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## LOCAL $L^2$ -REGULARITY OF RIEMANN'S FOURIER SERIES

by Stéphane SEURET & Adrián UBIS (\*)

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ABSTRACT. — We are interested in the convergence and the local regularity of the lacunary Fourier series  $F_s(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s}$ . In the 1850's, Riemann introduced the series  $F_2$  as a possible example of nowhere differentiable function, and the study of this function has drawn the interest of many mathematicians since then. We focus on the case when  $1/2 < s \leq 1$ , and we prove that  $F_s(x)$  converges when  $x$  satisfies a Diophantine condition. We also study the  $L^2$ -local regularity of  $F_s$ , proving that the local  $L^2$ -norms of  $F_s$  around a point  $x$  behave differently around different  $x$ , according again to Diophantine conditions on  $x$ .

RÉSUMÉ. — Dans cet article, nous nous intéressons aux propriétés de convergence et de régularité locale des séries de Fourier lacunaires  $F_s(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s}$ . Dans les années 1850, Riemann avait proposé la série  $F_2$  comme exemple possible de fonction continue nulle part dérivable. La non-dérivabilité de  $F_2$  et plus généralement sa régularité locale ont depuis lors été étudiées par de nombreux mathématiciens, soulevant des questions d'analyse harmonique, d'analyse complexe et d'approximation diophantienne. Nous considérons le cas  $1/2 < s \leq 1$ , et trouvons un critère diophantien sur  $x \in \mathbb{R}$  pour la convergence de  $F_s(x)$ . Nous étudions également la régularité locale de  $F_s$ , en démontrant que les  $L^2$ -exposants de  $F_s$  dépendent de conditions diophantiennes sur  $x$ . Les preuves utilisent des estimées locales sur la norme  $L^2$  des sommes partielles de  $F_s$ .

### 1. Introduction

Riemann introduced in 1857 the Fourier series

$$R(x) = \sum_{n=1}^{+\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

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as a possible example of continuous but nowhere differentiable function. Though it is not the case ( $R$  is differentiable at rationals  $p/q$  where  $p$  and  $q$  are both odd [5]), the study of this function has drawn the interest of many mathematicians, mainly because of its connections with several domains: complex analysis, harmonic analysis, Diophantine approximation, and dynamical systems [4, 5, 6, 7, 8, 9] and more recently [2, 3, 11].

In this article, we study the local regularity of the series

$$(1.1) \quad F_s(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s}$$

when  $s \in (1/2, 1]$ . In this case, the series converges in  $L^2([0, 1])$ ; nevertheless several questions considering its local behavior arise. First it does not converge everywhere, hence one needs to characterize its set of convergence points; this question was studied in [11], and we will first continue that work by finding slightly more precise conditions to decide the convergence. Then, if one wants to characterize the local regularity of a (real) function, one classically studies the pointwise Hölder exponent defined for a locally bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$  at a point  $x$  by using the functional spaces  $C^\alpha(x)$ :  $f \in C^\alpha(x)$  when there exist a constant  $C$  and a polynomial  $P$  with degree less than  $\lfloor \alpha \rfloor$  such that, locally around  $x$  (i.e. for small  $H$ ), one has

$$\begin{aligned} & \| (f(\cdot) - P(\cdot - x)) \mathbf{1}_{B(x, H)} \|_\infty \\ & := \sup(|f(y) - P(y - x)| : y \in B(x, H)) \leq CH^\alpha, \end{aligned}$$

where  $B(x, H) = \{y \in \mathbb{R} : |y - x| \leq H\}$ . Unfortunately these spaces are not appropriate for our context since  $F_s$  is nowhere locally bounded (for instance, it diverges at every irreducible rational  $p/q$  such that  $q \neq 2 \times \text{odd}$ ).

Following Calderon and Zygmund in their study of local behaviors of solutions of elliptic PDE's [1], it is natural to introduce in this case the pointwise  $L^2$ -exponent defined as follows.

**DEFINITION 1.1.** — *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function belonging to  $L^2(\mathbb{R})$ ,  $\alpha \geq 0$  and  $x \in \mathbb{R}$ . The function  $f$  is said to belong to  $C_2^\alpha(x)$  if there exist a constant  $C$  and a polynomial  $P$  with degree less than  $\lfloor \alpha \rfloor$  such that, locally around  $x$  (i.e. for small  $H > 0$ ), one has*

$$\left( \frac{1}{H} \int_{B(x, H)} |f(h) - P(h - x)|^2 dh \right)^{1/2} \leq CH^\alpha.$$

Then, the pointwise  $L^2$ -exponent of  $f$  at  $x$  is

$$\alpha_f(x) = \sup \{ \alpha \in \mathbb{R} : f \in C_2^\alpha(x) \}.$$

This definition makes sense for the series  $F_s$  when  $s \in (1/2, 1]$ , and is based on a natural generalization of the spaces  $C^\alpha(x)$  by replacing the  $L^\infty$  norm by the  $L^2$  norm. The pointwise  $L^2$ -exponent has been studied for instance in [10], and is always greater than  $-1/2$  as soon as  $f \in L^2$ .

Our goal is to perform the multifractal analysis of the series  $F_s$ . In other words, we aim at computing the Hausdorff dimension, denoted by  $\dim$  in the following, of the level sets of the pointwise  $L^2$ -exponents.

DEFINITION 1.2. — *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function belonging to  $L^2(\mathbb{R})$ . The  $L^2$ -multifractal spectrum  $d_f : \mathbb{R}^+ \cup \{+\infty\} \rightarrow \mathbb{R}^+ \cup \{-\infty\}$  of  $f$  is the mapping*

$$d_f(\alpha) := \dim E_f(\alpha),$$

where the level set  $E_f(\alpha)$  is

$$E_f(\alpha) := \{x \in \mathbb{R} : \alpha_f(x) = \alpha\}.$$

By convention one sets  $d_f(\alpha) = -\infty$  if  $E_f(\alpha) = \emptyset$ .

Performing the multifractal analysis consists in computing its  $L^2$ -multifractal spectrum. This provides us with a very precise description of the distribution of the local  $L^2$ -singularities of  $f$ . In order to state our result, we need to introduce some notations.

DEFINITION 1.3. — *Let  $x$  be an irrational number, with convergents  $(p_j/q_j)_{j \geq 1}$ . Let us define*

$$(1.2) \quad x - \frac{p_j}{q_j} = h_j, \quad |h_j| = q_j^{-r_j}$$

with  $2 \leq r_j < \infty$ . Then the approximation rate of  $x$  is defined by

$$r_{\text{odd}}(x) = \overline{\lim}\{r_j : q_j \neq 2 * \text{odd}\}.$$

This definition always makes sense because if  $q_j$  is even, then  $q_{j+1}$  and  $q_{j-1}$  must be odd (so cannot be equal to  $2 * \text{odd}$ ). Thus, we always have  $2 \leq r_{\text{odd}}(x) \leq +\infty$ . It is classical that one can compute the Hausdorff dimension of the set of points with the Hausdorff dimension of the points  $x$  with a given approximation rate  $r \geq 2$ :

$$(1.3) \quad \text{for all } r \geq 2, \quad \dim\{x \in \mathbb{R} : r_{\text{odd}}(x) = r\} = \frac{2}{r}.$$

When  $s > 1$ , the series  $F_s$  converges, and the multifractal spectrum of  $F_s$  was computed by S. Jaffard in [9]. For instance, the multifractal spectrum of the classical Riemann's series  $F_2$  (using the classical pointwise Hölder

exponents) reads

$$d_{F_2}(\alpha) = \begin{cases} 4\alpha - 2 & \text{if } \alpha \in [1/2, 3/4], \\ 0 & \text{if } \alpha = 3/2, \\ -\infty & \text{otherwise.} \end{cases}$$

Actually the  $L^2$ -multifractal spectrum is the same as the Hölder spectrum for  $F_2$ .

Here our aim is to somehow extend this result to the range  $1/2 < s \leq 1$ . The convergence of the series  $F_s$  is described by our first theorem.

**THEOREM 1.4.** — *Let  $s \in (1/2, 1]$ , and let  $x \in (0, 1)$  with convergents  $(p_j/q_j)_{j \geq 1}$ . We set for every  $j \geq 1$*

$$\delta_j = \begin{cases} 1 & \text{if } s \in (1/2, 1) \\ \log(q_{j+1}/q_j) & \text{if } s = 1, \end{cases}$$

and

$$(1.4) \quad \Sigma_s(x) = \sum_{j: q_j \neq 2 * \text{odd}} \delta_j \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}}.$$

- (1)  $F_s(x)$  converges whenever  $\frac{s-1+1/r_{\text{odd}}(x)}{2} > 0$ . In fact, it converges whenever  $\Sigma_s(x) < +\infty$ .
- (2)  $F_s(x)$  does not converge if  $\frac{s-1+1/r_{\text{odd}}(x)}{2} < 0$ . In fact, it does not converge whenever

$$\overline{\lim}_{j: q_j \neq 2 * \text{odd}} \delta_j \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}} > 0,$$

In the same way we could extend this results to rational points  $x = p/q$ , by proving that  $F_s(x)$  converges for  $q \neq 2 * \text{odd}$  and does not for  $q = 2 * \text{odd}$ . Observe that the convergence of  $\Sigma_s(x)$  implies that  $r_{\text{odd}}(x) \leq \frac{1}{1-s}$ . Our result asserts that  $F_s(x)$  converges as soon as  $r_{\text{odd}}(x) < \frac{1}{1-s}$ , and also when  $r_{\text{odd}}(x) = \frac{1}{1-s}$  when  $\Sigma_s(x) < +\infty$ .

