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LOCAL SPECTRAL DEFORMATION

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Abstract. — We develop an analytic perturbation theory for eigenvalues with finite multiplicities, embedded into the essential spectrum of a self-adjoint operator H. We assume the existence of another self-adjoint operator A for which the family $H_\theta = e^{\mathrm{i}\theta A} H e^{-\mathrm{i}\theta A}$ extends analytically from the real line to a strip in the complex plane. Assuming a Mourre estimate holds for $\mathrm{i}[H, A]$ in the vicinity of the eigenvalue, we prove that the essential spectrum is locally deformed away from the eigenvalue, leaving it isolated and thus permitting an application of Kato's analytic perturbation theory.

Résumé. — Nous construisons dans cet article une théorie de perturbation analytique pour des valeurs propres avec multiplicités finies, plongées dans le spectre essentiel d'un opérateur auto-adjoint H. Pour pouvoir faire ça on suppose l'existence d'un autre opérateur auto-adjoint A pour lequel la famille $H_\theta = e^{\mathrm{i}\theta A} H e^{-\mathrm{i}\theta A}$ a une extension analytique de la ligne réelle à une bande dans le plan complexe. En supposant que l'estimation de Mourre soit vraie pour $\mathrm{i}[H, A]$ au voisinage de la valeur propre, on montre que le spectre essentiel est localement déformé afin qu'il ne contienne plus la valeur propre permettant ainsi l'application de la théorie de la perturbation analytique de Kato.

1. Introduction

The investigation of the essential spectrum of a self-adjoint operator via spectral deformation techniques goes back to two papers by Aguilar–Combes and Balslev–Combes, see [1] and [3]. The starting point of the whole theory is the behavior of the Laplace operator under dilations. We define the unitary group of dilations on $\mathrm{L}^2(\mathbb{R}^d)$ by

$$U(\theta)\psi(x) = \mathrm{e}^{\frac{d}{2}\theta}\psi(\mathrm{e}^{\theta}x), \quad \text{for } \theta \in \mathbb{R}.$$

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Under conjugation with $U(\theta)$ the Laplace operator transforms into

$$U(\theta)\Delta U(\theta)^{-1} = \mathrm{e}^{-2\theta}\Delta .$$

The right-hand side extends by analytic continuation to $\theta \in \mathbb{C}$. Thus, the spectrum of $\mathrm{e}^{-2\theta}\Delta$ is a half-line starting at 0 which has an angle of $-2\operatorname{Im}\theta$ to the real line. The observation by Aguilar and Combes was that for certain one-body potentials V, the essential spectrum of the Schrödinger operator $H = -\Delta + V$ exhibits the same behavior, when conjugated with $U(\theta)$.

This idea is generalized by Balslev and Combes to the situation of many-body Schrödinger operators. After dilation, the essential spectrum consists of multiple half-lines, one starting at each threshold (eigenvalue of a subsystem Hamiltonian) protruding into the complex plane at a common angle $-2\operatorname{Im}\theta$. Any non-threshold embedded eigenvalue will remain on the real axis, as an isolated eigenvalue of finite rank for which Kato's analytic perturbation theory applies [17].

The class of (pair-)potentials for which this strategy works are called *dilation analytic*. The theory of dilation analytic potentials and its application to quantum mechanics is summed up in [26]. The method has been refined to include potentials that may be locally singular using so-called exterior complex scaling, which is needed to treat e.g. Born–Oppenheimer molecules [29].

In the paper [15], Hunziker and Sigal considered an abstract setup, where the unitary group $U(\theta)$ is, in principle, arbitrary and allowing for an analytic extension of $H_\theta = U(\theta)HU(\theta)^*$ into a strip around the real axis. Supposing that the continuous spectrum is locally deformed down into the lower half-plane, when $\operatorname{Im}\theta > 0$, leaving behind only isolated eigenvalues with finite rank Riesz projections, Hunziker and Sigal show that there is a one-one correspondence between embedded eigenvalues of H and real eigenvalues of the deformed Hamiltonian H_θ, in the region where the essential spectrum has been cleared away. This in turn permits an application of Kato's analytic perturbation theory for isolated eigenvalues of finite multiplicity, thus enabling an analytic perturbation theory of embedded eigenvalues as well as an analysis of resonances (poles of the resolvent are complex eigenvalues of the deformed Hamiltonian).

In the present paper, we provide a natural set of abstract conditions on a pair of self-adjoint operators H and A that ensures a local — in energy — deformation of the essential spectrum of H_θ, leaving embedded eigenvalues isolated behind. Here A drives the unitary group $U(\theta) = \mathrm{e}^{\mathrm{i}\theta A}$. Together with the results of [15], this allows for an analytic perturbation theory of "non-threshold" embedded eigenvalues. In fact, exterior complex scaling may be viewed as an example of our general result. We note that there are refinements of exterior scaling that does not fit into our framework, where $U(\theta)$ is not a group [14].

To elucidate the role of the Mourre estimate in the theory of analytic deformations, it is useful to expand H_θ as a formal power series:

$$H_\theta = \mathrm{e}^{\mathrm{i}\theta A} H \mathrm{e}^{-\mathrm{i}\theta A} = H - \theta \mathrm{i}[H,A] + \frac{\theta^2}{2!}\mathrm{i}^2[[H,A],A] - \dots \tag{1.1}$$

In fact, we shall in Subsect. 2.1 make sense out of this series strongly on $D(H)$.

Suppose $\lambda_0 \in \mathbb{R}$ is an (embedded) eigenvalue of H. Based on the expansion (1.1), it is reasonable to expect that a Mourre estimate

$$\mathrm{i}[H,A] \geqslant e - CE(|H-\lambda_0| \geqslant \kappa)\langle H\rangle - K \tag{1.2}$$

will force the essential spectrum of H_θ with $\operatorname{Im}\theta > 0$ down into the lower half-plane, at least near λ_0. Here, as usual, $e, \kappa, C > 0$ and K is a compact operator. The use of commutator estimates of this form goes back to Mourre [22]. The compact error in the Mourre estimate leaves room for finitely many eigenvalues to stay behind. In fact, λ_0 will stay behind, but resonances — eigenvalues with negative imaginary part — may appear as well. Exploiting the Mourre estimate in conjunction with the series (1.1) is not new, cf. e.g. [21, 16].

The main difficulty in establishing the spectral picture discussed in the preceding paragraph, comes from the fact that H_θ is not (in general) normal when $\operatorname{Im}\theta \neq 0$. In Subsect. 2.2, we assume a Mourre estimate and perform a Feshbach analysis to study the structure of the essential spectrum of H_θ. This puts us in a position to invoke [15]. In Subsect 2.3, we employ Kato's analytic perturbation theory, to conclude a theorem on analytic dependence on parameters of embedded eigenvalues of H.

The main result of this paper, Theorem 2.13, may be summed up succinctly as follows: Let H, A be a pair of self-adjoint operators. Put $H_\theta = \mathrm{e}^{\mathrm{i}\theta A} H \mathrm{e}^{-\mathrm{i}\theta A}$ for $\theta \in \mathbb{R}$, and assume

- $\forall\ \psi \in D(H)$, the map $\mathbb{R} \ni \theta \to H_\theta\psi$ (is well-defined and) extends to an analytic function in a strip around the real axis.
- A Mourre estimate is satisfied for the pair H, A in the vicinity, energetically, of an eigenvalue λ_0 of H.

Then, for θ with $\operatorname{Im}\theta > 0$ not too large, we have

$$\sigma_{\mathrm{ess}}(H_\theta) \cap \big\{z \in \mathbb{C} \,\big|\, \operatorname{Im} z > -e' \operatorname{Im}\theta/2,\ |\operatorname{Re} z - \lambda_0| < \kappa'\big\} = \emptyset.$$

Here $0 < e' < e$ and $0 < \kappa' < \kappa$ plays a role similar to e and κ in (1.2). Apart from the two main conditions itemized above, we have to impose some technical conditions on the pair H, A in order for the analysis to go through.

Combining our main result, Theorem 2.13, with Hunziker–Sigal [15] and Kato [17], yields an analytic perturbation theory for embedded "non-threshold" eigenvalues, summed up in Theorems 2.19 and 2.21.

In Section 3, we apply our analysis to two-body dispersive systems with real analytic one-body dispersion relations and a "dilation analytic" pair interaction. Such a system is translation invariant, and we study the analytic dependence of possible embedded non-threshold eigenvalues on total momentum.

The underlying motivation for this work in fact stems from three-body scattering for dispersive systems. While there are several unresolved issues surrounding scattering theory for three-body dispersive systems, one of them arises when dealing with scattering channels consisting of one incoming/outgoing free particle and one incoming/outgoing bound two-particle cluster. The free dynamics of the two-particle cluster is governed by an effective dispersion relation, which is in fact an eigenvalue of the two-body subsystem as a function of the total momentum of the two-particle cluster. If one cannot rule out the existence of embedded eigenvalues, then knowing that such effective dispersion relations are real analytic would allow one to argue that the associated threshold energies are nowhere dense. We can only say something about non-threshold energies, but that should in principle suffice, since the threshold set of two-body systems is well understood.

To conclude this introduction we discuss two examples. A trivial case of an operator which admits a band of embedded eigenvalues depending real-analytically on a parameter is provided in the following example.

Example 1.1 ([7]). — Let $H_0 = \Delta^2$ as an operator on $H^4(\mathbb{R}^d)$. Let $f \in C_0^\infty(\mathbb{R}^d)$ be nonnegative, $f \geqslant 0$.

Then, for any $\xi > 0$ and since $(-\Delta + \xi)^{-1}$ is positivity improving, we have $u(\xi) = (-\Delta+\xi)^{-1} f$ is Schwartz class and strictly positive everywhere.

Put
$$V(\xi) = -\frac{1}{u(\xi)}(-\Delta - \xi) f \in C_0^\infty(\mathbb{R}^d).$$
Then
$$V(\xi)u(\xi) = -(-\Delta - \xi) f = -(-\Delta - \xi)(-\Delta + \xi)u(\xi) = -\Delta^2 u(\xi) + \xi^2 u(\xi).$$
Hence $(H_0 + V(\xi))u(\xi) = \xi^2 u(\xi)$ and consequently, $H(\xi) = H_0 + V(\xi)$ has an embedded eigenvalue at the energy $E = \xi^2$.

One can choose to read $V(\xi)$ as a function of $\xi > 0$. The associated family of operators $H(\xi) = \Delta^2 + V(\xi)$ will now have a persistent (real analytic) band of embedded eigenvalues $E(\xi) = \xi^2$.

Since the strategy of the paper is to transform the Hamiltonian into a non-self-adjoint operator with receding essential spectrum in the area of interest, the question whether or not our assumptions are too strong arises. In particular, one could be tempted to hope that the minimal requirements of Kato's theory are sufficient. This however is not the case, since the following example illustrates that one cannot expect the usual conclusions of Kato to hold true, when one considers the behavior of embedded eigenvalues of self-adjoint operators under analytic perturbations. We recall from Kato [17] that for one-parameter holomorphic families of self-adjoint operators, isolated eigenvalues of finite multiplicity may split up while locally preserving total multiplicity and forming real analytic branches that (suitably ordered) are real analytic through crossings. For non-normal holomorphic families it is only the algebraic multiplicity that is locally conserved (in $\mathbb{C}$), and eigenvalue branches may have at most algebraic singularities at crossings.

Example 1.2. — Let $\mathcal{H} = L^2(\mathbb{R}^2) \oplus \mathbb{C}$ and define

$$H(\xi) = \begin{pmatrix} -\Delta - \xi^2 \mathbf{1}[|x| \leqslant 1] & 0 \\ 0 & 0 \end{pmatrix}$$

with domain $H^2(\mathbb{R}^2) \oplus \mathbb{C}$. There exist $\rho > 0$, such that For $\xi \in \mathbb{R}$ with $0 < |\xi| < \rho$, the operator $-\Delta - \xi^2 \mathbf{1}[|x| \leqslant 1]$ has a unique eigenvalue $\lambda(\xi)$, which is simple and depends real analytically on $0 < |\xi| < \rho$. See [28]. We may extend λ to a continuous function on $(-\rho, \rho)$ by setting $\lambda(0) = 0$. Hence, for $\xi \in (-\rho, \rho)$, $\sigma_{\mathrm{pp}}(H(\xi)) = \{\lambda(\xi), 0\}$ and $\xi \to H(\xi)$ is clearly analytic of Type (A). We observe two things:

(1) At $\xi = 0$, there is a single simple eigenvalue $\lambda = 0$. But we have two branches of eigenvalues coming out for $\xi \neq 0$. That is, the total multiplicity of the eigenvalue cluster is not upper semi-continuous.
(2) The lower of the two eigenvalue branches $\xi \to \lambda(\xi)$ does not continue analytically through $\xi = 0$, nor does it have an algebraic singularity at $\xi = 0$, more precisely; $|\lambda(\xi)| \leqslant e^{-(a\xi^2)^{-1}}$ for some $a > 0$, cf. [28].

For a closely related example, see [12, p. 585].

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2. General Theory

2.1. Generalized Dilations

In this subsection H and A denote two self-adjoint operators on a complex separable Hilbert space $\mathcal{H}$. The inner product $\langle\,\cdot\,,\cdot\,\rangle$ is assumed linear in the second variable and conjugate linear in the first variable. We associate to an (unbounded) operator T with domain $D(T)$, its graph norm $\|\psi\|_T = \|T\psi\| + \|\psi\|$, as a norm on the subspace $D(T)$. We shall frequently, for self-adjoint T, exploit the easy estimate $\frac{1}{2}\|\psi\|_T \leqslant \|(T+\mathrm{i})\psi\| \leqslant \|\psi\|_T$.

We work throughout Section 2 under the following condition:

Condition 2.1.

(1) *Abbreviating $U(t) = e^{\mathrm{i}tA}$ for $t \in \mathbb{R}$, we assume $U(t)D(H) \subset D(H)$ for all $t \in \mathbb{R}$ and*
$$\forall\, \psi \in D(H): \quad \sup_{|t|\leqslant 1} \|HU(t)\psi\| < \infty\,.$$

(2) *The quadratic form on $D(H)\cap D(A) \times D(H)\cap D(A)$ given by*
$$(\psi,\varphi) \mapsto \langle H\psi, A\varphi\rangle - \langle A\psi, H\varphi\rangle$$
is continuous w.r.t. the norm $\|(\psi,\varphi)\|_H := \|\psi\|_H + \|\varphi\|_H$.

(3) *There exists $R > 0$ such that for any $\psi \in D(H)$, the map*
$$\mathbb{R} \ni t \mapsto H_t\psi := U(t)HU(-t)\psi$$
extends to a strongly analytic $\mathcal{H}$-valued function $\{H_\theta\psi\}_{\theta\in S_R}$, where
$$S_R := \bigl\{z \in \mathbb{C} \bigm| |\mathrm{Im}(z)| < R\bigr\}. \tag{2.1}$$
This defines a collection of linear operators $\{H_\theta\}_{\theta\in S_R}$ with domain $D(H)$.

