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TITS ENDOMORPHISMS AND BUILDINGS OF TYPE F_4

by Tom DE MEDTS, Yoav SEGEV & Richard M. WEISS

Abstract. — The fixed point building of a polarity of a Moufang quadrangle of type F_4 is a Moufang set, as is the fixed point building of a semi-linear automorphism of order 2 of a Moufang octagon that stabilizes at least two panels of one type but none of the other. We show that these two classes of Moufang sets are, in fact, the same, that each member of this class can be constructed as the fixed point building of a group of order 4 acting on a building of type F_4 and that the group generated by all the root groups of any one of these Moufang sets is simple.

Résumé. — L'immeuble de points fixes d'une polarité d'un quadrangle de Moufang de type F_4 est un ensemble de Moufang. Il en va de même pour l'immeuble de points fixes d'un automorphisme semi-linéaire d'ordre 2 d'un octogone de Moufang qui stabilise au moins deux cloisons d'un type mais aucun de l'autre. Nous montrons que ces deux classes d'ensembles de Moufang sont en fait identiques, que chaque membre de cette classe peut être construit comme l'immeuble de points fixes d'un groupe d'ordre 4 agissant sur un immeuble de type F_4, et que pour chacun de ces ensembles de Moufang, le groupe engendré par tous les sous-groupes radiciels est un groupe simple.

1. Introduction

The notion of a building was introduced by J. Tits in order to give a uniform geometric/combinatorial description of the groups of rational points of an isotropic absolutely simple group. The buildings that arise in this context are spherical. In [19], Tits classified irreducible spherical buildings of rank at least 3 and this classification was extended to the rank 2 case in [23] under the assumption that the building is Moufang (which is automatic when the rank is at least 3).

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The result of this classification is that most Moufang buildings (as defined in Definition 2.5) are the spherical buildings associated with isotropic absolutely simple algebraic groups. The exceptions are buildings determined by algebraic data involving infinite dimensional structures, defective quadratic or pseudo-quadratic forms, inseparable field extension and/or the square root of a Frobenius endomorphism. Among these exceptions are the mixed buildings of type B_2, G_2 and F_4, the Moufang quadrangles of type F_4 and the Moufang octagons.

The classification results in [19] and [23] are proved by coordinatizing with an appropriate algebraic structure. These methods do not reveal the connection between the automorphism group of a Moufang building and an associated split group which is the central concern in the theory of Galois descent. In [11], this shortcoming is remedied with a theory of descent for buildings. This theory provides, in particular, a combinatorial interpretation of the Tits indices which appear in [18].

In Definition 2.10, we define the notion of a *descent group* of a building Δ and in Definition 3.9, we define the notion of a *Galois subgroup* of $\mathrm{Aut}(\Delta)$ in the case that Δ is Moufang. One of the fundamental results in the theory of Galois descent in buildings in [11] says that the set of residues fixed by a descent group Γ has, in a canonical way, the structure of a building, which we call the *fixed point building* of Γ (see Theorem 2.18). This result applies to arbitrary descent groups acting on arbitrary buildings. A second fundamental result (proved in [15]) says that if Δ is Moufang and Γ is a Galois subgroup of $\mathrm{Aut}(\Delta)$ acting with finite orbits on Δ and stabilizing at least one proper residue of Δ, then Γ is a descent group.

A third fundamental result says that if the fixed point building of a descent group of a Moufang building Δ has rank 1, then the fixed point building inherits from the Moufang condition on Δ the structure of a *Moufang set*. Moufang sets, which were first introduced in [21, 4.4], are a class of 2-transitive permutation groups. The notion of a Moufang set is closely related to the notion of a split BN-pair of rank 1. All known Moufang sets which are proper (i.e. not sharply 2-transitive) arise as fixed point buildings of Moufang buildings. For a survey of recent results in the study of Moufang sets, see [4].

Polarities (i.e. non-type-preserving automorphisms of order 2) of Moufang buildings of type B_2, G_2 and F_4 are a second source of descent groups. In the case that the building is *pseudo-split* (as defined in [11, 28.16]), polarities give rise to the Suzuki and Ree groups. In [20], Tits characterized

Moufang octagons as the fixed point building of an arbitrary polarity of a building of type F_4.

The Moufang quadrangles of type F_4 (a class of non-pseudo-split Moufang buildings of type B_2) were discovered in the course of classifying Moufang buildings of rank 2 in [23]. Subsequently, it was shown (in [12]) that these quadrangles can be constructed as the fixed point building of a type-preserving Galois involution acting on a building of type F_4. (It was due to this result that the designation "of type F_4" was chosen in [23].)

Among all *non*-pseudo-split Moufang buildings of type B_2, G_2 or F_4, the Moufang quadrangles of type F_4 are the only ones that can have a polarity. If a Moufang quadrangle Ξ of type F_4 has a polarity ρ, then the centralizer of ρ in $\mathrm{Aut}(\Xi)$ induces a Moufang set on the set $\Xi^{\langle\rho\rangle}$ of chambers fixed by ρ. The first examples of such Moufang sets were constructed in [13]. It is also possible for a Moufang octagon Ω to have a type-preserving Galois involution κ such that the centralizer of κ in $\mathrm{Aut}(\Omega)$ induces a Moufang set on the set $\Omega^{\langle\kappa\rangle}$ of panels fixed by κ. In [13], it was conjectured that these two classes of Moufang sets are the same. Our main goal here is to prove this conjecture.

In the course of verifying this conjecture, we show that each of these Moufang sets is, in fact, the fixed point building of an elementary abelian subgroup Γ of order 4 of the automorphism group of a building Δ of type F_4. The group Γ is not type-preserving. For this reason, we say (in Definition 4.5) that these Moufang sets are *of outer F_4-type*. The fixed point building of each of the two polarities in Γ is a Moufang octagon (but the two octagons are not, in general, isomorphic to each other) and the fixed point building of the third involution in Γ is a Moufang quadrangle of type F_4. This lattice of fixed point buildings is all the more interesting in light of the fact that there is no obvious connection between an arbitrary Moufang quadrangle of type F_4 and an arbitrary Moufang octagon except that they both "descend from" a building of type F_4 in characteristic 2.

The root groups of a Moufang set of outer F_4-type are nilpotent of class 3 as are the root groups in the Ree groups of type G_2. In every other known proper Moufang set, the root groups are either abelian or nilpotent of class 2.

In Sections 15–19, we show that the group generated by all the root groups of a Moufang set of outer F_4-type is simple and describe a number of other properties of these Moufang sets.

The basic invariant of a Moufang quadrangle Ξ of type F_4 is a pair of quadratic forms q and $\hat{q}$ of type F_4 (as defined in Definition 5.3 below), one defined over a field K of characteristic 2 and the other over a field F such that $F^2 \subset K \subset F$. The quadratic forms q and $\hat{q}$ are anisotropic but have a defect of dimension $\dim_{K^2} F$, respectively, $\dim_F K$. (We do not make any assumptions about these dimensions; in particular, either one or both can be infinite.)

Let V denote the vector space over K on which q is defined. We show that if the Moufang quadrangle Ξ has a polarity ρ, then the quadratic forms q and $\hat{q}$ are similar to each other and there is a *Tits endomorphism* θ of K (i.e. an endomorphism whose square is the Frobenius endomorphism) with image F and a non-associative algebra structure on V with respect to which the quadratic form q satisfies the identity

$$q(uv) = q(u)q(v)^\theta \tag{1.1}$$

for all $u, v \in V$ (see Propositions 6.23 and 7.9). Thus q is multiplicative "with a twist" (cf. [7], for instance).

We call the non-associative algebras that arise in this way *polarity algebras*. In Section 7, we describe polarity algebras in terms of a system of axioms and deduce from these axioms equation (1.1) along with a number of other intriguing identities. In Theorem 19.1, we use some of these identities to show that q is, up to similarity, an invariant not only of the quadrangle Ξ, but also of the Moufang set $\Xi^{\langle\rho\rangle}$. See also [17].

Tits endomorphisms and their extensions play a central role in the study of polarities. The first thorough study of this connection is in [22]; in fact, it was the influence of this paper which led to the common attribution of these endomorphisms to Tits. See, in particular, Section 8.

Convention 1.2. — *If x and y are two elements of a group, we set $x^y = y^{-1}xy$ and*

$$[x, y] = x^{-1}y^{-1}xy = (y^{-1})^x y = x^{-1}x^y$$

(as in [23]*). As a consequence of these conventions, the following two identities*

$$[xy, z] = [x, z]^y \cdot [y, z] \quad \text{and} \quad [x, yz] = [x, z] \cdot [x, y]^z \tag{1.3}$$

hold.

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2. Fixed Point Buildings and Moufang Sets

Before we can give precise formulations of our main results in Section 4, we need to introduce some basic notions.

Definition 2.1. — *Let E be a field of positive characteristic p. We will denote the Frobenius endomorphism $x \mapsto x^p$ of E by* Frob_E. *A* Tits endomorphism *of E is an endomorphism of E whose square is the Frobenius endomorphism. An* octagonal set *is a pair (E, θ), where E is a field of characteristic 2 and θ is a Tits endomorphism of E.*

Definition 2.2. — *Let Δ be a building. An* involution *of Δ is an automorphism of order 2. A* polarity *of Δ is an involution which is not type-preserving. (We will only use this term when Δ is of type B_2, F_4 or G_2.)*

Notation 2.3. — Let (E, θ) be an octagonal set. We denote by $\mathcal{O}(E, \theta)$ the Moufang octagon defined in [23, 16.9] and we denote by $F_4(E, \theta)$ the building of type F_4 called $F_4(E, E^\theta)$ in [26, 30.15].

Definition 2.4. — *A* Moufang set *is a pair $(X, \{U_x\}_{x \in X})$, where X is a set of cardinality at least 3, U_x is a subgroup of* $\mathrm{Sym}(X)$ *fixing x and acting sharply transitively on $X \backslash \{x\}$ for all $x \in X$ and $g^{-1}U_x g = U_{(x)g}$ for all $x \in X$ and all $g \in G^\dagger := \langle U_x \mid x \in X \rangle$. The subgroups U_x are called the* root groups *of the Moufang set. A Moufang set is* proper *if the group $G^\dagger$ does not act sharply 2-transitively on X.*

Definition 2.5. — *As in* [24, 11.2]*, we call a building* Moufang *if it is spherical, irreducible and of rank at least 2 and for each root α, the root group U_α (as defined in* [24, 11.1]*) acts transitively on the set of apartments containing α. (There are more general notions of a Moufang building, but they are not relevant in this paper.)*

Proposition 2.6. — *Let κ be an involution acting on a spherical building Δ. Then there exists an apartment of Δ stabilized by κ.*

Proof. — This holds by [11, 25.15]. □

Definition 2.7. — *Let Δ be a building and let Γ be a subgroup of* $\mathrm{Aut}(\Delta)$. *A* Γ-residue *is a residue of Δ stabilized by Γ. A* Γ-chamber *is a Γ-residue which is minimal with respect to inclusion. A* Γ-panel *is a Γ-residue P such that for some Γ-chamber C, P is minimal in the set of all Γ-residues containing C.*

Definition 2.8. — *Let Δ and Γ be as in Definition 2.7. A residue of Δ is* proper *if it is different from Δ itself. (In particular, chambers are proper* *residues.) The group* Γ *is* anisotropic *if* Γ *stabilizes no proper residues of* Δ*, and* Γ *is* isotropic *if this is not the case. Thus* Γ *is isotropic if and only if there exist* Γ*-panels. An automorphism* ξ *of* Δ *is isotropic (or anisotropic) if* $\langle\xi\rangle$ *is isotropic (or anisotropic).*

Notation 2.9. — Let Δ be a building and let Γ be an isotropic subgroup of $\mathrm{Aut}(\Delta)$. We denote by Δ^Γ the graph with vertex set the set of all Γ-chambers, where two Γ-chambers are joined by an edge of Δ^Γ if and only if there is a Γ-panel containing them both.

Definition 2.10. — *Let* Δ *be a building. A* descent group *of* Δ *is an isotropic subgroup* Γ *of* $\mathrm{Aut}(\Delta)$ *such that each* Γ*-panel contains at least three* Γ*-chambers.*

Proposition 2.11. — *Suppose that a building* Δ *is Moufang as defined in Definition 2.5. Let* R *be a residue of* Δ*, let* Σ *be an apartment containing chambers of* R *and let* U_R *denote the subgroup generated by the root groups* U_α *for all roots* α *of* Σ *containing* $R\cap\Sigma$*. Then* U_R *is independent of the choice of* Σ*.*

Proof. — This holds by [11, 24.17] □

Definition 2.12. — *Let* R *and* Δ *be as in Proposition 2.11. The group* U_R *is called the* unipotent radical *of* R*.*

Proposition 2.13. — *Let* R*,* Δ *and* U_R *be as in Definition 2.12. Then* U_R *acts sharply transitively on the residues of* Δ *opposite* R

Proof. — This holds by [11, 24.21]. □

Definition 2.14. — *Let* Π *be a Coxeter diagram and let* (W,S) *be the corresponding Coxeter system. Thus* S *is both a distinguished set of generators of the Coxeter group* W *and the vertex set of* Π*. Let* J *be a subset of* S *such that the subdiagram* Π_J *spanned by* J *is spherical and let* w_J *denote the longest element of the Coxeter system* (W_J,J)*, where* W_J *denotes the subgroup of* W *generated by* J*. By* [24, 5.11]*, the map* $s\mapsto w_Jsw_J$ *is an automorphism of* Π*. We denote this automorphism by* op_J*. This map is called the* opposite map *of the diagram* Π_J *(and ought, in fact, to be denoted by* op_{Π_J}*). This map stabilizes every connected component of* Π_J *and acts non-trivially on a given connected component if and only if it is isomorphic to the Coxeter diagram* A_n *for some* $n\geqslant 2$*, to* D_n *for some odd* $n\geqslant 5$*, to* E_6 *or to* $I_2(n)$ *for some odd* $n\geqslant 5$*. In particular, its order is at most* 2*.*

Definition 2.15. — *A* Tits index *is a triple* (Π, Θ, A) *where* Π *is a Coxeter diagram,* Θ *is a subgroup of* $\mathrm{Aut}(\Pi)$ *and* A *is a* Θ*-invariant subset of the vertex set* S *of* Π *such that for each* $s \in S\backslash A$*, the subdiagram of* Π *spanned by* $A \cup \Theta(s)$ *is spherical and* A *is invariant under the opposite map* $\mathrm{op}_{A\cup\Theta(s)}$ *defined in Definition 2.14. Here* $\Theta(s)$ *denotes the* Θ*-orbit containing* s*.*

Definition 2.16. — *Let* $T = (\Pi, \Theta, A)$ *be a Tits index. For each* $s \in S\backslash A$*, let* $\tilde{s} = w_A w_{A\cup\Theta(s)}$*, where* w_J *for* $J = A$ *and* $J = A \cup \Theta(s)$ *is as in Definition 2.14. Thus there is one element* $\tilde{s}$ *for each* Θ*-orbit in* $S\backslash A$*. Let* $\tilde{S}$ *be the set of all these elements* $\tilde{s}$*. By* [11, 20.32], $(\tilde{W}, \tilde{S})$ *is a Coxeter system. Let* $\tilde{\Pi}$ *be the corresponding Coxeter diagram. We call* Π *the* absolute Coxeter diagram *of* T *and* $\tilde{\Pi}$ *the* relative Coxeter diagram *of* T*. An algorithm for calculating the relative Coxeter diagram of a Tits index is described in* [23, 42.3.5 (c)].

Example 2.17. — Let $T = (\Pi, \Theta, A)$, where Π is the Coxeter diagram F_4. If $\Theta = \mathrm{Aut}(\Pi)$ and $A = \emptyset$, then T is a Tits index with relative Coxeter diagram is $I_2(8)$. If Θ is trivial and $A = \{2, 3\}$ with respect to the standard numbering of the vertex set of Π, then T is a Tits index with relative Coxeter diagram B_2.

Theorem 2.18. — *Let* Γ *be a descent group of a building* Δ*. Let* Π *be the Coxeter diagram of* Δ*, let* S *denote the vertex set of* Π *and let* Θ *denote the subgroup of* $\mathrm{Aut}(\Pi)$ *induced by* Γ*. Then the following hold:*

(1) *The graph* Δ^Γ *defined in Notation 2.9 is a building with respect to a canonical coloring of its edges.*
(2) *All* Γ*-chambers are residues of* Δ *of the same type* $A \subset S$ *and the rank of* Δ^Γ *is the number of* Θ*-orbits in* S *disjoint from* A*.*
(3) *If* A *is spherical, then the triple* $T := (\Pi, \Theta, A)$ *is a Tits index and* Δ^Γ *is a building of type* $\tilde{\Pi}$*, where* $\tilde{\Pi}$ *is the relative Coxeter diagram of* T*.*
(4) *If* Δ *is Moufang and the rank of* Δ^Γ *is at least* 2*, then* Δ^Γ *is also Moufang.*
(5) *Suppose that* Δ *is Moufang and that the rank of* Δ^Γ *is* 1 *and let* X *be the set of all* Γ*-chambers, For each* $C \in X$*, let* $\tilde{U}_C$ *denote the subgroup of* $\mathrm{Sym}(X)$ *induced by the centralizer* $C_{U_C}(\Gamma)$ *of* Γ *in the unipotent radical* U_C*. Then*

$$(X, \{\tilde{U}_C \mid C \in X\})$$

is a Moufang set.

Proof. — Assertions (1) and (2) hold by [11, 22.20 (v) and (viii)], assertion (3) holds by [11, 22.20 (iv) and (viii)] and the remaining two assertions hold by [11, 24.31]. □

Definition 2.19. — *The building* Δ^Γ *in 2.18*(1) *is called the* fixed point building *of* Γ. *The rank of* Δ^Γ *is called the* relative rank *of* Γ. *If the relative rank of* Γ *is* 1*, we interpret* Δ^Γ *to mean the Moufang set described in 2.18*(5).

3. Polarities and Galois subgroups

In this section we describe two ways in which descent groups arise.

Proposition 3.1. — *Let* Δ *be a Moufang building of type* Π*, where* Π *is the Coxeter diagram* B_2*,* G_2 *or* F_4*. Suppose that* σ *is a polarity of* Δ *as defined in Definition 2.2 and let* $\Gamma = \langle\sigma\rangle$*. Then* Γ *is a descent group of* Δ *and* Γ*-chambers are chambers of* Δ*. If* $\Pi = B_2$ *or* G_2*, the fixed point building* Δ^Γ *is a Moufang set and if* $\Pi = F_4$*, the fixed point building* Δ^Γ *is a Moufang octagon.*

Proof. — By Proposition 2.6, we can choose an apartment Σ stabilized by σ. By Definition 2.14, the automorphism of Σ sending each chamber to its unique opposite is color-preserving. By [11, 25.17], therefore, there exists a Γ-residue R containing chambers of Σ. We can assume that R is minimal with respect to containment.

Since Γ is not type-preserving, the type J of R is Θ-invariant, where Θ is the automorphism group of Π. Suppose that $J \neq \emptyset$. Since J is Θ-invariant, we must have $\Pi = F_4$ and J is either $\{1,4\}$ or $\{2,3\}$ (with respect to the standard numbering of the vertex set of the Coxeter diagram F_4). Thus $R \cap \Sigma$ is a thin building of type $A_1 \times A_1$ or B_2. By Definition 2.14, the map which sends each chamber of $R \cap \Sigma$ to its unique opposite is again color-preserving. Another application of [11, 25.17] thus implies that σ stabilizes a proper residue of R. This contradicts the choice of R. With this contradiction, we conclude that there are chambers of Σ fixed by σ.

Let c be a chamber of Σ fixed by σ and let d be the unique chamber of Σ opposite c. Since σ stabilizes Σ and c, it fixes d as well. Suppose that $\Pi = B_2$ or G_2. Among the roots of Σ containing c, there are two such that c is at maximal distance from the root that is opposite in Σ. We call these two roots α and β. They are interchanged by σ and by [23, 5.5 and 5.6], we have $U_\alpha \cap U_\beta = 1$ and $[U_\alpha, U_\beta] = 1$. Suppose, instead, that $\Pi = F_4$. Let

c_1 denote the unique chamber of Σ that is 1-adjacent to c, let α denote the unique root of Σ containing c but not c_1 and let $\beta = \alpha^\sigma$. By [24, 11.28 (i) and (iii)] with $\{i,j\} = \{1,4\}$, we have $U_\alpha \cap U_\beta = 1$ and $[U_\alpha, U_\beta] = 1$ also in this case. We now return to the assumption that Π is in any one of the three cases B_2, G_2 or F_4 and let u be a non-trivial element of U_α. Both U_α and U_β are contained in the unipotent radical U_c and hence $b := uu^\sigma \in U_c$. Since $U_\alpha \cap U_\beta = 1$, we have $b \neq 1$ and since $[U_\alpha, U_\beta] = 1$, we have $b^\sigma = b$. Thus by Proposition 2.13, $d \neq d^b$. We conclude that c, d and d^b are distinct chambers fixed by Γ.

Suppose that Π is B_2 or G_2. Since the type of a Γ-residue is Θ-invariant, Δ itself is the unique Γ-panel. Since there at least three Γ-chambers, Γ is a descent group. By Theorem 2.18, the Tits index of Γ is $(\Pi, \Theta, \emptyset)$ and Δ^Γ (interpreted as in Definition 2.19) is a Moufang set.

Suppose now that $\Pi = F_4$. In this case, we let R_{ij} be the unique $\{i,j\}$-residue containing the chamber c and we let U_{ij} denote the group induced on R_{ij} by the unipotent radical U_c for $\{i,j\} = \{1,4\}$ and $\{2,3\}$. Then R_{14} and R_{23} are the two Γ-panels containing c. Choose $ij = 14$ or 23 and let d_{ij} be the unique chamber of $R_{ij} \cap \Sigma$ opposite c. Just as above, we can choose a non-trivial element b_{ij} in U_{ij} centralized by σ. By [11, 24.8 (iii) and 24.33], U_{ij} acts sharply transitively on the set of chambers of R_{ij} opposite c. Thus c, d_{ij} and $d_{ij}^{b_{ij}}$ are distinct Γ-chambers contained in R_{ij}. We conclude that both Γ-panels R_{14} and R_{23} contain at least three Γ-chambers. By [11, 22.37], therefore, Γ is a descent group of Δ. By 2.18 (3), therefore, the fixed point building Δ^Γ is of type $\tilde{\Pi}$, where $\tilde{\Pi}$ is the relative Coxeter diagram of the Tits index $(\Pi, \Theta, \emptyset)$. By Examples 2.17, the relative Coxeter diagram of this Tits index is $I_2(8)$. By 2.18 (4), we conclude that Δ^Γ is a Moufang octagon. □

Notation 3.2. — Suppose that Δ is Moufang. Let G° denote the group of all type-preserving automorphisms of Δ, let $G = \mathrm{Aut}(\Delta)$ if Δ is simply laced and let $G = G^\circ$ if Δ is not simply laced. (Thus if Δ and ρ are as in Proposition 3.1, then $\rho \notin G$.) Let $G^\dagger$ denote the subgroup of $\mathrm{Aut}(\Delta)$ generated by all the root groups of Δ. Root groups are type-preserving, so $G^\dagger \subset G^\circ$.

Notation 3.3. — Let Π be a Coxeter diagram, let Δ be a building of type Π (as defined in [24, 7.1]) and suppose that Δ is Moufang as defined in Definition 2.5. Let c be a chamber of Δ and let Σ be an apartment containing c. Let B_Π be the set of ordered pairs of (i,j) such that $\{i,j\}$ is an edge of Π. For each $(i,j) \in B_\Pi$, let R_{ij} be the unique $\{i,j\}$-residue of Δ containing c and let Ω_{ij} be the root group sequence of R_{ij} based at

$(\Sigma \cap R_{ij}, c)$ as defined in [11, 3.1–3.3]. The first term of R_{ij} acts non-trivially on the i-panel of R_{ij} containing c.

Remark 3.4. — Let $(i,j) \in B_\Pi$. Interchanging i and j if necessary, there exists an isomorphism from Ω_{ij} to one of the root group sequences described in [23, 16.1–16.9] (by the classification of Moufang polygons). We say that an element (i,j) of B_Π is *standard* if there is such an isomorphism.

Notation 3.5. — Let F be a field of characteristic $p > 0$ and let E/F be an extension such that $E^p \subset F$. By identifying E with E^p via Frob_E, we can regard F as an extension of E. We can recover the extension E/F from the extension F/E by the same trick. We describe this situation by saying simply that we have a pair of extensions $\{E/F, F/E\}$. Let $\mathrm{Aut}(E,F)$ be the set of all elements of $\mathrm{Aut}(E)$ stabilizing F. This group is canonically isomorphic to the group $\mathrm{Aut}(F,E)$ of all elements of $\mathrm{Aut}(F)$ stabilizing E.

Notation 3.6. — Suppose that Δ is Moufang. By [11, 28.8], the building Δ has either a field of definition F or a pair $\{E/F, F/E\}$ of defining extensions as in Notation 3.5. In the first case, we set $A = \mathrm{Aut}(F)$ and in the second case, we let A denote the group $\mathrm{Aut}(E,F)$ defined in Notation 3.5.

Remark 3.7. — If the building Δ has a pair $\{E/F, F/E\}$ of defining extensions rather than a field of definition F, then $\Delta \cong B_2^{\mathcal{D}}(\Lambda)$ for some indifferent set Λ, $\Delta \cong B_2^{\mathcal{F}}(\Lambda)$ for some quadratic space Λ of type F_4, $\Delta \cong G_2(\Lambda)$ for some inseparable hexagonal system Λ or $\Delta \cong F_4(\Lambda)$ for some inseparable composition algebra Λ. (See [26, 30.15 and 30.23] for the definition of these terms.) By [23, 35.9, 35.12 and 35.13], the pair $\{E/F, F/E\}$ is an invariant of Δ in all these cases even though neither E nor F is an invariant. (By [23, 35.6-35.8, 35.10, 35.11 and 35.14], the field of definition F *is* an invariant of Δ in every other case. Thus the group A is an invariant of Δ in *every* case.)

Notation 3.8. — Suppose that Δ is Moufang, let G be as in Notation 3.2, let Σ, c, B_Π and Ω_{ij} be as in Notation 3.3 and let A be as in Notation 3.6. Let (s,t) be a standard element of B_Π as defined in Remark 3.4 and let φ be an isomorphism from Ω_{st} to Θ, where Θ is one of the root group sequences described in [23, 16.1–16.9]. For each $g \in G$ acting trivially on Σ, let g_{st} denote the automorphism of Ω_{st} induced by g and let g_{st}^* denote the automorphism $\varphi^{-1} g_{st} \varphi$ of Θ. By [11, (29.22) and 29.23–29.25], there exists a unique homomorphism ψ from G to A such that the following hold:

(1) $G^\dagger$ is contained in the kernel of ψ.

