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$\mathcal{L}$-invariants, partially de Rham families, and local-global compatibility

$\mathcal{L}$-INVARIANTS, PARTIALLY DE RHAM FAMILIES, AND LOCAL-GLOBAL COMPATIBILITY

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ABSTRACT. — Let $F_\wp$ be a finite extension of $\mathbb{Q}_p$. By considering partially de Rham families, we establish a Colmez–Greenberg–Stevens formula (on Fontaine–Mazur $\mathcal{L}$-invariants) for (general) 2-dimensional semi-stable non-crystalline representations of the group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$. As an application, we prove local-global compatibility results for completed cohomology of quaternion Shimura curves, and in particular the equality of Fontaine–Mazur $\mathcal{L}$-invariants and Breuil's $\mathcal{L}$-invariants, in critical case.

RÉSUMÉ. — Soit $F_\wp$ une extension finie de $\mathbb{Q}_p$. En étudiant des familles de représentations galoisiennes partiellement de Rham, on donne une formule de Colmez–Greenberg–Stevens (concernant les invariants $\mathcal{L}$ de Fontaine–Mazur) pour les représentations semi-stables non cristallines de dimension 2 de $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$. Comme application, on montre dans le cas critique des résultats de compatibilité local-global pour le H^1-complété d'une courbe de Shimura quaternionique, et en particulier l'égalité des invariants $\mathcal{L}$ de Fontaine–Mazur et Breuil.

Introduction

Let F be a totally real number field, B a quaternion algebra of center F such that there exists only one real place of F where B is split. One can associate to B a system of quaternion Shimura curves $\{M_K\}_K$, proper and smooth over F, indexed by compact open subgroups K of $(B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. We fix a prime number p, and suppose that there exists only one place $\wp$ of F above p, let $\quad_\wp$ be the set of embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. Suppose B is split at $\wp$, i.e. $(B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \cong \mathrm{GL}_2(F_\wp)$ (where $F_\wp$ denotes the completion of F at $\wp$). Let E be a finite extension of $\mathbb{Q}_p$ su ciently large containing all the

embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$, with $\mathcal{O}_E$ its ring of integers and ϖ_E a uniformizer of $\mathcal{O}_E$.

Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$ over E such that ρ appears in the étale cohomology of M_K for K su ciently small (so ρ is associated to certain Hilbert eigenforms). By Emerton's completed cohomology theory [27], one can associate to ρ a unitary admissible Banach representation $\widehat{}(\rho)$ of $\mathrm{GL}_2(F_\wp)$ as follows: put

$$\widetilde{H}^1(K^p, E) := \left(\varprojlim_n \varinjlim_{K'_p} H^1_{\text{ét}}(M_{K^p K'_p} \times_F \overline{F}, \mathcal{O}_E/\varpi_E^n) \right) \otimes_{\mathcal{O}_E} E$$

where K^p denotes the component of K outside p, and K'_p runs over open compact subgroups of $\mathrm{GL}_2(F_\wp)$. This is an E-Banach space equipped with a continuous action of $\mathrm{GL}_2(F_\wp) \times \mathrm{Gal}_F \times \mathcal{H}^p$, where $\mathcal{H}^p$ denotes certain commutative Hecke algebra outside p over E. Put

$$\widehat{}(\rho) := \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\rho, \widetilde{H}^1(K^p, E)).$$

This is an admissible unitary Banach representation of $\mathrm{GL}_2(F_\wp)$ over E, which plays an important role in p-adic Langlands program [11]. In [24], it's proved that if the local Galois representation $\rho_\wp := \rho|_{\mathrm{Gal}_{F_\wp}}$ (where $\mathrm{Gal}_{F_\wp} := \mathrm{Gal}(\overline{F_\wp}/F_\wp)$) is semi-stable non-crystalline and *non-critical*, then one could find the Fontaine–Mazur $\mathcal{L}$-invariants $(\mathcal{L}_\sigma)_{\sigma \in \Sigma_\wp}$ of $\rho_\wp$ (which are *invisible* in classical local Langlands correspondence) in $\widehat{}(\rho)$, generalizing some of Breuil's results in [13].

However, when $F_\wp$ is di erent from $\mathbb{Q}_p$, a new phenomenon is that there exist 2-dimensional semi-stable non-crystalline $\mathrm{Gal}_{F_\wp}$-representations which are *critical* (or more precisely, critical for some embeddings in $\wp$). We consider this case in this paper. Denote by $S_c(\rho_\wp)$ (resp. $S_n(\rho_\wp)$) the set of embeddings where $\rho_\wp$ is critical (resp. non-critical), one can associate to $\rho_\wp$ the Fontaine–Mazur $\mathcal{L}$-invariants $\mathcal{L}_\sigma$ but only for embeddings σ in $S_n(\rho_\wp)$. In this paper, we prove that these $\mathcal{L}$-invariants can be found in $\widehat{}(\rho)$, meanwhile, we prove a partial result on Breuil's locally analytic socle conjecture [15] for embeddings in $S_c(\rho_\wp)$.

One important ingredient in [24] is Zhang's generalization [44, Thm. 1.1] of Colmez–Greenberg–Stevens formula [22] (on $\mathcal{L}$-invariants) in $F_\wp$-case. But the results in [44] are only for non-critical case. The following theorem generalizes such a formula in general case, which is of interest in its own right.

THEOREM 0.1 (Corollary 2.3). — *Let A be an affinoid E-algebra, V be a locally free A-module of rank 2 equipped with a continuous A-linear action of $\mathrm{Gal}_{F_\wp}$, let z be an E-point of A, and suppose*

(1) *V is trianguline with a triangulation given by*

$$0 \longrightarrow \mathcal{R}_A(\delta_1) \longrightarrow D_{\mathrm{rig}}(V) \longrightarrow \mathcal{R}_A(\delta_2) \longrightarrow 0,$$

where δ_i are continuous characters of $F_\wp^\times$ in $A^\times$,

(2) *$V_z := z^*V$ is semi-stable non-crystalline with $\mathcal{L}_\sigma \in E$ for $\sigma \in S_n(V_z)$ the associated Fontaine–Mazur $\mathcal{L}$-invariants (cf. Section 1.3, where $S_n(V_z)$ denotes the set of embeddings where V_z is non-critical),*

(3) *V is $S_c(V_z)$-de Rham (cf. Section 2, where $S_c(V_z) = \wp \setminus S_n(V_z)$);*

then the differential form

$$d\log(\delta_1\delta_2^{-1}(p)) + \sum_{\sigma \in S_n(V_z)} \mathcal{L}_\sigma d(\mathrm{wt}(\delta_1\delta_2^{-1})_\sigma) \in ^1_{A/E}$$

vanishes at the point z.

Such formula was firstly established by Greenberg–Stevens in [29, Thm. 3.14] in the case of 2-dimensional ordinary $\mathrm{Gal}_{\mathbb{Q}_p}$-representations by Galois cohomology computations. In [22], Colmez generalized this theorem to 2-dimensional trianguline $\mathrm{Gal}_{\mathbb{Q}_p}$-representations case by Galois cohomology computations and computations in Fontaine's rings. Theorem 0.1 in non-critical case (i.e. $S_c(V_z) = \emptyset$) was obtained by Zhang in [44], by generalizing Colmez's method. In [37], Pottharst generalized [29, Thm. 3.14] to rank 2 triangulable $(\varphi,)$-modules (in $\mathbb{Q}_p$ case) by studying cohomology of $(\varphi,)$-modules.

The hypothesis (3) in Theorem 0.1 is new but crucial. In fact, the statement does *not* hold (in general) if the condition (3) is replaced by (only) fixing the Hodge–Tate weights for $\sigma \in S_c(V_z)$ (namely, replacing the $S_c(V_z)$-de Rham family by $S_c(V_z)$-Hodge–Tate family). Partially de Rham families appear naturally in the study of p-adic automorphic forms, e.g. one encounters such families when studying locally analytic vectors in completed cohomology of Shimura curves (e.g. see Proposition 4.14), or certain families of overconvergent Hilbert modular forms (e.g. see Appendix A, in particular Conjecture A.9). Note Theorem 0.1 also applies for families of $F_\wp$-*analytic* $\mathrm{Gal}_{F_\wp}$-representations (cf. [7], which in fact can be viewed as special cases of partially de Rham families). Indeed, this theorem also includes the case of parallel Hodge–Tate weights for some embeddings (and such embeddings would be contained in $S_c(V_z)$).

Return to the global setting before Theorem 0.1, and suppose moreover ρ is absolutely irreducible modulo ϖ_E, and $\rho_\wp$ is of Hodge–Tate weights $\underline{h}_{\Sigma_\wp} := (\frac{w-k_\sigma+2}{2}, \frac{w+k_\sigma}{2})_{\sigma\in\Sigma_\wp}$ with $w \in 2\mathbb{Z}$, $k_\sigma \in 2\mathbb{Z}_{\geqslant 1}$ (where we use the convention that the Hodge–Tate weight of the cyclotomic character is -1). Since $\rho_\wp$ is semi-stable non-crystalline, there exists $\alpha \in E^\times$, such that the eigenvalues of φ^{d_0} (where d_0 is the degree of the maximal un-ramified extension in $F_\wp$ over $\mathbb{Q}_p$) on $D_{\mathrm{st}}(\rho_\wp)$ are given by $\{\alpha, p^{d_0}\alpha\}$. Put $\mathrm{alg}(\underline{h}_{\Sigma_\wp}) := \otimes_{\sigma\in\Sigma_\wp}(\mathrm{Sym}^{k_\sigma-2} E^2 \otimes_E \det^{\frac{2-w-k_\sigma}{2}})^\sigma$, which is an algebraic representation of $\mathrm{Res}_{F_\wp/\mathbb{Q}_p} \mathrm{GL}_2$ over E, and

$$\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}) := \mathrm{unr}(\alpha) \circ \det \otimes_E \mathrm{St} \otimes_E \mathrm{alg}(\underline{h}_{\Sigma_\wp}),$$

which is in fact the locally algebraic representation of $\mathrm{GL}_2(F_\wp)$ associated to $\rho_\wp$ via classical local Langlands correspondence, where $\mathrm{unr}(\alpha)$ denotes the unramified character of $F_\wp^\times$ sending uniformizers to α, and St denotes the usual smooth Steinberg representation of $\mathrm{GL}_2(F_\wp)$. Moreover it's known $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}) \hookrightarrow \widehat{}(\rho)$. By Schraen's results ([41]) on Breuil's $\mathcal{L}$-invariants, one can associate to $\rho_\wp$ a locally $\mathbb{Q}_p$-analytic representation $(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n(\rho_\wp)})$ of $\mathrm{GL}_2(F_\wp)$ over E (cf. Section 3, as suggested by the notation, this rep-resentation is determined by α, $\underline{h}_{\Sigma_\wp}$ and $\mathcal{L}_{S_n(\rho_\wp)}$), whose socle is exactly $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp})$.

THEOREM 0.2 (Theorem 4.22 (2), Corollary 4.23). — *Keep the above notation,* $(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n(\rho_\wp)})$ *is a subrepresentation of the locally* $\mathbb{Q}_p$*-analytic representation* $\widehat{}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$. *Moreover,*

$$(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}'_{S_n(\rho_\wp)}) \longhookrightarrow \widehat{}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$$

if and only if $\mathcal{L}'_{S_n(\rho_\wp)} = \mathcal{L}_{S_n(\rho_\wp)}$.

This theorem shows the equality of Fontaine–Mazur $\mathcal{L}$-invariants and Breuil's $\mathcal{L}$-invariants. As in [24], we use p-adic family arguments on both Galois side and $\mathrm{GL}_2(F_\wp)$ side. The main objects are eigenvarieties, where live the locally analytic $T(F_\wp)$-representations and Gal_F-representations. On one hand, we use the global triangulation theory to relate the $\mathrm{Gal}_{F_\wp}$-representations and $T(F_\wp)$-representations; on the other hand, the locally analytic $T(F_\wp)$-representations and locally analytic $\mathrm{GL}_2(F_\wp)$-repre-sentations are linked by the theory of Jacquet–Emerton functor ([26, 28]). Roughly speaking, we get a picture as follows:

$$\left\{\begin{array}{c}\text{Locally analytic}\\ T(F_\wp)\text{-representations}\\ \text{(Eigenvarieties)}\end{array}\right\}$$

Triangulation

Adjunction formula

Jacquet-Emerton functor

$$\left\{\begin{array}{c}\text{Trianguline}\\ \mathrm{Gal}_{F_\wp}\text{-representations}\end{array}\right\} \qquad\qquad \left\{\begin{array}{c}\text{Locally analytic}\\ \mathrm{GL}_2(F_\wp)\text{-representations}\end{array}\right\}$$
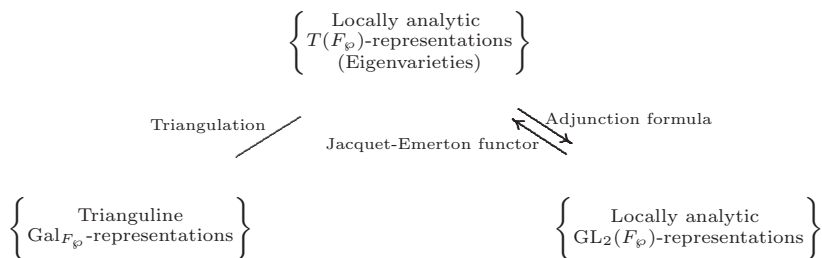
All these relations are in family. The global triangulation theory and Theorem 0.1 allow one to find the $\mathcal{L}$-invariants in the related $T(F_\wp)$-representations. Via the second arrow, one can thus find the $\mathcal{L}$-invariants on GL_2-side. A key fact is that the family of Galois representations associated to locally τ-analytic vectors of $\widehat{H}^1(K^p, E)$ is $\wp \setminus \{\tau\}$-*de Rham* (cf. Theorem 4.13), which ensures Theorem 0.1 to apply (this observation, together with Schraen's results [41] on Breuil's $\mathcal{L}$-invariants, in fact leads to the discovery of the hypothesis (3) in Theorem 0.1).

For the critical embeddings, by using global triangulation theory and Bergdall's result, we prove some results on Breuil's locally analytic socle conjecture ([15]). Namely, for each $\sigma \in S_c(\rho_\wp)$, one can associate a locally $\mathbb{Q}_p$-analytic representation $I^c_\sigma(\alpha, \underline{h}_{\Sigma_\wp})$ of $\mathrm{GL}_2(F_\wp)$ (see Section 3), which can be viewed as a σ-*companion* representation of $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp})$.

THEOREM 0.3 (Theorem 4.22 (1)). — *Keep the notation as in Theorem 0.2, $I^c_\sigma(\alpha, \underline{h}_{\Sigma_\wp})$ is a subrepresentation of $\widehat{}(\rho)$ if and only if $\sigma \in S_c(\rho_\wp)$.*

Thus from $\widehat{}(\rho)$, we can read out $S_c(\rho_\wp)$ by Theorem 0.3, and then $\underline{\mathcal{L}}_{S_n(\rho_\wp)}$ by Theorem 0.2. Since $\rho_\wp$ is determined by $\{\alpha, \underline{h}_{\Sigma_\wp}, S_n(\rho_\wp), \underline{\mathcal{L}}_{S_n(\rho_\wp)}\}$, we see:

COROLLARY 0.4. — *The local Galois representation $\rho_\wp$ is determined by $\widehat{}(\rho)_{\mathrm{an}}$.*

We refer the body of the text for more detailed and more precise statements.

In Section 1, we recall (and define) the Fontaine–Mazur $\mathcal{L}$-invariants in terms of B-pairs, and develop some partially de Rham Galois cohomology theory for B-pairs. We prove Theorem 0.1 in Section 2. In Section 3, we recall Schraen's theory on Breuil's $\mathcal{L}$-invariants of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$. The content of these three sections is purely local. In the last section, we prove Theorem 0.2 and Theorem 0.3. In Appendix A, we study some partially de Rham trianguline representations.

Acknowledgements

1. Fontaine–Mazur $\mathcal{L}$-invariants

In this section, we recall (and define) Fontaine–Mazur $\mathcal{L}$-invariants for 2-dimensional B-pairs (see Definition 1.20 below). Let $F_\wp$ be a finite extension of $\mathbb{Q}_p$ of degree d with $\mathcal{O}_\wp$ the ring of integers and ϖ a uniformizer, $\wp :=$ $\{\sigma : F_\wp \hookrightarrow \overline{\mathbb{Q}_p}\}$, $\mathrm{Gal}_{F_\wp} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$. We fix an embedding $\iota : F_\wp \hookrightarrow B_{\mathrm{dR}}^+$ (and hence embeddings $\iota : F_\wp \hookrightarrow \mathbb{C}_p, B_{\mathrm{dR}}$), and view $B_{\mathrm{dR}}^+, B_{\mathrm{dR}}, \mathbb{C}_p$ as $F_\wp$-algebras via ι. Let E be a finite extension of $\mathbb{Q}_p$ sufficiently large containing all the embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. For an $F_\wp$-algebra R and $\sigma \in \wp$, put $R_\sigma := R \otimes_{F_\wp,\sigma} E$ (e.g. we get E-algebras $B_{\mathrm{dR},\sigma}^+, B_{\mathrm{dR},\sigma}, \mathbb{C}_{p,\sigma}$); for an R-module M, put $M_\sigma := M \otimes_R R_\sigma$.

