$$\Psi_{w,\pm}\left(x, \frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}, \frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right) = \frac{\psi_{v,\pm}(x,y_1,y_2)}{x^a}. \tag{5.19}$$

Thus, according to Lemma 5.16 we have:

$$\begin{cases} \psi_{v,\lambda}(x,\mathbf{y}) = x^a \sum_{n\geqslant 0} \Psi_{w,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n \\ \psi_{v,-\lambda}(x,\mathbf{y}) = x^a \sum_{n\geqslant 0} \Psi_{w,-\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right)^n \end{cases}. \tag{5.20}$$

Besides we know that $\psi_{v,\pm\lambda}$ admits y_1y_2 as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$:

$$\psi_{v,\pm\lambda}(x,y_1,y_2) = y_1y_2 + \sum_{\mathbf{k}\in\mathbb{N}^2} \psi_{v,\pm\lambda,\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}}, \tag{5.21}$$

where for all $\mathbf{k} = (k_1,k_2) \in \mathbb{N}^2$ the mapping $\psi_{v,\pm\lambda,\mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. Let us compare both expressions of $\psi_{v,\pm\lambda}$ above. Looking at monomials $\mathbf{y}^{\mathbf{k}}$ with $k_1 = k_2$ in (5.21), and at terms corresponding to $n = 0$ on the right-hand side of (5.20), we must have for all $x \in S_{\pm\lambda}$ and $v \in \mathbb{C}$ with $|v| < r_1r_2$:

$$v + \sum_{k\geqslant 0} \psi_{v,\lambda,\mathbf{k}(k,k)}(x)v^k = x^a\Psi_{w,\lambda,0}\left(\frac{v}{x^a}\right). \tag{5.22}$$

Since $\Psi_{w,\pm\lambda,0}$ is analytic in $\mathbb{C}$, there exists $(\alpha_{\pm\lambda,k})_{k\in\mathbb{N}} \subset \mathbb{C}$ such that

$$\Psi_{w,\pm\lambda,0}\left(\frac{v}{x^a}\right) = \sum_{k\geqslant 0} \alpha_{\pm\lambda,k}\left(\frac{v}{x^a}\right)^k. \tag{5.23}$$

This can only happen if $\alpha_{\pm\lambda,k} = 0$ whenever $k = 1$, for $\psi_{v,\pm\lambda,\mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. A further immediate identification yields

$$\Psi_{v,\pm\lambda,0}(w) = w.$$

Thus

$$\begin{cases} \psi_{v,\lambda}(x,\mathbf{y}) = y_1y_2 + x^a \sum_{n\geqslant 1} \Psi_{w,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n \\ \psi_{v,-\lambda}(x,\mathbf{y}) = y_1y_2 + x^a \sum_{n\geqslant 1} \Psi_{w,-\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right)^n \end{cases}.$$

Let us prove that $\psi_{v,\pm\lambda}$ admits y_1y_2 as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times (\mathbb{C}^2,0)$. We have to show that $|\psi_{v,\pm\lambda}(x,y_1,y_2) - y_1y_2|$ is exponentially small with respect to $x \in S_{\pm\lambda}$, uniformly in $\mathbf{y} \in \mathbf{D}(\mathbf{0},\mathbf{r})$. As for the previous lemma, we perform the proof for $\psi_{v,\lambda}$ only (the same proof applies for $\psi_{v,-\lambda}$ by exchanging y_1 and y_2). From the computations above we derive

$$|\psi_{v,\lambda}(x,y_1,y_2) - y_1y_2| \leqslant \sum_{n\geqslant 1} \left| x^a \Psi_{w,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n \right|. \tag{5.24}$$

Let us fix $\tilde{r}_1, \tilde{r}_2, \delta > 0$ in such a way that

$$0 < \tilde{r}_j + \delta < r_j\ ,\ j \in \{1,2\}.$$

Let us take $|x|$, $|y_1|$ and $|y_2|$ small enough so that

$$2x \in S_\lambda$$

and

$$|y_1|y_2 < \frac{\tilde{r}_1\tilde{r}_2}{|2^a|} < r_1r_2.$$

According to Lemma 5.16, for all $\tilde{x} \in S_\lambda$ and all $w \in \mathbb{C}$:

$$|w\tilde{x}^a| \leqslant \tilde{r}_1\tilde{r}_2 \Longrightarrow |\Psi_{w,\lambda,n}(w)| \leqslant \frac{r_1r_2}{|\tilde{x}^a|}\left|\frac{f_1(\tilde{x},w)}{\tilde{r}_1+\delta}\right|^n. \tag{5.25}$$

In particular for $\tilde{x} = 2x$ and $w = \frac{y_1y_2}{x^a}$ we derive $|w\tilde{x}^a| < \tilde{r}_1\tilde{r}_2$, from which we conclude

$$\left|\Psi_{w,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\right| \leqslant \frac{r_1r_2}{|2^a x^a|}\left|\frac{f_1(2x,\frac{y_1y_2}{x^a})}{\tilde{r}_1+\delta}\right|^n. \tag{5.26}$$

Consequently, for all $(x,y_1,y_2) \in S_\lambda \times \mathbf{D}(\mathbf{0},\tilde{\mathbf{r}})$ with

$$\begin{cases} 2x \in S_\lambda \\ |y_1y_2| < \frac{\tilde{r}_1\tilde{r}_2}{|2^a|} < r_1r_2, \end{cases}$$

we have

$$\begin{aligned}|\psi_{v,\lambda}(x,y_1,y_2) - y_1y_2| &\leqslant \sum_{n\geqslant 1}\left|x^a\frac{r_1r_2}{2^a x^a}\left(\frac{f_1(2x,\frac{y_1y_2}{x^a})}{\tilde{r}_1+\delta}\right)^n\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n\right|\\ &\leqslant \frac{r_1r_2}{|2^a|}\sum_{n\geqslant 1}\left|\left(\frac{y_1}{\tilde{r}_1+\delta}\right)^n\left(\frac{f_1(2x,\frac{y_1y_2}{x^a})}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n\right|.\end{aligned}$$