(2) For each $g \in G$ acting trivially on Σ, $\psi(g)$ is equal to the element called $\lambda_\Omega(h)$ in [11, 29.5], where $\Omega = \Theta$ and $h = g_{st}^*$.

(3) [15, 4.7 (iii)] holds for all non-type-preserving elements $g \in G$.

(We are only interested here in applying ψ to type-preserving elements, so we do not take the trouble to state (3) more precisely.) A homomorphism $\psi\colon G \to A$ satisfying conditions (1)–(3) for some choice of (s, t) and φ is called a *Galois map* of Δ. If ψ and ψ_1 are two Galois maps of Δ, then there is an inner automorphism ι of A such that $\psi_1 = \psi \cdot \iota$ (by [11, 29.25]). Thus, in particular, all Galois maps of Δ have the same kernel.

DEFINITION 3.9. — *A* Galois subgroup *of* $\mathrm{Aut}(\Delta)$ *is a subgroup* Γ *of the group* G *defined in Notation 3.2 whose intersection with the kernel of a Galois map of* Δ *is trivial. Since two Galois maps differ by an inner automorphism of the group* A*, this notion is independent of the choice of the Galois map.*

THEOREM 3.10. — *Suppose that* Δ *is Moufang and that* Γ *is an isotropic Galois subgroup of* $\mathrm{Aut}(\Delta)$ *that acts on the set of chambers of* Δ *with finite orbits. Then* Γ *is a descent group of* Δ*.*

Proof. — This is [15, 12.2 (ii)]. □

Notation 3.11. — Let Δ be Moufang and let ψ be a Galois map of Δ. A *Galois involution* g of Δ is an element of order 2 in $\mathrm{Aut}(\Delta)$ such that $\langle g\rangle$ is a Galois subgroup. A χ*-involution* of Δ for some $\chi \in A$ is a Galois involution g such that $\chi = \psi(g)$.

PROPOSITION 3.12. — *Let* Δ *be a building of type* F_4 *and suppose that* Γ *is a type-preserving Galois subgroup of* $\mathrm{Aut}(\Delta)$ *acting on the set of chambers of* Δ *with finite orbits such that* Γ*-chambers are residues of type* $\{2, 3\}$ *with respect to the standard numbering of the vertex set of the Coxeter diagram* F_4*. Then* Γ *is a descent group and* Δ^Γ *is a Moufang quadrangle of type* F_4*.*

Proof. — By Theorem 3.10, Γ is a descent group. Let T denote the Tits index (Π, Θ, A), where Π is the Coxeter diagram F_4, Θ is the trivial subgroup of $\mathrm{Aut}(\Pi)$ and $A = \{2, 3\}$. By Example 2.17, the relative Coxeter diagram of this index is B_2. By 2.18 (3), the fixed point building Δ^Γ is of type B_2. By 2.18 (4), we conclude that Δ^Γ is a Moufang quadrangle. By 11.11 (2) below, Δ^Γ is, in fact, a Moufang quadrangle of type F_4. □

PROPOSITION 3.13. — *Let* $\Omega = \mathcal{O}(E, \theta)$ *for some octagonal set* (E, θ) *and let* Γ *be a Galois subgroup of* $\mathrm{Aut}(\Omega)$ *acting on the set of chambers of* Δ *with finite orbits and fixing panels of one type but none of the other. Then* Γ *is a descent group and* Ω^Γ *is a Moufang set.*

Proof. — This holds by 2.18 (5) and Theorem 3.10. □

4. Main Results

We can now state our main results. The proofs of Theorems 4.1 and 4.2 are in Section 14.

THEOREM 4.1. — *Let Ξ be a Moufang quadrangle of type F_4 with a polarity ρ. Then there exists an octagonal set (E, θ) and an automorphism χ of the field E of order 2 that commutes with the Tits endomorphism θ such that the following hold:*

(1) *The building*
$$\Delta := F_4(E, \theta)$$
possesses a type-preserving χ-involution ξ and a polarity σ such that
$$\Gamma := \langle \xi, \sigma \rangle \subset \mathrm{Aut}(\Delta)$$
is a descent group of order 4.

(2) *There exists an isomorphism from Ξ to the fixed point building $\Delta^{\langle \xi \rangle}$ which carries the polarity ρ to the restriction of σ to $\Delta^{\langle \xi \rangle}$.*

(3) *The fixed point buildings $\Delta^{\langle \sigma \rangle}$ and $\Delta^{\langle \sigma\xi \rangle}$ are Moufang octagons, one isomorphic to $\mathcal{O}(E, \theta)$ and the other to $\mathcal{O}(E, \theta\chi)$.*

(4) *The Moufang sets Δ^{Γ}, $(\Delta^{\langle \xi \rangle})^{\langle \sigma \rangle}$, $(\Delta^{\langle \sigma \rangle})^{\langle \xi \rangle}$ and $(\Delta^{\langle \sigma\xi \rangle})^{\langle \xi \rangle}$ are canonically isomorphic.*

(5) *The restriction of ξ to each of the two octagons in (3) is a χ-involution which fixes panels of one type and none of the other.*

THEOREM 4.2. — *Let (E, θ) be an octagonal set, let χ be an automorphism of E of order 2, let $\Omega = \mathcal{O}(E, \theta)$, let $\Delta = F_4(E, \theta)$ and suppose that there exists a χ-involution κ of Ω which fixes panels of one type but not of the other type. Then there is a type-preserving χ-involution ξ of Δ and a polarity σ of Δ such that the following hold:*

(1) $\Gamma = \langle \xi, \sigma \rangle \subset \mathrm{Aut}(\Delta)$ *is a descent group of Δ of order 4.*

(2) *There is an isomorphism from Ω to the fixed point building $\Delta^{\langle \sigma \rangle}$ which carries κ to the restriction of ξ to $\Delta^{\langle \sigma \rangle}$.*

(3) $\langle \xi \rangle$*-chambers are residues of type $\{2, 3\}$ with respect to the standard numbering of the vertex set of the Coxeter diagram F_4, the fixed point building $\Xi := \Delta^{\langle \xi \rangle}$ is a Moufang quadrangle of type F_4 and the polarity σ of Δ induces a polarity ρ on Ξ.*

(4) *The Moufang sets Δ^{Γ}, $(\Delta^{\langle \xi \rangle})^{\langle \sigma \rangle}$, $(\Delta^{\langle \sigma \rangle})^{\langle \xi \rangle}$ and $(\Delta^{\langle \sigma\xi \rangle})^{\langle \xi \rangle}$ are canonically isomorphic.*

(5) *The automorphism χ commutes with the Tits endomorphism θ and the fixed point building $\Delta^{\langle \xi\sigma \rangle}$ is isomorphic to $\mathcal{O}(E, \theta\chi)$.*

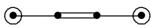
Corollary 4.3. — *The class of Moufang sets of the form $\Xi^{\langle\rho\rangle}$, where Ξ and ρ are as in Theorem 4.1, coincides with the class of Moufang sets of the form $\Omega^{\langle\kappa\rangle}$, where Ω and κ are as in Theorem 4.2.*

Proof. — Let Ξ, ρ, Δ and Γ be as in Theorem 4.1. By 4.1 (4), $\Xi^{\langle\rho\rangle} \cong \Delta^\Gamma$ and hence by 4.1 (5), $\Xi^{\langle\rho\rangle}$ is isomorphic to a Moufang set of the form $\Omega^{\langle\kappa\rangle}$, where Ω and κ are as in Theorem 4.2. Suppose, conversely, that Ω, κ, Δ and Γ are as in 4.2. By 4.2 (4), $\Omega^{\langle\kappa\rangle} \cong \Delta^\Gamma$. By 4.2 (3), it follows that $\Omega^{\langle\kappa\rangle}$ is isomorphic to a Moufang set of the form $\Xi^{\langle\rho\rangle}$, where Ξ and ρ are as in Theorem 4.1. □

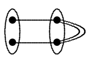
Remark 4.4. — In both Theorems 4.1 and 4.2, we are using the notion of a χ-involution (defined in 3.11) with respect to Galois maps ψ_Δ, ψ_Ω and ψ_{Ω_ξ} as in Propositions 13.4 and 13.8 (4).

Definition 4.5. — *We call a Moufang set of the form Δ^Γ, where Δ and Γ are as in Theorem 4.1 (or 4.2) a Moufang set* of outer F_4-type *(to distinguish them from other Moufang sets which arise as the fixed point buildings of type-preserving descent groups of a building of type F_4; see* [5]*).*

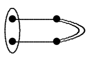
Remark 4.6. — With the conventions described in [11, 34.2],



is the Tits index of $\langle\xi\rangle$,



is the Tits index of $\langle\sigma\rangle$ and $\langle\sigma\xi\rangle$ and



is the Tits index of Γ, where $\Gamma = \langle\xi, \sigma\rangle$ is as in Theorem 4.1 (or 4.2).

The following is proved in Section 18.

Theorem 4.7. — *The group generated by all the root groups of a Moufang set of outer F_4-type is simple.*

5. Moufang Quadrangles of Type F_4

The Moufang quadrangles of type F_4 were first described in [23, Chapter 14 and 16.7]; see also [2], [3] and [25]. In [11, Chapter 16], it is shown that these quadrangles arise in the study of pseudo-reductive quotients of parahoric subgroups of groups of absolute type E_6, E_7 and E_8.

Definition 5.1. — *Let K be a field of characteristic 2 and let F be a subfield such that $K^2 \subset F$. We set $t * s = \mathrm{Frob}_K(t)s = t^2 s$ for all $t \in K$ and all $s \in F$ and $q_{F/K}(s) := s$ for all $s \in F$. The map $*$ makes F into a vector space over K on which the map $q_{F/K}$ is a quadratic form. We write $[F]_K$ to refer to F considered as a vector space over K with respect to $*$.*

Notation 5.2. — An F_4*-datum* is a 4-tuple $S = (E/K, F, \alpha, \beta)$, where E/K is a separable quadratic extension with $\mathrm{char}(K) = 2$, F is a subfield of K containing K^2, α is a non-zero element of F, and β is a non-zero element of K, such that the quadratic form on $V_S := E \oplus E \oplus [F]_K$ given by

$$(a, b, s) \mapsto \beta^{-1}(N(a) + \alpha N(b)) + q_{F/K}(s) = \beta^{-1}(N(a) + \alpha N(b)) + s$$

is anisotropic, where $N = N_{E/K}$ is the norm of the extension E/K and $q_{F/K}$ is as in Definition 5.1. We denote this quadratic form by q_S.

Definition 5.3. — *A quadratic form q over a field K is of* type F_4 *if q is similar to q_S for some F_4-datum $(E/K, F, \alpha, \beta)$. If this case, we will say that S is a* standard decomposition *of q and that E is a* splitting field *of q. We say that a quadratic space (K, V, q) is of type F_4 if the quadratic form q is of type F_4.*

Definition 5.4. — *Let $S = (E/K, F, \alpha, \beta)$ be an F_4-datum, let D denote the composite field FE^2 and let $[K]_F$ denote K regarded as a vector space over its subfield F in the standard way. By* [23, 14.8], *the quadratic form $\hat{q}_S$ on $\hat{V}_S := D \oplus D \oplus [K]_F$ over F given by*

$$(x, y, t) \mapsto \alpha(N(x) + \beta^2 N(y)) + t^2 \tag{5.5}$$

is a quadratic form of type F_4 with standard decomposition

$$(D/F, K^2, \beta^2, \alpha^{-1}).$$

Notation 5.6. — Let $S = (E/K, F, \alpha, \beta)$ be an F_4-datum, let $q := q_S$ and $V = V_S$ be as in Notation 5.2 and let $\hat{V} = \hat{V}_S$ and $\hat{q} := \hat{q}_S$ be as in Definition 5.4. Let f and $\hat{f}$ denote the bilinear forms ∂q and $\partial \hat{q}$ associated with q and $\hat{q}$ and let $x \mapsto \bar{x}$ be the non-trivial element of $\mathrm{Gal}(E/K)$.

We now introduce the Moufang quadrangle that corresponds to this data.

Notation 5.7. — Let S, $\hat{V}$, V, etc. be as in Notation 5.6, let

$$U_+ := U_1 U_2 U_3 U_4$$

be the group defined in terms of the isomorphisms $x_i \colon V \to U_i$ for $i = 2$ and 4 and $x_i \colon \hat{V} \to U_i$ for $i = 1$ and 3 the following commutator relations taken from [23, 16.7]):

$$\begin{aligned}
[x_1(x,y,t),x_3(x',y',t')] &= x_2\big(0,\ 0,\ \alpha\big(x\bar{x}'+x'\bar{x}+\beta^2(y\bar{y}'+y'\bar{y})\big)\big),\\
[x_2(u,v,s),x_4(u',v',s')] &= x_3\big(0,\ 0,\ \beta^{-1}\big(u\bar{u}'+u'\bar{u}+\alpha(v\bar{v}'+v'\bar{v})\big)\big),\\
[x_1(x,y,t),x_4(u,v,s)] &= x_2\big(tu+\alpha(\bar{x}v+\beta y\bar{v}),\ tv+xu+\beta y\bar{u},\\
&\qquad t^2s+s\alpha(x\bar{x}+\beta^2y\bar{y})\\
&\qquad\qquad +\alpha\big(u^2x\bar{y}+\bar{u}^2\bar{x}y+\alpha(\bar{v}^2xy+v^2\bar{x}\bar{y})\big)\big)\\
&\quad\cdot x_3\big(sx+\bar{u}^2y+\alpha v^2\bar{y},\ sy+\beta^{-2}(u^2x+\alpha v^2\bar{x}),\\
&\qquad st+t\beta^{-1}(u\bar{u}+\alpha v\bar{v})\\
&\qquad\qquad +\alpha\big(\beta^{-1}(xu\bar{v}+\bar{x}\bar{u}v)+y\bar{u}\bar{v}+\bar{y}uv\big)\big)
\end{aligned}$$

for all $(u,v,s),(u',v',s')\in V$ and all $(x,y,t),(x',y',t')\in\hat{V}$ and

$$[U_1,U_2]=[U_2,U_3]=[U_3,U_4]=1.$$

The group U_+, its subgroups $U_1,\dots,U_4$ and the isomorphisms $x_1,\dots,x_4$ depend only on the F_4-datum S. By [23, 16.7 and 32.11],

$$(U_+,U_1,U_2,U_3,U_4)$$

is a root group sequence. Let

$$\mathcal{Q}(S)$$

denote the Moufang quadrangle, Σ the apartment of $\mathcal{Q}(S)$ and c the chamber of Σ obtained by applying [23, 8.3] to this root group sequence. There is a canonical identification of $U_1,\dots,U_4$ with the root groups of $\Xi:=\mathcal{Q}(S)$ associated with the four roots of Σ containing c and we always identify U_+ with the subgroup of $\mathrm{Aut}(\Xi)$ generated by these four root groups.

Definition 5.8. — *A* Moufang quadrangle of type F_4 *is a Moufang quadrangle isomorphic to $\mathcal{Q}(S)$ (as defined in Notation 5.7) for some F_4-datum S.*

Notation 5.9. — Let $S=(E/K,F,\alpha,\beta)$ and $\Omega:=(U_+,U_1,\dots,U_4)$ be as in Notation 5.7. By [23, 28.45], there is an anti-isomorphism (as defined in [23, 8.9]) from Ω to the root group sequence obtained by applying Notation 5.7 to the F_4-datum $(D/F,K^2,\beta^2,\alpha^{-1})$ in Definition 5.4.

Notation 5.10. — Let S and (K,V,q) be as in Notation 5.6 and let

$$\Omega=(U_+,U_1,\dots,U_4)$$

and $\Xi=\mathcal{Q}(S)$ be as in Notation 5.7. The quadrangle Ξ is called $\mathcal{Q}_{\mathcal{F}}(K,V,q)$ in [23, 16.7] and $B_2^{\mathcal{F}}(K,V,q)$ in [26, 30.15]. Suppose that S' is any other

F_4-datum and let $\Omega' = (U'_+, U'_1, \ldots, U'_4)$ be the root group sequence obtained by applying Notation 5.7 to S'. Then by [23, 35.12], there is a type-preserving isomorphism from Ω to Ω' if and only if the quadratic form $q_{S'}$ is similar to q, where $q_{S'}$ is the quadratic form obtained by applying Notation 5.2 to S'. In particular, $\Xi \cong \mathcal{Q}(S')$ for every standard decomposition S' of q (as defined in Definition 5.3).

Remark 5.11. — Let $\Xi = \mathcal{Q}(S)$ for some F_4 datum $S = (E/K, F, \alpha, \beta)$. By [11, 28.4] (see Notation 3.6), $\{K/F, F/K\}$ is the pair of defining extensions of Ξ. By [23, 35.12], it is an invariant of Ξ.

Remark 5.12. — Let $s_0 \in F^*$ and let $S_1 = (E/K, F, \alpha, \beta/s_0)$. Then S_1 is an F_4-datum, $V_{S_1} = V$, $\hat{V}_{S_1} = \hat{V}$ and the maps $x_1(x, y, t) \mapsto x_1(x, s_0y, t)$, $x_3(x, y, t) \mapsto x_3(x/s_0, y, t/s_0)$ and $x_i(u, v, s) \mapsto x_i(u/s_0, v/s_0, s/s_0)$ for $i =$ 2 and 4 extend to an isomorphism from $\mathcal{Q}(S)$ to $\mathcal{Q}(S_1)$ (by [23, 7.5]). Thus by reparametrizing U_+, we can replace the element $x_4(0, 0, s_0)$ by $x_4(0, 0, 1)$ without changing the element $x_1(0, 0, 1)$.

Notation 5.13. — We define two maps, one from $V \times \hat{V}$ to V and the other from $\hat{V} \times V$ to $\hat{V}$, both denoted either by $\cdot$ or juxtaposition, so that

$$[x_1(\hat{v}), x_4(v)] = x_2(v \cdot \hat{v})x_3(\hat{v} \cdot v) \tag{5.14}$$

in U_+ for all $(\hat{v}, v) \in \hat{V} \times V$. (Note that we will also denote scalar multiplication by $\cdot$ or juxtaposition, but this should not cause any confusion.)

Remark 5.15. — Let (K, V, q) and f be as in Notation 5.6 and let d, e be linearly independent elements of V such that $f(d, e) = 1$. Then $q(d)x^2+x+q(e) = q(xd+e) \neq 0$ for all $x \in K$ since q is anisotropic, and the restriction of $q/q(d)$ to $\langle e, d\rangle$ is isometric to the norm of the quadratic extension L/K, where L is the splitting field of the polynomial $q(d)x^2 + x + q(e)$ over K. Note that L is also the splitting field of the polynomial $x^2 + x + q(d)q(e)$ over K.

Theorem 5.16. — *Let (K, V, q), $\hat{V}$ and f be as in Notation 5.6 and let*

$$(U_+, U_1, \ldots, U_4)$$

and $x_1, \ldots, x_4$ be as in Notation 5.7. Let d, e be two elements of V and let ξ be an element of $\hat{V}$ such that $f(d, e) = 1$ and $f(d, e\xi) = 0$. Let $\alpha_0 = f(d\xi, e\xi)$, let $\beta_0 = q(d)^{-1}$, let L be the splitting field of the polynomial $p(x) = q(d)x^2 + x + q(e)$ over K and let ω be a root of $p(x)$ in L. Then the following hold:

(1) *$S_0 := (L/K, F, \alpha_0, \beta_0)$ is a standard decomposition of q.*

(2) *There exists an isometry* π *from* (K, V, q) *to* (K, V_{S_0}, q_{S_0}) *sending the elements* d, e, $d\xi$ *and* $e\xi$ *to* $(1, 0, 0)$, $(\omega, 0, 0)$, $(0, 1, 0)$ *and* $(0, \omega, 0)$, *respectively, and* $(0, 0, s)$ *to* $(0, 0, s)$ *for all* $s \in F$.

(3) *There exists an isometry* $\hat{\pi}$ *from* $(F, \hat{V}, \hat{q})$ *to* $(F, \hat{V}_{S_0}, \hat{q}_{S_0})$ *sending the elements* ξ, $\xi e \cdot d^{-1}$, $\xi e \cdot d^{-1}$ *and* ξd^{-1} *to* $(1, 0, 0)$, $(\omega^2, 0, 0)$, $(0, 1, 0)$ *and* $(0, \omega^2, 0)$, *respectively, and* $(0, 0, t)$ *to* $(0, 0, t)$ *for all* $t \in K$.

(4) *Let* $(\tilde{U}_+, \tilde{U}_1, \ldots, \tilde{U}_4)$ *and* $\tilde{x}_1, \ldots, \tilde{x}_4$ *be the root group sequence and the isomorphisms obtained by applying Notation 5.7 to* S_0. *Then there is an isomorphism from* U_+ *to* $\tilde{U}_+$ *extending the maps* $x_i(v) \mapsto \tilde{x}_i(\pi(v))$ *for* $i = 1$ *and* 3 *and* $x_i(v) \mapsto \tilde{x}_i(\hat{\pi}(v))$ *for* $i = 2$ *and* 4.

Proof. — This is proved in [3, 8.98-8.106]. See, in particular, the equations at the top of page 77 in [3]. □

Notation 5.17. — Let $[s]_K = (0, 0, s) \in V$ for each $s \in F$ and $[t]_F = (0, 0, t) \in \hat{V}$ for each $t \in K$. Thus $t[s]_K = [t^2 s]_K$ and $s[t]_F = [st]_F$ for all $s \in F$ and all $t \in K$.

PROPOSITION 5.18. — *The following identities hold for all* $t \in K$, *all* $s \in F$, *all* $u, v, w \in V$ *and all* $\hat{u}, \hat{v}, \hat{w} \in \hat{V}$:

(F0) $x \mapsto x\hat{w}$ *and* $\hat{x} \mapsto \hat{x}w$ *are linear maps from* V *to* V *and from* $\hat{V}$ *to* $\hat{V}$.

(F1) $v[t]_F = tv$.

(F2) $\hat{v}[s]_K = s\hat{v}$.

(F3) $v \cdot s\hat{w} = v\hat{w} \cdot [s]_F$.

(F4) $\hat{v} \cdot tw = \hat{v}w \cdot [t^2]_K$.

(F5) $[t]_F v = [tq(v)]_F$.

(F6) $[s]_K \hat{v} = [s\hat{q}(\hat{v})]_K$.

(F7) $v\hat{w}\hat{w} = v \cdot [\hat{q}(\hat{w})]_F$

(F8) $\hat{v}ww = \hat{v} \cdot [q(w)^2]_K$.

(F9) $v \cdot \hat{w}v = q(v)v\hat{w}$.

(F10) $\hat{v} \cdot w\hat{v} = \hat{q}(\hat{v})\hat{v}w$.

(F11) $w(\hat{u} + \hat{v}) = w\hat{u} + w\hat{v} + [\hat{f}(\hat{u}w, \hat{v})]_K$.

(F12) $\hat{w}(u + v) = \hat{w}u + \hat{w}v + [f(u\hat{w}, v)]_F$.

Proof. — Comparing [23, 16.7] with (5.14), we can write the products $\hat{v} \cdot v$ and $v \cdot \hat{v}$ in terms of the functions given in [23, 14.15–14.16]. The identities (F0)–(F12) can then be verified with the help of the identities in [23, 14.18]. □

The identities (F0)–(F12) are the axioms of a *radical quadrangular system* as defined in [3, Appendix A.3.2]. These axioms can be used to characterize Moufang quadrangles of type F_4 (defined in Definition 5.8 above); see [3, §8.5] and [23, Chapter 28] for details.

Notation 5.19. — Using Notation 5.13, we can re-write the commutator relations in Notation 5.7 as follows:

$$[x_1(\hat{v}), x_3(\hat{u})] = x_2([\hat{f}(\hat{v}, \hat{u})]_K)$$
$$[x_2(v), x_4(u)] = x_3([f(v, u)]_F)$$
$$[x_1(\hat{v}), x_4(v)] = x_2(v\hat{v})x_3(\hat{v}v)$$

for all $u, v \in V$ and all $\hat{u}, \hat{v} \in \hat{V}$ as well as $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$.

6. The Polarity ρ

We continue with all the notation of the previous section.

HYPOTHESIS 6.1. — *We suppose now that the Moufang quadrangle $\Xi = \mathcal{Q}(S)$ introduced in Notation 5.7 has a polarity ρ.*

Remark 6.2. — By Proposition 2.6, the polarity ρ fixes an apartment. Since ρ is a non-type-preserving involution and apartments are circuits of length 8, ρ fixes two opposite chambers of this apartment. Since $\mathrm{Aut}(\Xi)$ acts transitively on incident pairs of apartments and chambers (by [24, 11.12]), we can assume that ρ fixes the apartment Σ and the chamber c in Notation 5.7. This means that

$$\rho U_i \rho = U_{5-i} \tag{6.3}$$

for all $i \in [1, 4]$.

Notation 6.4. — Let $\hat{\varphi},\ \hat{\varphi}_1 \colon \hat{V} \to V$ and $\varphi,\ \varphi_1 \colon V \to \hat{V}$ be the unique additive bijections such that

$$\rho(x_1(\hat{v})) = x_4(\hat{\varphi}(\hat{v}))$$
$$\rho(x_2(v)) = x_3(\varphi_1(v))$$
$$\rho(x_3(\hat{v})) = x_2(\hat{\varphi}_1(\hat{v}))$$
$$\rho(x_4(v)) = x_1(\varphi(v))$$

for all $v \in V$ and all $\hat{v} \in \hat{V}$. By Remark 5.12, we can assume that $\varphi([1]_K) = [1]_F$. Since ρ is of order 2, we have

$$\hat{\varphi} = \varphi^{-1} \text{ and } \hat{\varphi}_1 = \varphi_1^{-1}.$$

Lemma 6.5. — *The following hold:*

(1) $\varphi = \varphi_1$.
(2) $\varphi([\hat{q}(\varphi(v))]_K) = [q(v)]_F$ *for all* $v \in V$.

Proof. — Let $v \in V$. By (5.14), (F1) and (F5), we have

$$[x_1([1]_F), x_4(v)] = x_2(v)x_3([q(v)]_F).$$

Applying ρ, we obtain

$$[x_1(\varphi(v)), x_4([1]_K)] = x_2(\varphi_1^{-1}([q(v)]_F))x_3(\varphi_1(v)).$$

By (5.14), (F2) and (F6), on the other hand,

$$[x_1(\varphi(v)), x_4([1]_K)] = x_2([\hat{q}(\varphi(v))]_K)x_3(\varphi(v)).$$

Therefore $\varphi(v) = \varphi_1(v)$ and $\varphi_1^{-1}([q(v)]_F) = [\hat{q}(\varphi(v))]_K$. □

Lemma 6.6. — $\varphi^{-1}([f(u,v)]_F) = [\hat{f}(\varphi(u), \varphi(v))]_K$ *for all* $u, v \in V$ *and*

$$\varphi([\hat{f}(\hat{u}, \hat{v})]_K) = [f(\varphi^{-1}(\hat{u}), \varphi^{-1}(\hat{v}))]_F$$

for all $\hat{u}, \hat{v} \in \hat{V}$. *In particular,* $\varphi([F]_K) = [K]_F$ *and* $\varphi^{-1}([K]_F) = [F]_K$.

