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NOTE ON POINCARÉ TYPE FUTAKI CHARACTERS

by Hugues AUVRAY

ABSTRACT. — A Poincaré type Kähler metric on the complement $X\backslash D$ of a simple normal crossing divisor D, in a compact Kähler manifold X, is a Kähler metric on $X\backslash D$ with cusp singularity along D. We relate the *Futaki character for holomorphic vector fields parallel to the divisor*, defined for any fixed Poincaré type Kähler class, to the classical Futaki character for the relative smooth class. As an application we express a numerical obstruction to the existence of extremal Poincaré type Kähler metrics, in terms of mean scalar curvatures and Futaki characters.

RÉSUMÉ. — On appelle métrique kählérienne de type Poincaré, sur le complémentaire $X\backslash D$ d'un diviseur à croisements normaux simples D dans une variété kählérienne compacte X, une métrique kählérienne sur $X\backslash D$ à singularités *cusp* le long de D. On relie le *caractère de Futaki des champs de vecteurs holomorphes parallèles au diviseur*, défini pour toute classe de Kähler de métriques de type Poincaré fixée, au caractère de Futaki classique de la classe lisse sous-jacente. On donne en application une obstruction numérique à l'existence de métriques extrémales de type Poincaré, exprimée en termes de courbures scalaires moyennes et de caractères de Futaki.

Introduction

A basic fact in Kähler geometry is the independence of the de Rham class of the Ricci form from the background metric on a compact Kähler manifold: it is always $-2\pi c_1(K)$, with $c_1(K)$ the first Chern class of the canonical line bundle. This invariance turns out to constitute the first obstacle for a compact Kähler manifold to admit a Kähler–Einstein metric: the Chern class in question must then have a sign, which, if definite, forces Kähler–Einstein metrics to lie in a consequently fixed Kähler class.

When $c_1(K) > 0$, a (unique) Kähler–Einstein metric was obtained by Aubin and Yau, and Bochner's technique then rules out the existence

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of non-trivial holomorphic vector fields. Conversely, in the opposite case $c_1(K) < 0$, that is, when dealing with so-called *"Fano manifolds"*, non-trivial holomorphic vector fields may exist, and the existence of a Kähler–Einstein metric, which does not always hold, is noticeably more involved. More precisely, in this case (and, respectively, on any compact Kähler manifold) non-trivial holomorphic vector fields bring a constraint to the existence of a Kähler–Einstein metric (respectively, of a constant scalar curvature metric Kähler metric in some fixed Kähler class). If indeed such a canonical metric exists, a numerical function, the *Futaki character* [11], defined on the Lie algebra of holomorphic vector fields, and depending only on the Kähler class under study, has to vanish identically.

The Futaki character was later generalised by Donaldson to polarised manifolds, into a numerical function defined on *test-configurations*, that generalise the concept of (the action of) holomorphic vector fields [10]. In the lines of suggestions by Yau [23], and after Tian's *special degenerations* [19], test-configurations and their Donaldson–Futaki invariants are meant to reveal the link between algebro-geometric stability of the manifold, and existence of a Kähler–Einstein/constant scalar curvature Kähler metric:

Conjecture A (Yau–Tian–Donaldson). — *A polarised manifold (X, L) admits a constant scalar Kähler curvature metric in $2\pi c_1(L)$ if, and only if, (X, L) is "K-stable", that is: the Donaldson–Futaki invariant is nonpositive (negative) for any (non-trivial) test-configuration.*

The "only if" direction is now established [16, 17]; the "if" direction is still a very active area of research, and has been solved rather recently for Kähler–Einstein metrics in the Fano case, i.e. when $L = -K_X$ is ample, see [7, 8, 9] and [20].

In a related scope, the aim of this note is, after restricting to the relevant set of holomorphic vector fields, to generalise the Futaki character to a certain class of singular metrics on a compact manifold. Namely, fixing a simple normal crossing divisor D in a compact Kähler manifold (X, J, ω_X), we recall the definition of *Poincaré type Kähler metrics on $X \backslash D$*, following [4, 21, 22]:

Definition B. — *A smooth positive $(1,1)$-form ω on $X \backslash D$ is called a* Poincaré type Kähler metric on $X \backslash D$ *if: on every open subset U of coordinates $(z^1, \dots, z^m)$ in X, in which D is given by $\{z^1 \cdots z^j = 0\}$, ω is* *mutually bounded with*

$$\omega_U^{\mathrm{mdl}} := \frac{i\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} + \cdots + \frac{i\mathrm{d}z^j \wedge \mathrm{d}\overline{z^j}}{|z^j|^2 \log^2(|z^j|^2)} + i\mathrm{d}z^{j+1} \wedge \mathrm{d}\overline{z^{j+1}} + \cdots + i\mathrm{d}z^m \wedge \mathrm{d}\overline{z^m},$$

and has bounded derivatives at any order for this model metric.

We say moreover that ω is of class $[\omega_X]$ *if* $\omega = \omega_X + \mathrm{dd}^c\varphi$ *for some* φ *smooth on* $X\backslash D$*, with* $\varphi = \mathcal{O}\big(\sum_{\ell=1}^j \log[-\log(|z^\ell|^2)]\big)$*, and* $\mathrm{d}\varphi$ *bounded at any order for* ω_U^{mdl}*, in any of the above open set* U *of coordinates* (z^k)*. We then set:* $\omega \in \mathscr{M}^D_{[\omega_X]}$.

Metrics of $\mathscr{M}^D_{[\omega_X]}$ are complete, with finite volume (equal to that of X for smooth Kähler metrics of class $[\omega_X]$). They also share a common mean scalar curvature, $\overline{\mathbf{s}}^D$, which nonetheless differs from $\overline{\mathbf{s}}$, the mean scalar curvature attached to smooth Kähler metrics of class $[\omega_X]$: one has $\overline{\mathbf{s}}^D = \overline{\mathbf{s}} - 4\pi m \frac{c_1(D)[\omega_X]^{m-1}}{[\omega_X]^m}$ $(m = \dim_{\mathbb{C}} X)$.

Now, the (real) holomorphic vector fields bounded (at any order) for *any* given Poincaré type Kähler metric on $X\backslash D$ can easily be identified to the holomorphic vector fields on the whole X with their normal component along D vanishing identically. Thus, restricting to the set $\mathfrak{h}^D_{/\!/}$ of these vector fields, and mimicking the compact case $D = \varnothing$, we define *Poincaré type Futaki character* $\mathscr{F}^D_{[\omega_X]}$, as a map from $\mathfrak{h}^D_{/\!/}$ to $\mathbb{R}$. The definition still involves the integration of Riemannian potentials of vector fields against a scalar curvature, with the difference that these are now first computed with some metric in $\mathscr{M}^D_{[\omega_X]}$; namely, if $\omega \in \mathscr{M}^D_{[\omega_X]}$ has scalar curvature $\mathbf{s}(\omega)$, and if $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$ verifies $\mathsf{Z}^{\sharp_g} = \mathrm{d}f_\omega + \alpha$ with f_ω bounded and $\mathrm{d}^c\alpha = 0$ on $X\backslash D$ $(g = \omega(\,\cdot\,, J\,\cdot\,))$, we set:

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \int_{X\backslash D} f_\omega\big(\mathbf{s}(\omega) - \overline{\mathbf{s}}^D\big)\frac{\omega^m}{m!}\,.$$

As in the compact case, $\mathscr{F}^D_{[\omega_X]}$ turns out to be a character of $\mathfrak{h}^D_{/\!/}$ (which, indeed, is a Lie algebra), depending only on the (Poincaré) Kähler class.

Results. — The Poincaré type Futaki character generally differs from the usual smooth Futaki character restricted to $\mathfrak{h}^D_{/\!/}$. We hence establish a precise formula relating the different characters, involving a divisorial term, and then apply the Poincaré type Futaki character to the search for necessary conditions for existence of canonical Kähler metrics: calling *extremal* a Poincaré type Kähler metric having a scalar curvature with gradient in $\mathfrak{h}_{/\!/}^D$, our main result is a *numerical constraint to the existence of extremal metrics of Poincaré type on* $X\backslash D$, stating as:

Theorem C. — *Assume that there exists an extremal metric in* $\mathscr{M}^D_{[\omega_X]}$*, and denote by* K *the Riemannian gradient of its scalar curvature. For any* $j \in \{1,\dots,N\}$ *indexing an irreducible component* D_j *of* D*, set* $\hat{D}_j = \sum_{\ell\neq j} D_\ell$*, and* $E_j = \hat{D}_j|_{D_j} = \sum_{\ell\neq j}(D_\ell \cap D_j)$*. Then one has:*

$$\overline{\mathbf{s}}^D < \overline{\mathbf{s}}^{E_j} + \frac{1}{4\pi\,\mathrm{Vol}(D_j)}\big(\mathscr{F}^D_{[\omega_X]}(\mathsf{K}) - \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K})\big). \tag{A}$$

Here, $\mathscr{F}^{\hat{D}_j}_{[\omega_X]}$ *is the Futaki character for Poincaré type metrics on* $X\backslash\hat{D}_j$*, of class* $[\omega_X]$*, and* $\overline{\mathbf{s}}^D$ *(resp.* $\overline{\mathbf{s}}^{E_j}$*) is the mean scalar curvature attached to* $\mathscr{M}^D_{[\omega_X]}$ *(resp. to* $\mathscr{M}^{E_j}_{[\omega_X]|_{D_j}}$*, the space of Poincaré type metrics on* $D_j\backslash E_j$*, and of class* $[\omega_X]|_{D_j}$*).*

Constraint (A) is an “invariant-flavoured” reformulation of that of [3, Prop. 4.5], and thus extends the obstruction to the existence of constant scalar curvature metrics in $\mathscr{M}^D_{[\omega_X]}$ of [2], which states as $\overline{\mathbf{s}}^D < \overline{\mathbf{s}}^{E_j}$ for all $j = 1,\dots,N$: simply put $\mathsf{K} = 0$ in (A). As in the compact case moreover, by construction and invariance along $\mathscr{M}^D_{[\omega_X]}$, $\mathscr{F}^D_{[\omega_X]}$ vanishes identically if there exists a constant scalar curvature metric in $\mathscr{M}^D_{[\omega_X]}$. Conversely, its vanishing forces possible extremal metrics of $\mathscr{M}^D_{[\omega_X]}$ to have constant scalar curvature. By contrast, the interest of Theorem C is to provide constraints on extremal metrics independently of such a vanishing.