Jaffard's result is then extended in the following sense:

**THEOREM 1.5.** — *Let  $s \in (1/2, 1]$ .*

- (1) *For every  $x$  such that  $\Sigma_s(x) < +\infty$ , one has*

$$\alpha_{F_s}(x) = \frac{s - 1 + 1/r_{\text{odd}}(x)}{2}.$$

- (2) *For every  $\alpha \in [0, s/2 - 1/4]$*

$$d_{F_s}(\alpha) = 4\alpha + 2 - 2s.$$

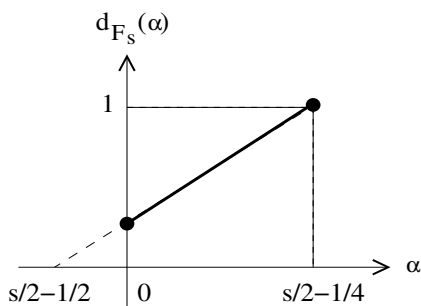


Figure 1.1.  $L^2$ -multifractal spectrum of  $F_s$

The second part of Theorem 1.5 follows directly from the first one. Indeed, using part (1) of Theorem 1.5 and (1.3), one gets

$$\begin{aligned} d_{F_s}(\alpha) &= \dim \left\{ x : \frac{s - 1 + 1/r_{\text{odd}}(x)}{2} = \alpha \right\} \\ &= \dim \{ x : r_{\text{odd}}(x) = (2\alpha + 1 - s)^{-1} \} \\ &= \frac{2}{(2\alpha + 1 - s)^{-1}} = 4\alpha + 2 - 2s. \end{aligned}$$

The paper is organized as follows. Section 2 contains some notations and preliminary results. In Section 3, we obtain other formulations for  $F_s$  based on Gauss sums, and we get first estimates on the increments of the partial sums of the series  $F_s$ . Using these results, we prove Theorem 1.4 in Section 4. Finally, in Section 5, we use the previous estimates to obtain upper and lower bounds for the local  $L^2$ -means of the series  $F_s$ , and compute in Section 6 the local  $L^2$ -regularity exponent of  $F_s$  at real numbers  $x$  whose Diophantine properties are controlled, namely we prove Theorem 1.5.

Finally, let us mention that theoretically the  $L^2$ -exponents of a function  $f \in L^2(\mathbb{R})$  take values in the range  $[-1/2, +\infty]$ , so they may have negative values. We believe that this is the case at points  $x$  such that  $r_{\text{odd}}(x) > \frac{1}{1-s}$ , so that in the end the entire  $L^2$ -multifractal spectrum of  $F_s$  would be  $d_{F_s}(\alpha) = 4\alpha + 2 - 2s$  for all  $\alpha \in [s/2 - 1/2, s/2 - 1/4]$ .

Another remark is that for a given  $s \in (1/2, 1)$ , there is an optimal  $p > 2$  such that  $F_s$  belongs locally to  $L^p$ , so that the  $p$ -exponents (instead of the 2-exponents) may carry some interesting information about the local behavior of  $F_s$ .

### 2. Notations and first properties

In all the proofs,  $C$  will denote a constant that does not depend on the variables involved in the equations.

For two real numbers  $A, B \geq 0$ , the notation  $A \ll B$  means that  $A \leq CB$  for some constant  $C > 0$  independent of the variables in the problem.

In Section 2 of [3] (also in [2]), the key point to study the local behavior of the Fourier series  $F_s$  was to obtain an explicit formula for  $F_s(p/q + h) - F_s(p/q)$  in the range  $1 < s < 2$ ; this formula was just a twisted version of the one known for the Jacobi theta function. In our range  $1/2 < s \leq 1$ , such a formula cannot exist because of the convergence problems, but we will get some truncated versions of it in order to prove Theorems 1.4 and 1.5.

Let us introduce the partial sum

$$F_{s,N}(x) = \sum_{n=1}^N \frac{e^{2i\pi n^2 x}}{n^s}.$$

For any  $H \neq 0$  let  $\tilde{\mu}_H$  be the probability measure defined by

$$(2.1) \quad \tilde{\mu}_H(g) = \int_{\mathcal{C}(H)} g(h) \frac{dh}{2H},$$

where  $\mathcal{C}(H)$  is the annulus  $\mathcal{C}(H) = [-2H, -H] \cup [H, 2H]$ .

LEMMA 2.1. — *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function in  $L^2(\mathbb{R})$ , and  $x \in \mathbb{R}$ . If  $\alpha_f(x) < 1$ , then*

$$(2.2) \quad \alpha_f(x) = \sup \left\{ \beta \in [0, 1) : \exists C > 0, \exists f_x \in \mathbb{R} \right. \\ \left. \|f(x + \cdot) - f_x\|_{L^2(\tilde{\mu}_H)} \leq C|H|^\beta \right\}.$$

*Proof.* — Assume that  $\alpha := \alpha_f(x) < 1$ . Then, for every  $\varepsilon > 0$ , there exist  $C > 0$  and a real number  $f_x \in \mathbb{R}$  such that for every  $H > 0$  small enough

$$\left( \frac{1}{H} \int_{B(x,H)} |f(h) - f_x|^2 dh \right)^{1/2} \leq CH^{\alpha-\varepsilon}.$$

Since  $\mathcal{C}(H) \subset B(x, 2H)$ , one has

$$(2.3) \quad \|f(x + \cdot) - f_x\|_{L^2(\tilde{\mu}_H)} = \left( \int_{\mathcal{C}(H)} |f(x + h) - f_x|^2 \frac{dh}{2H} \right)^{1/2} \\ \leq C|2H|^{\alpha-\varepsilon} \ll |H|^{\alpha-\varepsilon}.$$

Conversely, if (2.3) holds for every  $H > 0$ , then the result follows from the fact that  $B(x, H) = \{x\} \cup (x + \bigcup_{k \geq 1} \mathcal{C}(H/2^k))$ . □

So, we will use equation (2.2) as definition of the local  $L^2$  regularity.

It is important to notice that the frequencies in different ranges are going to behave differently. Hence, it is better to look at  $N$  within dyadic intervals. Moreover, it will be easier to deal with smooth pieces. This motivates the following definition.

DEFINITION 2.2. — *Let  $N \geq 1$  and let  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  function with support included in  $[1/2, 2]$ . One introduces the series*

$$(2.4) \quad F_{s,N}^\psi(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s} \psi\left(\frac{n}{N}\right),$$

and for  $R > 0$  one also sets

$$w_R^\psi(t) = e^{2i\pi R t^2} \psi(t)$$

and

$$E_N^\psi(x) = \frac{1}{N} \sum_{n=1}^{+\infty} w_{N^2 x}^\psi\left(\frac{n}{N}\right),$$

For every function  $\psi$ , we will use the notation

$$(2.5) \quad \psi_s(t) := t^{-s} \psi(t),$$

which is still  $C^\infty$  with support in  $(1/2, 2)$  when  $\psi$  is. It is immediate to check that

$$(2.6) \quad F_{s,N}^\psi(x) = N^{1-s} E_N^{\psi_s}(x).$$

### 3. Summation Formula for $F_{s,N}$ and $F_{s,N}^\psi$

#### 3.1. Poisson Summation

Let  $p, q$  be coprime integers, with  $q > 0$ . In this section we obtain some formulas for  $F_s(p/q + h) - F_s(p/q)$  with  $h > 0$ . This is not a restriction, since

$$(3.1) \quad F_s\left(\frac{p}{q} - h\right) = \overline{F_s\left(\frac{-p}{q} + h\right)}.$$

We are going to write a summation formula for  $E_N^\psi(p/q + h)$ , with  $h > 0$ .



PROPOSITION 3.1. — We have

$$(3.2) \quad E_N^\psi \left( \frac{p}{q} + h \right) = \frac{1}{\sqrt{q}} \sum_{m \in \mathbb{Z}} \theta_m \cdot \widehat{w_{N^2h}^\psi} \left( \frac{Nm}{q} \right).$$

where  $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2i\pi t\xi} dt$  stands for the Fourier transform of  $f$  and  $(\theta_m)_{m \in \mathbb{Z}}$  are some complex numbers whose modulus is bounded by  $\sqrt{2}$ .

*Proof.* — We begin by splitting the series into arithmetic progressions

$$\begin{aligned} E_N^\psi \left( \frac{p}{q} + h \right) &= \frac{1}{N} \sum_{n=1}^{+\infty} e^{2i\pi n^2 \frac{p}{q}} \cdot w_{N^2h}^\psi \left( \frac{n}{N} \right) \\ &= \sum_{b=0}^{q-1} e^{2i\pi b^2 \frac{p}{q}} \sum_{\substack{n=1: \\ n \equiv b \pmod q}}^{+\infty} \frac{1}{N} w_{N^2h}^\psi \left( \frac{n}{N} \right). \end{aligned}$$

Since  $w_{N^2h}^\psi$  is  $C^\infty$  and supported inside  $(1/2, 2)$ , we can apply Poisson Summation to get

$$\begin{aligned} \sum_{\substack{n=1: \\ n \equiv b \pmod q}}^{+\infty} \frac{1}{N} w_{N^2h}^\psi \left( \frac{n}{N} \right) &= \sum_{n \in \mathbb{Z}} \frac{1}{N} w_{N^2h}^\psi \left( \frac{b+nq}{N} \right) \\ &= \sum_{m \in \mathbb{Z}} \frac{1}{q} e^{2i\pi \frac{bm}{q}} \cdot \widehat{w_{N^2h}^\psi} \left( \frac{Nm}{q} \right). \end{aligned}$$

This yields

$$E_N^\psi \left( \frac{p}{q} + h \right) = \frac{1}{q} \sum_{m \in \mathbb{Z}} \tau_m \cdot \widehat{w_{N^2h}^\psi} \left( \frac{Nm}{q} \right)$$

with

$$\tau_m = \sum_{b=0}^{q-1} e^{2i\pi \frac{pb^2+mb}{q}}.$$

This term  $\tau_m$  is a Gauss sum. One has the following bounds:

- for every  $m \in \mathbb{Z}$ ,  $\tau_m = \theta_m \sqrt{q}$  with  $\theta_m := \theta_m(p, q)$  satisfying

$$0 \leq |\theta_m| \leq \sqrt{2}.$$

- if  $q = 2 * \text{odd}$ , then  $\theta_0 = 0$ .
- if  $q \neq 2 * \text{odd}$ , then  $1 \leq |\theta_0| \leq \sqrt{2}$ .