Proof. — This follows from 6.5 (2). □

Proposition 6.7. — *The following hold for all* $v \in V$ *and all* $\hat{v} \in \hat{V}$*:*

(1) $\varphi(v\hat{v}) = \varphi(v)\varphi^{-1}(\hat{v})$.
(2) $\varphi^{-1}(\hat{v}v) = \varphi^{-1}(\hat{v})\varphi(v)$.

Proof. — Applying ρ to the identity (5.14), we obtain both claims by 6.5 (1). □

Notation 6.8. — By Lemma 6.6, there exists a unique additive bijection θ from K to F such that $\varphi^{-1}([t]_F) = [t^\theta]_K$. Note that it means that

$$\hat{f}(\varphi(u), \varphi(v)) = f(u,v)^\theta \tag{6.9}$$

for all $u, v \in V$.

Proposition 6.10. — *The following hold:*

(1) *The map θ defined in Notation 6.8 is a Tits endomorphism of K.*
(2) $\varphi(tv) = t^\theta \varphi(v)$ *for all* $v \in V$ *and all* $t \in K$.
(3) $q(v)^\theta = \hat{q}(\varphi(v))$ *for all* $v \in V$.
(4) $u \cdot \varphi(tv) = t^\theta \cdot u\varphi(v)$ *for all* $u, v \in V$ *and all* $t \in K$.

Proof. — Choose $t \in K$ and $s \in F$. By Notation 5.17 and (F2), we have $[t]_F \cdot [s]_K = [st]_F$. Applying φ^{-1}, we obtain $[t^\theta]_K \cdot [s^{\theta^{-1}}]_F = [(st)^\theta]_K$.

By 5.17 and (F1), on the other hand, we have $[t^\theta]_K \cdot [s^{\theta^{-1}}]_F = [(s^{\theta^{-1}})^2 t^\theta]_K$. Therefore

$$(ts)^\theta = (s^{\theta^{-1}})^2 t^\theta. \tag{6.11}$$

Setting $t = 1$ in (6.11), we obtain

$$s^\theta = (s^{\theta^{-1}})^2. \tag{6.12}$$

Substituting this back into (6.11), we then have

$$(ts)^\theta = s^\theta t^\theta. \tag{6.13}$$

Let $x = s^{\theta^{-1}}$. Substituting x^θ for s in (6.12), we conclude that

$$(x^\theta)^\theta = x^2. \tag{6.14}$$

Thus

$$(x^2)^\theta = (x^{\theta^2})^\theta = (x^\theta)^{\theta^2} = (x^\theta)^2 \tag{6.15}$$

for all $x \in K$. Choose $u \in K$. Then u^2 and t^2 are in F, so

$$(t^2u^2)^\theta = (t^2)^\theta (u^2)^\theta$$

by (6.13). By (6.15), it follows that θ is multiplicative. By (6.14), therefore, θ is a Tits endomorphism. Thus (1) holds.

Now let $v \in V$ and $t \in K$. Then

$$\begin{aligned} \varphi(tv) &= \varphi(v[t]_F) && \text{by (F1)} \\ &= \varphi(v)\varphi^{-1}([t]_F) && \text{by 6.7(1)} \\ &= \varphi(v)[t^\theta]_K = t^\theta\varphi(v) && \text{by (F2)}, \end{aligned}$$

so (2) holds, and

$$\begin{aligned} [q(v)^\theta]_K &= \varphi^{-1}[q(v)]_F \\ &= [\hat{q}(\varphi(v))]_K && \text{by 6.5(2)}, \end{aligned}$$

so (3) holds. Now choose another $u \in V$. Then

$$\begin{aligned} u\varphi(tv) &= u \cdot t^\theta\varphi(v) && \text{by (2)} \\ &= u\varphi(v) \cdot [t^\theta]_F && \text{by (F3)} \\ &= t^\theta \cdot u\varphi(v) && \text{by (F1)}. \end{aligned}$$

Thus (4) holds. □

Notation 6.16. — Let $x_1, \dots, x_4$ be as in Notation 5.7. We replace x_i by $x_i \cdot \varphi$ for $i = 1$ and 3. After these replacements, we have $x_i \colon V \to U_i$ for all $i \in [1, 4]$,

$$[x_1(u), x_3(v)] = x_2([\hat{f}(\varphi(u), \varphi(v))]_K)$$

and

$$\begin{aligned}[x_2(u), x_4(v)] &= x_3(\varphi^{-1}([f(u,v)]_F)) \\ &= x_3([\hat{f}(\varphi(u), \varphi(v)]_K)\end{aligned}$$

for all $u, v \in V$ by Lemma 6.6,

$$\begin{aligned}[x_1(u), x_4(v)] &= x_2(v\varphi(u))x_3\big(\varphi^{-1}(\varphi(u)v)\big) \\ &= x_2(v\varphi(u))x_3(u\varphi(v))\end{aligned}$$

for all $u, v \in V$ by Proposition 6.7 and

$$x_i(u)^\rho = x_{5-i}(u) \tag{6.17}$$

for all $u \in V$ and for all $i \in [1, 4]$. We define a product on V by

$$uv = u\varphi(v) \tag{6.18}$$

and a symmetric map $g \colon V \times V \to [F]_K \subset V$ by

$$g(u, v) = [\hat{f}(\varphi(u), \varphi(v))]_K \tag{6.19}$$

for all $u, v \in V$. With this notation, we have

$$\begin{aligned}[x_1(u), x_3(v)] &= x_2(g(u,v)) \\ [x_2(u), x_4(v)] &= x_3(g(u,v)) \\ [x_1(u), x_4(v)] &= x_2(vu)x_3(uv)\end{aligned} \tag{6.20}$$

for all $u, v \in V$ as well as $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$.

Notation 6.21. — From now on, we set $[t] = [t^\theta]_K$ for all $t \in K$. Thus

$$[K] := \{[t] \mid t \in K\} = [F]_K$$

is a vector space over K with scalar multiplication given by $a[t] = [a^\theta t]$ for all $a, t \in K$, and

$$g(u, v) = [f(u, v)^\theta]_K = [f(u, v)] \tag{6.22}$$

for all $u, v \in V$ by (6.9) and (6.19).

Proposition 6.23. — *Let the multiplication on* $V = E \oplus E \oplus [K]$ *and the map* g *be as in* (6.18) *and* (6.22)*. Then the following hold for all* $u, v \in V$ *and all* $t \in K$*:*

(R1) *The map* $x \mapsto xv$ *from* V *to itself is* K*-linear.*
(R2) $v[t] = tv$.
(R3) $u \cdot tv = t^\theta \cdot uv$.
(R4) $[t]v = [tq(v)]$.
(R5) $uv \cdot v = q(v)^\theta \cdot u$.
(R6) $v \cdot uv = q(v) \cdot vu$.
(R7) $u(v + w) = uv + uw + g(vu, w)$.

Proof. — Just for this proof, we will denote by $*$ both the map from $V \times \hat{V}$ to V and the map from $\hat{V} \times V$ to V defined in Notation 5.13 to distinguish them from the multiplication on V defined in (6.18). Thus, in particular, $uv = u * \varphi(v)$ for all $u, v \in V$.

Let $u, v, w \in V$ and $t \in K$. The assertion (R1) is just a special case of (F0). We have

$$v[t] = v[t^\theta]_K = v * [t]_F = tv$$

by (F1). Thus (R2) holds. The assertion (R3) follows from 6.10(iv). To see that (R4) holds, we observe that

$$\begin{aligned} [t]v &= [t^\theta]_K * \varphi(v) && \\ &= [t^\theta \hat{q}(\varphi(v))]_K && \text{by (F6)} \\ &= [t^\theta q(v)^\theta]_K && \text{by 6.10(3)} \\ &= [tq(v)]. && \end{aligned} \tag{6.24}$$

Next we have

$$\begin{aligned} uv \cdot v &= u * [\hat{q}(\varphi(v))]_F && \text{by (F7)} \\ &= u * [q(v)^\theta]_F && \text{by 6.10(3)} \\ &= u[q(v)^\theta] && \\ &= q(v)^\theta \cdot u && \text{by (R2),} \end{aligned}$$

so (R5) holds, and

$$\begin{aligned} v \cdot uv &= v * \varphi\big(u * \varphi(v)\big) && \\ &= v * \big(\varphi(u) * v\big) && \text{by 6.7(1)} \\ &= q(v) \cdot vu && \text{by (F9),} \end{aligned}$$

so (R6) holds. Finally, we have

$$\begin{aligned} u(v+w) &= u * \varphi(v+w) \\ &= uv + uw + [\hat{f}(\varphi(v) * u, \varphi(w))]_K && \text{by (F11)} \\ &= uv + uw + [\hat{f}(\varphi(v * \varphi(u)), \varphi(w))]_K && \text{by 6.7(2)} \\ &= uv + uw + [\hat{f}(\varphi(vu), \varphi(w))]_K \\ &= uv + uw + g(vu, w) && \text{by (6.19).} \end{aligned}$$

Thus (R7) holds. □

7. Polarity Algebras

In this section we introduce polarity algebras and prove a series of identities. Some (for instance Proposition 7.9) we have included only because they are compelling, not because we will apply them later on.

Definition 7.1. — *A* polarity algebra *is a 6-tuple* $(K, V, q, \theta, t \mapsto [t], \cdot)$, *where* K *is a field of characteristic* 2, (K, V, q) *is an anisotropic quadratic space such that the bilinear form* $f := \partial q$ *is not identically zero,* θ *is a Tits endomorphism of* K, $t \mapsto [t]$ *is a* K*-linear embedding of the* K*-vector space* $[K] = [K^\theta]_K$ *defined in Definition 5.1 and Notation 6.21 into the radical of* f*, so*

$$a[t] = [a^\theta t] \tag{7.2}$$

for all $a, t \in K$*, and* $(u, v) \mapsto u \cdot v$ *is a multiplication on* V *(which we often denote by juxtaposition), satisfying the conditions* (R1)–(R7) *in Proposition 6.23 with*

$$g(u, v) = [f(u, v)]$$

for all $u, v \in V$ *in* (R7).

Throughout this section we assume that $(K, V, q, \theta, t \mapsto [t], \cdot)$ is a polarity algebra. We let f and g be as in Definition 7.1 and we set

$$v^{-1} = q(v)^{-1} v$$

for all non-zero $v \in V$. Since q is anisotropic, this is allowed.

Remark 7.3. — Let K and f be as in Definition 7.1. An anisotropic form over a finite field is either 1-dimensional or similar to the norm of a quadratic extension. Since f is not identically zero and its radical is non-trivial, we conclude that K is infinite.

Proposition 7.4. — *The following hold for all $u, v, w \in V$ and all $t \in K$:*

(1) $g(u, uw) = 0$.
(2) $g(u, vw) = g(uw, v)$.
(3) $g(u, v)w = g(q(w)u, v)$.
(4) $tg(u, v) = g(t^\theta u, v)$.

Proof. — Assertions (1) and (2) follow immediately from (R7) and assertion (3) follows immediately from (R4). We have $tg(u, v) = t[f(u, v)] = [t^\theta f(u, v)] = g(t^\theta u, v)$ for all $u, v \in V$ and all $t \in K$, so also assertion (4) holds. □

Proposition 7.5. — $v \cdot wu = f(u, vw)u + f(u, v)uw + q(u)vw$ *for all* $u, v, w \in V$.

Proof. — Let $t \in K$. By (R6), we have

$$
\begin{aligned}
(7.6)\quad & (u + tv) \cdot (w(u + tv)) \\
&= q(u + tv)(u + tv)w \\
&= (q(u) + tf(u, v) + t^2 q(v))(u + tv)w \\
&= q(u)uw + t\big(q(u)vw + f(u, v)uw\big) \\
&\qquad + t^2\big(q(v)uw + f(u, v)vw\big) + t^3 q(v)vw\,.
\end{aligned}
$$

By (R7) and (R3), on the other hand, we have

$$
\begin{aligned}
(7.7)\quad & (u + tv) \cdot (w(u + tv)) \\
&= (u + tv)\big(wu + t^\theta wv + g(u, tvw)\big) \\
&= u(wu + t^\theta wv + g(u, tvw)) + tv(wu + t^\theta wv + g(u, tvw)) \\
&= u(wu) + u \cdot t^\theta wv + ug(u, tvw) + g(wu \cdot u, t^\theta wv) \\
&\qquad + tv \cdot wu + t^3 v \cdot wv + tv \cdot g(u, tvw) + g(wu, t^\theta wv \cdot tv) \\
&= q(u)uw + t^2 u \cdot wv + ug(u, tvw) + g(wu \cdot u, t^\theta wv) \\
&\qquad + tv \cdot wu + t^3 q(v)vw + tv \cdot g(u, tvw) + g(wu, t^\theta wv \cdot tv)\,.
\end{aligned}
$$

Thus the sum of the expressions (7.6) and (7.7) is zero. By (R2) and 7.4 (4), this sum lies in $K[t]$ and the coefficient of t is

$$q(u)vw + f(u, v)uw + f(u, vw)u + v \cdot wu.$$

To verify this, we need only observe that $ug(u, tvw) = u[f(u, tvw)] = f(u, tvw)u$ by (R2) and $g(wu \cdot u, wv) = q(u)g(w, wv) = 0$ by (R5) and 7.4(1). It follows from Remark 7.3 and [23, 2.26] that the coefficient of each power of t in this sum is zero. □

PROPOSITION 7.8. — *$uv\cdot w + uw\cdot v = g(v, f(v, wu)w) + f(v, w)^\theta u$ for all $u, v, w \in V$.*

Proof. — By (R5), we have

$$\begin{aligned} u(v+w)\cdot(v+w) &= q(v+w)^\theta u \\ &= (q(v)+f(v,w)+q(w))^\theta u \\ &= q(v)^\theta u + f(v,w)^\theta u + q(w)^\theta u\,. \end{aligned}$$

By (R5), (R6), (R7) and 7.4(3), on the other hand, we have

$$\begin{aligned} u(v+w)\cdot(v+w) &= (uv+uw+g(v,wu))(v+w) \\ &= uv(v+w)+uw(v+w)+g(v,wu)(v+w) \\ &= uv\cdot v + uv\cdot w + g(v\cdot uv, w) \\ &\qquad + uw\cdot v + uw\cdot w + g(v, w\cdot uw) \\ &\qquad + g(q(v)v, wu) + g(q(w)v, wu) + g(f(v,w)v, wu) \\ &= q(v)^\theta u + uv\cdot w + \overbrace{g(q(v)vu, w)}^{1} \\ &\qquad + uw\cdot v + q(w)^\theta u + \overbrace{g(v, q(w)wu)}^{2} \\ &\qquad + \overbrace{g(q(v)v, wu)}^{1} + \overbrace{g(q(w)v, wu)}^{2} + g(v, f(v,wu)w)\,. \end{aligned}$$

Note that

$$\begin{aligned} g(f(v,w)v, wu) &= [f(v,w)f(v,wu)] \\ &= [f(v, f(v,wu)w)] = g(v, f(v,wu)w)\,. \end{aligned}$$

Therefore $f(v,w)^\theta u = uv\cdot w + uw\cdot v + g(v, f(v,wu)w)$. □

The following observation says that the quadratic form q is multiplicative "with a twist."

PROPOSITION 7.9. — *$q(uv) = q(u)q(v)^\theta$ for all $u, v \in V$.*

Proof. — Choose $u, w \in V$ and recall that $f(V, [K]) = 0$. Setting $v = [1]$ in Proposition 7.5, we obtain

$$[1]\cdot wu = f(u,[1]w)u + f(u,[1])uw + q(u)[1]w\,.$$

By (R3) and (R4), therefore, $[q(wu)] = q(u)[q(w)] = [q(w)q(u)^\theta]$. □

Proposition 7.10. — *$q([t]) = t^\theta$ for all $t \in K$.*

Proof. — If $t \in K$, then $[q([t])] = [1][t] = t[1] = [t^\theta]$ by (R2) and (R4). □

Proposition 7.11. — *The following hold for all $u, v, w \in K$:*

(1) $f(uv, uw) = f(v, wu)^\theta + q(u)f(v, w)^\theta$

(2) $f(uv, wv) = q(v)^\theta f(u, w)$.

(3) $(uv)^{-1} = u^{-1}v^{-1}$ *if* $u, v \neq 0$.

Proof. — Let $u, v, w \in K$. We have

$$\begin{aligned} q(u(v+w)) &= q(u)q(v+w)^\theta \\ &= q(u)(q(v) + q(w) + f(v,w))^\theta \\ &= q(uv) + q(uw) + q(u)f(v,w)^\theta \end{aligned}$$

by Proposition 7.9, whereas

$$\begin{aligned} q(u(v+w)) &= q(uv + uw + g(v, wu)) && \text{by (R7)} \\ &= q(uv) + q(uw) + q([f(v,wu)]) + f(uv,uw) && \text{by Def. 7.1} \\ &= q(uv) + q(uw) + f(v,wu)^\theta + f(uv,uw) && \text{by Prop. 7.10.} \end{aligned}$$

Thus (1) holds.

We have

$$\begin{aligned} q((u+w)v) &= q(u+w)q(v)^\theta \\ &= (q(u) + q(w) + f(u,w))q(v)^\theta \\ &= q(uv) + q(wv) + q(v)^\theta f(u,w), \end{aligned}$$

by Proposition 7.9, whereas

$$q((u+w)v) = q(uv + wv) = q(uv) + q(wv) + f(uv, wv)$$

by (R1). Thus (2) holds. Finally, we have

$$\begin{aligned} (uv)^{-1} &= q(uv)^{-1}uv \\ &= q(u)^{-1}q(v)^{-\theta}uv = q(u)^{-1}u \cdot q(v)^{-1}v = u^{-1}v^{-1} \end{aligned}$$

by (R3) and Proposition 7.9. Thus (3) holds. □

Proposition 7.12. — *The following hold for all $u, v, w \in K$:*

(1) $u^{-1} \cdot vu = uv$ *if* $u \neq 0$.

(2) $uv \cdot v^{-1} = u$ *if* $v \neq 0$.

Proof. — These identities follow immediately from (R3), (R5) and (R6). □

Remark 7.13. — Let $v \in V^*$. By (R1) and 7.12(2), the map $x \mapsto xv$ is an automorphism of V. By 7.9, This map is a similitude of q with similarity factor $q(v)^\theta$.

PROPOSITION 7.14. — *The following hold for all $u, v, z, w \in V$:*

(1) $g(uv \cdot w, zv) = f(w, v)g(uv, z) + q(v)g(uw, z)$.
(2) $g(uv \cdot w, uz) = f(vu, z)w + f(wu, v)z + f(wu, z)v + f(w, v)zu + f(w, z)vu + f(v, z)wu$.
(3) $f(uv, zw) + f(uw, zv) = f(u, z)f(v, w)^\theta$.
(4) $uv \cdot vu = q(uv)u$.
(5) $uv \cdot vw = q(wv)u + f(u, w)q(v)^\theta w + f(uv, w)wv$.
(6) $uv \cdot vw = (u \cdot vw) \cdot v$.
(7) $(vv)^{-1} \cdot vv = v$ *if* $v \neq 0$.
(8) $u^{-1}u \cdot u^{-1}u = u$ *if* $u \neq 0$.

Proof. — On the one hand,

$$\begin{aligned} w \cdot (u+z)v &= f(w(u+z), v)v + f(w, v)v(u+z) + q(v)w(u+z) && \text{by Prop. 7.5} \\ &= f(wu + wz, v)v + f(w, v)\big(vu + vz + g(uv, z)\big) \\ &\quad + q(v)\big(wu + wz + g(uw, z) && \text{by (R7),} \end{aligned}$$

and on the other,

$$\begin{aligned} w \cdot (u+z)v &= w \cdot (uv + zv) && \text{by (R1)} \\ &= w \cdot uv + w \cdot zv + g(uv \cdot w, zv) \\ &= f(v, wu)v + f(v, w)vw + q(v)wu \\ &\quad + f(v, wz)v + f(v, w)vz + q(v)wz \\ &\quad + g(uv \cdot w, zv) && \text{by Prop. 7.5.} \end{aligned}$$

Thus (1) holds.

Using Proposition 7.5, (R2) and (R7), we have

$$\begin{aligned} w \cdot u(v+z) &= w \cdot \big(uv + uz + g(vu, z)\big) \\ &= w(uv + uz) + w[f(vu, z)] \\ &= w(uv + uz) + f(vu, z)w \\ &= w \cdot uv + w \cdot uz + g(uv \cdot w, uz) + f(vu, z)w, \end{aligned}$$

whereas

$$\begin{aligned} w\cdot u(v+z) &= f(wu,v+z)(v+z)+f(w,v+z)(vu+zu)+q(v+z)wu\\ &= \big(f(wu,v)v+f(w,v)vu+q(v)wu\big)\\ &\qquad+\big(f(wu,z)z+f(w,z)zu+q(z)wu\big)\\ &\qquad+f(wu,v)z+f(wu,z)v+f(w,v)zu\\ &\qquad+f(w,z)vu+f(v,z)wu\\ &= w\cdot uv+w\cdot uz+f(wu,v)z+f(wu,z)v+f(w,v)zu\\ &\qquad+f(w,z)vu+f(v,z)wu. \end{aligned}$$

by several applications of Proposition 7.5. Thus (2) holds.

Using 7.4 (2) and Proposition 7.8, we obtain

$$uv\cdot w+uw\cdot v=g(v,f(v,wu)w)+f(v,w)^\theta u$$

and

$$\begin{aligned} f(uv,zw)+f(uw,zv) &= f(uv\cdot w,z)+f(uw\cdot v,z)\\ &= f(uv\cdot w+uw\cdot v,z)\\ &= f(f(v,w)^\theta u,z)=f(u,z)f(v,w)^\theta. \end{aligned}$$

Thus (3) holds.

We have

$$uv\cdot vu=f(u,uv\cdot v)u+f(u,uv)uv+q(u)uv\cdot v$$

by Proposition 7.5 and

$$uv\cdot v=uv\cdot q(v)v^{-1}=q(v)^\theta u$$

by 7.12 (2) and (R2). Hence

$$uv\cdot vu=q(u)q(v)^\theta v=q(uv)u$$

by 7.4 (1), Proposition 7.9 and 7.12 (2). Thus (4) holds.

By Propositions 7.5 and 7.9 and (R5), we have

$$\begin{aligned} uv\cdot vw &= f(uv\cdot v,w)w+f(uv,w)wv+q(w)uv\cdot v\\ &= f(u,w)q(v)^\theta w+f(uv,w)wv+q(wv)u. \end{aligned}$$

Thus (5) holds.

Using 7.8, we have

$$\begin{aligned} \overbrace{u}^{P}\overbrace{v}^{U}\cdot\overbrace{vw}^{V}+(u\cdot vw)\cdot v &= g(U,f(U,VP)V)+f(U,V)^\theta P\\ &= g(v,f(v,vw\cdot u)vw)+f(v,vw)^\theta u=0. \end{aligned}$$

Thus (6) holds.

By (4), we have $vv \cdot vv = q(vv) \cdot v$. Hence

$$v = q(vv)^{-1} vv \cdot vv = (vv)^{-1}(vv)$$

and thus (7) holds.

Using (R3), Proposition 7.9 and (4), finally, we have

$$u^{-1}u \cdot u^{-1}u = q(u)^{-1}q(u)^{-\theta} uu \cdot uu = q(uu)^{-1}q(uu)u = u.$$

Thus (8) holds. □

PROPOSITION 7.15. — *For each nonzero $a \in V$, there exists $b \in V$ such that $a = b^{-1}b$.*

Proof. — This holds by 7.14(7). □

8. Tits Endomorphisms

We begin this section by proving some elementary properties of arbitrary Tits endomorphisms in Propositions 8.2 and 8.3.

DEFINITION 8.1. — *Let (K, θ) be an octagonal set as defined in Definition 2.1. We will call an element of K a* Tits trace *(with respect to θ) if it is of the form $x^\theta + x$ for some $x \in K$.*

PROPOSITION 8.2. — *Let (K, θ) be an octagonal set. Then the following hold:*

(1) *The map $x \mapsto x^\theta + x$ from K to itself is additive.*
(2) *If $u^\theta = u$ for some $u \in K$, then $u = 0$ or 1.*
(3) *If $u^\theta + u = v^\theta + v$ for $u, v \in K$, then either $u = v$ or $u = v + 1$.*
(4) *If z^θ is a Tits trace, then so is z.*
(5) *$u^2 + u$ is a Tits trace for every $u \in K$.*
(6) 1 *is not a Tits trace.*

Proof. — Since θ is additive, (1) holds. Suppose that $u^\theta = u$ for some $u \in K$, then $u^2 = u$ and hence $u = 0$ or 1. Thus (2) holds, and (3) follows from (1) and (2). Suppose that $z^\theta = u^\theta + u$ for some $u \in K$. Let $v = u + z$. Then $v^\theta = u$ and hence $z = v^\theta + v$. Thus (4) holds. Applying the map $x \mapsto x^\theta + x$ twice to an element $u \in K$ yields $u^2 + u$. Thus (5) holds. Suppose, finally, that $u^\theta + u = 1$ for some $u \in K$. Then $u^2 = (u^\theta)^\theta = (u+1)^\theta = u^\theta + 1 = u$ and hence $u = 0$ or 1. Thus (6) holds. □

Proposition 8.3. — *Let θ be a Tits endomorphism of a field K, let $\delta \in K$, let L be the splitting field over K of the polynomial $x^2+x+\delta$ and let χ be the non-trivial element of* $\mathrm{Gal}(L/K)$*. Then the following hold:*

(1) *θ extends to a Tits endomorphism of L if and only if δ is a Tits trace.*
(2) *If θ extends to a Tits endomorphism of L, then there are exactly two extensions θ_1 and θ_2, both commute with χ and $\chi = \theta_2^{-1} \cdot \theta_1$.*

Proof. — Let γ be a root of $x^2+x+\delta$ in L. Suppose that $\delta = \lambda^\theta + \lambda$ for some $\lambda \in K$. We extend θ to an endomorphism θ_1 of L by setting $\gamma^{\theta_1} = \gamma + \lambda$ and

$$(a + b\gamma)^{\theta_1} = a^\theta + b^\theta \gamma^{\theta_1}$$

for all $a, b \in K$. We have

$$\begin{aligned}(\gamma+\lambda)^{\theta_1} &= \gamma^{\theta_1} + \lambda^\theta \\ &= \gamma + \lambda + \lambda^\theta \\ &= \gamma + \delta = \gamma^2.\end{aligned}$$

Hence $\theta_1^2 = \mathrm{Frob}_L$. Thus θ_1 is a Tits endomorphism of L.