Donaldson–Futaki invariants are already considered in [6] which take into account the contribution of a divisor. These are used in the context of *Kähler metrics with conical singularities on polarised manifolds*, making use of a divisorial term with coefficient $(1-\beta)$, with $2\pi\beta$ the angle of the cone singularity. In view of our formula between invariants (see Proposition 2.1 below), the Poincaré type Futaki invariant might thus be viewed, after switching from test-configurations to holomorphic vector fields, as the limit when the conical singularity angle goes to 0, that is, roughly speaking, when *cones* become *cusps*. In this spirit, and with help of the algebraic interpretation of the Poincaré–Futaki character, which we also explain in this paper, one may notice that our constraint is used to rule out the existence of Poincaré type Kähler metrics on Hirzebruch surfaces in [18, §5]; other examples of application, in the toric setting, are considered in [1].

Organisation of the article. This note is divided into three parts, plus an appendix. In the first part, we analyse holomorphic vector fields parallel to a divisor D, i.e., vector fields in $\mathfrak{h}^D_{/\!/}$. We see in particular that a Hodge decomposition with respect to Poincaré type Kähler metrics case still holds for such vector fields. This allows us in Section 1.2 to define the Poincaré type Futaki character, as an *invariant* of a given Poincaré type Kähler class.

The formula between smooth and Poincaré type Futaki characters is stated in Section 2.1 (Proposition 2.1). It is proven in Section 2.2, and the final section 2.3 of Part 2 is devoted to a key technical lemma (Lemma 2.2) used in Section 2.2.

In Part 3 we state and prove Theorem C: a useful technical extension (Proposition 3.1) of Proposition 2.1 is given in Section 3.1; Theorem C is then proven in Section 3.2 (Theorem 3.2), first in the smooth divisor case using Proposition 3.1, then in the simple normal crossing case. Notice that both steps use the asymptotic properties of extremal Poincaré type metrics obtained in [3].

We finally highlight with a few words, in the appendix, links between our definition of Poincaré-type Futaki character, and K-stability terminology, as developed in particular for triples including a divisor by G. Székelyhidi [18], motivated by an extension of Conjecture A to metrics with cusp singularities.

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Notation. — In all this note, X is a compact Kähler manifold, and $D \subset X$ a simple normal crossing divisor, the decomposition into irreducible smooth components of which we write as $\sum_{j=1}^N D_j$.

1. The Futaki character of a Poincaré class

1.1. Hodge decomposition of vector fields parallel to the divisor

Reminder: the compact case. Fix a smooth Kähler form ω_X on X, of associated Riemannian metric g_X. Given any real holomorphic vector field Z (we write: $\mathsf{Z} \in \mathfrak{h}$), it is well-known that its g_X-dual 1-form ξ^{Z}, that is, $\mathsf{Z}^{\sharp_{g_X}}$, enjoys the following decomposition:

$$\xi^{\mathsf{Z}} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + \mathrm{d}f^{\mathsf{Z}}_{\omega_X} + \mathrm{d}^c h^{\mathsf{Z}}_{\omega_X}, \tag{1.1}$$

into harmonic, d- and d^c-exact parts; these are uniquely determined, provided that $f^{\mathsf{Z}}_{\omega_X}$ and $h^{\mathsf{Z}}_{\omega_X}$ are taken with null mean against ω_X^m.

Decomposition (1.1) is called the *(dual) Hodge decomposition* of Z. Given moreover any other smooth metric $\tilde{\omega} = \omega_X + dd^c\varphi$ of $\mathscr{M}_{[\omega_X]}$, and setting $\tilde{\xi}^{\mathsf{Z}}$ for the dual 1-form of Z with respect to $\tilde{\omega}(\,\cdot\,, J\,\cdot\,)$, its Hodge decomposition is:

$$\tilde{\xi}^{\mathsf{Z}} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + d\big(f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi\big) + d^c\big(h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi\big)$$

see [13, Lem. 4.5.1]; notice in particular that the harmonic part remains unchanged at the level of 1-forms; recall that on compact Kähler manifolds, the space of harmonic 1-forms is *independent of the Kähler metric*.

Extension to Poincaré type Kähler metrics (vector fields parallel to a divisor). Consider now the simple normal crossing divisor $D = \sum_{j=1}^N D_j$ in X. The normal crossing assumption can be expressed as follows: given any $p \in (D_{j_1}\cap\cdots\cap D_{j_k})\backslash(D_{\ell_1}\cup\cdots\cup D_{\ell_{N-k}})$, with $\{j_1,\dots,j_k\}\sqcup\{\ell_1,\dots,\ell_{N-k}\} = \{1,\dots,N\}$, one can find in X an open subset U of holomorphic coordinates $(z^1,\dots,z^m)$ centred at p, such that $U\cap D_{j_s} = \{z^s = 0\}$ for $s = 1,\dots,k$ (in particular, $k \leqslant m$).

We define a restricted class of holomorphic vector fields on X, the use of which is natural when working with Poincaré type Kähler metrics on $X\backslash D$:

Definition 1.1. — *Let $\mathsf{Z}\in\mathfrak{h}$. We say that* Z is parallel to D, *denoted $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$, if: writing Z as $\mathfrak{Re}\big(f_1\frac{\partial}{\partial z^1}+\cdots+f_m\frac{\partial}{\partial z^m}\big)$ in local holomorphic coordinates as above, one has $f_1\equiv 0$ on $D_{j_1},\dots,f_k\equiv 0$ on D_{j_k}, that is: as soon as D is given by $z^1\cdots z^k = 0$ and the f_j's are the respective coefficient functions of Z along the $\frac{\partial}{\partial z^j}$'s, then f_1 factors through $z^1,\dots,f_k$ factors through z^k.*

Given $j\in\{1,\dots,N\}$, we then define the restriction $\mathsf{Z}|_{D_j}$ of Z to D_j *by setting locally $\mathsf{Z}|_{D_j} = \mathfrak{Re}\big(f_2|_{D_j}\frac{\partial}{\partial z^2}+\cdots+f_m|_{D_j}\frac{\partial}{\partial z^m}\big)$, whenever $j_1 = j$ in the above coordinates.*

One checks in particular that the definition of $\mathsf{Z}|_{D_j}$ is independent of the choice of holomorphic coordinates, as long as the first coordinate is a local equation of D_j. One also checks easily that $\mathfrak{h}^D_{/\!/}$ is a Lie subalgebra of $\mathfrak{h}$, namely, it is the Lie algebra of automorphisms of X globally preserving D, component by component.

Holomorphic vector fields parallel to D are relevant when working with Poincaré type Kähler metrics on D for that *any holomorphic vector field on $X\backslash D$ which is bounded (or merely L^2) with respect to a Poincaré type Kähler metric on $X\backslash D$ extends to a holomorphic vector field on X, parallel to D.* Indeed, working with a Poincaré type Kähler metric, an L^2 holomorphic vector field Z on $X\backslash D$ remains square-integrable on $X\backslash D$ for a *smooth* metric on X, and can thus be extended holomorphically to the whole X. Then, writing the Poincaré-type L^2 condition on Z in coordinates, one sees that the components of Z normal to D have to vanish along D, for the very reason that $\int_{\{0<|z|\leqslant 1/2\}} \frac{i\,\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(|z|^2\log(|z|)^2)^2} = +\infty$ (whereas $\int_{\{0<|z|\leqslant 1/2\}} \frac{i\,|z|^2\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(|z|^2\log(|z|)^2)^2} < \infty$). Conversely, any holomorphic vector field on X parallel to D gives on $X\backslash D$ a vector field bounded at any order for any Poincaré type metric on $X\backslash D$.

Before we develop more theoretic considerations, let us push further the latter observation, and state the following elementary asymptotic considerations, that shall be much helpful to keep in mind later on (e.g. in the proof of Lemma 2.2 below): let f be a smooth function on $X\backslash D$, with the singularity class of a Poincaré type Kähler potential, that is, $f = \mathcal{O}\big(\sum_{\ell=1}^{j} \log[-\log(|z^\ell|^2)]\big)$ in the usual holomorphic coordinates, and $\mathrm{d}f$ is bounded at any order for any (fixed) Poincaré type Kähler metric on $X\backslash D$; let moreover $\mathsf{Z} \in \mathfrak{h}_{/\!/}^{D}$. Then: $\mathsf{Z}\cdot f = \mathrm{d}f(\mathsf{Z})$ is bounded at any order in Poincaré type Kähler metric on $X\backslash D$. Moreover, assuming that D is reduced to one component, given in some open subset U of coordinates by $z^1 = 0$, where f has the asymptotics $f = c\log[-\log(|z^1|^2)] + \mathcal{O}\big(\big|\log(|z^1|)\big|^{-a}\big)$ ($c\in\mathbb{R}$, $a>0$, e.g. $a=1$) at any order for some Poincaré type Kähler metric on $X\backslash D$, then: $\mathsf{Z}\cdot f = \mathcal{O}\big(\big|\log(|z^1|)\big|^{-\inf(a,1)}\big)$ at any order on $U\backslash D$. This follows from the writing $\mathrm{d}f = 2c\mathfrak{Re}\big[\frac{\mathrm{d}z^1}{z^1\log(|z^1|^2)}\big] + \epsilon$ with $\epsilon = \mathcal{O}\big(\big|\log(|z^1|)\big|^{-a}\big)$, and $\mathsf{Z} = \mathfrak{Re}\big(z^1 g_1\frac{\partial}{\partial z^1} + \sum_{j=2}^{m} f_j\frac{\partial}{\partial z^j}\big)$ (g_1 and the f_j's holomorphic on U) giving the pairing $\mathrm{d}f(\mathsf{Z}) = \frac{c\mathfrak{Re}(g_1)}{\log(|z^1|^2)} + \epsilon(\mathsf{Z}) = \mathcal{O}\big(\big|\log(|z^1|)\big|^{-1}\big) + \mathcal{O}\big(\big|\log(|z^1|)\big|^{-a}\big)$.

Hodge decomposition. Holomorphic vector fields parallel to D turn out to admit *a Hodge decomposition with respect to Poincaré type Kähler metrics on* $X\backslash D$:

Proposition 1.2. — *Let $\mathsf{Z}\in\mathfrak{h}_{/\!/}^{D}$, and let $\omega = \omega_X + \mathrm{dd}^c\varphi \in \mathscr{M}^{D}_{[\omega_X]}$. Let $\xi^{\mathsf{Z}}_{\varphi}$ be the dual 1-form of Z with respect to $\omega(\,\cdot\,, J\,\cdot\,)$. Then*

$$\xi^{\mathsf{Z}}_{\varphi} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + \mathrm{d}\big(f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi\big) + \mathrm{d}^c\big(h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi\big) \tag{1.2}$$

on $X\backslash D$, with the same harmonic part $\xi^{\mathsf{Z}}_{\mathrm{harm}}$ as in the compact case, and this decomposition is unique. Moreover,

$$\int_{X\backslash D} \big(f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi\big)\omega^m = \int_{X\backslash D} \big(h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi\big)\omega^m = 0.$$

The uniqueness we state here is understood as follows: if $\xi^{\mathsf{Z}}_{\varphi} = \alpha + \mathrm{d}\beta + \mathrm{d}^c\gamma$ with α harmonic on $X\backslash D$, and α, β, γ bounded for ω of Poincaré type, then $\alpha = \xi^{\mathsf{Z}}_{\mathrm{harm}}$, and $\beta = f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi$ and $\gamma = h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi$ up to a constant. This justifies:

Notation 1.3. — With the notations of Proposition 1.2, we set

$$f^{\mathsf{Z}}_{\omega} = f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi \quad \text{and} \quad h^{\mathsf{Z}}_{\omega} = h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi.$$

Proof of Proposition 1.2. — Before starting, we recall the following "Gaffney–Stokes' theorem", that we use several times in the upcoming proof, and more generally along this article; we refer the reader to [12] for the proof.