Finally, we get the summation formula (3.2). □

**3.2. Behavior of the Fourier transform of  $w_R^\psi$**

To use formula (3.2), one needs to understand the behavior of the Fourier transform of  $w_R^\psi$

$$\widehat{w_R^\psi}(\xi) = \int_{\mathbb{R}} \psi(t)e^{2i\pi(Rt^2 - \xi t)} dt.$$

On one hand, we have the trivial bound  $|\widehat{w_R^\psi}(\xi)| \ll 1$  since  $\psi$  is  $C^\infty$ , bounded and compactly supported. On the other hand, one has

LEMMA 3.2. — *Let  $R > 0$  and  $\xi \in \mathbb{R}$ . Let  $\psi$  be a  $C^\infty$  function compactly supported inside  $[1/2, 2]$ . Let us introduce the mapping  $g_R^\psi : \mathbb{R} \rightarrow \mathbb{C}$*

$$g_R^\psi(\xi) = e^{i\pi/4} \frac{e^{-i\pi\xi^2/(2R)}}{\sqrt{2R}} \psi\left(\frac{\xi}{2R}\right).$$

Then one has

$$(3.3) \quad \widehat{w_R^\psi}(\xi) = g_R^\psi(\xi) + \mathcal{O}_\psi\left(\frac{\rho_{R,\xi}}{\sqrt{R}} + \frac{1}{(1 + R + |\xi|)^{3/2}}\right),$$

with

$$\rho_{R,\xi} = \begin{cases} 1 & \text{if } \xi/2R \in [1/2, 2] \text{ and } R < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, one has

$$(3.4) \quad \widehat{w_{2R}^\psi}(\xi) - \widehat{w_R^\psi}(\xi) = g_{2R}^\psi(\xi) - g_R^\psi(\xi) + \mathcal{O}_\psi\left(\rho_{R,\xi}\sqrt{R} + \frac{R}{(1 + R + |\xi|)^{5/2}}\right),$$

Moreover, the constant implicit in  $\mathcal{O}_\psi$  depends just on the  $L^\infty$ -norm of a finite number of derivatives of  $\psi$ .

*Proof.* — For  $\xi/2R \notin [1/2, 2]$  the upper bound (3.3) comes just from integrating by parts several times; for  $\xi, R \ll 1$  the bound (3.3) is trivial. The same properties hold true for the upper bound in (3.4).

Let us assume  $\xi/2R \in [1/2, 2]$ , and  $R > 1$ . The lemma is just a consequence of the stationary phase theorem. Precisely, Proposition 3 in Chapter VIII of [12] (and the remarks thereafter) implies that for some suitable functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , and if  $g$  is such that  $g'(t_0) = 0$  at a unique point  $t_0$ , if one sets

$$S(\lambda) = \int_{\mathbb{R}} f(t)e^{i\lambda g(t)} dt - \sqrt{\frac{2\pi}{-i\lambda g''(t_0)}} f(t_0)e^{i\lambda g(t_0)},$$

then  $|S(\lambda)| \ll \lambda^{-3/2}$  and also  $|S'(\lambda)| \ll \lambda^{-5/2}$ , where the implicit constants depend just on upper bounds for some derivatives of  $f$  and  $g$ , and also on a lower bound for  $g''$ .

In our case, we can apply it with  $f = \psi$ ,  $\lambda = R$ , and  $g(t) = 2\pi(t^2 - t\xi/\lambda)$  to get precisely (3.3). This is so because although  $g$  depends on  $\lambda$ , since in our range for  $R$  we have  $|\xi/\lambda| \leq 4$ , the trivial upper bounds for  $g$  and its derivatives do not (and also  $g''(t)$  does not depend on  $\lambda$ ).

With the same choices for  $f$  and  $g$ , by applying the Mean Value Theorem and our bound for  $S'(\lambda)$ , we finally obtain formula (3.4). □

### 3.3. Summation formula for the partial series $F_{s,N}$ .

A certain duality formula for the Riemann function  $R(x)$  has been at the heart of the proofs of many of its convergence and differentiability properties [6, 7, 9]. This formula in turn can be seen as coming from the symmetries of the Jacobi theta function, and can be derived by using Fourier Analysis.

There is a related duality formula for  $F_s(x)$ , but it does not work in our case due to convergence problems related to the range in which  $s$  moves. As a substitute we will use a truncated version of it that can be proved by similar techniques. This formula gives precise information about the behavior of the partial sum function  $F_{s,2N} - F_{s,N}$  near a rational number.

PROPOSITION 3.3. — *Let  $p, q$  be two coprime integers,  $q > 1$ . For  $N \geq q$  and  $0 \leq h \leq q^{-1}$ , we have*

$$\begin{aligned}
 F_{s,2N} \left( \frac{p}{q} + h \right) - F_{s,N} \left( \frac{p}{q} + h \right) &= \frac{\theta_0}{\sqrt{q}} \int_N^{2N} \frac{e^{2i\pi ht^2}}{t^s} dt \\
 &+ G_{s,2N}(h) - G_{s,N}(h) \\
 &+ \mathcal{O}(N^{\frac{1}{2}-s} \log q),
 \end{aligned}$$

where

$$(3.5) \quad G_{s,N}(h) = (2hq)^{s-\frac{1}{2}} e^{i\pi/4} \sum_{1 \leq m \leq 2Nhq} \frac{\theta_m}{m^s} e^{-i\pi \frac{m^2}{q^2 h}}.$$

Pay attention to the fact that  $G_{s,N}$  depends on  $p$  and  $q$ . We omit this dependence in the notation for clarity.

*Proof.* — We can write

$$F_{s,2N}(x) - F_{s,N}(x) = F_{s,N}^{\mathbb{1}_{(1,2)}}(x) + \mathcal{O}(N^{-s}).$$

Hence, we would like to use the formulas proved in the preceding section, but those formulas apply only to compactly supported  $C^\infty$  functions. We thus decompose the indicator function  $\mathbb{1}_{(1,2)}$  into a countable sum of  $C^\infty$  functions, as follows. Let us consider  $\eta$ , a  $C^\infty$  function with support  $[1/2, 2]$  such that

$$\eta(t) = 1 - \eta(t/2) \quad 1 \leq t \leq 2.$$

Then, the function

$$\psi(t) = \sum_{k \geq 2} \eta\left(\frac{t}{2^{-k}}\right)$$

has support in  $[0, 1/2]$ , equals 1 in  $(0, 1/4]$  and is  $C^\infty$  in  $(0, +\infty)$ . Therefore, we have

$$(3.6) \quad \mathbb{1}_{(1,2)}(t) = \psi(t-1) + \psi(2-t) + \tilde{\psi}(t)$$

with  $\tilde{\psi}$  some  $C^\infty$  function with support included in  $[5/4, 7/4] \subset [1, 2]$ .

In order to get a formula for  $F_{s,N}^{\mathbb{1}_{(1,2)}}(x)$ , we are going to use (3.6) and the linearity in  $\psi$  of the formula (2.4). Indeed, writing for all  $k \geq 2$

$$(3.7) \quad \eta^k(t) := \eta((t-1)/2^{-k}) \quad \text{and} \quad \zeta^k(t) := \eta((2-t)/2^{-k}),$$

one has the decomposition:

$$(3.8) \quad F_{s,N}^{\mathbb{1}_{(1,2)}}(x) = F_{s,N}^{\tilde{\psi}}(x) + \sum_{k \geq 2} F_{s,N}^{\eta^k}(x) + \sum_{k \geq 2} F_{s,N}^{\zeta^k}(x).$$

Our goal is to obtain several estimates to treat all these terms. The proof is decomposed into several lemmas, to deal with the different cases.

First, we obtain an estimate for  $F_{s,N}^\phi$ , where  $\phi$  is any  $C^\infty$  function supported in  $[1/2, 2]$ . In particular, this will apply to the term  $F_{s,N}^{\tilde{\psi}}(x)$ .

LEMMA 3.4. — *Let  $\phi$  be a  $C^\infty$  function supported in  $[1/2, 2]$ . Then,*

$$(3.9) \quad F_{s,N}^\phi\left(\frac{p}{q} + h\right) = \frac{N^{1-s}\theta_0}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 ht^2}}{t^s} \phi(t) dt + G_{s,N}^\phi(h) + \mathcal{O}_\phi\left(\frac{q}{N^{1/2+s}}\right),$$

where

$$(3.10) \quad G_{s,N}^\phi(h) = (2hq)^{s-\frac{1}{2}} e^{i\pi/4} \sum_{m \neq 0} \frac{\theta_m}{m^s} e^{-i\pi \frac{m^2}{2q^2 h}} \phi\left(\frac{m}{2Nhq}\right).$$

*Proof.* — Recall the notation (2.5):  $\phi_s(t) = t^{-s}\phi(t)$ .

First, by (2.6) one has  $F_{s,N}^\phi(x) = N^{1-s}E_N^{\phi_s}(x)$ . Further, by (3.2) one has

$$E_N^{\phi_s}\left(\frac{p}{q} + h\right) = \frac{1}{\sqrt{q}} \sum_{m \in \mathbb{Z}} \theta_m \cdot \widehat{w_{N^2h}^{\phi_s}}\left(\frac{Nm}{q}\right),$$

and then, applying Lemma 3.2 with  $\xi = \frac{Nm}{q}$  and  $R = N^2h$ , one gets

$$\begin{aligned} E_N^{\phi_s}\left(\frac{p}{q} + h\right) &= \theta_0 \frac{\hat{w}_{N^2h}^{\phi_s}(0)}{\sqrt{q}} + \frac{e^{i\pi/4}}{\sqrt{q}} \\ &\quad \times \sum_{m \neq 0} \left( \theta_m \phi_s\left(\frac{m}{2Nqh}\right) \frac{e^{-i\pi \frac{m^2}{2q^2h}}}{\sqrt{2N^2h}} + \mathcal{O}\left((Nm/q)^{-3/2}\right) \right). \end{aligned}$$

When  $\frac{\xi}{2R} = \frac{m}{2Nqh} > 2$ ,  $\phi_s\left(\frac{m}{2Nqh}\right) = 0$ . Recalling that  $N \geq q$ , since  $\phi_s(t) = \phi(t)t^{-s}$ , the above equation can be rewritten

$$\begin{aligned} (3.11) \quad F_{s,N}^\phi\left(\frac{p}{q} + h\right) &= \frac{N^{1-s}\theta_0}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2ht^2}}{t^s} \phi(t) dt + G_{s,N}^\phi(h) \\ &\quad + \frac{N^{1-s}}{\sqrt{q}} \sum_{m \geq 1} \mathcal{O}_\phi\left((Nm/q)^{-3/2}\right). \end{aligned}$$

The last term is controlled by

$$\frac{N^{1-s}}{\sqrt{q}} \left( \frac{1}{(N/q)^{3/2}} \right) = \frac{q}{N^{1/2+s}},$$

which yields (3.9). □

Second, one wants to obtain a comparable estimate for  $F_{s,N}^{\eta^k}(x)$ , for all  $k \geq 1$ . For large  $k$  the mass of the function  $\eta^k$  is very small, so the same should happen to  $F_{s,N}^{\eta^k}(p/q + h)$  and  $G_{s,N}^{\eta^k}(h)$ . Then the estimate we want will be true in a trivial way, namely because all the terms involved are small. That is the content of the following lemma and its proof (an equivalent way to proceed would be to say that all terms coming from large  $k$  are negligible).