Suppose, conversely, that θ extends to a Tits endomorphism θ_1 of L. Then $\gamma^{\theta_1} = a + b\gamma$ for some $a, b \in K$. Therefore

$$\begin{aligned}\gamma + \delta = \gamma^2 &= a^\theta + b^\theta \gamma^{\theta_1} \\ &= a^\theta + b^\theta (a + b\gamma) \\ &= a^\theta + ab^\theta + b^{\theta+1}\gamma,\end{aligned}$$

so $\delta = a^\theta + ab^\theta$ and $b^{\theta+1} = 1$. Hence $b = b^{\theta^2-1} = (b^{\theta+1})^{\theta-1} = 1$ and therefore

$$\delta = a^\theta + a, \tag{8.4}$$

so δ is a Tits trace and hence (1) holds. Furthermore

$$\gamma^{\chi\theta_1} = \gamma^{\theta_1} + 1 = a + 1 + \gamma = a + \gamma^\chi = \gamma^{\theta_1\chi},$$

so θ_1 commutes with χ and thus the product $\theta_2 := \chi\theta_1$ is a second Tits endomorphism of L extending θ. By 8.2(3) and (8.4), there are no others. □

We now go back to assuming that S, K, V, q, f and $\Xi = \mathcal{Q}(S)$ are as in Notations 5.6 and 5.7, that ρ, φ and θ are as is as in Hypothesis 6.1 and Notations 6.4 and 6.8, and that the product uv on V is as in (6.18). Our goal for the rest of this section is to prove Proposition 8.5.

Proposition 8.5. — *Let d be a non-zero element of V. Then there exists an element $e \in V$ such that $f(d,e) = 1$, $f(d,ed) = 0$,*

$$f(dd,ed) = q(d)^\theta \tag{8.6}$$

as well as

$$q(d)^\theta \cdot de = f(de,ed) \cdot dd + q(d) \cdot ed \tag{8.7}$$

and $q(d)q(e) = f(de,ed) + f(de,ed)^\theta$. In particular, $q(d)q(e)$ is a Tits trace.

Proof. — By [2, Theorem 2.1 (i)], there exists $e \in V$ such that

$$f(d,e) = 1 \text{ and } f(d,ed) = 0\,. \tag{8.8}$$

By 7.4 (2), it follows that

$$f(dd,e) = 0\,. \tag{8.9}$$

By 7.4 (1), we have

$$f(d,dd) = 0 \tag{8.10}$$

and by 7.4 (2), we have

$$f(d, dd \cdot d) = f(dd,dd) = 0\,. \tag{8.11}$$

By 7.11 (1) and (8.8), we have

$$f(dd,de) = f(d,ed)^\theta + q(d)f(d,e)^\theta = q(d) \tag{8.12}$$

and by 7.14 (2) and (8.8), we have

$$f(dd,ed) = q(d)^\theta \cdot f(d,e) = q(d)^\theta.$$

Thus (8.6) holds.

Choose $\lambda \in K$. The image of the map g is the radical of f. By (R1), (R3) and (R7), therefore, we have

$$\begin{aligned} f\big((e+\lambda\cdot dd)(e+\lambda\cdot dd),d\big) = f(ee,d) + \lambda f(dd\cdot e,d) + \lambda^\theta f(e\cdot dd,d) \\ + \lambda^{\theta+1} f(dd\cdot dd,d)\,. \end{aligned}$$

We observe that $f(dd\cdot e,d) = f(dd,de) = q(d)$ by 7.4 (2) and (8.12) as well as

$$\begin{aligned} f(e\cdot dd,d) &= f(e,d\cdot dd) && \text{by 7.4 (2)} \\ &= q(d)f(e,dd) && \text{by (R6)} \\ &= 0 && \text{by (8.9),} \end{aligned}$$

and

$$f(dd\cdot dd,d) = f(q(dd)\cdot d,d) = q(dd)f(d,d) = 0$$

by 7.14 (4). It follows that

$$f\big((e+\lambda\cdot dd)(e+\lambda\cdot dd),d\big)=f(ee,d)+\lambda q(d)\,.$$

Furthermore,

$$f(d,e+\lambda\cdot dd)=f(d,e)=1$$

by (8.8) and (8.10) and $f(d,(e+\lambda\cdot dd)d)=f(d,ed)+\lambda f(d,dd\cdot d)$ by (R1), so, in fact,

$$f\big(d,(e+\lambda\cdot dd)d\big)=0$$

by (8.8) and (8.11). Thus if we replace e by $e+\lambda\cdot dd$ and λ by $f(ee,d)/q(d)$, we can assume that $f(ee,d)=0$ while (8.8) and therefore also (8.6) remain valid. Hence also

$$f(e,de)=0 \tag{8.13}$$

by 7.4 (2). By 7.11 (1), it follows that

$$f(ed,ee)=f(e,de)^\theta+q(e)f(d,e)=q(e). \tag{8.14}$$

By 7.4 (1) and (8.13), we have $de\in\langle d,e\rangle^\perp$ (where $\langle d,e\rangle^\perp$ denotes the subspace orthogonal to $\langle d,e\rangle$ with respect to the bilinear form f). Setting $\xi=d$ in [2, Theorem 2.1], we obtain $\langle d,e\rangle^\perp=\langle dd,ed\rangle+[K]$. Hence

$$de=\kappa\cdot dd+\mu\cdot ed+[t] \tag{8.15}$$

for some $\kappa,\mu,t\in K$. Therefore

$$f(dd,de)=f(dd,\kappa\cdot dd+\mu\cdot ed)=\mu\cdot f(dd,ed)=\mu\cdot q(d)^\theta$$

and

$$f(de,ed)=f(\kappa\cdot dd+\mu\cdot ed,ed)=\kappa\cdot f(dd,ed)=\kappa\cdot q(d)^\theta$$

by (8.6), so

$$\mu=q(d)^{1-\theta} \tag{8.16}$$

by (8.12) and

$$\kappa=f(de,ed)/q(d)^\theta. \tag{8.17}$$

Substituting (8.16) but not (8.17) in (8.15) and applying q, we obtain

$$\begin{aligned}q(de)=q(d)q(e)^\theta=\kappa^2q(d)^{\theta+1}+q(d)^{2-2\theta}\cdot q(e)q(d)^\theta\\ +t^\theta+\kappa q(d)^{1-\theta}f(dd,ed)\end{aligned}$$

by Propositions 7.9 and 7.10. Multiplying by $q(d)^{\theta-1}$ and applying (8.6), we then obtain

$$q(d)^\theta q(e)^\theta=\kappa^2q(d)^{2\theta}+\kappa q(d)^\theta+q(d)q(e)+t^\theta q(d)^{\theta-1}.$$

Thus $\kappa q(d)^\theta$ is a root of the polynomial

$$p(x) := x^2 + x + q(d)q(e) + q(d)^\theta q(e)^\theta + t^\theta q(d)^{\theta-1}. \tag{8.18}$$

By 8.17, $\kappa q(d)^\theta = f(de, ed)$ and by 7.14 (3) and (8.8), we have

$$f(de, ed) + f(dd, ee) = f(d, e)f(e, d)^\theta = 1. \tag{8.19}$$

We conclude that $f(de, ed)$ and $f(dd, ee)$ are the two roots of the polynomial p.

Next we observe that

$$\begin{aligned} q(e)^\theta &= q(e)^\theta f(d, e) && \text{by (8.8)} \\ &= f(de, ee) && \text{by 7.11 (2)} \\ &= \kappa \cdot f(dd, ee) + \mu \cdot q(e) && \text{by (8.14) and (8.15)} \\ &= \big(f(de, ed) \cdot f(dd, ee) + q(d)q(e)\big)/q(d)^\theta && \text{by (8.16) and (8.17),} \end{aligned}$$

so

$$f(de, ed) \cdot f(dd, ee) = q(d)q(e) + q(d)^\theta q(e)^\theta.$$

Thus by (8.19), $f(de, ed)$ is a root of the polynomial.

$$x^2 + x + q(d)q(e) + q(d)^\theta q(e)^\theta.$$

Since $f(de, ed)$ is also a root of the polynomial $p(x)$ defined in (8.18), we conclude that $t = 0$. Hence

$$\begin{aligned} q(d)^\theta \cdot de &= q(d)^\theta \big(\kappa \cdot dd + \mu \cdot ed\big) && \text{by (8.15)} \\ &= f(de, ed) \cdot dd + q(d) \cdot ed && \text{by (8.16) and (8.17).} \end{aligned}$$

Thus (8.7) holds.

Let $s = f(de, ed)$. We multiply (8.7) on the left by d. Applying (R3) and (R7), we obtain

$$q(d)^2 d \cdot de = s^\theta d \cdot dd + q(d)^\theta d \cdot ed + g\big(s \cdot (dd \cdot d), q(d) \cdot ed\big). \tag{8.20}$$

By Proposition 7.5, (8.8) and (8.9), we have

$$d \cdot de = f(e, dd)e + f(e, d)ed + q(e)dd = ed + q(e)dd.$$

By (R6), we have $d \cdot dd = q(d)dd$ and $d \cdot ed = q(d)de$. We also have

$$g(dd \cdot d, ed) = g(q(d)^\theta d, ed) = g(q(d)^\theta dd, e) = 0$$

by (R5) and (8.9). Hence we can rewrite (8.20) as

$$q(d)^2 \cdot ed + q(d)^2 q(e) \cdot dd = s^\theta q(d) \cdot dd + q(d)^{\theta+1} \cdot de.$$

Dividing by $q(d)$ and rearranging terms, we obtain

$$q(d)^\theta \cdot de = \big(s^\theta + q(d)q(e)\big) \cdot dd + q(d) \cdot ed.$$

Comparing this equation with (8.7), we conclude that

$$s^\theta + q(d)q(e) = s\,.$$

In other words, $q(d)q(e) = f(de, ed) + f(de, ed)^\theta$. □

9. Polar Triples

We continue to assume that S, K, F, V, q, f, $\Xi = \mathcal{Q}(S)$, Σ, c and $x_1, \dots, x_4$ are as in Notations 5.6 and 5.7, that ρ, φ and θ are as is as in Hypothesis 6.1 and Notations 6.4 and 6.8, and that the product uv on V is as in (6.18). Thus $F = K^\theta$, θ is a Tits endomorphism of K by 6.10(1) and

$$\Xi = \mathcal{Q}_{\mathcal{F}}(K, V, q) = B_2^{\mathcal{F}}(K, V, q)$$

by Notation 5.10. The main results of this section are Theorem 9.12 and Corollary 9.26.

Proposition 9.1. — *Let d and e as in Proposition 8.5 and let $\xi = \varphi(d)$, so that $f(d, e) = 1$ and $f(d, e\xi) = 0$. Let*

$$S_0 := (L/K, K^\theta, \alpha_0, \beta_0)$$

be the standard decomposition of q obtained by applying Theorem 5.16 to the triple d, e and ξ. Then $\alpha_0 = \beta_0^{-\theta}$ and θ has an extension to a Tits endomorphism of L.

Proof. — By Remark 5.15, the field L is the splitting field of the polynomial

$$x^2 + x + q(d)q(e)$$

over K. Hence 8.3(1) and the last assertion in Proposition 8.5, θ has an extension to a Tits endomorphism of L. By Theorem 5.16, we have $\alpha_0 = f(d\xi, e\xi)$ and $\beta_0 = q(d)^{-1}$. Thus by (8.6), $\alpha_0 = q(d)^\theta = \beta_0^{-\theta}$. □

Hypothesis 9.2. — *We assume from now on that d and e are as in Proposition 8.5 and that $\xi = \varphi(d)$ and that the standard decomposition S in Notation 5.6 is the standard decomposition S_0 in Proposition 9.1. Thus E/K is now the extension called L/K and α and β are now the constants called α_0 and β_0 in Proposition 9.1, the Tits endomorphism θ has an extension to the field E and $\beta^\theta = \alpha^{-1}$. The group U_+ is unchanged by this assumption, but we assume that V, $\hat{V}$ and the isomorphisms $x_1, \dots, x_4$ are as in Notation 5.7 with respect to the new S.*

Notation 9.3. — By 8.3 (2)), θ has exactly two extensions to E. We denote these extensions by θ_1 and θ_2. Both commute with the non-trivial element χ of $\mathrm{Gal}(E/K)$ and $\theta_2 = \chi\theta_1$. Let $\bar{x} = x^\chi$ for all $x \in E$.

Remark 9.4. — Let θ_1, θ_2 and χ be as in Notation 9.3, let $\gamma \in E$ be a root of

$$x^2 + x + q(d)q(e)$$

and let $i = 1$ or 2. Since χ commutes with θ_i, we have $\gamma^{\theta_i} \notin K$. Thus $E = K(\gamma^{\theta_i})$ and hence

$$D = E^2F = F(\gamma^2) = K(\gamma^{\theta_i})^{\theta_i} = E^{\theta_i}.$$

Proposition 9.5. — *For either $i = 1$ or $i = 2$, the map φ from $V = E \oplus E \oplus [F]_K$ to $\hat{V} = D \oplus D \oplus [K]_F$ is given by*

$$\varphi(a, b, s) = (a^{\theta_i}, \beta^{-2}b^{\theta_i}, s^{\theta^{-1}})$$

for all $(a, b, s) \in V$.

Proof. — Let $\eta = \varphi(e)$. Then by (R5) and (8.7), we have

$$d\eta \cdot \xi \in \langle d, e\rangle.$$

Applying φ, we obtain

$$\xi e \cdot d \in \langle \xi, \eta\rangle \tag{9.6}$$

by 6.7 (1). By (6.9) and 7.4 (2), we have

$$\begin{aligned}\hat{f}(\xi e \cdot d, \xi) &= \hat{f}(\varphi(d\eta \cdot \xi), \varphi(d)) \\ &= f(d\eta \cdot \xi, d)^\theta = f(d\eta, d\xi)^\theta.\end{aligned}$$

By (8.12), it follows that $\hat{f}(\xi e \cdot d, \xi) \neq 0$. Thus $\xi e \cdot d$ and ξ are linearly independent. By (9.6), therefore,

$$\eta \in \langle \xi, \xi e \cdot d\rangle. \tag{9.7}$$

By 6.7 (1) again, we have $\varphi(e\xi) = \eta d$ and $\varphi(d\xi) = \xi d$. Hence by (R1), (R5) and (9.7),

$$\varphi(e\xi) \in \langle \xi d, \xi e\rangle.$$

By 6.10 (2), φ is an isomorphism of vector spaces. Thus

$$\varphi(\langle d, e\rangle) = \langle \xi, \xi e \cdot d\rangle \tag{9.8}$$

and

$$\varphi(\langle d\xi, e\xi\rangle) = \langle \xi d, \xi e\rangle. \tag{9.9}$$

Let γ, χ, θ_1 and θ_2 be as in Remark 9.4 and let $N(x) = x \cdot x^\chi$ and $T(x) = x + x^\chi$ for all $x \in E$. We set $\omega = \beta\gamma$. Thus

$$\hat{q}(\xi) = \hat{q}(\varphi(d)) = q(d)^\theta = \beta^{-\theta} = \alpha$$

and ω is a root in E of $q(d)x^2 + x + q(e)$. By Theorem 5.16, we can assume that $d = (1, 0, 0)$, $e = (\omega, 0, 0)$, $d\xi = (0, 1, 0)$ and $e\xi = (0, \omega, 0)$ in V and $\xi = (1, 0, 0)$, $\xi e \cdot d^{-1} = (\omega^2, 0, 0)$, $\xi d^{-1} = (0, 1, 0)$ and $\beta^2\xi e = (0, \omega^2, 0)$ in $\hat{V}$. Thus $\varphi(d) = \xi = (1, 0, 0)$.

By Notation 6.8, (9.8) and (9.9), there exist maps φ_1 and φ_2 from E to D such that

$$\varphi(a, b, s) = (\varphi_1(a), \varphi_2(b), s^{\theta^{-1}}) \tag{9.10}$$

for all $(a, b, s) \in V$. By 6.10 (2), φ_1 and φ_2 are θ-linear and since $\varphi(d) = \xi$, we have $\varphi_1(1) = 1$. Furthermore, $\hat{f}(\xi, \eta) = \hat{f}(\varphi(d), \varphi(e)) = f(d, e)^\theta = 1$ and $\hat{q}(\eta) = \hat{q}(\varphi(e)) = q(e)^\theta$. By (5.5), therefore, $T(\varphi_1(\omega)) = \alpha^{-1}$ and $N(\varphi_1(\omega)) = \alpha^{-1}q(e)^\theta$. Thus $\varphi_1(\omega)$ is a root of $q(d)^\theta x^2 + x + q(e)^\theta$. Hence we can choose $i \in \{1, 2\}$ such that

$$\varphi_1(\omega) = \omega^{\theta_i}.$$

Since $\varphi_1(1) = 1$ and φ_1 is θ-linear, it follows that

$$\varphi_1 = \theta_i\,. \tag{9.11}$$

Let $v \in E$. By [23, 16.7], we have

$$[x_1(1, 0, 0), x_4(0, v, 0)]_2 = x_2(\alpha v, 0, 0)$$

and

$$[x_4(1, 0, 0), x_1(0, \varphi_2(v), 0)]_3 = x_3(\beta^{-2}\varphi_2(v), 0, 0).$$

Applying ρ to the first of these equations, we obtain

$$[x_4(1, 0, 0), x_1(0, \varphi_2(v), 0)]_3 = x_3(\varphi_1(\alpha v), 0, 0).$$

Hence

$$\varphi_2(x) = \varphi_1(\alpha x) = \beta^{-2}\varphi_1(x) = \beta^{-2}x^{\theta_i}.$$

Thus by (9.10) and (9.11), we have

$$\varphi(a, b, s) = (a^{\theta_i}, \beta^{-2}b^{\theta_i}, s^{\theta^{-1}})$$

for all $(a, b, s) \in V$. □

We summarize our results as follows:

Theorem 9.12. — *Let $\Xi = B_2^{\mathcal{F}}(K,V,q)$ for some quadratic space*

$$(K,V,q)$$

of type F_4 and suppose that ρ is a polarity of Ξ. Then there exists a standard decomposition

$$S = (E/K, F, \alpha, \beta)$$

of q and a Tits endomorphism θ of E such that the following hold:

(1) $F = K^\theta$.
(2) $\alpha = \beta^{-\theta}$.
(3) *Ξ can be identified with $\mathcal{Q}(S)$ in such a way that ρ stabilizes Σ and c and $x_i(u,v,s)^\rho = x_{5-i}(u^\theta, \beta^{-2}v^\theta, s^{\theta^{-1}})$ for $i=2$ and 4 and all $(u,v,s) \in V$, where Σ, c, V and x_i for $i \in [1,4]$ are as in Notation 5.7 applied to S.*

Proof. — By Definition 5.3, we can choose a standard decomposition S of q and by Notation 5.10, $\Xi \cong \mathcal{Q}(S)$. Let Σ and c be as in Notation 5.7 applied to S. By Remark 6.2, there exists an isomorphism ξ_S from Ξ with $\mathcal{Q}(S)$ such that $\xi_S^{-1}\rho\xi_S$ stabilizes c and Σ. By Proposition 9.1, the standard decomposition S and a Tits endomorphism θ of E can be chosen so that (1) and (2) hold. If we identify Ξ with $\mathcal{Q}(S)$ via ξ_S and replace ρ by $\xi^{-1}\rho\xi$ for his choice of S, then (3) holds by Proposition 9.5. □

Remark 9.13. — In Example 10.1 we give an example of Ξ, ρ, $S = (E/K, F, \alpha, \beta)$ and θ satisfying the conditions (1) and (2) in Theorem 9.12 and a splitting field $\tilde{E}$ of q_S (as defined in Definition 5.3) such that the restriction of θ to K does *not* have an extension to a Tits endomorphism of $\tilde{E}$. See also Proposition 10.4. In Example 10.12, we give an example of a Moufang quadrangle of type F_4 that has non-type-preserving automorphisms but no polarity.

Notation 9.14. — Let Ξ, ρ, $S = (E/K, F, \alpha, \beta)$, θ, Σ, c and the identification of Ξ with $\mathcal{Q}(S)$ be as in Theorem 9.12, let

$$(U_+, U_1, \ldots, U_4)$$

and $x_1, \ldots, x_4$ be as in Notation 5.7 applied to S, let $x \mapsto \bar{x}$ be as in Notation 9.3 and let ι denote the map $(a,b,s) \mapsto (a,b,s^{\theta^{-1}})$ from $V_S = E \oplus E \oplus [F]_K$ to $E \oplus E \oplus [K]$, where $[K]$ is as in Notation 6.21. We identify $V = V_S$ with its image under ι and we reparametrize U_+ by replacing x_i by $\varphi \cdot x_i$ for $i = 1$ and 3 as in Notation 6.16 and then replacing x_i by

$\iota \cdot x_i$ for all $i \in [1,4]$. Thus x_i is an isomorphism from the additive group of $V = E \oplus E \oplus [K]$ to U_i for all $i \in [1,4]$ and the following identities hold:

$$[x_1(a,b,r), x_3(a',b',r')] = x_2\big(0,\ 0,\ \beta^{-1}\big(a\bar{a}' + \bar{a}a' + \alpha(b\bar{b}' + \bar{b}b')\big)\big),$$

$$[x_2(u,v,s), x_4(u',v',s')] = x_3\big(0,\ 0,\ \beta^{-1}\big(u\bar{u}' + \bar{u}u' + \alpha(v\bar{v}' + \bar{v}v')\big)\big),$$

$$\begin{aligned}[x_1(a,b,r), x_4(u,v,s)] &\\ = x_2\big(&ru + \alpha(\bar{a}^\theta v + \beta^{-1}b^\theta\bar{v}),\ rv + a^\theta u + \beta^{-1}b^\theta\bar{u},\\ &r^\theta s + \beta^{-1}s(a\bar{a} + \alpha b\bar{b})\\ &\quad + \alpha\beta^{-1}\big(u^\theta a\bar{b} + \bar{u}^\theta\bar{a}b + \beta^{-1}(v^\theta\bar{a}\bar{b} + \bar{v}^\theta ab)\big)\big)\\ \cdot\, x_3\big(&sa + \alpha(\bar{u}^\theta b + \beta^{-1}v^\theta\bar{b}),\ sb + u^\theta a + \beta^{-1}v^\theta\bar{a},\\ &s^\theta r + \beta^{-1}r(u\bar{u} + \alpha v\bar{v})\\ &\quad + \alpha\beta^{-1}\big(a^\theta u\bar{v} + \bar{a}^\theta\bar{u}v + \beta^{-1}(b^\theta\bar{u}\bar{v} + \bar{b}^\theta uv)\big)\big)\end{aligned}$$

for all $(a,b,r), (a',b',r'), (u,v,s), (u',v',s') \in V$,

$$[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$$

and

$$x_i(v)^\rho = x_{5-i}(v) \tag{9.15}$$

for all $i \in [1,4]$ and all $v \in V$.

Proposition 9.16. — *Let Ξ, $S = (E/K, F, \alpha, \beta)$, θ, ρ and the identification of Ξ with $\mathcal{Q}(S)$ be as in Theorem 9.12, let $\cdot$ be the multiplication on V defined in* (6.18)*, let $x \mapsto \bar{x}$ be as in Notation 9.3 and let V be identified with $E \oplus E \oplus [K]$ as in Notation 9.14. Then*

$$\begin{aligned}(a,b,r)\cdot(u,v,s) = \big(&sa + \alpha(\bar{u}^\theta b + \beta^{-1}v^\theta\bar{b}),\ sb + u^\theta a + \beta^{-1}v^\theta\bar{a},\\ &s^\theta r + \beta^{-1}r(u\bar{u} + \alpha v\bar{v})\\ &\quad + \alpha\beta^{-1}\big(a^\theta u\bar{v} + \bar{a}^\theta\bar{u}v + \beta^{-1}(b^\theta\bar{u}\bar{v} + \bar{b}^\theta uv)\big)\end{aligned}$$

for all $(a,b,r), (u,v,s) \in V$.

Proof. — This holds by (6.20) and Notation 9.14. □

Notation 9.17. — Let Ξ, (K,V,q), θ and $S = (E/K, F, \alpha, \beta)$ and the identification of Ξ with $\mathcal{Q}(S)$ be as in Theorem 9.12, so $F = K^\theta$ and $\alpha = \beta^{-\theta}$. We identify V with $E \oplus E \oplus [K]$ as in Notation 9.14, so that

$$q(u,v,t) = \beta^{-1}(N(u) + \alpha N(v)) + t^\theta \tag{9.18}$$

for all $(u,v,t) \in V$. Let $[t] = (0,0,t)$ for all $t \in K$ and let $\cdot$ be the multiplication on V given by the formula in Proposition 9.16. By Proposition 6.23,

$(K, V, q, \theta, t \mapsto [t], \cdot)$ is a polarity algebra. We denote this polarity algebra by $A = A(E/K, \theta, \beta)$.

Definition 9.19. — *A* polar triple *is a triple $(E/K, \theta, \beta)$, where E/K is a separable quadratic extension in characteristic 2, θ is a Tits endomorphism of E such that $F := K^\theta \subset K$ and β is an element of K such that the quadratic form on $E \oplus E \oplus [K]$ given by (9.18) is anisotropic, where $[K]$ is as defined in Notation 6.21.*

In the next result, we show that every polarity algebra is of the form $A(E/K, \theta, \beta)$ for some polar triple $(E/K, \theta, \beta)$ as defined in Definition 9.19. See also Theorem 19.1.

Theorem 9.20. — *Let $P = (K, V, q, \theta, t \mapsto [t], \cdot)$ be a polarity algebra as defined in Definition 7.1. Then q is a quadratic form of type F_4 and there exists:*

(1) *a standard decomposition $S = (E/K, F, \alpha, \beta)$ of q such that $\alpha = \beta^{-\theta}$ and $F = K^\theta$,*
(2) *an extension of θ to a Tits endomorphism of E and*
(3) *an identification of V with $E \oplus E \oplus [K]$ with respect to which $t \mapsto [t]$ is the map $t \mapsto (0, 0, t)$, $\cdot$ is given by the formula in Proposition 9.16 and*
$$q(u, v, t) = \beta^{-1}(N(u) + \beta^\theta N(v)) + t^\theta$$
for all $(u, v, t) \in E \oplus E \oplus [K]$, where N is the norm of the extension E/K.