1.1. Preliminaries on B-pairs

Let $B_e := B_{\mathrm{cris}}^{\varphi=1}$, recall

DEFINITION 1.1 ([6, §2]).
(1) A B-pair of $\mathrm{Gal}_{F_\wp}$ is a couple $W = (W_e, W_{\mathrm{dR}}^+)$ where W_e is a finite free B_e-module equipped with a semi-linear continuous action of $\mathrm{Gal}_{F_\wp}$, and W_{dR}^+ is a $\mathrm{Gal}_{F_\wp}$-stable B_{dR}^+-lattice of $W_{\mathrm{dR}} :=$ $W_e \otimes_{B_e} B_{\mathrm{dR}}^+$. Let $r \in \mathbb{Z}_{>0}$, we say that W is (a B-pair) of rank r if $\mathrm{rk}_{B_e} W_e = r$.
(2) Let W, W' be two B-pairs, a morphism $f : W \to W'$ is defined to be a B_e-linear $\mathrm{Gal}_{F_\wp}$-invariant map $f_e : W_e \to W'_e$ such that the induced B_{dR}-linear map $f_{\mathrm{dR}} := f_e \otimes \mathrm{id} : W_{\mathrm{dR}} \to W'_{\mathrm{dR}}$ sends W_{dR}^+ to $(W')_{\mathrm{dR}}^+$. Moreover, we say that f is strict if the B_{dR}^+-module $(W')_{\mathrm{dR}}^+/f_{\mathrm{dR}}^+(W_{\mathrm{dR}}^+)$ is torsion free, where $f_{\mathrm{dR}}^+ := f_{\mathrm{dR}}|_{W_{\mathrm{dR}}^+}$.

By [6, Thm. 2.2.7], there exists an equivalence of categories between the category of B-pairs and that of $(\varphi,\ )$-modules over the Robba ring $B^{\dagger}_{\mathrm{rig},F_{\wp}}$ (e.g. see [6, §1.1]).

Let A be a local artinian E-algebra with residue field E.

DEFINITION 1.2 ([34, Def. 2.11, Lem. 2.12]).

(1) *An A-B-pair is a B-pair $W = (W_e, W^{+}_{\mathrm{dR}})$ such that W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} A$-module, and W^{+}_{dR} is a $\mathrm{Gal}_{F_{\wp}}$-stable finite free $B^{+}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-submodule of $W_{\mathrm{dR}} := W_e \otimes_{B_e} B_{\mathrm{dR}}$, which generates W_{dR}. We say W is (an A-B-pair) of rank r if $\mathrm{rk}_{B_e \otimes_{\mathbb{Q}_p} A} W_e = r$.*

(2) *Let W, W' be two A-B-pairs, a morphism $f : W \to W'$ is defined to be a morphism of B-pairs such that $f_e : W_e \to W'_e$ (cf. Definition 1.1(2)) is moreover $B_e \otimes_{\mathbb{Q}_p} A$-linear.*

As in [33, Thm. 1.36], one can deduce from [6, Thm. 2.2.7] an equivalence of categories between the category of A-B-pairs and that of $(\varphi,\ )$-modules free over $\mathcal{R}_A := B^{\dagger}_{\mathrm{rig},F_{\wp}} \widehat{\otimes}_{\mathbb{Q}_p} A$.

Let W be an A-B-pair of rank r. By using the isomorphism

$$(1.1) \qquad F_{\wp} \otimes_{\mathbb{Q}_p} A \xrightarrow{\sim} \prod_{\sigma \in \Sigma_{\wp}} A, \ a \otimes b \mapsto (\sigma(a)b)_{\sigma \in \Sigma_{\wp}},$$

one gets $B^{*}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A \xrightarrow{\sim} \oplus_{\sigma \in \Sigma_{\wp}} B^{*}_{\mathrm{dR},\sigma}$ and $W^{*}_{\mathrm{dR}} \xrightarrow{\sim} \oplus_{\sigma \in \Sigma_{\wp}} W^{*}_{\mathrm{dR},\sigma}$ where $* \in \{\emptyset, +\}$. Put $D_e(W) := W_e^{\mathrm{Gal}_{F_{\wp}}}$ and $D_{\mathrm{dR}}(W) := W_{\mathrm{dR}}^{\mathrm{Gal}_{F_{\wp}}}$. The last one is thus a finite $F_{\wp} \otimes_{\mathbb{Q}_p} A$-module, and admits a decomposition (according to (1.1)) $D_{\mathrm{dR}}(W) \xrightarrow{\sim} \prod_{\sigma \in \Sigma_{\wp}} D_{\mathrm{dR}}(W)_{\sigma}$. For $\sigma \in \ _{\wp}$, one has in fact $D_{\mathrm{dR}}(W)_{\sigma} \cong W_{\mathrm{dR},\sigma}^{\mathrm{Gal}_{F_{\wp}}}$.

DEFINITION 1.3. — *Keep the above notation, let $\sigma \in \ _{\wp}$, W is called σ-de Rham if $D_{\mathrm{dR}}(W)_{\sigma}$ is a free A-module of rank r; for $S \subseteq \ _{\wp}$, W is called S-de Rham if W is σ-de Rham for all $\sigma \in S$ (thus W is de Rham if W is $\ _{\wp}$-de Rham).*

Remark 1.4. — Let W be an A-B-pair, for $\sigma \in \ _{\wp}$, W is σ-de Rham if and only if W is σ-de Rham as an E-B-pair. The "only if" part is trivial. Suppose W is σ-de Rham as an E-B-pair, denote by $\mathfrak{m}_A$ the maximal ideal of A, and $d_A := \dim_E A$, thus $\dim_E D_{\mathrm{dR}}(W)_{\sigma} = r d_A$. Consider the exact sequence $0 \to \mathfrak{m}_A D_{\mathrm{dR}}(W)_{\sigma} \to D_{\mathrm{dR}}(W)_{\sigma} \to D_{\mathrm{dR}}(W/\mathfrak{m}_A)_{\sigma}$, we deduce the last map is surjective and $\dim_E D_{\mathrm{dR}}(W/\mathfrak{m}_A)_{\sigma} = r$ by dimension calculation (since $\dim_E \mathfrak{m}_A D_{\mathrm{dR}}(W)_{\sigma} = \dim_E D_{\mathrm{dR}}(\mathfrak{m}_A W)_{\sigma} \leqslant (d_A - 1)r$), from which we deduce $D_{\mathrm{dR}}(W)_{\sigma}$ is a free A-module.

DEFINITION 1.5. — *An A-B-pair W of rank r is called triangulable if it's an successive extension of A-B-pairs of rank 1 , i.e. W admits an increasing filtration of sub-A-B-pairs W_i for $0 \leqslant i \leqslant r$ such that $W_0 = 0$, $W_r = W$, and W_i/W_{i-1} is an A-B-pair of rank 1.*

Denote by $B_A := (B_e \otimes_{\mathbb{Q}_p} A, B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A)$ the trivial A-B-pair. Let χ be a continuous character of $F_\wp^\times$ in $A^\times$, following [34, §2.1.2], one can associate to χ an A-B-pair of rank 1, denoted by $B_A(\chi)$ (and we refer to loc. cit. for details). By [34, Prop. 2.16], all the rank 1 A-B-pairs can be obtained in this way: let W be an A-B-pair of rank 1, then there exists a unique continuous character $\chi : F_\wp^\times \to A^\times$ such that $W \xrightarrow{\sim} B_A(\chi)$. For a continuous representation V of $\mathrm{Gal}_{F_\wp}$ over A, denote by $W(V) := (B_e \otimes_{\mathbb{Q}_p} V, B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$ the associated A-B-pair. The $\mathrm{Gal}_{F_\wp}$-representation V is called *trianguline* if $W(V)$ is triangulable.

1.2. Cohomology of B-pairs

Recall the cohomology of E-B-pairs (note that A-B-pairs can also be viewed as E-B-pairs). Let $W = (W_e, W_{\mathrm{dR}}^+)$ be an E-B-pair, following [33, §2.1], consider the following complex (of $\mathrm{Gal}_{F_\wp}$-modules)

$$C^\bullet(W) := W_e \oplus W_{\mathrm{dR}}^+ \xrightarrow{(x,y) \mapsto x-y} W_{\mathrm{dR}}.$$

Put $H^i(\mathrm{Gal}_{F_\wp}, W) := H^i(\mathrm{Gal}_{F_\wp}, C^\bullet(W))$ (cf. [33, Def. 2.1]). By definition, one has a long exact sequence

$$(1.2) \quad 0 \to H^0(\mathrm{Gal}_{F_\wp}, W) \to H^0(\mathrm{Gal}_{F_\wp}, W_e) \oplus H^0(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}}^+)$$

$$\to H^0(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}}) \xrightarrow{\delta} H^1(\mathrm{Gal}_{F_\wp}, W)$$

$$\to H^1(\mathrm{Gal}_{F_\wp}, W_e) \oplus H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}}^+) \to H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}}).$$

For an E-B-pair W, denote by $W^\vee$ the dual of W:

$$W^\vee := \Big(W_e^\vee := \mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, B_e \otimes_{\mathbb{Q}_p} E),$$

$$(W^\vee)_{\mathrm{dR}}^+ := \mathrm{Hom}_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E) \Big)$$

where $W_e^\vee$, $(W^\vee)_{\mathrm{dR}}^+$ are equipped with a natural $\mathrm{Gal}_{F_\wp}$-action. One can check $W^\vee$ is also an E-B-pair.

Remark 1.6. — As in [33, Def. 1 (3)], one can also consider the dual W' of W as B-pair with $W'_e := \mathrm{Hom}_{B_e}(W_e, B_e)$ and $(W')_{\mathrm{dR}}^+ := \mathrm{Hom}_{B_{\mathrm{dR}}^+}(W_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+)$ (equipped with a natural $\mathrm{Gal}_{F_\wp}$-action). Moreover, W'_e, $(W')_{\mathrm{dR}}^+$ can

be equipped with a natural E-action: $(a \cdot f)(v) := f(av)$. One can check this action realizes W' as an E-B-pair. Moreover, the trace map $\mathrm{tr}_{E/\mathbb{Q}_p} : E \to \mathbb{Q}_p$, induces bijections $W_e^\vee \xrightarrow{\sim} W_e'$ and $(W^\vee)_{\mathrm{dR}}^+ \xrightarrow{\sim} (W')_{\mathrm{dR}}^+ : f \mapsto \mathrm{tr}_{E/\mathbb{Q}_p} \circ f$, and these bijections give an isomorphism $W^\vee \xrightarrow{\sim} W'$ as E-B-pairs.

Denote by $W(1)$ the twist of W by $W(\chi_{\mathrm{cyc}})$ where χ_{cyc} is the cyclotomic character of $\mathrm{Gal}_{F_\wp}$ (base change to E):

$$W(1) := \left(W(1)_e := W_e \otimes_{B_e \otimes_{\mathbb{Q}_p} E} W(\chi_{\mathrm{cyc}})_e, \right.$$

$$\left. W(1)_{\mathrm{dR}}^+ := W_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E} W(\chi_{\mathrm{cyc}})_{\mathrm{dR}}^+ \right).$$

By [33, §2] and [34, §5], one has

PROPOSITION 1.7.

  (1) $H^i(\mathrm{Gal}_{F_\wp}, W) = 0$ if $i \notin \{0, 1, 2\}$, and

  $$\sum_{i=0}^2 (-1)^i \dim_E H^i(\mathrm{Gal}_{F_\wp}, W) = -d(\mathrm{rk}\, W).$$

  (2) *There exists a natural isomorphism* $H^1(\mathrm{Gal}_{F_\wp}, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W)$, *where* $\mathrm{Ext}^1(B_E, W)$ *denotes the group of extensions of E-B-pairs of B_E by W.*

  (3) *Let V be a finite dimensional continuous $\mathrm{Gal}_{F_\wp}$-representation over E, then we have natural isomorphisms* $H^i(\mathrm{Gal}_{F_\wp}, W(V)) \cong H^i(\mathrm{Gal}_{F_\wp}, V)$ *for all $i \in \mathbb{Z}_{\geqslant 0}$.*

  (4) *The cup-product (see [34, §5] for details)*

(1.3) $\cup : H^i(\mathrm{Gal}_{F_\wp}, W) \times H^{2-i}(\mathrm{Gal}_{F_\wp}, W^\vee(1))$

$$\longrightarrow H^2(\mathrm{Gal}_{F_\wp}, B_E(1)) \cong H^2(\mathrm{Gal}_{F_\wp}, \chi_{\mathrm{cyc}}) \cong E$$

  is a perfect pairing for $i = 0, 1, 2$.

Remark 1.8.
  (1) In fact, in [34, §5], it's shown that the cup-product $H^i(\mathrm{Gal}_{F_\wp}, W) \times H^{2-i}(\mathrm{Gal}_{F_\wp}, W'(1)) \to H^2(\mathrm{Gal}_{F_\wp}, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ is a perfect pairing (see Remark 1.6 for W'). By discussions in Remark 1.6, identifying $W^\vee$ and W', this pairing then is equal to the composition of (1.3) with the trace map $\mathrm{tr}_{E/\mathbb{Q}_p}$, from which one deduces (1.3) is also perfect.
  (2) Let W be an E-B-pair, for an exact sequence of E-B-pairs

$$0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow 0,$$

one has the following commutative diagram

$$\begin{array}{ccc}
H^i(\mathrm{Gal}_{F_\wp}, W_3) \times H^j(\mathrm{Gal}_{F_\wp}, W) & \xrightarrow{\;\cup\;} & H^{i+j}(\mathrm{Gal}_{F_\wp}, W_3 \otimes W) \\
\delta \downarrow \qquad\qquad & \| & \delta \downarrow \qquad\quad \\
H^{i+1}(\mathrm{Gal}_{F_\wp}, W_1) \times H^j(\mathrm{Gal}_{F_\wp}, W) & \xrightarrow{\;\cup\;} & H^{i+j+1}(\mathrm{Gal}_{F_\wp}, W_1 \otimes W)\,,
\end{array}$$

where the δ's denote the connecting maps, $\cup$ the cup-products, and $W_i \otimes W$ is the E-B-pair given by $(W_i \otimes W)_e := (W_i)_e \otimes_{B_e \otimes_{\mathbb{Q}_p} E} W_e$, $(W_i \otimes W)^+_{\mathrm{dR}} := (W_i)^+_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} W^+_{\mathrm{dR}}$.

(3) If W is moreover an A-B-pair, by the same argument as in [33, §2.1], one can show there exists a natural isomorphism $H^1(\mathrm{Gal}_{F_\wp}, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_A, W)$ as A-modules, where $\mathrm{Ext}^1(B_A, W)$ denotes the group of extensions of A-B-pairs of B_A by W.

Put (cf. [33, Def. 2.4])

$$H^1_g(\mathrm{Gal}_{F_\wp}, W) := \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, W) \to H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}})]\,,$$
$$H^1_e(\mathrm{Gal}_{F_\wp}, W) := \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, W) \to H^1(\mathrm{Gal}_{F_\wp}, W_e)]\,,$$

where the above maps are induced from the natural maps

$$C^\bullet(W) \longrightarrow [W_e \to 0] \longrightarrow [W_{\mathrm{dR}} \to 0]\,.$$

Note that by (1.2), the map $H^1(\mathrm{Gal}_{F_\wp}, W) \to H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}})$ factors through (up to ± 1) the natural map $H^1(\mathrm{Gal}_{F_\wp}, W) \to H^1(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR}})$. If W is a de Rham A-B-pair, let $[X] \in H^1(\mathrm{Gal}_{F_\wp}, W) \cong \mathrm{Ext}^1(B_A, W)$, then X is de Rham if and only if $[X] \in H^1_g(\mathrm{Gal}_{F_\wp}, W)$. Moreover, in this case, by [33, Lem. 2.6], the natural map $H^1(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR}}) \to H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR}})$ is injective, thus $H^1_g(\mathrm{Gal}_{F_\wp}, W) = \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, W) \to H^1(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR}})]$. One has as in [33, Prop. 2.10]

PROPOSITION 1.9. — *Suppose W is de Rham, the perfect pairing* (1.3) *induces an isomorphism*

$$H^1_g(\mathrm{Gal}_{F_\wp}, W) \xrightarrow{\sim} H^1_e(\mathrm{Gal}_{F_\wp}, W^\vee(1))^\perp.$$

For $J \subseteq \Sigma_\wp$, $J \neq \emptyset$, put

$$H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W) := \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma})],$$

where the map is induced by

$$C^\bullet(W) \longrightarrow [W_{\mathrm{dR}} \to 0] \longrightarrow [\oplus_{\sigma \in J} W_{\mathrm{dR},\sigma} \to 0]\,.$$

Thus we have $H^1_{g,\Sigma_\wp}(\mathrm{Gal}_{F_\wp}, W) = H^1_g(\mathrm{Gal}_{F_\wp}, W)$,

$$H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W) \cong \cap_{\sigma \in J} H^1_{g,\sigma}(\mathrm{Gal}_{F_\wp}, W)\,,$$

and $H^1(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma})$ factors through (up to ± 1) the natural map $H^1(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+)$ (see the discussion above Proposition 1.9). Moreover, suppose W is a J-de Rham A-B-pair, for $[X] \in H^1(\mathrm{Gal}_{F_\wp}, W) \cong \mathrm{Ext}^1(B_A, W)$, X is J-de Rham if and only if $[X] \in H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W)$. By the same argument as in [33, Lem. 2.6], one has

LEMMA 1.10. — *Let $J \subseteq \wp$, $J \neq \emptyset$, suppose W is J-de Rham, then the map*

$$\oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+) \longrightarrow \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma})$$

is injective.