Since $\tilde{c}(v)$ is the germ of an analytic function near the origin which is null at the origin, we can take r_1,$r_2 > 0$ small enough in order that for all closed sub-sector $S \subset S_\lambda$, for all $\tilde{r}_1 \in]0, r_1[$ and $\tilde{r}_2 \in]0, r_2[$, there exist $A, B > 0$ satisfying:

$$(5.27)\quad \forall(x,y_1,y_2)\in S\times\mathbf{D}(\mathbf{0},\tilde{\mathbf{r}}) \Longrightarrow |\psi_{v,\lambda}(x,y_1,y_2)-y_1y_2| \leqslant A\exp\left(-\frac{B}{|x|}\right).$$

Let us prove this. We need here to estimate the quantity:

$$(5.28)\quad \begin{aligned}&\left|\frac{f_1(2x,\frac{y_1y_2}{x^a})}{f_1(x,\frac{y_1y_2}{x^a})}\right|\\ &= \left|2^{a_1}\exp\left(-\frac{\lambda}{2x} - c_m\frac{(y_1y_2)^m}{x}\log(2) - \frac{\tilde{c}(y_1y_2 2^a)}{2x} + \frac{\tilde{c}(y_1y_2)}{x}\right)\right|.\end{aligned}$$

On only have tot deal with the case where $x \in S$ is such that $2x \in S$ (otherwise, x is "far from the origin", and we conclude without difficulty). We have:

$$(5.29)\quad \begin{aligned}&\forall(x,y_1,y_2)\in S\times\mathbf{D}(\mathbf{0},\tilde{\mathbf{r}}) \text{ et } 2x\in S\\ &\Longrightarrow \left|\frac{f_1(2x,\frac{y_1y_2}{x^a})}{f_1(x,\frac{y_1y_2}{x^a})}\right| \leqslant |2^{a_1}|\exp\left(-\frac{B}{|x|}\right) < 1.\end{aligned}$$

Hence

$$\begin{aligned}|\psi_{v,\lambda}(x,y_1,y_2) - y_1y_2| &\leqslant \frac{r_1r_2}{|2^a|}\sum_{n\geqslant 1}\left|\frac{2^{a_1}y_1}{\tilde{r}_1+\delta}\exp\left(-\frac{B}{|x|}\right)\right|^n\\ &\leqslant \frac{r_1r_2}{|2^a|}\frac{|\frac{2^{a_1}y_1}{\tilde{r}_1+\delta}\exp(-\frac{B}{|x|})|}{1-|\frac{2^{a_1}y_1}{\tilde{r}_1+\delta}\exp(-\frac{B}{|x|})|}\\ &\leqslant A\exp\left(-\frac{B}{|x|}\right),\end{aligned}$$

for a convenient $A > 0$.

The latter lemma implies $\Psi_{v,\pm\lambda,0}(w) = w$, having for consequence the next result.

Corollary 5.18. — *For all closed sub-sector $S \subset S_{\pm\lambda}$ and for all $\tilde{r}_1 \in]0, r_1[$ and $\tilde{r}_2 \in]0, r_2[$, there exists $A, B > 0$ such that for all $x \in S$:*

$$\left\{\begin{array}{l} |h_1| \leqslant \dfrac{\tilde{r}_1}{|f_1(x, h_1h_2)|} \\ |h_2| \leqslant \dfrac{\tilde{r}_2}{|f_2(x, h_1h_2)|} \end{array}\right. \Longrightarrow \quad |\Psi_{w,\pm}(x, h_1, h_2) - h_1h_1| \leqslant \frac{A\exp(-\frac{B}{|x|})}{|x^a|} .$$

In particular, there exists $C > 0$ such that:

$$\left\{\begin{array}{l} |h_1| \leqslant \dfrac{\tilde{r}_1}{|f_1(x, h_1h_2)|} \\ |h_2| \leqslant \dfrac{\tilde{r}_2}{|f_2(x, h_1h_2)|} \end{array}\right.$$

$$\Longrightarrow \quad \left|\frac{\exp\left(c_m(h_1h_2)^m \log(x) + \frac{\tilde{c}(x^a(h_1h_2)^m)}{x}\right)}{\exp\left(c_m(\Psi_w(x, h_1, h_2))^m \log(x) + \frac{\tilde{c}(x^a(\Psi_w(x,h_1,h_2))^m)}{x}\right)}\right| < C.$$

5.2.5. Power series expansion of sectorial isotropies in the space of leaves

Now, we give a power series expansion of $\Psi_{1,\pm\lambda}$ and $\Psi_{2,\pm\lambda}$ in the space of leaves. Let us introduce the following notations:

$$\left\{\begin{array}{l} N(1, +) := N(2, -) := 1 \\ N(1, -) := N(2, +) := -1 . \end{array}\right.$$

Lemma 5.19. — *With the notations and assumptions above, there exists entire functions (i.e. analytic over $\mathbb{C}$) denoted by $\Psi_{j,\pm\lambda,n}$, $j \in \{1, 2\}$, $n \geqslant N(j, \pm)$, such that for $j \in \{1, 2\}$:*

$$\left\{\begin{array}{l} \Psi_{j,\lambda}(h_1, h_2) = \displaystyle\sum_{n \geqslant N(j,+)} \Psi_{j,\lambda,n}(h_1h_2)h_1^n \\ \Psi_{j,-\lambda}(h_1, h_2) = \displaystyle\sum_{n \geqslant N(j,-)} \Psi_{j,\lambda,n}(h_1h_2)h_2^n . \end{array}\right.$$