Proof. — Let $F := K^\theta$, and let $\hat{V}$ be the set consisting of the symbols $\hat{v}$ for all $v \in V$, i.e. the map $v \mapsto \hat{v}$ is a bijection from V to $\hat{V}$. We make $\hat{V}$ into an F-vector space by defining
$$s \cdot \hat{v} := \widehat{s^{\theta^{-1}} v}$$
for all $s \in F$ and all $\hat{v} \in \hat{V}$, or equivalently,
$$t^\theta \cdot \hat{v} := \widehat{tv} \tag{9.21}$$
for all $t \in K$ and all $v \in V$. The map $\hat{q} \colon \hat{V} \to F$ given by
$$\hat{q}(\hat{v}) := q(v)^\theta \tag{9.22}$$
for all $\hat{v} \in \hat{V}$ is a quadratic form over F. For each $t \in K$, we define
$$[t]_F := \widehat{[t]} \in \widehat{[K]} \subset \hat{V}, \tag{9.23}$$
and for each $s \in F$, we define
$$[s]_K := [s^{\theta^{-1}}] \in [K] \subset V. \tag{9.24}$$

Next we define maps from $V \times \hat{V}$ to V and from $\hat{V} \times V$ to $\hat{V}$ (both denoted by juxtaposition) by

$$v\hat{w} = vw \qquad \text{and} \qquad \hat{v}w = \widehat{vw} \tag{9.25}$$

for all $v, w \in V$, where the multiplication on the right hand side of both equations is the multiplication of the polarity algebra. We claim that these data satisfy the axioms (F0)–(F12) of 5.18. To illustrate this, we will prove (F2), (F4) and (F7) and leave the verification of the other axioms to the reader.

So let $v, w \in V$, $s \in F$, and $t \in K$; then using (9.24), (9.25), (R2) and (9.21), we obtain

$$\hat{v}[s]_K = \hat{v}[s^{\theta^{-1}}] = \widehat{v[s^{\theta^{-1}}]} = \widehat{s^{\theta^{-1}}v} = s\hat{v}.$$

Thus (F2) holds. Next, by (9.25), (R3), (9.21) again and (F2), we obtain

$$\hat{v} \cdot tw = \widehat{v \cdot tw} = \widehat{t^\theta \cdot vw} = (t^\theta)^\theta \cdot \widehat{vw} = t^2\hat{v}w = \hat{v}w \cdot [t^2]_K.$$

Thus (F4) holds. By (9.25), (R5), (9.22) and (F1), we obtain

$$v\hat{w} \cdot \hat{w} = vw \cdot w = q(w)^\theta \cdot v = \hat{q}(\hat{w}) \cdot v = v \cdot [\hat{q}(\hat{w})]_F.$$

Thus (F7) holds. This (together with the proof of the remaining identities) shows that V and $\hat{V}$, together with the maps we just defined, form a radical quadrangular system as defined in [3, Appendix A.3.2]. We denote this quadrangular system by Θ.

Let $\Omega = (U_+, U_1, \ldots, U_4)$ and $x_1, \cdots, x_4$ be as in Notation 5.19. By [3, Chapter 4], Ω is a root group sequence and thus Ω determines a unique Moufang quadrangle Ξ by [23, 7.5 and 8.5]. It follows from (9.22), (9.23), (9.24) and (9.25) that there is a unique anti-automorphism ρ of Ω extending the maps $x_i(\hat{v}) \mapsto x_{5-i}(v)$ for $i = 1$ and 3, and $x_i(v) \mapsto x_{5-i}(\hat{v})$ for $i = 2$ and 4. The square ρ^2 centralizes U_+. Thus ρ induces a polarity of the Moufang quadrangle Ξ (by [23, 7.5]).

By Definition 7.1, ∂f is not identically zero. By Notation 5.19, therefore, $[U_2, U_4] \neq 1$. By [23, 17.4], $[U_2, U_4] \neq 1$ and the existence of an anti-automorphism of Ω imply that Ξ is a quadrangle of type F_4. In other words, Ξ is isomorphic to the quadrangle $\mathcal{Q}_{\mathcal{F}}(\tilde{\Lambda}) = B_2^{\mathcal{F}}(\tilde{\Lambda})$ obtained by applying [23, 16.7] to some quadratic space $\tilde{\Lambda} = (\tilde{K}, \tilde{V}, \tilde{q})$. of type F_4. Let $Y_1 = C_{U_1}(U_3)$, let $Y_3 = C_{U_3}(U_1)$ and let $Y_+ = Y_1U_2Y_3U_4$. By (F5) and Notation 5.19, $Y_i = x_i([K]_F)$ for $i = 1$ and 3, Y_+ is a subgroup of U_+ and $(Y_+, Y_1, U_2, Y_3, U_4)$ is a root group sequence isomorphic to the root group sequence $\mathcal{Q}_{\mathcal{Q}}(K, V, q)$ obtained by applying [23, 16.3] to (K, V, q). By [23, 16.7], on the other hand, $(Y_+, Y_1, U_2, Y_3, U_4)$ is a root group sequence isomorphic to $\mathcal{Q}_{\mathcal{Q}}(\tilde{\Lambda})$. By [23, 35.8], it follows that $K \cong \tilde{K}$ and q is similar to $\tilde{q}$. Thus q is of type F_4 and $\Xi \cong B_2^{\mathcal{F}}(K, V, q)$ (by [23, 35.12]). We conclude that the quadrangular system Θ is an extension of the quadrangular system associated with (K, V, q); see the beginning of [3, Chapter 8] for the definition of these terms. This is exactly the situation investigated in [3, §8.5] (and [23, Chapter 28]). By [3, Theorem 8.107], there exists a standard decomposition S of q and an isomorphism ξ from Ω to the root group sequence Ω_S obtained by applying Notation 5.7 to S extending the maps $x_i([t]_F) \mapsto x_i(0,0,t)$ for $i = 1$ and 3 and $x_i([s]_K) \mapsto x_i(0,0,s)$ for $i = 2$ and 4. We now replace ρ by the unique automorphism of $\mathcal{Q}(S)$ obtained by applying [23, 7.5] to the automorphism $\xi^{-1}\rho\xi$ of Ω_S. By (9.23) and (9.24), we have $x_i(0,0,t)^\rho = x_i(0,0,t^\theta)$ for all $t \in K$. Thus θ is as in Notation 6.8. By Proposition 9.1, we conclude that (1)–(3) hold. □

COROLLARY 9.26. — *Every polarity algebra is of the form $A(E/K, \theta, \beta)$ for some polar triple $(E/K, \theta, \beta)$ as defined in Definition 9.19.*

Proof. — This holds by Notation 9.17 and Theorem 9.20. □

10. Two Examples

In this section, we give two examples illustrating the results of the previous section; see Remark 9.13.

Example 10.1. — Let $K = \mathbb{F}_2(\alpha, \beta)$ be a purely transcendental extension of the field $\mathbb{F}_2$, let E be the splitting field of the polynomial

$$p(x) = x^2 + x + \alpha + \beta^2$$

over K, let $\gamma \in E$ be a root of $p(x)$, let θ denote the unique Tits endomorphism of K that maps β to α and let $F = K^\theta$. By 8.3(1), θ has an extension to a Tits endomorphism of E. We leave it to the reader to check that $S := (E/K, F, \alpha^{-1}, \beta)$ is an F_4-datum as defined in Notation 5.2. Let $q = q_S$ be the quadratic form of type F_4 on $E \oplus E \oplus [F]_K$ as defined in Notation 5.2. We claim that β is not a Tits trace of K (with respect to θ). To show this, we assume that

$$\beta = g(\alpha, \beta) + g(\alpha, \beta)^\theta = g(\alpha, \beta) + g(\beta^2, \alpha)$$

for some rational function $g(\alpha, \beta) \in K$. Let $k = \deg_\alpha(g)$ and $m = \deg_\beta(g)$. (If $g = g_1/g_2$ for polynomials g_1 and g_2 in $\mathbb{F}_2[\alpha, \beta]$ and $u = \alpha$ or β, then $\deg_u(g) = \deg_u(g_1) - \deg_u(g_2)$.) We have

$$1 = \deg_\beta(\beta) = \deg_\beta\big(g(\alpha, \beta) + g(\beta^2, \alpha)\big) \leqslant \max(2k, m) \tag{10.2}$$

and

$$1 = \max(2k, m) \quad \text{if } m \neq 2k \tag{10.3}$$

and

$$0 = \deg_\alpha(\beta) = \deg_\beta\big(g(\alpha,\beta) + g(\beta^2,\alpha)\big) \leqslant \max(k,m)$$

and $0 = \max(k,m)$ if $k \neq m$. By (10.2), we have $0 \neq \max(k,m)$. Thus $k = m \neq 0$. Hence $m \neq 2k$ and $\max(2k,m) \geqslant 2$, which is impossible by (10.3). We conclude that β is not a Tits trace in K as claimed. By 8.2 (4), it follows that also β^2 is not a Tits trace in K. Let L be the splitting field of the polynomial

$$p_1(x) = x^2 + x + \beta^2$$

over K. By 8.3 (1), θ does not have an extension to a Tits endomorphism of L. Let $d = (\beta, 0, 0)$ and $e = (\gamma, \alpha, 0)$ in V. Then $q(d) = q(e) = \beta$ and $f(d,e) = 1$, so L is also the splitting field of $q(d)x^2 + x + q(e)$ over K. Applying Theorem 5.16 with $\xi = (1,0,0) \in \hat{V}$, we conclude that L is a splitting field of q. Thus the Tits endomorphism θ of K has an extension to a Tits endomorphism of some of the splitting fields of q but there are also splitting fields of q to which θ does not have an extension to a Tits endomorphism.

Proposition 10.4. — *Let $S = (E/K, F, \alpha, \beta)$ be an F_4-datum, let $\Xi = \mathcal{Q}(S)$ and suppose that θ is a Tits endomorphism of K such that $F = K^\theta$ and $\alpha = \beta^{-\theta}$. Choose $\lambda \in K$ such that E is the splitting field of the polynomial $x^2 + x + \lambda$ over K. Then Ξ has a polarity if and only if there exists $u \in K$ such that*

$$\lambda + \alpha u^2$$

is a Tits trace with respect to θ.

Proof. — Let $q = q_S$, let $f = \partial q$ and let $\gamma \in E$ be a root of $x^2 + x + \lambda$. Let V, D and $\hat{V}$ be as in Notation 5.6, let $d = (1,0,0)$ and $e = (\beta\gamma, 0, 0)$ in V and let $\xi = (1,0,0) \in \hat{V}$. Then $q(d) = \beta^{-1}$ and $q(e) = \beta\lambda$. Hence $\omega := \beta\gamma$ is a root of $q(d)x^2 + x + q(e)$.

We suppose now that u is an element of K such that $\lambda + \alpha u^2$ is a Tits trace and let E' be the splitting field of $x^2 + x + \lambda + \alpha u^2$ over K. By 8.3 (1) and the choice of u, we can choose a Tits endomorphism θ_1 of E' extending θ. As in Remark 9.4, we have $(E')^{\theta_1} = (E')^2 F$. Since $\alpha u^2 \in F$, we can set $e' = e + (0, 0, \alpha u^2)$. Thus $f(d, e') = 1$ and, by (F0) and (F6), $f(d, e'\xi) = 0$. Applying Theorem 5.16 with e' in place of e, it follows that we can assume that $E' = E$. Now let $\varphi\colon V \to \hat{V}$ be given by the formula in Proposition 9.5 and let ρ be defined by the equations in Notation 6.4 with $\varphi_1 = \varphi$ and

$\hat{\varphi} = \hat{\varphi}_1 = \varphi^{-1}$. Then ρ is an automorphism of U_+ of order 2 mapping U_i to U_{5-i} for each $i \in [1,4]$. Thus Ξ has a polarity.

Suppose, conversely, that Ξ has a polarity ρ. Our goal is to find an element $u \in K$ such that $\lambda + \alpha u^2$ is a Tits trace. By Proposition 9.1, we can choose $e' \in V$ and $\xi' \in \hat{V}$ such that $f(d, e') = 1$, $f(d, e'\xi') = 0$ and $\hat{q}(\xi') = \alpha$ such that θ has an extension to the splitting field of $x^2 + x + q(d)q(e')$ over K. Thus $q(d)q(e')$ is a Tits trace. Since $f(d, e') = 1$, we have $e' = (t + \beta\gamma, y + z\gamma, s)$ for some $t, y, z \in K$ and some $s \in F$. Let $e'' = e' + (t, 0, 0)$. By [2, Lemma 2.1], we have $f(d, e''\xi') = 0$. We also have $q(e'') = q(e') + \beta^{-1}t^2 + t$ and hence $q(d)q(e'') + q(d)q(e') = \beta^{-2}t^2 + \beta^{-1}t$. By 8.2 (5), this expression is a Tits trace. It follows that we can assume that

$$e' = (\beta\gamma, y + z\gamma, s)\,. \tag{10.5}$$

Hence

$$\begin{aligned} q(d)q(e') &= \beta^{-2}\big(N(\beta\gamma) + \alpha(y^2 + yz + \lambda z^2)\big) + \beta^{-1}s \\ &= \lambda + \alpha\beta^{-2}(y^2 + yz + \lambda z^2) + \beta^{-1}s\,. \end{aligned}$$

We have $s = x^\theta$ for some $x \in K$ and thus

$$(\beta^{-1}s)^\theta = \alpha s^\theta = \alpha x^2.$$

By 8.2 (1), therefore,

$$\begin{aligned} p :&= \lambda + \alpha\beta^{-2}(y^2 + yz + \lambda z^2) + \alpha x^2 \\ &= \lambda + \alpha\beta^{-2}\big((y + \beta x)^2 + z(y + \lambda z)\big) \end{aligned} \tag{10.6}$$

is a Tits trace.

By (F12) and the choice of e' and ξ', we have

$$f(d\xi', e') = f(d, e'\xi') = 0. \tag{10.7}$$

By (F12), we also have $f(d\xi', d) = 0$, from which it follows that there exist $w, u, v, r \in K$ such that

$$d\xi' = (w, u + v\gamma, r^\theta)\,. \tag{10.8}$$

By [3, 8.95], we have $q(d\xi') = q(d)\hat{q}(\xi') = \beta^{-1}\alpha$. Hence

$$w^2 + \alpha(u^2 + uv + \lambda v^2 + 1) + \beta r^\theta = 0\,. \tag{10.9}$$

By (10.5), (10.7) and (10.8), we have

$$\beta w + \alpha(zu + yv) = 0\,. \tag{10.10}$$

Suppose that $v = 0$. Then $w^2 + \alpha(u^2 + 1) + \beta r^\theta = 0$ by (10.9), hence $\beta r^\theta \in F$ and therefore, $r = 0$ since $\beta \notin F$. Hence $\alpha(u + 1)^2 \in K^2$ and therefore $u = 1$ since $\alpha \notin K^2$. Hence $w = 0$. By (10.10), therefore, $z = 0$. Thus by (10.6), we have $p = \lambda + \alpha a^2$ for $a = \beta^{-1}(y + \beta x)$.

Suppose, finally, that $v \neq 0$. Then $y = v^{-1}(\alpha^{-1}\beta w + zu)$ by (10.10). Hence

$$\begin{aligned}
&\alpha\beta^{-2}z(y + \lambda z)\\
&\quad = \alpha\beta^{-2}zv^{-1}(\alpha^{-1}\beta w + zu + \lambda vz)\\
&\quad = \beta^{-1}zv^{-1}w + \beta^{-2}z^2v^{-2} \cdot \alpha(uv + \lambda v^2)\\
&\quad = \beta^{-1}zv^{-1}w + \beta^{-2}z^2v^{-2}(\alpha(u^2 + 1) + w^2 + \beta r^\theta) \qquad \text{by (10.9)}\\
&\quad = \alpha\big(\beta^{-1}zv^{-1}(u + 1)\big)^2\\
&\qquad\qquad + \beta^{-1}zv^{-1}w + (\beta^{-1}zv^{-1}w)^2 + \beta^{-1}z^2v^{-2}r^\theta.
\end{aligned}$$

By 8.2 (5) and (10.6), it follows that

$$\lambda + \alpha b^2 + \beta^{-1}z^2v^{-2}r^\theta \tag{10.11}$$

is a Tits trace for $b = \beta^{-1}\big(zv^{-1}(u + 1) + (y + \beta x)\big)$. Adding the Tits trace

$$\beta^{-1}z^2v^{-2}r^\theta + (\beta^{-1}z^2v^{-2}r^\theta)^\theta$$

to the expression (10.11), we conclude that

$$\lambda + \alpha b^2 + (\beta^{-1}z^2v^{-2}r^\theta)^\theta$$

is also a Tits trace. Finally, we observe that

$$(\beta^{-1}z^2v^{-2}r^\theta)^\theta = \alpha c^2$$

for $c = z^\theta v^{-\theta}r$. Thus $\lambda + \alpha(b + c)^2$ is a Tits trace. □

Example 10.12. — Let $K = \mathbb{F}_2(\alpha, \beta)$ be a purely transcendental extension of the field $\mathbb{F}_2$, let E be the splitting field of the polynomial

$$p(x) = x^2 + x + 1$$

over K, let $\gamma \in E$ be a root of $p(x)$, let θ denote the unique Tits endomorphism of K that maps β to α and let $F = K^\theta$. By [23, 14.25], $S := (E/K, F, \alpha^{-1}, \beta)$ is an F_4-datum, so we can set $\Xi = \mathcal{Q}(S)$. There are exactly three elements of E^* of finite order. Let $\hat\theta$ be the unique extension of θ to an endomorphism of E which acts trivially on these three elements and let χ denote the non-trivial element of $\mathrm{Gal}(E/K)$. The endomorphism $\hat\theta$ is, of course, not a Tits endomorphism of E. (By 8.2 (6) and 8.3 (1), θ does not have an extension to a Tits endomorphism of E.) Let V, $\hat V$,

$\Omega := (U_+, U_1, \ldots, U_4)$ and $x_1, \ldots, x_4$ be as in Notation 5.7, let φ denote the map from V to $\hat{V}$ given by

$$\varphi(u, v, s) = (u^{\hat{\theta}}, \beta^{-2} v^{\hat{\theta}}, s^{\theta^{-1}})$$

for all $(u, v, s) \in V$ and let ψ denote the automorphism of V given by

$$\psi(u, v, s) = (u^{\chi}, v^{\chi}, s)$$

for all $(u, v, s) \in V$. There is a unique anti-automorphism κ of Ω extending the maps $x_i(b) \mapsto x_{5-i}(\varphi(\psi(b))$ for $i = 2$ and 4 and $x_i(a) \mapsto x_{5-i}(\varphi^{-1}(a))$ for $i = 1$ and 3. The square of κ is an involution. By [23, 7.5], therefore, κ gives rise to a non-type-preserving automorphism of Ξ of order 4.

We claim that Ξ does not, however, have any polarities. Let Γ be the additive group

$$\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\},$$

let k denote the field of Hahn series $\mathbb{F}_2(t^{\Gamma})$ and let $\nu\colon k^* \to \Gamma$ be the canonical valuation on k (as described, for example, in [8, 3.5.6]). There is a unique embedding π from K to k which sends β to t and α to $t^{\sqrt{2}}$. We identify K with its image under π. Modulo this identification, there is a unique extension of θ to a Tits endomorphism of k which we also denote by θ. If $u, v \in k$, then the constant coefficient of $t^{\sqrt{2}} u^2$ is 0 and the constant coefficient of v is the same as the constant coefficient of v^{θ}. It follows that there do not exist $u, v \in k$ such that

$$1 + t^{\sqrt{2}} u^2 = v + v^{\theta}.$$

By Proposition 10.4 with α^{-1} in place of α, it follows that our Moufang quadrangle Ξ does not have a polarity, as claimed.

11. Buildings of Type F_4

The main results of this section are Theorems 11.10 and 11.11.

Notation 11.1. — Let L/E be a field extension such that $\mathrm{char}(E) = 2$ and $L^2 \subset E$ and let $\Delta = \mathsf{F}_4(L, E)$ as defined in [26, 30.15]. Let Φ be a root system of type F_4, let Σ be an apartment of Δ, let c be a chamber of Σ and for each $\alpha \in \Phi$, let s_α denote the corresponding reflection. Let $\alpha_1, \ldots, \alpha_4$ be a basis of Φ ordered so that α_1 and α_2 are long and $|s_{\alpha_2} s_{\alpha_3}| = 4$, let S be the set of reflections s_{α_i} for $i \in [1, 4]$ and let $W = \langle S \rangle$ be the Weyl group of Φ. We think of the map $i \mapsto \alpha_i$ as a bijection from the vertex set of the Coxeter diagram F_4 to S. There is a unique action of W on Σ with respect to which s_{α_i} interchanges c with the unique chamber of

Σ that is i-adjacent to c, there is a unique chamber C of Φ contained in the half-space determined by α_i for all $i \in [1,4]$ and there is a unique W-equivariant bijection ι from the set of chambers of Σ to the set of chambers of Φ mapping c to C. The bijection ι induces a bijection from the set of roots of Σ to Φ and its inverse induces an injection from $\mathrm{Aut}(\Phi)$ to $\mathrm{Aut}(\Sigma)$. From now on, we identify $\mathrm{Aut}(\Phi)$ with its image under this injection and we identify the roots of Σ with the corresponding elements of Φ. Thus for each $\beta \in \Phi$, we have a root group U_β of Δ.

Theorem 11.2. — *There exists a collection of isomorphisms $x_\beta\colon E \to U_\beta$, one for each long root β of Φ, and a collection of isomorphisms $x_\beta\colon L \to U_\beta$, one for each short root β of Φ, such that for all $\alpha, \beta \in \Phi$ at an angle $\omega < 180°$ to each other and for all s in the domain of x_α and all t in the domain of x_β, the following hold:*

(1) *If $\omega = 120°$, then $\alpha+\beta \in \Phi$ and $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)$.*
(2) *If $\omega = 135°$, then α and β have different lengths; if α is long, then $\alpha+\beta \in \Phi$, $\alpha+2\beta \in \Phi$ and $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)x_{\alpha+2\beta}(st^2)$.*
(3) *$[x_\alpha(s), x_\beta(t)] = 1$ if ω is neither $120°$ nor $135°$.*

Proof. — This holds by [1, 5.2.2] and [19, 10.3.2]. □

Definition 11.3. — *We call a set $\{x_\beta\}_{\beta\in\Phi}$ satisfying the three conditions in Theorem 11.2 a* coordinate system *for Δ.*

Theorem 11.4. — *Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ, let $\gamma \in \mathrm{Aut}(\Phi)$, let λ_1, λ_2 be non-zero elements of E, let λ_3, λ_4 be non-zero elements of L and let χ be an element of $\mathrm{Aut}(L)$ stabilizing E. Then the following hold:*

(1) *There exists a unique automorphism*

$$g = g_{\gamma,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\chi}$$

of Δ that stabilizes the apartment Σ such that

$$x_{\alpha_i}(t)^g = x_{\gamma(\alpha_i)}(\lambda_i t^\chi)$$

for all t in the domain of x_{α_i} and for all $i \in [1,4]$.

(2) *If*

$$\beta = \sum_{i=1}^4 c_i\alpha_i \in \Phi,$$

then

$$x_\beta(t)^g = x_{\gamma(\beta)}(\lambda_\beta t^\chi)$$

for all t in the domain of x_β, where

$$\lambda_\beta = \prod_{i=1}^{4} \lambda_i^{c_i}.$$

Proof. — Inserting χ into [16, Lemma 58] and restricting scalars to E in the long root groups, we obtain the existence assertion in (1); (see also [16, Theorem 29]). The uniqueness assertion holds by [24, 9.7]. By 11.2 (1)–(3), [9, §10.2, Lemma A] and induction, it follows that (2) holds for all $\beta \in \Phi^+$ (i.e. for all $\beta \in \Phi$ that are positive with respect to the basis $\{\alpha_1, \ldots, \alpha_4\}$). For each $i \in [1, 4]$, there exists a unique $j \in [1, 4]$ such that the angle between α_i and α_j is 120°. By 11.2 (1), $\beta := \alpha_i + \alpha_j \in \Phi$ and $[x_\beta(1), x_{-\alpha_j}(t)] = x_{\alpha_i}(t)$ for all t in the domain of $x_{-\alpha_j}$ (which is the same as the domain of x_{α_j}). Conjugating by g and applying (1), we conclude that $x_{-\alpha_j}(t)^g = x_{\gamma(-\alpha)}(\lambda_j^{-1} t)$ for all t in the domain of x_{α_j}. Thus by 11.2 (1)–(3), [9, §10.2, Lemma A] and induction again, (2) holds for all $\beta \in \Phi^-$. □

Proposition 11.5. — *Every type-preserving automorphism of Δ that stabilizes Σ is of the form*

$$g_{\gamma,\lambda_1,\ldots,\lambda_4,\chi}$$

for some $\gamma \in \mathrm{Aut}(\Phi)$, some $\lambda_1, \lambda_2 \in E$, some $\lambda_3, \lambda_4 \in L$ and some $\chi \in \mathrm{Aut}(L, E)$.

Proof. — By 11.4 (1), it suffices to show that every type-preserving automorphism of Δ that stabilizes Σ pointwise is of the desired form. Let g be such an element. By [24, 9.7], g is uniquely determined by its restrictions to the irreducible rank 2 residues containing c. These are isomorphic to $A_2(E)$, $B_2^{\mathcal{D}}(\Lambda)$ and $A_2(L)$, where Λ is the indifferent set (L, L, E). By [23, 37.13], it follows that there exist $\lambda_1, \lambda_2 \in E^*$, $\lambda_3, \lambda_4 \in L$, $\chi_E \in \mathrm{Aut}(E)$ and $\chi_L \in \mathrm{Aut}(L)$ such that $x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^{\chi_E})$ for all $t \in E$ if $i = 1$ or 2 and $x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^{\chi_L})$ for all $t \in L$ if $i = 3$ or 4. By [23, 37.32] applied to the indifferent set (L, L, E), $\chi_L \in \mathrm{Aut}(L, E)$ and the restriction of χ_L to E equals χ_E. Thus $g = g_{\mathrm{id},\lambda_1,\ldots,\lambda_4,\chi}$ for $\chi = \chi_L$. □

Remark 11.6. — By [11, 28.8], $\{L/E, E/L\}$ is the pair of defining extensions of Δ; see Notation 3.5. Let G° and $G^\dagger$ be as in Notation 3.2. By [24, 2.8 and 11.12], the stabilizer $G^\dagger_\Sigma$ induces the same group as the stabilizer G°_Σ on Σ. Thus every element in G°_Σ is conjugate by an element in $G^\dagger$ to one which fixes the chamber c of Σ. By 3.8 (1)–(2), therefore, we can choose a Galois map ψ of Δ such that

$$\psi(g_{\gamma,\lambda_1,\ldots,\lambda_4,\chi}) = \chi$$

for all $\gamma \in \mathrm{Aut}(\Phi)$, for all $\lambda_1, \lambda_2 \in E$, for all $\lambda_3, \lambda_4 \in L$ and for all $\chi \in \mathrm{Aut}(L, E)$.