Thus if W is J-de Rham, then one has

(1.4) $H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W) \cong \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+)]$.

LEMMA 1.11. — *Let $J \subseteq \wp$, $J \neq \emptyset$, suppose W is J-de Rham, if $H^0(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+) = 0$ for all $\sigma \in J$, then $H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, W)$.*

Proof. — It's sufficient to prove $H^1_{g,\sigma}(\mathrm{Gal}_{F_\wp}, W) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, W)$ for all $\sigma \in J$. Since W is σ-de Rham and $H^0(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+) = 0$, we see $W_{\mathrm{dR},\sigma}^+ \cong \oplus_{i \in \mathbb{Z}_{\geqslant 1}} (t^i B_{\mathrm{dR},\sigma}^+)^{n_i}$ where $n_i = 0$ for all but finite many i. However, for $i \in \mathbb{Z}_{\geqslant 1}$, $H^1(\mathrm{Gal}_{F_\wp}, t^i B_{\mathrm{dR},\sigma}^+) = 0$, thus $H^1(\mathrm{Gal}_{F_\wp}, W_{\mathrm{dR},\sigma}^+) = 0$, from which (and (1.4)) the lemma follows. $\square$

For an E-B-pair W, $\delta : F_\wp^\times \to E^\times$, put $W(\delta) := W \otimes B_E(\delta)$ (see Remark 1.8 (2) for tensor products of E-B-pairs, and Section 1.1 for $B_E(\delta)$). If there exist $k_\sigma \in \mathbb{Z}$ for all $\sigma \in \wp$ such that $\delta = \prod_{\sigma \in \Sigma_\wp} \sigma^{k_\sigma}$, then by [33, Lem. 2.12], one has natural isomorphisms

(1.5) $W(\delta)_e \cong W_e$, $W(\delta)_{\mathrm{dR}}^+ \cong \oplus_{\sigma \in \Sigma_\wp} W(\delta)_{\mathrm{dR},\sigma}^+ \cong \oplus_{\sigma \in \Sigma_\wp} t^{k_\sigma} W_{\mathrm{dR},\sigma}^+$.

Thus if $k_\sigma \in \mathbb{Z}_{\geqslant 0}$ for all $\sigma \in \wp$, one gets a natural morphism

(1.6) $\mathrm{j} : W(\delta) \longrightarrow W$

with $\mathrm{j}_e = \mathrm{id}$ and $\mathrm{j}_{\mathrm{dR}}^+$ the natural injection $\oplus_{\sigma \in \Sigma_\wp} t^{k_\sigma} W_{\mathrm{dR},\sigma}^+ \hookrightarrow \oplus_{\sigma \in \Sigma_\wp} W_{\mathrm{dR},\sigma}^+$.

Let $J \subseteq \wp$, $J \neq \emptyset$, W be a J-de Rham E-B-pair, let $k_\sigma \in \mathbb{Z}_{\geqslant 0}$, such that $(t^{k_\sigma} W_{\mathrm{dR},\sigma}^+)^{\mathrm{Gal}_{F_\wp}} = 0$ for $\sigma \in J$ (thus $t^{k_\sigma} W_{\mathrm{dR},\sigma}^+ \cong \oplus_{i \in \mathbb{Z}_{\geqslant 1}} (t^i B_{\mathrm{dR},\sigma}^+)^{\oplus n_i}$ with $n_i = 0$ for all but finite many i for $\sigma \in J$), let $\delta := \prod_{\sigma \in J} \sigma^{k_\sigma}$. The morphism (1.6) induces an exact sequence of $\mathrm{Gal}_{F_\wp}$-complexes

$$0 \to [W(\delta)_e \oplus W(\delta)_{\mathrm{dR}}^+ \to W(\delta)_{\mathrm{dR}}] \to [W_e \oplus W_{\mathrm{dR}}^+ \to W_{\mathrm{dR}}]$$
$$\to [\oplus_{\sigma \in J} (W_{\mathrm{dR},\sigma}^+ / t^{k_\sigma} W_{\mathrm{dR},\sigma}^+) \to 0].$$

Taking $\mathrm{Gal}_{F_\wp}$-cohomology, one gets

$$0 \to H^0(\mathrm{Gal}_{F_\wp}, W(\delta)) \to H^0(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^0(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR},\sigma}/t^{k_\sigma})$$
$$\to H^1(\mathrm{Gal}_{F_\wp}, W(\delta)) \to H^1(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^1(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR},\sigma}/t^{k_\sigma}).$$

By our assumption on k_σ, $H^0(\mathrm{Gal}_{F_\wp}, W(\delta)) = 0$ and $H^i(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR},\sigma}) \xrightarrow{\sim} H^i(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR},\sigma}/t^{k_\sigma})$ for $i = 0, 1$, from which and Lemma 1.10 (and the discussion above it), one gets

$$(1.7) \quad 0 \to H^0(\mathrm{Gal}_{F_\wp}, W) \to \oplus_{\sigma \in J} H^0(\mathrm{Gal}_{F_\wp}, W^+_{\mathrm{dR},\sigma})$$
$$\to H^1(\mathrm{Gal}_{F_\wp}, W(\delta)) \to H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W) \to 0,$$

which would be useful to calculate $H^1_{g,J}(\mathrm{Gal}_{F_\wp}, W)$. At last, note that a morphism of E-B-pairs $W_1 \to W_2$ induces a map $H^1(\mathrm{Gal}_{F_\wp}, W_1) \to H^1(\mathrm{Gal}_{F_\wp}, W_2)$ which restricts to maps $H^1_*(\mathrm{Gal}_{F_\wp}, W_1) \to H^1_*(\mathrm{Gal}_{F_\wp}, W_2)$ with $* \in \{e, g, \{g, J\}\}$.

1.3. Fontaine–Mazur $\mathcal{L}$-invariants

Let χ be a continuous character of $F_\wp^\times$ in $E^\times$, χ is called *special* if there exist $k_\sigma \in \mathbb{Z}$ for all $\sigma \in \ _\wp$ such that

$$\chi = \mathrm{unr}(q^{-1}) \prod_{\sigma \in \Sigma_\wp} \sigma^{k_\sigma} = \chi_{\mathrm{cyc}} \prod_{\sigma \in \Sigma_\wp} \sigma^{k_\sigma - 1}$$

where $\mathrm{unr}(z)$ denotes the unramified character of $F_\wp^\times$ sending uniformizers to z. In this section, we associate to $[X] \in H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi))$ the so-called *Fontaine–Mazur $\mathcal{L}$-invariants* for special characters χ.

Let $\chi = \chi_{\mathrm{cyc}} \prod_{\sigma \in \Sigma_\wp} \sigma^{k_\sigma - 1}$, by [33, Lem. 2.12], $B_E(\chi)_e \cong (B_e \otimes_{\mathbb{Q}_p} E)t$, $B_E(\chi)^+_{\mathrm{dR}} \cong \oplus_{\sigma \in \Sigma_\wp} t^{k_\sigma} B^+_{\mathrm{dR},\sigma}$. Put $\eta := \prod_{\sigma \in \Sigma_\wp} \sigma^{1-k_\sigma}$, thus $B_E(\eta) \cong B_E(\chi)^\vee(1)$, $B_E(\eta)_e \cong B_e \otimes_{\mathbb{Q}_p} E$ and $B_E(\eta)^+_{\mathrm{dR}} \cong \oplus_{\sigma \in \Sigma_\wp} t^{1-k_\sigma} B^+_{\mathrm{dR},\sigma}$. Put

$$(1.8) \quad \begin{cases} S_c(\chi) := \{\sigma \in \ _\wp \mid k_\sigma \in \mathbb{Z}_{\leqslant 0}\}, \\ S_n(\chi) := \{\sigma \in \ _\wp \mid k_\sigma \in \mathbb{Z}_{\geqslant 1}\}, \end{cases}$$

thus $B_E(\chi)$ is *non-$S_n(\chi)$-critical* (cf. Definition A.2). By [33, Prop. 2.15, Lem. 4.2 and Lem. 4.3], one has

LEMMA 1.12. — *Keep the above notation.*

(1) *If $S_c(\chi) = \emptyset$, then*

$$\dim_E H^0(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = \dim_E H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = 1,$$

and

$$\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = \dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = d + 1 \,;$$

if $S_c(\chi) \neq \emptyset$, then

$$\dim_E H^i(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = \dim_E H^i(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = 0$$

for $i = 0, 2$, and

$$\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = \dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta))$$
$$= d \, (= [F_\wp : \mathbb{Q}_p]) \,.$$

(2) We have

$$\dim_E H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = d - |S_c(\chi)|$$

and

$$\dim_E H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = d + 1 - |S_c(\chi)| \,.$$

If $S_c(\chi) = \emptyset$,

$$\dim_E H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = 0 \,;$$

if $S_c(\chi) \neq \emptyset$,

$$\dim_E H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = |S_c(\chi)| - 1 \,.$$

Suppose first $S_c(\chi) = \emptyset$ (thus $H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)))$, we would use the cup-product

$$(1.9) \quad \langle \cdot, \cdot \rangle : H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E)$$
$$\longrightarrow H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \cong E$$

to define $\mathcal{L}$-invariants for elements in $H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$.

LEMMA 1.13. — *The cup-product* (1.9) *is a perfect pairing.*

Proof. — The natural morphism $\mathrm{j} : B_E(\chi) \to B_E(1)$ (cf. (1.6)) induces an exact sequence of $\mathrm{Gal}_{F_\wp}$-complexes

$$(1.10) \quad 0 \longrightarrow [B_E(\chi)_e \oplus B_E(\chi)^+_{\mathrm{dR}} \to B_E(\chi)_{\mathrm{dR}}]$$
$$\longrightarrow [B_E(1)_e \oplus B_E(1)^+_{\mathrm{dR}} \to B_E(1)_{\mathrm{dR}}]$$
$$\longrightarrow [\oplus_{\sigma \in \Sigma_\wp} t B^+_{\mathrm{dR}, \sigma} / t^{k_\sigma} B^+_{\mathrm{dR}, \sigma} \to 0] \longrightarrow 0.$$

Since $H^i(\mathrm{Gal}_{F_\wp}, tB_{\mathrm{dR},\sigma}^+/t^{k_\sigma}B_{\mathrm{dR},\sigma}^+) = 0$ for any $i \in \mathbb{Z}_{\geqslant 0}$, we see j induces isomorphisms $H^i(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \xrightarrow{\sim} H^i(\mathrm{Gal}_{F_\wp}, B_E(1))$ for $i \in \mathbb{Z}_{\geqslant 0}$. Moreover, the following diagram commutes

$$
\begin{array}{ccc}
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E) & \xrightarrow{\ \cup\ } & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \\
\sim\downarrow \qquad\qquad\qquad & \| & \sim\downarrow \\
H^1(\mathrm{Gal}_{F_\wp}, B_E(1)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E) & \xrightarrow{\ \cup\ } & H^2(\mathrm{Gal}_{F_\wp}, B_E(1)).
\end{array}
$$

Since the cup-product below is perfect by Proposition 1.7(4), so is the above one. $\qquad\square$

Recall that $H^1(\mathrm{Gal}_{F_\wp}, B_E) \cong H^1(\mathrm{Gal}_{F_\wp}, E) \cong \mathrm{Hom}(\mathrm{Gal}_{F_\wp}, E)$, where the last denotes the E-vector space of continuous additive characters of $\mathrm{Gal}_{F_\wp}$ in E. Before going any further, we recall some facts on additive characters of $\mathrm{Gal}_{F_\wp}$.

1.3.1. A digression: additive characters of $\mathrm{Gal}_{F_\wp}$

Let $W_{F_\wp}$ denote the Weil group of $F_\wp$. We fix a local Artin map $\mathrm{Art}_{F_\wp}: F_\wp^\times \xrightarrow{\sim} W_{F_\wp}^{\mathrm{ab}}$ sending uniformizers to geometric Frobenius. One has thus

$$
\begin{aligned}
H^1(\mathrm{Gal}_{F_\wp}, B_E) &\cong H^1(\mathrm{Gal}_{F_\wp}, E) \cong \mathrm{Hom}(\mathrm{Gal}_{F_\wp}, E) \cong \mathrm{Hom}(\mathrm{Gal}_{F_\wp}^{\mathrm{ab}}, E) \\
&\cong \mathrm{Hom}(W_{F_\wp}^{\mathrm{ab}}, E) \xrightarrow[\sim]{\mathrm{Art}_{F_\wp}} \mathrm{Hom}(F_\wp^\times, E)
\end{aligned}
$$
(1.11)

where the fourth isomorphism follows from the fact that any character of $\mathbb{Z}$ in E gives rise to a continuous character of $\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ in E. We would identify these E-vector spaces via (1.11) with no mention.

For a uniformiser $\varpi \in F_\wp^\times$ one gets a character $\varepsilon_\varpi : F_\wp^\times \longrightarrow \mathcal{O}_\wp^\times$ which is identity on $\mathcal{O}_\wp^\times$ and sends ϖ to 1 . Let $\psi_{\sigma,\varpi} := \sigma \circ \log \circ \varepsilon_\varpi : F_\wp^\times \to E$ for $\sigma \in \Sigma_\wp$, and $\psi_{\mathrm{ur}} : F_\wp^\times \to \mathbb{Z}$ be the unramified character sending p to 1 (thus sending ϖ to e^{-1}).

LEMMA 1.14. — $\{\psi_{\sigma,\varpi}\}_{\sigma\in\Sigma_\wp}$ and ψ_{ur} form a basis of $\mathrm{Hom}(F_\wp^\times, E)$.

Proof. — One has isomorphisms

(1.12) $\quad \mathrm{Hom}(\mathcal{O}_\wp^\times, E) \cong \mathrm{Hom}_{\mathbb{Q}_p}(F_\wp, E)$

$$
\cong \mathrm{Hom}_E(F_\wp \otimes_{\mathbb{Q}_p} E, E) \cong \mathrm{Hom}_E\left(\prod_{\sigma\in\Sigma_\wp} E, E\right),
$$

where the first isomorphism is induced by the log map. For $\tau \in \Sigma_\wp$, one sees $\tau \circ \log : \mathcal{O}_\wp^\times \to E$ corresponds to the map $\prod_{\sigma\in\Sigma_\wp} E, \to E$, $(a_\sigma)_{\sigma\in\Sigma_\wp} \mapsto a_\tau$.

