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GAPS IN SUMSETS OF s PSEUDO s-TH POWERS

by Javier CILLERUELO† & Jean-Marc DESHOUILLERS (*)

Abstract. — We study the length of the gaps between consecutive members in the sumset sA when A is a pseudo s-th power sequence, with $s \geqslant 2$. We show that, almost surely, $\limsup(b_{n+1} - b_n)/\log b_n = s^s s!/\Gamma^s(1/s)$, where b_n are the elements of sA.

Résumé. — On étudie la taille des différences entre les termes consécutifs de la suite sA où A est une suite de pseudo-puissances s-ièmes avec $s \geqslant 2$. On montre qu'on a presque sûrement $\limsup(b_{n+1} - b_n)/\log b_n = s^s s!/\Gamma^s(1/s)$, où les b_n sont les éléments de la suite sA.

1. Introduction

Erdős and Rényi [3] proposed in 1960 a probabilistic model for sequences A growing like the s-th powers: they build a probability space $(\mathcal{U}, \mathcal{T}, P)$ and a sequence of independent random variables $(\xi_n)_{n\in\mathbb{N}}$ with values in $\{0, 1\}$ and $P(\xi_n = 1) = \frac{1}{s}n^{-1+1/s}$; to any $u \in \mathcal{U}$, they associate the sequence of positive integers $A = A_u$ such that $n \in A_u$ if and only if $\xi_n(u) = 1$. In short, the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s}n^{-1+1/s}$. The counting function of these random sequences A satisfies almost surely the asymptotic relation $|A \cap [1, x]| \sim x^{1/s}$, whence the terminology *pseudo s-th powers*. Erdős and Rényi studied the random variable $r_s(A, n)$ which counts the number of representations of n in the form $n = a_1 + \cdots + a_s$,

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$a_1 \leqslant \cdots \leqslant a_s,\ a_i \in A$. For the simplest case $s = 2$ they proved that $r_2(A, n)$ converges to a Poisson distribution with parameter $\pi/8$, when $n \to \infty$. They also claimed the analogous result for $s > 2$ but their analysis did not take into account the dependence of some events. J. H. Goguel [4] proved indeed that for each integer d, the sequence of the integers n such that $r_s(A, n) = d$ has almost surely the density $\lambda_s^d e^{-\lambda_s}/d!$, where $\lambda_s = \Gamma^s(1/s)/(s^s s!)$. B. Landreau [5] gave a proof of this result based on correlation inequalities and also showed that the sequence of random variables $(r_s(A, n))_n$ converges in law towards the Poisson distribution with parameter λ_s.

In particular, both the sets of the integers belonging, or not belonging, to $sA = \{a_1 + \cdots + a_s\ :\ a_i \in A\}$ have almost surely a positive density and it makes sense to study the length of the gaps in sA. The aim of the paper is to obtain a precise estimate for the maximal length of such gaps.

Theorem 1.1. — *For any $s \geqslant 2$ the sequence $sA = (b_n)_n$, sum of s copies of a pseudo s-th power sequence A, satisfies almost surely*

$$\limsup_{n\to\infty} \frac{b_{n+1} - b_n}{\log b_n} = \frac{s^s s!}{\Gamma^s(1/s)}. \tag{1.1}$$

We remark that this result is heuristically consistent with the fact that for a random sequence S with $P(n \in S) = 1 - e^{-\lambda}$, we have $\limsup(s_{m+1} - s_m)/\log s_m = 1/\lambda$, an exercise on Borel–Cantelli Lemma.

2. Notation, hint of the proof and general lemmas

2.1. Notation

We retain the notation of the introduction, for the probability space $(\mathcal{U}, \mathcal{T}, P)$ and the definition of the random sequences $A = A_u$, where the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s} n^{-1+1/s}$. We further use the following notation.

(1) We write ω to denote a set of distinct positive integers and we denote by E_ω and E_ω^c the events

$$E_\omega = \{\omega \subset A\} \qquad \text{and} \qquad E_\omega^c = \{\omega \not\subset A\}$$

respectively. We write $\omega \sim \omega'$ to mean that $\omega \cap \omega' \neq \emptyset$ but $\omega \neq \omega'$; we remark that $\omega \sim \omega'$ if and only if the events E_ω and $E_{\omega'}$ are distinct and dependent.

