



# ANNALES

DE

# L'INSTITUT FOURIER

Emmanuel ULLMO & Andrei YAFAEV

**Holomorphic curves in compact Shimura varieties**

Tome 68, n° 2 (2018), p. 647-659.

[http://aif.cedram.org/item?id=AIF\\_2018\\_\\_68\\_2\\_647\\_0](http://aif.cedram.org/item?id=AIF_2018__68_2_647_0)



© Association des Annales de l'institut Fourier, 2018,  
*Certains droits réservés.*



Cet article est mis à disposition selon les termes de la licence  
CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE.  
<http://creativecommons.org/licenses/by-nd/3.0/fr/>

L'accès aux articles de la revue « Annales de l'institut Fourier »  
(<http://aif.cedram.org/>), implique l'accord avec les conditions générales  
d'utilisation (<http://aif.cedram.org/legal/>).

cedram

Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>

# HOLOMORPHIC CURVES IN COMPACT SHIMURA VARIETIES

by Emmanuel ULLMO & Andrei YAFAEV

---

ABSTRACT. — We prove a hyperbolic analogue of the Bloch–Ochiai theorem about the Zariski closure of holomorphic curves in abelian varieties.

RÉSUMÉ. — On démontre un analogue hyperbolique du théorème de Bloch–Ochiai sur l’adhérence de Zariski d’une courbe holomorphe dans une variété abélienne.

## 1. Introduction

The following theorem of Bloch–Ochiai (see [5, Chap. 9, Thm. 3.9.19]) is classically proved using the Nevanlinna theory.

**THEOREM 1.1** (Bloch–Ochiai). — *Let  $A$  be an abelian variety and  $f: \mathbb{C} \rightarrow A$  be a non-constant holomorphic map. Then the Zariski closure of  $f(\mathbb{C})$  is a translate of an abelian subvariety.*

In this paper we formulate and prove an analogue of this theorem for a certain type of locally symmetric varieties, namely the compact Shimura varieties.

For notations and facts about Shimura varieties and weakly special subvarieties, we refer to [10] and references therein. Recall that any hermitian symmetric domain  $X$ , admits a realisation  $X \subset \mathbb{C}^n$  (with  $n = \dim(X)$ ) as a bounded symmetric domain. See [6, Chap. 4] for details.

Recall also that given an arithmetic lattice  $\Gamma \subset \text{Aut}(X)^+$ , such that the quotient  $\Gamma \backslash X$  is compact, there exists a fundamental domain  $\mathcal{F}$  for the action of  $\Gamma$  on  $X$  which is an open subset of  $X$  such that  $\overline{\mathcal{F}}$  is compact. For a bounded hermitian symmetric domain  $X \subset \mathbb{C}^n$ , we denote by  $\partial X$  the boundary of  $X$ , i.e.  $\partial X = \overline{X} \setminus X$  where  $\overline{X}$  denotes the topological closure of  $X$  in  $\mathbb{C}^n$ .

For notions of Shimura data, Shimura varieties and their weakly special subvarieties we refer to [2], [10] and references contained therein. We just recall that weakly special subvarieties are defined in terms of Shimura sub-data, but as shown in [7], they are exactly the totally geodesic subvarieties of  $\Gamma \backslash X$  and terms “weakly special” and “totally geodesic” are used in the literature interchangeably.

Let  $(G, X)$  be a Shimura datum with  $G$  anisotropic over  $\mathbb{Q}$ , let  $X^+$  be a connected component of  $X$  and  $K$  a compact open subgroup of  $G(\mathbb{A}_f)$ . As above,  $X^+ \subset \mathbb{C}^n$  is a bounded symmetric domain.

We let  $\Gamma$  be the intersection of  $K$  with the stabiliser of  $X^+$  in  $G(\mathbb{Q})$ . Then  $\Gamma$  is an arithmetic congruence group acting on  $X^+$ .

Then  $S = \Gamma \backslash X^+$  is compact. Let  $\pi: X^+ \rightarrow \Gamma \backslash X^+$  be the quotient map.

**THEOREM 1.2.** — *Let  $f: \mathbb{C} \rightarrow \mathbb{C}^n$  be a holomorphic map such that  $C = f(\mathbb{C}) \cap X^+$  is non-empty. The following holds:*

- (1) *Let  $C'$  be an analytic irreducible component of  $f(\mathbb{C}) \cap X^+$ . Then the Zariski closure  $Zar(\pi(C'))$  is weakly special.*
- (2) *The components of the Zariski closure  $Zar(\pi(C))$  of  $\pi(C)$  are weakly special subvarieties of  $S$ .*

An easy analytic continuation argument shows that if  $Y$  is an irreducible analytic subset of  $X^+$ , then  $Zar(\pi(Y))$  is (algebraic) irreducible. It follows that in the statement (1) of Theorem 1.2, one does not need to consider irreducible components of  $Zar(\pi(C'))$ .