Notation 11.7. — Let $w_1 = (s_{\alpha_2} s_{\alpha_3})^2 \in \mathrm{Aut}(\Phi)$, where s_{α_2} and s_{α_3} are as in Notation 11.1.

Notation 11.8. — Let χ be an involution in the group $\mathrm{Aut}(L, E)$ defined as in Notation 3.5, let $F_0 = \mathrm{Fix}_L(\chi)$, let $K = F_0 \cap E$ and let N be the norm of the extension L/F_0. Thus F_0/K is a purely inseparable extension such that $F_0^2 \subset K$, the restriction of N to E is the norm of the extension E/K and L is the composite EF_0.

Notation 11.9. — Let χ, F_0 and K be as in Notation 11.8, let $F = F_0^2$ and suppose that

$$S = (E/K, F, \alpha, \beta)$$

is an F_4-datum for some $\alpha \in F$ and some $\beta \in K$. Let $\lambda_1 = \alpha\beta^{-1}$, let $\lambda_2 = \alpha^{-1}$, let $\lambda_3 = \beta$, let λ_4 be the unique element of F_0 such that $\lambda_4^2 = \beta^{-2}\alpha$ and let

$$\xi = g_{w_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\chi}\,,$$

where w_1 is as in Notation 11.7.

In the next two results, we use the term "χ-involution" (as defined in 3.11) with respect to the Galois map ψ chosen in Remark 11.6.

Theorem 11.10. — *Let Δ be as in Notation 11.1, let S and ξ be as in Notation 11.9 and let $\Gamma = \langle\xi\rangle$. Then ξ is a type-preserving isotropic χ-involution of Δ, Γ-chambers are residues of type $\{2, 3\}$ and*

$$\Delta^\Gamma \cong \mathcal{Q}(S).$$

Proof. — This holds by [12, p. 368 at the bottom]. See [14, 17.14] for a shorter proof. See also Remark 12.14. □

Theorem 11.11. — *Let Δ be as in Notation 11.1, let ξ be an arbitrary type-preserving χ-involution of Δ for some $\chi \in \mathrm{Aut}(L, E)$, let $\Gamma = \langle\xi\rangle$ and suppose that Γ-chambers are residues of type $\{2, 3\}$. Then the following hold:*

(1) *There exist $\alpha \in F$ and $\beta \in K$ such that ξ is conjugate by an element in $G^\dagger$ to*

$$g_{w_1,\alpha\beta^{-1},\alpha^{-1},\beta,\beta^{-1}\sqrt{\alpha},\chi}.$$

(2) *Δ^Γ is a Moufang quadrangle of type F_4.*

Proof. — By [12, Lemma 3.2], for every Γ-chamber R, there exists an apartment that is stabilized by ξ and contains chambers of R. By [24, 11.12], there exists an element δ in the group $G^\dagger$ such that ξ^δ stabilizes the apartment Σ and the unique $\{2,3\}$-residue containing c, where Σ and c are as in Notation 11.1. By [11, 25.17], ξ^δ induces the automorphism w_1 on Σ and by 3.8 (1), ξ^δ is also a χ-involution. By Proposition 11.5 and [12, Lemma 4.3], it follows that there exist $\alpha \in F$ and $\beta \in K$ such that $\xi^\delta = g_{w_1,\alpha\beta^{-1},\alpha^{-1},\beta,\beta^{-1}\sqrt{\alpha},\chi}$. Thus (1) holds. By Theorem 11.10, (2) follows from (1). □

12. F_4-Buildings with Polarity

The goal of this section is to prove Proposition 12.15.

Notation 12.1. — Suppose now that Ξ, ρ, $S = (E/K, F, \alpha, \beta)$, θ and the identification of Ξ with $\mathcal{Q}(S)$ are as in Theorem 9.12. Let $F_0 = F^{1/2}$ in the algebraic closure of E. Thus $K \subset F_0$ and $F_0^2 = F$. Let L be the composite field EF_0. Choose $\gamma \in K$ such that $E = K(\gamma)$. Then $L = F_0(\gamma)$. In particular, L/F_0 is a separable quadratic extension. Let χ be the generator of $\mathrm{Gal}(L/F_0)$. The map $x \mapsto ((x^2)^\theta)^{1/2}$ is the unique extension of θ to a Tits endomorphism of L. We denote this extension by the same letter θ. Since $K^\theta = F$, we have $F_0^\theta = K$. Thus $K = F_0^\theta \neq L^\theta = K(\gamma^\theta)$. Since $E^\theta \subset E$, it follows that $E = K(\gamma^\theta)$. Hence $L^\theta = E$. By 8.3 (2), θ commutes with χ. We set $\Delta = F_4(L, E)$.

Let c, Σ, Φ, $\{\alpha_1, \ldots, \alpha_4\}$, the identification of Φ with the set of roots of Σ, etc., be as in Notation 11.1 applied to $\Delta = F_4(L, E)$, let $\{x_\alpha\}_{\alpha\in\Phi}$ be as in Theorem 11.2, let

$$|\Phi| = \{\alpha/|\alpha| \mid \alpha \in \Phi\}$$

and let π be denote the bijection $\alpha \mapsto \alpha/|\alpha|$ from Φ to $|\Phi|$. We now identify the set of roots of Σ with $|\Phi|$ via π.

Notation 12.2. — Let $\dot{x}_{\pi(\alpha)}(t) = x_\alpha(t)$ for all $t \in E$ and all long $\alpha \in \Phi$, let $\dot{x}_{\pi(\alpha)}(t) = x_\alpha(t^{\theta^{-1}})$ for all $t \in E$ and all short $\alpha \in \Phi$ and let $U_{\pi(\alpha)} = U_\alpha$ for all $\alpha \in \Phi$. Thus $\dot{x}_\alpha$ is an isomorphism from the additive group of E to U_α for each $\alpha \in |\Phi|$ and by Theorem 11.2, if $s, t \in E$ and α and β are elements of $|\Phi|$ with an angle $\omega < 180°$ between them, then the following hold:

(1) If $\omega = 120°$, then $\alpha + \beta \in |\Phi|$ and $[\dot{x}_\alpha(s), \dot{x}_\beta(t)] = \dot{x}_{\alpha+\beta}(st)$

(2) If $\omega = 135°$, then $\sqrt{2}\alpha+\beta \in |\Phi|$, $\alpha+\sqrt{2}\beta \in |\Phi|$ and $[\dot{x}_\alpha(s), \dot{x}_\beta(t)] = \dot{x}_{\sqrt{2}\alpha+\beta}(s^\theta t)\dot{x}_{\alpha+\sqrt{2}\beta}(st^\theta)$.

(3) $[\dot{x}_\alpha(s), \dot{x}_\beta(t)] = 1$ if ω is neither $120°$ nor $135°$.

Let

$$B := \{\eta_1, \dots, \eta_4\}$$

be the image of the basis $\{\alpha_1, \dots, \alpha_4\}$ of Φ under π. We set $m' = \sqrt{2}m$ for each positive integer m and

$$abcd = a\eta_1 + b\eta_2 + c\eta_3 + d\eta_4$$

for all $a, b, c, d \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$. Thus, for example,

$$1'2'21 = \sqrt{2}\eta_1 + 2\sqrt{2}\eta_2 + 2\eta_3 + \eta_4 .$$

We then set

$$\begin{aligned}
W_0 &= \{0100, 0010, 011'0, 01'10\},\\
W_1 &= \{0001, 0011, 011'1', 01'11, 01'21\},\\
W_2 &= \{111'1', 121'1', 1'2'32, 122'1', 132'1'\},\\
W_3 &= \{1'1'11, 1'1'21, 232'1', 1'2'21, 1'2'31\},\\
W_4 &= \{1000, 1100, 1'1'10, 111'0, 121'0\}.
\end{aligned}$$

Let $|\Phi^+|$ denote the image under π of the set of positive roots of Φ with respect to the basis $\{\alpha_1, \dots, \alpha_4\}$. Then

$$|\Phi^+| = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4 .$$

Notation 12.3. — Let R_1 be the unique $\{2, 3, 4\}$-residue of Δ containing c, let R_4 be the unique $\{1, 2, 3\}$-residue containing c, let $R = R_1 \cap R_4$ and for $i = 1$ and 4, let R'_i be the unique residue such that $R'_i \cap \Sigma$ is opposite $R \cap \Sigma$ in $R_i \cap \Sigma$. Then W_i is the set of roots of Σ that contain $R \cap \Sigma$ but are disjoint from $R'_i \cap \Sigma$ for $i = 1$ and 4.

Notation 12.4. — There exists a unique set X of $\{2, 3\}$-residues of Σ containing $R \cap \Sigma$ with the property that there exists a bijection $i \mapsto T_i$ from $\mathbb{Z}_8$ to X such that for each $i \in \mathbb{Z}_8$, T_{i-1} and T_i are opposite residues of a residue of rank 3 of Σ. We denote by Λ the graph with vertex set X, where T_i is adjacent to T_j whenever $i - j = \pm 1$. Thus the residues $R'_1 \cap \Sigma$ and $R'_4 \cap \Sigma$ are the two vertices adjacent to $R \cap \Sigma$ in Λ.

Notation 12.5. — Let $\tilde{X}$ be the graph obtained from the set X in Notation 12.4 by replacing each vertex T_i by the unique residue $\tilde{T}_i$ of Δ such that $\tilde{T}_i \cap \Sigma = T_i$. Let $\tilde{\Sigma}$ be the graph with vertex set $\tilde{X}$, where $\tilde{T}_i$ is adjacent to $\tilde{T}_j$ whenever $i - j = \pm 1$.

Notation 12.6. — Let κ denote the unique involutory permutation of $|\Phi|$ which interchanges *abcd* with *dcba* for all $abcd \in |\Phi|$. Note that $W_0^\kappa = W_0$ and $W_i^\kappa = W_{5-i}$ for each $i \in [1,4]$. By [20, 1.2], there is a unique polarity of Δ stabilizing c and Σ and interchanging $\dot{x}_\alpha(t)$ and $\dot{x}_{\kappa(\alpha)}(t)$ for all $\alpha \in |\Phi|$ and all $t \in E$. We denote this polarity by σ.

Notation 12.7. — Let $[abcd]$ denote the reflection associated with the vector *abcd* for all $abcd \in |\Phi|$. Let $r_1 = [011'1']$ and $r_4 = [1'1'10]$ and let R, R_1, R_4, R_1' and R_4' be as in Notation 12.3. Then $|r_1r_4| = 4$. The reflection r_1 stabilizes $R_1 \cap \Sigma$ and interchanges $R \cap \Sigma$ with $R_1' \cap \Sigma$ as well as W_2 and W_4. The reflection r_4 stabilizes $R_4 \cap \Sigma$ and interchanges $R \cap \Sigma$ with $R_4' \cap \Sigma$ as well as W_1 and W_3. In particular, r_1 induces the reflection on the graph Λ defined in Notation 12.4 that interchanges $R \cap \Sigma$ and $R_1' \cap \Sigma$ and r_4 induces the reflection that interchanges $R \cap \Sigma$ and $R_4' \cap \Sigma$.

Notation 12.8. — We denote by r the square of the product

$$[0100] \cdot [0010].$$

The element r is an involution commuting with κ and with r_1 and r_4. It stabilizes the residue R and hence acts trivially on the graph Λ. It stabilizes the four sets $W_1, \ldots, W_4$ and fixes the vectors $011'1' \in W_1$, $1'2'32 \in W_2$, $232'1' \in W_3$ and $1'1'10 \in W_4$, but does not fix any other elements of $W_1 \cup W_2 \cup W_3 \cup W_4$.

By 11.4(1), there exists a unique automorphism ζ of Δ stabilizing Σ such that

$$\dot{x}_v(t)^\zeta = \dot{x}_{r(v)}(t) \tag{12.9}$$

for all $t \in E$.

We set

$$\lambda^{m'} = \lambda^{m\theta}$$

for all $\lambda \in K$ and all $m \in \mathbb{N}$ and let

$$h_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = g_{1,\lambda_1,\lambda_2,\lambda_3^{\theta-1},\lambda_4^{\theta-1},1}$$

for all $\lambda_1, \ldots, \lambda_4 \in E^*$. Let $h = h_{\lambda_1,\ldots,\lambda_4}$ for some choice of $\lambda_1, \ldots, \lambda_4 \in E^*$. By 11.4(2), we have

$$\dot{x}_{abcd}(t)^h = \dot{x}_{abcd}(\lambda t) \tag{12.10}$$

for all $abcd \in |\Phi|$ and all $t \in E$, where

$$\lambda = \lambda_1^a \lambda_2^b \lambda_3^c \lambda_4^d.$$

Thus, for example,

$$\dot{x}_{1'2'21}(t)^h = \dot{x}_{1'2'21}(\lambda t)$$

for all $t \in L$, where

$$\lambda = \lambda_1^\theta \lambda_2^{2\theta} \lambda_3^2 \lambda_4 .$$

Notation 12.11. — We set

$$\xi = g_{w_1, \beta^{-(\theta+1)}, \beta^\theta, \beta, \beta^{-(\theta+1)\theta^{-1}}, \chi},$$

where w_1 is as in Notation 11.7. By 9.12 (2) and Notation 12.1, we have $\alpha = \beta^{-\theta}$; thus ξ is the same as the element ξ in Notation 11.9. Note that

$$\dot{x}_{abcd}(t)^\xi = \left(\dot{x}_{abcd}(t^\chi)^h\right)^\zeta$$

for all $abcd \in |\Phi|$, where ζ is as in (12.9) and

$$h = h_{\beta^{-(\theta+1)}, \beta^\theta, \beta^\theta, \beta^{-(\theta+1)}} .$$

Notation 12.12. — By Theorem 11.10, we already know that the automorphism ξ is a type-preserving χ-involution of Δ and that $\tilde{\Xi} := \Delta^{\langle\xi\rangle}$ is isomorphic to Ξ. The polarity σ defined in Notation 12.6 commutes with ξ and thus induces a polarity of $\tilde{\Xi}$ which we denote by $\tilde{\rho}$. Our goal in Proposition 12.15 is to show that there is an isomorphism from $\tilde{\Xi}$ to Ξ which carries $\tilde{\rho}$ to ρ.

Remark 12.13. — Since (by Theorem 11.10) the minimal residues stabilized by ξ are of type $\{2, 3\}$, the residue R in Notation 12.3 is a chamber of $\tilde{\Xi}$. Since ξ stabilizes Σ, the graph $\tilde{\Sigma}$ defined in Notation 12.5 is an apartment of $\tilde{\Xi}$ containing R. The polarity $\tilde{\rho}$ stabilizes both R and $\tilde{\Sigma}$.

Remark 12.14. — It might appear that we are giving a new proof of Theorem 11.10 in Proposition 12.15. In fact, however, the proof of Proposition 12.15 we give relies on Remark 12.13 which, in turn, relies on the fact that the minimal residues stabilized by ξ are of type $\{2, 3\}$. It is exactly in the proof of this fact that the proof of Theorem 11.10 in [14] differs from the proof in [12].

Proposition 12.15. — *Let Ξ, Σ, c and ρ be as Notation 9.14 and let $\tilde{\Xi}$, $\tilde{\Sigma}$, R and $\tilde{\rho}$ be as in Notation 12.3, Notation 12.12 and Remark 12.13. Then there is an isomorphism from $\tilde{\Xi}$ to Ξ mapping the pair $(\tilde{\Sigma}, R)$ to the pair (Σ, c) that carries the polarity $\tilde{\rho}$ to ρ.*

Proof. — We define maps $X_1, \ldots, X_4$ from $V = E \oplus E \oplus [K]$ to $\mathrm{Aut}(\Delta)$ as follows:

$$X_1(u,v,t) = \dot{x}_{0011}(u)\dot{x}_{01'11}(\beta^{-1}\bar{u}) \cdot \dot{x}_{0001}(v)\dot{x}_{01'21}(\beta^{-(\theta+1)}\bar{v}) \cdot \dot{x}_{011'1'}(t)$$
$$X_2(u,v,t) = \dot{x}_{121'1'}(u)\dot{x}_{122'1'}(\beta^{-1}\bar{u}) \cdot \dot{x}_{111'1'}(v)\dot{x}_{132'1'}(\beta^{-(\theta+1)}\bar{v}) \cdot \dot{x}_{1'2'32}(t)$$
$$X_3(u,v,t) = \dot{x}_{1'1'21}(u)\dot{x}_{1'2'21}(\beta^{-1}\bar{u}) \cdot \dot{x}_{1'1'11}(v)\dot{x}_{1'2'31}(\beta^{-(\theta+1)}\bar{v}) \cdot \dot{x}_{232'1'}(t)$$
$$X_4(u,v,t) = \dot{x}_{1100}(u)\dot{x}_{111'0}(\beta^{-1}\bar{u}) \cdot \dot{x}_{1000}(v)\dot{x}_{121'0}(\beta^{-(\theta+1)}\bar{v}) \cdot \dot{x}_{1'1'10}(t)$$

for all $(u,v,t) \in V$, where $\bar{x} = x^{\chi}$ for all $x \in E$. Note that

$$X_i(u,v,t)^{\tilde{\rho}} = X_{5-i}(u,v,t) \tag{12.16}$$

for all $(u,v,t) \in V$. Let $M_i = X_i(V)$ for all $i \in [1,4]$, let M_+ denote the subgroup generated by $M_1, \ldots, M_4$ and let

$$\tilde{\Psi} := (M_+, M_1, M_2, M_3, M_4).$$

We have $M_i = C_{\langle U_\alpha | \alpha \in W_i \rangle}(\xi)$ for each $i \in [1,4]$. By Notation 12.3, Remark 12.13 and [11, 24.32], M_1 and M_4 are root groups of Δ corresponding to the two roots of the apartment $\tilde{\Sigma}$ containing R but not some chamber of $\tilde{\Sigma}$ adjacent to R. By Notation 12.7, we conclude that $\tilde{\Psi}$ is a root group sequence of the Moufang quadrangle $\tilde{\Xi}$.

It follows from 12.2 (1)–(3) that the map X_i is additive and thus M_i is abelian for all $i \in [1,4]$, that

$$[M_1, M_2] = [M_2, M_3] = [M_3, M_4] = [M_1, M_3] = 1,$$

and that

$$\begin{aligned}[X_2(a,b,s), X_4(u,v,t)] &= X_3(0,0,\beta^{-1}(u\bar{a} + a\bar{u}) + \beta^{-(\theta+1)}(v\bar{b} + b\bar{v})) \\ &= X_3(0,0,f((a,b,s),(u,v,t))\end{aligned}$$

for all $(a,b,s),(u,v,t) \in V$, where $f = \partial q$. Applying also the identities (1.3), we find that

$$\begin{aligned}
[X_1(a,0,0), X_4(u,0,0)] &= X_2(0, a^\theta u, 0)X_3(0, u^\theta a, 0)\\
[X_1(a,0,0), X_4(0,v,0)] &= X_2(\beta^{-\theta}\bar{a}^\theta v, 0, 0)X_3(0, \beta^{-1}v^\theta \bar{a}, 0)\\
[X_1(a,0,0), X_4(0,0,s)] &= X_2(0,0,s\beta^{-1}N(a))X_3(sa,0,0)\\
[X_1(0,b,0), X_4(0,v,0)] &= X_2(\beta^{-(\theta+1)}b^\theta \bar{v}, 0, 0)X_3(\beta^{-(\theta+1)}v^\theta \bar{v}, 0, 0)\\
[X_1(0,b,0), X_4(0,0,s)] &= X_2(0,0,s\beta^{-(\theta+1)}N(b))X_3(0, sb, 0)\\
[X_1(0,0,r), X_4(0,0,s)] &= X_2(0,0,r^\theta s)X_3(0,0,s^\theta r)
\end{aligned}$$

for all $(a,b,r),(u,v,s) \in V$. The commutator of an arbitrary element of M_i with an arbitrary element of M_j for $(i,j) = (1,3)$ and $(1,4)$ is uniquely determined by the identities (1.3), (12.16) and the identities above. It follows that there is an isomorphism ω from $\tilde{\Psi}$ to the root group sequence $\Psi := (U_+, U_1, \dots, U_4)$ defined by the commutator relations in Notation 9.14 sending $X_i(u,v,t)$ to $x_i(u,v,t)$ for all $(u,v,t) \in V$ and all $i \in [1,4]$. By (9.15) and (12.16), ω carries the anti-automorphism of $\tilde{\Psi}$ induced by $\tilde{\rho}$ to the anti-automorphism of Ψ induced by ρ. By [23, 7.5], there exists a unique isomorphism ω_1 from $\tilde{\Xi}$ to Ξ mapping the pair $(\tilde{\Sigma}, R)$ to the pair (Σ, c) and inducing the map ω from $\tilde{\Psi}$ to Ψ. By [23, 3.7], ω_1 carries $\tilde{\rho}$ to ρ. □

13. Moufang Octagons

Let Ω be a Moufang octagon. By [23, 17.7], $\Omega = \mathcal{O}(E,\theta)$ for some octagonal set (E,θ) as defined in Definition 2.1. Let $\Delta = F_4(E,\theta) = F_4(E,E^\theta)$ as defined in Notation 2.3.

Notation 13.1. — Let L/E and the extension of θ to L be as in Notation 12.1. Then θ maps the pair (L,E) to the pair (E,E^θ) and hence induces an isomorphism from $F_4(L,E)$ to Δ. Let c, Σ, Φ and the identification of Φ with the set of roots of Σ be as in Notation 11.1, let $\{x_\alpha\}_{\alpha\in\Phi}$ be as in Theorem 11.2 and let σ be the polarity of $F_4(L,E)$ defined in Notation 12.6. We identify $F_4(L,E)$ with Δ via the isomorphism induced by θ.

Proposition 13.2. — $\Omega \cong \Delta^{\langle\sigma\rangle}$.

Proof. — This holds by [20, Theorem (on p. 540)]. □

For the rest of this section, we will simply identify Ω with $\Delta^{\langle\sigma\rangle}$.

Proposition 13.3. — *Every automorphism of Ω has a unique extension to a type-preserving automorphism of Δ, and all of these extensions commute with σ.*

Proof. — Let G° denote the group of type-preserving automorphisms of Δ (as in Notation 3.2 applied to Δ). By [20, 1.6 and 1.13.1 (ii)], every automorphism of Ω can be extended to an element in the centralizer $C_{G^\circ}(\sigma)$. Suppose that g is a type-preserving automorphism of Δ that acts trivially on Ω. It remains only to show that g is trivial. Opposite chambers of Ω are opposite chambers of Δ and opposite chambers of Δ are contained in a unique apartment of Δ. We can thus assume that g fixes the apartment Σ and chamber c in Notation 13.1. Since g is type-preserving, it acts trivially on Σ and hence normalizes the root group U_α for all $\alpha \in \Phi$. By [20, 1.5], the map from the additive group of E to $\mathrm{Aut}(\Omega)$ which sends $t \in E$ to the element of $\mathrm{Aut}(\Omega)$ induced by $x_{\alpha_1}(t^\theta)x_{\alpha_4}(t)$ is injective as is the map from the additive group of E to $\mathrm{Aut}(\Omega)$ which sends $t \in E$ to the element of $\mathrm{Aut}(\Omega)$ induced by $x_{\alpha_2}(t^\theta)x_{\alpha_3}(t)x_{\alpha_2+2\alpha_3}(t^{\theta+2})$. It follows that g centralizes U_{α_i} for each $i \in [1,4]$. By 11.4(1), therefore, $g = 1$. □

By [11, 28.8], E is the defining field of Ω and $\{E/E^\theta, E^\theta/E\}$ is the pair of defining extensions of Δ.

Proposition 13.4. — *Let $A = \mathrm{Aut}(E, E^\theta)$ be as in Notation 3.6, let ι denote the inclusion map from A to $\mathrm{Aut}(E)$ and let ψ_Δ denote the Galois map of Δ in Remark 11.6. Then there is a unique Galois map ψ_Ω of Ω such that*

$$\psi_\Omega(\kappa) = \iota(\psi_\Delta(\zeta)) \tag{13.5}$$

for all pairs (κ, ζ), where $\kappa \in \mathrm{Aut}(\Omega)$ and ζ is the unique extension of κ to a type-preserving automorphism of Δ.

Proof. — Let G° and $G^\dagger$ be as in Notation 3.2 applied to Δ and let $H = \mathrm{Aut}(\Omega)$. By Proposition 13.3, there is a unique homomorphism $\psi = \psi_\Omega$ from H to $\mathrm{Aut}(E)$ such that (13.5) holds. Let $\kappa \in \mathrm{Aut}(\Omega)$ and let ζ be its unique extension to an element of G°. If κ lies in a root group of Ω, then by [11, 24.32], $\zeta \in G^\dagger$ and hence $\psi(\kappa) = 1$. Thus ψ satisfies 3.8(1). Let d be the chamber opposite c in Σ. Then d is a chamber of Ω opposite c. Hence there exists a unique apartment Σ_Ω containing c and d. Suppose that κ acts trivially on Σ_Ω. Then ζ acts trivially on Σ since it is type-preserving. By Proposition 11.5, therefore,

$$\zeta = g_{\gamma,\lambda_1,\ldots,\lambda_4,\chi}$$

for some $\gamma \in \mathrm{Aut}(\Phi)$, some $\lambda_1, \lambda_2 \in E^\theta$, some $\lambda_3, \lambda_4 \in E$ and some $\chi \in \mathrm{Aut}(E, E^\theta)$. By Remark 11.6, $\psi_\Delta(\zeta) = \chi$. Let $B = B_\Pi$ be as in Notation 3.3 for $\Pi = I_2(8)$, let (s,t) be the standard element of B as defined in Remark 3.4 and let $\Theta = (\hat{U}_+, \hat{U}_1, \ldots, \hat{U}_8)$ be the root group sequence and $\hat{x}_1, \ldots, \hat{x}_8$ the isomorphisms obtained by applying [23, 16.9] to the octagonal set (E, θ). By [20, 1.5–1.7], there exists an isomorphism $\varphi\colon \Omega_{st} \to \Theta$ such that there exist $\delta_1, \ldots, \delta_8 \in E^*$ so that for each $i \in [1,8]$, $\hat{x}_i(u)^h = \hat{x}_i(\delta_i u^\chi)$ for all $u \in E$ if i is odd and $\hat{x}_i(u,v)^h = \hat{x}_i(\delta_i u^\chi, \lambda_i^{\theta+1} v^\chi)$ for all $u, v \in E$ if i is even, where $h := \varphi^{-1}\kappa\varphi$. Thus $\chi = \psi_\Omega(\kappa)$ equals the element called $\lambda_\Omega(h)$ in [11, 29.5] with $\Omega = \Theta$. By Notation 3.8, ψ_Ω is the unique Galois map of Ω determined by φ. □

Remark 13.6. — Let $\kappa \in \mathrm{Aut}(\Omega)$ and let ζ be the unique extension κ to a type-preserving automorphism of Δ. By Proposition 13.3, κ is an involution if and only if ζ is. If we choose Galois maps as in Proposition 13.4, it follows that ζ is, in fact, a χ-involution for some $\chi \in A$ if and only if κ is a $\iota(\chi)$-involution, where A and ι are as in Proposition 13.4; see Notation 3.11.