LEMMA 3.5. — *For any  $k \geq 1$  and  $0 < h \leq 1/q$ , one has*

$$F_{s,N}^{\eta^k}\left(\frac{p}{q} + h\right) = \frac{\theta_0 N^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2ht^2}}{t^s} \eta^k(t) dt + G_{s,N}^{\eta^k}(h) + \mathcal{O}_\eta(N^{1-s}2^{-k}).$$

*Proof.* — We are going to bound each sum and integral trivially, that is by introducing absolute values inside them. First, since  $\eta$  has support in  $[1/2, 2]$ , one has directly:

- by (2.4):

$$|F_{s,N}^{\eta^k}(x)| \leq \sum_{n=1}^{+\infty} \frac{1}{n^s} \left| \eta \left( \frac{\frac{n}{N} - 1}{2^{-k}} \right) \right| \leq \sum_{n=N+N2^{-k-1}}^{N+N2^{-k+1}} \frac{1}{n^s} \ll N^{1-s} 2^{-k}$$

- by (3.10):

$$\begin{aligned} |G_{s,N}^{\eta^k}(h)| &\ll (qh)^{s-\frac{1}{2}} \sum_{m: \eta^k(\frac{m}{2Nhq}) \neq 0} \frac{1}{m^s} \\ &\ll (qh)^{s-\frac{1}{2}} \frac{2Nhq2^{-k}}{(2Nhq)^s} \ll \sqrt{qh} 2^{-k} N^{1-s} \end{aligned}$$

- and

$$\begin{aligned} \left| \frac{\theta_0 N^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 ht^2}}{t^s} \eta^k(t) dt \right| &\ll \frac{N^{1-s}}{q^{1/2}} \int_{1+2^{-k-1}}^{1+2^{-k+1}} \frac{dt}{t^s} \\ &\ll 2^{-k} N^{1-s} q^{-1/2}, \end{aligned}$$

hence the result by (3.11), where we used that  $qh \leq 1$ .

□

For small  $k$  the function  $F_{s,N}^{\eta^k}$  should behave as  $F_{s,N}^{\phi}$ , but the estimates needed in order to prove the estimate for it are more delicate than the ones used in the proof of Lemma 3.4. The proof can probably be skipped at first reading.

LEMMA 3.6. — For every  $k \geq 2$  and  $0 < h \leq 1$ , one has

$$\begin{aligned} F_{s,N}^{\eta^k} \left( \frac{p}{q} + h \right) &= \frac{\theta_0 N^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 ht^2}}{t^s} \eta^k(t) dt + G_{s,N}^{\eta^k}(h) \\ &\quad + \mathcal{O}_{\eta} \left( \frac{\sqrt{q}}{N^s} + N^{1/2-s} \gamma_k \right), \end{aligned}$$

where the sequence  $(\gamma_k)_{k \geq 1}$  is positive and satisfies  $\sum_{k \geq 1} \gamma_k \ll 1$ .

Proof. — The proof starts as the one of Lemma 3.4. Recalling the notation (2.5) and the definition (3.7) of  $\eta_k$ , one has  $(\eta^k)_s(t) = \frac{\eta((t-1)/2^{-k})}{t^s}$ . For this function, one has

$$\begin{aligned} \widehat{w_R^{(\eta^k)_s}}(\xi) &= \int_{\mathbb{R}} (\eta^k)_s(t) e^{2i\pi(Rt^2-t\xi)} dt = \int_{\mathbb{R}} \frac{\eta(\frac{t-1}{2^{-k}})}{t^s} e^{2i\pi(Rt^2-t\xi)} dt \\ &= 2^{-k} e^{2i\pi(R-\xi)} \int_{\mathbb{R}} \frac{\eta(u)}{(1+u2^{-k})^s} e^{2i\pi(R2^{-2k}u^2-2^{-k}(\xi-2R)u)} du \\ &= 2^{-k} e^{2i\pi(R-\xi)} \widehat{w_{R2^{-2k}}^{\eta^k}}(2^{-k}(\xi-2R)), \end{aligned}$$

where  $\widetilde{\eta}^k(u) = \frac{\eta(u)}{(1+u2^{-k})^s}$ . Hence,

$$\begin{aligned}
 E_N^{(\eta^k)_s} \left( \frac{p}{q} + h \right) &= \frac{1}{\sqrt{q}} \sum_{m \in \mathbb{Z}} \theta_m \cdot \widehat{w_{N^2 h}^{(\eta^k)_s}} \left( \frac{Nm}{q} \right) \\
 (3.12) \qquad &= \frac{2^{-k}}{\sqrt{q}} \sum_{m \in \mathbb{Z}} \theta_m \cdot e^{2i\pi(N^2 h - \frac{Nm}{q})} \\
 &\qquad \qquad \qquad \times \widehat{w_{N^2 h 2^{-2k}}^{\widetilde{\eta}^k}} \left( 2^{-k} \left( \frac{Nm}{q} - 2N^2 h \right) \right)
 \end{aligned}$$

Here we apply again Lemma 3.2 and we obtain

$$\begin{aligned}
 &E_N^{(\eta^k)_s} \left( \frac{p}{q} + h \right) \\
 &= \frac{2^{-k}}{\sqrt{q}} \sum_{m \neq 0} \theta_m e^{2i\pi(N^2 h - \frac{Nm}{q})} \\
 &\quad \times \left( e^{i\pi/4} \widetilde{\eta}^k \left( \frac{2^{-k} \left( \frac{Nm}{q} - 2N^2 h \right)}{2N^2 h 2^{-2k}} \right) \frac{e^{-i\pi \frac{2^{-2k} \left( \frac{Nm}{q} - 2N^2 h \right)^2}{2N^2 h 2^{-2k}}}}{\sqrt{2N^2 h 2^{-2k}}} \right. \\
 &\quad \left. + \mathcal{O}_{\widetilde{\eta}^k} \left( \frac{\rho_{R_k, \xi_k}}{\sqrt{2^{-2k} N^2 h}} + \left( 1 + 2^{-k} \left| \frac{Nm}{q} - 2N^2 h \right| + N^2 h 2^{-2k} \right)^{-3/2} \right) \right).
 \end{aligned}$$

with  $R_k = 2^{-2k} N^2 h$  and  $\xi_k = 2^{-k} |Nm/q - 2N^2 h|$ . Finally, after simplification, one gets

$$\begin{aligned}
 F_{s,N}^{\eta^k} \left( \frac{p}{q} + h \right) &= N^{1-s} E_N^{(\eta^k)_s} \left( \frac{p}{q} + h \right) \\
 (3.13) \qquad &= \frac{(2qh)^{s-\frac{1}{2}}}{e^{-i\pi/4}} \sum_{m \in \mathbb{Z}} \frac{\theta_m}{m^s} e^{-2i\pi \frac{m^2}{q^2 h}} \eta^k \left( \frac{m}{2N h q} \right) \\
 (3.14) \qquad &= \frac{\theta_0 N^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 h t^2}}{t^s} \eta^k(t) dt + G_{s,N}^{\eta^k}(h) + L_N^k,
 \end{aligned}$$

where by Lemmas 3.4 and 3.5 one has

$$\begin{aligned}
 (3.15) \quad L_N^k &= \mathcal{O}_{\widetilde{\eta}^k} \left( \sum_{m \in J_N^k \cap \mathbb{Z}^*} \frac{2^{-k} N^{1-s} / \sqrt{q}}{\sqrt{2^{-2k} N^2 h}} \right. \\
 &\qquad \qquad \qquad \left. + \sum_{m \in \mathbb{Z}^*} \frac{2^{-k} N^{1-s} / \sqrt{q}}{\left( 1 + 2^{-k} \left| \frac{Nm}{q} - 2N^2 h \right| + N^2 h 2^{-2k} \right)^{3/2}} \right),
 \end{aligned}$$

with  $J_N^k = [(2 + 2^{-k-1})Nqh, (2 + 2^{-k+1})Nqh]$ .

First, as specified in Lemma 3.2, the constants involved in the  $\mathcal{O}_{\widetilde{\eta}^k}$  depend on upper bounds for some derivatives of  $\widetilde{\eta}^k$ , and then by the definition of  $\widetilde{\eta}^k$  we can assume they are fixed and independent on both  $k$  and  $s$ .

Let  $\{x\}$  stand for the distance from the real number  $x$  to the nearest integer.

The first sum in  $L_N^k$  is bounded above by:

- $\sqrt{q}N^{-s} + N^{1/2-s}$  when  $2^{-k} \in [\{2Nhq\}/4Nhq, \{2Nhq\}/Nhq]$ ,
- $\sqrt{q}N^{-s}$  otherwise.

In particular,  $x$  being fixed, the term  $N^{1/2-s}$  may appear only a finite number of times when  $k$  ranges in  $\mathbb{N}$ .