(4) *For H_θ defined above, note that $H_\theta(H+\mathrm{i})^{-1} \in \mathcal{B}(\mathcal{H})$ by the closed graph theorem.*[(1)] *We suppose that*
$$M := \sup_{\theta\in B_R^{\mathbb{C}}(0)} \|H_\theta(H+\mathrm{i})^{-1}\| < \infty\,.$$

Remarks 2.2.

(1) In Condition 2.1 (1), the demand that $\sup_{|t|\leqslant 1}\|HU(t)\psi\| < \infty$, for $\psi \in D(H)$, is in fact a consequence of the assumption that $U(t)D(H) \subset D(H)$ for all t. For a proof of this fact, we refer the reader to [2, Prop. 3.2.5].

(1) H_θ is closable, since $H_{\bar\theta} \subset H_\theta^*$.

(2) The Conditions 2.1 (1) and (2) go back to Mourre [22] and are equivalent to saying that H is of class $C^1(A)$ with commutator $[H, A]^\circ$ bounded as an operator from $D(H)$ into $\mathcal{H}$. See [19, Prop. B.11].

(3) Another consequence of Conditions 2.1 (1) and (2) is the density of $D(H) \cap D(A)$ in both $D(H)$ and $D(A)$, equipped with their respective graph norms. See [19, Lemma B.10].

(4) It suffices that the map $\theta \mapsto H_\theta\psi$ extends from $(-R, R)$ to $B_R^{\mathbb{C}}(0)$ in order to obtain an extension into S_R. Indeed, since we assume that $U(t)D(H) \subset D(H)$, the composition $H_\theta U(t)$ makes sense on $D(H)$ for all $\theta \in B_R^{\mathbb{C}}(0)$. Let $t \in \mathbb{R}$ and $\theta \in (t-R, t+R)$, then

$$H_\theta\psi = U(t)H_{\theta-t}U(-t)\psi$$

extends from $(t-R, t+R)$ to an analytic function on $B_R^{\mathbb{C}}(t)$ for all $\psi \in D(H)$. Sliding t along the real axis produces an analytic continuation of $H_\theta\psi$ to the whole strip S_R.

We recall from [17] that if $U \subset \mathbb{C}$ is open then a family $\{T_\theta\}_{\theta\in U}$ of closed operators is said to be *analytic of Type (A)* if the domain of T_θ does not depend on θ and the map $U \ni \theta \to T_\theta\psi$ is analytic for any ψ in the common domain. If $U \subset \mathbb{C}^d$, then $\{T_\theta\}_{\theta\in U}$ is said to be analytic of Type (A), if it is separately analytic of Type (A) in each of its d variables.

Lemma 2.3. — *Assume Conditions 2.1*(1) *and 2.1*(2)*. The following holds:*

(1) *For any $\psi \in D(H)$, $\theta \in \mathbb{C}$ and $m \in \mathbb{N}$, we have*

$$\psi_m(\theta) := e^{-A^2/(2m)+\mathrm{i}\theta A}\psi \in D(H).$$

(2) *If $\theta \in \mathbb{R}$, we have $\lim_{m\to\infty}\psi_m(\theta) = U(\theta)\psi$ in the topology of $D(H)$. In particular ($\theta = 0$), the set of vectors in $D(H)$ that are analytic vectors for A are dense in $D(H)$.*

(3) *For all $\psi \in D(H)$ and $m \in \mathbb{N}$, the map $\theta \to H\psi_m(\theta)$ is entire.*

Proof. — Put $\psi_m(\theta) = e^{-A^2/(2m)+\mathrm{i}\theta A}\psi$. Using the Fourier transform, we may write

$$\psi_m(\theta) = \sqrt{\frac{m}{2\pi}}e^{-m\theta^2/2}\int_{\mathbb{R}} e^{-mt^2/2+m\theta t}U(t)\psi\,\mathrm{d}t.$$

Note that for any $m \in \mathbb{N}$, the integral converges absolutely in $D(H)$, since $\|U(t)\psi\|_{D(H)} \leqslant e^{c|t|}$, for all $t \in \mathbb{R}$, where $c > 0$ is some constant. This is a consequence of Condition 2.1 (1) and implies (1).

Let $\theta \in \mathbb{R}$. To show that $\psi_m(\theta) \to U(\theta)\psi$ in $D(H)$, it suffices to argue that $H\psi_m(\theta) \to HU(\theta)\psi$ in $\mathcal{H}$. Since $U(\theta)\psi \in D(H)$ for real θ, it suffices to prove this with $\theta = 0$. Here we observe that

$$\sqrt{m}\int_{|t|\geqslant 1} e^{-mt^2/2}U(t)\psi\,\mathrm{d}t \to 0\,,$$

due to the estimate $\|U(t)\psi\|_{D(H)} \leqslant e^{c|t|}$ from before. Furthermore, the estimate

$$\|(HU(t) - U(t)H)\psi\| = \|(H_{-t} - H)\psi\| \leqslant C|t|\,, \tag{2.2}$$

valid for $|t| \leqslant 1$ with some $C > 0$, follows from Condition 2.1 (3) and finally yields (2).

We now establish (3). Since the map $\theta \to \psi_m(\theta)$ is entire it suffices, by Vitali–Porter's theorem, to show that $n\|H(H + \mathrm{i}n)^{-1}\psi_m(\theta)\|$ is bounded locally uniformly in $\theta \in \mathbb{C}$. But this follows easily from the estimates already invoked above. □

It turns out that under the assumption in Condition 2.1 (1), the remaining three items are equivalent to the statement that all iterated commutators of H with A are H-bounded and satisfy a certain growth bound. If these bounds are satisfied the analytic continuation of the family H_θ can be written as a power series in a neighborhood of 0. More precisely, we can prove

Proposition 2.4. — *Assume Condition 2.1* (1). *Then the following two properties are equivalent:*

(1) *Conditions 2.1* (2)–(4).
(2) *There exists a constant* $C > 0$ *such that: the iterated commutators* $\mathrm{ad}_A^k(H)$ *exist as* H*-bounded operators for all* $k \in \mathbb{N}$ *and*

$$\left\|\mathrm{ad}_A^k(H)(H + \mathrm{i})^{-1}\right\| \leqslant C^k k!\,. \tag{2.3}$$

If Condition 2.1 holds, then $\{H_\theta\}_{\theta\in B^{\mathbb{C}}_{(3C)^{-1}}(0)}$ *with common domain* $D(H)$ *is an analytic family of Type (A), and for all* $\theta \in B^{\mathbb{C}}_{(3C)^{-1}}(0)$ *and* $\psi \in D(H)$, *we have*

$$H_\theta\psi = \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!}\mathrm{i}^k\,\mathrm{ad}_A^k(H)\psi \tag{2.4}$$

and

$$\frac{1}{2}\|\psi\|_H \leqslant \|\psi\|_{H_\theta} \leqslant 2\|\psi\|_H\,. \tag{2.5}$$

Remark 2.5. — If one supposes (1) with given R and M coming from Condition 2.1 (3) and (4), respectively, then one may choose $C = \max\{1, M\}/R$ in (2.3).

Conversely, if one assumes (2) with a given C, then one may choose $R = (3C)^{-1}$ and $M = 3$.

Since we have elected to state our assumptions in terms of an analytic extension of H, we shall below employ the estimate (2.3) with

$$C = \frac{\max\{1, M\}}{R}\,. \tag{2.6}$$

The expansion (2.4) of H_θ and the relative bounds (2.5) will then hold true for $\theta \in B^{\mathbb{C}}_{R'}(0)$, where

$$R' = \frac{1}{3C} = \frac{R}{3\max\{1, M\}}\,. \tag{2.7}$$

Proof. — We begin with (2) $\Rightarrow$ (1). Therefore, we assume that for all k, the iterated commutators exist as H-bounded operators $\mathrm{ad}^k_A(H)$ and that (2.3) holds.

That Condition 2.1 (2) follows is obvious (take $k = 1$).

Note that Condition 2.1 (1) ensures that H_θ is well-defined for real θ as an operator with domain $D(H)$.

Exploiting (2.3), we may for $\psi \in D(H)$ and $|\theta| < 1/C$ estimate

$$\sum_{k=0}^{\infty} \left\| \frac{\theta^k}{k!} \mathrm{ad}^k_A(H)\psi \right\| \leqslant \frac{\|(H + \mathrm{i})\psi\|}{1 - C|\theta|}\,. \tag{2.8}$$

Hence, the prescription

$$S_\theta \psi := \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} \mathrm{i}^k \,\mathrm{ad}^k_A(H)\psi \tag{2.9}$$

defines an analytic $\mathcal{H}$-valued function defined in the disc $B^{\mathbb{C}}_{1/C}(0)$. It is now easy to check that the map $\psi \mapsto S_\theta\psi$ defines — for each $\theta \in B^{\mathbb{C}}_{1/C}(0)$ — a linear operator with domain $D(H)$.

The estimate (2.8) implies that

$$\forall\, \psi \in D(H): \quad \|S_\theta \psi\| \leqslant (1 - C|\theta|)^{-1} \|\psi\|_H\,, \tag{2.10}$$

and in particular that S_θ is H-bounded.

We proceed to show that H and S_θ (for θ in a sufficiently small disc centered at 0) define equivalent graph norms on $D(H)$. Note that (2.10) already establishes that there exists a constant $C_1 > 0$ independent of $\theta \in B^{\mathbb{C}}_{1/(3C)}(0)$ such that

$$\|\psi\|_{S_\theta} \leqslant C_1 \|\psi\|_H\,.$$

In complete analogy to the first estimate, we estimate for $\psi \in D(H)$ and $\theta \in B^{\mathbb{C}}_{1/C}(0)$:

$$\begin{aligned}\|\psi\|_H = \|\psi\| + \|H\psi\| &\leqslant \|\psi\|_{S_\theta} + \sum_{k=1}^{\infty} (C|\theta|)^k \|\psi\|_H \\ &= \|\psi\|_{S_\theta} + \frac{C|\theta|}{1-C|\theta|}\|\psi\|_H\,. \end{aligned} \tag{2.11}$$

Hence, for $\theta \in B^{\mathbb{C}}_{1/(3C)}(0)$ we have

$$\|\psi\|_H \leqslant 2\|\psi\|_{S_\theta}\,.$$

This proves the claimed equivalence of graph norms and thus that S_θ is closed as an operator with domain $D(H)$ for all $\theta \in B^{\mathbb{C}}_{1/(3C)}(0)$. Abbreviating $R = 1/(3C)$, we have now proved that $\{S_\theta\}_{\theta \in B^{\mathbb{C}}_R(0)}$ is an analytic family of Type (A). (Note that redoing the estimate (2.8) using $|\theta| \leqslant 1/(3C)$ yields $\|\psi\|_{S_\theta} \leqslant 2\|\psi\|_H$ as well.)

It remains, recalling Remark 2.2 (4), to argue that $S_\theta = H_\theta$ for $\theta \in (-R, R)$. Let $\psi, \phi \in D(H)$ and put $\psi_m = e^{-A^2/(2m)}\psi$ and $\phi_m = e^{-A^2/(2m)}\phi$. Then, with the notation of Lemma 2.3, we have

$$f_m(\theta) = \langle \psi_m, H_\theta \phi_m\rangle = \langle \psi_m(\bar{\theta}), H\phi_m(\theta)\rangle$$

a priori for real θ, but extending to an entire function of θ. Here we used Lemma 2.3 (3).

We may use the assumption on the existence of iterated H-bounded commutators $\mathrm{ad}^k_A(H)$ to compute

$$\left.\frac{\mathrm{d}^k f_m}{\mathrm{d}\theta^k}\right|_{\theta=0} = \langle \psi_m, (-\mathrm{i})^k \,\mathrm{ad}^k_A(H)\phi_m\rangle\,.$$

Since analytic functions in $B^{\mathbb{C}}_R(0)$ are determined by their derivatives at zero, we may conclude that

$$\langle \psi_m, H_\theta\phi_m\rangle = \langle \psi_m, S_\theta \phi_m\rangle$$

for all $\theta \in B^{\mathbb{C}}_R(0)$. Finally, we exploit Lemma 2.3 once more to compute the limit $m \to \infty$ in the above identity and conclude that for all $\theta \in (-R, R)$ and $\psi, \phi \in D(H) \cap D(A)$, we have $\langle \psi, H_\theta\phi\rangle = \langle \psi, S_\theta\phi\rangle$. By density of $D(H) \cap D(A)$ in $D(H)$, we conclude that $H_\theta = S_\theta$ for $\theta \in (-R, R)$ as desired. It now follows from (2.10) that we may choose $M = 3$ in Condition 2.1 (4).

In order to prove that (1) $\Rightarrow$ (2), we assume that Conditions 2.1 (2)–(4) holds true. Let $\eta, \psi \in D(H)$. By Condition 2.1 (1) and the analyticity of $\theta \mapsto H_\theta\psi$, we may use [20, Prop. 2.2] to argue that all iterated commutators of A with H exists and are implemented by H-bounded operators, provided we can establish that for every $j \in \mathbb{N}$ there exist H-bounded operators $H_0^{(j)}$, such that

$$\forall\, \theta \in (-R, R): \quad \frac{\mathrm{d}^j}{\mathrm{d}\theta^j}\, \langle \eta, H_\theta \psi\rangle|_{\theta=0} = \langle \eta, H_0^{(j)}\psi\rangle\,.$$

As a starting point we use the analyticity of $\theta \mapsto H_\theta\psi$ to obtain a power series expansion for $|\theta| < r < R$, that is

$$\langle \eta, H_\theta\psi\rangle = \sum_{k=0}^{\infty} \theta^k b_k(\eta,\psi)\,, \quad b_k(\eta,\psi) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_r} \theta^{-k-1}\langle \eta, H_\theta\psi\rangle\,\mathrm{d}\theta\,, \tag{2.12}$$

where $\eta \in \mathcal{H}$ and Γ_r is the circle in the complex plane with radius r centered at 0. Observe that the $b_k(\eta,\psi)$'s define sesquilinear forms.

Using Condition 2.1 (4), we get an $M > 0$ such that

$$|b_k(\eta,\psi)| \leqslant \|\eta\|\|\psi\|_H \frac{M}{R^k}\,,$$

where we also took the limit $r \to R$. For every $\psi \in D(H)$ (and $k \in \mathbb{N}$) there thus exists a vector $\tilde{\psi}$ such that $b_k(\eta,\psi) = \langle \eta, \tilde{\psi}\rangle$ for all $\eta \in D(H)$. It follows that the assignment $B_k\psi := \tilde{\psi}$ defines an H-bounded linear operator on $D(H)$. With this construction, we have

$$\frac{\mathrm{d}^j}{\mathrm{d}\theta^j}\, \langle \eta, H_\theta\psi\rangle|_{\theta=0} = \langle \eta, k! B_k\psi\rangle$$

and [20, Prop. 2.2] now implies that (2.3) holds with $C := \max\{1, M\}/R$. □

In the following we abbreviate

$$W_\theta := H_\theta - H = \sum_{k=1}^{\infty} \frac{(-\theta)^k}{k!}\mathrm{i}^k\, \mathrm{ad}_A^k(H) \tag{2.13}$$

as an operator with domain $D(H)$. Observe for $\theta \in B^{\mathbb{C}}_{R'}(0)$ the estimate

$$\|W_\theta(H+\mathrm{i})^{-1}\| \leqslant \frac{C|\theta|}{1 - C|\theta|} \leqslant \frac{3C}{2}|\theta|\,, \tag{2.14}$$

We have the following — rough but sufficient — spectral localization result.