Lemma 1.4 (Gaffney). — *Let (M,g) be a complete orientable Riemannian manifold of real dimension $n \geqslant 1$, and let α be a differential form of degree $(n-1)$ on M. Assume that α and $\mathrm{d}\alpha$ are L^1 on (M,g), that is: $\int_M |\alpha|_g \,\mathrm{vol}^g$ and $\int_M |\mathrm{d}\alpha|_g \,\mathrm{vol}^g$ are finite. Then: $\int_M \mathrm{d}\alpha = 0$.*

With the notations of the statement, we first prove that equality (1.2) holds on $X\backslash D$. This identity is purely local; it is thus sufficient to establish it for any Kähler metric equal to ω in the neighbourhood of any given point of $X\backslash D$. More concretely, as $\omega = \omega_X + \mathrm{dd}^c\varphi$ is of Poincaré type, local analysis provides that $\varphi \to -\infty$ near D. Consider a convex function $\chi : \mathbb{R} \to \mathbb{R}$, with $\chi(t) = 0$ if $t \leqslant -1$, $\chi(t) = t$ if $t \geqslant 1$ – and thus $0 \leqslant \chi'(t) \leqslant 1$ for all t. Given $K \in \mathbb{R}$, one now easily checks that

$$\omega_K := \omega_X + \mathrm{dd}^c\big(\chi\circ(\varphi+K)\big)$$

is a smooth metric on X, equal to ω on $\{\varphi \geqslant 1-K\}$ (compact in $X\backslash D$), and to ω_X on $\{\varphi \leqslant -(K+1)\}$. Now (1.2) follows on $\{\varphi > 1-K\}$ by the smooth case of Hodge decomposition applied to ω_K, thus on all $X\backslash D$ by letting $K\to\infty$.

Observe that $\xi^{\mathsf{Z}}_{\mathrm{harm}}$ is still harmonic with respect to ω; again, this condition is local, implied, thanks to the Kähler identities, by the closedness and the d^c-closedness of $\xi^{\mathsf{Z}}_{\mathrm{harm}}$. These latter conditions are independent of the Kähler metric, and indeed implied by the harmonicity of $\xi^{\mathsf{Z}}_{\mathrm{harm}}$ for the smooth ω_X, as X is compact.

As $\xi^{\mathsf{Z}}_{\mathrm{harm}}$ is bounded for ω_X, it is so for ω, which dominates ω_X. Similarly, $f^{\mathsf{Z}}_{\omega_X}$ and $h^{\mathsf{Z}}_{\omega_X}$ are bounded at any order for ω_X hence for ω, and as Z is parallel to D, it is bounded at any order for ω, as well as $\mathrm{d}\varphi$ by definition; consequently, f^{Z}_{ω} and h^{Z}_{ω} are bounded at any order for ω. From this the uniqueness of Hodge decomposition easily follows. Write $\xi^{\mathsf{Z}}_{\varphi} = \alpha + \mathrm{d}\beta + \mathrm{d}^c\gamma$ with α, β, γ as above. As $\mathrm{d}\alpha = \mathrm{d}^c\alpha = 0$ (α is bounded and harmonic for ω hence bounded at any order by uniform ellipticity in quasi-coordinates, and one can thus integrate by parts without boundary terms by Lemma 1.4, with $M = X\backslash D$ and $g = \omega(\,\cdot\,, J\,\cdot\,)$), one gets $\mathrm{dd}^c(f^{\mathsf{Z}}_\omega - \beta) = \mathrm{dd}^c(h^{\mathsf{Z}}_\omega - \gamma) = 0$ on $X\backslash D$. Therefore $f^{\mathsf{Z}}_\omega - \beta$ and $h^{\mathsf{Z}}_\omega - \gamma$ are constant (use e.g. Yau's maximum principle [22, p. 406]) as wanted, and thus $\alpha = \xi^{\mathsf{Z}}_{\mathrm{harm}}$.

We are left with the mean assertion on f^{Z}_ω and h^{Z}_ω. For $t \in [0,1]$, set $\omega_t = \omega_X + t\mathrm{dd}^c\varphi$, $f_t = f^{\mathsf{Z}}_{\omega_X} + t(\mathsf{Z}\cdot\varphi)$, and consider the function $t \mapsto \int_{X\backslash D} f_t\,\omega_t^m$. Thanks to the growths near D, this function is smooth, with derivative $\int_{X\backslash D}(\mathsf{Z}\cdot\varphi)\,\omega_t^m + m\int_{X\backslash D} f_t\mathrm{dd}^c\varphi\wedge\omega_t^{m-1}$. Now,

$$m\int_{X\backslash D} f_t\mathrm{dd}^c\varphi\wedge\omega_t^{m-1} = -m\int_{X\backslash D}\mathrm{d}f_t\wedge\mathrm{d}^c\varphi\wedge\omega_t^{m-1} = -\int_{X\backslash D}\langle\mathrm{d}f_t, d\varphi\rangle_{\omega_t}\,\omega_t^m$$

(no boundary terms by Lemma 1.4). On the other hand, we now know that for all t, $\xi^{\mathsf{Z}}_{t\varphi} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + \mathrm{d}\big(f^{\mathsf{Z}}_{\omega_X} + t(\mathsf{Z}\cdot\varphi)\big) + \mathrm{d}^c\big(h^{\mathsf{Z}}_{\omega_X} - t(J\mathsf{Z})\cdot\varphi\big)$. Notice that $\int_{X\backslash D}\langle\xi^{\mathsf{Z}}_{\mathrm{harm}}, \mathrm{d}\varphi\rangle_{\omega_t}\,\omega_t^m = \int_{X\backslash D}\varphi(\delta_{\omega_t}\xi^{\mathsf{Z}}_{\mathrm{harm}})\,\omega_t^m = 0$ ($\xi^{\mathsf{Z}}_{\mathrm{harm}}$ is coclosed for ω_t, as $\delta_{\omega_t} = \Lambda_{\omega_t}\mathrm{d}^c$ on 1-forms), and $\int_{X\backslash D}\langle\mathrm{d}^c\big(h^{\mathsf{Z}}_{\omega_X} - t(J\mathsf{Z})\cdot\varphi\big), \mathrm{d}\varphi\rangle_{\omega_t}\,\omega_t^m = -m\int_{X\backslash D}\mathrm{d}\big(h^{\mathsf{Z}}_{\omega_X} - t(J\mathsf{Z})\cdot\varphi\big)\wedge\mathrm{d}\varphi\wedge\omega_t^{m-1} = 0$ (again by Lemma 1.4). This way $\int_{X\backslash D}\langle\mathrm{d}f_t, \mathrm{d}\varphi\rangle_{\omega_t}\,\omega_t^m = \int_{X\backslash D}\langle\xi^{\mathsf{Z}}_{t\varphi}, \mathrm{d}\varphi\rangle_{\omega_t}\,\omega_t^m = \int_{X\backslash D}(\mathsf{Z}\cdot\varphi)\,\omega_t^m$, hence: $\int_{X\backslash D} f_t\,\omega_t^m$ is constant, which gives (take $t = 0, 1$): $\int_{X\backslash D} f^{\mathsf{Z}}_\omega\,\omega^m = \int_{X\backslash D} f^{\mathsf{Z}}_{\omega_X}\,\omega_X^m = 0$. The mean of h^{Z}_ω against ω^m is seen to vanish likewise. □

1.2. The Poincaré type Futaki character

Definition. We can now generalise to Poincaré type Kähler metrics/classes, and holomorphic vector fields parallel to the divisor, a well-known invariant [11] of compact Kähler manifolds:

Definition 1.5. — *For* $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$ *and* $\omega \in \mathscr{M}^D_{[\omega_X]}$, *we call* Poincaré type Futaki character of Z with respect to D *the quantity*

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \int_{X\backslash D}\mathbf{s}(\omega) f^{\mathsf{Z}}_\omega\,\frac{\omega^m}{m!}\,. \tag{1.3}$$

Here, $\mathbf{s}(\omega)$ denotes the *(Riemannian) scalar curvature* of ω, that one can compute for instance via: $\mathbf{s}(\omega)\frac{\omega^m}{m!} = 2\varrho(\omega)\wedge\frac{\omega^{m-1}}{(m-1)!}$, with $\varrho(\omega)$ the Ricci form of ω.

Independence from the reference metric. As terminology and notation suggest, this Poincaré type Futaki character does not depend on ω of class $[\omega_X]$, *provided it is of Poincaré type*:

Proposition 1.6. — *Let $\tilde\omega$ be any Poincaré type metric in $\mathscr{M}^D_{[\omega_X]}$, and $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$. Then $\mathscr{F}^D_{[\omega_X]}(\mathsf{Z})=\int_{X\setminus D}\mathbf{s}(\tilde\omega)f^{\mathsf{Z}}_{\tilde\omega}\frac{\tilde\omega^m}{m!}$.*

Observe nonetheless that we take $\tilde\omega$ of Poincaré type in this proposition; the relation between the usual smooth Futaki character, and our Poincaré type Futaki character, is the purpose of next part. For now, let us address the proof of Proposition 1.6.