In the second sum, there is at most one integer  $m$  for which  $|Nm/q - 2N^2h| < N/2q$ , and the corresponding term is bounded above by

$$\begin{aligned} 2^{-k}N^{1-s}q^{-1/2}(1 + N/q2^{-k} + |N^2h|^{-2k})^{-3/2} &\leq 2^{-k}N^{1-s}q^{-1/2}(1 + N/q2^{-2k})^{-3/2} \\ &= N^{1/2-s} \frac{N^{1/2}q^{-1/2}2^{-k}}{(1 + N/q2^{-2k})^{-3/2}} \\ &\leq N^{1/2-s}\gamma_k, \end{aligned}$$

where  $\gamma_k = \sqrt{\frac{u_k}{(1+u_k)^3}}$  and  $u_k = 2^{-2k}N/q$ . The sum over  $k$  of this upper bound is finite, and this sum can be bounded above independently on  $N$  and  $q$ .

The rest of the sum is bounded, up to a multiplicative constant, by

$$\int_{u=0}^{+\infty} \frac{\sqrt{q}N^{-s} (2^{-k}N/q)du}{(1 + N^2h2^{-2k} + 2^{-k}|\frac{Nu}{q} - 2N^2h|)^{3/2}} \ll \frac{\sqrt{q}N^{-s}}{(1 + N^2h2^{-2k})^{1/2}},$$

which is less than  $\sqrt{q}N^{-s}$ . Hence the result. □

Now we are ready to prove Proposition 3.3.

Recall that we have the decomposition (3.8). The first term  $F_{s,N}^{\widetilde{\psi}}(x)$  is dealt with by Lemma 3.4. Now we deal with the sum  $\sum_{k \geq 1} F_{s,N}^{\eta^k}(x)$ . The third term  $\sum_{k \geq 1} F_{s,N}^{\zeta^k}(x)$  is absolutely similar to the second one (Lemmas 3.5 and 3.6 must be adapted in a straightforward way; we let them to the reader).

Recall that  $N \geq q$  and  $0 \leq h \leq q^{-1}$ . Let  $K$  be the unique integer such that  $2^{-K} \leq \frac{\sqrt{q}}{N} < 2^{-(K+1)}$ . If we apply Lemma 3.5 to  $F_{s,N}^{\eta^k}$  for every  $k > K$ , the error terms coming from that lemma give a contribution bounded, up



to a constant, by

$$\sum_{k>K} N^{1-s}2^{-k} = N^{1-s}2^{-K} \ll N^{1-s} \frac{\sqrt{q}}{N} = \frac{\sqrt{q}}{N^s} \ll \frac{1}{N^{s-1/2}}.$$

If we apply Lemma 3.6 to  $F_{s,N}^{\eta^k}$  for every  $k \leq K$ , the error terms coming from that lemma give a contribution bounded, up to a constant, by

$$\sum_{k \leq K} \left( \frac{\sqrt{q}}{N^s} + N^{1/2-s} \gamma_k \right) \ll \log N \frac{\sqrt{q}}{N^s} + N^{1/2-s} \ll \log q \frac{\sqrt{N}}{N^s} = \frac{\log q}{N^{s-1/2}},$$

where we use that the mapping  $x \mapsto \frac{\sqrt{x}}{\log x}$  is increasing for large  $x$ .

Gathering all the informations, and recalling that  $\sum_{k=2}^{+\infty} \eta^k(t) = \psi(t-1)$ , we have that

$$F_{s,N}^{\psi(\cdot-1)} \left( \frac{p}{q} + h \right) = \frac{N^{1-s} \theta_0}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 h t^2}}{t^s} \psi(t-1) dt + G_{s,N}^{\psi(\cdot-1)}(h) + \mathcal{O}_\eta \left( \frac{\log q}{N^{s-1/2}} \right).$$

As said above, the same inequalities remain true if we use the functions  $\zeta^k(t) = \eta((2-t)/2^{-k})$ , so the last inequality also holds for  $\psi(2-\cdot)$ .

Finally, recalling the decomposition (3.6) and (3.8) expressing  $\mathbb{1}_{(1,2)}$  and  $F_{s,N}^{\mathbb{1}_{(1,2)}}$  in terms of smooth terms, we get

$$(3.16) \quad F_{s,N}^{\mathbb{1}_{(1,2)}} \left( \frac{p}{q} + h \right) = \frac{N^{1-s} \theta_0}{\sqrt{q}} \int_1^2 \frac{e^{2i\pi N^2 h t^2}}{t^s} dt + G_{s,N}^{\mathbb{1}_{(1,2)}}(h) + \mathcal{O}_\eta \left( \frac{\log q}{N^{s-1/2}} \right),$$

and the result follows from the formula

$$G_{s,N}^{\mathbb{1}_{(1,2)}}(h) = G_{s,2N}(h) - G_{s,N}(h) + O(N^{-s+1/2}). \quad \square$$

## 4. Proof of the convergence Theorem 1.4

### 4.1. Convergence part: item (1)

Let  $x$  be such that  $\Sigma_s(x) < +\infty$ .

Recall the Definition 1.3 of the convergents of  $x$ . We begin by bounding  $F_{s,M}(x) - F_{s,N}(x)$  for any

$$(4.1) \quad 1/2 \leq q_j/4 \leq N < M < q_{j+1}/4.$$

We apply Proposition 3.3 with  $p/q = p_j/q_j$  and  $h = h_j$ , so that  $x = p/q + h$ . Due to (3.1), we can assume that  $h_j > 0$ . It is known that  $\frac{1}{2q_j q_{j+1}} \leq h_j = |x - p_j/q_j| < \frac{1}{q_j q_{j+1}}$ . Let  $N' \in [N, 2N]$ . Since  $4N h_j q_j < 4N/q_{j+1} < 1$ , the sums (3.5) appearing in  $G_{s,N'}(h_j)$  and  $G_{s,N}(h_j)$  have no terms, hence are equal to zero. This yields

$$F_{s,N'}(x) - F_{s,N}(x) = \frac{\theta_0}{\sqrt{q_j}} \int_N^{N'} \frac{e^{2i\pi h_j t^2}}{t^s} dt + O(N^{\frac{1}{2}-s} \log q_j).$$

Using the trivial bound directly or after writing

$$t^{-s} e^{2i\pi t^2} = \frac{1}{4i\pi} t^{-s-1} (e^{2i\pi t^2})'$$

and integrating by parts we get the bound

$$\left| \int_a^{a'} t^{-s} e^{2i\pi t^2} dt \right| \leq a^{-s} \min\left(\frac{s+1}{a}, a\right),$$

true for every  $0 < a \leq a' \leq 2a$  and every  $s > 0$ . So,

$$\begin{aligned} \left| \int_N^{N'} \frac{e^{2i\pi h_j t^2}}{t^s} dt \right| &= |h_j|^{s/2-1/2} \left| \int_{N\sqrt{h_j}}^{N'\sqrt{h_j}} \frac{e^{2i\pi u^2}}{u^s} du \right| \\ &\ll |h_j|^{-1/2} N^{-s} \min(|N\sqrt{h_j}|^{-1}, |N\sqrt{h_j}|). \end{aligned}$$

One deduces (using that  $q_j h_j$  is equivalent to  $q_{j+1}^{-1}$ ) that

$$(4.2) \quad |F_{s,N'}(x) - F_{s,N}(x)| \ll |\theta_0| \frac{\sqrt{q_{j+1}}}{N^s} \min\left(\frac{N}{\sqrt{q_j q_{j+1}}}, \frac{\sqrt{q_j q_{j+1}}}{N}\right) + N^{\frac{1}{2}-s} \log q_j.$$

Thus, writing  $F_{s,M} - F_{s,N}$  as a dyadic sum and using (4.2) for each term we have

$$|F_{s,M}(x) - F_{s,N}(x)| \ll |\theta_0| \delta_j \frac{\sqrt{q_{j+1}}}{(\sqrt{q_j q_{j+1}})^s} + \frac{\log q_j}{q_j^{s-1/2}}$$

for every  $M, N$  as in (4.1). Finally, recalling that  $\theta_0$  is equal to zero when  $q_j \neq 2 \times \text{odd}$ , fixing an integer  $j_0 \geq 3$ , for any  $M > N > q_{j_0}$ , one has

$$|F_{s,M}(x) - F_{s,N}(x)| \ll \sum_{j \geq j_0, q_j \neq 2 * \text{odd}} \delta_j \frac{\sqrt{q_{j+1}}}{(\sqrt{q_j q_{j+1}})^s} + \sum_{j \geq j_0} \frac{\log q_j}{q_j^{s-1/2}}.$$

The second series goes to 0 when  $j_0 \rightarrow \infty$  and the first does when  $\sum_s(x) < \infty$ .

**4.2. Divergence part: item (2)**

Let  $0 < \varepsilon < 1/2$  a small constant. Let  $N_j = \varepsilon q_j$  and  $M_j = 2\varepsilon\sqrt{q_j q_{j+1}}$ . Proceeding exactly as in the previous proof we get

$$F_{s,M_j}(x) - F_{s,N_j}(x) = \frac{\theta_0}{\sqrt{q_j}} \int_{N_j}^{M_j} \frac{e^{2i\pi h_j t^2}}{t^s} dt + \mathcal{O}\left(q_j^{\frac{1}{2}-s} \log q_j\right).$$

Since  $e^{2i\pi h_j t^2} = 1 + \mathcal{O}(\varepsilon)$  inside the integral, as soon as  $q_j \neq 2 * \text{odd}$ , one has

$$\begin{aligned} |F_{s,M_j}(x) - F_{s,N_j}(x)| &\geq \frac{|\theta_0|}{\sqrt{q_j}} \frac{M_j - N_j}{2 \cdot M_j^s} \geq |\theta_0| \varepsilon \frac{2\sqrt{q_{j+1}} - \sqrt{q_j}}{2^{1+s} \cdot \varepsilon^s \cdot (q_j q_{j+1})^{s/2}} \\ &\gg \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}}, \end{aligned}$$

which is infinitely often large by our assumption. Hence the divergence of the series.

**5. Local  $L^2$  bounds for the function  $F_s$**

Further intermediary results are needed to study the local regularity of  $F_s$ .