Another comment is that (2) does not follow directly from (1) since  $f(\mathbb{C}) \cap X^+$  can have infinitely analytic irreducible components. Additional arguments are necessary. They are given in Section 2.

This result is partly inspired the so-called hyperbolic Ax–Lindemann theorem whose slightly different but equivalent formulation is proven in [10, Thm. 1.3] in the co-compact case and in [4] for all Shimura varieties. The statement of the hyperbolic Ax–Lindemann theorem is as follows.

**THEOREM 1.3.** — *Keep the notations of Theorem 1.2.*

*Let  $Y$  be an irreducible algebraic subset of  $X^+$ . Then the Zariski closure of  $\pi(Y)$  is a weakly special subvariety of  $S$ .*

The proof of Theorem 1.2 relies on the theory of o-minimality and the Pila–Wilkie theorem and is inspired by the proof of the hyperbolic Ax–Lindemann theorem in the co-compact case as in [10]. The proof also uses in an essential way the hyperbolic Ax–Lindemann theorem itself and the results of [9].

An analogous question in the context of abelian varieties has been investigated in [11]. In that paper we have not been able to re-prove Bloch–Ochiai theorem using o-minimal techniques. We however obtained a result analogous to Theorem 1.2 in the abelian context for certain sets definable in the usual o-minimal structures. Our result in [11] is in some ways more general than the Bloch–Ochiai theorem. It is surprising and interesting that the obstructions to prove the Bloch–Ochiai theorem using o-minimality do not occur in the hyperbolic case we consider here, however additional serious difficulties arise which we overcome in Section 3.

The strategy of the proof is as follows. We start by decomposing  $f^{-1}(f(\mathbb{C}) \cap X^+)$  as a union of connected components  $U_i \subset \mathbb{C}$ . For a given  $i$  we prove that for some  $R_i > 0$ , it is in fact enough to prove the conclusion for  $\pi \circ f(U_i \cap B(0, R_i))$  where  $B(0, R_i)$  is the open ball centered at the origin of radius  $R_i$ . This is done in Section 2.

We now set  $C_i = f(U_i \cap B(0, R_i))$ . Section 3 is the technical heart of the proof. The analytic curve  $C_i$  in  $X^+$  is definable in the o-minimal structure  $\mathbb{R}_{an}$  (here  $\mathbb{C}^n$  is identified with  $\mathbb{R}^{2n}$ ). For o-minimality, related notions and results we refer to [3]. We fix a fundamental domain  $\mathcal{F}$  for the action of  $\Gamma$  on  $X^+$ .

We let  $V_i$  be the Zariski closure of  $\pi(C_i)$  and  $\tilde{V}_i$  be  $\pi^{-1}(V_i) \cap \mathcal{F}$ . We associate to  $C_i$  a certain definable (in  $\mathbb{R}_{an}$ ) set  $\Sigma \subset G(\mathbb{R})$  and show that  $\Sigma \cdot C_i \subset \tilde{V}_i$ . The main technical work is to prove that  $\Sigma$  contains a lot of points of  $G(\mathbb{Q})$  of height up to  $T$ . The Pila–Wilkie theorem then allows us to conclude that  $\Sigma$  contains a positive dimensional semi-algebraic subset  $W$  and the hyperbolic Ax–Lindemann theorem allows us to conclude that  $V_i$  contains a Zariski dense set of weakly special subvarieties. Using results from [9] and some additional arguments, we conclude the proof of Theorem 1.2.

We do not know whether conclusions of Theorem 1.2 remain true without the assumption that  $S$  is compact. The main difficulty in removing the compactness assumption lies in proving that  $C_i$  (with the above notations) intersects “many” translates of a fundamental set. We in fact do not know whether it is possible for  $C_i$  to be contained in a union of finitely many such translates.

**Acknowledgements.** The second author is very grateful to the IHES for hospitality during his visits in May and September 2016. The second author gratefully acknowledges financial support of the ERC, Project 307364\_SPGSV.

We would like to thank the referees for their helpful comments.