These series converge normally in every subset of $\Gamma_{\pm\lambda}$ of the form $\Gamma_{\pm\lambda}(\tilde{\mathbf{r}})$ with $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$ (cf. Definition 5.14). More precisely, for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ such that

$$0 < \tilde{r}_j + \delta < r_j \ , \ j \in \{1, 2\}$$

there exists $C > 0$ such that for all $x \in S_{\pm\lambda}$ and for all $w \in \mathbb{C}$, we have:

$$|wx^a| \leqslant \tilde{r}_1\tilde{r}_2 \Rightarrow \begin{cases} |\Psi_{1,\lambda,n}(w)| < Cr_1\dfrac{|f_1(x,w)|^{n-1}}{(\tilde{r}_1+\delta)^n}, & n \geqslant 1 \\ |\Psi_{2,\lambda,n}(w)| < \dfrac{Cr_2}{|x^a|}\dfrac{|f_1(x,w)|^{n+1}}{(\tilde{r}_1+\delta)^n}, & n \geqslant -1 \\ |\Psi_{1,-\lambda,n}(w)| < \dfrac{Cr_1}{|x^a|}\dfrac{|f_2(x,w)|^{n+1}}{(\tilde{r}_2+\delta)^n}, & n \geqslant -1 \\ |\Psi_{2,-\lambda,n}(w)| < Cr_2\dfrac{|f_2(x,w)|^{n-1}}{(\tilde{r}_2+\delta)^n}, & n \geqslant 1\,. \end{cases}$$

Moreover:

$$\Psi_{1,-\lambda,-1}(0) = \Psi_{2,\lambda,-1}(0) = 0\,.$$

Proof. — We use the same notations as in the proof of Lemma 5.16, and as usual, we give the proof only for Ψ_λ (the proof for $\Psi_{-\lambda}$ is analogous, by exchanging the role played by h_1 and h_2). For fixed $w \in \mathbb{C}$, the maps

$$\varphi_1 : h_1 \mapsto \Psi_{1,\lambda}\left(h_1, \frac{w}{h_1}\right)$$

and

$$\varphi_2 : h_1 \mapsto \Psi_{2,\lambda}\left(h_1, \frac{w}{h_1}\right)$$

are analytic in

$$M_{w,1} = \bigcup_{\substack{x \in S_\lambda \\ |wx^a| < r_1r_2}} \Omega_{x,w}$$

(see the proof of Lemma 5.16). In particular, φ_1 and φ_2 admit Laurent expansions

$$\begin{cases} \varphi_1(h_1) = \Psi_{1,\lambda}\left(h_1, \dfrac{w}{h_1}\right) = \displaystyle\sum_{n \geqslant -L_1} \Psi_{1,\lambda,n}(w)h_1^n \\ \varphi_2(h_1) = \Psi_{2,\lambda}\left(h_1, \dfrac{w}{h_1}\right) = \displaystyle\sum_{n \geqslant -L_2} \Psi_{2,\lambda,n}(w)h_1^n \end{cases}$$

in every annulus $\Omega_{x,w}$, with $x \in S_\lambda$ such that $|wx^a| < r_1r_2$. Using the same method as in the proof of Lemma 5.16, we prove without additional difficulties that for all $n \in \mathbb{Z}$, $\Psi_{1,\lambda,n}$ and $\Psi_{2,\lambda,n}$ are analytic in any point $w \in \mathbb{C}$, and thus are entire functions (i.e. analytic over $\mathbb{C}$). Moreover, we also show in the same way as earlier that for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ with

$$0 < \tilde{r}_j + \delta < r_j\ ,\ j \in \{1,2\}\,,$$

for all $n \in \mathbb{Z}$ and for all $(x,w) \in S_\lambda \times \mathbb{C}$ such that $|wx^{\mathbf{a}}| \leqslant \tilde{r}_1\tilde{r}_2$, we have:

$$\begin{cases} |\Psi_{1,\lambda,n}(w)| < \dfrac{r_1}{|f_1(x, \Psi_{w,\lambda}(x, h_1, \frac{w}{h_1}))|}\left(\dfrac{f_1(x,w)}{\tilde{r}_1+\delta}\right)^n \\ |\Psi_{2,\lambda,n}(w)| < \dfrac{r_2}{|f_2(x, \Psi_{w,\lambda}(x, h_1, \frac{w}{h_1}))|}\left(\dfrac{f_1(x,w)}{\tilde{r}_1+\delta}\right)^n . \end{cases}$$

According to Corollary 5.18, there exists $C>0$ such that for all $(x,w) \in S_\lambda \times \mathbb{C}$ with $|wx^{\mathbf{a}}| \leqslant \tilde{r}_1\tilde{r}_2$, we have:

$$\begin{cases} |\Psi_{1,\lambda,n}(w)| < Cr_1\dfrac{|f_1(x,w)|^{n-1}}{(\tilde{r}_1+\delta)^n} \\ |\Psi_{2,\lambda,n}(w)| < \dfrac{Cr_2}{|x^{\mathbf{a}}|}\dfrac{|f_1(x,w)|^{n+1}}{(\tilde{r}_1+\delta)^n} . \end{cases}$$

According to the statement in Lemma 5.13, for a fixed value $w \in \mathbb{C}$, if we look at the limit as x tends to 0 in S_λ of the right hand-sides above we deduce that:

$$\begin{cases} |\Psi_{1,\lambda,n}(w)| = 0, & n \leqslant 0 \\ |\Psi_{2,\lambda,n}(w)| = 0, & n \leqslant -2 . \end{cases}$$