So $\{\sigma \circ \log\}_{\sigma \in \Sigma_\wp}$ form a basis of the E-vector space $\mathrm{Hom}(\mathcal{O}_\wp^\times, E)$, and hence $\{\psi_{\sigma,\varpi}\}_{\sigma \in \Sigma_\wp}$ form a basis of the E-vector subspace of $\mathrm{Hom}(F_\wp^\times, E)$ generated by characters sending ϖ to 0. The lemma follows. $\qquad\square$

The cyclotomic character χ_{cyc} of $\mathrm{Gal}_{F_\wp}$ corresponds (via $\mathrm{Art}_{F_\wp}$) to the character $F_\wp^\times \xrightarrow{\mathcal{N}_{F_\wp/\mathbb{Q}_p}} \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$ with the last map being identity on $\mathbb{Z}_p^\times$ and sending p to 1. Consider the restriction of $\mathcal{N}_{F_\wp/\mathbb{Q}_p}$ to $\mathcal{O}_\wp^\times$, which corresponds (via (1.12)) to the map $\mathrm{tr} \in \mathrm{Hom}_E(\prod_{\sigma \in \Sigma_\wp} E, E) : (a_\sigma)_\sigma \mapsto \sum_{\sigma \in \Sigma_{F_\wp}} \sigma(a_\sigma)$. This map is in fact a generator of $\mathrm{Hom}_E(\prod_{\sigma \in \Sigma_\wp} E, E)$ over $F_\wp \otimes_{\mathbb{Q}_p} E$. For any $f \in F_\wp \otimes_{\mathbb{Q}_p} E$, denote by $\psi_{f,p}$ the character $F_\wp^\times \to E$ such that $\psi_{f,p}|_{\mathcal{O}_\wp^\times}$ coincides with the preimage of $f \cdot \mathrm{tr} \in \mathrm{Hom}_E(\prod_{\sigma \in \Sigma_\wp} E, E)$ in $\mathrm{Hom}(\mathcal{O}_\wp^\times, E)$ via (1.12) and that $\psi_{f,p}(p) = 1$. For $\tau \in \quad_\wp$, denote by $1_\tau \in F_\wp \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_\wp} E$ with $(1_\tau)_\tau = 1$ and $(1_\tau)_\sigma = 0$ for $\sigma \neq \tau$. Let $\psi_{\tau,p} := \psi_{1_\tau,p}$ to simplify, we see $\psi_{\tau,\varpi} = \psi_{\tau,p} + \tau(\log(p/\varpi^e))\psi_{\mathrm{ur}}$ (by comparing their values of p and $\mathcal{O}_\wp^\times$). In particular, $\{\psi_{\sigma,p}\}_{\sigma \in \Sigma_\wp}$ and ψ_{ur} also form a basis of $\mathrm{Hom}_{\mathbb{Q}_p}(F_\wp^\times, E)$.

For $\tau \in \quad_\wp$, the embedding $\iota : F_\wp \hookrightarrow B_{\mathrm{dR}}^+$ induces $\iota_\tau : E \hookrightarrow B_{\mathrm{dR},\tau}^+ \twoheadrightarrow \mathbb{C}_{p,\tau}$. One gets

$$\iota_\tau : H^1(\mathrm{Gal}_{F_\wp}, E) \longrightarrow H^1(\mathrm{Gal}_{F_\wp}, B_{\mathrm{dR},\tau}^+) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, \mathbb{C}_{p,\tau}).$$

For $\psi \in H^1(\mathrm{Gal}_{F_\wp}, E)$, ψ is mapped to zero if and only if there exists $x \in \mathbb{C}_{p,\tau}$ such that $\chi(g) = g(x) - x$. It's known that for any $\iota' \neq \iota : F_\wp \hookrightarrow \mathbb{C}_p$, there exists $u_{\iota'} \in \mathbb{C}_p^\times$, such that $g(u_{\iota'}) = (\iota' \circ \varepsilon_\varpi(g)) \cdot u_{\iota'}$ (where ε_ϖ is viewed as a character of $\mathrm{Gal}_{F_\wp}$ via $\mathrm{Art}_{F_\wp}$), put $x_{\iota'} := \log(u_{\iota'})$, we have $g(x_{\iota'}) - x_{\iota'} = \log \circ \varepsilon_\varpi(g)$. From which we deduce that for any $\tau' \neq \tau$, $\iota_\tau(\psi_{\tau',\varpi}) = 0$. Similarly, we have $\iota_\tau(\psi_{\mathrm{ur}}) = 0$. So $\psi_{\mathrm{ur}} \in H_g^1(\mathrm{Gal}_{F_\wp}, B_E) = H_g^1(\mathrm{Gal}_{F_\wp}, E) = \ker[H^1(\mathrm{Gal}_{F_\wp}, E) \to H^1(\mathrm{Gal}_{F_\wp}, \oplus_{\sigma \in \Sigma_\wp} B_{\mathrm{dR},\sigma})]$, and is a generator of $H_g^1(\mathrm{Gal}_{F_\wp}, B_E)$ (which is 1-dimensional over E). For $S \subseteq \quad_\wp$, recall $H_{g,S}^1(\mathrm{Gal}_{F_\wp}, E) = \mathrm{Ker}[H^1(\mathrm{Gal}_{F_\wp}, E) \to \oplus_{\sigma \in S} H^1(\mathrm{Gal}_{F_\wp}, B_{\mathrm{dR},\sigma})]$, by the above discussion, one has

LEMMA 1.15. — *The E-vector space $H_{g,S}^1(\mathrm{Gal}_{F_\wp}, E)$ is of dimension $|\quad_\wp \setminus S| + 1$, and is generated by $\{\psi_{\sigma,\varpi}\}_{\sigma \in \Sigma_\wp \setminus S}$ and ψ_{ur} (thus can also be generated by $\{\psi_{\sigma,p}\}_{\sigma \in \Sigma_\wp \setminus S}$ and ψ_{ur}).*

1.3.2. $\mathcal{L}$-invariants

Return to the situation before Section 1.3.1 (thus χ is a special character with $S_c(\chi) = \emptyset$). Let $[X] \in H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H_g^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, by Proposition 1.9 and Lemma 1.15, $[X] \in H_e^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$ if and only if $\langle [X], \psi_{\mathrm{ur}} \rangle = 0$ (cf. (1.9)).

DEFINITION 1.16 (non-critical case). — *Keep the above notation, if*
$[X] \notin H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, *for* $\sigma \in$ $_\wp$, *put* $\mathcal{L}(X)_\sigma := \langle [X], \psi_{\sigma,p} \rangle / \langle [X], \psi_{\mathrm{ur}} \rangle \in$
E (*cf.* (1.9)), *and the* $\{\mathcal{L}(X)_\sigma\}_{\sigma \in \Sigma_\wp}$ *are called the Fontaine–Mazur* $\mathcal{L}$-
invariants of X; *if* $[X] \in H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, *we define the Fontaine–Mazur*
$\mathcal{L}$-*invariants of* X *to be* $(\mathcal{L}(X)_\sigma)_{\sigma \in \Sigma_\wp} := (\langle [X], \psi_{\sigma,p} \rangle)_{\sigma \in \Sigma_\wp} \in \mathbb{P}^{d-1}(E)$.

Remark 1.17. — Let $\chi' = \mathrm{unr}(q^{-1}) \prod_{\sigma \in \Sigma_\wp} \sigma^{k'_\sigma}$ with $1 \leqslant k'_\sigma \leqslant k_\sigma$ for all
$\sigma \in$ $_\wp$. The natural morphism $\mathrm{j} : B_E(\chi) \to B_E(\chi')$ induces isomorphisms
$\mathrm{j} : H^i(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \xrightarrow{\sim} H^i(\mathrm{Gal}_{F_\wp}, B_E(\chi'))$ for $i \in \mathbb{Z}_{\geqslant 0}$ (by the same
argument as in the proof of Lemma 1.13), moreover, the following diagram
commutes

$$
\begin{array}{ccc}
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E) & \xrightarrow{\cup} & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \\
\sim \downarrow \qquad\qquad & \parallel & \sim \downarrow \\
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi')) \times H^1(\mathrm{Gal}_{F_\wp}, B_E) & \xrightarrow{\cup} & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi')) .
\end{array}
$$

We see by definition $(\mathcal{L}(X')_\sigma)_{\sigma \in \Sigma_\wp} = (\mathcal{L}(X)_\sigma)_{\sigma \in \Sigma_\wp}$ if $[X'] = \mathrm{j}([X])$ (up to
scalars).

Consider now the case $S_c(\chi) \neq \emptyset$ (i.e. the *critical* case). Let

$$
\chi^\sharp := \chi \prod_{\sigma \in S_c(\chi)} \sigma^{1-k_\sigma} = \mathrm{unr}(q^{-1}) \prod_{\sigma \in S_n(\chi)} \sigma^{k_\sigma} \prod_{\sigma \in S_c(\chi)} \sigma \, ,
$$

thus $\chi^\sharp$ is also special and $S_c(\chi^\sharp) = \emptyset$. The natural morphism of $\mathrm{j} :$
$B_E(\chi^\sharp) \to B_E(\chi)$ (cf. (1.6)) induces a map $\mathrm{j} : H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) \to$
$H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$.

LEMMA 1.18. — $\mathrm{Im}(\mathrm{j}) = H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi))$.

Proof. — By (1.7), $\mathrm{Im}(\mathrm{j}) = H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi))$. By Lemma 1.11,
$H^1_{g,\sigma}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$ for $\sigma \in S_n(\chi)$, and hence

$$
H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \, .
$$

The lemma follows.   □

Let $\eta' := \prod_{\sigma \in S_c(\chi)} \sigma^{1-k_\sigma}$, so $\chi^\sharp = \chi \eta'$. We claim the cup-product

$$(1.13) \quad H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta')) \longrightarrow H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp))$$

is a perfect pairing. Indeed, similarly as in the proof of Lemma 1.13, this follows from the commutative diagram (recall $\eta = \chi^{-1}\chi_{\mathrm{cyc}}$):

$$
\begin{array}{ccc}
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta')) & \xrightarrow{\ \cup\ } & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) \\
\| \qquad\qquad \sim\downarrow & & \sim\downarrow \\
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta)) & \xrightarrow{\ \cup\ } & H^2(\mathrm{Gal}_{F_\wp}, B_E(1)).
\end{array}
$$

One deduces from Proposition 1.9 that this pairing induces isomorphisms

(1.14)
$$
\begin{aligned}
H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi)) &\cong H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta'))^\perp, \\
H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\chi)) &\cong H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\eta'))^\perp.
\end{aligned}
$$

Let j′ denote the natural morphism $B_E(\eta') \to B_E$, the following diagram commutes

(1.15)
$$
\begin{array}{ccc}
H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta')) \times H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) & \longrightarrow & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) \\
\text{j}'\downarrow \qquad\qquad \text{j}\uparrow & & \| \\
H^1(\mathrm{Gal}_{F_\wp}, B_E) \times H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) & \longrightarrow & H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)).
\end{array}
$$

By (1.7), $\mathrm{Im}(\mathrm{j}') = H^1_{g, S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E)$.

LEMMA 1.19. — *Denote by $\langle\cdot,\cdot\rangle_n$ the bottom (perfect) pairing in (1.15), then the following pairing*

(1.16) $\langle\cdot,\cdot\rangle : H^1_{g, S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E) \times H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi))$
$$
\longrightarrow H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) \cong E,
$$

with $\langle x, y\rangle := \langle x, y^\sharp\rangle_n$, where $y^\sharp$ is a preimage of y in $H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp))$, is independent of the choice of $y^\sharp$ and is a perfect pairing. Moreover, this pairing induces an isomorphism $H^1_g(\mathrm{Gal}_{F_\wp}, B_E)^\perp \xrightarrow{\sim} H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\chi))$.

Proof. — The independence of the choice of $y^\sharp$ follows from the commutativity of (1.15) and the fact $\mathrm{Im}(\mathrm{j}') = H^1_{g, S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E)$. Indeed, for $y' \in H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp))$, if $\mathrm{j}(y') = 0$, by (1.15), $\mathrm{Im}(\mathrm{j}') \subseteq (E \cdot y')^\perp$.

By (1.14), the top pairing in (1.15) induces a perfect pairing

$$
H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta'))/H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta')) \times H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \longrightarrow E.
$$

We claim j′ induces an isomorphism

$$
H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta'))/H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta')) \xrightarrow{\sim} H^1_{g, S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E),
$$

from which one can easily deduce (1.16) is perfect. Since $H^1_e(\mathrm{Gal}_{F_\wp}, B_E) = \{0\}$, $H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta')) \subseteq \mathrm{Ker}(\mathrm{j}')$ (note j′ sends $H^1_e(\mathrm{Gal}_{F_\wp}, B_E(\eta'))$ to

$H_e^1(\mathrm{Gal}_{F_\wp}, B_E) = 0)$. By Lemma 1.15,

$$\dim_E \mathrm{Im}(\mathrm{j}') = \dim_E H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E) = |S_n(\chi)| + 1\,;$$

by Lemma 1.12,

$$\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\eta')) = d$$

and

$$\dim_E H_e^1(\mathrm{Gal}_{F_\wp}, B_E(\eta')) = |S_c(\chi)| - 1\,.$$

By dimension calculation, the claim follows.

The second part follows from (1.14) and the fact that the map j' sends $H_g^1(\mathrm{Gal}_{F_\wp}, B_E(\eta'))$ to $H_g^1(\mathrm{Gal}_{F_\wp}, B_E)$.   □

Using this pairing and Lemma 1.15, one can now define Fontaine–Mazur $\mathcal{L}$-invariants in general case:

DEFINITION 1.20 (general case). — *Let χ be a special character of $F_\wp^\times$, $[X] \in H_g^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, if $[X] \notin H_e^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, for $\sigma \in S_n(\chi)$, put $\mathcal{L}(X)_\sigma := \langle[X], \psi_{\sigma,p}\rangle / \langle[X], \psi_{\mathrm{ur}}\rangle \in E$ (cf. (1.16)), and $\{\mathcal{L}(X)_\sigma\}_{\sigma\in S_n(\chi)}$ are called the Fontaine–Mazur $\mathcal{L}$-invariants of X; if $[X] \in H_e^1(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, we define the Fontaine–Mazur $\mathcal{L}$-invariants of X to be $(\mathcal{L}(X)_\sigma)_{\sigma\in S_n(\chi)} := (\langle[X], \psi_{\sigma,p}\rangle)_{\sigma\in S_n(\chi)} \in \mathbb{P}^{|S_n(\chi)|-1}(E)$ (cf. (1.16)).*

Remark 1.21. — Keep the above notation, and let

$$[X] \in H_g^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)), \quad [X^\sharp] \in H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp))$$

such that $\mathrm{j}([X^\sharp]) = [X]$, then we have

$$(\mathcal{L}(X))_{\sigma\in S_n(\chi)} = (\mathcal{L}(X^\sharp))_{\sigma\in S_n(\chi)}\,.$$

Let X be a 2-dimensional triangulable E-B-pair with a triangulation given by

$$0 \to B_E(\chi_1) \to X \to B_E(\chi_2) \to 0\,.$$

We denote by (X, χ_1, χ_2) a such triangulation. The E-B-pair X is called *special* if $\chi_1\chi_2^{-1}$ is special. Suppose X is special, let

$$[X_0] \in H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi_1\chi_2^{-1}))$$

be the image of $[X]$ via the isomorphism

$$\mathrm{Ext}^1(B_E(\chi_1), B_E(\chi_2)) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi_1\chi_2^{-1}))\,.$$

If $[X_0] \in H_g^1(\mathrm{Gal}_{F_\wp}, B_E(\chi_1\chi_2^{-1}))$, we define the $\mathcal{L}$-invariants of (X, χ_1, χ_2) to be the $\mathcal{L}$-invariants of $[X_0]$; if moreover $[X_0] \notin H_e^1(\mathrm{Gal}_{F_\wp}, B_E(\chi_1\chi_2^{-1}))$, these are called $\mathcal{L}$-invariants of X (since in this case, X admits a unique triangulation, cf. [33, Thm. 3.7]).

Let V be a 2-dimensional semi-stable representation of $\text{Gal}_{F_\wp}$ over E, and

$$0 \to B_E(\chi_1) \to W(V) \to B_E(\chi_2) \to 0$$

a triangulation of $W(V)$. Suppose $\chi_1\chi_2^{-1}$ is special, which is equivalent to that the eigenvalues α_1, α_2 of the E-linear operator φ^{d_0} on $D_{\text{st}}(V)$ satisfy $\alpha_1\alpha_2^{-1} = q$ or q^{-1}. One defines the $\mathcal{L}$-invariants of (V, χ_1, χ_2) to be the $\mathcal{L}$-invariants of $(W(V), \chi_1, \chi_2)$, which are called the *Fontaine–Mazur $\mathcal{L}$-invariants* of V if V is moreover non-crystalline.

2. $\mathcal{L}$-invariants and partially de Rham families

Let χ be a special character of $F_\wp^\times$ in $E^\times$, $\widetilde{\chi}$ be a character of $F_\wp^\times$ in $(E[\epsilon]/\epsilon^2)^\times$ such that $\widetilde{\chi} \equiv \chi \pmod{\epsilon}$. So there exists an additive character ψ of $F_\wp^\times$ in E such that $\widetilde{\chi} = \chi(1+\epsilon\psi)$. By results in Section 1.3.1, there exist $a_\sigma \in E$ for all $\sigma \in \Sigma_\wp$ and $a_{\text{ur}} \in E$ such that $\psi = a_{\text{ur}}\psi_{\text{ur}} + \sum_{\sigma \in \Sigma_\wp} a_\sigma \psi_{\sigma,p}$.