Proposition 13.7. — *Let κ be a Galois involution of Ω that fixes panels of one type but none of the other type. Then κ has a unique extension to a type-preserving Galois involution ζ of Δ, $\langle\zeta\rangle$ is a descent group of Δ and $\langle\zeta\rangle$-chambers are residues of type $\{2,3\}$.*

Proof. — By Remark 13.6, κ has a unique extension to a type-preserving Galois involution ζ of Δ. By Theorem 3.10, $\langle\zeta\rangle$ is a descent group of Δ. By Proposition 13.2, some panels of Ω are $\{1,4\}$-residues of Δ and the others are $\{2,3\}$-residues (with respect to the standard numbering of the vertex set of the Coxeter diagram F_4). Since κ fixes panels of Ω, we can choose a J-residue R of Δ stabilized by $\langle\kappa,\sigma\rangle$, where J is either $\{1,4\}$ or $\{2,3\}$. Let R_1 be a minimal $\langle\zeta\rangle$-residue contained in R and let J_1 be its type. Since ζ commutes with σ, $R_1 \cap R_1^\sigma$ is also stabilized by ζ. By the choice of R_1, it follows that R_1 is stabilized by σ. Thus J_1 is a subset of J invariant under the non-trivial automorphism of the Coxeter diagram of Δ, so either $J_1 = \emptyset$ or $J_1 = J$. Suppose that $J_1 = \emptyset$. Then R_1 is contained in a unique J'-residue R_2, where J' denotes the complement of J in the vertex set of the Coxeter diagram F_4. Since $\langle\sigma,\zeta\rangle$ stabilizes R_1, it must stabilize R_2 as well. This contradicts the assumption, however, that κ does not fix panels of Ω of both types. We conclude that $J_1 = J$ and hence $R_1 = R$. Thus R is a $\langle\zeta\rangle$-chamber.

Let Π be the Coxeter diagram of type F_4 and let Θ denote the trivial subgroup of $\mathrm{Aut}(\Pi)$. By Definitions 2.14 and 2.15, the triple $(\Pi,\Theta,\{1,4\})$ is not a Tits index. By 2.18 (3), we conclude that $J = \{2,3\}$. □

Proposition 13.8. — *Let $\chi = \psi_\Delta(\xi)$, where ψ_Δ is as in Proposition 13.4 and ξ is as in Proposition 13.7. Then the following hold:*

(1) *$\chi\theta = \theta\chi$ and $\chi\theta$ is a Tits endomorphism of E.*
(2) *$\Omega_\xi := \Delta^{\langle\xi\sigma\rangle}$ is isomorphic to $\mathcal{O}(E,\chi\theta)$.*
(3) *Proposition 13.3 holds with Ω_ξ and $\xi\sigma$ in place of Ω and σ.*
(4) *There exists a Galois map ψ_{Ω_ξ} of Ω_ξ such that* (13.5) *holds with ψ_{Ω_ξ} in place of ψ_Ω.*

Proof. — By 11.11 (1) and the choice of ψ_Δ in Proposition 13.4, we can assume that

$$\xi = g_{w_1,\lambda_1,\dots,\lambda_4,\chi}$$

for some $\lambda_1, \lambda_2 \in E^\theta$ and some $\lambda_3, \lambda_4 \in E$. For each $\alpha \in \Phi$, we denote by s_α the corresponding reflection of Φ (as in Notation 11.1). Let $s = s_{\alpha_3} s_{\alpha_2+2\alpha_3}$, let $\beta_i = \alpha_i^s$ and let $d = c^s$ with respect to the action of W on Σ described in Notation 11.1. Then $\beta_1, \ldots, \beta_4$ is a basis of Φ and $\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$, $\beta_2 = -\alpha_2 - 2\alpha_3$, $\beta_3 = \alpha_2 + \alpha_3$ and $\beta_4 = \alpha_3 + \alpha_4$. The restriction of $\sigma\xi$ to Σ induces the unique automorphism of Φ that interchanges β_i and β_{5-i} for all $i \in [1, 4]$ (via the identification of the roots of Σ with Φ in Notation 11.1). By 11.4(2), there exist non-zero $\varepsilon_3, \varepsilon_3', \varepsilon_4 \in E^\theta$ such that

$$x_{\beta_i}(t)^{\xi\sigma} = x_{\beta_{5-i}}(\varepsilon_i t^{\chi\theta}) \tag{13.9}$$

for $i = 3$ and 4 and all $t \in E^\theta$ and

$$x_{\beta_3}(t)^{\sigma\xi} = x_{\beta_2}(\varepsilon_3' t^{\theta\chi})$$

for all $t \in E^\theta$. Since σ and ξ commute, we conclude that $\varepsilon_3 = \varepsilon_3'$ and $\theta\chi = \chi\theta$. Thus (1) holds.

Let $h = g_{1,\varepsilon_3\varepsilon_4^{-1},\varepsilon_3,\varepsilon_3^{-1},\varepsilon_3,1}$. By 11.4(2) again, h centralizes U_{β_3} and U_{β_4} and $x_{\beta_1}(t)^h = x_{\beta_1}(\varepsilon_4^{-1}t)$ and $x_{\beta_2}(t)^h = x_{\beta_2}(\varepsilon_3^{-1}t)$ for all $t \in F$. Let $\ddot{x}_\alpha = x_\alpha \cdot h_\alpha$ for all $\alpha \in \Phi$, where h_α denotes the automorphism $a \mapsto a^h$ of U_α. Then $\{\ddot{x}_\alpha\}_{\alpha\in\Phi}$ is a coordinate system for Δ (as defined in Definition 11.3), $\ddot{x}_{\beta_i} = x_{\beta_i}$ for $i = 3$ and 4 and $\ddot{x}_{\beta_1}(\varepsilon_4 t) = x_{\beta_1}(t)$ and $\ddot{x}_{\beta_2}(\varepsilon_3 t) = x_{\beta_2}(t)$ for all $t \in E$. By (13.9), therefore, $\ddot{x}_{\beta_i}(t)^{\sigma\xi} = \ddot{x}_{\beta_{5-i}}(t^{\chi\theta})$ for all $i \in [1, 4]$ and for all $t \in E$. We can thus apply Propositions 13.2–13.4 with $\xi\sigma$ and $\{\ddot{x}_\alpha\}_{\alpha\in\Phi}$ in place of σ and $\{x_\alpha\}_{\alpha\in\Phi}$ to conclude that (2)–(4) hold. □

14. Proofs of Theorems 4.1 and 4.2

We first prove Theorem 4.1. Suppose that Ξ and ρ satisfy the hypotheses, let

$$S = (E/K, F, \alpha, \beta)$$

and θ be as in Theorem 9.12 and let $\Delta = F_4(L, E)$ and χ be as in Notation 12.1. The Tits endomorphism θ commutes with χ; it also maps the pair (L, E) to the pair (E, E^θ) and hence induces an isomorphism from Δ to $F_4(E, \theta)$. Let σ be as in Notation 12.6 and let ξ be as in Notation 12.11. By Notation 12.12, $[\sigma, \xi] = 1$, ξ is a type-preserving χ-involution of Δ and σ induces a polarity on $\tilde{\Xi} := \Delta^{\langle\xi\rangle}$. By Proposition 3.1, the restriction of $\langle\sigma\rangle$ to $\tilde{\Xi}$ is a descent group of relative rank 1. It follows that $\langle\xi, \sigma\rangle$ is a descent group of Δ. Thus (1) holds. By Proposition 12.15, (2) holds. By Proposition 3.1 again, $\Delta^{\langle\sigma\rangle}$ and $\Delta^{\langle\sigma\xi\rangle}$ are Moufang octagons. By Notation 13.1

and Proposition 13.2, the first of these octagons is isomorphic to $\mathcal{O}(E,\theta)$ and by 13.8 (2), the second is isomorphic to $\mathcal{O}(E,\chi\theta)$. Thus (3) holds.

By Proposition 3.1, Δ^Γ, $(\Delta^{\langle\xi\rangle})^{\langle\sigma\rangle}$, $(\Delta^{\langle\sigma\rangle})^{\langle\xi\rangle}$ and $(\Delta^{\langle\sigma\xi\rangle})^{\langle\xi\rangle}$ are all Moufang sets. The underlying set of each of them is the set X of all Γ-chambers and by 2.18 (5) and [11, 24.32], the root group corresponding to a Γ-chamber R is the permutation group induced by $C_\Gamma(U_R)$ on X, where U_R is the unipotent radical of R in Δ. Thus (4) holds. By 2.18 (2) and Theorem 11.10, there are $\{2,3\}$-residues of Δ stabilized by ξ but none of type $\{1,4\}$. By Proposition 13.4 and 13.8 (4), therefore, (5) holds. This concludes the proof of Theorem 4.1.

We turn now to Theorem 4.2. Suppose that χ, (E,θ), Δ, Ω and κ satisfy the hypotheses, let σ be as in Notation 13.1 and let ξ be the type-preserving automorphism of Δ obtained by applying Proposition 13.3 to κ. Then ξ and σ commute and by Remark 13.6, ξ is a χ-involution. Let $\Gamma = \langle\xi,\sigma\rangle$. Then $\Delta^\Gamma = \Omega^{\langle\kappa\rangle}$. By Proposition 3.13, it follows that Γ is a descent group of Δ. Thus (1) holds. Assertion (2) holds by Proposition 13.2 and the choice of ξ. By Proposition 13.7, $\langle\xi\rangle$-chambers are of type $\{2,3\}$ and by Proposition 3.12, $\Xi := \Delta^{\langle\sigma\rangle}$ is a Moufang quadrangle of type F_4. Since ξ and σ commute, σ induces a polarity on Ξ. Thus (3) holds. Assertion (4) holds for the same reason that 4.1 (4) holds and assertion (5) holds by Proposition 13.8. This concludes the proof of Theorem 4.2.

15. Moufang Sets of Outer F_4-Type

Our goal in the remaining sections is to determine a few essential properties of the Moufang sets of outer F_4-type defined in Definition 4.5.

Notation 15.1. — Let S, $V = E \oplus E \oplus [K]$, U_+, $x_1,\ldots,x_4$ and θ be as in Notation 9.14, let θ_K be the restriction of θ to K, let $\Xi = \mathcal{Q}(S)$, Σ and c be as in Notation 5.7, let $q = q_S$, let $f = \partial q$, let g be as in Notation 6.21, let ρ be as in (9.15) and let $M = (X, \{U_x\}_{x\in X})$ be the Moufang set $\Xi^{\langle\rho\rangle}$ obtained by applying 2.18 (5) with $\langle\rho\rangle$ in place of Γ. Note that $c \in X$.

Notation 15.2. — We have $c \in X$ and by 2.18 (5), the centralizer $C_{U_+}(\rho)$ equals the root group U_c of the Moufang set M. We set $U = U_c$ and write U additively even though it is not, as we will see, abelian. In this section we use (6.20) and Proposition 6.23 to compute a few basic properties of U.

Let $\eta \in U_+$. Thus

$$\eta = x_1(b)x_2(w)x_3(v)x_4(u)$$

for some $b, w, u, v \in V$. Note that $f(a, g(y, z)) = 0$ for all $a, y, z \in V$ by Notations 5.17 and 6.21 (and, of course, that the characteristic of K is 2). Thus

$$\begin{aligned}\eta^\rho &= x_4(b)x_3(w)x_2(v)x_1(u)\\ &= x_4(b)x_1(u)x_3(w)x_2(v + g(u, w))\\ &= x_1(u)x_2(bu)x_3(ub)x_4(b)x_3(w)x_2(v + g(u, w))\\ &= x_1(u)x_2(bu)x_3(ub)x_3(w)x_3(g(b, v))x_2(v + g(u, w))x_4(b)\\ &= x_1(u)x_2(bu + v + g(u, w))x_3(ub + w + g(b, v))x_4(b)\,,\end{aligned}$$

so $\eta \in U$ if and only if $b = u$, $w = bu + v + g(u, w)$ and $v = ub + w + g(b, v)$. Note that $g(u, w) = g(u, uu) + g(u, v) = g(u, v)$ by 7.4(1). It follows that

$$U := \{x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u) \mid u, w \in V\}. \tag{15.3}$$

Let

$$\{u, w\} = x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u) \tag{15.4}$$

for all $u, w \in V$. Then

$$\begin{aligned}\{u, w\} + \{a, b\} &= x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u)\\ &\qquad\qquad \cdot x_1(a)x_2(b)x_3(aa + b + g(a, b))x_4(a)\\ &\qquad \in x_1(u + a)x_2\big(w + b + ua + g(a, uu + w + g(u, w))\big)U_{[3,4]}\end{aligned}$$

and thus

$$\{u, w\} + \{a, b\} = \{u + a, w + b + ua + g(a, w) + g(a, uu)\} \tag{15.5}$$

for all $u, w, a, b \in V$. It follows that

$$-\{u, w\} = \{u, w + uu + g(u, w)\}$$

and the commutator $-\{u, w\} - \{a, b\} + \{u, w\} + \{a, b\}$ equals

$$\{0,\ ua + au + g(u, b) + g(a, w) + g(u, aa) + g(a, uu)\} \tag{15.6}$$

for all $u, w, a, b \in V$.

Proposition 15.7. — *$U' = \{0, V\}$ and $[U, U'] = Z(U) = \{0, [K]\}$. In particular, U is nilpotent and has nilpotency class 3.*

Proof. — Setting $a = [1]$ in (15.6), we obtain

$$\{0, u + [q(u) + f(u, b)]\} \in U'$$

for all $u, b \in V$. For each $u \in V\backslash[K]$, there exists b such that $q(u) = f(u, b)$. Therefore $\{0, u\}$ is in the commutator group U' of U for all $u \in V\backslash[K]$. Hence $U' = \{0, V\}$. If we set $u = 0$ in (15.6), we are left with only

$\{0, g(a,w)\}$. It follows that $[U', U] = \{0, [K]\} \subset Z(U)$, $Z(U) \subset \{[K], V\}$ and

$$Z(U) \cap \{0, V\} = \{0, [K]\}. \tag{15.8}$$

Let $t \in K^*$. By Remark 7.3, we can choose $s \in K$ with $s \neq 0$ and $s \neq t$. Since $(x^{\theta-1})^{\theta+1} = x$ for all $x \in K^*$, it follows that $s^{\theta-1} \neq t^{\theta-1}$ and hence $s^\theta t + st^\theta \neq 0$. Setting $u = [s]$ and $a = [t]$ in (15.6), we obtain $\{0, [s^\theta t + st^\theta]\}$. It follows that $Z(U) \subset \{0, V\}$. By (15.8), we conclude that $Z(U) = \{0, [K]\}$. □

16. The Element τ

Let Ξ, Σ, c, U_+, $x_1, \ldots, x_4$, ρ, X, etc., be as in Notation 15.1 and let ϕ and $\Omega = \mathcal{G}(\Theta, U_+, \phi)$ be as in [23, 7.2] with $n = 4$, where Θ is a circuit of length 8 whose vertex set $V(\Theta)$ has been numbered by the integers modulo 8 so that the vertex x is adjacent to the vertex $x - 1$ for all x. The vertex set of Ω consists of pairs (x, B), where $x \in V(\Theta)$ and B is a right coset in U_+ of the subgroup $\phi(x)$. The vertices $(x, \phi(x))$ span an apartment of Ω which we identify with Θ via the map $x \mapsto (x, \phi(x))$. We set $\bullet = (4, \phi(4))$ and $\star = (5, \phi(5))$. Thus $e := \{\bullet, \star\}$ is an edge of Ω. For all vertices (x, B) of Ω other than $\bullet$ and $\star$, the vertex x of Θ is uniquely determined by B and we can denote the vertex (x, B) simply by B. The elements of U_+ fix $\bullet$ and $\star$ and acts on all other vertices by right multiplication.

By [23, 8.11], Ω is a Moufang quadrangle. We identify Ξ with the corresponding bipartite graph as described in [24, 1.8] and let π be an isomorphism Σ to Θ mapping the chamber c to the edge $\{\bullet, \star\}$. By [23, 7.5], π extends to a unique U_+-equivariant isomorphism from Ξ to Ω. We identify Ξ with Ω via this extension, so that $\Sigma = \Theta$, $c = e$ and the polarity ρ is an element of $\mathrm{Aut}(\Xi)$ stabilizing Σ and interchanging the vertices $\bullet$ and $\star$. In particular, c and d are in X, where $d = \{U_1, U_4\}$ is the chamber of Σ opposite $e = \{\bullet, \star\}$. Let $U = U_e = U_c$ be as in Notation 15.2.

Let $m_1 = \mu_\Sigma(x_1(0,0,1))$ and $m_4 = \mu_\Sigma(x_4(0,0,1))$ be as in [24, 11.22]. By (9.15), conjugation by the polarity ρ interchanges $x_1(0,0,1)$ and $x_4(0,0,1)$. By [24, 11.23], therefore, conjugation by ρ interchanges m_1 and m_4. By the identities in [23, 14.18 and 32.11], m_1 and m_4 both have order 2. By [23, 6.9], therefore, $\langle m_1, m_4 \rangle$ is a dihedral group of order 8. In particular, $(m_1 m_4)^2 = (m_4 m_1)^2$ and hence

$$\nu := (m_1 m_4)^2 \tag{16.1}$$

is centralized by ρ. The element m_1 fixes the vertices $\star$ and U_1 and reflects Σ onto itself and the element m_4 fixes the vertices $\bullet$ and U_4 and reflects Σ onto itself. Thus, in particular, $d = c^\nu$. Hence $U_d = \nu U_c \nu = \nu U \nu$ and thus $\nu U \nu$ acts sharply transitively on $X \backslash \{d\}$. The map $u \mapsto d^u$ is a bijection from the root group U of M to the set $X \backslash \{e\}$. Hence there exists a unique permutation τ of U^* such that

$$d^{(u)\tau} = c^{\nu u \nu} \tag{16.2}$$

for all $u \in U^*$.

The Moufang set M is isomorphic to the Moufang set $\mathbb{M}(U, \tau)$ defined in [6, §3], where τ is as in (16.2). As the results [6, Theorems 3.1 and 3.2] indicate, τ is an essential structural feature of the Moufang set M. Our goal in this section is to compute the formula for τ in Theorem 16.9.

Using the definition of the graph Ω in [23, 7.1], one can check that the permutation of the vertex set of Ω given in Table 1 (where $U_{ij} := U_{[i,j]}$ is as in [23, 5.1]) is an automorphism of Ω which, like m_1, fixes the vertices $\star$ and U_1 and reflects Σ onto itself. It follows from [23, 32.11] (even though we have reparametrized U_+) that m_1 centralizes U_3 and $x_4(u)^{m_1} = x_2(u)$ for all $u \in V$. Since m_1 maps the vertex U_{13} to the vertex U_{34}, it maps the image $U_{13}x_4(u)$ of the vertex U_{13} under the element $x_4(u) \in U_+$ to the image $U_{34}x_2(u)$ of the vertex U_{34} under the element $x_4(u)^{m_1} = x_2(u)$ of U_+ for all $u \in V$. Similarly, it maps $U_{12}x_3(u)$ to $U_4x_3(u)$ for all $u \in V$. It follows by [23, 3.7] that the automorphism in Table 1 is m_1. By similar arguments, the action of m_4 on the vertex set of Ω is as in Table 2.

In Table 16.3, which is derived from Tables 1 and 2, we have displayed the action of the product m_1m_4. We consider the vertex

$$U_1\{w, u\} = U_1\, x_2(u)\, x_3(ww + u + g(u, w))\, x_4(w)$$

with $u, w \in V$, where $\{w, u\}$ is as in (15.5), and compute the image of this vertex under $\nu = (m_1m_4)^2$ in Theorem 16.9 using Table 16.3. First, though, we need to make a few preparations.

Lemma 16.3. — *Let $u, w \in V$ with $u \neq 0$, and let $v = ww + u + g(u, w) \in V$. Then $u^{-1}v + w \neq 0$.*

Proof. — Suppose by contradiction that $w = u^{-1}v$. By Proposition 7.4(1),

$$g(u, w) = g(u, q(u)^{-1}uv) = 0\,, \tag{16.4}$$

$$\begin{aligned}
\star &\longleftrightarrow \star \\
\bullet &\longleftrightarrow U_{24} \\[1ex]
U_1\, x_2(u)\, x_3(v)\, x_4(w) &\longleftrightarrow U_1\, x_2(w)\, x_3(v+g(u,w))\, x_4(u) \\
U_{12}\, x_3(v)\, x_4(w) &\longleftrightarrow U_4\, x_2(w)\, x_3(v) \\
U_{13}\, x_4(w) &\longleftrightarrow U_{34}\, x_2(w) \\[1ex]
U_4\, x_1(u)\, x_2(v)\, x_3(w) &\overset{u\neq 0}{\longleftrightarrow} U_4\, x_1(u^{-1})\, x_2(vu^{-1})\, x_3(u^{-1}v+w) \\
U_4\, x_2(v)\, x_3(w) &\longleftrightarrow U_{12}\, x_3(w)\, x_4(v) \\
U_{34}\, x_1(u)\, x_2(v) &\overset{u\neq 0}{\longleftrightarrow} U_{34}\, x_1(u^{-1})\, x_2(vu^{-1}) \\
U_{34}\, x_2(v) &\longleftrightarrow U_{13}\, x_4(v) \\
U_{24}\, x_1(u) &\overset{u\neq 0}{\longleftrightarrow} U_{24}\, x_1(u^{-1})
\end{aligned}$$

Table 16.1. The Involution m_1

$$\begin{aligned}
\star &\longleftrightarrow U_{13} \\
\bullet &\longleftrightarrow \bullet \\[1ex]
U_1\, x_2(u)\, x_3(v)\, x_4(w) &\overset{w\neq 0}{\longleftrightarrow} U_1\, x_2(w^{-1}v+u+w^{-1}g(u,w))\, x_3(vw^{-1})\, x_4(w^{-1}) \\
U_1\, x_2(u)\, x_3(v) &\longleftrightarrow U_{34}\, x_1(v)\, x_2(u) \\
U_{12}\, x_3(v)\, x_4(w) &\overset{w\neq 0}{\longleftrightarrow} U_{12}\, x_3(vw^{-1})\, x_4(w^{-1}) \\
U_{12}\, x_3(v) &\longleftrightarrow U_{24}\, x_1(v) \\
U_{13}\, x_4(w) &\overset{w\neq 0}{\longleftrightarrow} U_{13}\, x_4(w^{-1}) \\[1ex]
U_4\, x_1(u)\, x_2(v)\, x_3(w) &\longleftrightarrow U_4\, x_1(w)\, x_2(v+g(u,w))\, x_3(u) \\
U_{34}\, x_1(u)\, x_2(v) &\longleftrightarrow U_1\, x_2(v)\, x_3(u) \\
U_{24}\, x_1(u) &\longleftrightarrow U_{12}\, x_3(u)
\end{aligned}$$

Table 16.2. The Involution m_4

so $v = ww + u$. Then $u^{-1} = wv^{-1}$ by 7.12 (2), so by 7.11 (3), $u = w^{-1}v = w^{-1}(ww+u)$. By (R7), 7.12 (1) and (16.4),

$$u = w^{-1}\cdot ww + w^{-1}u + g(ww\cdot w^{-1}, u) \\ = ww + w^{-1}u + g(w,u) = ww + w^{-1}u\,.$$

Hence $v = ww + u = w^{-1}u$, so 7.12 (1) implies $w = u^{-1}v = u^{-1}\cdot w^{-1}u = uw^{-1}$. Then $u = ww$, and hence $v = 0$, so $w = u^{-1}v = 0$, and then $u = ww = 0$, a contradiction. We conclude that $u^{-1}v + w \neq 0$. $\square$

$$\begin{aligned}
\star &\longmapsto U_{13}\\
\bullet &\longmapsto U_{12}\\
U_{34} &\longmapsto \star\\
U_{24} &\longmapsto \bullet
\end{aligned}$$

$$\begin{aligned}
U_1\, x_2(u)\, x_3(v)\, x_4(w) &\xmapsto{u\neq 0} U_1\, x_2(u^{-1}v+w)\, x_3\big(vu^{-1}+g(u^{-1},w)\big)\, x_4(u^{-1})\\
U_1\, x_3(v)\, x_4(w) &\longmapsto U_{34}\, x_1(v)\, x_2(w)\\
U_{12}\, x_3(v)\, x_4(w) &\longmapsto U_4\, x_1(v)\, x_2(w)\\
U_{13}\, x_4(w) &\longmapsto U_1\, x_2(w)
\end{aligned}$$

$$\begin{aligned}
U_4\, x_1(u)\, x_2(v)\, x_3(w) &\xmapsto{u\neq 0} U_4\, x_1(u^{-1}v+w)\, x_2\big(vu^{-1}+g(u^{-1},w)\big)\, x_3(u^{-1})\\
U_4\, x_2(v)\, x_3(w) &\xmapsto{v\neq 0} U_{12}\, x_3(wv^{-1})\, x_4(v^{-1})\\
U_4\, x_3(w) &\longmapsto U_{24}\, x_1(w)\\
U_{34}\, x_1(u)\, x_2(v) &\xmapsto{u\neq 0} U_1\, x_2(vu^{-1})\, x_3(u^{-1})\\
U_{34}\, x_2(v) &\xmapsto{v\neq 0} U_{13}\, x_4(v^{-1})\\
U_{24}\, x_1(u) &\xmapsto{u\neq 0} U_{12}\, x_3(u^{-1})
\end{aligned}$$

Table 16.3. The Product m_1m_4

Notation 16.5. — We set

$$\mathsf{N}(\{w,u\}) := \begin{cases} q(u)q\big(u^{-1}(ww+u+g(u,w))+w\big) & \text{if } u\neq 0,\\ q(w)^{\theta+2} & \text{if } u=0\end{cases}$$

for all $\{w,u\}\in U$. By Lemma 16.3, $\mathsf{N}(\{w,u\})=0$ only if $w=u=0$. We call N the *norm* of M.