Proof of Proposition 1.6. — Take $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$. Fix $\omega=\omega_X+dd^c\varphi$ and $\tilde\omega=\omega_X+dd^c\tilde\varphi$ in $\mathscr{M}^D_{[\omega_X]}$, and for $t\in[0,1]$, set $\omega_t=(1-t)\omega+t\tilde\omega=\omega_X+dd^c\varphi_t$, $\varphi_t=(1-t)\varphi+t\tilde\varphi$; the ω_t are metrics of Poincaré type, uniformly bounded below by $c\omega$, say. As a consequence, the $\mathbf{s}(\omega_t)$ are uniformly bounded, at any order for ω, and for all $t_0\in[0,1]$, $\mathbf{s}(\omega_t)=\mathbf{s}(\omega_{t_0})+(t-t_0)\dot{\mathbf{s}}_{t_0}+(t-t_0)^2w_{t_0,t}$, with $\dot{\mathbf{s}}_{t_0}=-\frac12\Delta^2_{\omega_{t_0}}(\tilde\varphi-\varphi)-\langle\varrho(\omega_{t_0}),dd^c(\tilde\varphi-\varphi)\rangle_t$, and $w_{t_0,t}$ (uniformly) bounded at any order ; with the local formula for the linearisation $\dot{\mathbf{s}}_{t_0}$ of the scalar curvature [5, Lem. 2.158] at hand, it is an easy exercise to track the non-linear terms in $\mathbf{s}(\omega_t)-\mathbf{s}(\omega_{t_0})$ and establish the announced bounds on $w_{t_0,t}$. Uniform bounds at any order hold as well for the $f^{\mathsf{Z}}_{\omega_t}=f^{\mathsf{Z}}_{\omega_X}+\mathsf{Z}\cdot\varphi_t=f^{\mathsf{Z}}_{\omega}+t\mathsf{Z}\cdot(\tilde\varphi-\varphi)$; these growth conditions near D thus ensure us that

$$t\longmapsto\mathscr{F}_t:=\int_{X\setminus D}\mathbf{s}(\omega_t)f^{\mathsf{Z}}_{\omega_t}\frac{\omega_t^m}{m!}$$

is a smooth function of t, with derivative

$$t\longmapsto\dot{\mathscr{F}}_t=\int_{X\setminus D}\big(\dot{\mathbf{s}}_tf^{\mathsf{Z}}_{\omega_t}+\mathbf{s}(\omega_t)\big[\mathsf{Z}\cdot(\tilde\varphi-\varphi)\big]\big)\frac{\omega_t^m}{m!}+\int_{X\setminus D}\mathbf{s}(\omega_t)f^{\mathsf{Z}}_{\omega_t}\,dd^c(\tilde\varphi-\varphi)\wedge\frac{\omega_t^{m-1}}{(m-1)!},$$

just as in the compact case. And as in the compact case, integrations by parts can be performed without boundary terms, again thanks to the bounds at hand and Lemma 1.4 ; one thus ends with $\dot{\mathscr{F}}_t=0$ for all $t\in[0,1]$ (see e.g. [13, Prop. 4.12.1]), hence the result. □

Remark 1.7. — The word "character" for the map $\mathscr{F}^D_{[\omega_X]}:\mathfrak{h}^D_{/\!/}\to\mathbb{R}$ might appear slightly abusive, as long as we have not checked that $\mathscr{F}^D_{[\omega_X]}([\mathsf{Z}_1,\mathsf{Z}_2])=0$ for all $\mathsf{Z}_1,\mathsf{Z}_2\in\mathfrak{h}^D_{/\!/}$. As in the compact case, this identity however follows at once from the invariance of $\mathscr{F}^D_{[\omega_X]}$ along $\mathscr{M}^D_{[\omega_X]}$, and the stability of this class under automorphisms of X homotopic to id_X and parallel to D.

Remark 1.8. — It is clear from what precedes that if $\mathscr{M}^D_{[\omega_X]}$ admits a metric of constant scalar curvature, then $\mathscr{F}^D_{[\omega_X]}\equiv0$, and that conversely, if $\mathscr{F}^D_{[\omega_X]} \equiv 0$, any possible extremal metric in $\mathscr{M}^D_{[\omega_X]}$ has constant scalar curvature (the extremal vector field lying automatically in $\mathfrak{h}^D_{/\!/}$ in this case).

2. Link between smooth and Poincaré type Futaki characters

2.1. Statement

We now establish a formula relating the usual Futaki character to the Poincaré type Futaki character advertised in the Introduction. We keep the notations of the previous part; in particular, ω_X is a smooth Kähler metric on X compact, and $\mathscr{F}^D_{[\omega_X]} : \mathfrak{h}^D_{/\!/} \to \mathbb{R}$ denotes the Futaki character associated to the space $\mathscr{M}^D_{[\omega_X]}$ of Poincaré type Kähler metrics on $X\backslash D$ of class $[\omega_X]$.

Recall moreover that if $\mathsf{Z} \in \mathfrak{h}$, we set $f^{\mathsf{Z}}_{\omega_X}$ for the normalised potential of its (Riemannian) gradient part, relatively to ω_X. In order to establish the desired formula, we use, as intermediates, Futaki characters of Poincaré type *with respect to sub-divisors of* D, e.g. $\hat{D}_j := \sum_{\ell=1,\ell\neq j}^N D_\ell$ (recall $D = \sum_{\ell=1}^N D_\ell$); the Poincaré type Futaki character is denoted by $\mathscr{F}^{\hat{D}_j}_{[\omega_X]}$ in this case, and is still defined on $\mathfrak{h}^D_{/\!/} \subset \mathfrak{h}^{\hat{D}_j}_{/\!/}$. We denote by $\mathscr{F}_{[\omega_X]}$ the usual Futaki character on X.

Proposition 2.1. — *For all* $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$ *and for all* $j = 1, \ldots, N$, *one has:*

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{Z}) - 4\pi \int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}\,. \tag{2.1}$$

Consequently, for all $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$,

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \mathscr{F}_{[\omega_X]}(\mathsf{Z}) - 4\pi \sum_{j=1}^N \int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}\,. \tag{2.2}$$

2.2. Proof of Proposition 2.1

Identity (2.2) clearly follows from an inductive use of identity (2.1), the proof of which we focus on for the rest of this part.

Fix $Z \in \mathfrak{h}^D_{/\!/}$. To compute $\mathscr{F}^D_{[\omega_X]}(Z)$, we first fix a Poincaré type Kähler metric $\omega \in \mathscr{M}^D_{[\omega_X]}$ as follows. We take $\omega = \omega_X - dd^c \sum_{j=1}^N \log\big(-\log(|\sigma_j|_j^2)\big)$, with $\sigma_j \in \mathscr{O}([D_j])$ such that $D_j = \{\sigma_j = 0\}$, and the $|\cdot|_j$ are smooth hermitian metrics on the $[D_j]$, chosen so that ω is indeed a (Poincaré type) metric on $X\backslash D$ – see [4, §1.1.1] for details.

Fix now $j \in \{1,\dots,N\}$, set $\varphi_j = -\log\big(-\log(|\sigma_j|_j^2)\big)$, $\psi_j = -\sum_{\ell\neq j}\log\big(-\log(|\sigma_j|_j^2)\big)$, and define $\omega_t = \omega_X + dd^c(\psi_j + t\varphi_j)$ for $t\in[0,1]$. Notice that *these are metrics of Poincaré type on $X\backslash D$ for $t\in(0,1]$ only*, as $\omega_0 = \omega_{t=0} = \omega_X - dd^c\sum_{\ell\neq j}\log\big(-\log(|\sigma_j|_j^2)\big)$ *is of Poincaré type on* $X\backslash\hat{D}_j$ (assuming a good choice of the $|\cdot|_\ell$ for the positivity assertion). Now by Proposition 1.6,

$$\mathscr{F}^D_{[\omega_X]}(Z) = \int_{X\backslash D}\mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t}\,\frac{\omega_t^m}{m!} \tag{2.3}$$

for all $t\in(0,1]$. Observe however that the integrand tends uniformly to $\mathbf{s}(\omega_0)f^{\mathsf{Z}}_{\omega_0}\,\frac{\omega_0^m}{m!}$ away from D_j, as t goes to 0. Our strategy is hence to show that, for the price of the correction $-4\pi\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}$, the formal limit $\int_{X\backslash\hat{D}_j}\mathbf{s}(\omega_0)f^{\mathsf{Z}}_{\omega_0}\,\frac{\omega_0^m}{m!}$ is the limit of (2.3) as t goes to 0. In other words, we want to show that:

$$\lim_{t\searrow 0}\int_{X\backslash D}\mathbf{s}(\omega_t)f^{\mathsf{Z}}_{\omega_t}\,\frac{\omega_t^m}{m!} = \int_{X\backslash\hat{D}_j}\mathbf{s}(\omega_0)f^{\mathsf{Z}}_{\omega_0}\,\frac{\omega_0^m}{m!} - 4\pi\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}, \tag{2.4}$$

which provides (2.1), by the definitions of $\mathscr{F}^D_{[\omega_X]}$ and of $\mathscr{F}^{\hat{D}_j}_{[\omega_X]}$.

Set $E_j := \sum_{\ell\neq j}(D_\ell\cap D_j)$, or, more geometrically, $E_j = \hat{D}_j|_{D_j}$. Then E_j is a simple normal crossing divisor in D_j; admitting momentarily that

$$\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!} = \int_{D_j\backslash E_j} f^{\mathsf{Z}}_{\omega_0}\frac{(\omega_0|_{D_j\backslash E_j})^{m-1}}{(m-1)!}, \tag{2.5}$$

our aim is to prove (2.4) with $\int_{D_j\backslash E_j} f^{\mathsf{Z}}_{\omega_0}\frac{(\omega_0|_{D_j\backslash E_j})^{m-1}}{(m-1)!}$ instead of $\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}$. The key point is the following technical lemma:

Lemma 2.2. — *Let $f\in C^\infty\big(X\backslash\hat{D}_j\big)$, and $w\in C^\infty_1(X\backslash D_j)$. Then*

$$\lim_{t\searrow 0}\int_{X\backslash D}\mathbf{s}(\omega_t)(f+w)\,\frac{\omega_t^m}{m!} = \int_{X\backslash\hat{D}_j}\mathbf{s}(\omega_0)(f+w)\,\frac{\omega_0^m}{m!} - 4\pi\int_{D_j\backslash E_j} f\,\frac{(\omega_0|_{D_j\backslash E_j})^{m-1}}{(m-1)!}. \tag{2.6}$$

By "$f \in C^\infty\big(X\backslash\hat{D}_j\big)$", we mean: f is smooth on $X\backslash\hat{D}_j$, with derivatives bounded at any order with respect to any Poincaré type metric on $X\backslash\hat{D}_j$, e.g. ω_0; by "$w \in C_1^\infty(X\backslash D_j)$", we mean: w smooth on $X\backslash D_j$, with derivatives at any order $\mathcal{O}\big(\big|\log|\sigma_j|_j\big|^{-1}\big)$ with respect to any Poincaré type metric on $X\backslash D_j$, e.g. $\omega_X + dd^c\varphi_j$.