Proposition 2.6. — *Assume Condition 2.1. Then*

$$\forall\, \theta \in B^{\mathbb{C}}_{R'}(0): \quad \sigma(H_\theta) \subset \big\{x + \mathrm{i}y \,\big|\, |y| \leqslant 4C|\theta|(|x|+1)\big\}.$$

Proof. — Let $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ and compute on $D(H)$:

$$H_\theta - z = \left[1 + W_\theta (H-z)^{-1}\right](H-z).$$

Hence, $H_\theta - z$ is invertible if $\|W_\theta(H-z)^{-1}\| < 1$ due to the Neumann series. The norm appearing in the previous inequality can be estimated trivially by

$$\|W_\theta(H-z)^{-1}\| \leqslant \|W_\theta(H+\mathrm{i})^{-1}\| \sup_{p\in\mathbb{R}} \frac{|p+\mathrm{i}|}{|p-z|}.$$

Let $c>0$. Suppose $z = x+\mathrm{i}y$ with $|y| \geqslant c(|x|+1)$. Then $|p+\mathrm{i}|^2/|p-z|^2 \leqslant (p^2+1)/((p-x)^2+c^2x^2+c^2) \leqslant 4/c^2$ uniformly in p, x and y. Using (2.14), we have:

$$\|W_\theta(H-z)^{-1}\| \leqslant \frac{3C|\theta|}{c}.$$

for $z = x+\mathrm{i}y$ with $|y| \geqslant c|x|$ The choice $c = 4C|\theta|$ ensures convergence of the Neumann series. ☐

Lemma 2.7. — *Assume Condition 2.1 and let $\theta \in B^{\mathbb{C}}_{R'}(0)$. We have*

$$D(H_\theta^*) = D(H) \quad \textit{and} \quad H_\theta^* = H_{\overline{\theta}}.$$

Proof. — Let $\psi, \phi \in D(H)$. We compute

$$\begin{aligned}\langle \psi, H_\theta \phi\rangle &= \sum_{k=0}^\infty \left\langle \psi, \frac{(-\theta)^k}{k!} \mathrm{i}^k \operatorname{ad}_A^k(H)\phi \right\rangle\\ &= \sum_{k=0}^\infty \left\langle \frac{(-\overline{\theta})^k}{k!} \mathrm{i}^k \operatorname{ad}_A^k(H)\psi, \phi \right\rangle = \left\langle H_{\overline{\theta}}\psi, \phi\right\rangle.\end{aligned}$$

Hence $H_{\overline{\theta}} \subset H_\theta^*$. Conversely, let $\phi \in D(H)$, $\psi \in D(H_\theta^*)$ and set

$$y = \max\{1, 8CR'\}. \tag{2.15}$$

Observe that $\mathrm{i}y \in \rho(H_\theta)\setminus\mathbb{R}$, due to Proposition 2.6. We compute, using the notation from (2.13)

$$\begin{aligned}|\langle\psi, H\phi\rangle| &\leqslant |\langle \psi, H_\theta\phi\rangle| + |\langle\psi, W_\theta\phi\rangle|\\ &= |\langle \psi, H_\theta\phi\rangle| + |\langle\psi, (H_\theta-\mathrm{i}y)(H_\theta-\mathrm{i}y)^{-1}W_\theta\phi\rangle|\\ &\leqslant \|H_\theta^*\psi\|\|\phi\| + \|(H_\theta^*+\mathrm{i}y)\psi\|\|(H_\theta-\mathrm{i}y)^{-1}W_\theta\phi\|.\end{aligned}$$

Note that

$$H_\theta - \mathrm{i}y = (H-\mathrm{i}y)(1+(H-\mathrm{i}y)^{-1}W_\theta)$$

and that, recalling (2.7), (2.14) and (2.15),

$$\left\|(H-\mathrm{i}y)^{-1}W_\theta\right\| \leqslant \left(\sup_{x\in\mathbb{R}} \frac{x^2+1}{x^2+y^2}\right)^{1/2} \left\|(H-\mathrm{i})^{-1}W_\theta\right\| \leqslant \frac{1}{2}.$$

Abbreviating $B_\theta = (1 + (H - \mathrm{i}y)^{-1} W_\theta)^{-1}$, we may estimate

$$\|(H_\theta - \mathrm{i}y)^{-1} W_\theta \phi\| \leqslant \|B_\theta\| \|(H - \mathrm{i}y)^{-1} W_\theta \phi\| \leqslant C\|\phi\|.$$

Hence, there exists a $C_\psi > 0$ such that

$$\forall\, \phi \in D(H): \quad |\langle \psi, H\phi\rangle| \leqslant C_\psi \|\phi\|,$$

and therefore we may conclude that $\psi \in D(H^*) = D(H)$, exploiting the self-adjointness of H. This shows that $D(H_\theta^*) = D(H)$ and that $H_\theta^* = H_{\overline{\theta}}$. □

2.2. The Mourre Estimate

At this stage we will single out a specific energy $\lambda_0 \in \mathbb{R}$, where we shall assume that H has an eigenvalue. In order for the dilated Hamiltonian to have its essential spectrum out of the way of the eigenvalue, we shall impose a Mourre estimate locally around λ_0. To formulate the requirement, we need the notation $E_H(B)$ for the spectral projection associated with a Borel set $B \subset \mathbb{R}$ and the self-adjoint operator H.

Condition 2.8. — *Let $\lambda_0 \in \mathbb{R}$. For a pair of self-adjoint operators H and A satisfying Condition 2.1, we further assume:*

(1) $\lambda_0 \in \sigma_{\mathrm{pp}}(H)$.

(2) *There exist $e, C, \kappa > 0$ and a compact operator K, such that*

$$\mathrm{i}\,\mathrm{ad}_A(H) \geqslant e - CE_H(\mathbb{R} \setminus [\lambda_0 - \kappa, \lambda_0 + \kappa])\langle H\rangle - K \tag{2.16}$$

in the sense of quadratic forms on $D(H)$.

Notation 2.9. — We write $P_0 = E_H(\{\lambda_0\})$ for the orthogonal projection onto the eigenspace of H associated with the eigenvalue λ_0. Furthermore, we abbreviate $\overline{P}_0 = 1 - P_0$ for the projection onto the orthogonal complement of the eigenspace.

Remarks 2.10.

(1) Observe that it is a consequence of Conditions 2.1 (1), (2), 2.8 (2), and the Virial Theorem [9] that P_0 is a finite rank projection.

(2) Choosing κ possibly smaller, one may replace the compact operator K in (2.16) with a positive multiple of the eigenprojection P_0. More precisely,

$$\mathrm{i}\,\mathrm{ad}_A(H) \geqslant e' - C'\left(E_H(\mathbb{R} \setminus [\lambda_0 - \kappa', \lambda_0 + \kappa'])\langle H\rangle + P_0\right), \tag{2.17}$$

for suitably chosen constants $e' \in (0, e]$, $\kappa' \in (0, \kappa]$ and $C' \geqslant C$. It is in this form that we shall use the Mourre estimate, and for convenience we assume $\kappa' \leqslant \sqrt{3}$.

(3) Equation (2.17) differs from the more usual version of Mourre's estimate:

$$E_H(I) \mathrm{i}\, \mathrm{ad}_A(H) E_H(I) \geqslant e E_H(I) - K,$$

where $I = [\lambda_0 - \kappa, \lambda_0 + \kappa]$. Under Condition 2.1, most notably the consequence that $\mathrm{ad}_A(H)$ is an H-bounded operator, the two estimates are equivalent. This would be false if the factor $\langle H \rangle$ is replaced by 1 on the right-hand side of (2.17).

As a preparation for a Feshbach analysis, we have:

Lemma 2.11. — *For $\lambda_0 \in \mathbb{R}$ and a pair of self-adjoint operators H and A, we assume Conditions 2.1 and 2.8. The following three statements are true for all $\theta \in B^{\mathbb{C}}_{R'}(0)$:*

(1) *$\overline{P}_0 H_\theta \overline{P}_0$ is a closed operator with dense domain $\overline{P}_0 D(H)$.*
(2) *$[\overline{P}_0 H_\theta \overline{P}_0]^* = \overline{P}_0 H_{\bar{\theta}} \overline{P}_0$ on the domain $\overline{P}_0 D(H)$, considered as an operator on $\overline{P}_0 \mathcal{H}$.*
(3) *For all $\theta \in B^{\mathbb{C}}_{R'}(0)$: $\sigma(\overline{P}_0 H_\theta \overline{P}_0) \subset \big\{ x + \mathrm{i} y \,\big|\, |y| \leqslant 4C|\theta|(|x| + 1) \big\}$.*

Proof. — As for (1), note first that $H_\theta \overline{P}_0$ with domain $D(H)$ is closed, since $\overline{P}_0 D(H) \subset D(H)$ and H_θ with domain $D(H)$ is a closed operator (Proposition 2.4). To conclude, observe that the graph of $\overline{P}_0 H_\theta \overline{P}_0$ is the range of the open map $\mathcal{H} \oplus \mathcal{H} \ni (\psi, \varphi) \to (\overline{P}_0 \psi, \overline{P}_0 \varphi) \in \overline{P}_0 \mathcal{H} \oplus \overline{P}_0 \mathcal{H}$ applied to the graph of $H_\theta \overline{P}_0$.

We turn to the claim (2). Clearly, $\overline{P}_0 H_{\bar{\theta}} \overline{P}_0 \subset [\overline{P}_0 H_\theta \overline{P}_0]^*$. Let $\varphi \in D([\overline{P}_0 H_\theta \overline{P}_0]^*)$ viewed as an element of $\overline{P}_0 \mathcal{H} \subset \mathcal{H}$, and compute for $\psi \in D(H)$:

$$\begin{aligned} \langle \varphi, H_\theta \psi \rangle &= \langle \overline{P}_0 \varphi, H_\theta (\overline{P}_0 + P_0) \psi \rangle \\ &= \langle \varphi, \overline{P}_0 H_\theta \overline{P}_0 \psi \rangle + \langle \overline{P}_0 \varphi, H_\theta P_0 \psi \rangle. \end{aligned}$$

Since P_0 is finite rank operator and H_θ is closed, it follows from the Closed Graph Theorem that $H_\theta P_0$ is bounded. Hence, there exists $C > 0$ such that

$$|\langle \varphi, H_\theta \psi \rangle| \leqslant C \|\psi\|,$$

which implies that $\varphi \in D((H_\theta)^*) = D(H_{\bar{\theta}}) = D(H)$. Here we used Lemma 2.7. Since $\overline{P}_0 \mathcal{H} \cap D(H) = \overline{P}_0 D(H)$, we are done.

The last claim (3) may be established by repeating the proof of Proposition 2.6. $\square$

In formulating the following proposition and in its proof, we make use of the eigenvalue λ_0 from Condition 2.8 and the constants e' and κ' from (2.17). The radius R' was defined in (2.7). For the open upper half-plane, we use the notation

$$\mathbb{C}^+ := \big\{ z \in \mathbb{C} \,\big|\, \mathrm{Im}(z) > 0 \big\}. \tag{2.18}$$

Proposition 2.12. — *For a real number λ_0, and a pair of self-adjoint operators H and A, we assume Conditions 2.1 and 2.8. Abbreviate for $\sigma, \rho > 0$ and $\theta \in \mathbb{C}^+$:*

$$\mathcal{R}_\theta(\sigma,\rho) = \big\{ z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) \in (\lambda_0 - \rho, \lambda_0 + \rho), \mathrm{Im}(z) \in (-\sigma\,\mathrm{Im}(\theta), \infty) \big\}, \tag{2.19}$$

There exist constants $R'', \rho > 0$ with $R'' \leqslant R'$, such that

$$\forall\, \theta \in B^{\mathbb{C}}_{R''}(0) \cap \mathbb{C}^+ : \quad \mathcal{R}_\theta(e'/2, \rho) \cap \sigma(\overline{P}_0 H_\theta \overline{P}_0) = \emptyset. \tag{2.20}$$

Proof. — Using the constants from (2.17), we define a bounded operator

$$L := C' E_H(\mathbb{R} \setminus [\lambda_0 - \kappa', \lambda_0 + \kappa']) \langle H \rangle (H - \lambda_0)^{-1}. \tag{2.21}$$

Note that $\|L\| \leqslant 4C'\langle \lambda_0 \rangle / \kappa'$, where we used $\kappa' \leqslant \sqrt{3}$ and that $\langle \lambda \rangle \leqslant 2\langle \lambda_0 \rangle \langle \lambda - \lambda_0 \rangle$. We claim suitable choices

$$\begin{aligned} \rho &= \min\left\{ 1, \frac{e'}{16 C' \langle \lambda_0 \rangle / \kappa'} \right\}, \\ R'' &= \min\left\{ R', \frac{e'}{24 C (5|\lambda_0| + 11)(C' \langle \lambda_0 \rangle / \kappa' + C)} \right\}, \end{aligned} \tag{2.22}$$

where C and R' were defined in (2.6) and (2.7), respectively. Recall that $R''C \leqslant R'C \leqslant 1/3$.