## 2. Preliminaries

Keep notations as in Theorem 1.2. For simplicity of notation, we write  $X$  for  $X^+$ . Let

$$f^{-1}(f(\mathbb{C}) \cap X) = \coprod_{i \in I} U_i$$

be the decomposition of  $f^{-1}(f(\mathbb{C}) \cap X)$  into connected components. By definition of the  $U_i$ , for each  $i$ , we have

$$\overline{f(U_i)} \cap \partial X \neq \emptyset.$$

For  $R_i > 0$  large enough, we have

$$\overline{f(B(0, R_i) \cap U_i)} \cap \partial X \neq \emptyset$$

(where  $B(0, R_i)$  is the open ball of radius  $R_i$  centered at the origin). For each  $i$ , we fix an  $R_i$  with this property.

PROPOSITION 2.1. — *We have*

$$\text{Zar}(\pi \circ f(U_i)) = \text{Zar}(\pi \circ f(B(0, R_i) \cap U_i)).$$

*Proof.* — One inclusion is obvious. Write  $\text{Zar}(\pi \circ f(B(0, R_i) \cap U_i)) \subset \mathbb{P}^m$  for some  $m$  and let  $s \in H^0(\mathbb{P}^m, \mathcal{O}(l))$  for  $l \geq 1$  such that  $s$  is zero on  $\pi \circ f(B(0, R_i) \cap U_i)$ . Then the function  $s \circ f \circ \pi: U_i \rightarrow \mathbb{C}$  is zero on  $B(0, R_i) \cap U_i$ . Since  $U_i$  is connected, by analytic continuation, the function  $s \circ f \circ \pi$  is zero on  $U_i$ . It follows that  $s$  is zero on  $\pi \circ f(U_i)$ . This proves the other inclusion.  $\square$

In this paper we will prove the following:

THEOREM 2.2. — *The Zariski closure of  $\pi \circ f(B(0, R_i) \cap U_i)$  contains a Zariski dense subset of weakly special subvarieties.*

Let  $V = V_i$  be the Zariski closure of  $\pi \circ f(B(0, R_i) \cap U_i)$ . Theorem 2.2 will be deduced from the following:

THEOREM 2.3. — *There exists a positive dimensional semialgebraic set  $W$  in  $G(\mathbb{R})$  such that*

$$W \cdot f(B(0, R_i) \cap U_i) \subset \pi^{-1}(V).$$

To deduce Theorem 2.2 from Theorem 2.3, let  $P \in f(B(0, R_i) \cap U_i)$ . For the notion of algebraic subset of  $X$ , we refer to Appendix B of [4]. There exists a complex algebraic subset  $Y_P \subset \pi^{-1}(V)$  such that  $W \cdot P \subset Y_P$  (see [4, Lem. B.3]). By Ax–Lindemann Theorem 1.3, the Zariski closure of  $\pi(Y_P) \subset V$  is weakly special. Therefore, through each point of  $\pi f(B(0, R_i) \cap U_i)$  there passes a weakly special subvariety and hence  $V$  contains a dense set of weakly special subvarieties.

We will now prove that Theorem 1.2 follows from Theorem 2.2. Let  $V$  be a component of the Zariski closure of  $\pi(f(\mathbb{C}) \cap X)$ . By Theorem 2.2,  $V$  contains a Zariski dense set of weakly special subvarieties.

If  $V$  is a weakly special subvariety, then we are done. Assume that  $V$  is not weakly special. By [9, Cor. 1.4], there exists a special subvariety  $S' \subset S$  containing  $V$  and such that  $S' = S_1 \times S_2$  ( $S_i$  special and positive dimensional) and such that

$$V = S_1 \times V'$$

where  $V'$  is a subvariety of  $S_2$ .

There exists a sub-Shimura datum  $(G', X')$  of  $(G, X)$  and a decomposition

$$(G'^{ad}, X'^{ad}) = (G_1, X_1) \times (G_2, X_2)$$

such that  $S_1 = \Gamma_1 \backslash X_1$  and  $S_2 = \Gamma_2 \backslash X_2$  (where as usual we omit the superscript  $+$ ) and  $\Gamma_1$  and  $\Gamma_2$  are suitable arithmetic lattices in  $G_1(\mathbb{Q})^+$  and  $G_2(\mathbb{Q})^+$ .

Let  $\mathfrak{p}_1 \cong \mathbb{C}^{r_1}$  and  $\mathfrak{p}_2 \cong \mathbb{C}^{r_2}$  be the holomorphic tangent spaces to  $X_1$  and  $X_2$ . Then  $\mathfrak{p}_1 \times \mathfrak{p}_2$  is a subspace of the holomorphic tangent space  $\mathfrak{p} \cong \mathbb{C}^n$  to  $X$ . Let

$$f^{-1}(f(\mathbb{C}) \cap X) = \coprod_{i \in I} U_i.$$

be as before, the connected component decomposition. There exists a  $U_i$  such that the restriction,  $f: U_i \rightarrow \mathbb{C}^n$  factors through  $\mathbb{C}^{r_1} \times \mathbb{C}^{r_2}$ . By analytic continuation  $f: \mathbb{C} \rightarrow \mathbb{C}^n$  factors through  $\mathbb{C}^{r_1} \times \mathbb{C}^{r_2}$ .