Consequently:

$$\begin{cases} \Psi_{1,\lambda}(h_1,h_2) = \displaystyle\sum_{n \geqslant 1} \Psi_{1,\lambda,n}(h_1h_2)h_1^n \\ \Psi_{2,\lambda}(h_1,h_2) = \displaystyle\sum_{n \geqslant -1} \Psi_{2,\lambda,n}(h_1h_2)h_1^n . \end{cases}$$

These function series converges normally (and are analytic) in every domain of the form $\Gamma_\lambda(\tilde{\mathbf{r}})$ with $\tilde{\mathbf{r}} := (\tilde{r}_1, \tilde{r}_2)$ and

$$0 < \tilde{r}_j + \delta < r_j \ , \ j \in \{1,2\}$$

(cf. Definition 5.14). Moreover, for any fixed value of h_2, on the one hand the function series

$$h_1 \mapsto \Psi_{2,\lambda}(h_1,h_2) = \sum_{n \geqslant -1} \Psi_{2,\lambda,n}(h_1h_2)h_1^n$$

is analytic in a punctured disc, since

$$|f_2(x,h_1,h_2)| \underset{\substack{x \to 0 \\ x \in S_\lambda}}{\longrightarrow} 0 ,$$

and on the other hand, we already know that the function $h_1 \mapsto \Psi_{2,\lambda}(h_1,h_2)$ is analytic in a neighborhood of the origin. Thus, we must have $\Psi_{2,\lambda,-1}(0) = 0$.

5.2.6. Sectorial isotropies: proof of Proposition 5.2

The following lemma is a more precise version of Proposition 5.2. We recall the notations:

$$\begin{cases} N(1,+) = N(2,-) = 1 \\ N(1,-) = N(2,+) = -1. \end{cases}$$

Lemma 5.20. — *With the notations and assumptions above, we consider $\psi_{\pm\lambda} \in \Lambda_{\pm\lambda}^{(\text{weak})}(Y_{\text{norm}})$, with*

$$\psi_{\pm\lambda}(x,\mathbf{y}) = (x, \psi_{1,\pm\lambda}(x,\mathbf{y}), \psi_{2,\pm\lambda}(x,\mathbf{y})) . \tag{5.30}$$

Then, for $i \in \{1,2\}$, $\psi_{i,\lambda}$ and $\psi_{i,-\lambda}$ can be written as power series as follows:

$$\begin{cases} \psi_{i,\lambda}(x,\mathbf{y}) = y_i + f_i\left(x, \frac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right) \left(\sum_{n > N(i,+)+1} \Psi_{i,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}\right)^n\right) \\ \psi_{i,-\lambda}(x,\mathbf{y}) = y_i + f_i\left(x, \frac{\psi_{v,-\lambda}(x,\mathbf{y})}{x^a}\right) \left(\sum_{n > N(i,-)+1} \Psi_{i,-\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right)^n\right) \end{cases}$$

which are normally convergent in every subset of $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$ of the form $S_{\pm\lambda} \times \overline{\mathbf{D}}(\mathbf{0},\tilde{\mathbf{r}})$, where $\overline{\mathbf{D}}(\mathbf{0},\tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1,\tilde{r}_2)$ such that $0 < \tilde{r}_j < r_j$, $j \in \{1,2\}$. Here $\Psi_{i,\lambda,n}$, $\Psi_{i,-\lambda,n}$ (for $i = 1,2$ and $n \in \mathbb{N}$) are given in Lemma 5.19. Moreover, for all closed sub-sector $S \subset S_{\pm\lambda}$ and for all closed poly-disc $\overline{\mathbf{D}} \subset \mathbf{D}(\mathbf{0},\mathbf{r})$, there exists $A, B > 0$ such that for $j = 1,2$:

$$|\psi_{j,\pm\lambda}(x,y_1,y_2) - y_j| \leqslant A\exp\left(-\frac{B}{|x|}\right), \quad \forall (x,\mathbf{y}) \in S \times \overline{\mathbf{D}} . \tag{5.31}$$

As a consequence, $\psi_{j,\pm\lambda}$ admits y_j as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$.

Remark 5.21. — In particular,we have $\Psi_{1,\lambda,1}(w) = \Psi_{2,-\lambda,1}(w) = 1$ and $\Psi_{1,-\lambda,-1}(w) = \Psi_{2,\lambda,-1}(w) = w$.

Proof. — By definition, we have

$$\Psi_{\pm\lambda} \circ H_{\pm\lambda} = H_{\pm\lambda} \circ \psi_{\pm} .$$

In particular, for $j = 1,2$ and all $(x,\mathbf{y}) \in S_{\pm\lambda} \times \mathbf{D}(\mathbf{0},\mathbf{r})$:

$$\Psi_{j,\pm\lambda}\left(x, \frac{y_1}{f_1(x,\frac{y_1y_2}{x^a})}, \frac{y_2}{f_2(x,\frac{y_1y_2}{x^a})}\right) = \frac{\psi_{j,\pm\lambda}(x,y_1,y_2)}{f_j(x,\frac{\psi_{v,\pm}(x,y_1,y_2)}{x^a})} . \tag{5.32}$$

Thus, according to Lemma 5.19 we have for $i = 1, 2$:

(5.33)
$$\begin{cases} \psi_{i,\lambda}(x,\mathbf{y}) = f_i\left(x, \dfrac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right) \displaystyle\sum_{n>N(i,+)} \Psi_{i,\lambda,n}\left(\dfrac{y_1y_2}{x^a}\right)\left(\dfrac{y_1}{f_1(x, \frac{y_1y_2}{x^a})}\right)^n \\ \psi_{i,-}(x,\mathbf{y}) = f_i\left(x, \dfrac{\psi_{v,-\lambda}(x,\mathbf{y})}{x^a}\right) \displaystyle\sum_{n>N(i,-)} \Psi_{i,-\lambda,n}\left(\dfrac{y_1y_2}{x^a}\right)\left(\dfrac{y_2}{f_2(x, \frac{y_1y_2}{x^a})}\right)^n \end{cases}$$

and these series are normally convergent (and then define analytic functions) in any domain of the form $S \times \overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$, where S is a closed sub-sector of $S_{\pm\lambda}$ and $\overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2)$ such that

$$0 < \tilde{r}_j < r_j\ ,\ j \in \{1, 2\}\ .$$

Let us compare the different expressions of $\psi_{j,\pm\lambda}$, $j = 1, 2$. We know that $\psi_{j,\pm\lambda}(x, y_1, y_2)$ admits y_j as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. Thus, we can write:

$$\psi_{j,\pm\lambda}(x, y_1, y_2) = y_j + \sum_{\mathbf{k} \in \mathbb{N}^2} \psi_{j,\pm\lambda,\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}}\ , \tag{5.34}$$

where for all $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$, $\psi_{j,\pm\lambda,\mathbf{k}}$ is analytic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. As usual, let us deal with the case of $\psi_{1,\lambda}$ and $\psi_{2,\lambda}$ (the other one being similar by exchanging y_1 and y_2). According to the expressions given by Lemmas 5.16 and 5.19, we can be more precise on the index sets in the sums above:

$$\begin{cases} \psi_{1,\lambda}(x, y_1, y_2) = y_1 + \displaystyle\sum_{\substack{\mathbf{k}=(k_1,k_2) \in \mathbb{N}^2 \\ k_1 > k_2+1}} \psi_{1,\lambda,\mathbf{k}}(x) y_1^{k_1} y_2^{k_2} \\ \psi_{2,\lambda}(x, y_1, y_2) = y_2 + \displaystyle\sum_{\substack{\mathbf{k}=(k_1,k_2) \in \mathbb{N}^2 \\ k_1 > k_2}} \psi_{2,\lambda,\mathbf{k}}(x) y_1^{k_1} y_2^{k_2}\ . \end{cases} \tag{5.35}$$

Let us deal with $\psi_{1,\lambda}$ (a similar proof holds for $\psi_{2,\lambda}$). Looking at terms for $n = 1$ in (5.33) and at monomials terms $\mathbf{y}^{\mathbf{k}}$ such that $k_1 \leqslant k_2 + 1$ in (5.35), we must have for all $x \in S_\lambda$, $y_1, y_2 \in \mathbb{C}$ with $|y_1| < r_1$, $|y_2| < r_2$:

$$1 + \sum_{k>0} \psi_{1,\lambda,(k+1,k)}(x) y_1^k y_2^k = \frac{f_1(x, \frac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a})}{f_1(x, \frac{y_1y_2}{x^z})} \Psi_{1,\lambda,1}\left(\frac{y_1y_2}{x^a}\right)\ . \tag{5.36}$$

According to Lemma 5.17 and Corollary5.18, we have:

$$\begin{aligned}\frac{f_1\left(x,\frac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right)}{f_1\left(x,\frac{y_1y_2}{x^z}\right)} &= 1+\sum_{j_1>j_2+1>1} F_{j_1,j_2}(x)y_1^{j_1}y_2^{j_2}\\ &= 1+\underset{(x,\mathbf{y})\in S_\lambda\times\mathbf{D}(\mathbf{0},\mathbf{r})}{\underset{(x,\mathbf{y})\to 0}{\mathrm{O}}}\left(|y_1|\right),\end{aligned}$$

for some analytic and bounded functions $F_{j_1,j_2}(x)$, $j_1 > j_2$. As in the proof of Lemma 5.17, using the fact that ψ_λ admits the identity as weak Gevrey-1 asymptotic expansion, we deduce that $\Psi_{1,\lambda,1}(w)=1$, and then:

$$\begin{aligned}\psi_{1,\lambda}(x,\mathbf{y}) &= y_1+f_1\left(x,\frac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right)\left(\sum_{n>2}\Psi_{1,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1\left(x,\frac{y_1y_2}{x^a}\right)}\right)^n\right)\\ &= y_1+\sum_{\substack{\mathbf{k}=(k_1,k_2)\in\mathbb{N}^2\\ k_1>k_2+2}}\psi_{1,\lambda,\mathbf{k}}(x)y_1^{k_1}y_2^{k_2}.\end{aligned}$$

It remains to show that $\psi_{1,\lambda}$ admits y_1 as Gevrey-1 asymptotic expansion in $S_\lambda\times\mathbf{D}(\mathbf{0},\mathbf{r})$. From the computations above, we deduce:

$$\begin{aligned}&\left|\psi_{1,\lambda}(x,y_1,y_2)-y_1\right|\\ &\qquad\leqslant\sum_{n>2}\left|\Psi_{1,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1\left(x,\frac{y_1y_2}{x^a}\right)}\right)^{n-1}\frac{f_1\left(x,\frac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right)}{f_1\left(x,\frac{y_1y_2}{x^a}\right)}y_1\right|.\end{aligned}$$

Using Lemma 5.19, Corollary 5.18 and the same method as at the end of the proof of Lemma 5.17, we can show the following: we can take $r_1,r_2>0$ small enough such that for all closed sub-sector S of S_λ for all $\tilde{r}_1\in\,]0,r_1[$ and $\tilde{r}_2\in\,]0,r_2[$, there exists $A,B>0$ satisfying:

$$(5.37)\quad (x,y_1,y_2)\in S\times\mathbf{D}(\mathbf{0},\tilde{\mathbf{r}})\Longrightarrow\left|\psi_{1,\lambda}(x,y_1,y_2)-y_1\right|\leqslant A\exp\left(-\frac{B}{|x|}\right).$$

A similar proof holds for $\psi_{2,\lambda},\psi_{2,-\lambda}$ and $\psi_{1,-\lambda}$.