LEMMA 16.6. — *Let $\{w,u\}\in U$. Then*

$$\mathsf{N}(\{w,u\}) = q(u)^\theta + q(u)q(w) + q(w)^{\theta+2} + f(u,ww)^\theta \\ + f(u,wu) + q(w)f(u,ww).$$

Proof. — This is obvious if $u=0$, so assume that $u\neq 0$. Let $v=ww+u+g(u,w)$. Then

$$\mathsf{N}(\{w,u\}) = q(u)q(u^{-1}v+w) = q(v)^\theta + f(uv,w) + q(u)q(w) \tag{16.7}$$

by Proposition 7.9. We have

$$\begin{aligned} q(v) &= q(ww)+q(u)+f(u,w)^\theta + f(ww,u)\\ &= q(w)^{\theta+1} + q(u) + f(u,w)^\theta + f(ww,u), \end{aligned}$$

and hence

$$q(v)^\theta = q(w)^{\theta+2} + q(u)^\theta + f(u,w)^2 + f(ww,u)^\theta. \tag{16.8}$$

We also have

$$\begin{aligned} f(uv,w) = f(u,wv) &= f\big(u, w\cdot(ww+u+g(u,w))\big)\\ &= f\big(u, w\cdot ww + wu + f(u,w)w + g(uw,ww)\big)\\ &= q(w)f(u,ww) + f(u,wu) + f(u,w)^2. \end{aligned}$$

Combining this with (16.7) and (16.8), we obtain the required formula. □

Theorem 16.9. — *Let $\{w,u\}\in U^*$. Then*

$$\{w,u\}^\tau = \left\{\frac{q(u)w + f(u,w)u + u(ww+u)}{\mathsf{N}(\{w,u\})}, \frac{q(w)u + w(ww+u)}{\mathsf{N}(\{w,u\})}\right\}.$$

Proof. — Assume first that $u=0$. Then

$$U_1\{w,u\} = U_1\{w,0\} = U_1\, x_3(ww)\, x_4(w).$$

Since $\{w,u\}\in U^*$, we have $w\neq 0$ and hence $ww\neq 0$. Using Table 16.3, we obtain

$$\begin{aligned} U_1\, x_3(ww)\, x_4(w) &\xmapsto{m_1m_4} U_{34}\, x_1(ww)\, x_2(w)\\ &\xmapsto{m_1m_4} U_1\, x_2(w\cdot(ww)^{-1})\, x_3((ww)^{-1}). \end{aligned}$$

By 7.12(1), we have $w\cdot(ww)^{-1} = w\cdot w^{-1}w^{-1} = w^{-1}w^{-1} = (ww)^{-1}$ and hence

$$\begin{aligned} U_1\, x_2(w\cdot(ww)^{-1})\, x_3((ww)^{-1}) &= U_1\, x_2((ww)^{-1})\, x_3((ww)^{-1})\\ &= U_1\{0,(ww)^{-1}\}. \end{aligned}$$

By (16.2), therefore,

$$\{w,0\}^\tau = \{0,(ww)^{-1}\}.$$

Since

$$\frac{w\cdot ww}{\mathsf{N}(\{w,0\})} = q(w)ww/q(w)^{\theta+2} = (ww)^{-1},$$

we obtain the required formula.

Assume now that $u\neq 0$, and let $v := ww+u+g(u,w)$. By Lemma 16.3, $u^{-1}v+w\neq 0$. Let

$$\begin{aligned} a &= u^{-1}v + w,\\ b &= vu^{-1} + g(u^{-1},w) \text{ and}\\ c &= u^{-1}, \end{aligned}$$

so that

$$U_1\{w,u\} = U_1\, x_2(u)\, x_3(v)\, x_4(w) \xmapsto{m_1 m_4} U_1\, x_2(a)\, x_3(b)\, x_4(c),$$

and hence

$$U_1\{w,u\} \xmapsto{(m_1 m_4)^2} U_1\{a^{-1}, a^{-1}b + c\}.$$

Thus

$$\{w,u\}^\tau = \{a^{-1}, a^{-1}b + c\}$$

by (16.2). Observe that

$$a = u^{-1}\big(ww + u + g(u,w)\big) + w = w + f(u,w)u^{-1} + u^{-1}(ww + u) \tag{16.10}$$

by (R2) and

$$b = (ww+u)u^{-1} + g(u,w)u^{-1} + g(u^{-1},w) = (ww+u)u^{-1} \tag{16.11}$$

by 7.4 (3). Also notice that

$$q(a) = q(u^{-1}v + w) = q(u)^{-1}\mathsf{N}(\{w,u\}) \tag{16.12}$$

by Notation 16.5, and hence

$$a^{-1} = \frac{q(u)a}{\mathsf{N}(\{w,u\})} = \frac{q(u)w + f(u,w)u + u(ww+u)}{\mathsf{N}(\{w,u\})}\,. \tag{16.13}$$

To compute $a^{-1}b + c$, we first notice that, by Proposition 7.5,

$$\begin{aligned} w\cdot(ww+u)u^{-1} = f(u^{-1}, w(ww+u))u^{-1} \\ + f(u^{-1},w)u^{-1}(ww+u) + q(u^{-1})w(ww+u), \end{aligned}$$

and hence

$$\begin{aligned} &q(u)w\cdot(ww+u)u^{-1} \\ &\quad = f(u^{-1}, w(ww+u))u + f(u,w)u^{-1}(ww+u) + w(ww+u)\,. \end{aligned} \tag{16.14}$$

Next, by 7.12 (1),

$$f(u,w)u\cdot(ww+u)u^{-1} = f(u,w)u^{-1}(ww+u)\,. \tag{16.15}$$

Finally, by 7.14 (4),

$$\begin{aligned} u(ww+u)\cdot(ww+u)u^{-1} &= q(u)u^{-1}(ww+u)\cdot(ww+u)u^{-1} \\ &= q(u)q\big(u^{-1}(ww+u)\big)u^{-1} = q\big(u^{-1}(ww+u)\big)u\,. \end{aligned} \tag{16.16}$$

Also observe that

$$c = u^{-1} = q(u)^{-1}u = q(a)u/\mathsf{N}(\{w,u\}) \tag{16.17}$$

by (16.12). By (16.11), (16.13), (16.14), (16.15), (16.16) and (16.17), we obtain

$$a^{-1}b+c=\frac{f(u^{-1},w(ww+u))u+w(ww+u)+q\big(u^{-1}(ww+u)\big)u+q(a)u}{\mathsf{N}(\{w,u\})}.$$

It remains to show that

$$f(u^{-1},w(ww+u))+q\big(u^{-1}(ww+u)\big)+q(a)=q(w)\,. \tag{16.18}$$

By (16.10), however, we have

$$\begin{aligned}q(a)=q(w)+f(u,w)^2q(u^{-1})+q\big(u^{-1}(ww+u)\big)+f(u,w)f(w,u^{-1})\\ +f\big(w,u^{-1}(ww+u)\big)+f(u,w)f\big(u^{-1},u^{-1}(ww+u)\big).\end{aligned}$$

Notice that the last term is 0 by 7.4(1) and that

$$f(u,w)^2q(u^{-1})=f(u,w)f(w,u^{-1})\,.$$

We conclude that

$$\begin{aligned}q(a)&=q(w)+q\big(u^{-1}(ww+u)\big)+f\big(w,u^{-1}(ww+u)\big)\\ &=q(w)+q\big(u^{-1}(ww+u)\big)+f\big(u^{-1},w(ww+u)\big)\end{aligned}$$

(by 7.4(2)). Thus (16.18) holds. □

17. Moufang Subsets

Our next goal is to identify three Moufang subsets of the Moufang set M. We continue with the notation in Notation 15.1. Let $G^\dagger$ be as in Definition 2.4 applied to M.

Remark 17.1. — Let U and τ be as in Section 16 and suppose that R is a subgroup of U such that R^* is τ-invariant. Let τ_R denote the restriction of τ to R. By [4, 6.2.2(1)], $M_R := \mathbb{M}(R,\tau_R)$ (as defined in [6, §3]) is a Moufang set. Let $G_R^\dagger$ be as in Definition 2.4 applied to M_R. Let c, d and ν be as in (16.2), let $X_R=\{c\}\cup d^R$ and let $N=\langle R,R^\nu\rangle$. By [4, 6.2.6], X_R is an N-orbit, R acts faithfully on X_R and the map from the underlying set $\{\infty\}\cup R$ of M_R to X_R sending ∞ to c and u to d^u for all $u\in R$ is a bijection which induces an isomorphism from the group induced by N on X_R to $G_R^\dagger$ with kernel $Z(N)$.

Notation 17.2. — Let $\Lambda=(L,\kappa)$ be an arbitrary octagonal set as defined in Definition 2.1. We denote by $\mathrm{MouSu}(\Lambda)$ the Moufang set corresponding to the group $\mathrm{Suz}(\Lambda)$. The root groups of $\mathrm{MouSu}(\Lambda)$ are the root groups of Suz(Λ). Each of them is isomorphic to the group P_Λ with underlying set $L \times L$, where

$$(a,b)\cdot(u,v) = (a+u, b+v+au^\kappa) \tag{17.3}$$

for all $(a,b),(u,v) \in L \times L$. For each $z \in L^*$ and for each automorphism σ of L commuting with κ (possibly trivial), let

$$(a,b)^{\tau_{z,\sigma}} = \left(z\Big(\frac{b}{a^{\kappa+2}+ab+b^\kappa}\Big)^\sigma, z^{\kappa+1}\Big(\frac{a}{a^{\kappa+2}+ab+b^\kappa}\Big)^\sigma \right)$$

for all $(a,b) \in P_\Lambda^*$. By [22, Exemple 2], we have $\mathrm{MouSu}(\Lambda) \cong \mathbb{M}(P_\Lambda, \tau_{z,\sigma})$ for all $z \in L^*$ and all $\sigma \in \mathrm{Aut}(L)$ that commute with κ.

Remark 17.4. — Let $\Lambda = (L,\kappa)$ be as in Notation 17.2 and suppose that $|L| > 2$. Let $\mathrm{MouSu}(\Lambda) = (X, \{U_x\}_{x\in X})$ and let $B^\dagger$ be the group obtained by applying Definition 2.4 to MouSu(Λ). By [23, 33.17], we have $U_x = [B_x^\dagger, U_x]$ for all $x \in X$.

Notation 17.5. — Let χ, θ and θ_K be as in Notations 9.3 and 15.1, let $\theta_1 = \theta$ and let $\theta_2 = \chi\theta_1$. Thus χ commutes with θ_1, and θ_2 is also a Tits endomorphism of E.

Proposition 17.6. — *Let θ_K, θ_1, θ_2 and χ be as in Notation 17.5.*

$$\begin{aligned} R_0 &:= \{\{(0,0,s),(0,0,t)\} \mid s,t \in K\},\\ R_1 &:= \{\{(a,0,0),(0,b,0)\} \mid a,b \in E\} \text{ and}\\ R_2 &:= \{\{(0,b,0),(a,0,0)\} \mid a,b \in E\}\end{aligned}$$

and let τ_i denote the restriction of τ to R_i for $i \in [0,2]$. Then θ_2 is a Tits endomorphism of E, R_0, R_1 and R_2 are τ-invariant subgroups of U, $\mathbb{M}(R_0,\tau_0) \cong \mathrm{MouSu}(K,\theta_K)$ and $\mathbb{M}(R_i,\tau_i) \cong \mathrm{MouSu}(E,\theta_i)$ for $i \in [1,2]$.

Proof. — We calculate using Proposition 9.16. First note that

$$\begin{aligned}\{[s],[t]\}\cdot\{[a],[b]\} &= \{[s]+[a],[t]+[b]+[s]\cdot[a]\}\\ &= \{[s+a],[t+b+sa^\theta]\}\end{aligned}$$

for all $s,t,a,b \in K$ by (15.5). Let $w = [s]$ and $u = [t]$ for some $s,t \in K$. Then $f(u,w) = 0$, $q(u) = t^\theta$ and $q(w) = s^\theta$, $ww + u = [s^{\theta+1}+t]$. Hence $\mathsf{N}(\{w,u\}) = \big(s^{\theta+2}+st+t^\theta\big)^\theta$ by Lemma 16.6. Thus

$$\{[s],[t]\}^\tau = \left\{ \Big[\frac{t}{s^{\theta+2}+st+t^\theta}\Big], \Big[\frac{s}{s^{\theta+2}+st+t^\theta}\Big] \right\}$$

by Theorem 16.9. Therefore R_0 is τ-invariant subgroup of U and by Notation 17.2, the map $(s,t) \mapsto \{[s],[t]\}$ is an isomorphism from $P_{(K,\theta_K)}$ to R_0 which carries the map $\tau_{1,1}$ to τ_0. Hence $\mathbb{M}(R_0,\tau_0) \cong \mathrm{MouSu}(K,\theta_K)$.

Let $w = (a, 0, 0) \in V$ and $u = (0, b, 0) \in V$ for some $a, b \in E$. Then $q(w) = \beta^{-1}N(a)$ and $q(u) = \beta^{-\theta_1-1}N(b)$, where N is the norm of the extension E/K, and $v := ww + u = \big(0,\ a^{\theta_1+1} + b,\ 0\big)$. Hence

$$q(u)w + u(ww + u) = \big(\beta^{-\theta_1-1}(a^{\theta_1+2} + ab + b^{\theta_1})b^{\chi},\ 0,\ 0\big)$$

and

$$q(w)u + w(ww + u) = \big(0,\ \beta^{-1}(a^{\theta_1+2} + ab + b^{\theta_1})a^{\chi},\ 0\big).$$

By (16.7), we also have

$$\mathsf{N}(\{w, u\}) = q(u)^{-1}q\big(uv + q(u)w\big) = \beta^{-\theta_1-2}N\big(a^{\theta_1+2} + ab + b^{\theta_1}\big).$$

Setting $[a, b] = \{(a, 0, 0), (0, b, 0)\}$ for all $a, b \in E$, we obtain

$$[a, b]^{\tau} = \left[\beta\Big(\frac{b}{a^{\theta_1+2} + ab + b^{\theta_1}}\Big)^{\chi}, \beta^{\theta_1+1}\Big(\frac{a}{a^{\theta_1+2} + ab + b^{\theta_1}}\Big)^{\chi}\right]$$

for all $a, b \in E$ by Theorem 16.9. By (15.5), we have

$$[a, b] \cdot [u, v] = [a + u, b + v + au^{\theta_1}]$$

for all $a, b, u, v \in E$. Hence R_1 is a τ-invariant subgroup of U and by Notation 17.2, $(a, b) \mapsto [a, b]$ is an isomorphism from $P_{(E,\theta_1)}$ to R_1 that carries the map $\tau_{\beta,\chi}$ to τ_1. Hence $\mathbb{M}(R_1, \tau_1) \cong \mathrm{MouSu}(E, \theta_1)$.

The computations for R_2 are similar. Setting

$$[a, b] = \{(0, \beta a^{\chi}, 0), (b, 0, 0)\},$$

we find that

$$[a, b] \cdot [u, v] = [a + u, b + v + au^{\theta_2+1}]$$

for all $a, b, u, v \in E$ and

$$[a, b]^{\tau} = \left[\beta^{\theta_2-1}\Big(\frac{b}{a^{\theta_2+2} + ab + b^{\theta_2}}\Big)^{\chi}, \beta\Big(\frac{a}{a^{\theta_2+2} + ab + b^{\theta_2}}\Big)^{\chi}\right]$$

for all $a, b \in E$. Hence R_2 is a τ-invariant subgroup of U and by Notation 17.2, the map $(a, b) \mapsto [a, b]$ is an isomorphism from $P_{(E,\theta_2)}$ to R_2 that carries the map $\tau_{\beta^{\theta_2-1},\chi}$ to τ_2. Hence $\mathbb{M}(R_2, \tau_2) \cong \mathrm{MouSu}(E, \theta_2)$. □

18. Simplicity

We can now deduce Theorem 4.7 as a corollary of Proposition 17.6. Let M, X, c, Σ and U_c be as in Notation 15.1, let $G^{\dagger}$ be as in Definition 2.4 applied to M and let R_0, R_1 and R_2 be as in Proposition 17.6. Let $i \in [0, 2]$ and let N be as in with R_i in place of R. By Remarks 7.3, 17.1, 17.4 and Proposition 17.6, we have $R_i \subset [N_c, R_i] \subset [G^{\dagger}, G^{\dagger}]$. Thus $\langle R_0, R_1, R_2\rangle \subset$

$[G^\dagger, G^\dagger]$. By Proposition 9.16 and (15.5) and some calculation, on the other hand, $U_c = \langle R_0, R_1, R_2 \rangle$. Since $G^\dagger$ is generated by conjugates of U_c, it follows that $G^\dagger$ is perfect. To finish the proof, we proceed with a standard argument which goes back to [10]: Let I be a non-trivial normal subgroup of $G^\dagger$. Since I is normal, the product IU_c is a subgroup of $G^\dagger$. Since $G^\dagger$ acts 2-transitively on X, the subgroup I acts transitively. Hence the subgroup IU_c contains all the root groups of $G^\dagger$. Therefore, $G^\dagger = IU_c$. Thus

$$G^\dagger/I \cong IU_c/I \cong U_c/U_c \cap I.$$

Since U_c is nilpotent, it follows that $G^\dagger/I$ is nilpotent. Since $G^\dagger$ is perfect, the quotient $G^\dagger/I$ is also perfect. A perfect nilpotent group must be trivial. It follows that $I = G^\dagger$. Thus $G^\dagger$ is simple. This concludes the proof of Theorem 4.7.

19. Invariants

In this last section, we show that q is an invariant of M (where q and M are as in Notation 15.1).

THEOREM 19.1. — *Let* $(K, V, q, \theta, t \mapsto [t], \cdot)$ *and* $(\tilde{K}, \tilde{V}, \tilde{q}, \tilde{\theta}, t \mapsto [t], *)$ *be two polarity algebras as defined in Definition 7.1 and assume that the corresponding Moufang sets* M *and* $\tilde{M}$ *are isomorphic. Then there is a field isomorphism* $\psi\colon K \to \tilde{K}$*, an additive bijection* $\zeta\colon V \to \tilde{V}$ *and an element* $e \in \tilde{K}^\times$ *such that*

(1) $\zeta(v \cdot w) = e^{-1}\zeta(v) * \zeta(w)$ *for all* $v, w \in V$*;*
(2) $\zeta(tv) = \psi(t)\zeta(v)$ *for all* $t \in K$ *and all* $v \in V$*;*
(3) $\tilde{q}(\zeta(v)) = e^{\tilde{\theta}}\psi(q(v))$ *for all* $v \in V$*; and*
(4) $\psi(t^\theta) = \psi(t)^{\tilde{\theta}}$ *for all* $t \in K$*.*

In particular, the quadratic forms q *and* $\tilde{q}$ *are similar.*

Proof. — Let π be an isomorphism from $M = (X, \{U_x\}_{x\in X})$ to $\tilde{M} = (\tilde{X}, \{\tilde{U}_{\tilde{x}}\}_{\tilde{x}\in\tilde{X}})$, i.e. a bijection from X to $\tilde{X}$ such that $\pi^{-1}U_z\pi = \tilde{U}_{(z)\pi}$ for all $z \in X$. We can assume that M, c and Σ are as in Notation 15.1. Thus by Notation 15.2, U_x is the group U described in Section 16. Let $d \in \Sigma$ be the unique chamber of Σ opposite c (as in Section 16) and let $\tilde{c}$ and $\tilde{d}$ be the images of c and d under π. Let H denote the pointwise stabilizer of $\{c, d\}$ in M and let $\tilde{H}$ denote the pointwise stabilizer of $\{\tilde{c}, \tilde{d}\}$ in $\tilde{M}$. For each $b \in U_c^*$, we denote by m_b the unique element in $U_d b U_d$ interchanging c and d and by μ_b be the unique permutation of U_c^* such that

$$d^{(a)\mu_b} = c^{m_b^{-1} u m_b} \qquad (19.2)$$

for each $a \in U_c^*$. We define $\tilde{\mu}_{\tilde{b}}$ for each $\tilde{b} \in \tilde{U}_{\tilde{c}}$ analogously. Let $\varphi\colon U_c \to \tilde{U}_{\tilde{c}}$ be the isomorphism induced by π. Then

$$\varphi((a)\mu_b) = (\varphi(a))\tilde{\mu}_{\varphi(b)} \text{ for all } a, b \in U \text{ with } b \neq 1. \tag{19.3}$$

Let $\{u, v\}$ for $u, v \in V$ be as in (15.4); we define $\{\tilde{u}, \tilde{v}\}$ for $\tilde{u}, \tilde{v} \in \tilde{V}$ analogously. Thus $U = \{V, V\}$ and $\tilde{U} = \{\tilde{V}, \tilde{V}\}$. Recall from Proposition 15.7 that $U' = \{0, V\}$ and $\tilde{U}' = \{0, \tilde{V}\}$. Therefore

$$\varphi(\{0, V\}) = \{0, \tilde{V}\} \tag{19.4}$$

and

$$\{0, \tilde{V}\} \text{ is } \tilde{H}\text{-invariant.} \tag{19.5}$$

Let $\tilde{c} = \varphi(\{0, [1]\})$, $\tau_1 = \mu_{\{0,[1]\}}$, let ν be as in (16.1), let τ be as in (16.2) and let $\tilde{\tau} = \varphi^{-1}\tau\varphi$. The product $m_{\{0,[1]\}}\nu$ fixes c and d and hence lies in H. By (16.2) and (19.2), it follows that $\tau_1 \in H^\circ\tau$, where H° denotes the permutation group

$$\{u \mapsto h^{-1}uh \mid h \in H\}$$

on U_c^*. Similarly, $\tilde{\mu}_c \in \tilde{H}^\circ\tilde{\tau}$, where $\tilde{H}^\circ$ is defined analogously. By Theorem 16.9 and (19.5), it follows that $\{0, V^*\}\tau_1 = \{V^*, 0\}$ and $\{0, \tilde{V}^*\}\tilde{\mu}_c = \{\tilde{V}^*, 0\}$. By (19.3), (19.4) and (19.5), therefore,

$$\varphi(\{V^*, 0\}) = \varphi(\{0, V^*\}\tau_1) = \varphi(\{0, V^*\})\tilde{\mu}_c = \{\tilde{V}^*, 0\}. \tag{19.6}$$

By (19.4) and (19.6), we conclude that there exist maps $\zeta\colon V \to \tilde{V}$ and $\gamma\colon V \to \tilde{V}$ such that $\varphi(\{u, 0\}) = \{\zeta(u), 0\}$ and $\varphi(\{0, v\}) = \{0, \gamma(v)\}$ for all $u, v \in V$. Since $\{u, v\} = \{u, 0\} + \{0, v\}$ by (15.5), we have

$$\varphi(\{u, v\}) = \{\zeta(u), \gamma(v)\}$$

for all $u, v \in V$. Therefore ζ is additive and

$$\begin{aligned}\gamma\big(w + b + ua + g(a, w) + g(a, uu)\big) &= \gamma(w) + \gamma(b) + \zeta(u) * \zeta(a) \\ &\quad + \tilde{g}\big(\zeta(a), \gamma(w)\big) + \tilde{g}\big(\zeta(a), \zeta(u) * \zeta(u)\big)\end{aligned} \tag{19.7}$$

for all $u, w, a, b \in V$ by (15.5) since ϕ is a homomorphism. Setting $a = 0$ in (19.7), we see that also γ is additive. By (19.7), therefore,

$$\begin{aligned}\gamma(ua) + \gamma\big(g(a, w) + g(a, uu)\big) \\ = \zeta(u) * \zeta(a) + \tilde{g}\big(\zeta(a), \gamma(w)\big) + \tilde{g}\big(\zeta(a), \zeta(u) * \zeta(u)\big)\end{aligned}$$

for all $u, w, a \in V$. Substituting uu for w in this identity, we obtain

$$\gamma(ua) = \zeta(u) * \zeta(a) + \tilde{g}\big(\zeta(a), \gamma(uu)\big) + \tilde{g}\big(\zeta(a), \zeta(u) * \zeta(u)\big) \tag{19.8}$$

and if we set $a = u$ in this identity, we have

$$\gamma(uu) = \zeta(u) * \zeta(u) + \tilde{g}\big(\zeta(u), \gamma(uu)\big) + \tilde{g}\big(\zeta(u), \zeta(u) * \zeta(u)\big).$$

Applying the map $\tilde{x} \mapsto \tilde{g}(\zeta(a), \tilde{x})$ to this last identity, we obtain

$$\tilde{g}\big(\zeta(a), \gamma(uu)\big) = \tilde{g}\big(\zeta(a), \zeta(u) * \zeta(u)\big).$$

Substituting this back into (19.8) now yields

$$\gamma(ua) = \zeta(u) * \zeta(a) \tag{19.9}$$

for all $u, a \in V$. In particular,

$$\gamma(u) = \zeta(u) * \zeta([1]). \tag{19.10}$$

Now observe that by Proposition 15.7 again, γ maps $[K]$ onto $[\tilde{K}]$, so by (19.10) and (R4), also $\zeta([K]) = [\tilde{K}]$. In particular, there exists an $e \in \tilde{K}^\times$ such that $\zeta([1]) = [e]$, and hence $\gamma(u) = e\zeta(u)$ for all $u \in V$ by (R2). Substituting this back into (19.9) now yields (1).

Since $\zeta([K]) = [\tilde{K}]$, there is a unique map $\psi\colon K \to \tilde{K}$ such that

$$\zeta([t]) = [e\psi(t)] \tag{19.11}$$

for all $t \in K$. Substituting $[t]$ for w in (1) and applying (R2), we obtain $\zeta(tv) = \zeta(v[t]) = e^{-1}\zeta(v)[e\psi(t)] = \psi(t)\zeta(v)$. Thus (2) holds.

Since ζ is additive, so is ψ. By (2), we have

$$\psi(st)\zeta(u) = \zeta(stu) = \psi(s)\zeta(tu) = \psi(s)\psi(t)\zeta(u)$$

for all $s, t \in K$ and all $u \in V$, so ψ is multiplicative. By the definition of e, we have $\psi(1) = 1$. Thus ψ is a field isomorphism.

We have $[1]v = [q(v)]$ for all $v \in V$ by (R1). Applying ζ, we obtain using (1) that $[e] * \zeta(v) = e\zeta([q(v)])$. By (7.2), (19.11) and (R1), this implies that $[e\tilde{q}(\zeta(v))] = e[e\psi(q(v))] = [e^{\tilde{\theta}+1}\psi(q(v))]$ for all $v \in V$. Thus (3) holds.

Finally, we have $t[1] = [t^\theta]$ for all $t \in K$ by (7.2). Applying ζ again, we obtain using (2) that $\psi(t)\zeta([1]) = \zeta([t^\theta])$ and hence, by (19.11), $\psi(t)[e] = [e\psi(t^\theta)]$. Another application of (7.2) now yields $[\psi(t)^{\tilde{\theta}}e] = [e\psi(t^\theta)]$ for all $t \in K$. We conclude that (4) holds. □

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