Let  $f_1$  and  $f_2$  be the holomorphic functions from  $\mathbb{C}$  to  $\mathbb{C}^{r_1}$  and  $\mathbb{C}^{r_2}$  respectively such that  $f = (f_1, f_2)$ .

Similarly, write

$$f_2^{-1}(f_2(\mathbb{C}) \cap X_2) = \coprod_{j \in J} V_j.$$

the connected component decomposition.

From the definition of  $U_i$ s and  $V_j$ s, it follows that any  $i \in I$ , there exists an  $j \in J$  such that  $U_i \subset V_j$ . It follows that for any  $i \in I$ , there exists  $j \in J$

such that:

$$\text{Zar}(\pi_2 \circ f_2(U_i)) = \text{Zar}(\pi_2 \circ f_2(V_j)).$$

Note that  $V'$  is the Zariski closure of the union of the  $\pi_2 \circ f_2(U_i)$ . Therefore  $V'$  is the Zariski closure of

$$\bigcup_{i \in I} \text{Zar}(\pi_2 \circ f_2(U_i)).$$

By Theorem 2.2,  $\text{Zar}(\pi_2 \circ f_2(U_i)) = \text{Zar}(\pi_2 \circ f_2(V_j))$  contains a Zariski dense set of weakly special subvarieties.

It follows that  $V'$  contains a Zariski dense set of weakly special subvarieties of  $S_2$ . An inductive argument finishes the proof of Theorem 1.2 assuming Theorem 2.2.

### 3. Counting lattice elements

In this section we show that  $f(U_i)$  (as in the previous section) in  $X$  intersects “exponentially many” (in a suitable sense)  $\Gamma$ -translates of a fixed fundamental domain. This section constitutes the technical heart of the paper.

Recall the following notations from [10]. Let  $X$  be a connected Hermitian symmetric domain (as usual we omit the superscript  $+$ ), realised as a bounded symmetric domain in some  $\mathbb{C}^n$ . We let  $C$  to be  $f(B(0, R_i) \cap U_i)$  with  $R_i$  and  $U_i$  as in the previous section.

Let  $\Gamma$  be a cocompact arithmetic lattice in the group  $G$  of holomorphic isometries of  $X$ . For a point  $x_0 \in X$ , we let  $\mathcal{F}$  be a fundamental domain for the action of  $\Gamma$  on  $X$  such that  $x_0 \in \mathcal{F}$ . We assume that  $\mathcal{F}$  is an open connected set such that  $\overline{\mathcal{F}}$  is compact. The set

$$\mathcal{S}_{\mathcal{F}} = \{\gamma \in \Gamma : \gamma \overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}$$

is finite and generates  $\Gamma$ .

The “word metric”  $l: \Gamma \rightarrow \mathbb{N}$  with respect to  $\mathcal{S}_{\mathcal{F}}$  is defined as follows:  $l(1) = 0$  and for  $\gamma \neq 1$ ,  $l(\gamma)$  is the minimal length of a word in the elements of  $\mathcal{S}_{\mathcal{F}}$  representing  $\gamma$ .

We also let  $K(Z, W)$  be the Bergmann kernel on  $X$  and we let

$$\omega = \sqrt{-1} \partial \bar{\partial} K(Z, Z)$$

be the associated Kähler form. We refer to [6, 4.1] for details on this.

We define the following functions:

$$N_C(n) = |\{\gamma \in \Gamma : \dim(\gamma \mathcal{F} \cap C) = 1, l(\gamma) \leq n\}|$$

and

$$N'_C(n) = |\{\gamma \in \Gamma : \dim(\gamma\mathcal{F} \cap C) = 1, l(\gamma) = n\}|.$$

The main result of this section is the following theorem:

**THEOREM 3.1.** — *There is a positive constant  $c$  such that for all  $n \gg 0$ , we have*

$$N_C(n) \geq e^{cn}.$$

Let  $b$  be a point of the boundary of  $\overline{C} \cap \partial X$  and a neighbourhood  $V_b$  of  $b$  such that  $\overline{C} \cap \partial X \cap V_b$  is a real analytic curve.