PROPOSITION 5.1. — *Let  $h > 0$ ,  $1/2 < s \leq 1$  and  $q^2 h \ll 1$ . We have*

$$\begin{aligned} (5.1) \quad F_{s,N}\left(\frac{p}{q} + 2h\right) - F_{s,N}\left(\frac{p}{q} + h\right) &= \frac{\theta_0}{\sqrt{q}} \int_0^N \frac{e^{2i\pi 2ht^2} - e^{2i\pi ht^2}}{t^s} dt \\ &\quad + G_{s,N}(2h) - G_{s,N}(h) + \mathcal{O}\left(|qh|^{s-1/2}\right). \end{aligned}$$

*Proof.* — First, one writes

$$(5.2) \quad F_{s,N}(x) = F_{s,N}^{1(0,1]}(x) = \sum_{1 \leq m \leq 1 + \log_2 N} F_{s,N/2^m}^{1(1,2]}(x).$$

As we did in the previous sections, we are going to estimate (5.1) using the

decomposition (5.2). So we start to work with the functions

$$\begin{aligned}
 (5.3) \quad F_{s, \tilde{N}}^{\mathbb{1}_{(1,2]}} \left( \frac{p}{q} + 2h \right) - F_{s, \tilde{N}}^{\mathbb{1}_{(1,2]}} \left( \frac{p}{q} + h \right) \\
 = F_{s, \tilde{N}}^{\mathbb{1}_{(1,2)}} \left( \frac{p}{q} + 2h \right) - F_{s, \tilde{N}}^{\mathbb{1}_{(1,2)}} \left( \frac{p}{q} + h \right) \\
 + \mathcal{O}(\tilde{N}^{-s} \min(1, \tilde{N}^2 h))
 \end{aligned}$$

with  $\tilde{N} \leq N$ , and for them we will get the estimation given by equation (5.5). Then, we will use that for  $\tilde{N} = N/2^m$  and (5.2) will give the result in the proposition.

In order to prove (5.5) the argument starts as before: we use the decompositions (3.7) and (3.8), so that our goal is to find estimates for terms like  $F_{s, \tilde{N}}^{\eta^k} \left( \frac{p}{q} + 2h \right) - F_{s, \tilde{N}}^{\eta^k} \left( \frac{p}{q} + h \right)$  and  $F_{s, \tilde{N}}^{\zeta^k} \left( \frac{p}{q} + 2h \right) - F_{s, \tilde{N}}^{\zeta^k} \left( \frac{p}{q} + h \right)$ . As above, we detail the calculations for  $\eta^k$ , the ones associated with  $\zeta^k$  are similar, and omitted.

We start from equation (3.12) applied with  $h$  and  $2h$ , and then we apply Lemma 3.2, but this time equation (3.4) instead of (3.3). Let us introduce for all integers  $k$  the quantity

$$\begin{aligned}
 (5.4) \quad E_{\tilde{N}}^k := F_{s, \tilde{N}}^{\eta^k} \left( \frac{p}{q} + 2h \right) - F_{s, \tilde{N}}^{\eta^k} \left( \frac{p}{q} + h \right) \\
 + \frac{\theta_0 \tilde{N}^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi \tilde{N}^2 2ht^2} - e^{2i\pi \tilde{N}^2 ht^2}}{t^s} \eta^k(t) dt \\
 + G_{s, \tilde{N}}^{\eta^k}(2h) - G_{s, \tilde{N}}^{\eta^k}(h).
 \end{aligned}$$

By the exact same computations as in Lemma 3.6, one obtains the upper bound

$$\begin{aligned}
 |E_{\tilde{N}}^k| \ll \beta_{\tilde{N}}^k \sum_{m \in J_k \cap \mathbb{Z}^*} \frac{2^{-k} \tilde{N}^{1-s} / \sqrt{q}}{(2^{-2k} \tilde{N}^2 h)^{-1/2}} \\
 + \sum_{m \in \mathbb{Z}^*} \frac{(2^{-k} \tilde{N}^{1-s} / \sqrt{q}) \tilde{N}^2 h 2^{-2k}}{(1 + \tilde{N}^2 h 2^{-2k} + 2^{-k} |\frac{\tilde{N}m}{q} - 2\tilde{N}^2 h|)^{5/2}},
 \end{aligned}$$

with  $J_{\tilde{N}}^k = [(2 + 2^{-k-1})\tilde{N}qh, (2 + 2^{-k+1})\tilde{N}qh]$  and

$$\beta_{\tilde{N}}^k = \begin{cases} 1 & \text{if } 2^{-2k} \tilde{N}^2 h \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as at the end of the proof of Lemma 3.6, since  $h \ll q^{-2}$ , we can bound the sums by

$$|E_{\tilde{N}}^k| \ll \beta_{\tilde{N}}^k \delta_{\tilde{N}qh} 2^{-2k} \tilde{N}^{2-s} \frac{\sqrt{h}}{\sqrt{q}} + \frac{(\tilde{N}^{1/2-s})\sqrt{(\tilde{N}/q)2^{-2k}\tilde{N}^2h2^{-2k}}}{(1 + \tilde{N}^2h2^{-2k} + (\tilde{N}/q)2^{-2k})^{5/2}} + \frac{(\sqrt{q}\tilde{N}^{-s})\tilde{N}^2h2^{-2k}}{(1 + \tilde{N}^2h2^{-2k})^{3/2}}$$

with  $\delta_t = \begin{cases} 0 & \text{when } t < 1/4 \\ 1 & \text{when } t \geq 1/4 \end{cases}$ . Adding up in  $k \geq 1$ , we get  $\sum_{k=1}^{\infty} |E_{\tilde{N}}^k| \ll \tilde{E}_{\tilde{N}}$   
with

$$\tilde{E}_{\tilde{N}} := \frac{\sqrt{1/hq}}{\tilde{N}^s} \min(1, \tilde{N}qh)^2 + \frac{\sqrt{\tilde{N}}}{\tilde{N}^s} \min(1, \tilde{N}qh) + \frac{\sqrt{q}}{\tilde{N}^s} \min(1, \tilde{N}^2h).$$

The same holds true for the functions  $\zeta^k(t) = \eta((2-t)/2^{-k})$ .

In addition, the estimate for  $F_{s,\tilde{N}}^{\psi} \left(\frac{p}{q} + 2h\right) - F_{s,\tilde{N}}^{\psi} \left(\frac{p}{q} + h\right)$  goes exactly like the one for  $\eta^1$ , since they both are compactly supported, regular functions. Finally, by the decomposition (3.6), using (5.3) and a similar estimation for  $G_{s,\tilde{N}}^{\mathbb{1}_{(1,2]}} - G_{s,\tilde{N}}^{\mathbb{1}_{(1,2)}}$  we finally obtain that

$$(5.5) \quad F_{s,\tilde{N}}^{\mathbb{1}_{(1,2]}} \left(\frac{p}{q} + 2h\right) - F_{s,\tilde{N}}^{\mathbb{1}_{(1,2]}} \left(\frac{p}{q} + h\right) = \frac{\theta_0}{\sqrt{q}} \int_0^{\tilde{N}} \frac{e^{2i\pi 2ht^2} - e^{2i\pi ht^2}}{t^s} dt + G_{s,\tilde{N}}^{\mathbb{1}_{(1,2]}}(2h) - G_{s,\tilde{N}}^{\mathbb{1}_{(1,2]}}(h) + \mathcal{O}(\tilde{E}_{\tilde{N}}).$$

Now, applying (5.5) for  $\tilde{N} = N/2^m$  and (5.2), by using the formula for the sum of a geometric progression we get that

$$\sum_{1 \leq m \leq 1 + \log_2 N} \tilde{E}_{N/2^m} \leq \sum_{m \in \mathbb{Z}} \tilde{E}_{N/2^m} \ll \tilde{E}_{1/qh} + \tilde{E}_{1/\sqrt{h}},$$

since the three summands defining  $\tilde{E}_{\tilde{N}}$  have a unique maximum as functions in the variable  $\tilde{N}$  (for any  $s$  in  $(1/2, 1]$ ), two of them at  $\tilde{N} = 1/qh$  and the other at  $\tilde{N} = 1/\sqrt{h}$ . The result of the proposition follows from that inequality and the fact that  $\tilde{E}_{1/qh} + \tilde{E}_{1/\sqrt{h}} \ll (hq)^{s-1/2}$  in the range  $h \ll q^{-2}$ ,  $1/2 \leq s \leq 1$ . □

We also need to control the  $L^2$  norm of the main term.

LEMMA 5.2. — Let  $0 < s \leq 1$  and fix  $0 < H < 1$ . Let

$$f_{s,N}(\cdot) = \int_0^N \frac{e^{2i\pi t^2(\delta+2\cdot)} - e^{2i\pi t^2(\delta+)}}{t^s} dt.$$

Then for any  $N > 0$ ,  $\|f_{s,N}(\cdot)\|_{L^2(\tilde{\mu}_H)} \ll \min(H^{(s-1)/2}, H|\delta|^{(s-3)/2})$ .

*Proof.* — Let us treat first the case  $|\delta| < H/4$ . Using a change of variable, one has

$$f_{s,N}(h) = H^{(s-1)/2} \int_0^{N\sqrt{H}} \frac{e^{2i\pi t^2 \frac{\delta+2h}{H}} - e^{2i\pi t^2 \frac{\delta+h}{H}}}{t^s} dt.$$

We are interested in the range  $H < h < 2H$ , and in this case the ratios  $\frac{\delta+2h}{H}, \frac{\delta+h}{H}$  are bounded, so that the integral is bounded by a constant independent of  $N$ . One deduces that  $\|f_{s,N}(\cdot)\|_{L^2(\tilde{\mu}_H)} \ll H^{(s-1)/2}$ .