Let X be an $E[\epsilon]/\epsilon^2$-B-pair of rank 2 such that

$$[X] \in H^1(\text{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) \cong \text{Ext}^1(B_{E[\epsilon]/\epsilon^2}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})).$$

Denote by $X_0 := X \pmod{\epsilon}$, which is a triangulable E-B-pair and lies in $H^1(\text{Gal}_{F_\wp}, B_E(\chi))$. Suppose X_0 is de Rham (i.e. $[X_0] \in H^1_g(\text{Gal}_{F_\wp}, B_E(\chi)))$, and denote by $\underline{\mathcal{L}}_{S_n(\chi)} = (\mathcal{L}_\sigma)_{\sigma \in S_n(\chi)}$ the associated $\mathcal{L}$-invariants (cf. Definition 1.20). This section is devoted to prove the following theorem.

THEOREM 2.1. — *Keep the above notation, and suppose X is $S_c(\chi)$-de Rham (cf. Definition 1.3), then*

$$(2.1) \qquad \begin{cases} a_{\text{ur}} + \sum_{\sigma \in S_n(\chi)} a_\sigma \mathcal{L}_\sigma = 0 & \text{if } X_0 \text{ is non-crystalline,} \\ \sum_{\sigma \in S_n(\chi)} a_\sigma \mathcal{L}_\sigma = 0 & \text{if } X_0 \text{ is crystalline.} \end{cases}$$

Remark 2.2.
(1) Such formula was firstly established by Greenberg–Stevens [29, Thm. 3.14] in the case of 2-dimensional ordinary $\text{Gal}_{\mathbb{Q}_p}$-representations by Galois cohomology computations. In [22], Colmez generalized [29, Thm. 3.14] to 2-dimensional trianguline $\text{Gal}_{\mathbb{Q}_p}$-representations case by Galois cohomology computations and computations in Fontaine's rings. Theorem 2.1 in non-critical case (i.e. $S_c(\chi) = \emptyset$) was obtained by Zhang in [44], by generalizing Colmez's method. In [37], Pottharst generalized [29, Thm. 3.14] to rank 2 triangulable $(\varphi,)$-modules (in $\mathbb{Q}_p$ case) by studying cohomology of $(\varphi,)$-modules.

(2) The hypothesis X being $S_c(\chi)$-de Rham would imply that $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$. In fact, X being $S_c(\chi)$-de Rham implies $B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})$ being $S_c(\chi)$-de Rham. We claim that $B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})$ is $S_c(\chi)$-de Rham if an only if $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$. Indeed, it's easy to see $B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})$ is $S_c(\chi)$-de Rham if and only if $B_{E[\epsilon]/\epsilon^2}(1 + \psi\epsilon)$ is $S_c(\chi)$-de Rham. Viewing $B_{E[\epsilon]/\epsilon^2}(1 + \psi\epsilon)$ as an extension of B_E by B_E defined by $\psi \in H^1(\mathrm{Gal}_{F_\wp}, B_E)$ (cf. Section 1.3.1), we see $B_{E[\epsilon]/\epsilon^2}(1 + \psi\epsilon)$ is $S_c(\chi)$-de Rham if and only $\psi \in H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E)$, which is equivalent to that $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$ by Lemma 1.15. However, the converse is not true. This is a new subtlety: the formulas in (2.1) do *not* hold (in general) if one only assumes $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$ (e.g. see the discussion before Lemma 2.7 below).

We translate this theorem in terms of families of Galois representations. Let A be an a noid E-algebra, V be a locally free A-module of rank 2 equipped with a continuous $\mathrm{Gal}_{F_\wp}$-action. Thus $D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \widehat{\otimes}_{\mathbb{Q}_p} V)^{\mathrm{Gal}_{F_\wp}}$ is an $A \otimes_{\mathbb{Q}_p} F_\wp$-module. Using $A \otimes_{\mathbb{Q}_p} F_\wp \xrightarrow{\sim} \prod_{\sigma \in \Sigma_\wp} A$, $a \otimes b \mapsto (a\sigma(b))_\sigma$, one can decompose $D_{\mathrm{dR}}(V) \xrightarrow{\sim} \oplus_{\sigma \in \Sigma_\wp} D_{\mathrm{dR}}(V)_\sigma$. For $\sigma \in _\wp$, we say V is σ-*de Rham* if $D_{\mathrm{dR}}(V)_\sigma$ is locally free of rank 2 over A. Let $\mathcal{R}_A := \mathcal{R}_E \widehat{\otimes}_E A$, one can associate to V a $(\varphi,)$-module $D_{\mathrm{rig}}(V)$ (cf. [31, Thm. 2.2.17]) over $\mathcal{R}_A$. Suppose $D_{\mathrm{rig}}(V)$ sits in an exact sequence of $(\varphi,)$-modules over $\mathcal{R}_A$ as follows:

$$0 \longrightarrow \mathcal{R}_A(\delta_1) \longrightarrow D_{\mathrm{rig}}(V) \longrightarrow \mathcal{R}_A(\delta_2) \longrightarrow 0,$$

where $\delta_i : F_\wp \to A^\times$ are continuous characters, and we refer to [31, Const. 6.2.4] for rank 1 $(\varphi,)$-modules associated to characters. For a continuous character χ of $F_\wp^\times$ in $A^\times$, χ induces a $\mathbb{Q}_p$-linear map

$$d\chi : F_\wp \longrightarrow A, \quad a \mapsto \frac{d}{dx}\chi(\exp(ax))|_{x=0},$$

and thus an E-linear map $d\chi : F_\wp \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_\wp} E \to A$. So there exists $(\mathrm{wt}(\chi)_\sigma)_{\sigma \in \Sigma_\wp} \in A^{|\Sigma_\wp|}$, called the weight of χ, such that $d\chi((a_\sigma)_{\sigma \in \Sigma_\wp}) = \sum_{\sigma \in \Sigma_\wp} a_\sigma \mathrm{wt}(\chi)_\sigma$. Let z be an E-point of A, and $\delta_{i,z} := z^*\delta_i$, suppose

- $V_z := z^*V$ is semi-stable;
- $\delta_{1,z}\delta_{2,z}^{-1}$ is special.

Put $S_n(V_z) := S_n(\delta_{1,z}\delta_{2,z}^{-1})$, $S_c(V_z) := S_c(\delta_{1,z}\delta_{2,z}^{-1})$ (cf. (1.8)), and $\underline{\mathcal{L}}_{S_n(V_z)}$ the Fontaine–Mazur $\mathcal{L}$-invariants of V_z. By Theorem 2.1, one has

COROLLARY 2.3. — *Keep the above notation, suppose moreover V is $S_c(V_z)$-de Rham, then the differential form in $\Omega^1_{A/E}$*

$$\begin{cases} d\log(\delta_1\delta_2^{-1}(p)) + \sum_{\sigma\in S_n(V_z)} \mathcal{L}_\sigma d(\mathsf{wt}(\delta_1\delta_2^{-1})_\sigma) & \text{if } V_z \text{ is non-crystalline} \\ \sum_{\sigma\in S_n(V_z)} \mathcal{L}_\sigma d(\mathsf{wt}(\delta_1\delta_2^{-1})_\sigma) & \text{if } V_z \text{ is crystalline} \end{cases}$$

vanishes at z.

Remark 2.4. — Partially de Rham families would appear naturally in the study of p-adic automorphic forms, e.g. one encounters such families when studying locally analytic vectors in completed cohomology of Shimura curves (see Proposition 4.14 below), or certain families of overconvergent Hilbert modular forms (see App. A below). Note that this formula also applies for families of $F_\wp$-analytic $\mathrm{Gal}_{F_\wp}$-representations (cf. [7], which can be viewed as special cases of partially de Rham families).

The rest of this section is devoted to the proof of Theorem 2.1. We use Pottharst's method [37] (but in terms of B-pairs). It's clear that X being $S_c(\chi)$-de Rham is equivalent to saying that $B_{E[\epsilon]/\epsilon^2}(\tilde{\chi})$ is $S_c(\chi)$-de Rham and $[X] \in H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\tilde{\chi}))$. As discussed in Remark 2.2 (2), $B_{E[\epsilon]/\epsilon^2}(\tilde{\chi})$ being $S_c(\chi)$-de Rham is equivalent to that $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$. Thus it's sufficient to prove

PROPOSITION 2.5. — *Suppose $a_\sigma = 0$ for all $\sigma \in S_c(\chi)$ and $[X] \in H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\tilde{\chi}))$, then the formulas in (2.1) hold.*

Let $k_\sigma \in \mathbb{Z}$ for all $\sigma \in \Sigma_\wp$ such that $\chi = \mathrm{unr}(q^{-1})\prod_{\sigma\in\Sigma_\wp}\sigma^{k_\sigma}$. One has a natural exact sequence of E-B-pairs

$$(2.2) \qquad 0 \to B_E(\chi) \to B_{E[\epsilon]/\epsilon^2}(\tilde{\chi}) \to B_E(\chi) \to 0,$$

by taking cohomology, one gets an exact sequence

$$(2.3) \quad 0 \to H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \to H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\tilde{\chi}))$$
$$\xrightarrow{\kappa} H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \xrightarrow{\delta} H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi)).$$

Note $\kappa([X]) = [X_0]$. We suppose $\psi \neq 0$ (since the case $\psi = 0$ is trivial).

First consider non-critical case (i.e. $S_c(\chi) = \emptyset$), thus $k_\sigma \in \mathbb{Z}_{\geqslant 1}$ for all $\sigma \in \Sigma_\wp$. In this case, one has $H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi))$, which is of dimension $d + 1$ over E. One also has

LEMMA 2.6.
$$\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\tilde{\chi})) = 2d + 1.$$

Proof. — Let $W := B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})$ for simplicity, one has

$$\sum_{i=0}^{2}(-1)^i H^1(\mathrm{Gal}_{F_\wp}, W) = -2d.$$

It's easy to see $H^0(\mathrm{Gal}_{F_\wp}, W) = 0$ (cf. [33, Prop. 2.14]); moreover one has $\dim_E H^0(\mathrm{Gal}_{F_\wp}, W^\vee(1)) = 1$: let $\eta := \prod_{\sigma \in \Sigma_\wp} \sigma^{1-k_\sigma}$, then $W^\vee(1)$ is an extension of $B_E(\eta)$ by $B_E(\eta)$ (defined by ψ), by [33, Prop. 2.14], $\dim_E H^0(\mathrm{Gal}_{F_\wp}, B_E(\eta)) = 1$, which together with the fact $\psi \neq 0$ deduces $\dim_E H^0(\mathrm{Gal}_{F_\wp}, W^\vee(1)) = 1$. By the duality between $H^0(\mathrm{Gal}_{F_\wp}, W^\vee(1))$ and $H^2(\mathrm{Gal}_{F_\wp}, W)$, one sees $\dim_E H^2(\mathrm{Gal}_{F_\wp}, W) = 1$, so $H^1(\mathrm{Gal}_{F_\wp}, W) = 2d + 1$.   □

In particular, the map κ is not surjective. On the other hand, by Remark 1.8(1) (applied to $W_1 = W_3 = B_E$, $W_2 = B_{E[\epsilon]/\epsilon^2}(1 + \epsilon\psi)$, $W = B_E(\chi)$), we see the map δ is given (up to scalars) by $x \mapsto \langle x, \psi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the cup-product

$$H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \times H^1(\mathrm{Gal}_{F_\wp}, B_E) \to H^2(\mathrm{Gal}_{F_\wp}, B_E(\chi))$$

(cf. (1.9)). So one has (since $[X_0] = \kappa([X])$)

$$(2.4) \qquad \delta([X_0]) = \sum_{\sigma \in \Sigma_\wp} a_\sigma \langle [X_0], \psi_{\sigma,p} \rangle + a_{\mathrm{ur}} \langle [X_0], \psi_{\mathrm{ur}} \rangle = 0,$$

from which we deduces Proposition 2.5 in non-critical case by the definition of $\mathcal{L}$-invariants of X_0 (cf. Definition 1.16).

Suppose now $S_c(\chi) \neq \emptyset$, in this case $\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = d$. One can show as in the proof of Lemma 2.6 that $\dim_E H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) = 2d$, so κ is surjective (cf. (2.3)). Consequently, one can *not* expect any formula without further condition on X.

As in Section 1.3.2, put $\chi^\sharp := \chi \prod_{\sigma \in S_c(\chi)} \sigma^{1-k_\sigma}$, and $\widetilde{\chi}^\sharp := \chi^\sharp(1 + \epsilon\psi) = \widetilde{\chi} \prod_{\sigma \in S_c(\chi)} \sigma^{1-k_\sigma}$. Note $B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})$ and $B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}^\sharp)$ are both $S_c(\chi)$-de Rham (see Remark 2.2(2)). By (1.7), one has an exact sequence

$$(2.5) \ 0 \to H^0(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) \ (= 0)$$

$$\to \oplus_{\sigma \in S_c(\chi)} H^0(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})_{\mathrm{dR},\sigma}^+/t^{1-k_\sigma})$$

$$\to H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}^\sharp)) \to H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) \to 0,$$

from which (and Lemma 2.6) one calculates:

LEMMA 2.7.

$$\dim_E H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) = 2d + 1 - 2|S_c(\chi)|.$$

The commutative diagram of E-B-pairs

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_E(\chi^\sharp) & \longrightarrow & B_E(\widetilde{\chi}^\sharp) & \longrightarrow & B_E(\chi^\sharp) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & B_E(\chi) & \longrightarrow & B_E(\widetilde{\chi}) & \longrightarrow & B_E(\chi) & \longrightarrow & 0
\end{array}
$$

induces a commutative diagram (by (1.7))

$$
\begin{array}{ccccc}
H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) & \longrightarrow & H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}^\sharp)) & \xrightarrow{\kappa} & H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\
H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) & \longrightarrow & H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi})) & \xrightarrow{\kappa_g} & H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) \,,
\end{array}
$$

where all the vertical arrows are surjective, the two horizontal maps on the left are injective, and the top sequence is exact. Note that by Lemma 1.18 and Lemma 1.12, $H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) = H^1_g(\mathrm{Gal}_{F_\wp}, B_E(\chi))$ is of dimension $d + 1 - |S_c(\chi)|$. Since $\dim_E \mathrm{Im}(\kappa) = d = \dim_E H^1(\mathrm{Gal}_{F_\wp}, B_E(\chi^\sharp)) - 1$, one has $\dim_E \mathrm{Im}(\kappa_g) \geqslant \dim_E H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) - 1 = d - |S_c(\chi)|$, which together with Lemma 2.7 shows that the bottom sequence is also exact and that $\dim_E \mathrm{Im}(\kappa_g) = \dim_E H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_E(\chi)) - 1$ (in particular, κ_g is not surjective).

Denote by

$$[X^\sharp] \in H^1(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}^\sharp))$$

a preimage of $[X] \in H^1_{g,S_c(\chi)}(\mathrm{Gal}_{F_\wp}, B_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}))$ via j, $[X_0^\sharp] := \kappa([X^\sharp])$, thus $j([X_0^\sharp]) = [X_0]$. By (2.4) applied to $[X_0^\sharp]$, one has (note that $a_\sigma = 0$ for $\sigma \in S_c(\chi)$)

$$\sum_{\sigma \in \Sigma_\wp} a_\sigma \langle [X_0^\sharp], \psi_{\sigma,p} \rangle + a_{\mathrm{ur}} \langle [X_0^\sharp], \psi_{\mathrm{ur}} \rangle = 0,$$

from which, together with the definition of $\mathcal{L}$-invariants for $[X_0]$ (Definition 1.20, see in particular Remark 1.21), Proposition 2.5 follows.

3. Breuil's $\mathcal{L}$-invariants

In [9], to a 2-dimensional semi-stable non-crystalline representation V of $\mathrm{Gal}_{\mathbb{Q}_p}$, Breuil associated a locally analytic representation (V) of $\mathrm{GL}_2(\mathbb{Q}_p)$ (Breuil also considered Banach representations, but we only focus on locally analytic representations in this paper), which can determine V and in particular contains the information on the Fontaine–Mazur $\mathcal{L}$-invariant of V. Roughly speaking, Breuil found that certain extensions of locally

analytic representations (of $\mathrm{GL}_2(\mathbb{Q}_p)$) can be parametrized by some in-variants (which are referred to as *Breuil's $\mathcal{L}$-invariants*), and by matching these invariants with Fontaine–Mazur $\mathcal{L}$-invariants, one could get a one-to-one correspondence (in p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$) in semi-stable non-crystalline case. In [41], generalizing Breuil's theory, Schraen associ-ated a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ to a 2-dimensional semi-stable non-crystalline representation of $\mathrm{Gal}_{F_\wp}$ (although only the non-critical case was considered in loc. cit., Schraen's construction can easily generalize to critical case). We recall some results of loc. cit. in this section.