If $\omega = \{x_1, \ldots, x_r\}$ we write

$\sigma(\omega) = \{a_1x_1 + \cdots + a_rx_r : \ a_1 + \cdots + a_r = s, \ a_i \geqslant 1\}$

for the set of all integers which can be written as a sum of s integers using all the integers $x_1, \ldots, x_r$. For an integer z we let

$$\Omega_z = \{\omega : \ z \in \sigma(\omega)\}.$$

(2) Given $\alpha > 0$, we denote by I_i the interval $[i, i + \alpha \log i]$ and we denote by $F_i^{(\alpha)}$, or simply F_i when the context is clear, the event

$$F_i = F_i^{(\alpha)} = \{sA \cap I_i = \emptyset\}.$$

We denote by Ω_{I_i} the family of sets

$$\Omega_{I_i} = \{\omega : \ \sigma(\omega) \cap I_i \neq \emptyset\}.$$

(3) We let $\lambda_s = \dfrac{\Gamma^s(1/s)}{s!s^s}$.

(4) We use Vinogradov's notation $\ll$, where $f \ll g$ is equivalent to Landau's notation $f = O(g)$.

2.2. Hints for the proof to Theorem 1.1

We wish to prove that for $\alpha > \lambda_s^{-1}$, the event $F_i^{(\alpha)}$, defined in Section 2.1, occurs — almost surely — for only finitely many i's and that for $\alpha < \lambda_s^{-1}$ it occurs — almost surely — for infinitely many i's. There is a flavour of Borel–Cantelli and indeed a key point in the proof is Lemma 3.5 which asserts relation (3.2), namely

$$P(F_i^{(\alpha)}) = i^{-\alpha\lambda_s + o(1)}. \tag{2.1}$$

Let us first see how we can obtain that relation. Here α is fixed and we do not mention it anylonger. By the definition, the event F_i occurs if and only if for any family of s non necessarily distinct integers which sum up to an integer in I_i, at least one of them is not in A; with our notation, this leads to

$$F_i = \bigcap_{\omega \in \Omega_{I_i}} E_\omega^c.$$

If the ω's which are involved had pairwise empty intersections, the events E_ω^c would be independent and we would have

$$P(F_i) = \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c).$$

Although this is not the case, the structure of the events E_ω, which are finite intersections of events taken from from an independent family, permits us to use Harris' inequality (or FKG inequality, cf. Theorem 2.2 below) to get the lower bound

$$P(F_i) \geqslant \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c).$$

It also permits us, thanks to Janson's Correlation Inequality (cf. Theorem 2.2 below), to get the upper bound

$$P(F_i) \leqslant \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) \times \exp\left(2\sum_{\omega\sim\omega'} P(E_\omega \cap E_{\omega'})\right),$$

where the notation $\omega \sim \omega'$ is defined in Section 2.1. It is then a matter of computation, based on Lemma 2.3, to get the central inequality (2.1).

When $\alpha > \lambda_s^{-1}$ the series $\sum_i P(F_i^{(\alpha)})$ converges and the Borel–Cantelli lemma immediately implies that for such α the events $F_i^{(\alpha)}$ almost surely occur for only finitely many i's.

When $\alpha < \lambda_s^{-1}$ the series $\sum_i P(F_i)$ diverges, but this is not enough to conclude directly since the events F_i's are not independent. However, P. Erdős and A. Rényi proved that a weak dependence among the F_i's is sufficient for obtaining an "inverse Borel–Cantelli" result (cf. Theorem 2.1 below). It is thus important to have a small upper bound for $P(F_i \cap F_j) - P(F_i)P(F_j)$ in average. With our notation, we have

$$F_i \cap F_j = \bigcap_{\omega\in I_i\cup I_j} E_\omega^c,$$

and here again Janson's inequality will help us to obtain a suitable bound.

2.3. Probabilistic results

We use the following generalization of the Borel–Cantelli Lemma, proved indeed by P. Erdős and A. Rényi in 1959 [2].

Theorem 2.1 (Borel–Cantelli Lemma). — *Let $(F_i)_{i\in\mathbb{N}}$ be a sequence of events and let $Z_n = \sum_{i\leqslant n} P(F_i)$.*

If the sequence $(Z_n)_n$ is bounded, then, with probability 1, only finitely many of the events F_i occur.