Assume then that  $|\delta| > 4H$ . Assume that  $\delta > 0$  (the same holds true with negative  $\delta$ 's). Using a change of variable, one has

$$f_{s,N}(h) = |\delta|^{(s-1)/2} \int_0^{N\sqrt{|\delta|}} \frac{e^{2i\pi t^2(1+\frac{2h}{\delta})} - e^{2i\pi t^2(1+\frac{h}{\delta})}}{t^s} dt.$$

The integral between 0 and 1 is clearly  $\mathcal{O}(h/|\delta|)$ . For the other part, one has (after integration by parts)

$$\int_1^{N\sqrt{\delta}} \frac{e^{2i\pi t^2(1+\frac{2h}{\delta})} - e^{2i\pi t^2(1+\frac{h}{\delta})}}{t^s} dt = \mathcal{O}(h/|\delta|),$$

so that  $|f_{s,N}(h)| \ll H|\delta|^{(s-3)/2}$  for any  $H < h < 2H$ . Hence one concludes that  $\|f_{s,N}(\cdot)\|_{L^2(\tilde{\mu}_H)} \ll H|\delta|^{(s-3)/2}$ .

It remains us to deal with the case  $H/4 < \delta \leq 4H$ . One observes that

$$f_{s,N}(h) = D(\delta+2h) - D(\delta+h) + \mathcal{O}(H), \quad \text{where } D(v) = \int_1^N t^{-s} e^{2i\pi vt^2} dt.$$

It is enough to get the bound

$$\int_0^H |D(v)|^2 dv \ll H^s,$$

which follows from the fact that  $|D(v)| \ll |v|^{(s-1)/2}$  when  $s < 1$  and  $|D(v)| \ll 1 + \log(1/|v|)$  when  $s = 1$ . □

Finally, the oscillating behavior of  $G_{s,N}(h)$  gives us the following.

PROPOSITION 5.3. — Let  $0 < H \leq q^{-2}$  and  $|\delta| \leq \sqrt{H}/q$ . Let

$$g_{s,N}(\cdot) = F_{s,N} \left( \frac{p}{q} + \delta + 2\cdot \right) - F_{s,N} \left( \frac{p}{q} + \delta + \cdot \right) - \frac{\theta_0}{\sqrt{q}} f_{s,N}(\cdot).$$

One has  $\|g_{s,N}\|_{L^2(\tilde{\mu}_H)} \ll H^{\frac{s-1/2}{2}}$ .

*Proof.* — We consider  $\mu_H = (\tilde{\mu}_H)|_{\mathbb{R}^+}$ . By (3.1), it is enough to treat the case  $\delta + h > 0$  and  $\delta + 2h > 0$ .

One writes the decompositions

$$F_{s,N} \left( \frac{p}{q} + \delta + 2h \right) = F_{s,N} \left( \frac{p}{q} \right) + \sum_{n \geq 1} F_{s,N} \left( \frac{p}{q} + 2h_n \right) - F_{s,N} \left( \frac{p}{q} + h_n \right)$$

and

$$F_{s,N} \left( \frac{p}{q} + \delta + h \right) = F_{s,N} \left( \frac{p}{q} \right) + \sum_{n \geq 1} F_{s,N} \left( \frac{p}{q} + 2\tilde{h}_n \right) - F_{s,N} \left( \frac{p}{q} + \tilde{h}_n \right),$$

where  $h_n := 2^{-n}(\delta + 2h)$  and  $\tilde{h}_n := 2^{-n}(\delta + h)$ .

Further, we apply Proposition 5.1 to each term of the form

$$F_{s,N} \left( \frac{p}{q} + 2h_n \right) - F_{s,N} \left( \frac{p}{q} + h_n \right)$$

(and similarly with  $\tilde{h}_n$ ). Summing over  $n \geq 1$ , one obtains

$$\begin{aligned} F_{s,N} \left( \frac{p}{q} + \delta + 2h \right) &= F_{s,N} \left( \frac{p}{q} \right) + \sum_{n \geq 1} G_{s,N}(2h_n) - G_{s,N}(h_n) \\ &\quad + \mathcal{O} \left( |qh_n|^{s-1/2} \right) \\ &= F_{s,N} \left( \frac{p}{q} \right) + G_{s,N}(\delta + 2h) + \sum_{n \geq 1} \mathcal{O} \left( |qh_n|^{s-1/2} \right) \\ &= F_{s,N} \left( \frac{p}{q} \right) + G_{s,N}(\delta + 2h) \\ &\quad + O \left( (q(|\delta| + |2h|))^{s-1/2} \right). \end{aligned}$$

The same estimate holds true with  $\tilde{h}_n$ . Finally, we deduce that

$$(5.6) \quad g_{s,N}(h) = G_{s,N}(\delta + 2h) - G_{s,N}(\delta + h) + O \left( (q(|\delta| + |h|))^{s-1/2} \right).$$

Thus, since  $q(|\delta| + |h|) \ll \sqrt{H}$ , it is enough to show that

$$(5.7) \quad \|G_{s,N}(\delta + \cdot)\|_{L^2(\mu_H)} \ll H^{\frac{s-1/2}{2}}.$$

Assume first that  $|\delta| \geq 3H$ . By expanding the square and changing the order of summation, and using that  $\delta + 2H \leq 2|\delta|$ , we have for some

$$c_{n,m} \geq 0$$

$$\begin{aligned} \|G_{s,N}(\delta + \cdot)\|_{L^2(\mu_H)}^2 &\ll (q|\delta|)^{2s-1} \sum_{n,m=1}^{2\lfloor 2N|\delta|q \rfloor} \frac{|\theta_m|}{m^s} \frac{|\theta_n|}{n^s} \left| \int_{\delta+H+c_{n,m}}^{\delta+2H} e^{2i\pi \frac{n^2-m^2}{q^2h}} \frac{dh}{H} \right| \\ &\ll (q|\delta|)^{2s-1} \sum_{n,m=1}^{2\lfloor 2N|\delta|q \rfloor} \frac{|\theta_m|}{m^s} \frac{|\theta_n|}{n^s} \left| \int_{\delta+H}^{\delta+2H} e^{2i\pi \frac{n^2-m^2}{q^2h}} \frac{dh}{H} \right|. \end{aligned}$$

Observe that for  $|M| \geq 1$  and  $0 < \varepsilon \ll 1$ , one has

$$\int_1^{1+\varepsilon} e^{2i\pi \frac{M}{t}} dt = \int_1^{1+\varepsilon} \frac{-t^2}{2i\pi M} (e^{2i\pi \frac{M}{t}})' dt \ll \frac{1}{|M|}$$

where the last inequality comes from integrating by parts. Hence, the previous sum is bounded above by

$$(q|\delta|)^{2s-1} \left[ \sum_{m \geq 1} \frac{1}{m^{2s}} + \frac{|\delta|}{H} q^2 |\delta| \sum_{m \geq 1} \frac{1}{m^{1+s}} \sum_{j \geq 1} \frac{1}{j^{1+s}} \right],$$

with  $j = |n - m|$ . The term between brackets is bounded by a universal constant (since  $q^2\delta^2/H \leq 1$ ), hence (5.7) holds true. It is immediate that the same holds true with  $(\tilde{\mu}_H)_{\mathbb{R}^-}$ .

Further, assume that  $|\delta| < 3H$ . Setting  $H_k = 2^{-k}H$ , one has

$$\|G_{s,N}(\delta + \cdot)\|_{L^2(\tilde{\mu}_H)}^2 \leq \int_0^{5H} |G_{s,N}(h)|^2 \frac{dh}{H} \leq \sum_{k \geq -2} 2^{-k} \|G_{s,N}(\cdot)\|_{L^2(\mu_{H_k})}^2.$$

Now, observing that  $[H_k, H_{k-1}] \subset 3H_k + ([-2H_k, -H_k] \cup [H_k, 2H_k])$ , one can apply (5.7) with  $H = H_k$  and  $\delta = 3H_k$  to get

$$\|G_{s,N}(\cdot)\|_{L^2(\mu_{H_k})}^2 \leq \|G_{s,N}(\delta_k + \cdot)\|_{L^2(\mu_{H_k})} \leq H_k^{\frac{s-1/2}{2}} = H^{\frac{s-1/2}{2}} 2^{-k \frac{s-1/2}{2}}.$$

Summing over  $k$  yields the result. □

## 6. Proof of Theorem 1.5

### 6.1. Lower bound for the local $L^2$ -exponent $\alpha_{F_s}$

Assume that  $\Sigma_s(x) < \infty$  (see equation (1.4)), so that the series  $F_{s,N}(x)$  converges to  $F_s(x)$ . Recall that  $p_j/q_j$  stands for the convergents of  $x$ .



Pick  $N$  such that  $0 \leq |F_s(x) - F_{s,N}(x)| < H$  and  $N^{\frac{1}{2}-s} \leq H^2$ . Since

$$\begin{aligned} \|F_s(x + \cdot) - F_{s,N}(x + \cdot)\|_{L^2(\tilde{\mu}_H)} &\leq \frac{\|F_s(x + \cdot) - F_{s,N}(x + \cdot)\|_{L^2([0,1])}}{H/2} \\ &\ll \frac{N^{\frac{1}{2}-s}}{H} \leq H, \end{aligned}$$

and since one has

$$\begin{aligned} F_s(x + \cdot) - F_s(x) &= F_s(x + \cdot) - F_{s,N}(x + \cdot) \\ &\quad + F_{s,N}(x + \cdot) - F_{s,N}(x) + F_{s,N}(x) - F_s(x), \end{aligned}$$

one deduces that

$$\|F_s(x + \cdot) - F_s(x)\|_{L^2(\tilde{\mu}_H)} = \|F_{s,N}(x + \cdot) - F_{s,N}(x)\|_{L^2(\tilde{\mu}_H)} + \mathcal{O}(H).$$

Thus, it is enough to take care of the local  $L^2$ -norm of  $F_{s,N}(x+h) - F_{s,N}(x)$ . One has

$$\begin{aligned} (6.1) \quad \|F_{s,N}(x + \cdot) - F_{s,N}(x)\|_{L^2(\tilde{\mu}_H)} &\leq \sum_{k \geq 1} \left\| F_{s,N}(x + 2\frac{\cdot}{2^k}) - F_{s,N}(x + \frac{\cdot}{2^k}) \right\|_{L^2(\tilde{\mu}_H)} \\ &\leq \sum_{k \geq 1} \|F_{s,N}(x + 2\cdot) - F_{s,N}(x + \cdot)\|_{L^2(\tilde{\mu}_{H_k})}, \end{aligned}$$

where  $H_k = H2^{-k}$ . Let us introduce the function  $f(h) = F_{s,N}(x + 2h) - F_{s,N}(x + h)$ .