Let V be a 2-dimensional semi-stable non-crystalline representation of $\mathrm{Gal}_{F_\wp}$ over E of distinct Hodge–Tate weights $\underline{h}_{\Sigma_p} := (k_{1,\sigma}, k_{2,\sigma})_{\sigma \in \Sigma_\wp}$ (where $k_{1,\sigma} < k_{2,\sigma}$, and we use the convention that the Hodge–Tate weight of the cyclotomic character is -1), denote by α, $q\alpha$ the eigenvalues of φ^{d_0} on $D_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_{F_\wp}}$. By [33, §4.3], the E-B-pair $W(V)$ admits a unique triangulation:

$$0 \to B_E(\delta_1) \to W(V) \to B_E(\delta_2) \to 0$$

where

$$\delta_1 = \mathrm{unr}(\alpha) \prod_{\sigma \in S_n} \sigma^{-k_{1,\sigma}} \prod_{\sigma \in S_c} \sigma^{-k_{2,\sigma}},$$

$$\delta_2 = \mathrm{unr}(q\alpha) \prod_{\sigma \in S_n} \sigma^{-k_{2,\sigma}} \prod_{\sigma \in S_c} \sigma^{-k_{1,\sigma}},$$

S_n is a subset of $\quad_\wp$, and $S_c = {}_\wp \setminus S_n$. In fact, $S_c = S_c(\delta_1\delta_2^{-1})$ is the set of embeddings where V is critical (cf. Definition A.2 below). Since V is semi-stable non-crystalline, so is $W(V)$, one can thus associate to V the Fontaine–Mazur $\mathcal{L}$-invariants $\underline{\mathcal{L}}_{S_n} \in E^{|S_n|}$ (see the end of Section 1.3).

For $S \subseteq {}_\wp$, let $\underline{h}_S := (k_{1,\sigma}, k_{2,\sigma})_{\sigma \in S}$ and put

$$(3.1) \quad \mathrm{alg}(\underline{h}_S) := \otimes_{\sigma \in S}(\mathrm{Sym}^{k_{2,\sigma}-k_{1,\sigma}-1} E^2 \otimes_E \det^{-k_{2,\sigma}+1})^\sigma$$

$$\cong \left(\otimes_{\sigma \in S} (\mathrm{Sym}^{k_{2,\sigma}-k_{1,\sigma}-1} E^2 \otimes_E \det^{k_{1,\sigma}})^\sigma \right)^\vee,$$

which is an irreducible algebraic representation of $\mathrm{Res}_{F_\wp/\mathbb{Q}_p} \mathrm{GL}_2$ with the action of $\mathrm{GL}_2(F_\wp)$ on $(\cdot)^\sigma$ induced by the natural action of $\mathrm{GL}_2(E)$ via σ. Put

$$(3.2) \quad \chi(\alpha, \underline{h}_S) := \mathrm{unr}(\alpha) \prod_{\sigma \in S} \sigma^{-k_{1,\sigma}} \otimes \mathrm{unr}(\alpha) \prod_{\sigma \in S} \sigma^{-k_{2,\sigma}+1},$$

which is a locally S-analytic character of $T(F_\wp)$ over E. Consider the locally S-analytic parabolic induction (cf. [41, §2.3], where $\overline{B}(F_\wp)$ denotes the

lower triangular subgroup)

$$I(\alpha, \underline{h}_S) := (\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\alpha, \underline{h}_S))^{S-\mathrm{an}},$$

by [12, Thm. 4.1] (see also [41, §2.3]), we have

(1) $\mathrm{soc}_{\mathrm{GL}_2(F_\wp)} I(\alpha, \underline{h}_S) \cong (\mathrm{unr}(\alpha) \circ \det) \otimes_E \mathrm{alg}(\underline{h}_S) =: F(\alpha, \underline{h}_S)$;

(2) put $\quad(\alpha, \underline{h}_S) := I(\alpha, \underline{h}_S)/F(\alpha, \underline{h}_S)$, then

$$\mathrm{soc}_{\mathrm{GL}_2(F_\wp)} \quad(\alpha, \underline{h}_S) \cong (\mathrm{unr}(\alpha) \circ \det) \otimes_E \mathrm{St} \otimes_E \mathrm{alg}(\alpha, \underline{h}_S) =: \mathrm{St}(\alpha, \underline{h}_S),$$

which is also the maximal locally algebraic subrepresentation of $(\alpha, \underline{h}_S)$, where St denotes the standard smooth Steinberg representation;

(3) let $\sigma \in \quad_\wp$,

$$\chi(\alpha, \underline{h}_\sigma)^c := \mathrm{unr}(\alpha)\sigma^{-k_{2,\sigma}} \otimes \mathrm{unr}(\alpha)\sigma^{-k_{1,\sigma}+1},$$

and $I_\sigma^c(\alpha, \underline{h}_\sigma) := (\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\alpha, \underline{h}_\sigma)^c)^{\sigma-\mathrm{an}}$ (which is irreducible by [12, Thm. 4.1]), then one has a non-split exact sequence

$$0 \to \mathrm{St}(\alpha, \underline{h}_\sigma) \to \quad(\alpha, \underline{h}_\sigma) \to I_\sigma^c(\alpha, \underline{h}_\sigma) \to 0.$$

For $\mathcal{L} \in E$ and $\sigma \in \quad_\wp$, let $\log_{\sigma, \mathcal{L}} := \psi_{\sigma, p} + \mathcal{L}\psi_{\mathrm{ur}}$ (cf. Section 1.3.1) which is thus the additive character of $F_\wp^\times$ in E satisfying that $\log_{\sigma, \mathcal{L}}|_{\mathcal{O}_\wp^\times} = \sigma \circ \log$ and that $\log_{\sigma, \mathcal{L}}(p) = \mathcal{L}$.
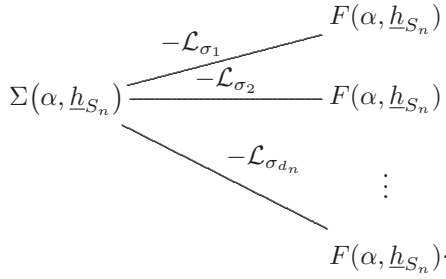
Let $d_n := |S_n|$, and $\psi(\underline{\mathcal{L}}_{S_n})$ be the following $(d_n + 1)$-dimensional representation of $T(F_\wp)$ over E

$$(3.3) \quad \psi(\underline{\mathcal{L}}_{S_n}) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \log_{\sigma_1, -\mathcal{L}_{\sigma_1}}(ad^{-1}) & \log_{\sigma_2, -\mathcal{L}_{\sigma_2}}(ad^{-1}) & \cdots & \log_{\sigma_{|d_n|}, -\mathcal{L}_{\sigma_{d_n}}}(ad^{-1}) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $\sigma_i \in S_n$. One gets thus an exact sequence of locally S_n-analytic representations of $\mathrm{GL}_2(F_\wp)$:

$$(3.4) \quad 0 \longrightarrow I(\alpha, \underline{h}_{S_n}) \longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\alpha, \underline{h}_{S_n}) \otimes_E \psi(\underline{\mathcal{L}}_{S_n}) \right)^{S_n-\mathrm{an}}$$

$$\xrightarrow{s} I(\alpha, \underline{h}_{S_n})^{\oplus d_n} \longrightarrow 0.$$

Put $\quad(\alpha, \underline{h}_{S_n}, \underline{\mathcal{L}}_{S_n}) := s^{-1}(F(\alpha, \underline{h}_{S_n})^{\oplus d_n})/F(\alpha, \underline{h}_{S_n})$, which is thus an extension of d_n-copies of $F(\alpha, \underline{h}_{S_n})$ by $\quad(\alpha, \underline{h}_{S_n})$:

$$\Sigma(\alpha,\underline{h}_{S_n}) \xrightarrow[\substack{-\mathcal{L}_{\sigma_1} \\ -\mathcal{L}_{\sigma_2}}]{\quad\quad} \begin{array}{l} F(\alpha,\underline{h}_{S_n}) \\ \\ F(\alpha,\underline{h}_{S_n}) \\ \\ \vdots \\ \\ F(\alpha,\underline{h}_{S_n}). \end{array}$$

Remark 3.1.

(1) Let $\underline{\mathcal{L}}'_{S_n} \in E^{d_n}$, as in [41, Prop. 4.13], one can show $(\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}'_{S_n}) \cong (\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n})$ if and only if $\underline{\mathcal{L}}'_{S_n} = \underline{\mathcal{L}}_{S_n}$. In particular, one can recover the data $\{\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n}\}$ from $(\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n})$.

(2) Let $\underline{h}'_{S_n} = (k'_{1,\sigma},k'_{2,\sigma})_{\sigma\in S_n} \in \mathbb{Z}^{2|S_n|}$ with $k'_{1,\sigma} - k_{1,\sigma} = k'_{2,\sigma} - k_{2,\sigma} = n_\sigma$, thus $\mathrm{alg}(\underline{h}'_{S_n}) \cong \mathrm{alg}(\underline{h}_{S_n}) \otimes_E \otimes_{\sigma\in S_n} \sigma \circ \det^{n_\sigma}$. It's straightforward to see $(\alpha,\underline{h}'_{S_n},\underline{\mathcal{L}}_{S_n}) \cong (\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n}) \otimes_E (\otimes_{\sigma\in S_n}\sigma\circ\det^{n_\sigma})$.

(3) By replacing the terms $\log_{\sigma_i,-\mathcal{L}_{\sigma_i}}(ad^{-1})$ in $\psi(\underline{\mathcal{L}}_{S_n})$ by $\log_{\sigma_i,-\mathcal{L}_{\sigma_i}}(ad^{-1}) + \chi_i \circ \det$ with an arbitrary locally σ_i-analytic (additive) character χ_i of $F_\wp^\times$ in E, one can get a locally $\mathbb{Q}_p$-analytic representation $(\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n})'$ in the same way as $(\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n})$. By some cohomology arguments in [41, §4.3] (see [24, Lem. 5]), one can actually prove

(3.5) $$(\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n})' \cong (\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n}).$$

(4) For $\sigma \in S_n$, denote by $\psi(\mathcal{L}_\sigma)$ the following 2-dimensional representation of $T(F_\wp)$:

$$\psi(\mathcal{L}_\sigma)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & \log_{\sigma,-\mathcal{L}_\sigma}(ad^{-1}) \\ 0 & 1 \end{pmatrix}.$$

One has thus an exact sequence

$$0 \longrightarrow I(\alpha,\underline{h}_{S_n}) \longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\alpha,\underline{h}_{S_n}) \otimes_E \psi(\mathcal{L}_\sigma)\right)^{S_n-\mathrm{an}}$$

$$\xrightarrow{s_\sigma} I(\alpha,\underline{h}_{S_n}) \longrightarrow 0.$$

Put $(\alpha,\underline{h}_{S_n},\mathcal{L}_\sigma) := s_\sigma^{-1}(F(\alpha,\underline{h}_{S_n}))/F(\alpha,\underline{h}_{S_n})$, the following isomorphism is straightforward:

$$(\alpha,\underline{h}_{S_n},\mathcal{L}_{\sigma_1}) \oplus_{\Sigma(\alpha,\underline{h}_{S_n})} (\alpha,\underline{h}_{S_n},\mathcal{L}_{\sigma_2}) \oplus_{\Sigma(\alpha,\underline{h}_{S_n})}$$

$$\cdots \oplus_{\Sigma(\alpha,\underline{h}_{S_n})} (\alpha,\underline{h}_{S_n},\mathcal{L}_{\sigma_{d_n}}) \xrightarrow{\sim} (\alpha,\underline{h}_{S_n},\underline{\mathcal{L}}_{S_n}).$$

Put $F(\alpha, \underline{h}_{\Sigma_\wp}) := F(\alpha, \underline{h}_{S_n}) \otimes_E \mathrm{alg}(\underline{h}_{S_c})$, $(\alpha, \underline{h}_{\Sigma_\wp}) := (\alpha, \underline{h}_{S_n}) \otimes_E \mathrm{alg}(\underline{h}_{S_c})$, and $(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n}) := (\alpha, \underline{h}_{S_n}, \mathcal{L}_{S_n}) \otimes_E \mathrm{alg}(\underline{h}_{S_c})$ which is thus an extension of d_n-copies of $F(\alpha, \underline{h}_{\Sigma_\wp})$ by $(\alpha, \underline{h}_{S_n})$, and carries the information of $\{\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n}\}$. For $\sigma \in {}_\wp$, put $I_\sigma^c(\alpha, \underline{h}_{\Sigma_\wp}) := I_\sigma^c(\alpha, \underline{h}_\sigma) \otimes_E \mathrm{alg}(\underline{h}_{\Sigma_\wp \setminus \{\sigma\}})$.

4. Local-global compatibility

We prove some local-global compatibility results for completed cohomology of quaternion Shimura curves in semi-stable non-crystalline case.

4.1. Setup and notations

Let F be a totally real field of degree d over $\mathbb{Q}$, denote by ${}_\infty$ the set of real embeddings of F. For a finite place $\mathfrak{l}$ of F, we denote by $F_\mathfrak{l}$ the completion of F at $\mathfrak{l}$, $\mathcal{O}_\mathfrak{l}$ the ring of integers of $F_\mathfrak{l}$ with $\varpi_\mathfrak{l}$ a uniformiser of $\mathcal{O}_\mathfrak{l}$. Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_F$ the ring of adeles of F. For a set S of places of $\mathbb{Q}$ (resp. of F), we denote by $\mathbb{A}^S$ (resp. by $\mathbb{A}_F^S$) the ring of adeles of $\mathbb{Q}$ (resp. of F) outside S, S_F the set of places of F above that in S, and $\mathbb{A}_F^S := \mathbb{A}_F^{S_F}$.

Let p be a prime number, suppose there exists only one prime $\wp$ of F lying above p. Denote by ${}_\wp$ the set of $\mathbb{Q}_p$-embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$; let ϖ be a uniformizer of $\mathcal{O}_\wp$, $F_{\wp,0}$ the maximal unramified extension of $\mathbb{Q}_p$ in $F_\wp$, $d_0 := [F_{\wp,0} : \mathbb{Q}_p]$, $e := [F_\wp : F_{\wp,0}]$, $q := p^{d_0}$ and $v_\wp$ a p-adic valuation on $\overline{\mathbb{Q}_p}$ normalized by $v_\wp(\varpi) = 1$. Let E be a finite extension of $\mathbb{Q}_p$ big enough such that E contains all the $\mathbb{Q}_p$-embeddings of F in $\overline{\mathbb{Q}_p}$, $\mathcal{O}_E$ the ring of integers of E and ϖ_E a uniformizer of $\mathcal{O}_E$.

Let B be a quaternion algebra of center F with $S(B)$ the set (of even cardinality) of places of F where B is ramified, suppose $|S(B) \cap {}_\infty| = d-1$ and $S(B) \cap {}_\wp = \emptyset$, i.e. there exists $\tau_\infty \in {}_\infty$ such that $B \otimes_{F,\tau_\infty} \mathbb{R} \cong M_2(\mathbb{R})$, $B \otimes_{F,\sigma} \mathbb{R} \cong \mathbb{H}$ for all $\sigma \in {}_\infty \setminus \{\tau_\infty\}$, where $\mathbb{H}$ denotes the Hamilton algebra, and $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(F_\wp)$. We associate to B a reductive algebraic group G over $\mathbb{Q}$ with $G(R) := (B \otimes_{\mathbb{Q}} R)^\times$ for any $\mathbb{Q}$-algebra R. Set $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, and denote by h the morphism

$$h : \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times \longrightarrow G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^*)^{d-1},$$

$$a + bi \mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, \cdots, 1 \right).$$

The space of $G(\mathbb{R})$-conjugacy classes of h has a structure of complex manifold, and is isomorphic to $\mathfrak{h}^{\pm} := \mathbb{C} \setminus \mathbb{R}$ (i.e. 2 copies of the Poincaré's upper half plane). We get a projective system of Riemann surfaces indexed by open compact subgroups of $G(\mathbb{A}^{\infty})$:

$$M_K(\mathbb{C}) := G(\mathbb{Q}) \setminus (\mathfrak{h}^{\pm} \times (G(\mathbb{A}^{\infty})/K))$$

where $G(\mathbb{Q})$ acts on $\mathfrak{h}^{\pm}$ via $G(\mathbb{Q}) \hookrightarrow G(\mathbb{R})$ and the transition map is given by

$$
\begin{aligned}
G(\mathbb{Q}) \setminus (\mathfrak{h}^{\pm} \times (G(\mathbb{A}^{\infty})/K_1)) &\longrightarrow G(\mathbb{Q}) \setminus (\mathfrak{h}^{\pm} \times (G(\mathbb{A}^{\infty})/K_2)), \\
(x, g) &\mapsto (x, g),
\end{aligned}
$$
(4.1)

for $K_1 \subseteq K_2$. It's known that $M_K(\mathbb{C})$ has a canonical proper smooth model over F (via the embedding τ_{∞}), denoted by M_K, and these $\{M_K\}_K$ form a projective system of proper smooth algebraic curves over F. Note that one has a natural isomorphism $G(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_2(F_{\wp})$. For an open compact subgroup K of $G(\mathbb{A}^{\infty})$, let $K_p := K \cap G(\mathbb{Q}_p)$, and $K^p := K \cap G(\mathbb{A}^{\infty,p})$, so one has $K = K^p K_p$.