Remark 5.22. — It should be noticed that in the expressions

$$\begin{cases}\psi_{1,\lambda}(x,\mathbf{y})=y_1+f_1\left(x,\dfrac{\psi_{v,\lambda}(x,\mathbf{y})}{x^a}\right)\left(\displaystyle\sum_{n>2}\Psi_{1,\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_1}{f_1\left(x,\frac{y_1y_2}{x^a}\right)}\right)^n\right)\\ \psi_{1,-\lambda}(x,\mathbf{y})=y_1+f_1\left(x,\dfrac{\psi_{v,-\lambda}(x,\mathbf{y})}{x^a}\right)\left(\displaystyle\sum_{n>0}\Psi_{1,-\lambda,n}\left(\frac{y_1y_2}{x^a}\right)\left(\frac{y_2}{f_2\left(x,\frac{y_1y_2}{x^a}\right)}\right)^n\right)\end{cases}$$

given by Lemma 5.20, the expansion of $\psi_{1,\lambda}$ with respect to $\mathbf{y}=(y_1,y_2)$ starts with a term of order 1, namely y_1, followed by terms of order at least 2, while in the expansion of $\psi_{1,-\lambda}$, the term of lowest order is a constant, namely $\Psi_{1,-\lambda,0}(0)$. Similarly, the expansion of $\psi_{2,-\lambda}$ (with respect to $\mathbf{y}=(y_1,y_2)$) starts with y_2, while the expansion of $\psi_{1,-\lambda}$ starts with the constant $\Psi_{2,\lambda,0}(0)$.

5.3. Description of the moduli space and some applications

From Lemmas 5.19 and 5.20, we can give a description of the moduli space $\Lambda_\lambda(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}(Y_{\mathrm{norm}})$ of a fixed analytic normal form Y_{norm}.

5.3.1. A power series presentation of the moduli space

We use the notations introduced in Section 4. We denote by $\mathcal{O}(\mathbb{C})$ the set of entire functions, i.e. of functions holomorphic in $\mathbb{C}$. We consider the functions f_1 and f_2 defined in (5.7) and introduce four subsets of $(\mathcal{O}(\mathbb{C}))^{\mathbb{N}}$, denoted by $E_{1,\lambda}(Y_{\mathrm{norm}})$, $E_{2,\lambda}(Y_{\mathrm{norm}})$, $E_{1,-\lambda}(Y_{\mathrm{norm}})$ and $E_{2,-\lambda}(Y_{\mathrm{norm}})$, defined as follows. On remind the notations

$$\begin{cases} N(1,+)=N(2,-)=1 \\ N(1,-)=N(2,+)=-1. \end{cases}$$

Definition 5.23. — *For $j \in \{1,2\}$, a sequence $(\psi_n(w))_{n\geqslant N(j,\pm)+1} \in (\mathcal{O}(\mathbb{C}))^{\mathbb{N}}$ belongs to $E_{j,\pm\lambda}(Y_{\mathrm{norm}})$ if there exists an open polydisc $\mathbf{D}(\mathbf{0},\mathbf{r})$ and an open asymptotic sector*

$$S_{\pm\lambda} \in \mathcal{AS}_{\arg(\pm\lambda),2\pi} \tag{5.38}$$

such that for all $\tilde r_1, \tilde r_2, \delta > 0$ with

$$0<\tilde r_i+\delta<r_i \ , \quad i \in \{1,2\}$$

there exists $C>0$ such that for all $x \in S_\lambda$ (resp. $x \in S_{-\lambda}$) and for all $w \in \mathbb{C}$, if $|wx^{\mathbf{a}}| \leqslant \tilde r_1\tilde r_2$ then:

$$\begin{cases} |\psi_n(w)| < C\dfrac{|f_1(x,w)|^{n-1}}{(\tilde r_1+\delta)^n}, & n \geqslant 2, \text{ if } (\psi_n(w))_{n\geqslant 2} \in E_{1,\lambda}(Y_{\mathrm{norm}}) \\ |\psi_n(w)| < \dfrac{C}{|x^{\mathbf{a}}|}\dfrac{|f_1(x,w)|^{n+1}}{(\tilde r_1+\delta)^n}, & n \geqslant 0, \text{ if } (\psi_n(w))_{n\geqslant 2} \in E_{2,\lambda}(Y_{\mathrm{norm}}) \end{cases}$$

$$\begin{cases} |\psi_n(w)| < \dfrac{C}{|x^{\mathbf{a}}|}\dfrac{|f_2(x,w)|^{n+1}}{(\tilde r_2+\delta)^n}, & n \geqslant 0, \text{ if } (\psi_n(w))_{n\geqslant 2} \in E_{1,-\lambda}(Y_{\mathrm{norm}}) \\ |\psi_n(w)| < C\dfrac{|f_2(x,w)|^{n-1}}{(\tilde r_2+\delta)^n}, & n \geqslant 2, \text{ if } (\psi_n(w))_{n\geqslant 2} \in E_{2,-\lambda}(Y_{\mathrm{norm}}) . \end{cases}$$