We parametrise  $\overline{C} \cap \partial X \cap V_b$  as follows. For  $0 < \alpha, \beta < 2\pi$ , let  $\Delta_{\alpha, \beta}$  be the sector of the unit disc  $\Delta$  defined as follows:

$$\Delta_{\alpha, \beta} = \{z = r e^{i\theta} : 0 \leq r \leq 1, \alpha \leq \theta \leq \beta\}.$$

Let  $C_{\alpha, \beta}$  be the subset of  $\partial\Delta_{\alpha, \beta}$  defined as

$$C_{\alpha, \beta} = \{z = e^{i\theta} : \alpha \leq \theta \leq \beta\}.$$

We can find  $\alpha, \beta$  and a real analytic map  $\psi$  from a neighbourhood of  $\Delta_{\alpha, \beta}$  to  $\mathbb{C}^n$  such that  $\psi(\Delta_{\alpha, \beta}) \subset \overline{C} \cap X$  and  $\psi(C_{\alpha, \beta}) \subset \overline{C} \cap \partial X$ .

Let  $\Delta$  be the open unit disk. We let  $\omega_\Delta$  be the usual Poincaré  $(1, 1)$ -form on  $\Delta$  ( $\omega_\Delta = \sqrt{-1} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$ ). By Lemma 2.8 of [10], there exists a smooth  $(1, 1)$ -form  $\eta$  on  $\Delta_{\alpha, \beta}$  such that

$$\psi^* \omega = s\omega_\Delta + \eta$$

for some integer  $s > 0$ .

Let  $\gamma \in \Gamma$  be such that  $\dim(\gamma\mathcal{F} \cap C) = 1$  and  $\gamma\mathcal{F} \cap C \subset \psi(\Delta_{\alpha, \beta})$ , then

$$(3.1) \quad \int_{\gamma\mathcal{F} \cap C} \omega = s \int_{\psi^{-1}(\gamma\mathcal{F} \cap C)} \omega_\Delta + \int_{\psi^{-1}(\gamma\mathcal{F} \cap C)} \eta.$$

**PROPOSITION 3.2.** — *There exists a constant  $B$  such that for any  $\gamma \in \Gamma$  such that  $\dim(\gamma\mathcal{F} \cap C) = 1$ , we have*

$$\int_{\gamma\mathcal{F} \cap C} \omega \leq B.$$

*Proof.* — We consider the compact dual  $X_c$  of  $X$  which is a closed algebraic subvariety of some projective  $\mathbb{P}^m$ . Let  $\mathcal{L}$  be the dual of the canonical line bundle endowed with the Fubini-Study metric  $\|\cdot\|_{FS}$ . We let  $\omega_{FS}$  the associated  $(1, 1)$ -form:  $\omega_{FS} = c_1(\mathcal{L}, \|\cdot\|_{FS})$ .

By Harish-Chandra embedding theorem, there is a biholomorphism  $\lambda$  from  $\mathfrak{p} \cong \mathbb{C}^n$  to an open dense subset of  $X_c$ . For details, see [6, §5.2, Thm. 1].



Let  $\gamma \in \Gamma$  be such that  $\gamma\mathcal{F} \cap C \neq \emptyset$ . Since  $\omega$  is  $\Gamma$ -invariant, we have

$$\int_{\gamma\mathcal{F} \cap C} \omega = \int_{\gamma^{-1}(\gamma\mathcal{F} \cap C)} \omega.$$

On the compact set  $\overline{\mathcal{F}}$ , the two forms  $\omega$  and  $\lambda^*(\omega_{FS})$  are positive holomorphic forms, therefore there is a constant  $B_1$  such that on  $\mathcal{F}$ , we have

$$\omega \leq B_1 \lambda^*(\omega_{FS}).$$

We have

$$\int_{\gamma\mathcal{F} \cap C} \omega \leq B_1 \int_{\gamma^{-1}(\gamma\mathcal{F} \cap C)} \lambda^*(\omega_{FS}).$$

Furthermore,

$$\int_{\gamma^{-1}(\gamma\mathcal{F} \cap C)} \lambda^*(\omega_{FS}) \leq \int_{\gamma^{-1}\lambda(C)} \omega_{FS}.$$

The conclusion of Proposition 3.2 follows from the following lemma that will be proven in the following section. :

LEMMA 3.3. — *There is a constant  $B_2$  such that for all  $\gamma \in \Gamma$ , we have*

$$\int_{\gamma^{-1}\lambda(C)} \omega_{FS} \leq B_2.$$

□

### 3.1. Proof of Lemma 3.3

The volume of the analytic curve  $\lambda(C)$  is defined as

$$\text{Vol}(\lambda(C)) = \int_{\lambda(C)} \omega_{FS}.$$

Let  $\mathbb{P}^{m\vee}$  be the dual projective space, the set of hyperplanes in  $\mathbb{P}^m$ . Let  $dJ$  be the invariant volume element on  $\mathbb{P}^{m\vee}$  normalised to have total mass one.