Let $\theta \in \mathbb{C}^+ \cap B^{\mathbb{C}}_{R''}(0)$ and $\mu \in \mathcal{R}_\theta(e'/2, \rho) \cap \sigma(\overline{P}_0 H_\theta \overline{P}_0)$. Note that due to Lemma 2.11 (3), we may estimate

$$|\mu| \leqslant (|\lambda_0| + \rho + 1) \left(1 + 16 R'^2 C^2 \right)^{1/2} \leqslant 2|\lambda_0| + 4. \tag{2.23}$$

By Lemma 2.11 (1), we may apply Lemma A.1 to the operator $T = \overline{P}_0 H_\theta \overline{P}_0$ acting in $\overline{P}_0 \mathcal{H}$. Assume first that there exists a sequence $\psi_n \in \overline{P}_0 D(H)$ with $\|\psi_n\| = 1$, such that

$$o_n := \big\| \overline{P}_0 (H_\theta - \mu) \overline{P}_0 \psi_n \big\| \to 0, \quad \text{for } n \to \infty. \tag{2.24}$$

We estimate for all n using (2.5) and (2.14) (recalling that $C|\theta| \leqslant CR'' \leqslant 1/3$)

$$\begin{aligned}\left\|\overline{P}_0\psi_n\right\|_H &\leqslant 2\left\|\overline{P}_0\psi_n\right\|_{H_\theta}\\ &\leqslant 2\left(\left\|\overline{P}_0H_\theta\overline{P}_0\psi_n\right\| + \left\|P_0W_\theta\overline{P}_0\psi_n\right\| + 1\right)\\ &\leqslant 2\left(o_n + |\mu| + \frac{1}{2}(|\lambda_0|+1)+1\right)\\ &= 2o_n + 5|\lambda_0| + 11\,. \end{aligned}\tag{2.25}$$

Here we used (2.23) in the last step. Exploiting the power series expansion (2.4) of H_θ, the Mourre estimate (2.17) and simplifying for real expectation values, we obtain for any n

$$\begin{aligned}\operatorname{Im}(\mu) &= \operatorname{Im}\left\langle\overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n\right\rangle + \operatorname{Im}\left\langle\overline{P}_0\psi_n, H_\theta\overline{P}_0\psi_n\right\rangle\\ &= \operatorname{Im}\left\langle\overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n\right\rangle - \operatorname{Im}\left\langle\overline{P}_0\psi_n, \theta\mathrm{i}\operatorname{ad}_A(H)\overline{P}_0\psi_n\right\rangle\\ &\qquad - \operatorname{Im}\left\langle\overline{P}_0\psi_n, \sum_{k=2}^{\infty}\frac{(-\theta)^k}{k!}\mathrm{i}^k\operatorname{ad}_A^k(H)\overline{P}_0\psi_n\right\rangle\\ &= \operatorname{Im}\left\langle\overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n\right\rangle - \operatorname{Im}(\theta)\left\langle\overline{P}_0\psi_n, \mathrm{i}\operatorname{ad}_A(H)\overline{P}_0\psi_n\right\rangle\\ &\qquad - \sum_{k=2}^{\infty}\frac{\operatorname{Im}\left((-\theta)^k\right)}{k!}\left\langle\overline{P}_0\psi_n, \mathrm{i}^k\operatorname{ad}_A^k(H)\overline{P}_0\psi_n\right\rangle\\ &\leqslant o_n - \operatorname{Im}(\theta)\left[e' - C'\left\langle\overline{P}_0\psi_n, E(|H-\lambda|\geqslant\kappa')\langle H\rangle\overline{P}_0\psi_n\right\rangle\right]\\ &\qquad - \sum_{k=2}^{\infty}\frac{\operatorname{Im}\left((-\theta)^k\right)}{k!}\left\langle\overline{P}_0\psi_n, \mathrm{i}^k\operatorname{ad}_A^k(H)\overline{P}_0\psi_n\right\rangle. \end{aligned}\tag{2.26}$$

Note that for all k, we have $|\operatorname{Im}((-\theta)^k)| \leqslant 2^k|\operatorname{Im}(\theta)||\theta|^{k-1}$. Therefore,

$$\begin{aligned}&\left|\sum_{k=2}^{\infty}\frac{\operatorname{Im}\left((-\theta)^k\right)}{k!}\left\langle\overline{P}_0\psi_n, \mathrm{i}^k\operatorname{ad}_A^k(H)\overline{P}_0\psi_n\right\rangle\right|\\ &\qquad\qquad \leqslant \sum_{k=2}^{\infty}\left|\operatorname{Im}\left((-\theta)^k\right)\right| C^k\left\|\overline{P}_0\psi_n\right\|_H\\ &\qquad\qquad \leqslant C\left|\operatorname{Im}(\theta)\right|\sum_{k=2}^{\infty}2^{k-1}\left|\theta\right|^{k-1}C^{k-1}\left\|\overline{P}_0\psi_n\right\|_H\\ &\qquad\qquad = C\left|\operatorname{Im}(\theta)\right|\frac{2C|\theta|}{1-2C|\theta|}\left\|\overline{P}_0\psi_n\right\|_H\\ &\qquad\qquad \leqslant 6\left|\operatorname{Im}(\theta)\right| R''C^2\left(2o_n + 5|\lambda_0| + 11\right), \end{aligned}\tag{2.27}$$

where we used (2.25) and that $C|\theta| \leqslant 1/3$ in the last step. We estimate using (2.14), recalling the definition (2.21) of the bounded self-adjoint operator L,

$$\begin{aligned}
&C' \left|\langle \overline{P}_0\psi_n, E_H(\mathbb{R}\setminus[\lambda_0-\kappa',\lambda_0+\kappa'])\langle H\rangle \overline{P}_0\psi_n\rangle\right| \\
&\quad= \left|\langle \overline{P}_0\psi_n, L\overline{P}_0(H-\lambda_0)\overline{P}_0\psi_n\rangle\right| \\
&\quad\leqslant \left|\langle \overline{P}_0\psi_n, L\overline{P}_0(H-\mu)\overline{P}_0\psi_n\rangle\right| + |\lambda_0-\mathrm{Re}(\mu)| \left|\langle \overline{P}_0\psi_n, L\overline{P}_0\psi_n\rangle\right| \\
&\quad\leqslant \|L\| \left\|\overline{P}_0(H_\theta-\mu)\overline{P}_0\psi_n\right\| + \|L\| \left\|W_\theta\overline{P}_0\psi_n\right\| + |\lambda_0-\mathrm{Re}(\mu)|\,\|L\| \\
&\quad\leqslant \|L\|\, o_n + \|L\| \left\|W_\theta(H+\mathrm{i})^{-1}\right\| \left\|\overline{P}_0\psi_n\right\|_H + \rho\,\|L\| \\
&\quad\leqslant \|L\| o_n + \frac{3}{2}CR''\|L\|\|\overline{P}_0\psi_n\|_H + \rho\|L\| \\
&\quad\leqslant \|L\|\,(1+3CR'')\,o_n + \frac{3}{2}\,(5|\lambda_0|+11)\,CR''\,\|L\| + \rho\,\|L\|\,, \qquad (2.28)
\end{aligned}$$

where we used (2.25) in the final step.

Combining (2.26), (2.27) and (2.28) we obtain

$$\begin{aligned}
\mathrm{Im}(\mu) \leqslant -\,\mathrm{Im}(\theta)\left(e' - \frac{3}{2}CR''(5|\lambda_0|+11)(\|L\|+4C) - \rho\|L\|\right)& \\
+ \left(1+|\,\mathrm{Im}(\theta)|(12R''C^2+\|L\|(1+3CR''))\right) o_n\,.&
\end{aligned}$$

By the choices of ρ and R'' from (2.22) and the estimate $\|L\| \leqslant 4C'\langle\lambda_0\rangle/\kappa'$, we observe that

$$\frac{3}{2}R''C(5|\lambda_0|+11)(\|L\|+4C) + \rho\|L\| \leqslant \frac{e'}{2}$$

and thus, taking the limit $n\to\infty$ using (2.24), we arrive at

$$\mathrm{Im}(\mu) \leqslant -\,\mathrm{Im}(\theta)\frac{e'}{2}\,. \qquad (2.29)$$

This estimate contradicts the choice of $\mu \in \mathcal{R}_\theta(e'/2,\rho)$, cf. (2.19).

If (2.24) does not hold, then by Lemma 2.11 (2) and Lemma A.1, there exists a sequence $\phi_n \in \overline{P}_0 D(H_{\bar\theta})$, with $\|\phi_n\| = 1$ all n and

$$o_n := \|\overline{P}_0(H_{\bar\theta}-\bar\mu)\overline{P}_0)\phi_n\| \to 0\,, \quad \text{for } n\to\infty\,. \qquad (2.30)$$

We now repeat the estimates (2.25)–(2.29), replacing μ by $\bar\mu$ and θ by $\bar\theta$, and recalling that $\mathrm{Im}(\bar\theta) > 0$. This results in the estimate

$$\mathrm{Im}(\bar\mu) \geqslant -\,\mathrm{Im}(\bar\theta)\frac{e'}{2}\,.$$

Hence $\mathrm{Im}(\mu) \leqslant -\,\mathrm{Im}(\theta)e'/2$, which completes the proof. □

In the following, we use the definition $\sigma_{\mathrm{ess}}(H) = \sigma(H) \setminus \sigma_{\mathrm{disc}}(H)$, where $\sigma_{\mathrm{disc}}(H)$ is the set of all isolated points $\lambda \in \sigma(H)$ such that when Γ is a counterclockwise loop around λ, which separates λ from the rest of the spectrum, the Riesz projection

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - H)^{-1} \mathrm{d}z$$

onto the generalized eigenspace has finite rank.

The following theorem is proven using the Feshbach reduction method, for which Proposition 2.12 above is an essential prerequisite.

THEOREM 2.13. — *For a real number λ_0, and a pair of self-adjoint operators H and A, we assume Conditions 2.1 and 2.8. Then*

$$\forall\, \theta \in B^{\mathbb{C}}_{R''}(0) \cap \mathbb{C}^+ : \quad \sigma_{\mathrm{ess}}(H_\theta) \cap \mathcal{R}_\theta(e'/2, \rho) = \emptyset .$$

The constants ρ, R'' and the sets $\mathcal{R}_\theta$ come from Proposition 2.12.

Proof. — By Proposition 2.12 there exist $R'', \rho > 0$ such that for all $|\theta| < R''$ the closed operator $\overline{P}_0 H_\theta \overline{P} - z\overline{P}_0$ is invertible on $\overline{P}_0\mathcal{H}$ for all $z \in \mathcal{R} := \mathcal{R}_\theta(e'/2, \rho)$. Define reduced resolvents

$$\overline{R}_\theta(z) := \big(\overline{P}_0 H_\theta \overline{P}_0 - z\overline{P}_0\big)^{-1}$$

on $\overline{P}_0\mathcal{H}$ for $z \in \mathcal{R}$. Recall that W_θ is defined in (2.13). For $z \in \mathcal{R}$ we can construct the Feshbach map on the finite dimensional subspace $P_0\mathcal{H}$:

$$\begin{aligned} F_{P_0}(z) &= P_0(H_\theta - z)P_0 - P_0 H_\theta \overline{P}_0 \overline{R}_\theta(z) \overline{P}_0 H_\theta P_0 \\ &= P_0(W_\theta + \lambda_0 - z)P_0 - P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 . \end{aligned}$$

Clearly, $F_{P_0}(z)$ is a finite rank operator, which can be interpreted as a matrix, and hence; by isospectrality of the Feshbach reduction [6, 11],

$$\mu \in \sigma(H_\theta) \cap \mathcal{R} \Leftrightarrow \det\big(F_{P_0}(\mu)\big) = 0 .$$

Since for $|\operatorname{Re}\mu - \lambda_0| < e'/2$ and $\operatorname{Im}\mu$ large it holds that $\mu \notin \sigma(H_\theta)$ (cf. Proposition 2.6), we conclude by the Unique Continuation Theorem for holomorphic functions that the set $\sigma(H_\theta) \cap \mathcal{R}$ is locally finite. Note that $\mu \in \sigma(H_\theta) \cap \mathcal{R}$ is necessarily an eigenvalue for H_θ. In order to establish the theorem, it remains to prove that the Riesz projections pertaining to the eigenvalues in $\mathcal{R}$ are of finite rank. Let $\mu \in \mathcal{R} \cap \sigma(H_\theta)$ and choose $r > 0$, such that $D \subset \mathcal{R} \setminus \sigma(H_\theta)$, where $D = \{z \in \mathbb{C} \,|\, 0 < |z - \mu| \leqslant r\}$ denotes a closed punctured disc.

The inverse of $F_{P_0}(z)$ for $z \in D$ has a Laurent expansion

$$F_{P_0}(z)^{-1} = \sum_{k=1}^{N} B_{-k}(z-\mu)^{-k} + \sum_{k=0}^{\infty} B_k(z-\mu)^k$$

convergent in the punctured disc D. Here $N \geqslant 1$ and $\{B_k\}_{k=-N}^{\infty}$ denote linear operators on $P_0\mathcal{H}$. See [24, §6.1]. Note that the inverse has no essential singularities since we are in finite dimension.

By [6, 11], for $z \in \mathcal{R}\backslash\sigma(H_\theta)$, the inverse $R_\theta(z)$ of $H_\theta - z$ can be recovered from the inverse Feshbach operator and the reduced resolvent via the block decomposition

$$\begin{aligned}
P_0R_\theta(z)P_0 &= F_{P_0}(z)^{-1},\\
P_0R_\theta(z)\overline{P}_0 &= -F_{P_0}(z)^{-1}P_0W_\theta\overline{P}_0\overline{R}_\theta(z),\\
\overline{P}_0R_\theta(z)P_0 &= -\overline{R}_\theta(z)\overline{P}_0W_\theta P_0F_{P_0}(z)^{-1},\\
\overline{P}_0R_\theta(z)\overline{P}_0 &= \overline{R}_\theta(z) + \overline{R}_\theta(z)\overline{P}_0W_\theta P_0F_{P_0}(z)^{-1}P_0W_\theta\overline{P}_0\overline{R}_\theta(z).
\end{aligned}$$

Note that the map $z \mapsto \overline{R}_\theta(z)$ is analytic in $\mathcal{R}$, so the only singularities are those in $\sigma(H_\theta)$, coming from the inverse Feshbach operator.

Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the closed curve $\gamma(t) = \mu + re^{\mathrm{i}t}$ parametrizing the (outer) boundary of D, encircling μ. Recall the construction of the Riesz projection

$$P_\theta(\mu) = -\frac{1}{2\pi\mathrm{i}}\int_\gamma R_\theta(z)\,\mathrm{d}z$$

associated with the eigenvalue μ. The block decomposition of $R_\theta(z)$ induces a block decomposition of $P_\theta(\mu)$ and the Riesz projection has finite rank, provided $\overline{P}_0P_\theta(\mu)\overline{P}_0$ is of finite rank. To check this, we compute

$$\begin{aligned}
&-\overline{P}_0P_\theta(\mu)\overline{P}_0\\
&\qquad= \frac{1}{2\pi\mathrm{i}}\int_\gamma \left[\overline{R}_\theta(z) + \overline{R}_\theta(z)\overline{P}_0W_\theta P_0F_{P_0}(z)^{-1}P_0W_\theta\overline{P}_0\overline{R}_\theta(z)\right]\mathrm{d}z\\
&\qquad= \sum_{k=1}^{N}\frac{1}{2\pi\mathrm{i}}\int_\gamma (z-\mu)^{-k}\overline{R}_\theta(z)\overline{P}_0W_\theta P_0B_{-k}P_0W_\theta\overline{P}_0\overline{R}_\theta(z)\,\mathrm{d}z\\
&\qquad\qquad+\frac{1}{2\pi\mathrm{i}}\int_\gamma\sum_{k=0}^{\infty}(z-\mu)^k\overline{R}_\theta(z)\overline{P}_0W_\theta P_0B_kP_0W_\theta\overline{P}_0\overline{R}_\theta(z)\,\mathrm{d}z,
\end{aligned}$$

where we have used that the function $\mathcal{R} \ni z \mapsto \overline{R}_\theta(z)$ is analytic. Moreover, the integral in the last line of the equation above is carried out over an analytic function, once again, and thus equals 0. The remaining N singular integrals can be evaluated by Cauchy's Integral Formula:

$$
\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_\gamma & (z-\mu)^{-k} \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z)\, \mathrm{d}z \\
&= \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) \Big|_{z=\mu} \\
&= \frac{1}{(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{k-1} j!(k-1-j)! \\
&\qquad \times \overline{R}_\theta(\mu)^{1+j} \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(\mu)^{k-j} \\
&= (-1)^{k-1} \sum_{j=0}^{k-1} \overline{R}_\theta(\mu)^{j+1} \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(\mu)^{k-j}.
\end{aligned}
$$

Since each term in the sum above is a finite rank operator, we conclude that $\overline{P}_0 P_\theta(\mu) \overline{P}_0$ is of finite rank.