As explained in Section 4, we can associate to any pair

$$(\psi_\lambda, \psi_{-\lambda}) \in \Lambda_\lambda(Y_{\text{norm}}) \times \Lambda_{-\lambda}(Y_{\text{norm}})$$

two germs of sectorial biholomorphisms of the space of leaves corresponding to each "narrow" sector, which we denote by Ψ_λ and $\Psi_{-\lambda}$, defined by:

$$\Psi_{\pm\lambda} := H_{\pm\lambda} \circ \psi_{\pm\lambda} \circ H_{\pm\lambda}^{-1}\,, \tag{5.39}$$

where $H_{\pm\lambda}$ is given by Corollary 5.9. According to Lemmas 5.19 and 5.20, if we write $\Psi_{\pm\lambda} = (x, \Psi_{1,\pm\lambda}, \Psi_{2,\pm\lambda})$, then for $j = 1, 2$ we have:

$$\begin{aligned} \Psi_{j,\lambda}(h_1, h_2) &= h_j + \sum_{n > N(j,+)+1} \Psi_{j,\lambda,n}(h_1 h_2) h_1^n \\ \Psi_{j,-\lambda}(h_1, h_2) &= h_j + \sum_{n > N(j,-)+1} \Psi_{j,-\lambda,n}(h_1 h_2) h_2^n \end{aligned} \tag{5.40}$$

$(\Psi_{j,\pm\lambda,n})_n \in E_{j,\pm\lambda}$. Conversely, given $(\Psi_{j,\pm\lambda})_n \in E_{j,\pm\lambda}$ for $j = 1, 2$, the estimates made in Section 4 show that

$$\psi_{\pm\lambda} := H_{\pm\lambda}^{-1} \circ \Psi_{\pm\lambda} \circ H_{\pm\lambda}\,,$$

where $\Psi_{\pm\lambda}(x, \mathbf{h}) = (x, \Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$, belongs to $\Lambda_{\pm\lambda}(Y_{\text{norm}})$. Consequently, we can state:

Proposition 5.24. — *We have the following bijections:*

$$\begin{aligned} \Lambda_\lambda(Y_{\text{norm}}) &\longrightarrow E_{1,\lambda}(Y_{\text{norm}}) \times E_{2,\lambda}(Y_{\text{norm}}) \\ \psi_\lambda &\longrightarrow (\Psi_{1,\lambda}, \Psi_{2,\lambda}) \end{aligned}$$

and

$$\begin{aligned} \Lambda_{-\lambda}(Y_{\text{norm}}) &\longrightarrow E_{1,-\lambda}(Y_{\text{norm}}) \times E_{2,-\lambda}(Y_{\text{norm}}) \\ \psi_{-\lambda} &\longrightarrow (\Psi_{1,-\lambda}, \Psi_{2,-\lambda}) \end{aligned}$$

(notice that we identify here $\Psi_{\pm\lambda}(x, \mathbf{h}) = (x, \Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$ *with* $(\Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$*).*

5.3.2. Analytic invariant varieties and two-dimensional saddle-nodes

We can give a necessary and sufficient condition for the existence of analytic invariant varieties in terms of the moduli space described above. We recall that for any vector field $Y \in [Y_{\text{norm}}]$ as in (1.1) (cf. Definition 1.14), there always exist three formal invariant varieties:

- $\mathcal{C} = \{(y_1, y_2) = (g_1(x), g_2(x))\}$,
- $\mathcal{H}_1 = \{y_1 = f_1(x, y_2)\}$
- and $\mathcal{H}_2 = \{y_2 = f_2(x, y_1)\}$,

where g_1, g_2, f_1, f_2 are formal power series with null constant term. The first one is classically called the *center variety,* and we have $\mathcal{C} = \mathcal{H}_1 \cap \mathcal{H}_2$. If $Y = Y_{\mathrm{norm}}$, then:

$$\begin{cases} \mathcal{C} = \{y_1 = y_2 = 0\} \\ \mathcal{H}_1 = \{y_1 = 0\} \\ \mathcal{H}_2 = \{y_2 = 0\}. \end{cases}$$

Proposition 5.25. — *Let $Y \in [Y_{\mathrm{norm}}]$ and $(\Phi_\lambda, \Phi_{-\lambda}) \in \Lambda_\lambda(Y_{\mathrm{norm}}) \times \Lambda_{-\lambda}(Y_{\mathrm{norm}})$ be its Stokes diffeomorphisms. We consider*

$$\Psi_\pm = H_{\pm\lambda} \circ \Phi_{\pm\lambda} \circ H_{\pm\lambda}^{-1}$$

as above. Then:

1. *the center variety $\mathcal{C}$ is convergent (analytic in the origin) if and only if $\Psi_{2,\lambda,0}(0) = \Psi_{1,-\lambda,0}(0) = 0$;*
2. *the invariant hypersurface $\mathcal{H}_1$ is convergent (analytic in the origin) if and only if for all $n > 0$, we have $\Psi_{1,-\lambda,n}(0) = 0$;*
3. *the invariant hypersurface $\mathcal{H}_2$ is convergent (analytic in the origin) if and only if for all $n > 0$, we have $\Psi_{2,\lambda,n}(0) = 0$.*

Proof. — It is a direct consequence of the power series representation (5.40) of the Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$. Let us explain item (2) (the same arguments hold for (1) and (3) with minor adaptation). The fact that $\Psi_{1,-\lambda,n}(0) = 0$ for all $n > 0$ means that $\Psi_{1,-\lambda}$ is divisible by h_1. Equivalently, both $\Phi_{1,\lambda}$ and $\Phi_{1,-\lambda}$ are divisible by y_1, so that the analytic hypersurface $\{y_1 = 0\}$ has the same pre-image by the sectorial normalizing maps Φ_+ and Φ_-. These pre-images glue together in order to define an analytic invariant hypersurface $\mathcal{H}_1$.

Notice that if we consider the restriction of a formal normal form Y_{norm} to one of the formal invariant hypersurfaces, we obtain precisely the normal form for two-dimensional saddle-nodes as given in [17]. When one of these hypersurfaces is convergent (i.e. analytic), we recover the Martinet–Ramis invariants by restriction to this hypersurface, as we present below.