By Generalised Crofton’s formula (see [1] and references therein), we have

$$\text{Vol}(\gamma^{-1}\lambda(C)) = \alpha \int_{\mathbb{P}^{m\vee}} n_{\gamma^{-1}\lambda(C)}(J) dJ.$$

where  $\alpha$  is a uniformisation constant and  $n_{\gamma^{-1}\lambda(C)}(J)$  is the number of points (counted with multiplicity) of the intersection of  $\gamma^{-1}\lambda(C)$  with  $J$ . Note that the function  $n_{\gamma^{-1}\lambda(C)}(J)$  is a function defined on the open subset of  $\mathbb{P}^{m\vee}$  consisting of hyperplanes  $J$  such that  $\gamma^{-1}\lambda(C)$  is not contained  $J$ . The complement of this open set is of measure zero, therefore the integral is well defined.

LEMMA 3.4. — *Let  $J$  be a hyperplane in  $\mathbb{P}^m$  and  $\gamma \in \Gamma$ . There exists a hyperplane  $J'$  such that*

$$n_{\gamma^{-1}\lambda(C)}(J) = n_{\lambda(C)}(J').$$

*Proof.* — Recall that  $\mathcal{L}$  is  $\Gamma$ -invariant and  $\mathcal{L}$  is very ample i.e.  $\mathcal{L} = \mathcal{O}(1)|_{X_c}$ . Write  $s$  the section of  $\mathcal{L}$  such that  $J \cap X_c = \text{div}(s)$ . Let  $s' = \gamma^*s$ . Then  $s'$  is a restriction of a section of  $\mathcal{O}(1)$  corresponding to some hyperplane  $J'$  and we thus have

$$\gamma(J \cap X_c) = J' \cap X_c.$$

Therefore

$$\lambda(C) \cap \gamma(J \cap X_c) = \lambda(C) \cap (J' \cap X_c).$$

We also have

$$\lambda(C) \cap \gamma(J \cap X_c) = \gamma(\gamma^{-1}\lambda(C) \cap J).$$

We conclude using the fact that

$$|\gamma(\gamma^{-1}\lambda(C) \cap J)| = n_{\gamma^{-1}\lambda(C)}(J). \quad \square$$

We finish by proving a general lemma:

LEMMA 3.5. — *Let  $f: \mathbb{C} \rightarrow \mathbb{P}^m(\mathbb{C})$  be a holomorphic map. Let  $R > 0$ . There exists a constant  $\Theta = \Theta(R, f)$  such that for any hyperplane  $H$  of  $\mathbb{P}^m(\mathbb{C})$  such that  $f(\mathbb{C})$  is not contained in  $H$ ,*

$$|\{f(B(0, R)) \cap H\}| \leq \Theta.$$

*Proof.* — A reference for notions of Nevanlinna–Cartan theory is [5, Chap. 3, §B]. We use notations from this reference.

Let  $N(R, f, H)$  be the counting function associated to  $f, R$  and  $H$ . Let  $\alpha_1, \dots, \alpha_t \in B(0, R)$  be complex numbers such that  $f(\alpha_i) \in H$ .

Let  $\nu(f, \alpha_i, H)$  be the multiplicity of  $f$  in  $H$  at  $\alpha_i$ . We have

$$N(R, f, H) = \sum_{i=1}^t \nu(f, \alpha_i, H) \log \left( \frac{R}{|\alpha_i|} \right).$$

Therefore

$$N(2R, f, H) \geq \sum_{i=1}^t \nu(f, \alpha_i, H) \log \left( \frac{2R}{|\alpha_i|} \right).$$

We have  $\log(\frac{2R}{|\alpha_i|}) \geq \log(2)$ , therefore a bound on  $N(2R, f, H)$  implies a bound on  $\sum_{i=1}^t \nu(f, \alpha_i, H) = |\{f(B(0, R)) \cap H\}|$ . It is hence enough to bound  $N(2R, f, H)$ .

The first main theorem of Cartan–Nevanlinna theory ([5, 3.B.16]), we have

$$N(2R, f, H) \leq T(2R, f) + c$$

where  $c$  is a uniform constant and  $T(2R, f)$  is the order function defined in [5, 3.B.2].