Let $K_{\wp,0} := \mathrm{GL}_2(\mathcal{O}_{\wp})$, and in the following, we fix an open compact subgroup K^p of $G(\mathbb{A}^{\infty,p})$ of the form $\prod_{v \nmid p} K_v$ small enough such that $K^p K_{\wp,0}$ is neat (e.g. see [35, Def. 4.11]). Denote by $S(K^p)$ the set of finite places $\mathfrak{l}$ of F such that $p \nmid \mathfrak{l}$, that B is split at $\mathfrak{l}$, i.e. $B \otimes_F F_{\mathfrak{l}} \xrightarrow{\sim} M_2(F_{\mathfrak{l}})$, and that $K^p \cap \mathrm{GL}_2(F_{\mathfrak{l}}) \cong \mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}})$. Denote by $\mathcal{H}^p$ the commutative $\mathcal{O}_E$-algebra generated by the double coset operators $[\mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}}) g_{\mathfrak{l}} \, \mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}})]$ for all $g_{\mathfrak{l}} \in \mathrm{GL}_2(F_{\mathfrak{l}})$ with $\det(g_{\mathfrak{l}}) \in \mathcal{O}_{\mathfrak{l}}$ and for all $\mathfrak{l} \in S(K^p)$. Set

$$T_{\mathfrak{l}} := \left[\mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}}) \begin{pmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}}) \right],$$

$$S_{\mathfrak{l}} := \left[\mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}}) \begin{pmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & \varpi_{\mathfrak{l}} \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{l}}) \right],$$

then $\mathcal{H}^p$ is the polynomial algebra over $\mathcal{O}_E$ generated by $\{T_{\mathfrak{l}}, S_{\mathfrak{l}}\}_{\mathfrak{l} \in S(K^p)}$.

Denote by Z_0 the kernel of the norm map $\mathcal{N} : \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \to \mathbb{G}_m$ which is a subgroup of $Z = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. We set $G^c := G/Z_0$.

For a Banach representation V of $\mathrm{GL}_2(F_{\wp})$ over E (cf. [39, §2]), denote by $V_{\mathbb{Q}_p-\mathrm{an}}$ the E-vector subspace of locally $\mathbb{Q}_p$-analytic vectors of V, which is stable by $\mathrm{GL}_2(F_{\wp})$ and hence is a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_{\wp})$. If V is moreover admissible, by [40, Thm. 7.1], $V_{\mathbb{Q}_p-\mathrm{an}}$ is an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_{\wp})$ and dense in V. For $J \subseteq \wp$, denote by $V_{J-\mathrm{an}}$ the subrepresentation generated by the locally J-analytic vectors of $V_{\mathbb{Q}_p-\mathrm{an}}$ (cf. [41, §2]), put $V_{\infty} := V_{\emptyset-\mathrm{an}}$.

Let A be a local artinian E-algebra, for a locally $\mathbb{Q}_p$-analytic character $\chi = \chi_1 \otimes \chi_2$ of $T(L)$ over A, let $\mathrm{wt}(\chi) := (\mathrm{wt}(\chi)_{1,\sigma}, \mathrm{wt}(\chi)_{2,\sigma})_{\sigma \in \Sigma_\wp} := (\mathrm{wt}(\chi_1)_\sigma, \mathrm{wt}(\chi_2)_\sigma)_{\sigma \in \Sigma_\wp} \in A^{2|d|}$ be the weight of χ. For an integer weight $\lambda \in \mathbb{Z}^{2|d|}$, denote by δ_λ the algebraic character of $T(L)$ over E with weight λ.

Let V be an E-vector space equipped with an E-linear action of A (with A a set of operators), χ a system of eigenvalues of A, denote by $V^{A=\chi}$ the χ-eigenspace, $V[A = \chi]$ the generalized χ-eigenspace, V^A the vector space of A-fixed vectors.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\overline{\mathfrak{b}}$, $\mathfrak{t}$ denote the Lie algebra (over $F_\wp$) of $\mathrm{GL}_2(F_\wp)$, $B(F_\wp)$, $\overline{B}(F_\wp)$, $T(F_\wp)$ respectively. For a Lie algebra $\mathfrak{g}'$ over $F_\wp$, put $\mathfrak{g}'_{\Sigma_\wp} := \mathfrak{g}' \otimes_{\mathbb{Q}_p} E \cong \oplus_{\sigma \in \Sigma_\wp} \mathfrak{g}' \otimes_{F_\wp,\sigma} E =: \oplus_{\sigma \in \Sigma_\wp} \mathfrak{g}'_\sigma$; let $J \subseteq \quad_\wp$, put $\mathfrak{g}'_J := \oplus_{\sigma \in J} \mathfrak{g}'_\sigma$.

4.2. Completed cohomology and eigenvarieties

We recall the construction of eigenvarieties from the completed cohomology of quaternion Shimura curves and survey some properties.

4.2.1. Completed cohomology of quaternion Shimura curves

Let W be a finite dimensional algebraic representation of G^c over E. As in [18, §2.1], one can associate to W a local system $\mathcal{V}_W$ of E-vector spaces over M_K. Let W_0 be an $\mathcal{O}_E$-lattice of W, and denote by $\mathcal{S}_{W_0}$ the set (ordered by inclusions) of open compact subgroups of $G(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_\wp)$ which stabilize W_0. For any $K_p \in S_{W_0}$, one can associate to W_0 (resp. to W_0/ϖ_E^s for $s \in \mathbb{Z}_{\geqslant 1}$) a local system $\mathcal{V}_{W_0}$ (resp. $\mathcal{V}_{W_0/\varpi_E^s}$) of $\mathcal{O}_E$-modules (resp. of $\mathcal{O}_E/\varpi_E^s$-modules) over $M_{K_p K^p}$. Following Emerton ([27, §2.1]), we put

$$H^i_{\text{ét}}(K^p, W_0) := \varinjlim_{K_p \in \mathcal{S}_{W_0}} H^i_{\text{ét}}(M_{K_p K^p, \overline{\mathbb{Q}}}, \mathcal{V}_{W_0})$$

$$\cong \varinjlim_{K_p \in \mathcal{S}_{W_0}} \varprojlim_s H^i_{\text{ét}}(M_{K_p K^p, \overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi_E^s});$$

$$\widetilde{H}^i_{\text{ét}}(K^p, W_0) := \varprojlim_s \varinjlim_{K_p \in \mathcal{S}_{W_0}} H^i_{\text{ét}}(M_{K_p K^p, \overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi_E^s});$$

$$H^i_{\text{ét}}(K^p, W_0)_E := H^i_{\text{ét}}(K^p, W_0) \otimes_{\mathcal{O}_E} E;$$

$$\widetilde{H}^i_{\text{ét}}(K^p, W_0)_E := \widetilde{H}^i_{\text{ét}}(K^p, W_0) \otimes_{\mathcal{O}_E} E.$$

All these groups ($\mathcal{O}_E$-modules or E-vector spaces) are equipped with a natural topology induced from the discrete topology on the finite group $H^i_{\text{ét}}(M_{K_pK^p,\overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi^s_E})$, and equipped with a natural continuous action of $\mathcal{H}^p \times \text{Gal}_F$ and of $K_p \in \mathcal{S}_{W_0}$. Moreover, for any $\mathfrak{l} \in S(K^p)$, the action of $\text{Gal}_{F_{\mathfrak{l}}}$ (induced by that of Gal_F) is unramified and satisfies the Eichler–Shimura relation:

$$\text{Frob}_{\mathfrak{l}}^{-2} - T_{\mathfrak{l}}\,\text{Frob}_{\mathfrak{l}}^{-1} + \ell^{f_{\mathfrak{l}}} S_{\mathfrak{l}} = 0$$

where $\text{Frob}_{\mathfrak{l}}$ denotes the arithmetic Frobenius, ℓ the prime number lying below $\mathfrak{l}$, $f_{\mathfrak{l}}$ the degree of the maximal unramified extension in $F_{\mathfrak{l}}$ over $\mathbb{Q}_{\ell}$ (thus $\ell^{f_{\mathfrak{l}}} = |\mathcal{O}_{\mathfrak{l}}/\varpi_{\mathfrak{l}}|$). Note that $\widetilde{H}^i_{\text{ét}}(K^p, W_0)_E$ is an E-Banach space with the norm defined by the $\mathcal{O}_E$-lattice $\widetilde{H}^i_{\text{ét}}(K^p, W_0)$.

Consider the ordered set (by inclusion) $\{W_0\}$ of $\mathcal{O}_E$-lattices of W. Following [27, Def. 2.2.9], we put

$$H^i_{\text{ét}}(K^p, W) := \varinjlim_{W_0} H^i_{\text{ét}}(K^p, W_0)_E \,,$$

$$\widetilde{H}^i_{\text{ét}}(K^p, W) := \varinjlim_{W_0} \widetilde{H}^i_{\text{ét}}(K^p, W_0)_E \,,$$

where all the transition maps are topological isomorphisms (cf. [27, Lem. 2.2.8]). These E-vector spaces are moreover equipped with a natural continuous action of $\text{GL}_2(F_{\wp})$ (cf. [27, Lem. 2.2.10]).

THEOREM 4.1 ([27, Thm. 2.2.11 (i), Thm. 2.2.17]).

(1) *The E-Banach space $\widetilde{H}^i_{\text{ét}}(K^p, W)$ is an admissible Banach representation of $\text{GL}_2(F_{\wp})$. If W is the trivial representation, the representation $\widetilde{H}^i_{\text{ét}}(K^p, W)$ is unitary.*

(2) *One has a natural isomorphism of Banach representations of $\text{GL}_2(F_{\wp})$ invariant under the action of $\mathcal{H}^p \times \text{Gal}(\overline{F}/F)$:*

$$\widetilde{H}^i_{\text{ét}}(K^p, W) \xrightarrow{\sim} \widetilde{H}^i_{\text{ét}}(K^p, E) \otimes_E W.$$

(3) *One has a natural $\text{GL}_2(F_{\wp}) \times \mathcal{H}^p \times \text{Gal}_F$-invariant map*

$$H^i_{\text{ét}}(K^p, W) \longrightarrow \widetilde{H}^i_{\text{ét}}(K^p, W).$$

Let ρ be a 2-dimensional continuous representation of Gal_F over E such that ρ is unramified at all $\mathfrak{l} \in S(K^p)$ and that the reduction $\overline{\rho}$ over k_E (up to semi-simplification *a priori*) is absolutely irreducible. To $\overline{\rho}$, one can associate a maximal ideal $\mathfrak{m}(\overline{\rho})$ of $\mathcal{H}^p$ as the kernel of the following morphism

$$\mathcal{H}^p \longrightarrow k_E := \mathcal{O}_E/\varpi_E,$$
$$T_{\mathfrak{l}} \mapsto \text{tr}(\text{Frob}_{\mathfrak{l}}^{-1}), \ S_{\mathfrak{l}} \mapsto \ell^{-f_{\mathfrak{l}}} \det(\text{Frob}_{\mathfrak{l}}^{-1}), \ \forall\, \mathfrak{l} \in S(K^p).$$

For an $\mathcal{H}^p$-module M, denote by $M_{\bar{\rho}}$ the localisation of M at $\mathfrak{m}(\bar{\rho})$.

Put $Z_1 := 1 + 2\varpi\mathcal{O}_\wp \subseteq Z(\mathrm{GL}_2(F_\wp))$ (where the latter one denotes the center of $\mathrm{GL}_2(F_\wp)$). Put

$$U_1 := \{g_\wp \in 1 + 2\varpi M_2(\mathcal{O}_\wp) \mid \det(g_\wp) = 1\},$$

and $H_\wp := Z_1 U_1$ which is a pro-p open compact subgroup of $\mathrm{GL}_2(\mathcal{O}_\wp)$.

PROPOSITION 4.2. — *Let W be an irreducible algebraic representation of G^c, and suppose that $\widetilde{H}^1_{\text{ét}}(K^p, W)_{\bar{\rho}} \neq 0$.*

(1) *The natural morphism*

$$H^1_{\text{ét}}(K^p, W)_{\bar{\rho}} \longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, W)_{\bar{\rho}, \infty}$$

is an isomorphism, where ∞ denotes the smooth vectors for the action of $\mathrm{GL}_2(F_\wp)$.

(2) *Let ψ be a continuous character of Z_1 such that $\psi|_{\overline{(Z(\mathbb{Q}) \cap K^p H_\wp)_p}} = 1$, then one has an isomorphism of $H_\wp$-representations*

$$\widetilde{H}^1_{\text{ét}}(K^p, W)^{Z_1 = \psi}_{\bar{\rho}} \xrightarrow{\sim} \mathcal{C}(U_1, E)^{\oplus r}$$

where Z_1 acts on $\mathcal{C}(U_1, E)^{\oplus r}$ by the character ψ, and U_1 by the right regular action.

Proof. — Part (1) follows from [35, Prop. 5.2]. Part (2) follows by the same arguments as in [35, §5] (see also [24, Cor. 1]).   □

Remark 4.3. — One can check $\overline{(Z(\mathbb{Q}) \cap K^p H_\wp)_p} \subseteq Z_0(\mathbb{Q}_p)$ (cf. Section 4.1), in particular, any continuous character of Z_1 factoring through $Z_1/(Z_1 \cap Z_0(\mathbb{Q}_p))$ satisfies the assumption in Proposition 4.2 (2).

4.2.2. Eigenvarieties

Let $J \subseteq \Sigma_\wp$, $k_\sigma \in 2\mathbb{Z}_{\geqslant 1}$ for all $\sigma \in J$ and $w \in 2\mathbb{Z}$, we set

$$W(\underline{k}_J, w) := \otimes_{\sigma \in J} \left(\mathrm{Sym}^{k_\sigma - 2} E^2 \otimes_E \det^{\frac{w - k_\sigma + 2}{2}} \right)^\sigma \otimes_E \left(\otimes_{\sigma \in \Sigma_\wp \setminus J} (\det^{\frac{w}{2}})^\sigma \right),$$

which is an irreducible algebraic representation of G over E with the action of $\mathrm{GL}_2(F_\wp)$ on $(*)^\sigma$ induced from the standard action of $\mathrm{GL}_2(E)$ via σ : $\mathrm{GL}_2(F_\wp) \hookrightarrow \mathrm{GL}_2(E)$. Note the central character of $W(\underline{k}_J, w)$ is given by $\mathcal{N}^w$ (where $\mathcal{N}$ denotes the norm map), thus $W(\underline{k}_J, w)$ can be viewed as an algebraic representation of G^c. One has $W(\underline{k}_J, w') = W(\underline{k}_J, w) \otimes_E W(2_J, w' - w)$ (where $w' \in 2\mathbb{Z}$), and $W(\underline{k}_J, w) = W(\underline{k}_{J'}, w) \otimes_E W(\underline{k}_{J \setminus J'}, 0)$ for $J' \subseteq J$.

Let ρ be a 2-dimensional continuous representation of Gal_F over E, absolutely irreducible modulo ϖ_E, such that ρ is unramified at all $\mathfrak{l} \in S(K^p)$

and that $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}} \neq 0$. By Theorem 4.1 (2), $\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w)) \cong \widetilde{H}^1_{\text{ét}}(K^p, E) \otimes_E W(\underline{k}_J, w)$, thus $\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}} \neq 0$ for $J \subseteq \Sigma_\wp$. Put $\quad := \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}, \mathbb{Q}_p-\text{an}}$, and for $J \subseteq \Sigma_\wp$, put

$$(\underline{k}_J, w) := \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}, \Sigma_\wp \setminus J-\text{an}} \otimes_E W(\underline{k}_J, w)^\vee,$$

which is in fact a closed subrepresentation of $\quad$ (cf. Corollary B.2 below). Indeed, we have

$$\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}, \Sigma_\wp \setminus J-\text{an}} \otimes_E W(\underline{k}_J, w)^\vee$$
$$\xrightarrow{\sim} (\quad \otimes_E W(\underline{k}_J, w))_{\Sigma_\wp \setminus J-\text{an}} \otimes_E W(\underline{k}_J, w)^\vee$$
$$\xrightarrow{\sim} \left(\quad \otimes_E \otimes_{\sigma \in J}(\text{Sym}^{k_\sigma - 2} E^2 \otimes_E \det^{\frac{w-k_\sigma+2}{2}})^\sigma \right)_{\Sigma_\wp \setminus J-\text{an}}$$
$$\otimes_E (\otimes_{\sigma \in \Sigma_\wp \setminus J}(\det^{\frac{w}{2}})^\sigma) \otimes_E W(\underline{k}_J, w)^\vee$$
$$\xrightarrow{\sim} \left(\quad \otimes_E \otimes_{\sigma \in J}(\text{Sym}^{k_\sigma - 2} E^2 \otimes_E \det^{\frac{w-k_\sigma+2}{2}})^\sigma \right)_{\Sigma_\wp \setminus J-\text{an}}$$
$$\otimes_E (\otimes_{\sigma \in J} \text{Sym}^{k_\sigma - 2} E^2 \otimes_E \det^{\frac{w-k_\sigma+2}{2}})^\vee \hookrightarrow \quad,$$

where the first isomorphism is from Theorem 4.1 (2), and the last injection follows from Proposition B.1 below. Similarly, for $J' \supseteq J$, we have a natural closed embedding invariant under the action of $\text{GL}_2(F_\wp) \times \mathcal{H}^p$:

$$(4.2) \qquad\qquad (\underline{k}_{J'}, w) \hookrightarrow (\underline{k}_J, w).$$

Note $\quad (\underline{k}_\emptyset, w) \cong \quad$, and by Proposition 4.2 (1),

$$(\underline{k}_{\Sigma_\wp}, w) \cong H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_p}, w))_{\overline{\rho}} \otimes_E W(\underline{k}_{\Sigma_p}, w)^\vee.$$

We have the following easy lemma.