Lemma 2.2 is proven in next section. Let us see for now how it applies to our situation. One has: $f_{\omega_t}^{\mathsf{Z}} = f_{\omega_0}^{\mathsf{Z}} + t(\mathsf{Z}\cdot\varphi_j)$; we already know that $f_{\omega_0}^{\mathsf{Z}} \in C^\infty\big(X\backslash\hat{D}_j\big)$, and we check easily that $(\mathsf{Z}\cdot\varphi_j) \in C_1^\infty(X\backslash D_j)$ thanks to the assumption that Z be parallel to D_j and the fact that $\varphi_j + \log[-\log(|z^1|^2)] = \mathcal{O}\big(\big|\log(|z^1|^2)\big|^{-1}\big)$ at any order in Poincaré type metric on $X\backslash D_j$ on any open subset of coordinates where $D_j = \{z^1 = 0\}$; see the third paragraph after Definition 1.1. This way, by Lemma 2.2, $\int_{X\backslash D}\mathbf{s}(\omega_t)f_{\omega_0}^{\mathsf{Z}}\,\omega_t^m/m!$ tends to the difference $\int_{X\backslash D}\mathbf{s}(\omega_0)f_{\omega_0}^{\mathsf{Z}}\,\omega_0^m/m!$ $-4\pi\int_{D_j\backslash E_j} f_{\omega_0}^{\mathsf{Z}}\,(\omega_0|_{D_j\backslash E_j})^{m-1}/(m-1)!$, and $\int_{X\backslash D}\mathbf{s}(\omega_t)(\mathsf{Z}\cdot\varphi_j)\,\omega_t^m/m!$ tends to $\int_{X\backslash D}\mathbf{s}(\omega_0)(\mathsf{Z}\cdot\varphi_j)\,\omega_0^m/m!$ as t goes to 0 (all that matters here is actually this latter t-depending integral being $o(t^{-1})$). As a result,

$$\begin{aligned}\int_{X\backslash D}\mathbf{s}(\omega_t)f_{\omega_t}^{\mathsf{Z}}\,\frac{\omega_t^m}{m!} &= \int_{X\backslash D}\mathbf{s}(\omega_t)f_{\omega_0}^{\mathsf{Z}}\,\frac{\omega_t^m}{m!} + t\int_{X\backslash D}\mathbf{s}(\omega_t)(\mathsf{Z}\cdot\varphi_j)\,\frac{\omega_t^m}{m!}\\ &\xrightarrow{t\searrow 0}\int_{X\backslash\hat{D}_j}\mathbf{s}(\omega_0)f_{\omega_0}^{\mathsf{Z}}\,\frac{\omega_0^m}{m!} - 4\pi\int_{D_j\backslash E_j} f_{\omega_0}^{\mathsf{Z}}\,\frac{(\omega_0|_{D_j\backslash E_j})^{m-1}}{(m-1)!},\end{aligned}$$

as wanted.

Apart from the proof of Lemma 2.2, we are left with that of equality (2.5). We work on $D_j\backslash E_j$ (recall that $E_j = \hat{D}_j|_{D_j} = \sum_{\ell\neq j}(D_\ell\cap D_j)$), where we set $\varpi_s = (1-s)(\omega_X|_{D_j}) + s(\omega_0|_{D_j\backslash E_j})$. For $s>0$, these are Poincaré type metrics on $D_j\backslash E_j$. Just as in the proof of Proposition 1.2, growths near E_j allow us to say that

$$s \longmapsto \int_{D_j\backslash E_j}\big(f_{\omega_X}^{\mathsf{Z}} + s(\mathsf{Z}\cdot\psi_j)\big)\varpi_s^{m-1}$$

is smooth, with derivative

$$\int_{D_j\backslash E_j}(\mathsf{Z}\cdot\psi_j)\,\varpi_s^{m-1} + (m-1)\int_{D_j\backslash E_j}\big(f_{\omega_X}^{\mathsf{Z}} + s(\mathsf{Z}\cdot\psi_j)\big)dd^c\psi_j\wedge\varpi_s^{m-2}.$$

In order to conclude as in the proof of Proposition 1.2, since $(\mathsf{Z}\cdot\psi_j)|_{D_j\backslash E_j} = (\mathsf{Z}|_{D_j\backslash E_j})\cdot(\psi_j|_{D_j\backslash E_j})$ as Z is parallel to D_j, we check that the Hodge decomposition on $X\backslash D$ induces a Hodge decomposition on D_j, up to the mean of the Riemannian/symplectic gradient potentials. Namely, we check that

$$\xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} := (\mathsf{Z}|_{D_j})^{\sharp(g_X|_{D_j})} = \xi_{\mathrm{harm}}|_{D_j} + \mathrm{d}(f^{\mathsf{Z}}_{\omega_X}|_{D_j}) + \mathrm{d}^c(h^{\mathsf{Z}}_{\omega_X}|_{D_j}), \tag{2.7}$$

the extension to couples (Poincaré type metric ϖ on $X\backslash\hat{D}_j$, restriction of ϖ on $D_j\backslash E_j$) being dealt with as in Proposition 1.2. Now, as harmonic 1-forms are exactly $d-$ and d^c-closed 1-forms on compact Kähler manifolds, (2.7) is immediate from $\xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} = \xi^{\mathsf{Z}}_{\omega_X}|_{D_j}$, and this latter identity follows at once from the definition of $\mathsf{Z}|_{D_j}$. Indeed, in local holomorphic coordinates $(z^1,\dots,z^m)$ such that D_j is given by $z^1=0$, write $\mathsf{Z} = \mathsf{Z}^k\frac{\partial}{\partial z^k} + \overline{\mathsf{Z}^k}\frac{\partial}{\partial \overline{z^k}}$, and thus $\mathsf{Z}|_{D_j} = \mathsf{Z}^\alpha|_{D_j}\frac{\partial}{\partial z^\alpha} + \overline{\mathsf{Z}}^\alpha|_{D_j}\frac{\partial}{\partial \overline{z^\alpha}}$ (we implicitly sum on repeated Latin indices over $\{1,\dots,m\}$, and on Greek indices over $\{2,\dots,m\}$). The dual 1-forms are given by:

$$\begin{aligned}
\xi^{\mathsf{Z}}_{\omega_X} &= \overline{\mathsf{Z}^\ell}(g_X)_{k\bar\ell}\mathrm{d}z^k + \mathsf{Z}^\ell(g_X)_{\ell\bar k}\mathrm{d}\overline{z^k},\\
\xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} &= \overline{\mathsf{Z}^\beta}|_{D_j}(g_X|_{D_j})_{\alpha\bar\beta}\mathrm{d}z^\alpha + \mathsf{Z}^\beta|_{D_j}(g_X|_{D_j})_{\beta\bar\alpha}\mathrm{d}\overline{z^\alpha},
\end{aligned}$$

hence the result after restriction to D_j of $\xi^{\mathsf{Z}}_{\omega_X}$, as $\mathsf{Z}^1|_{D_j} \equiv \overline{\mathsf{Z}^1}|_{D_j} \equiv 0$.

2.3. Main technical argument: proof of Lemma 2.2

Localisation of the problem. Recall that $\omega_0 = \omega_X + \mathrm{dd}^c\psi_j$ is of Poincaré type on $X\backslash\hat{D}_j$, and that the $\omega_t = \omega_X + \mathrm{dd}^c(t\varphi_j+\psi_j)$, $t\in(0,1]$, are of Poincaré type on $X\backslash D$. Now for all $t\in[0,1]$, $\mathbf{s}(\omega_t)\omega_t^m = 2m\varrho(\omega_0)\wedge\omega_t^{m-1} - m\mathrm{dd}^c\log\big(\frac{\omega_t^m}{\omega_0^m}\big)\wedge\omega_t^{m-1}$. On the one hand, for f and w as in the statement of the Lemma, since $(f+w)\varrho(\omega_0)\wedge\omega_t^m$ is uniformly dominated by ω^m,

$$\begin{aligned}
2m\int_{X\backslash D}(f+w)\varrho(\omega_0)\wedge\omega_t^{m-1} \to 2m&\int_{X\backslash D}(f+w)\varrho(\omega_0)\wedge\omega_0^{m-1}\\
&= \int_{X\backslash D}\mathbf{s}(\omega_0)(f+w)\,\omega_0^m
\end{aligned}$$

as t tends to 0; one recognises the first term in the right-hand side of (2.6).

On the other hand, thanks to the uniform convergence of $\mathrm{dd}^c\log\big(\frac{\omega_t^m}{\omega_0^m}\big)\wedge\omega_t^{m-1}$ to 0 far from D_j (for ω_0, say), as t tends to 0, we can restrict to f and w with compact supports in a neighbourhood U of holomorphic coordinates $(z^1,\dots,z^m)$ centred at any point of D_j; we also assume that $|z_\ell|\leqslant e^{-1}$ on U for all ℓ, that $D_j\cap U=\{z^1=0\}$, and that the possible other components of D intersecting U are respectively given by $z^2=0,\dots,z^k=0$ for some appropriate $k\in\{2,\dots,m\}$.

For fixed $t>0$, we can write $\omega_t^m/\omega_0^m = v_t/[|z^1|^2\log^2(|z^1|^2)]$ on $U\backslash D$, with v_t positively bounded below, and bounded up to order 2, for $\omega=\omega_{t=1}$; these bounds are not uniform in t though, as $(\omega_t^m/\omega_0^m)\to 1$ far from D_j when $t\searrow 0$. We rather write $|\log(\omega_t^m/\omega_0^m)|\leqslant C+\log\big(1+t/[|z^1|^2\log^2(|z^1|^2)]\big)$ for a control uniform in t, with $C>0$ independent of t.

Both controls come from the expansion $\omega_t=\omega_0+t\frac{i\mathrm{d}z^1\wedge\mathrm{d}\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}+\varepsilon_t$, with $|\varepsilon_t|_\omega, |\nabla^\omega\varepsilon_t|_\omega, |(\nabla^\omega)^2\varepsilon_t|_\omega\leqslant Ct\big|\log|z^1|\big|^{-1}$, where $C>0$ is independent of t.

Integration by parts. Now as $\mathrm{dd}^c\log(|z^1|^2)=0$ in $U\backslash D_j$, for fixed $t>0$,

$$\begin{aligned}\int_{U\backslash D}(f+w)\mathrm{dd}^c\log\Big(\frac{\omega_t^m}{\omega_0^m}\Big)\wedge\omega_t^{m-1}\\ &=\int_{U\backslash D}(f+w)\mathrm{dd}^c\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\wedge\omega_t^{m-1}\\ &=\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\omega_t^{m-1}\end{aligned}$$

(the absence of boundary terms is justified by Lemma 1.4; notice that the integrand in the intermediate step of the above integration by parts is $\mathrm{d}(f+w)\wedge\mathrm{d}^c\log\big(\frac{v_t}{\log^2(|z^1|^2)}\big)\wedge\omega_t^{m-1}$).