Since  $T(2R, f)$  does not depend on  $H$ , this concludes the proof. □

### 3.2. End of proof of Theorem 3.1

As  $\eta$  is smooth on  $\Delta_{\alpha,\beta}$ , the integral  $\int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \eta$  is bounded independently of  $\gamma$ . Equation (3.1) and Lemma 3.2 imply that  $\int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \omega_\Delta$  is bounded by a constant  $B'$ , independent of  $\gamma$ .

Recall the following lemma ([10, Lem. 21]). Note that this lemma is proved in [10] for  $C$  algebraic but the algebraicity assumption is not used, the statement and proof remain the same in our situation. In fact the proof is a combination of some general facts about hermitian symmetric domains and word metrics.

LEMMA 3.6. — *There exist positive constants  $\lambda_1$  and  $\lambda_2$  and  $D$  such that for all  $z \in \Delta_{\alpha,\beta}$  with  $z \in \psi^{-1}(\gamma\mathcal{F} \cap C)$ ,*

$$\lambda_1 l(\gamma) \leq -\log(1 - z\bar{z}) \leq \lambda_2 l(\gamma) + D.$$

We now follow the end of Section 2 of [10].

For  $n > 0$ , let

$$I_n = \{z \in \Delta_{\alpha,\beta}, e^{-(n+1)} \leq 1 - |z|^2 \leq e^{-n}\}.$$

The hyperbolic volume of  $I_n$  satisfies

$$\text{Vol}(I_n) \geq \delta_1 e^n$$

where  $\delta_1$  is a positive constant.

The set  $I_n$  is covered by the  $\phi^{-1}(\gamma\mathcal{F} \cap C)$ . For each  $n$  large enough and for all  $z \in I_n$ , by Lemma 3.6, there exists a  $\gamma$  such that  $\psi(z) \in \gamma\mathcal{F}$  with  $\gamma$  satisfying

$$c_1 n \leq l(\gamma) \leq c_2 n$$

with uniform constants  $c_1$  and  $c_2$ .

On the other hand, for all  $z \in \Delta_{\alpha,\beta}$ , such that  $\psi(z) \in \gamma\mathcal{F}$  for some  $\gamma \in \Gamma$ ,

$$\text{Vol}(\psi^{-1}(\gamma\mathcal{F} \cap C)) \leq B'.$$

Therefore, by the computation of  $\text{Vol}(I_n)$  above, there exists a  $\delta_1 > 0$  such that

$$\sum_{c_1 n \leq k \leq c_2 n} N'_C(k) \geq \delta_1 e^n.$$

This finishes the proof of Theorem 3.1.

#### 4. A definable set and application of Pila–Wilkie theorem

In this section we prove Theorem 2.3 and hence our main theorem. We follow Section 5 of [10] with appropriate modifications.

Let  $U$  be as before a connected component of  $f^{-1}(f(\mathbb{C}) \cap X)$  and  $R$  such that  $f(U \cap B(0, R)) \cap \partial X \neq \emptyset$ . Note that  $C = f(B(0, R) \cap U)$  is definable in  $\mathbb{R}_{an}$ . Let  $\mathcal{F}$  be as in the previous section. Recall (see [10, Prop. 4.2]) that  $\pi$  restricted to  $\mathcal{F}$  is definable in  $\mathbb{R}_{an}$ .

Consider

$$\Sigma(C) = \{g \in G(\mathbb{R}) : \dim(gC \cap \pi^{-1}(V) \cap \mathcal{F}) = 1\}.$$

The set  $\Sigma(C)$  is definable in  $\mathbb{R}_{an}$ .

We prove the following.

LEMMA 4.1.

- (1) For all  $g \in \Sigma(C)$ ,  $gC \subset \pi^{-1}(V)$ .
- (2) Define

$$\Sigma'(C) = \{g \in G(\mathbb{R}) : g^{-1}\mathcal{F} \cap C \neq \emptyset\}.$$

Then

$$\Sigma(C) \cap \Gamma = \Sigma'(C) \cap \Gamma.$$

*Proof.* — Let  $g \in \Sigma(C)$ , then

$$gC \cap \mathcal{F} \subset \pi^{-1}(V).$$

By analytic continuation, this implies that  $gC \subset \pi^{-1}(V)$ .

The proof of (2) is exactly identical to the proof of [10, Lem. 5.2], and relies on the fact that  $\pi^{-1}(V)$  is  $\Gamma$ -invariant. □

From the previous lemma and Theorem 3.1, we obtain the following.