Since $\sigma(H) \cap \mathcal{R}$ is locally finite and all the associated Riesz projections have finite rank, we have shown that $\sigma_{\mathrm{ess}}(H_\theta) \cap \mathcal{R} = \emptyset$. This completes the proof. □

Note that

$$D(U(\theta)) = \left\{ \psi \in \mathcal{H} \,\Big|\, \int_{\mathbb{R}} e^{2\,\mathrm{Im}(\theta) x}\, \mathrm{d}E_\psi(x) < \infty \right\},$$

where E_ψ is the spectral measure for A associated with the state ψ. Motivated by this we abbreviate for $r \geqslant 0$:

$$D_r(A) = \left\{ \psi \in \mathcal{H} \,\Big|\, \int_{\mathbb{R}} e^{2r|x|}\, \mathrm{d}E_\psi(x) < \infty \right\}.$$

Having established Theorem 2.13, we may conclude the following theorem by invoking a general result of Hunziker and Sigal [15, Thm. 5.2].

Theorem 2.14. — *For a real number λ_0, and a pair of self-adjoint operators H and A, we assume Conditions 2.1 and 2.8. Let $\theta \in B^{\mathbb{C}}_{R''}(0) \cap \mathbb{C}^+$. Then the dilated Hamiltonian H_θ has an isolated eigenvalue at λ_0. Denote by P_θ the associated Riesz projection. The following statements hold true:*

(1) $\mathrm{Range}(P_\theta)$ *is the eigenspace of H_θ pertaining to the eigenvalue λ_0.*
(2) *$P_0 = U(-\theta) P_\theta U(\theta)$ as a form identity on $D_{|\,\mathrm{Im}(\theta)|}(A)$.*
(3) $\mathrm{Rank}(P_0) = \mathrm{Rank}(P_\theta)$.
(4) *Let $r < R''$. Then* $\mathrm{Range}(P_0) \subset D_r(A)$.

Remark 2.15. — The above theorem implies that eigenfunctions pertaining to the eigenvalue λ_0 are analytic vectors for the operator A. This result was previously established by brute force in [20] under a slightly weaker condition.

2.3. Analytic Perturbation Theory

Condition 2.16. — *Let $\lambda_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}^d$, and $U \subset \mathbb{R}^d$ an open (connected) neighborhood of ξ_0, A a self-adjoint operator on $\mathcal{H}$ and $\{H(\xi)\}_{\xi \in U}$ a family of self-adjoint operators on $\mathcal{H}$.*

1. *$D(H(\xi)) = D(H(\xi_0)) =: \mathcal{D}$ for all $\xi \in U$.*
2. *For all ξ in U, the operator $H(\xi)$ satisfies Condition 2.1 with the same constants R and M.*
3. *The triple λ_0, A and $H(\xi_0)$ satisfies Condition 2.8.*
4. *There exists $\theta_0 \in B^{\mathbb{C}}_R(0)$ with $\mathrm{Im}(\theta_0) > 0$, such that the map $\xi \to H_{\theta_0}(\xi)$ extends from U to an analytic family of Type (A) defined for $\xi \in U_{\mathbb{C}} \subset \mathbb{C}^d$, an open (connected) set with $U \subset U_{\mathbb{C}} \cap \mathbb{R}$.*

Remark 2.17. — Suppose one strengthens Condition 2.16 and assumes that $\xi \to H_\theta(\xi)$ extends to an analytic family of Type (A) not just for one θ_0 but for all θ in a complex disc of radius $\Theta < R'$ around 0. Then one may use Morera's theorem to conclude that for any $\psi \in \mathcal{D}$ and n, we have

$$\mathrm{i}^n \,\mathrm{ad}_A^n(H(\xi))\psi = \frac{(-1)^n}{2\pi\mathrm{i}} \int_{|\theta|=\Theta/2} n!\theta^{-n-1} H_\theta(\xi)\psi \,\mathrm{d}\theta\,,$$

a priori for real ξ, but since the right-hand side extends analytically to ξ in a complex neighborhood of ξ_0, so does the left-hand side. This will in particular permit one to conclude that also for complex ξ does the closed operator $H(\xi)$ iteratively admit commutators with A of arbitrary order. (Note that $H(\bar{\xi}) \subset H(\xi)^*$, by unique continuation.) Furthermore, the iterated commutators must coincide (strongly) with the analytic extension from real ξ of $\mathrm{ad}_A^n(H(\xi))\psi$, obtained above.

Recall the notation λ_0 for the eigenvalue of $H(\xi_0)$ with eigenprojection P_0. By Theorem 2.14, we know that λ_0 is an isolated eigenvalue of $H_{\theta_0}(\xi_0)$ with finite rank eigenprojection P_{θ_0}. Denote by n_0 the common rank of P_0 and P_{θ_0}.

Fix $0 < \rho' < \rho$, such that

$$\sigma\left(H_{\theta_0}(\xi_0)\right) \cap B^{\mathbb{C}}_{2\rho'}(\lambda_0) = \{\lambda_0\}. \tag{2.31}$$

Remark 2.18. — We may choose $r' > 0$, such that for all $\xi \in B^{\mathbb{R}^d}_{r'}(\xi_0)$, we have

$$\sigma(H_{\theta_0}(\xi)) \cap B^{\mathbb{C}}_{\rho'}(\lambda_0) = \sigma_{\mathrm{pp}}(H_{\theta_0}(\xi)) \cap B^{\mathbb{C}}_{\rho'}(\lambda_0)$$

and the total algebraic multiplicity of the eigenvalues in $B^{\mathbb{C}}_{\rho'}(\lambda_0)$ equals n_0 ([17, §IV.4]).

By [15, Thm. 5.2], we may now conclude — just as we did with Theorem 2.14 — that for all $\xi \in B^{\mathbb{R}^d}_{r'}(\xi_0)$:

$$\sigma_{\mathrm{pp}}(H(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho') = \sigma(H_{\theta_0}(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho'). \tag{2.32}$$

If the perturbation parameter ξ is one-dimensional, we may in light of Theorem 2.13 and Condition 2.16, invoke Kato, in the form of [17, Thm. VII.1.8], and conclude the following theorem.

Theorem 2.19. — *Suppose Condition 2.16 and that $d = 1$. There exist*

- $r > 0$ *with* $(\xi_0 - r, \xi_0 + r) \subset U$,
- *natural numbers* $0 \leqslant m_\pm \leqslant n_0$ *and* $n_1^\pm, \dots, n_{m_\pm}^\pm \geqslant 1$ *with* $n_1^\pm + \cdots + n_{m_\pm}^\pm \leqslant n_0$,
- *real analytic functions* $\lambda_1^\pm, \dots, \lambda_{m_\pm}^\pm \colon I_\pm \to \mathbb{R}$, *where* $I_- = (\xi_0 - r, \xi_0)$ *and* $I_+ = (\xi_0, \xi_0 + r)$, *satisfying* $\lambda_i^\pm(\xi) \neq \lambda_j^\pm(\xi)$ *for all* $1 \leqslant i < j \leqslant m_\pm$ *and* $\xi \in I_\pm$,

such that (recalling ρ' *from* (2.31)*)*

1. *for any* $\xi \in I_\pm$, *we have* $\sigma_{\mathrm{pp}}(H(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho') = \{\lambda_1^\pm(\xi), \dots, \lambda_{m_\pm}^\pm(\xi)\}$,
2. *for all* $j = 1, \dots, m_\pm$, *we have* $\lim_{I_\pm \ni \xi \to \xi_0} \lambda_j^\pm(\xi) = \lambda_0$,
3. *the eigenvalue branches* $I_\pm \ni \xi \to \lambda_\pm(\xi)$ *have constant algebraic and geometric multiplicity* $n_j^\pm$.
4. *Each eigenvalue branch* $\lambda_j^\pm \colon I_\pm \to \mathbb{R}$ *can be expanded in a convergent Puiseux series near* ξ_0, *that is; a convergent power series expansion in* $(\pm(\xi - \xi_0))^{1/\ell}$ *for some integer* $\ell \geqslant 1$.

Remark 2.20. — If λ_0 is an isolated eigenvalue, then we know from Kato [17] that the perturbed eigenvalues $\lambda_i^\pm$ are analytic at $\xi = \xi_0$, that is; in 4 one may choose $\ell = 1$ (and the continuation through ξ_0 yields one of the other branches $\lambda_j^\mp$). Here we cannot exclude algebraic singularities at ξ_0, since the eigenvalues come from $H_{\theta_0}(\xi)$, which may not be a normal operator. We do not know of an example of an embedded eigenvalue where one cannot choose $\ell = 1$.

In the case of multiple parameters, the structure of the point spectrum becomes more complicated, and the right setting here is that of semi-analytic sets, the definition of which is recalled in Appendix B. More precisely, we are interested in the analytic structure of the set

$$\Sigma_{\mathrm{pp}} := \big\{(\lambda, \xi) \in \mathbb{R} \times U \,\big|\, \lambda \in \sigma_{\mathrm{pp}}(H(\xi))\big\}.$$

In the following section, we explore an example where ξ is a total momentum variable, in which case Σ_{pp} is the energy-momentum point spectrum. The reader may wish to consult Definitions B.1 (1) and B.1 (2) before proceeding to the main theorem of this subsection:

Theorem 2.21. — *Suppose Condition 2.16. There exists $r > 0$ and $\rho > 0$, such that with $W = (\lambda_0 - \rho', \lambda_0 + \rho') \times B_r^{\mathbb{R}^d}(\xi_0)$, we have that $\Sigma_{\mathrm{pp}} \cap W \in \mathcal{O}(W)$. In particular, $\Sigma_{\mathrm{pp}} \cap W$ is a semi-analytic subset of W.*

Proof. — Let r' be chosen as in Remark 2.18. The projection onto the generalized eigenspace is the Riesz projection,

$$P(\xi) \equiv P_{\theta_0}(\lambda_0;\xi) = \frac{1}{2\pi\mathrm{i}} \int_{|z-\lambda_0|=\rho'} (z - H_{\theta_0}(\xi))^{-1}\,\mathrm{d}z,$$

which depends analytically on $\xi \in B_{r'}^{\mathbb{C}}(\xi_0)$. Write $V(\xi) = \mathrm{Range}(P(\xi))$ for the generalized eigenspace of dimension n_0 and $\Pi\colon \mathbb{C}^{n_0} \to \Pi(\xi_0)$ a linear isomorphism identifying the unperturbed eigenspace with $\mathbb{C}^{n_0}$. Following [10], we choose $r \in (0, r']$ such that $\|P(\xi) - P(\xi_0)\| \leqslant 1/2$ for $|\xi - \xi_0| \leqslant r$. Then $\Theta(\xi) := P(\xi)|_{V(\xi_0)}$ defines a linear isomorphism from $V(\xi_0)$ onto $V(\xi)$ with inverse $\Theta^{-1}(\xi) = (1 + P(\xi_0)(P(\xi) - P(\xi_0)))^{-1}P(\xi_0)$ and

$$\forall\, \xi \in B_r^{\mathbb{C}^d}(\xi_0): \quad T(\xi) = \Pi^{-1}\Theta(\xi)^{-1}H_{\theta_0}(\xi)\Theta(\xi)\Pi$$

defines a family of linear operators on $\mathbb{C}^d$ depending analytically on ξ and satisfying that $\sigma(T(\xi)) = \sigma(H_{\theta_0}(\xi)) \cap B_{\rho'}^{\mathbb{C}}(\lambda_0)$. Hence, recalling (2.32),

$$\Sigma_{\mathrm{pp}} \cap W = \{(\lambda,\xi) \in W \mid \det(T(\xi) - \lambda) = 0\}.$$

Here W is defined in the statement of the theorem. Split into real and imaginary parts $\det(T(\xi)-\lambda) = u(\lambda,\xi) + \mathrm{i}v(\lambda,\xi)$, to obtain two real analytic real-valued functions. Then

$$\Sigma_{\mathrm{pp}} \cap W = \big\{(\lambda,\xi) \in W \,\big|\, u(\lambda,\xi) = 0\big\} \cap \big\{(\lambda,\xi) \in W \,\big|\, v(\lambda,\xi) = 0\big\}.$$

Since the right-hand side is an element of $\mathcal{O}(W)$, we are done. □

Remark 2.22. — In the one dimensional setup, the Puiseux expansion of the eigenvalue branches in particular ensures that two distinct branches separate as $c|\xi - \xi_0|^{p/q}$ for some $c \neq 0$ and some natural numbers p, q. (This behavior is violated in Example 1.2) It is not apparent in the higher dimensional setup that something like this holds true. There is however a remnant in the theory of semi-analytic sets called the Lojasiewicz inequality, which can be interpreted as a statement that two distinct strata that meet at a common boundary stratum, does so with an algebraic lower bound in the distance to the boundary. See [4, §7].

3. Example

We introduce a two-particle Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ by

$$H'_V = \omega_1(p_1) + \omega_2(p_2) + V(x_1 - x_2)MK,$$

where $p_i = -\mathrm{i}\nabla_{x_i}$, $x_i \in \mathbb{R}^d$.

We impose the following set of conditions on ω_1, ω_2 and V:

Condition 3.1 (Properties of ω_1, ω_2 and V).

(1) *The ω_i's are real-valued, real analytic functions on $\mathbb{R}^d$ and there exists $\widetilde{R} > 0$, such that the ω_i's extend to analytic functions in the d-dimensional strip*

$$S^d_{2\widetilde{R}} := \Big\{ (z_1, \dots, z_d) \in \mathbb{C}^d \,\Big|\, |\operatorname{Im}(z_i)| < 2\widetilde{R}, i = 1, \dots, d \Big\}.$$

We denote the analytic continuations of these functions by the same symbols.

(2) *There exist real numbers $s_2 \geqslant s_1 > 0$ and a constant $\widetilde{C} > 0$ such that*

$$|\partial^\alpha \omega_j(k)| \leqslant \widetilde{C}\langle k\rangle^{s_j}, \quad |\omega_j(k)| \geqslant \frac{1}{\widetilde{C}}\langle k\rangle^{s_j} - \widetilde{C} \tag{3.1}$$

for every multi-index $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leqslant 1$ and all $k \in S^d_{2\widetilde{R}}$.