Let  $j_H$  be the smallest integer such that  $q_j^{-2} \leq H$ . For every  $k \geq 1$ , and let  $j$  be the unique integer such that  $q_{j+1}^{-2} \leq H_k < q_j^{-2}$  (necessarily  $j \geq j_H - 1$ ). Using that  $|x - p_j/q_j| = |h_j| \leq q_j^{-2}$ , one sees that

$$\|f\|_{L^2(\tilde{\mu}_{H_k})} = \left\| F_{s,N}\left(\frac{p_j}{q_j} + h_j + 2\cdot\right) - F_{s,N}\left(\frac{p_j}{q_j} + h_j + \cdot\right) \right\|_{L^2(\tilde{\mu}_{H_k})}.$$

Since  $|h_j| < 1/q_j q_{j+1} \leq \sqrt{H_k}/q_j$ , we can apply Proposition 5.3 and Lemma 5.2 with  $H_k$  and  $\delta = h_j$  to get

$$\begin{aligned} \|f\|_{L^2(\tilde{\mu}_{H_k})} &\ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \min\left(H_k^{(s-1)/2}, H_k |h_j|^{(s-3)/2}\right) \\ &\ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} H_k^{(s-1)/2} \min\left(1, \left|\frac{h_j}{H_k}\right|^{(s-3)/2}\right). \end{aligned}$$

In order to finish the proof we are going to consider three different cases:

Case 1:  $s - 1 + 1/2r_{\text{odd}}(x) > 0$ . — Since  $h_j = q_j^{-r_j}$  we have

$$\|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + |\theta_0| H_k^{(s-1)/2} \min\left(|h_j|^{\frac{1}{2r_j}}, \frac{H_k^{(3-s)/2}}{|h_j|^{\frac{3-s}{2} - \frac{1}{2r_j}}}\right),$$

and optimizing in  $|h_j|$  we get

$$\|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + |\theta_0| H_k^{\frac{s-1+1/2r_j}{2}} \ll H_k^{(s-1+1/2r_{\text{odd}}(x)+o(H_k))/2}$$

by the definition of  $r_{\text{odd}}(x)$ . Adding up in  $k$  finishes the proof in this case.

Case 2:  $s - 1 + 1/2r_{\text{odd}}(x) = 0$  and  $s = 1$ . — In this case it is enough to show that  $\sum_{k \geq 1} \|f\|_{L^2(\tilde{\mu}_{H_k})} < \infty$ . We have

$$\|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}}$$

which implies

$$\sum_{q_{j+1}^{-2} \leq H_k \leq q_j^{-2}} \|f\|_{L^2(\tilde{\mu}_{H_k})} \ll \sum_{q_{j+1}^{-2} \leq H_k \leq q_j^{-2}} H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \log(q_{j+1}/q_j).$$

This yields

$$\sum_{k \geq 1} \|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H^{\frac{s-1/2}{2}} + \sum_{j: q_j \neq 2 * \text{odd}} \frac{1}{\sqrt{q_j}} \log \frac{q_{j+1}}{q_j} \ll 1 + \Sigma_s(x) < +\infty.$$

Case 3:  $s - 1 + 1/2r_{\text{odd}}(x) = 0$  and  $s < 1$ . — Since  $h_j \asymp 1/q_j q_{j+1}$ , we have

$$\|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \min\left(H_k^{(s-1)/2}, \frac{H_k}{(q_j q_{j+1})^{(s-3)/2}}\right),$$

so

$$\sum_{q_{j+1}^{-2} \leq H_k \leq q_j^{-2}} \|f\|_{L^2(\tilde{\mu}_{H_k})} \ll \left( \sum_{q_{j+1}^{-2} \leq H_k \leq q_j^{-2}} H_k^{\frac{s-1/2}{2}} \right) + \frac{|\theta_0|}{\sqrt{q_j}} \left( \frac{1}{q_j q_{j+1}} \right)^{(s-1)/2}.$$

Finally,

$$\sum_{k \geq 1} \|f\|_{L^2(\tilde{\mu}_{H_k})} \ll H^{\frac{s-1/2}{2}} + \sum_{j, q_j \neq 2 * \text{odd}} \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}} \ll 1 + \Sigma_s(x) < \infty.$$

**6.2. Upper bound for the local  $L^2$ -exponent**

Let  $K$  be a large constant. Let  $0 < H \leq (1/K)q^{-2}$ , with  $q \neq 2 * \text{odd}$  and  $N > H^{-2} > 1$ . We apply Propositions 5.1 and 5.3 to get

$$\begin{aligned} \left\| F_{s,N} \left( \frac{p}{q} + 2 \cdot \right) - F_{s,N} \left( \frac{p}{q} + \cdot \right) \right\|_{L^2(\tilde{\mu}_H)} &= \frac{|\theta_0|}{\sqrt{q}} \left\| \tilde{F}_s(\cdot) \right\|_{L^2(\tilde{\mu}_H)} + \mathcal{O}(H^{\frac{s-1}{2}}) \end{aligned}$$

with

$$\tilde{F}_s(h) = \int_0^N \frac{e^{4i\pi ht^2} - e^{2i\pi ht^2}}{t^s} dt.$$

Using a change of variable, and then after integrating by parts, one obtains

$$\begin{aligned} \tilde{F}_s(h) &= h^{\frac{s-1}{2}} \int_0^{N\sqrt{h}} \frac{e^{4i\pi t^2} - e^{2i\pi t^2}}{t^s} dt \\ &= h^{\frac{s-1}{2}} \left( \int_0^{+\infty} \frac{e^{4i\pi t^2} - e^{2i\pi t^2}}{t^s} dt + \mathcal{O} \left( (N\sqrt{|h|})^{-s-1} \right) \right). \end{aligned}$$

For  $0 \leq s \leq 1$ , by complex integration one can check that

$$\int_0^{+\infty} \frac{e^{4i\pi t^2} - e^{2i\pi t^2}}{t^s} dt = \left( \frac{i}{2\pi} \right)^{\frac{1-s}{2}} \Gamma\left(\frac{3-s}{2}\right) \frac{2^{-\frac{1-s}{2}} - 1}{1-s} \neq 0,$$

where for the case  $s = 1$  the right hand of the identity is understood as the limit when  $s \rightarrow 1$ . For  $N > H^{-2} > 1$  we have  $(N\sqrt{H})^{-s-1} \leq H^{\frac{3}{2}(1+s)} \leq H$ , hence

$$\left\| \tilde{F}_s(\cdot) \right\|_{L^2(\tilde{\mu}_H)} = C_s H^{\frac{s-1}{2}} (1 + \mathcal{O}(H))$$

for some non-zero constant  $C_s$ . Since  $0 < H \leq q^{-2}/K$ , we deduce that

$$(6.2) \quad \left\| F_{s,N} \left( \frac{p}{q} + 2 \cdot \right) - F_{s,N} \left( \frac{p}{q} + \cdot \right) \right\|_{L^2(\tilde{\mu}_H)} \geq \frac{H^{\frac{s-1}{2}}}{\sqrt{q}}$$

when  $H$  becomes small enough.

Now, pick a convergent  $p_j/q_j$  of  $x$  with  $q_j \neq 2 * \text{odd}$ , and take  $H_j = (1/K)|h_j|$ . One can check that

$$H_j \leq (1/K) \frac{1}{q_j q_{j+1}} \leq (1/K) \frac{1}{q_j^2}.$$

Then, we apply (6.2) to obtain that for every  $N \geq H_j^{-2}$ , one has

$$\left\| F_{s,N} \left( \frac{p_j}{q_j} + 2 \cdot \right) - F_{s,N} \left( \frac{p_j}{q_j} + \cdot \right) \right\|_{L^2(\tilde{\mu}_{H_j})} \geq \frac{H_j^{\frac{s-1}{2}}}{\sqrt{q_j}} = H_j^{\frac{s-1}{2}} h_j^{1/(2r_j)},$$

which is greater than  $\gg H_j^{\frac{s-1+1/r_j}{2}}$ .

On the other hand, by the triangular inequality,

$$\left| F_{s,N} \left( \frac{p_j}{q_j} + 2h \right) - F_{s,N} \left( \frac{p_j}{q_j} + h \right) \right| \leq \left| F_{s,N} \left( \frac{p_j}{q_j} + 2h \right) - F_{s,N}(x) \right| + \left| F_{s,N} \left( \frac{p_j}{q_j} + h \right) - F_{s,N}(x) \right|,$$

which implies that for  $\tilde{H}_j = H_j$  or  $\tilde{H}_j = 2H_j$ , one has

$$\begin{aligned} & \|F_{s,N}(x + \cdot) - F_{s,N}(x)\|_{L^2(\tilde{\mu}_{\tilde{H}_j})} \\ & \geq \frac{1}{2} \left\| F_{s,N} \left( \frac{p_j}{q_j} + 2 \cdot \right) - F_{s,N} \left( \frac{p_j}{q_j} + \cdot \right) \right\|_{L^2(\tilde{\mu}_{\tilde{H}_j})} \geq H_j^{\frac{s-1+1/r_j}{2}} \end{aligned}$$

Now, we can choose  $N$  so large that

$$\|F_{s,N}(x + \cdot) - F_{s,N}(x)\|_{L^2(\tilde{\mu}_{\tilde{H}_j})} = \|F_s(x + \cdot) - F_s(x)\|_{L^2(\tilde{\mu}_{\tilde{H}_j})} + \mathcal{O}(\tilde{H}_j),$$

and we finally obtain

$$\|F_s(x + \cdot) - F_s(x)\|_{L^2(\tilde{\mu}_{\tilde{H}_j})} \gg \tilde{H}_j^{\frac{s-1+1/r_j}{2}}.$$

Since this occurs for an infinite number of  $j$ , i.e. for an infinite number of small real numbers  $\tilde{H}_j$  converging to zero, one concludes that

$$\alpha_{F_s}(x) \leq \liminf_{j \rightarrow +\infty} \frac{s-1+1/r_j}{2} = \frac{s-1+1/r_{\text{odd}}(x)}{2}.$$

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