If the sequence $(Z_n)_n$ tends to infinity and

$$\lim_{n\to\infty} \frac{\sum_{1\leqslant i<j\leqslant n} P(F_i\cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

then, with probability 1, infinitely many of the events F_i occur.

The next result, which combines Harris' inequality and Janson's Correlation Inequality, can be found in [1].

Theorem 2.2. — *Let $(E_\omega)_{\omega\in\Omega}$ be a finite collection of events which are intersections of elementary independent events and assume that $P(E_\omega)\leqslant 1/2$ for any $\omega\in\Omega$. Then*

$$\prod_{\omega\in\Omega} P(E_\omega^c)\leqslant P\Big(\bigcap_{\omega\in\Omega} E_\omega^c\Big)\leqslant \prod_{\omega\in\Omega} P(E_\omega^c)\times\exp\Big(2\sum_{\omega\sim\omega'} P(E_\omega\cap E_{\omega'})\Big),$$

where $\omega\sim\omega'$ means that the events E_ω and $E_{\omega'}$ are dependent events.

2.4. A technical lemma

Lemma 2.3. — *Given $1\leqslant t\leqslant s-1$ and positive integers $a_1,\dots,a_t$ we have, as z tends to infinity:*

$$\sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\dots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s}\ll z^{-1+t/s}. \tag{1}$$

$$\sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\dots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s}\big(z-(a_1x_1+\dots+a_tx_t)\big)^{-2t/s}\ll z^{-t/s}\log z. \tag{2}$$

$$\sum_{\substack{1\leqslant x_1<\dots<x_s\\ x_1+\dots+x_s=z}} (x_1\cdots x_s)^{-1+1/s}\sim s^s\lambda_s. \tag{3}$$

Proof. — (1) We have

$$\begin{aligned}\sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\dots+a_tx_t=z}} &(x_1\cdots x_t)^{-1+1/s}\\ &=(a_1\cdots a_t)^{1-1/s}\sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\dots+a_tx_t=z}} (a_1x_1\cdots a_tx_t)^{-1+1/s}\\ &\leqslant (a_1\cdots a_t)^{1-1/s}\sum_{\substack{1\leqslant y_1,\dots,y_t\\ y_1+\dots+y_t=z}} (y_1\cdots y_t)^{-1+1/s}.\end{aligned}$$

If $y_1 + \cdots + y_t = z$ then at least one of them, say y_t, is greater than z/t and is determined by $y_1, \ldots, y_{t-1}$. Thus,

$$\begin{aligned}\sum_{\substack{1\leqslant x_1,\ldots,x_t\\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s} &\ll z^{-1+1/s} \sum_{1\leqslant y_1,\ldots,y_{t-1}<z} (y_1\cdots y_{t-1})^{-1+1/s}\\ &\ll z^{-1+1/s}\left(\sum_{1\leqslant y<z} y^{-1+1/s}\right)^{t-1}\\ &\ll z^{-1+1/s}(z^{1/s})^{t-1}\\ &\ll z^{-1+t/s}.\end{aligned}$$

(2) We have

$$\begin{aligned}&\sum_{\substack{1\leqslant x_1,\ldots,x_t\\ a_1x_1+\cdots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s}\big(z-(a_1x_1+\cdots+a_tx_t)\big)^{-2t/s}\\ &\qquad= \sum_{1\leqslant m<z}(z-m)^{-2t/s} \sum_{\substack{1\leqslant x_1,\ldots,x_t\\ a_1x_1+\cdots+a_tx_t=m}} (x_1\cdots x_t)^{-1+1/s}\\ (\text{by (1)})&\qquad\ll \sum_{1\leqslant m<z}(z-m)^{-2t/s}m^{-1+t/s}\\ &\qquad\ll \sum_{1\leqslant m\leqslant z/2}(z-m)^{-2t/s}m^{-1+t/s} + \sum_{z/2<m<z}(z-m)^{-2t/s}m^{-1+t/s}\\ &\qquad\ll z^{-2t/s}z^{t/s} + z^{-1+t/s}\sum_{z/2<m<z}(z-m)^{-2t/s}\\ &\qquad\ll z^{-t/s} + z^{-1+t/s}\left(z^{1-2t/s}\log z\right)\\ &\qquad\ll z^{-t/s}\log z.\end{aligned}$$

Remark 2.4. — Except in the case when $s = 2$ and $t = 1$, the upper bound in (2) may be replaced by $z^{-t/s}$.