(3) *Let $d' = 2[d/2] + 2$. We suppose that $V \in C^{d'}(\mathbb{R}^d)$ and there exists $a > 0$, such that for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant d'$, we have $\sup_{x\in\mathbb{R}^d} \mathrm{e}^{a|x|}|\partial_x^\alpha V(x)| < \infty$.*

Conjugating with the Fourier transform, we see that H'_V is unitarily equivalent to

$$H_V = \omega_1(k_1) + \omega_2(k_2) + t_V,$$

where t_V is the partial convolution operator

$$(t_V f)(k_1, k_2) := \int_{\mathbb{R}^d} \widehat{V}(u) f(k_1 - u, k_2 + u)\,\mathrm{d}u$$

and

$$\widehat{V}(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} V(x)\,\mathrm{d}x\,.$$

In order to fibrate H_V w.r.t. total momentum $\xi = k_1 + k_2$, we introduce a unitary operator $I\colon L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(\mathbb{R}^d; L^2(\mathbb{R}^d))$ by setting

$$(If)(\xi) = f(\xi - \cdot, \cdot)\,.$$

Under this transformation, we find that the Hamiltonian takes the form

$$IH_V I^* = \int_{\mathbb{R}^d}^{\oplus} H(\xi)\,\mathrm{d}\xi\,, \quad \text{where} \quad H(\xi) = \omega_\xi + T_V\,,$$

and

$$\omega_\xi(k) = \omega_1(\xi - k) + \omega_2(k), \qquad (T_V f)(k) = (\check{V} * f)(k). \tag{3.2}$$

Here $\check{V}(k) = \widehat{V}(-k)$ is the inverse Fourier transform of V and $\check{V} * f$ denotes the convolution product.

We are now in a position to formulate our main result of this section. We introduce the joint energy-momentum point spectrum

$$\Sigma_{\mathrm{pp}} = \left\{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, \lambda \in \Sigma_{\mathrm{pp}}(\xi)\right\}, \quad \Sigma_{\mathrm{pp}}(\xi) = \sigma_{\mathrm{pp}}(H(\xi))$$

and the energy-momentum threshold set

$$\mathcal{T} = \left\{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, \lambda \in \mathcal{T}(\xi)\right\},$$
$$\mathcal{T}(\xi) = \left\{\lambda \in \mathbb{R} \,\middle|\, \exists\, k \in \mathbb{R}^d :\ \omega_\xi(k) = \lambda \text{ and } \nabla_k \omega_\xi(k) = 0\right\}.$$

The main result of this section is the following.

Theorem 3.2. — *Suppose Condition 3.1. Then we have*

1. $\mathcal{T}$ *is a closed sub-analytic subset of* $\mathbb{R} \times \mathbb{R}^d$.
2. *For each* $\xi \in \mathbb{R}^d$*, the set* $\mathcal{T}(\xi)$ *is a locally finite subset of* $\mathbb{R}$.
3. $\Sigma_{\mathrm{pp}} \setminus \mathcal{T}$ *is a semi-analytic subset of* $(\mathbb{R} \times \mathbb{R}^d) \setminus \mathcal{T}$.
4. *For each* $\xi \in \mathbb{R}^d$*, the set* $\Sigma_{\mathrm{pp}}(\xi) \setminus \mathcal{T}(\xi)$ *is a locally finite subset of* $\mathbb{R} \setminus \mathcal{T}(\xi)$.

Remark 3.3.

1. It is the claim (3), which is of interest here. The other properties are more or less immediate. See the proof, which is located at the end of this section.
2. It remains an open question how bands of non-threshold eigenvalues, as functions of total momentum, may approach the threshold set. For example, under what conditions is Σ_{pp} a semi- (or sub-) analytic subset of $\mathbb{R} \times \mathbb{R}^d$? Example 1.2 provides an example where such a result fails to hold true.
3. In [13], Herbst and Skibsted study exponential decay of eigenfunctions pertaining to non-threshold eigenvalues of one-body operators of the form $\omega(p) + V$, assuming ω is a polynomial. Their results apply to $H(\xi)$.

We define a self-adjoint operator for every total momentum $\xi \in \mathbb{R}^n$ by

$$A_\xi = \frac{\mathrm{i}}{2}\left(v_\xi \cdot \nabla_k + \nabla_k \cdot v_\xi\right), \tag{3.3}$$

where the vector field v_ξ is given by

$$v_\xi(k) = \mathrm{e}^{-k^2 - \xi^2} (\nabla_k \omega_\xi)(k). \tag{3.4}$$

Dilation in momentum space — of the type considered here — was previously employed for Schrödinger operators $-\Delta + V$ in [23].

In the following, we make frequent use of the estimate

$$\forall\, p \in \mathbb{R}, k, k' \in \mathbb{R}^d : \qquad \langle k + k' \rangle^p \leqslant 2^{|p|} \langle k \rangle^{|p|} \langle k' \rangle^p, \tag{3.5}$$

referred to as Peetre's inequality in [27, Lemma 1.18].

Remark 3.4.

(1) By Condition 3.1 (2) $\widetilde{C}^{-1}\langle k\rangle^{s_j} - \widetilde{C} \leqslant |\omega_j(k)| \leqslant \widetilde{C}\langle k\rangle^{s_j}$ and thus $D(M_{\omega_j}) = D(M_{\langle\,\cdot\,\rangle^{s_j}})$. Consequently, cf. (3.5), $D(M_{\omega_\xi}) = D(M_{\langle\,\cdot\,\rangle^{s_2}}) =: \mathcal{D}$, since $s_2 \geqslant s_1$. Thus all operators $H(\xi)$ have the common domain $\mathcal{D}$.

(2) The two estimates in Condition 3.1 (2) are satisfied by the functions $f(k) = (k_1^2 + \cdots + k_d^2)^q$ for $q \in \mathbb{N}$ and $g(k) = (1 + k_1^2 + \cdots + k_d^2)^s$ for $s > 0$. As for the choice of $\widetilde{R}$, for the function f any $\widetilde{R} > 0$ will do, whereas for g one must choose $\widetilde{R} < d^{-1/2}/2$.

(3) Condition 3.1 (2) and (3.5) imply the existence of $C_\omega, C'_\omega > 0$ such that

$$\forall\, \xi, k \in S^d_{\widetilde{R}} : \qquad |v_\xi(k)| \leqslant C_\omega < \infty. \tag{3.6}$$

and

$$\forall\, \xi, k \in S^d_{\widetilde{R}} : \qquad \|Dv_\xi(k)\| \leqslant C'_\omega < \infty. \tag{3.7}$$

(4) Condition 3.1 (3) on the potential ensures that $\widehat{V}$ extends analytically to the strip S^d_a. Fix an $a' \in (0, a)$. The assumed decay and smoothness permits to argue — using integration by parts — that for some $C_V > 0$, we have

$$\forall\, k \in S^d_{a'} : \qquad \left|\widehat{V}(k)\right| \leqslant C_V \left(1 + |k|^{d'}\right)^{-1}. \tag{3.8}$$

The choice of d' is made to ensure that this estimate implies $\widehat{V} \in L^1(\mathbb{R}^d)$.

The next well-known lemma expresses the action of the unitary group generated by A_ξ in terms of solutions of the ODE generated by the vector field v_ξ. See e.g. the PhD thesis of one of the authors [25, Chap. 2, Prop. 2.3]. Since $v_\xi \colon S^d_{\widetilde{R}} \to \mathbb{C}^n$, we are led to study the parameter dependent autonomous initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_\xi(y), \quad y(0) = k. \tag{3.9}$$

The ODE is defined for $y \in S^d_{\widetilde{R}}$. That is, solutions are understood to take values in $S^d_{\widetilde{R}}$.

Lemma 3.5. — *For $k, \xi \in \mathbb{R}^d$, the initial value problem* (3.9) *admits a (unique) solution $t \mapsto \gamma^t_\xi(k)$ defined for all time $t \in \mathbb{R}$. Abbreviating*

$$J^t_\xi(k) = e^{\int_0^t \nabla \cdot v_\xi(\gamma^s_\xi(k))\mathrm{d}s}, \tag{3.10}$$

we have for $f \in L^2(\mathbb{R}^d)$ the formula

$$\left(\mathrm{e}^{\mathrm{i}tA_\xi} f\right)(k) = \sqrt{J^t_\xi(k)}\, f(\gamma^t_\xi(k)). \tag{3.11}$$

One may — by direct computation — verify the useful relation

$$\forall\, t \in \mathbb{R}, \forall\, k, \xi \in \mathbb{R}^d : \qquad J^{-t}_\xi(\gamma^t_\xi(k)) = \frac{1}{J^t_\xi(k)}. \tag{3.12}$$

Furthermore, (3.6) and the Fundamental Theorem of Calculus ensure that

$$\forall\, t \in \mathbb{R}, \forall\, k, \xi \in \mathbb{R}^d : \qquad |\gamma^t_\xi(k) - k| \leqslant C_\omega |t|. \tag{3.13}$$

Lemma 3.6. — *Let $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and $t \in \mathbb{R}$. Then $f^t := f \circ \gamma^t_\xi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$,*

$$\mathrm{e}^{\mathrm{i}tA_\xi} D(M_f) = D(M_{f^t}) \qquad \textit{and} \qquad \mathrm{e}^{\mathrm{i}tA_\xi} M_f = M_{f^t} \mathrm{e}^{\mathrm{i}tA_\xi}.$$

Proof. — That $f \circ \gamma^t_\xi$ is locally essentially bounded follows from (3.13). We may also observe that $\mathrm{e}^{\mathrm{i}tA_\xi} C^\infty_0(\mathbb{R}^d) \subset C^\infty_0(\mathbb{R}^d)$, cf. (3.11) and (3.13). Finally, for $\psi \in C^\infty_0(\mathbb{R}^d)$ we have $M_{f^t} \mathrm{e}^{\mathrm{i}tA_\xi}\psi = \mathrm{e}^{\mathrm{i}tA_\xi} M_f \psi$. The lemma now follows, since M_{f^t} is closed and $C^\infty_0(\mathbb{R}^d)$ is a core for M_f. □

In the following Lemma, we need the strip width $\widetilde{R}$ from Condition 3.1 (1) and the bound C_ω on v_ξ from (3.6).

Lemma 3.7. — *Assume Conditions 3.1*(1) *and 3.1*(2)*. For $k \in \mathbb{R}^d$ and $\xi \in S^d_{\widetilde{R}}$, the solution of* (3.9)*, $t \mapsto \gamma^t_\xi(k)$, admits an analytic continuation into the strip S_r, where $r = \widetilde{R}/(C_\omega + 1)$. Moreover, for $0 < r' \leqslant r$, we have $\{\gamma^z_\xi(k) \,|\, z \in S_{r'}\} \subset S^d_{r'C_\omega} \subset S^d_{\widetilde{R}}$.*

Proof. — By definition, $t \to \gamma^t_\xi(k)$ solves the ODE (3.9).

Since v_ξ is analytic in $S^d_{\widetilde{R}} \subset S^d_{2\widetilde{R}}$ as a function of k, the function $t \to \gamma^t_\xi(k)$ admits — by Cauchy–Kowaleskaya — an analytic continuation into some region $G = G(k;\xi) \subset \mathbb{C}$ containing the real axis. Hence, for each $x \in \mathbb{R}$, we may choose $r(x,k;\xi) > 0$, such that $B^{\mathbb{C}}_{r(x,k;\xi)}(x) \subset G(k;\xi)$. By possible decreasing $r(x,k,\xi)$, we may assume that $\gamma^z_\xi(k) \in S^d_{\widetilde{R}}$ for $|z - x| < r(x,k;\xi)$. Therefore, $\gamma^z_\xi(k) - \xi \in S^d_{2\widetilde{R}}$ and by Condition 3.1 (1) and the definition of v_ξ we may form $v_\xi(\gamma^z_\xi(k))$.

For each $x \in \mathbb{R}$, (and $k \in \mathbb{R}^d$, $\xi \in S^d_{\widetilde{R}}$), the function $y_x(t) = \gamma_\xi^{x+\mathrm{i}t}(k)$ solves the ODE

$$\frac{\mathrm{d}y_x}{\mathrm{d}t}(t) = \mathrm{i}v_\xi(y_x(t))$$

with the initial condition $y_x(0) = \gamma_\xi^x(k)$. The solution is a priori defined for $|t| < r(x,k;\xi)$.

The estimate (for $t > 0$)

$$|\mathrm{Im}(y_x(t))| = |\mathrm{Im}(y_x(t) - y_x(0))| \leqslant \int_0^t |v_\xi(y_x(s))| \,\mathrm{d}s \leqslant C_\omega t, \tag{3.14}$$

ensures that the solution may be extended beyond $|t| = r(x,k;\xi)$ at least until $|t| = r = \widetilde{R}/(C_\omega + 1)$. (A similar estimate holds for $t < 0$). This defines an extension of $\gamma_\xi^z(k)$ from $z \in G(k,\xi)$ to $z \in S_r$ for all $k \in \mathbb{R}^d$, $\xi \in S^d_{\widetilde{R}}$. Note that $\gamma_\xi^z(k) \in S^d_{C_\omega r} \subset S^d_{\widetilde{R}}$ for all $z \in S_r$. It remains to argue that the extension is analytic.

By Cauchy–Kowaleskaya, for each $x \in \mathbb{R}$, $k \in \mathbb{R}^d$ and $\xi \in S^d_{\widetilde{R}}$, the solution $t \mapsto y_x(t)$ extends to an analytic function in a complex neighborhood $O(x,k;\xi)$ of $[-r,r]$. Let $\delta = \delta(x,k;\xi) \in (0, r(x,k;\xi))$ be such that $[-r,r] \times \mathrm{i}[-\delta,\delta] \subset O(x,k;\xi)$.

By Unique Continuation, the extension $y_x(z)$ equals $\gamma_\xi^{x+\mathrm{i}z}(k)$ for $z \in ([-r,r] \times \mathrm{i}[-\delta,\delta]) \cap B^{\mathbb{C}}_{r(x,k;\xi)}(0)$. This implies that $t \to y_x(t + \mathrm{i}x')$, for $|x'| < \delta(x,k;\xi)$, solves the same initial value problem as $y_{x'}(t)$ and hence they must coincide. This proves that $z \mapsto \gamma_\xi^z(k)$ is analytic in $(x - \delta(x,k;\xi), x + \delta(x;k;\xi)) \times \mathrm{i}(-r,r)$. Since $x \in \mathbb{R}$ was arbitrary, we conclude that $z \mapsto \gamma_\xi^z(k)$ is analytic in S_r.

The last claim about the range of $z \mapsto \gamma_\xi^z(k)$ for $|\operatorname{Im} z| \leqslant r' \leqslant r$, follows from (3.14). $\square$

Remark 3.8.