Proposition 5.26. — *Suppose that the formal invariant hypersurface $\mathcal{H}_1$ is convergent (i.e. analytic in the origin). Then, the Martinet–Ramis invariants for the saddle-node $Y_{|\mathcal{H}_1}$ are given by:*

$$\begin{cases} \Psi_{2,\lambda}(0, h_2) = h_2 + \Psi_{2,\lambda,0}(0) \in Aff(\mathbb{C}) \\ \Psi_{2,-\lambda}(0, h_2) = h_2 + \sum_{n>2} \Psi_{2,-\lambda,n}(0) h_2^n \in \mathrm{Diff}(\mathbb{C}, 0). \end{cases}$$

Similar result holds for the hypersurface $\mathcal{H}_2$.

5.3.3. The transversally symplectic case and quasi-linear Stokes phenomena in the first Painlevé equation

Let us now focus on the transversally symplectic case studied in Theorem 1.24. Let $Y_{\text{norm}} \in \mathcal{SN}_{\text{diag},0}$ be transversally symplectic (i.e. its residue is $\text{res}(Y_{\text{norm}}) = 1$). Using the notations introduced in paragraph 5.3.1, we define the following sets:

$$
\begin{aligned}
&(\mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}))_\omega \\
&\qquad := \left\{ \begin{array}{c} \Psi_\lambda = (\Psi_{1,\lambda}, \Psi_{2,\lambda}) \in \mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}) \\ \text{such that: } \det(\mathrm{D}\Psi_\lambda) = 1 \end{array} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}))_\omega \\
&\qquad := \left\{ \begin{array}{c} \Psi_{-\lambda} = (\Psi_{1,-\lambda}, \Psi_{2,-\lambda}) \in \mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}) \\ \text{such that: } \det(\mathrm{D}\Psi_{-\lambda}) = 1 \end{array} \right\}.
\end{aligned}
$$

According to Proposition 5.24, the map

$$
\begin{aligned}
\Lambda_{\pm\lambda}(Y_{\text{norm}}) &\longrightarrow \mathcal{E}_{1,\pm\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\pm\lambda}(Y_{\text{norm}}) \\
\psi_{\pm\lambda} &\longmapsto \Psi_{\pm\lambda} := \mathcal{H}_{\pm\lambda} \circ \psi_{\pm\lambda} \circ \mathcal{H}_{\pm\lambda}^{-1}
\end{aligned}
$$

given in (5.39) is a bijection (notice that again, we identify here $\Psi_{\pm\lambda}(x, \mathbf{h}) = (x, \Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$ with $(\Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$). An easy computation based on (5.3) gives:

$$
(\mathcal{H}_{\pm\lambda}^{-1})^* \left(\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x} \right) = \mathrm{d}h_1 \wedge \mathrm{d}h_2 + \quad \mathrm{d}x \quad . \tag{5.41}
$$

This means in particular that $\psi_{\pm\lambda}$ is transversally symplectic with respect to $\omega = \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x}$, i.e.

$$
(\psi_{\pm\lambda})^* (\omega) = \omega + \quad \mathrm{d}x \quad , \tag{5.42}
$$

if and only if $\Psi_{\pm\lambda} = (\Psi_{1,\pm\lambda}, \Psi_{2,\pm\lambda})$ preserves the standard symplectic form $\mathrm{d}h_1 \wedge \mathrm{d}h_2$ in the space of leaves, i.e. $\det(\mathrm{D}\Psi_{\pm\lambda}) = 1$. In other words:

Proposition 5.27. — *We have the following bijections:*

$$
\begin{aligned}
\Lambda_\lambda^\omega(Y_{\text{norm}}) &\longrightarrow (\mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}))_\omega \\
\psi_\lambda &\longmapsto (\Psi_{1,\lambda}, \Psi_{2,\lambda})
\end{aligned}
$$

and

$$\Lambda^{\omega}_{-\lambda}(Y_{\mathrm{norm}}) \longrightarrow (E_{1,-\lambda}(Y_{\mathrm{norm}}) \times E_{2,-\lambda}(Y_{\mathrm{norm}}))_{\omega}$$
$$\psi_{-\lambda} \longrightarrow (\Psi_{1,-\lambda}, \Psi_{2,-\lambda})$$

(notice that we identify here $\Psi_{\pm\lambda}(x,\mathbf{h}) = (x, \Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$ *with* $(\Psi_{1,\pm\lambda}(\mathbf{h}), \Psi_{2,\pm\lambda}(\mathbf{h}))$*).*

5.3.4. Quasi-linear Stokes phenomena in the first Painlevé equation

In [2], we link the study of *quasi-linear Stokes phenomena* (see [13] for the first Painlevé equation) to our Stokes diffeomorphisms. For instance, in the case of the first Painlevé equation, we show that the quasi-linear Stokes phenomena formula found by Kapaev in [13] allows to compute the terms $\Psi_{2,\lambda,0}(0)$ and $\Psi_{1,-\lambda,0}(0)$ in (5.40). More precisely, elementary computations (using Kapaev's connection formula) give:

$$\Psi_{2,\lambda,0}(0) = i\Psi_{1,-\lambda,0}(0) = \frac{e^{\frac{i\pi}{8}}}{\sqrt{\pi}} 2^{\frac{3}{8}} 3^{\frac{1}{8}} .$$

Moreover, our description of the Stokes diffeomorphisms implies a more precise estimate of the order of the remaining terms in Kapaev's formula. In a forthcoming paper, we will use the study of some *non-linear Stokes phenomena* for the second Painlevé equations (see e.g. [6]) in order to compute coefficients of the $\Psi_{i,\pm\lambda}$'s.

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Manuscrit reçu le 10 novembre 2016,
révisé le 9 mai 2017,
accepté le 14 septembre 2017.

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