(3) It follows from Lemma 3 of [5]. □

3. Proof of Theorem 1.1

3.1. Combinatorial lemmas

LEMMA 3.1. — *We have*

$$\sum_{\omega\in\Omega_z} P(E_\omega) \sim \lambda_s$$

as $z \to \infty$*.*

Proof. — We have

$$\sum_{\omega\in\Omega_z} P(E_\omega) = \sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega) + \sum_{\substack{\omega\in\Omega_z\\|\omega|\leqslant s-1}} P(E_\omega). \tag{3.1}$$

The main contribution comes from the first sum.

$$\sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega) = \frac{1}{s^s} \sum_{\substack{1\leqslant x_1<\cdots<x_s\\x_1+\cdots+x_s=z}} (x_1\cdots x_s)^{-1+1/s} \sim \lambda_s$$

as $z \to \infty$, by Lemma 2.3 (3). For the second sum we have

$$\sum_{\substack{\omega\in\Omega_z\\|\omega|\leqslant s-1}} P(E_\omega) \leqslant \sum_{1\leqslant r\leqslant s-1} \sum_{\substack{a_1,\dots,a_r\\a_1+\cdots+a_r=s}} \sum_{\substack{1\leqslant x_1,\dots,x_r\\a_1x_1+\cdots+a_rx_r=z}} (x_1\cdots x_r)^{-1+1/s}$$

$$(\text{Lem. } 2.3\,(1)) \quad \ll \sum_{1\leqslant r\leqslant s-1} z^{\frac{r}{s}-1} \ll z^{-1/s}. \qquad \square$$

LEMMA 3.2. — *For any* $z \leqslant z'$ *we have*

$$\sum_{\substack{\omega\sim\omega'\\\omega\in\Omega_z,\ \omega'\in\Omega_{z'}}} P(E_\omega \cap E_{\omega'}) \ll z^{-1/s}\log z.$$

Proof. — If $\omega \in \Omega_z$ then there exist some $r \leqslant s$ and some positive integers $a_1,\dots,a_r$ with $a_1+\cdots+a_r = s$ such that $a_1x_1+\cdots+a_rx_r = z$. Thus, any pair of sets $\omega \sim \omega'$ with $\omega \in \Omega_z,\ \omega' \in \Omega_{z'},\ z \leqslant z'$ is of the form

$$\omega = \{x_1,\dots,x_t,u_{t+1},\dots,u_r\}$$
$$\omega' = \{x_1,\dots,x_t,v_{t+1},\dots,v_{r'}\}$$

with $1 \leqslant t \leqslant r \leqslant r' \leqslant s$ and positive integers $a_1,\dots,a_r$ and $b_1,\dots,b_{r'}$ with

$$a_1x_1+\cdots+a_tx_t\ + a_{t+1}u_{t+1}+\cdots+a_ru_r = z$$
$$b_1x_1+\cdots+b_tx_t\ + b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'} = z'.$$

Of course if $r = t$ then $\omega = \{x_1,\dots,x_r\}$ and $r' \geqslant t+1$. Otherwise $\omega = \omega'$. Similarly, when $r' = t$, we have $r \geqslant t+1$.

Given positive integers $z, z', t, r, r', a_1, \dots, a_r, b_1, \dots, b_{r'}$ we estimate the sum

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'})$$

where the symbol $\sum^*$ means that the sum is extended to the pairs $\omega \sim \omega'$ satisfying the above conditions. We distinguish several cases according to the values of r and r'.

Case $r \geqslant t+1$ and $r' \geqslant t+1$. — We have

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'}) \leqslant \sum_{\substack{1\leqslant x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \times \sum_{\substack{1\leqslant u_{t+1},\dots,u_r, \\ a_{t+1}u_{t+1}+\cdots+a_ru_r \\ =z-(a_1x_1+\cdots+a_tx_t)}} (u_{t+1}\cdots u_r)^{-1+1/s} \times \sum_{\substack{1\leqslant v_{t+1},\dots,v_{r'}, \\ b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'} \\ =z'-(b_1x_1+\cdots+b_tx_t)}} (v_{t+1}\cdots v_{r'})^{-1+1/s}.$$