(1) Let $z \in S_r$ and write $u = z/|z|$. The above lemma allows us to estimate

$$\begin{aligned} \left|\gamma_\xi^z(k) - k\right| &= \left|\int_0^{|z|} \frac{\mathrm{d}}{\mathrm{d}r}\left(\gamma_\xi^{ru}(k) - k\right)\mathrm{d}r\right| \\ &= \left|\int_0^{|z|} uv_\xi\left(\gamma_\xi^{ru}(k)\right)\mathrm{d}r\right| \leqslant C_\omega |z|. \end{aligned} \tag{3.15}$$

(2) Let $k,k',\xi\in\mathbb{R}^d$. Abbreviate $\beta_\xi^z(k,k')=\gamma_\xi^z(k')-\gamma_\xi^z(k)$ and $u(z)=|\beta_\xi^z(k,k')|^2$ for $z\in S_r$. Then

$$\begin{aligned}\frac{\mathrm{d}u(z)}{\mathrm{d}z}&=2\,\mathrm{Re}\left\{\overline{\beta_\xi^z(k,k')}\cdot\frac{\mathrm{d}\beta_\xi^z(k,k')}{\mathrm{d}z}\right\}\\&=2\,\mathrm{Re}\left\{\overline{\beta_\xi^z(k,k')}\cdot\big(v_\xi(\gamma_\xi^z(k'))-v_\xi(\gamma_\xi^z(k))\big)\right\}.\end{aligned}$$

Estimating $|v_\xi(\gamma_\xi^z(k'))-v_\xi(\gamma_\xi^z(k))|\leqslant C'_\omega|\beta_\xi^z(k,k')|$, using (3.7), we arrive at the differential inequalities $-2C'_\omega u\leqslant\dot u\leqslant 2C'_\omega u$. Therefore, we may conclude the estimate

$$\forall\, z\in S_r:\qquad |\beta_\xi^z(k,k')|\geqslant|k-k'|\mathrm{e}^{-C'_\omega|z|}. \tag{3.16}$$

The previous two lemmata allow us to explicitly compute how conjugation by the unitary group generated by A effects the fiber Hamiltonians and argue that the so obtained expressions admit analytic continuations.

Lemma 3.9. — *Assume Condition 3.1. Then the map*

$$\mathbb{R}\ni t\mapsto \mathrm{e}^{\mathrm{i}tA_\xi}T_V\mathrm{e}^{-\mathrm{i}tA_\xi}:=T_V^t$$

extends to an analytic $B(\mathcal{H})$-valued function defined in S_R, where

$$R=\min\left\{r,\frac{a'}{C_\omega+1},\frac{\pi}{dC'_\omega+1}\right\}. \tag{3.17}$$

We furthermore have the estimate $\|T_V^z\|\leqslant C_VC_d\mathrm{e}^{(d+d')C'_\omega|z|}$ for $z\in S_R$, where $C_d=\int_{\mathbb{R}^d}(1+|k|^{d'})^{-1}\,dk$.

Proof. — We begin by computing for $t\in\mathbb{R}$

$$\begin{aligned}&(\mathrm{e}^{\mathrm{i}A_\xi t}T_V\mathrm{e}^{-\mathrm{i}A_\xi t}g)(k)\\&\quad=\sqrt{J_\xi^t(k)}\int_{\mathbb{R}^d}\widehat V\left(k'-\gamma_\xi^t(k)\right)\sqrt{J_\xi^{-t}(k')}g(\gamma_\xi^{-t}(k'))\,\mathrm{d}k'\\&\quad=\sqrt{J_\xi^t(k)}\int_{\mathbb{R}^d}\widehat V\left(\gamma_\xi^t(k'')-\gamma_\xi^t(k)\right)\sqrt{J_\xi^{-t}(\gamma_\xi^t(k''))}J_\xi^t(k'')g(k'')\,\mathrm{d}k''\\&\quad=\sqrt{J_\xi^t(k)}\int_{\mathbb{R}^d}\widehat V\left(\gamma_\xi^t(k'')-\gamma_\xi^t(k)\right)\sqrt{J_\xi^t(k'')}g(k'')\,\mathrm{d}k''.\end{aligned} \tag{3.18}$$

Here we used substitution $k'=\gamma_\xi^t(k'')$ in the second equality and the identity (3.12) in the final equality.

By Condition 3.1 (3) and Lemma 3.7, the map $t\mapsto\widehat V(\gamma_\xi^t(k')-\gamma_\xi^t(k))$ extends to an analytic function on the strip S_R.

It follows from Condition 3.1 (1), (3.10) and Lemma 3.7 that $t \mapsto J^t_\xi(k)$ extends analytically to $z \in S_r$. The estimate (3.7) implies:

$$\forall\, z \in S_r, k, \xi \in \mathbb{R}^d : \quad |J^z_\xi(k)| \leqslant \mathrm{e}^{dC'_\omega |z|} \quad \text{and} \quad |\arg(J^z_\xi(k))| \leqslant dC'_\omega |\operatorname{Im} z|, \tag{3.19}$$

where the second estimate — on $\operatorname{Im}(\int_0^z \nabla \cdot v_\xi(\gamma^s_\xi(k))\,\mathrm{d}s)$ — is most easily derived by choosing a piecewise linear integration contour from 0 to z, which runs along the real axis from 0 to $\operatorname{Re}(z)$ (contributing nothing to the argument), and then from $\operatorname{Re}(z)$ to z. Here $\arg(\zeta)$ denotes the principal argument of the complex number ζ. The estimate on the argument shows that $\sqrt{J^t_\xi(k)}$ extends analytically to S_R (reading the square root as the principal square root).

It remains to show that, if these analytic extensions are substituted into the right-hand side of (3.18), it still defines a bounded operator on $\mathrm{L}^2(\mathbb{R}^d)$. We recall the notation $\beta^z_\xi(k,k') := \gamma^z_\xi(k') - \gamma^z_\xi(k)$ and estimate using (3.8), (3.16) and (3.19)

$$\begin{aligned}
&\int_{\mathbb{R}^d} |J^z_\xi(k)| \left| \int_{\mathbb{R}^d} \widehat{V}(\beta_z(k,k'))\sqrt{J^z_\xi(k')}g(k')\,\mathrm{d}k' \right|^2 \mathrm{d}k \\
&\qquad \leqslant C_V^2 \mathrm{e}^{2dC'_\omega|z|} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(1 + |\beta^z_\xi(k,k')|^{d'}\right)^{-1} |g(k')|\,\mathrm{d}k' \right)^2 \mathrm{d}k \\
&\qquad \leqslant C_V^2 \mathrm{e}^{2(d+d')C'_\omega|z|} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(1 + |k-k'|^{d'}\right)^{-1} |g(k')|\,\mathrm{d}k' \right)^2 \mathrm{d}k \\
&\qquad \leqslant C_V^2 C_d^2 \mathrm{e}^{2(d+d')C'_\omega|z|} \|g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,.
\end{aligned} \tag{3.20}$$

In the last step, we used (the sharp) Young's inequality [18, Thm. 4.2]. Here $C_d = \int_{\mathbb{R}^d}(1+|k|^{d'})^{-1}\,dk$. The above estimate shows that the right-hand side of (3.18) defines a bounded operator

for $z \in S_R$.

That the extension to complex $z \in S_R$ is analytic now follows from Morera's Theorem. □

Proposition 3.10. — *Assume Condition 3.1 and let $\xi, \xi_0 \in \mathbb{R}^d$. Then the pair of operators $H(\xi)$, A_{ξ_0} satisfies Condition 2.1 with R given by* (3.17) *and $M = \widetilde{M}\langle\xi\rangle^{s_1}$ for some $\widetilde{M}$, which does not depend on ξ.*

Remark 3.11. — Below, the proposition above is proven by checking each of the four assumptions in Condition 2.1 directly. In view of Proposition 2.4, it can also be proven by proving Condition 2.1 (1) and the commutator bound in Proposition 2.4 (2). In fact, two of the authors have persued this idea, see [8]. However, their proof is much longer than the one presented here.

Proof of Proposition 3.10. We begin by establishing Condition 2.1 (1). Since $D(H(\xi)) = \mathcal{D} = D(M_{\langle k\rangle^{s_2}})$, it suffices to show that $\mathrm{e}^{\mathrm{i}tA_{\xi_0}} : \mathcal{D} \to \mathcal{D}$ for $t \in [0,1]$ and that

$$\forall\, \psi \in \mathcal{D}\colon \qquad \sup_{t\in[0,1]} \|M_{\langle k\rangle^{s_2}} \mathrm{e}^{\mathrm{i}tA_{\xi_0}}\psi\| < \infty\,. \tag{3.21}$$

But this follows immediately from (3.5), (3.13) and Lemma 3.6.

As for Condition 2.1 (2), we abbreviate $w_0 := \frac{1}{2}\nabla\cdot v_{\xi_0}$, note that

$$A_{\xi_0} = \mathrm{i}\nabla_k\cdot v_{\xi_0} - \mathrm{i}w_0 = \mathrm{i}v_{\xi_0}\cdot\nabla_k + \mathrm{i}w_0$$

and compute as a commutator form on $C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} H(\xi)A_{\xi_0} - A_{\xi_0}H(\xi) &= -\mathrm{i}v_{\xi_0}\cdot\nabla_k\omega_\xi - \mathrm{i}M_{w_0}T_V - \mathrm{i}T_VM_{w_0} \\ &\qquad - \mathrm{i}\sum_{\sigma=1}^d \Big(T_{\mathrm{i}x_\sigma V}M_{(v_{\xi_0})_\sigma} - M_{(v_{\xi_0})_\sigma}T_{\mathrm{i}x_\sigma V}\Big). \end{aligned} \tag{3.22}$$

The right-hand side defines a bounded operator. Condition 2.1 (2) now follows from [22, Prop. II.1], since $\mathrm{e}^{\mathrm{i}tA_{\xi_0}}C_0^\infty(\mathbb{R}^d) \subset C_0^\infty(\mathbb{R}^d)$.

By Lemma 3.6, Lemma 3.7 and Lemma 3.9, we may conclude that the map $t\mapsto H_t(\xi)\psi$, for $\psi\in D(H(\xi))$, admits an analytic continuation into the strip S_R. Here R is defined in (3.17). The extension is given by $H_z(\xi) = M_{\omega_\xi\circ\gamma^z_{\xi_0}} + T^z_V$. This establishes Condition 2.1 (3).

It thus remains to examine whether or not Condition 2.1 (4) holds. Using Lemma 3.9, we may estimate for $\psi\in\mathcal{H}$ and $z\in S_R$:

$$\begin{aligned} &\|H_z(\xi)(H(\xi)+\mathrm{i})^{-1}\psi\| \\ &\quad\leqslant C_VC_d\mathrm{e}^{(d+d')C'_\omega|z|}\|(H(\xi)+\mathrm{i})^{-1}\psi\| + \|M_{\omega_\xi\circ\gamma^z_{\xi_0}}(H(\xi)+\mathrm{i})^{-1}\psi\| \\ &\quad\leqslant C_VC_d\mathrm{e}^{(d+d')C'_\omega|z|}\|\psi\| \\ &\qquad + C_\omega(2^{s_1}\langle\xi\rangle^{s_1}+1)2^{s_2}\langle C_\omega z\rangle^{s_2}\|M_{\langle k\rangle^{s_2}}(H(\xi)+\mathrm{i})^{-1}\psi\|\,. \end{aligned} \tag{3.23}$$

Here we used that $s_2\geqslant s_1$, Condition 3.1 (2) (from where the constant C_ω comes), (3.5) and (3.15) to estimate

$$\begin{aligned} |\omega_\xi(\gamma^z_{\xi_0}(k))| &\leqslant C_\omega\langle\xi-\gamma^z_{\xi_0}(k)\rangle^{s_1} + C_\omega\langle\gamma^z_{\xi_0}(k)\rangle^{s_2} \\ &\leqslant C_\omega(2^{s_1}\langle\xi\rangle^{s_1}+1)\langle\gamma^z_{\xi_0}(k)\rangle^{s_2} \\ &\leqslant C_\omega(2^{s_1}\langle\xi\rangle^{s_1}+1)2^{s_2}\langle C_\omega z\rangle^{s_2}\langle k\rangle^{s_2}\,. \end{aligned}$$

The proposition now follows since $D(H(\xi)) = D(M_{\langle k\rangle^{s_2}})$. ☐

Proposition 3.12. — *Assume Condition 3.1. Let* $(\lambda,\xi)\in\mathbb{R}\times\mathbb{R}^d\setminus\mathcal{T}$. *Then there exist* $e,\kappa,C>0$ *and a compact self-adjoint operator* K, *such that*

$$\mathrm{i}[H(\xi),A_\xi]\geqslant e-CE_{H(\xi)}(\mathbb{R}\setminus[\lambda-\kappa,\lambda+\kappa])-K,$$

in the sense of forms on $D(H(\xi))$. *If* $(\lambda,\xi)\notin\Sigma_{\mathrm{pp}}$, *then one may choose* $K=0$.

Proof. — Note first that it suffices to establish the form estimate on $C_0^\infty(\mathbb{R}^d)$.

We abbreviate $g_\xi(k):=\mathrm{e}^{-k^2-\xi^2}|\nabla_k\omega_\xi(k)|^2$ and write

$$\mathrm{i}[A_\xi,H(\xi)]=M_{g_\xi}+K',\tag{3.24}$$

where K' is i times the sum of the terms on the right-hand side in (3.22) involving V.

Using the $\lambda\notin\mathcal{T}(\xi)$ and that $\mathcal{T}(\xi)$ is closed, there exists $\kappa>0$ such that $[\lambda-2\kappa,\lambda+2\kappa]\subset\mathbb{R}\setminus\mathcal{T}(\xi)$. Put

$$e:=\inf\left\{g_\xi(k)\,\middle|\,|\,k\in\mathbb{R}^d\text{ with }|\omega_\xi(k)-\lambda|\leqslant2\kappa\right\}>0$$

and $C:=\sup\{|g_\xi(k)\,|\,k\in\mathbb{R}^d\}<\infty$.

Choose an $f\in C_0^\infty(\mathbb{R})$ satisfying: $\mathrm{supp}(f)\subset[\lambda-2\kappa,\lambda+2\kappa]$, $0\leqslant f\leqslant1$, and $f(\lambda')=1$ for $|\lambda'-\lambda|\leqslant\kappa$.

We may now estimate using the just chosen e,C and f :

$$\begin{aligned}M_{g_\xi}&=M_{g_\xi}f(M_{\omega_\xi})^2+M_{g_\xi}(1-f^2(M_{\omega_\xi}))\\&\geqslant ef(M_{\omega_\xi})^2\\&\geqslant e-2e(1-f(M_{\omega_\xi})).\end{aligned}\tag{3.25}$$

To pass from $1-f(M_{\omega_\xi})$ to $1-f(H(\xi))$ we compute

$$\begin{aligned}f(H(\xi))-f(M_{\omega_\xi})&=\frac{1}{\pi}\int_{\mathbb{C}}\bar\partial_z\tilde f(z)((H(\xi)-z)^{-1}-(M_{\omega_\xi}-z)^{-1})\,\mathrm{d}z\\&=\frac{1}{\pi}\int_{\mathbb{C}}\bar\partial_z\tilde f(z)(H(\xi)-z)^{-1}T_V(M_{\omega_\xi}-z)^{-1}\,\mathrm{d}z.\end{aligned}$$

Here $\tilde f$ is an almost analytic extension of f. The operator $T_V(M_{\omega_\xi}-z)^{-1}$ is compact (for the same reason K' was compact). This shows that $K''=f(H(\xi))-f(M_{\omega_\xi})$ is a compact operator.