Expand now ω_t^{m-1} as $\omega_0^{m-1}+(m-1)t\frac{i\mathrm{d}z^1\wedge\mathrm{d}\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}\wedge\omega_0^{m-2}+\tilde\varepsilon_t$, with $|\tilde\varepsilon_t|_\omega\leqslant Ct\big|\log|z^1|\big|^{-1}$; this way,

$$\begin{aligned}&\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\omega_t^{m-1}\\ &\quad=\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\omega_0^{m-1}\\ &\qquad+(m-1)\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\frac{t\,i\mathrm{d}z^1\wedge\mathrm{d}\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}\wedge\omega_0^{m-2}\\ &\qquad+\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\tilde\varepsilon_t\,.\end{aligned}\tag{2.8}$$

We deal with the three summands of the right-hand side separately, in the order 1-3-2; the aim is to show that when t goes to 0, the first summand provides the "$\int_{D_j}$-term" of (2.6), whereas the other two tend to 0.

First summand. — As $w|_{D_j} = 0$, (an easy adaptation[1] of) the classical Lelong formula [14, p. 387] yields: $\int_{U\backslash D}\log(|z^1|^2)\mathrm{dd}^c(f+w)\wedge\omega_0^{m-1} = 4\pi\int_{U\cap(D_j\backslash E_j)} f(\omega_0|_{D_j\backslash E_j})^{m-1}$. Consequently, for $t>0$, as $\omega_t^m/\omega_0^m = v_t/[|z^1|^2\log^2(|z^1|^2)]$,

$$\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\omega_0^{m-1} = \int_{U\backslash D}\log\Big(\frac{\omega_t}{\omega_0}\Big)\mathrm{dd}^c(f+w)\wedge\omega_0^{m-1} - 4\pi\int_{U\cap(D_j\backslash E_j)} f(\omega_0|_{D_j\backslash E_j})^{m-1}.$$

The uniform controls $|\log(\omega_t^m/\omega_0^m)| \leqslant C + \log\big(1+1/[|z^1|^2\log^2(|z^1|^2)]\big)$, $|(\mathrm{dd}^c f\wedge\omega_0^{m-1})/\omega_0^m| \leqslant C$, $\big|(\mathrm{dd}^c w\wedge\omega_0^{m-1})/\omega^m\big| \leqslant C\big|\log|z^1|\big|^{-1}$ now allow us[2] to argue by dominated convergence on the first summand of the right-hand side in the latter identity; since the integrand tends to 0 as $x\searrow 0$, we get:

$$\lim_{t\searrow 0}\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\omega_0^{m-1} = -4\pi\int_{U\cap(D_j\backslash E_j)} f(\omega_0|_{D_j\backslash E_j})^{m-1}.$$

Third summand of the right-hand side of (2.8). — Use the control on $\tilde\varepsilon_t$ to write:

$$\Big|\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\mathrm{dd}^c(f+w)\wedge\tilde\varepsilon_t\Big| \leqslant Ct\|\mathrm{dd}^c(f+w)\|_\omega\int_{U\backslash D}\Big|\log\Big(\frac{\omega_t^m}{\omega_0^m}\Big)+\log(|z^1|^2)\Big|\frac{\omega^m}{|\log(|z^1|^2)|};$$

the integral of the right-hand side is indeed finite (same argument as in the footnote above), and the left-hand side thus tends to 0 as $t\searrow 0$.

(1) As ω_0 and f are smooth on $D_j\backslash E_j$, one gets the result by considering the integrals $I_K := \int_{U\backslash D}\log(|z^1|^2)\mathrm{dd}^c[f+\chi(s-K)w]\wedge\omega_0^{m-1}$ (with $s=\log[-\log(|z^1|^2)]$, $K\geqslant 0$, and χ a smooth function with $\chi(\sigma)=1$ if $\sigma\leqslant 0$ and $\chi(\sigma)=0$ if $\sigma\geqslant 1$), which converge to $\int_{U\backslash D}\log(|z^1|^2)\mathrm{dd}^c[f+w]\wedge\omega_0^{m-1}$ when letting K go to ∞, thanks to the decay of w and its derivatives up to order 2, while $I_K = 4\pi\int_{U\cap(D_j\backslash E_j)} f(\omega_0|_{D_j\backslash E_j})^{m-1}$ for all K by Lelong formula.

(2) the worst term to deal with is $\int_{U\backslash D}\log\big(1+1/[|z^1|^2\log^2(|z^1|^2]\big)\big/\big|\log|z^1|\big|\,\omega^m$, which is finite, as $\log\big(1+1/[|z^1|^2\log^2(|z^1|^2]\big)\big/\big|\log|z^1|\big| = 1+o(1)$ for z^1 small

Second summand of the right-hand side of (2.8). — This might be the most delicate. We rewrite the integral in play as

$$\int_{0<|z^1|\leqslant 1/e} \frac{t\, i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \int_{V_{z^1}} \log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}$$

where the V_{z^1} are the slices $\{z^1 = \text{constant}\}$ of $U \backslash D$. On each such slice, (the restrictions of) $f+w$, $\mathrm{d}(f+w)$ and $\mathrm{dd}^c(f+w)$ are bounded, with respect to the restriction of ω_0, hence $\int_{V_{z^1}} \big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2} = 0$ for all $z^1 \neq 0$. Our integral can thus be rewritten as

$$\int_{0<|z^1|\leqslant 1/e} \frac{t\, i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \int_{V_{z^1}} \log(v_t)\big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2},$$

that is:

$$\int_{0<|z^1|\leqslant 1/e} i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z^1} \cdot \int_{V_{z^1}} \frac{t \log[|z^1|^2 \log^2(|z^1|^2) \cdot \omega_t^m/\omega_0^m]}{|z^1|^2 \log^2(|z^1|^2)} \big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}.$$

Now for all $z^1 \neq 0$, $t \in (0,1]$,

$$\Big|\int_{V_{z^1}} \frac{t \log[|z^1|^2 \log^2(|z^1|^2) \cdot \omega_t^m/\omega_0^m]}{|z^1|^2 \log^2(|z^1|^2)} \big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}\Big|$$
$$\leqslant C\big\|\big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}}\big\|_{\omega_0|_{V_{z^1}}} \operatorname{Vol}(V_{z^1}) \frac{t\big[1 + \big|\log[t + |z^1|^2 \log^2(|z^1|^2)]\big|\big]}{|z^1|^2 \log^2(|z^1|^2)},$$

where $\operatorname{Vol}(V_{z^1}) = \int_{V_{z^1}} \omega_0^{m-1}$. This volume, as well as the supremums $\big\|\big(\mathrm{dd}^c(f+w)\big)|_{V_{z^1}}\big\|_{\omega_0|_{V_{z^1}}}$, are bounded above independently of z^1 and of t (notice that we restrict to directions parallel to D_j, along which ω_0 and ω are comparable). Now,

$$\begin{aligned} I_t &:= \int_{\{0<|z^1|\leqslant 1/e\}} \frac{t\, i\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \Big[1 + \big|\log[t + |z^1|^2 \log^2(|z^1|^2)]\big|\Big] \\ &= t\int_{\{0<|z^1|\leqslant 1/e\}} \frac{i\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} + t|\log t| \int_{\{0<|z^1|\leqslant 1/e\}} \frac{i\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \\ &\quad - \int_{\{0<|z^1|\leqslant 1/e\}} \frac{t}{|z^1|^2 \log^2(|z^1|^2)} \log\Big(1 + \frac{|z^1|^2 \log^2(|z^1|^2)}{t}\Big) i\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1}. \end{aligned}$$

As $t \searrow 0$, the first two summands of the right-hand side clearly tend to 0; as for the third summand, the integrand is non-negative, and this only helps us in proving that I_t, and hence the whole second summand of (2.8), tend to 0 as $t \searrow 0$.

Summing up the above analysis of the three summands of the right-hand side of (2.8) yields:

$$\int_{U\backslash D}(f+w)\mathrm{dd}^c\log\left(\frac{\omega_t^m}{\omega_0^m}\right)\wedge\omega_t^{m-1} \xrightarrow{t\searrow 0} -4\pi\int_{U\cap(D_j\backslash E_j)} f\left(\omega_0|_{(D_j\backslash E_j)}\right)^{m-1},$$

and we saw that this is equivalent to Lemma 2.2 for our (localised) f and w. □

3. Application to extremal metrics of Poincaré type

3.1. Extension of Proposition 2.1 (smooth divisor)

Notice that the integral term in (2.1) does not depend on the smooth metric $\omega_X \in \mathscr{M}_{[\omega_X]}$, as neither $\mathscr{F}^D_{[\omega_X]}(\mathsf{Z})$ nor $\mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{Z})$ do. Considerations similar to those invoked when proving (2.5) tell us moreover that for the price of replacing D_j by $D_j\backslash E_j$, one can replace ω_X by *any* $\omega \in \mathscr{M}^{\hat{D}_j}_{[\omega_X]}$, $\omega|_{D_j\backslash E_j}$ being in that case an element of $\mathscr{M}^{E_j}_{[\omega_X]|_{D_j}}$.

One can go a step further, at least when the divisor is smooth, and take an $\omega \in \mathscr{M}^D_{[\omega_X]}$ which is asymptotically a product near D_j, i.e. for which there exist $a>0$, $\omega_j \in \mathscr{M}_{[\omega_X]|_{D_j}}$ and $\delta>0$ such that as soon as $D_j=\{z^1=0\}$ in local holomorphic coordinates $(z^1,\dots,z^m)$, then

$$\omega = \frac{a\,i\mathrm{d}z^1\wedge\mathrm{d}\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)} + p^*\omega_j + \mathcal{O}\big(\big|\log|z^1|\big|^{-\delta}\big),$$

where $p(z^1,\dots,z^m)=(z^2,\dots z^m)$, and with the $\mathcal{O}$ understood at any order for ω. This way $\omega|_{D_j}$ still makes sense as an element of $\mathscr{M}_{[\omega_X]|_{D_j}}$, as well as $f^{\mathsf{Z}}_\omega|_{D_j}$, and:

Proposition 3.1 (D smooth). — *Let $\omega \in \mathscr{M}^D_{[\omega_X]}$, and assume that ω is asymptotically a product near D_j, for $j\in\{1,\dots,N\}$. Then for all $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$, one has:*

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{Z}) - 4\pi\int_{D_j} f^{\mathsf{Z}}_\omega \frac{(\omega|_{D_j})^{m-1}}{(m-1)!}\,. \tag{3.1}$$

Proof. — Assume that ω is asymptotically a product as above; then $\omega = \omega_X + dd^c(\varphi + \tilde{\psi})$, with $\varphi = -a\log\big(-\log(|z^1|^2)\big)$, and in local holomorphic coordinates $(z^1,\dots,z^m)$ such that $D_j = \{z^1 = 0\}$, $\tilde{\psi} = p^*\psi + \mathcal{O}\big(\big|\log|z^1|\big|^{-\delta}\big)$, where the $\mathcal{O}$ is understood at any order for ω, and where $\psi \in C^\infty(D_j)$ is such that $\omega_{D_j}^{\psi} := \omega_X|_{D_j} + dd^c\psi \in \mathscr{M}_{[\omega_X]|_{D_j}}$, and $\omega|_{D_j} = \omega_{D_j}^{\psi}$.