By Lemma 2.3 (1) we have

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'}) \ll \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \times \Big(z-(a_1x_1+\cdots+a_tx_t)\Big)^{\frac{r-t}{s}-1} \times \Big(z'-(b_1x_1+\cdots+b_tx_t)\Big)^{\frac{r'-t}{s}-1}.$$

Using the inequality $AB \leqslant A^2+B^2$, we get

$$\begin{aligned}\sum_{\omega\sim\omega'}^{*} & P(E_\omega \cap E_{\omega'}) \\ &\leqslant \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s}\Big(z-(a_1x_1+\cdots+a_tx_t)\Big)^{2(r-t-s)/s} \\ &\quad + \sum_{\substack{x_1,\dots,x_t \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s}\Big(z'-(b_1x_1+\cdots+b_tx_t)\Big)^{2(r'-t-s)/s} \\ &\text{(Lem. 2.3 (2))} \\ &\ll z^{-t/s}\log z \ll z^{-1/s}\log z.\end{aligned}$$

Case $r=t$ and $r'\geqslant t+1$. — In this case we have

$$\begin{aligned}\sum_{\omega\sim\omega'}^{*} &P(E_\omega\cap E_{\omega'})\\ &\leqslant \sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z\\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s}\\ &\qquad\times \sum_{\substack{1\leqslant v_{t+1},\dots,v_{r'}\\ b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'}\\ =z'-(b_1x_1+\cdots+b_tx_t)}} (v_{t+1}\cdots v_{r'})^{-1+1/s}\\ (\text{Lem. } 2.3\,(1))\ &\leqslant \sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z\\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s}\Big(z'-(b_1x_1+\cdots+b_tx_t)\Big)^{\frac{r'-t}{s}-1}\\ &\leqslant \sum_{\substack{1\leqslant x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s}\\ &\ll z^{\frac{t}{s}-1}\ll z^{-1/s}.\end{aligned}$$

Case $r'=t$ and $r\geqslant t+1$. — This case is similar to the previous one. □

Lemma 3.3. — *Let $\alpha>0$ and the let I_i be the interval $[i,i+\alpha\log i]$. For any $i\leqslant j$ we have*

$$\sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'})\ll i^{-1/s}(\log i)^2(\log j).$$

Proof. — We have

$$\begin{aligned}\sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) &\leqslant \sum_{z\in I_i,\ z'\in I_j}\ \sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_z,\ \omega'\in\Omega_{z'}}} P(E_\omega\cap E_{\omega'})\\ (\text{Lem. } 3.2)\qquad &\ll \sum_{z\in I_i,\ z'\in I_j} z^{-1/s}\log z\\ &\ll (\log i)^2(\log j)i^{-1/s}.\end{aligned}$$
□

Lemma 3.4. — *We have*

$$\prod_{\omega\in\Omega_{I_i}} P(E_\omega^c)=i^{-\alpha\lambda_s+o(1)}.$$

Proof. — We observe that

$$\prod_{z\in I_i}\prod_{\omega\in\Omega_z} P(E_\omega^c) \leqslant \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) \leqslant \prod_{\substack{\omega\in\Omega_{I_i}\\ |\omega|=s}} P(E_\omega^c) = \prod_{z\in I_i}\prod_{\substack{\omega\in\Omega_z\\ |\omega|=s}} P(E_\omega^c).$$

Writing $P(E_\omega^c) = 1 - P(E_\omega)$ and taking logarithms we have

$$\log\left(\prod_{z\in I_i}\prod_{\omega\in\Omega_z} P(E_\omega^c)\right) = \sum_{z\in I_i}\sum_{\omega\in\Omega_z} \log(1-P(E_\omega)) \sim -\sum_{z\in I_i}\sum_{\omega\in\Omega_z} P(E_\omega)$$
$$(\text{Lem. } 3.1) \qquad \sim -\sum_{z\in I_i}\lambda_s \sim -\alpha\lambda_s \log i.$$

On the other hand,

$$\log\left(\prod_{z\in I_i}\prod_{\substack{\omega\in\Omega_z\\ |\omega|=s}} P(E_\omega^c)\right) = \sum_{z\in I_i}\sum_{\substack{\omega\in\Omega_z\\ |\omega|=s}} \log(1-P(E_\omega)) \sim -\sum_{z\in I_i}\sum_{\substack{\omega\in\Omega_z\\ |\omega|=s}} P(E_\omega)$$
$$= -\sum_{z\in I_i}\sum_{\substack{x_1<\cdots<x_s\\ x_1+\cdots+x_s=z}} \frac{1}{s^s}(x_1\cdots x_s)^{-1+1/s}$$
$$(\text{Lem. } 2.3(3)) \qquad \sim -\lambda_s\alpha\log i. \qquad \square$$