We may now combine (3.24), (3.25) and the estimate $1-f(H(\xi))\leqslant E_{H(\xi)}(\mathbb{R}\setminus[\lambda-\kappa,\lambda+\kappa])$ to arrive at the Mourre estimate

$$\mathrm{i}[A_\xi,H(\xi)]\geqslant e-2eE_{H(\xi)}(\mathbb{R}\setminus[\lambda-\kappa,\lambda+\kappa])-K,$$

where $K=-K'+2eK''$ is compact.

Finally, if $\lambda \notin \Sigma_{\mathrm{pp}}(\xi)$, then $KE_{H(\xi)}([\lambda-\kappa,\lambda+\kappa]) \to 0$ for $\kappa \to 0$ in operator norm. This implies the remaining claim. □

Proposition 3.13. — *Assume Condition 3.1. Suppose* $(\lambda_0,\xi_0) \in \Sigma_{\mathrm{pp}} \setminus \mathcal{T}$. *Let* $U := B^{\mathbb{R}^d}_{\widetilde{R}}(\xi_0)$ *and* $U_{\mathbb{C}} := B^{\mathbb{C}^d}_{\widetilde{R}}(\xi_0)$. *Then Condition 2.16 is satisfied with* $A = A_{\xi_0}$ *and* M *replaced by* $M\sup_{\xi\in U}\langle\xi\rangle^{s_1}$.

Proof. — Let $\xi \in U$ and put $U(t) := \exp(\mathrm{i}A_{\xi_0}t)$ for $t\in\mathbb{R}$. That Condition 2.16 (1) holds, that is that $D(H(\xi)) =: \mathcal{D}$ is independent of ξ, was discussed in Remark 3.4 (1).

Condition 2.16 (2) follows from Proposition 3.10 (with $M=\sup_{\xi\in U}\langle\xi\rangle^{s_1}\widetilde{M}$ and the R from (3.17)). Hence it remains to establish Condition 2.16 (3) and 4 (with the same R).

Condition 2.8 (1) is satisfied by assumption, and Condition 2.8 (2) follows from Proposition 3.12, since $(\lambda_0,\xi_0) \notin \mathcal{T}$ and $\langle H(\xi_0)\rangle \geqslant 1$. This verifies Condition 2.16 (3).

Put

$$r_0 = \frac{R}{1+C_\omega},$$

where $R>0$ is as in (3.17) and the constants C_ω and C'_ω are from (3.6) and (3.7), respectively. In order to verify Condition 2.16 (4), choose $z_0 \in B^{\mathbb{C}}_{r_0}(0) \subset B^{\mathbb{C}}_R(0)$ with $\mathrm{Im}(z_0)>0$.

Due to Lemma 3.7, $\gamma^{z_0}_\xi(k) \in S^d_{r_0C_\omega} \subset S^d_R \subset S^d_{\tilde{R}}$ for $k\in\mathbb{R}^d$ and $\xi \in U_{\mathbb{C}} \subset S^d_{\widetilde{R}}$. We therefore have $\xi - \gamma^{z_0}_\xi(k) \in S^d_{2\widetilde{R}}$ for all $\xi\in U_{\mathbb{C}}$ and $k\in\mathbb{R}^d$.

By Condition 3.1 (1), the map $U \ni \xi \to M_{\omega_\xi\circ\gamma^{z_0}_\xi}\psi$ extends to an analytic function in $U_{\mathbb{C}}$ for every $\psi\in\mathcal{D}$. To see that $\xi \to T^{z_0}_V\psi$ extends from U to an analytic function defined for $\xi\in U_{\mathbb{C}}$, we observe from (3.18) (with t replaced by z_0 on the right-hand side) that it suffices to ensure that $|\arg(J^{z_0}_\xi(k))| < \pi$ for all $k\in\mathbb{R}^d$ and $\xi\in U_{\mathbb{C}}$. But this follows from (3.7), (3.10) and the estimate $|\int_0^{z_0}\nabla\cdot v_\xi(\gamma^s_\xi(k))\,\mathrm{d}s| \leqslant |z_0|dC'_\omega$, valid for all $k\in\mathbb{R}^d$ and $\xi\in S^d_{\widetilde{R}}$. Here we used that $r_0 \leqslant R \leqslant \pi/(1+dC'_\omega)$, cf. (3.17).

It follows that $U_{\mathbb{C}} \ni \xi \mapsto H_{z_0}(\xi)\psi$ is analytic for all $\psi\in\mathcal{D}$. This verifies Condition 2.16 (4), since all operators have a common domain by Proposition 3.10. □

Proof of Theorem 3.2. — Let $M=\mathbb{R}\times\mathbb{R}^d$ and $N=\mathbb{R}^d$, both real analytic manifolds. To show that $\mathcal{T}$ is sub-analytic, we write it as the image under a real analytic map of the semi-analytic subset

$$Y = \big\{(\lambda,\xi,k)\in M\times N \,\big|\, \omega_\xi(k)-\lambda=0\big\} \cap \big(\cap_{j=1}^d \big\{(\lambda,\xi,k)\in M\times N \,\big|\, \partial_{k_j}w_\xi(k)=0\big\}\big)$$

of $M \times N = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. Here we used that the semi-analytic subsets of M form a ring. Let $\pi\colon M \times N \to M$ be the projection $\pi(\lambda, \xi, k) = (\lambda, \xi)$. Then $\mathcal{T} = \pi(Y)$. Since $|\omega_\xi(k)| \to \infty$ as $|k| \to \infty$, this implies that $\mathcal{T}$ is sub-analytic ($\pi_{|Y}$ is a proper map). Since $\mathcal{T}$ is a closed subset of M, this finishes the proof of (1).

Fix $\xi \in \mathbb{R}^d$. To see that $\mathcal{T}(\xi)$ is locally finite, consider the map π_ξ from $M_\xi = \{(\omega_\xi(k), k) \,|\, k \in \mathbb{R}^d\}$ into $\mathbb{R}$ defined by setting $\pi_\xi(\lambda, k) = \lambda$. Since M_ξ and $\mathbb{R}$ are real analytic manifolds and π_ξ is a real analytic proper map, we get from Theorem B.3 sub-analytic stratifications $\{S_\alpha\}_{\alpha\in A}$ of M_ξ and $\{T_\beta\}_{\beta\in B}$ of $\mathbb{R}$.

Let $\lambda \in \mathcal{T}(\xi)$. Then there exists $k \in \mathbb{R}^d$, such that $\omega_\xi(k) = \lambda$ and $\nabla_k \omega_\xi(k) = 0$. Let $\alpha \in A$ be such that $(\lambda, k) \in S_\alpha$. Since $\mathrm{d}\pi_{\xi|S_\alpha}(\lambda, k)(\eta, u) = \eta$ and $\eta = u \cdot \nabla_k \omega_\xi(k)$ for $(\eta, u) \in T_{(\lambda,k)} S_\alpha$, we conclude that $\operatorname{rank}(\mathrm{d}\pi_{\xi|S_\alpha}(\lambda, k)) = 0$. Let $\beta \in B$ be such that $\pi_\xi(S_\alpha) = T_\beta$. Then T_β is a zero-stratum, hence a singleton, which must be $T_\beta = \{\lambda\}$. By local finiteness of the stratification $\{T_\beta\}_{\beta\in B}$, we may therefore conclude that $\mathcal{T}(\xi)$ is locally finite. This concludes the proof of (2).

We now turn to (3). In order to verify that $\Sigma_{\mathrm{pp}} \setminus \mathcal{T}$ is a (relatively) closed and semi-analytic subset of $\mathbb{R} \times \mathbb{R}^d \setminus \mathcal{T}$, pick $(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d \setminus \mathcal{T}$.

If $(\lambda, \xi) \notin \Sigma_{\mathrm{pp}}$, then we obtain from Proposition 3.12 constants $e, \kappa, C > 0$ such that

$$\mathrm{i}[H(\xi), A_\xi] \geqslant e - C E_{H(\xi)}([\lambda - \kappa, \lambda + \kappa]).$$

By a continuity argument, there exists a sufficiently small open neighborhood $W \subset \mathbb{R} \times \mathbb{R}^d \setminus \mathcal{T}$ of (λ, ξ), perhaps new smaller positive constants $e' < e$, $\kappa' < \kappa$ and a bigger $C' > C$, such that

$$\forall\, (\lambda', \xi') \in W: \qquad \mathrm{i}[H(\xi'), A_{\xi'}] \geqslant e' - C' E_{H(\xi')}([\lambda' - \kappa', \lambda' + \kappa']).$$

Here we used that the map $\xi' \to [H(\xi'), A_{\xi'}] \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^d))$ is continuous, cf. (3.22). Hence, by the Virial Theorem [9], $\Sigma_{\mathrm{pp}} \cap W = \emptyset$. This shows that $\Sigma_{\mathrm{pp}} \setminus \mathcal{T}$ is a closed subset of $\mathbb{R} \times \mathbb{R}^d \setminus \mathcal{T}$, and that $W \cap (\Sigma_{\mathrm{pp}} \setminus \mathcal{T}) \in \mathcal{O}(W)$. Recall from Definition B.1(1), the definition of $\mathcal{O}(W)$.

As for the case $(\lambda, \xi) \in \Sigma_{\mathrm{pp}}$, we obtain from Theorem 2.21 a neighborhood W of (λ, ξ), such that $\Sigma_{\mathrm{pp}} \cap W \in \mathcal{O}(W)$. Here we used Proposition 3.13 to ensure the applicability of Theorem 2.21. This completes the proof of (3).

The last property (4) is a standard consequence of having a Mourre estimate satisfied for energies $\lambda \in \mathbb{R} \setminus \mathcal{T}(\xi)$. □

Appendix A. Approximate Eigenstates for Closed Operators

Lemma A.1. — *Let T be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Let $\mu \in \sigma(T)$. At least one of the following two properties hold true:*

(1) *There exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(T)$ with $\|\psi_n\| = 1$ for all n and*
$$\lim_{n\to\infty} \|(T-\mu)\psi_n\| = 0\,,$$
(2) *There exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset D(T^*)$ with $\|\phi_n\| = 1$ for all n and*
$$\lim_{n\to\infty} \|(T^*-\bar{\mu})\phi_n\| = 0\,.$$

Proof. — Note that $\mu \in \sigma(T)$ implies that $\bar{\mu} \in \sigma(T^*)$. Furthermore, if $T-\mu$ has a bounded left-inverse L, then L^* is a bounded right-inverse of $T^*-\bar{\mu}$. Conversely, if L is a bounded left-inverse of $T^*-\bar{\mu}$, then L^* is a bounded right-inverse of $T-\mu$. In particular, it is not possible for $T-\mu$ and $T-\bar{\mu}$ to both have a bounded left-inverse. Therefore we may suppose that $T-\mu$ does not have a bounded left-inverse.

Suppose (1) is false. Then there exists $c>0$, such that for all $\psi \in D(T)$ we have $\|(T-\mu)\psi\| \geqslant c\|\psi\|$. This coercive estimate ensures that $T-\mu$ is injective and that $V = \mathrm{Range}(T-\mu)$ is closed. If $V = \mathcal{H}$, then $T-\mu$ has a bounded left inverse by the Closed Graph Theorem, which we assumed it did not have, and hence $V \subsetneq \mathcal{H}$. We may now pick $\phi \in V^{\perp}$ with $\|\phi\| = 1$. Then, for $\psi \in D(T)$, $|\langle \phi, T\psi\rangle| = |\langle \phi, \mu\psi\rangle| \leqslant |\mu|\|\psi\|$. This shows that $\phi \in D(T^*)$. For any $\psi \in D(T)$, we may now compute $\langle T^*\phi, \psi\rangle = \langle \phi, T\psi\rangle = \langle \phi, \mu\psi\rangle = \langle \bar{\mu}\phi, \psi\rangle$, and conclude that $T^*\phi = \bar{\mu}\phi$. This concludes the proof, since (2) will hold with the constant sequence $\phi_n = \phi$ for all n. □

Appendix B. Semi- and Sub-analytic Geometry

In this appendix we rather briefly summarize the notions of *semi-analytic* and *sub-analytic* sets. For further background, we refer the reader to [4, 5] and references therein.

Definition B.1. — *Let M be a real analytic manifold.*

(1) *Let $W \subset M$ be an open nonempty set. We write $\mathcal{O}(W)$ for the smallest ring*[(2)] *of subsets of W containing sets of the form*

(2) collection of sets stable under complement as well as under finite intersections and unions.

$\{y \in W \,|\, f(y) > 0\}$ and $\{y \in W \,|\, f(y) = 0\}$ where f ranges over real analytic functions $f\colon W \to \mathbb{R}$.

(2) *A subset $X \subset M$ is called a semi-analytic subset of M if: for any $x \in M$, there exists an open neighborhood $W \subset M$ of x, such that $X \cap W \in \mathcal{O}(W)$.*

(3) *A subset $X \subset M$ is called a sub-analytic subset of M if: for each point $x \in M$, there exists an open neighborhood $x \in W \subset M$, a real analytic manifold N and a semi-analytic subset $Y \subset M \times N$, such that*
   - *The closure $\overline{Y}$ inside $M \times N$ is compact.*
   - *$\pi(Y) = X \cap W$, where $\pi\colon M \times N \to \mathbb{R}^n$ is the projection onto the first coordinate.*

The semi-analytic as well as the sub-analytic subsets of M form rings of subsets of M. Semi-analytic subsets are of course sub-analytic as well. The converse is in general false, but if $\dim(M) \leqslant 2$, the two notions are the same.

Definition B.2. — *Let M be a real analytic manifold and $X \subset M$ a semi-analytic (sub-analytic) subset. Let $\{S_\alpha\}_{\alpha\in A}$ be a collection of subsets of X. We say that $\{S_\alpha\}_{\alpha\in A}$ is a* semi-analytic stratification *(*sub-analytic stratification*) if*

- *Each S_α is a connected real analytic manifold.*
- *$\cup_{\alpha\in A} S_\alpha = X$ and $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$.*
- *For any $K \subset M$ compact, the set $\{\alpha \in A \,|\, S_\alpha \cap K \neq \emptyset\}$ is finite.* (Local finiteness.)
- *If $\alpha \neq \beta$ and $\overline{S}_\alpha \cap S_\beta \neq \emptyset$, then $S_\beta \subset \partial S_\alpha$.* (Frontier condition.)
- *Each S_α is semi-analytic (sub-analytic) as a subset of M.*

The sets S_α are called strata, more specifically k-strata if $\dim(S_\alpha) = k$. We note that any semi-analytic (sub-analytic) subset $X \subset M$ admits a semi-analytic (sub-analytic) stratification.

In this paper we need the following result:

Theorem B.3. — *Let M and N be real analytic manifolds and $\pi\colon M \to N$ a proper real analytic map. Then there exist sub-analytic stratifications $\{S_\alpha\}_{\alpha\in A}$ of M and $\{T_\beta\}_{\beta\in B}$ of N, such that for each $\alpha \in A$, there exists $\beta \in B$ with*

- *$\pi(S_\alpha) = T_\beta$.*
- $\mathrm{rank}(\mathrm{d}\pi_{|S_\alpha}(s)) = \dim(T_\beta)$ *for all $s \in S_\alpha$.*

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