Taking $\mathsf{Z} \in \mathfrak{h}_{/\!/}^{D}$, $\mathsf{Z}\cdot\varphi = \mathcal{O}\big(\big|\log|z^1|\big|^{-1}\big)$ in coordinates as above, so that $f_\omega^{\mathsf{Z}} = f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\big(\varphi + \tilde{\psi}\big)$ restricts to $f_\omega^{\mathsf{Z}}|_{D_j} + (\mathsf{Z}|_{D_j})\cdot\psi$ on D_j. Now we know from the treatment of equality (2.5) in the proof of Proposition 2.1 that $d\big(f_\omega^{\mathsf{Z}}|_{D_j} + (\mathsf{Z}|_{D_j})\cdot\psi\big)$ is the gradient part in the Hodge decomposition of the dual 1-form of $(\mathsf{Z}|_{D_j})$ for $\omega_{D_j}^{\psi}$. The analogue moreover holds when replacing ψ by $t\psi$ for $t \in [0,1]$; setting $\omega_{D_j}^{t} = \omega_X|_{D_j} + t\,dd^c\psi$ and $f_t = f_\omega^{\mathsf{Z}}|_{D_j} + t(\mathsf{Z}|_{D_j})\cdot\psi$, we thus see that the derivative of $\int_{D_j} f_t\,(\omega_{D_j}^{t})^{m-1}$ vanishes thanks to the usual integration by parts, hence the result, in view of (2.1). □

3.2. A numerical constraint on extremal metrics of Poincaré type

We apply what precedes to reformulate the numerical obstruction of [3, §4.2.2], which is a constraint on *extremal Poincaré type metrics* of class $[\omega_X]$; this is Theorem C of the Introduction, which we recall now:

Theorem 3.2. — *Assume that there exists an extremal metric of Poincaré type of class $[\omega_X]$ on $X\backslash D$, and let $\mathsf{K} \in \mathfrak{h}_{/\!/}^{D}$ be the Riemannian gradient of its scalar curvature. Then for all $j = 1,\dots,N$, if $E_j = \hat{D}_j|_{D_j} = \sum_{\ell=1,\ell\neq j}^{N}(D_\ell \cap D_j)$,*

$$\overline{\mathbf{s}}^D < \overline{\mathbf{s}}^{E_j} + \frac{1}{4\pi\,\mathrm{Vol}(D_j)}\big(\mathscr{F}_{[\omega_X]}^{D}(\mathsf{K}) - \mathscr{F}_{[\omega_X]}^{\hat{D}_j}(\mathsf{K})\big), \tag{3.2}$$

where $\overline{\mathbf{s}}^D$ (resp. $\overline{\mathbf{s}}_j$) denotes the mean scalar curvature attached to $\mathscr{M}_{[\omega_X]}^{D}$ (resp. to $\mathscr{M}_{[\omega_X]|_{D_j}}^{E_j}$).

Proof. — Assume for a start that D is smooth, i.e. has disjoint components, hence $E_j = 0$. Let $\omega \in \mathscr{M}_{[\omega_X]}^{D}$ be extremal, and let $\mathsf{K} = \nabla\mathbf{s}(\omega) \in \mathfrak{h}_{/\!/}^{D}$, where the gradient ∇ is computed with respect to the Riemannian metric $\omega(\,\cdot\,, J\,\cdot\,)$ associated to ω. According to [3, Thm. 4], ω is asymptotically a product near the divisor, and induces an extremal metric $\omega_j \in \mathscr{M}_{[\omega_X]|_{D_j}}$ for all $j=1,\dots,N$. We fix one of these j; as $f^{\mathsf{K}}_{\omega}=\mathbf{s}(\omega)-\bar{\mathbf{s}}^D$, Proposition 3.1 implies:

(3.3)

$$\begin{aligned}\mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) &= \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\int_{D_j}(\mathbf{s}(\omega)-\bar{\mathbf{s}}^D)\frac{\omega_j^{m-1}}{(m-1)!}\\ &= \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) + 4\pi\operatorname{Vol}(D_j)\bar{\mathbf{s}}^D - 4\pi\int_{D_j}\Big(\mathbf{s}(\omega_j)-\frac{2}{a_j}\Big)\frac{\omega_j^{m-1}}{(m-1)!}\\ &= \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) + 4\pi\operatorname{Vol}(D_j)\Big(\bar{\mathbf{s}}^D-\bar{\mathbf{s}}_j+\frac{2}{a_j}\Big),\end{aligned}$$

where $a_j\in(0,\infty)$ is such that: given a neighbourhood of holomorphic coordinates $(z^1,\dots,z^m)$ in X of any point of D_j such that D_j locally corresponds to $z^1=0$, then $\omega=a_j\frac{i dz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}+p^*\omega_j+\mathcal{O}\big(\frac{1}{|\log(|z^1|)|^{\delta}}\big)$ for some $\delta>0$, and with $p(z^1,\dots,z^m)=(z^2,\dots,z^m)$; in particular, the asymptotic product structure descends to the level of scalar curvature to give

$$\begin{aligned}\mathbf{s}(\omega) &= a_j^{-1}\mathbf{s}\left(\frac{i dz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}\right)+\mathbf{s}(p^*\omega_j)+\mathcal{O}\left(\frac{1}{|\log(|z^1|)|^{\delta}}\right)\\ &= -2a_j^{-1}+p^*\mathbf{s}(\omega_j)+\mathcal{O}\left(\frac{1}{|\log(|z^1|)|^{\delta}}\right)\end{aligned}$$

near D_j, and thus $\mathbf{s}(\omega)$ continuously extends to D_j as $-2a_j^{-1}+\mathbf{s}(\omega_j)$, as used in (3.3). As a_j is positive, one gets from the last line of (3.3):

$$\mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) > \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) + 4\pi\operatorname{Vol}(D_j)(\bar{\mathbf{s}}^D-\bar{\mathbf{s}}_j)\,,$$

of which (3.2) is simply a rewriting.

The simple normal crossing case. The asymptotically product behaviour of the extremal metric ω is not clear anymore when the divisor admits (simple normal) crossings; we thus content ourselves with applying Proposition 2.1, with $\mathsf{Z}=\mathsf{K}$, and ω_X smooth, and adapt our argument as follows. Let φ so that $\omega=\omega_X+dd^c\varphi$; then $f^{\mathsf{K}}_{\omega}=f^{\mathsf{K}}_{\omega_X}+\mathsf{K}\cdot\varphi$, that is, $f^{\mathsf{K}}_{\omega_X}=f^{\mathsf{K}}_{\omega}-\mathsf{K}\cdot\varphi=\mathbf{s}(\omega)-\bar{\mathbf{s}}^D-\mathsf{K}\cdot\varphi$. Remember that $f^{\mathsf{K}}_{\omega_X}$ is smooth on X, and set for the upcoming lines $\omega_{D_j}=\omega_X|_{D_j}$. To compute $f^{\mathsf{K}}_{\omega_X}|_{D_j}$, notice that by Remarks 4.4 and 4.7 in [3], one can find arbitrarily thin tubular shells around D_j in $X\backslash D$ such that: $\mathbf{s}(\omega)$ and $\mathsf{K}\cdot\varphi$ tend uniformly on compact subsets of these tubes, respectively to $\mathbf{s}(\omega_{D_j}+dd^c\psi)-2/a_j$ and $\mathsf{K}_{D_j}\cdot\psi$, and where: ψ is smooth on $D_j\backslash E_j$, such that $\omega^{\psi}_{D_j}:=\omega_{D_j}+dd^c\psi\in\mathscr{M}^{E_j}_{[\omega_{D_j}]}$, and $a_j>0$ is the inverse of the left-hand side of inequality (30) in [3, Prop. 4.5].

In other words, even if there is a priori some ambiguity on the Poincaré metric on $D_j \backslash E_j$ playing the role of the (extremal) metric induced by ω, for the price of picking one of them, $\omega^{\psi}_{D_j}$ here, and working on, say, a sequence $(K_\ell)_{\ell\geqslant 0}$ of compact subsets of tubular shells $S_\ell\backslash p_\ell^{-1}(E_j)$ of $D_j\backslash E_j$ (with some circle fibrations $p_\ell : S_\ell \to D_j$ such that S_ℓ lies an ϵ_ℓ-neighbourhood for ω_X of D_j, $\epsilon_\ell \xrightarrow{\ell\to\infty} 0$, and that $\big(p_\ell(K_\ell)\big)_{\ell\geqslant 0}$ is an exhaustive sequence for $D_j\backslash E_j$), an asymptotic splitting of the scalar curvature

$$\mathbf{s}(\omega) = -2a_j^{-1} + p_\ell^*\mathbf{s}(\omega^{\psi}_{D_j}) + \varepsilon_\ell, \qquad \sup_{K_\ell}|\varepsilon_\ell| \xrightarrow{\ell\to\infty} 0$$

on the K_ℓ's remains valid. Moreover a_j is independent of the underlying above choices, as it can be expressed via an integral quantity on $X\backslash D$ depending only on ω. Likewise, the quantity $\mathsf{K}\cdot\varphi$ can be said to converge to $\mathsf{K}_{D_j}\cdot\psi$ along $(K_\ell)_{\ell\geqslant 0}$.