LEMMA 3.5. — *We have*

$$P(F_i) = i^{-\alpha\lambda_s + o(1)}. \tag{3.2}$$

Proof. — As we noticed it in Section 2.2, we have

$$F_i = \bigcap_{\omega\in\Omega_{I_i}} E_\omega^c.$$

Since $P(E_\omega) \leqslant 1/2$ for any ω, Theorem 2.2 applies and we have

$$\prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) \leqslant P(F_i) \leqslant \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) \times \exp\left(2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'})\right).$$

After Lemma 3.4 we only need to prove

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'}) = o(1).$$

But it is a consequence of Lemma 3.3 with $j = i$.

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\ \omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'}) \ll i^{-1/s+o(1)}. \qquad \square$$

Lemma 3.6. — *If $i < j$ and $I_i \cap I_j = \emptyset$ then*

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \leqslant P(F_i)P(F_j)(1 + O(j^{-1/s} \log j)).$$

Proof. — It is clear that

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) = \left(\prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \right) \left(\prod_{\omega \in \Omega_{I_j}} P(E_\omega^c) \right) \left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1}.$$

Harris' inequality, applied to the first two products, gives

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \leqslant P(F_i)P(F_j) \left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1}.$$

The logarithm of the last factor is

$$-\sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} \log(1 - P(E_\omega)) \sim \sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega)$$

Since $I_i \cap I_j = \emptyset$, if $\omega \in \Omega_{I_i} \cap \Omega_{I_j}$ then $|\omega| \leqslant s-1$. Thus,

$$\begin{aligned} \sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega) &\leqslant \sum_{\substack{\omega \in \Omega_{I_j} \\ |\omega| \leqslant s-1}} P(E_\omega) \\ &\leqslant \sum_{z \in I_j} \sum_{1 \leqslant r \leqslant s-1} \sum_{a_1 + \cdots + a_r = s} \sum_{\substack{1 \leqslant x_1 < \cdots \leqslant x_r \\ a_1 x_1 + \cdots + a_r x_r = z}} (x_1 \cdots x_r)^{-1+1/s} \\ (\text{Lem. } 2.3\,(1)) \quad &\ll j^{-1/s} \log j. \end{aligned}$$

Thus,

$$\left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1} \leqslant 1 + O(j^{-1/s} \log j)$$

which ends the proof of the Lemma. □

3.2. End of the proof

After those Lemmas we are ready to finish the proof of Theorem 1.1. If $\alpha > 1/\lambda_s$ then

$$\sum_i P(F_i) = \sum_i i^{-\alpha\lambda_s + o(1)} < \infty$$

and Theorem 2.1 implies that with probability 1 only finitely many events F_i occur. This proves that

$$\limsup_{k\to\infty} \frac{b_{k+1}-b_k}{\log b_k} \leqslant 1/\lambda_s.$$

If $\alpha < 1/\lambda_s$ then

$$Z_n = \sum_{i\leqslant n} P(F_i) = \sum_{i\leqslant n} i^{-\alpha\lambda_s+o(1)} = n^{1-\alpha\lambda_s+o(1)} \to \infty.$$

If in addition

$$\lim_{n\to\infty} \frac{\sum_{1\leqslant i<j\leqslant n} P(F_i\cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0, \tag{3.3}$$

Theorem 2.1 implies that with probability 1 infinitely many events F_i occur and

$$\limsup_{k\to\infty} \frac{b_{k+1}-b_k}{\log b_k} \geqslant 1/\lambda_s.$$

Note that $P(F_i\cap F_j) \geqslant P(F_i)P(F_j)$ for all $1\leqslant i<j\leqslant n$, so the limit (3.3) is not negative.