LEMMA 4.2. — *Let*

$$N_{\Sigma(C)}(n) = |\{\gamma \in \Gamma \cap \Sigma(C) : l(\gamma) \leq n\}|.$$

For all  $n$  large enough,

$$N_{\Sigma(C)}(n) \geq e^{cn}.$$

The height  $H(\gamma)$  of an element  $\gamma$  of  $\Gamma$  is defined by viewing  $\Gamma$  as a subgroup of some  $\mathrm{GL}_m(\mathbb{Z})$  and taking the maximum of the absolute values of the entries. If  $l(\gamma) \leq n$ , then  $H(\gamma) \leq (mA)^n$  where  $A$  is the maximum of heights of elements of  $\mathcal{S}_{\mathcal{F}}$ .

Let now

$$\Theta(\Sigma(C), T) = \{g \in G(\mathbb{Q}) \cap \Sigma(C) : H(g) \leq T\}$$

and

$$N(\Sigma(C), T) = |\Theta(\Sigma(C), T)|$$

LEMMA 4.3. —  $N(\Sigma(C), T) \geq T^{c_1}$ .

We now appeal to the Pila–Wilkie theorem (see [8, Thm. 1.8]).

For a definable (in some o-minimal structure) subset  $\Theta \subset \mathbb{R}^n$ , we define  $\Theta^{alg}$  to be the union of all positive dimensional semi-algebraic subsets contained in  $\Theta$ . We define  $\Theta^{tr}$  to be  $\Theta \setminus \Theta^{alg}$ .

THEOREM 4.4 (Pila–Wilkie). — *Let  $\Theta$  be a subset of  $\mathbb{R}^n$  definable in an o-minimal structure. Let  $\epsilon > 0$ . There exists a constant  $c = c(\Theta, \epsilon)$  such that for any  $T \geq 0$ ,*

$$|\{x \in \Theta^{tr} \cap \mathbb{Q}^n : H(x) \leq T\}| \geq cT^\epsilon.$$

In view of Lemma 4.3, by Pila–Wilkie theorem, there exists a positive dimensional semi-algebraic subset  $W \subset \Sigma(C)$  and by (1) of Lemma 4.1, we have  $W \cdot C \subset \pi^{-1}(V)$ . This finishes the proof of Theorem 2.3.

## BIBLIOGRAPHY

- [1] H. ALEXANDER, “Volumes of varieties in projective space and in Grassmannians”, *Trans. Am. Math. Soc.* **289** (1974), p. 237-249.
- [2] P. DELIGNE, “Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques”, in *Automorphic forms, representations and L-functions (Corvallis, 1977)*, Proceedings of Symposia in pure Mathematics, vol. 33, American Mathematical Society, 1979, p. 247-289.
- [3] L. VAN DEN DRIES, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998, x+180 pages.
- [4] B. KLINGLER, E. ULLMO & A. YAFAEV, “The hyperbolic Ax-Lindemann-Weierstrass conjecture”, *Publ. Math., Inst. Hautes Étud. Sci.* **123** (2016), p. 333-360.
- [5] S. KOBAYASHI, *Hyperbolic Complex Spaces*, Grundlehren der Mathematischen Wissenschaften, vol. 318, Springer, 1998, xiii+471 pages.
- [6] N. MOK, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Mathematics, vol. 6, World Scientific, 1989, xiv+278 pages.
- [7] B. MOONEN, “Linearity properties of Shimura varieties. I.”, *J. Algebr. Geom.* **7** (1998), no. 3, p. 539-567.

- [8] J. PILA & A. J. WILKIE, “The rational points of a definable set”, *Duke Math. J.* **133** (2006), no. 3, p. 591-616.
- [9] E. ULLMO, “Applications du théorème d’Ax-Lindemann hyperbolique”, *Compos. Math.* **150** (2014), no. 2, p. 175-190.
- [10] E. ULLMO & Y. ANDREI, “Hyperbolic Ax-Lindemann theorem in the co-compact case”, *Duke Math. J.* **163** (2014), no. 2, p. 433-463.
- [11] ———, “o-minimal flows on abelian varieties”, *Q. J. Math.* **163** (2017), no. 2, p. 359-367.

Manuscrit reçu le 8 septembre 2016,  
révisé le 29 juin 2017,  
accepté le 14 septembre 2017.

Emmanuel ULLMO  
IHES and Université Paris-Saclay  
35 Route de Chartres,  
91440 Bures-sur-Yvette (France)  
ullmo@ihes.fr

Andrei YAFAEV  
UCL,  
Department of Mathematics,  
Gower street,  
WC1E 6BT, London (UK)  
yafaev@math.ucl.ac.uk