As a consequence, $f^{\mathsf{K}}_{\omega_X} = \mathbf{s}(\omega) - \overline{\mathbf{s}}^D - \mathsf{K}\cdot\varphi$ restricts to $f^{\mathsf{K}}_{\omega_X}|_{D_j\backslash E_j} = \mathbf{s}(\omega^{\psi}_{D_j}) - \overline{\mathbf{s}}^D - 2/a_j - \mathsf{K}_{D_j}\cdot\psi$ on $D_j\backslash E_j$, and Proposition 2.1 yields

$$\begin{aligned}(3.4)\quad & \mathscr{F}^D_{[\omega_X]}(\mathsf{K}) \\ & = \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\int_{D_j\backslash E_j}\Big(\mathbf{s}(\omega^{\psi}_{D_j}) - \overline{\mathbf{s}}^D - \frac{2}{a_j} - \mathsf{K}_{D_j}\cdot\psi\Big)\frac{(\omega_{D_j})^{m-1}}{(m-1)!}.\end{aligned}$$

We can be more specific when analysing ω near D_j, and see that $\omega^{\psi}_{D_j}$ is extremal, with $\mathsf{K}_{D_j} = \nabla^{\psi}\mathbf{s}(\omega^{\psi}_{D_j})$ and ∇^{ψ} the Riemannian gradient with respect to $\omega^{\psi}_{D_j}$. In other words, $f^{\mathsf{K}_{D_j}}_{\omega^{\psi}_{D_j}} = \mathbf{s}(\omega^{\psi}_{D_j}) - \overline{\mathbf{s}}^{E_j}$, hence

$$f^{\mathsf{K}_{D_j}}_{\omega_{D_j}} = \mathbf{s}(\omega^{\psi}_{D_j}) - \overline{\mathbf{s}}^{E_j} - \mathsf{K}_{D_j}\cdot\psi. \tag{3.5}$$

As $\int_{D_j} f^{\mathsf{K}_{D_j}}_{\omega_{D_j}}\omega^{m-1}_{D_j} = 0$ by definition of the normalised holomorphic potential, using (3.5), we can rewrite equation (3.4) as:

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{K}) = \mathscr{F}^{\hat{D}_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\,\mathrm{Vol}(D_j)\Big(\overline{\mathbf{s}}^{E_j} - \overline{\mathbf{s}}^D - \frac{2}{a_j}\Big).$$

We conclude as in the smooth divisor case, using the positivity of a_j. ☐

Appendix A. Poincaré–Futaki character as a Donaldson–Futaki invariant

We detail here an algebraic interpretation of the Poincaré–Futaki character, in the same lines as what is done in [10] (compact smooth case), and in [15] (conical case).

A.1. Donaldson-Futaki invariants

We first recall the now-classical relation between the Futaki character and Donaldson–Futaki invariants, as observed in [10]. Assume ω_X comes from a polarisation (L, h_L), i.e. ω_X is the curvature of h_L; in particular, $[\omega_X] = 2\pi c_1(L)$. Assume moreover that $\mathbb{C}^*$ acts on (X, L). Restricting the action to the base X provides an infinitesimal action $\mathsf{Z} \in \mathfrak{h}$, in the sense that for all $x \in X$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(e^t \cdot x) = \mathsf{Z}_x \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(e^{is} \cdot x) = (J\mathsf{Z})_x \,.$$

In the total space of L, denote the infinitesimal action by $\hat{\mathsf{Z}}$; in terms of *complex* vector fields, one thus has

$$\hat{\mathsf{Z}}^{1,0} = \widetilde{\mathsf{Z}^{1,0}} + 2(f + ih)i\xi, \tag{A.6}$$

with $\widetilde{\mathsf{Z}^{1,0}}$ the horizontal lift of $\mathsf{Z}^{1,0}$, ξ the tautological field on L, and f, $h \in C^\infty(X)$, such that

$$\mathrm{d}f = \omega_X(\,\cdot\,, J\mathsf{Z}) \qquad \text{and} \qquad \mathrm{d}h = -\omega_X(\,\cdot\,, \mathsf{Z})\,.$$

In our notations, f thus equals $f^{\mathsf{Z}}_{\omega_X}$ up to normalisation, i.e. $f^{\mathsf{Z}}_{\omega_X} = f - \bar{f}$, if $\bar{f}$ denotes $\frac{1}{\mathrm{Vol}(X)} \int_X f \, \frac{\omega_X^m}{m!}$.

For p large, denote by $d_p(X, L)$ the dimension of $H^0(X, L^p)$; denote also by $w_p(X, L)$ the weight of the $\mathbb{C}^*$-action induced on this latter space. Then

$$\frac{w_p(X, L)}{p d_p(X, L)} = F_0 + F_1 p^{-1} + \mathcal{O}(p^{-2}) \qquad \text{as } p \text{ goes to } \infty\,, \tag{A.7}$$

with F_0 and F_1 in $\mathbb{Q}$.

Seeing (X, L) as the central fibre of the *product test-configuration* $(\mathcal{X}, \mathcal{L}) := (X \times \mathbb{C}, \pi_X^* L)$ (endowed with the product $\mathbb{C}^*$-action and where $\pi_X : X \times \mathbb{C} \to X$ is the obvious projection), F_1 becomes the *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$. Moreover, with our conventions,

$$F_1 = -\frac{2}{\mathrm{Vol}(X)} \mathscr{F}_{[\omega_X]}(\mathsf{Z})\,.$$

For further purpose, recall that this identity readily follows from the respective asymptotics of $d_p(X,L)$ and $w_p(X,L)$, that can be written as:

$$\begin{aligned} d_p(X,L) &= \frac{\mathrm{Vol}(X)}{(2\pi)^m}p^m + \frac{\overline{\mathbf{s}}\,\mathrm{Vol}(X)}{4\cdot(2\pi)^m}p^{m-1} + \mathcal{O}(p^{m-2}) \\ &=: \gamma p^m + \delta p^{m-1} + \mathcal{O}(p^{m-2}) \end{aligned}$$

and

$$\begin{aligned} w_p(X,L) &= -\frac{8}{(2\pi)^m}\int_X f\frac{\omega_X^m}{m!}p^{m+1} - \frac{2}{(2\pi)^m}\int_X f\mathbf{s}(\omega_X)\frac{\omega_X^m}{m!}p^m + \mathcal{O}(p^{m-1}) \\ &=: \alpha p^{m+1} + \beta p^m + \mathcal{O}(p^{m-1}). \end{aligned}$$

A.2. The case of a triple (X,D,L)

Assume moreover that the $\mathbb{C}^*$-action on (X,L) globally preserves a divisor $D\subset X$, which, for simplicity, we suppose smooth. In any case, this implies $\mathsf{Z}\in\mathfrak{h}_{/\!/}^D$. Seeing the triple (X,D,L) as the central fibre of the product test-configuration(3) $(\mathcal{X},\mathcal{D},\mathcal{L}) := (X\times\mathbb{C}, D\times\mathbb{C}, \pi_X^*L)$, and following G. Székelyhidi's suggestion [18, §3.2], we now let $\tilde{d}_p(X,D,L)$ be the dimension of $H^0\big(X, L^p\otimes\mathscr{O}(-D)\big)$, and $\tilde{w}_p(X,D,L)$ be the weight of the induced $\mathbb{C}^*$-action on this space. We further set

$$\hat{d}_p(X,D,L) = \frac{d_p(X,L)+\tilde{d}_p(X,D,L)}{2}$$

and

$$\hat{w}_p(L,D) = \frac{w_p(X,L)+\tilde{w}_p(X,D,L)}{2}.$$

Then the quotients $\frac{\hat{w}_p(X,D,L)}{p\hat{d}_p(X,D,L)}$ admit an asymptotic with shape

$$\frac{\hat{w}_p(L,D)}{p\hat{d}_p(X,D,L)} = \hat{F}_0 + \hat{F}_1p^{-1} + \mathcal{O}(p^{-2}),$$

and $\hat{F}_1$ is called the *Donaldson–Futaki* invariant of the test-configuration $(\mathcal{X},\mathcal{D},\mathcal{L})$. The link with our Poincaré–Futaki character is given by:

Proposition A.3. — *Recall that the infinitesimal base action Z is in $\mathfrak{h}_{/\!/}^D$. With our conventions,*

$$\hat{F}_1 = -\frac{2}{\mathrm{Vol}(X)}\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}).$$

(3) This is all we need to introduce about test-configurations, and we refer the reader to [10] and [18] for the general definition.

Proof. — The proof is based on the computation of the asymptotics of $\hat{d}_p(X,D,L)$ and $\hat{w}_p(X,D,L)$, hence of those of $\tilde{d}_p(X,D,L)$ and $\tilde{w}_p(X,D,L)$. Now, for $p \gg 1$, one has short exact sequences

$$0 \longrightarrow H^0\big(X, L^p \otimes \mathscr{O}(-D)\big) \longrightarrow H^0(X, L^p) \longrightarrow H^0\big(D, (L|_D)^p\big) \longrightarrow 0,$$

from which one deduces

$$\tilde{d}_p(X,D,L) = d_p(X,L) - d_p(D, L|_D)$$

and

$$\tilde{w}_p(X,D,L) = w_p(X,L) - w_p(D, L|_D).$$

As

$$d_p(D, L|_D) = \frac{\mathrm{Vol}(D)}{(2\pi)^{m-1}} p^{m-1} + \mathcal{O}(p^{m-2})$$

and

$$w_p(D, L|_D) = -\frac{8}{(2\pi)^{m-1}} \int_D f \frac{(\omega_X|_D)^{m-1}}{(m-1)!} p^m + \mathcal{O}(p^{m-1}) \text{ (restrict (A.6) to } D),$$

one gets, after simplifications, $\hat{d}_p(X,D,L) = \hat{\gamma} p^m + \hat{\delta}_{X,L} p^{m-1} + \mathcal{O}(p^{m-2})$, and $\hat{w}_p(X,D,L) = \hat{\alpha} p^{m+1} + \hat{\beta} p^m + \mathcal{O}(p^{m-1})$, with

$$\hat{\alpha} = \alpha, \qquad \hat{\beta} = \beta - \frac{1}{2}\Big(- \frac{8}{(2\pi)^{m-1}} \int_D f \frac{(\omega_X|_D)^{m-1}}{(m-1)!} \Big) =: \beta - \frac{1}{2}\alpha_D,$$

$$\hat{\gamma} = \gamma, \qquad \hat{\delta} = \delta - \frac{1}{2} \frac{\mathrm{Vol}(D)}{(2\pi)^{m-1}} =: \delta - \frac{1}{2}\gamma_D.$$

Recall that $f^{\mathsf{Z}}_{\omega_X} = f - \bar{f}$, with $\bar{f} = \frac{1}{\mathrm{Vol}(X)} \int_X f \frac{\omega_X^m}{m!}$; therefore:

$$\begin{aligned}
\hat{F}_1 = \frac{\hat{\beta}\hat{\gamma} - \hat{\alpha}\hat{\delta}}{(\hat{\gamma})^2} &= F_1 + \frac{1}{2\gamma}\Big(\frac{\alpha}{\gamma}\gamma_D - \alpha_D\Big) \\
&= -\frac{2}{\mathrm{Vol}(X)} \mathscr{F}_{[\omega_X]}(\mathsf{Z}) + \frac{8\pi}{\mathrm{Vol}(X)} \int_D (f - \bar{f}) \frac{(\omega_X|_D)^{m-1}}{(m-1)!} \\
&= -\frac{2}{\mathrm{Vol}(X)} \Big(\mathscr{F}_{[\omega_X]}(\mathsf{Z}) - 4\pi \int_D f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_D)^{m-1}}{(m-1)!} \Big),
\end{aligned}$$

and thus $\hat{F}_1 = -\frac{2}{\mathrm{Vol}(X)} \mathscr{F}^D_{[\omega_X]}(\mathsf{Z})$, according to Proposition 2.1. □

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