We next prove (3.3). We observe that

$$F_i\cap F_j = \bigcap_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} E_\omega^c,$$

so we can use Janson inequality to get

$$P(F_i\cap F_j) \leqslant \prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P\big(E_\omega^c\big) \times \exp\left(2 \sum_{\substack{\omega\sim\omega' \\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'})\right).$$

Observe that

$$\sum_{\substack{\omega\sim\omega' \\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) \leqslant \sum_{\substack{\omega\sim\omega' \\ \omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'}) + \sum_{\substack{\omega\sim\omega' \\ \omega,\omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) + \sum_{\substack{\omega\sim\omega' \\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}).$$

Applying Lemma 3.3 to the three sums we have

$$\sum_{\substack{\omega\sim\omega' \\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) \ll i^{-1/s}(\log i)^3 + j^{-1/s}(\log j)^3 + i^{-1/s}(\log i)^2(\log j),$$

and so

$$\exp\left(2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'})\right)\leqslant 1+O\left(i^{-1/s}(\log i)^2(\log j)\right). \tag{3.4}$$

Thus,

$$P(F_i\cap F_j)\leqslant \prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P\big(E_\omega^c\big)\times\big(1+O(i^{-1/s}(\log i)^2(\log j))\big). \tag{3.5}$$

Since $\alpha<\lambda_s$, the number $\beta=(1-\alpha\lambda_s)/2$ is positive. Now we split the sum in (3.3) into three sums:

$$\Delta_{1n}=\sum_{\substack{1\leqslant i<j\leqslant n\\ n^\beta<i<j-\alpha\log j}} P(F_i\cap F_j)-P(F_i)P(F_j)$$

$$\Delta_{2n}=\sum_{\substack{1\leqslant i<j\leqslant n\\ i\leqslant n^\beta}} P(F_i\cap F_j)-P(F_i)P(F_j)$$

$$\Delta_{3n}=\sum_{\substack{1\leqslant i<j\leqslant n\\ j-\log j\leqslant i\leqslant j}} P(F_i\cap F_j)-P(F_i)P(F_j)$$

(1) Estimate of Δ_{1n}. Since in this case we have $I_i\cap I_j=\emptyset$, we can apply Lemma 3.6 to (3.5) to get

$$\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P\big(E_\omega^c\big)\leqslant P(F_i)P(F_j)(1+O(j^{-1/s}\log j)).$$

This inequality and (3.5) gives

$$P(F_i\cap F_j)\leqslant P(F_i)P(F_j)\times\big(1+O(i^{-1/s}(\log i)^2(\log j))\big),$$

so

$$\begin{aligned}P(F_i\cap F_j)-P(F_i)P(F_j)&\ll P(F_i)P(F_j)i^{-1/s}(\log i)^2(\log j)\\ &\ll n^{-\beta/s+o(1)}P(F_i)P(F_j).\end{aligned}$$

Thus

$$\Delta_{1n}\ll n^{-\beta/s+o(1)}\sum_{i,j\leqslant n}P(F_i)P(F_j)\ll n^{-\beta/s+o(1)}Z_n^2. \tag{3.6}$$

(2) Estimate of Δ_{2n}. In this case we use the crude estimate

$$P(F_i \cap F_j) - P(F_i)P(F_j) \leqslant P(F_j). \tag{3.7}$$

We have

$$\Delta_{2n} \leqslant \sum_{j\leqslant n}\sum_{i\leqslant j^\beta} P(F_j) \leqslant \sum_{j\leqslant n} j^\beta P(F_j) \leqslant n^\beta Z_n \leqslant n^{-\beta+o(1)} Z_n^2, \tag{3.8}$$

since $Z_n = n^{1-\alpha\lambda_s+o(1)} = n^{2\beta+o(1)}$.

(3) Estimate of Δ_{3n}. Again we use (3.7) and we have

$$\Delta_{3n} \leqslant \sum_{j\leqslant n}\ \sum_{j-\alpha\log j\leqslant i\leqslant j} P(F_j) \leqslant \alpha \log n \sum_{j\leqslant n} P(F_j) \leqslant n^{-2\beta+o(1)} Z_n^2. \tag{3.9}$$

Finally, using the estimates in (3.6), (3.8) and (3.9) we have

$$\frac{\sum_{1\leqslant i<j\leqslant n} P(F_i\cap F_j) - P(F_i)P(F_j)}{Z_n^2} \ll n^{-\beta/s+o(1)} + n^{-\beta+o(1)} + n^{-2\beta+o(1)} \to 0.$$

This ends the proof of (3.3) and hence that of Theorem 1.1. □

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