LEMMA 4.4. — *Keep the above notation, and let V be a locally $\Sigma_\wp \setminus J$-analytic representation of $\text{GL}_2(F_\wp)$, then*

$$(4.3) \quad \text{Hom}_{\text{GL}_2(F_\wp)} \left(V, \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}, \Sigma_\wp \setminus J-\text{an}} \right)$$
$$\xrightarrow{\sim} \text{Hom}_{\text{GL}_2(F_\wp)}(V \otimes_E W(\underline{k}_J, w)^\vee, \quad (\underline{k}_J, w))$$
$$\xrightarrow{\sim} \text{Hom}_{\text{GL}_2(F_\wp)}(V \otimes_E W(\underline{k}_J, w)^\vee, \quad),$$

where the first map is given by $f \mapsto f \otimes \text{id}$, and the second is induced by the injection $\quad (\underline{k}_J, w) \hookrightarrow \quad$.

Proof. — Given a morphism $g : V \otimes_E W(\underline{k}_J, w)^\vee \to \quad$, consider the composition

$$V \to V \otimes_E W(\underline{k}_J, w)^\vee \otimes_E W(\underline{k}_J, w) \xrightarrow{g \otimes \text{id}} \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}, \mathbb{Q}_p-\text{an}}$$

whose image is contained in $\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))_{\overline{\rho}, \Sigma_\wp \backslash J - \text{an}}$ since V is locally $_\wp \backslash J$-anlaytic, and it's straightforward to check this gives an inverse (up to non-zero scalars) of the composition (4.3). The lemma follows. $\qquad\square$

Let $J \subsetneq \,_\wp$, $\underline{k}_J \in 2\mathbb{Z}_{\geqslant 1}^{|J|}$ and $w \in 2\mathbb{Z}$, consider

$$(4.4) \qquad (\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}} \cong \widetilde{H}^1(K^p, W(\underline{k}_J, w))^{Z_1}_{\overline{\rho}, \Sigma_\wp \backslash J - \text{an}} \otimes_E W(\underline{k}_J, w)^\vee,$$

which is an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E, equipped with a continuous action of $\mathcal{H}^p$ commuting with $\mathrm{GL}_2(F_\wp)$. Applying Jacquet–Emerton functor (for the upper triangular subgroup $B(F_\wp)$, cf. [26]), we get an essentially admissible locally $\mathbb{Q}_p$-analytic representation $J_B((\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}})$ of $T(F_\wp)$ over E, whose strong dual corresponds to a coherent sheaf $\mathcal{M}_0(\underline{k}_J, w)$ over $\widehat{T}$ (which denotes the rigid space over E parametrizing continuous characters of $T(F_\wp)$) such that

$$(\widehat{T}, \mathcal{M}_0(\underline{k}_J, w)) \xrightarrow{\sim} J_B((\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}})^\vee$$

as coadmissible modules over the Fréchet-Stein algebra $\mathcal{O}(\widehat{T})$. By funtoriality, $\mathcal{M}_0(\underline{k}_J, w)$ is equipped with an $\mathcal{O}(\widehat{T})$-linear action of $\mathcal{H}^p$. Following Emerton [27, §2.3], we can construct an eigenvariety from the triplet $\{\mathcal{M}_0(\underline{k}_J, w), \widehat{T}, \mathcal{H}^p\}$:

THEOREM 4.5. — *There exists a rigid analytic space* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ *over E together with a finite morphism of rigid spaces* $i : \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \to \widehat{T}$ *and a morphism of E-algebras with dense image*

$$(4.5) \qquad \mathcal{H}^p \otimes_{\mathcal{O}_E} \mathcal{O}(\widehat{T}) \longrightarrow \mathcal{O}(\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}})$$

such that

(1) *a point z of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ is uniquely determined by its image χ in $\widehat{T}(\overline{E})$ and the induced morphism $\lambda : \mathcal{H}^p \longrightarrow \overline{E}$, called a system of eigenvalues of $\mathcal{H}^p$; hence z would be denoted by (χ, λ));*

(2) *for a finite extension L of E, $(\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(L)$ if and only if the corresponding eigenspace*

$$J_B((\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}} \otimes_E L)^{T(F_\wp) = \chi, \mathcal{H}^p = \lambda}$$

is non-zero;

(3) *there exists a coherent sheaf, denoted by $\mathcal{M}(\underline{k}_J, w)$, over $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, such that $i_* \mathcal{M}(\underline{k}_J, w) \cong \mathcal{M}_0(\underline{k}_J, w)$ and that for an L-point $z = (\chi, \lambda)$, the fiber $\mathcal{M}(\underline{k}_J, w)\big|_z$ is naturally dual to the (finite dimensional) L-vector space*

$$J_B((\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}} \otimes_E L)^{T(F_\wp) = \chi, \mathcal{H}^p = \lambda}.$$

By (4.4), one has an isomorphism

$$(4.6) \quad J_B(\ (\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}})$$
$$\cong J_B(\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_J, w))^{Z_1}_{\Sigma_\wp \setminus J - \text{an}}) \otimes_E \chi(\underline{k}_J, w),$$

where $\chi(\underline{k}_J, w) := (\prod_{\sigma \in J}(\sigma^{\frac{k_\sigma-2}{2}} \otimes \sigma^{\frac{2-k_\sigma}{2}}))(\prod_{\sigma \in \Sigma_\wp}(\sigma^{-w/2} \otimes \sigma^{-w/2}))$ is a character of $T(F_\wp)$ over E. Thus, by Theorem 4.5 (2), if $(\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, then $\text{wt}(\chi)_{1,\sigma} + \text{wt}(\chi)_{2,\sigma} = -w$ for all $\sigma \in \ _\wp$, and $\text{wt}(\chi)_{1,\sigma} - \text{wt}(\chi)_{2,\sigma} = k_\sigma - 2$ for all $\sigma \in J$.

Denote by $\widehat{T}_{\Sigma_\wp \setminus J}$ the rigid space over E parametrizing the locally $_\wp \setminus J$-analytic characters of $T(F_\wp)$, and denote by $\widehat{T}(\underline{k}_J, w)$ the image of the following closed embedding

$$\widehat{T}_{\Sigma_\wp \setminus J} \hookrightarrow \widehat{T}, \ \chi \mapsto \chi(\underline{k}_J, w)\chi,$$

which parametrizes locally $\mathbb{Q}_p$-analytic characters of $T(F_\wp)$ with fixed weights $(\frac{k_\sigma-w-2}{2}, \frac{2-k_\sigma-w}{2})$ for $\sigma \in J$. By the isomorphism (4.6), it's easy to see the action of $\mathcal{O}(\widehat{T})$ on $\mathcal{M}_0(\underline{k}_J, w)$ factors through $\mathcal{O}(\widehat{T}(\underline{k}_J, w))$, consequently, the morphism $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \to \widehat{T}$ factors through $\widehat{T}(\underline{k}_J, w)$. Denote by $\widehat{T}(\underline{k}_J, w)_0$ the closed subspace of $\widehat{T}(\underline{k}_J, w)$ consisting of the points χ with $\chi|_{Z_1} = \mathcal{N}^{-w}$, thus the morphism $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \to \widehat{T}(\underline{k}_J, w)$ factors through $\widehat{T}(\underline{k}_J, w)_0$. Denote by $Z_1' := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \ \middle| \ a \in 1 + 2\varpi\mathcal{O}_\wp \right\}$, and $\mathcal{W}_1$ the rigid space over E parametrizing continuous characters of $1 + 2\varpi\mathcal{O}_\wp$ (thus of Z_1'), and $\mathcal{W}_1(\underline{k}_J)$ the closed subspace of $\mathcal{W}_1$ of characters χ with $\text{wt}(\chi)_\sigma = k_\sigma - 2$ for all $\sigma \in J$. One has thus a natural projection

$$j : \widehat{T}(\underline{k}_J, w)_0 \twoheadrightarrow \mathcal{W}_1(\underline{k}_J) \times \mathbb{G}_m, \ \chi \mapsto (\chi|_{Z_1'}, \chi(z_\wp)),$$

where $z_\wp := \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$. By Proposition 4.2 (2) and (the proof of [26, Prop. 4.2.36]), $J_B(\ (\underline{k}_J, w)^{Z_1 = \mathcal{N}^{-w}})^\vee$ is a coadmissible module over $\mathcal{O}(\mathcal{W}_1(\underline{k}_J) \times \mathbb{G}_m)$, in other words, $j_* \mathcal{M}_0(\underline{k}_J, w)$ is a coherent sheaf over $\mathcal{W}_1(\underline{k}_J) \times \mathbb{G}_m$.

PROPOSITION 4.6.

(1) *The support $\mathcal{Z}(\underline{k}_J, w)$ of $j_*\mathcal{M}_0(\underline{k}_J, w)$ on $\mathcal{W}_1(\underline{k}_J) \times \mathbb{G}_m$ is a Fredholm hypersurface in $\mathcal{W}_1(\underline{k}_J) \times \mathbb{G}_m$, and there exists an admissible covering $\{U_i\}$ of $\mathcal{Z}(\underline{k}_J, w)$ by affinoids U_i such that the natural morphism $U_i \to \mathcal{W}_1$ induces a finite surjective map from U_i to an affinoid open W_i of $\mathcal{W}_1(\underline{k}_J)$, and that U_i is a connected component*

of the preimage of W_i. Moreover, $(U_i, j_*\mathcal{M}_0(\underline{k}_J, w))$ is a finite projective $\mathcal{O}(W_i)$-module.

(2) Denote by g the natural morphism $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \to \mathcal{Z}(\underline{k}_J, w)$, and let $\{U_i\}$ as in (1), then $g^{-1}(U_i)$ is an affinoid open in $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, and we have $(g^{-1}(U_i), \mathcal{M}(\underline{k}_J, w)) \cong (j^{-1}(U_i), \mathcal{M}_0(\underline{k}_J, w)) \cong M_i$. Let B_i be the affinoid algebra with $\mathrm{Spm}\, B_i \cong g^{-1}(U_i)$, then B_i is the $\mathcal{O}(W_i)$-subalgebra of $\mathrm{End}_{\mathcal{O}(W_i)}(M_i)$ generated by the $\mathcal{O}(W_i)$-linear operators in $T(F_\wp) \times \mathcal{H}^p$.

Proof. — As discussed in the proof of [26, Prop. 4.2.36] (which can apply by Proposition 4.2 (2)) (see also [16, Lem. 3.10], [23, §5.4] and the discussion before [24, Prop. 5]), one can reconstruct $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ by the method of Coleman–Mazur–Buzzard, and then the proposition follows from [17, §4, §5]. □

Denote by κ the composition

$$\kappa : \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \longrightarrow \mathcal{Z}(\underline{k}_J, w) \longrightarrow \mathcal{W}_1(\underline{k}_J),$$

which also equals the composition $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}} \to \widehat{T}(\underline{k}_J, w)_0 \to \mathcal{W}_1(\underline{k}_J)$.

4.2.3. Classicality

Let $z = (\chi, \lambda)$ be a point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, z is called *classical* if there exist $k_\sigma \in 2\mathbb{Z}_{\geqslant 1}$ for all $\sigma \in {}_\wp \setminus J$ such that

$$(J_B(\ (k_{\Sigma_\wp}, w)) \otimes_E \overline{E})^{\mathcal{H}^p = \lambda, T(F_\wp) = \chi} \neq 0.$$

Note $(k_{\Sigma_\wp}, w)$ is a locally algebraic subrepresentation of $(\underline{k}_J, w)$ by (4.2). In fact, by the description of locally algebraic vectors of ([35, Thm. 5.3]), one sees z is classical (for $z \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$) if and only if

$$(J_B(\ _{\mathrm{lalg}}) \otimes_E \overline{E})^{\mathcal{H}^p = \lambda, T(F_\wp) = \chi} \neq 0,$$

where "lalg" denotes the locally algebraic vectors.

For a locally analytic character χ of $T(F_\wp)$ over E, put

$$C(\chi) := \{\sigma \in {}_\wp \mid \mathrm{wt}(\chi)_{1,\sigma} - \mathrm{wt}(\chi)_{2,\sigma} \in \mathbb{Z}_{\geqslant 0}\};$$

for $S \subseteq C(\chi)$, put

$$\chi_S^c := \chi \prod_{\sigma \in S} \left(\sigma^{\mathrm{wt}(\chi)_{2,\sigma} - \mathrm{wt}(\chi)_{1,\sigma} - 1} \otimes \sigma^{\mathrm{wt}(\chi)_{1,\sigma} - \mathrm{wt}(\chi)_{2,\sigma} + 1} \right).$$

Let

$$I(\chi) := \mathrm{soc} \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi \right)^{\mathbb{Q}_p - \mathrm{an}}.$$

Note $I(\chi)$ is locally algebraic if and only if χ is locally algebraic and dominant, and we refer to [41, §2.3] for more description of $I(\chi)$.

DEFINITION 4.7. — *Let $z = (\chi, \lambda)$ be a point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, for $S \subseteq C(\chi) \cap J$, we say z admits an S-companion point if $z_S^c := (\chi_S^c, \lambda)$ is also a point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$.*

Denote by $\delta_B = \mathrm{unr}(q^{-1}) \otimes \mathrm{unr}(q)$ the modulus character of $B(F_{\wp})$.

LEMMA 4.8.

(1) *Let $z = (\chi, \lambda)$ be a point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ with χ locally algebraic and dominant, suppose for any $\emptyset \neq S \subseteq {}_{\wp} \setminus J$, $I(\chi_S^c \delta_B^{-1})$ is not a subrepresentation of $(\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$, then the point z is classical. We call the points satisfying this assumption ${}_{\wp} \setminus J$-very classical.*

(2) *Let $z = (\chi, \lambda)$ be a point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ with χ locally algebraic and dominant, then z is ${}_{\wp} \setminus J$-very classical if and only if z does not have S-companion point for all $\emptyset \neq S \subseteq {}_{\wp} \setminus J$.*

(3) *Let z be a ${}_{\wp} \setminus J$-very classical point of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, then the natural injection*

(4.7) $\quad J_B((\underline{k}_J, w)_{\mathrm{lalg}})^{T(\mathcal{O}_\wp) = \chi}[\mathcal{H}^p = \lambda, T(F_\wp) = \chi]$

$$\hookrightarrow J_B((\underline{k}_J, w))^{T(\mathcal{O}_\wp) = \chi}[\mathcal{H}^p = \lambda, T(F_\wp) = \chi]$$

is an isomorphism (where $T(\mathcal{O}_\wp) = \mathcal{O}_\wp^\times \times \mathcal{O}_\wp^\times \hookrightarrow T(F_\wp)$).

Proof. — The lemma follows from the same arguments in [23, §6.2.2], while we give an alternative proof using Breuil's adjunction formula [14, Thm. 4.3].

Let us first prove (1). By [14, Thm. 4.3], a non-zero vector $v \in J_B((\underline{k}_J, w))^{T(F_\wp) = \chi, \mathcal{H}^p = \lambda}$ would induce a non-zero morphism of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$

(4.8) $\quad \mathcal{F}_{\overline{B}}^{\mathrm{GL}_2}((U(\mathfrak{g}_{\Sigma_\wp}) \otimes_{U(\overline{\mathfrak{b}}_\wp)} (- \mathrm{wt}(\chi)))^\vee, \psi_z \delta_B^{-1}) \longrightarrow (\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$

where "$\vee$" denotes the dual in the BGG category $\mathcal{O}^{\overline{\mathfrak{b}}}{}_\wp$, and $\psi_z := \chi \delta_{\mathrm{wt}(\chi)}^{-1}$ is a smooth character of $T(F_\wp)$. By the structure of the Verma module, [36, Thm. (iv)] and [41, §2.3], one sees that any non locally algebraic irreducible constituent of the left object in (4.8) has the form $I(\chi_S^c \delta_B^{-1})$ with $\emptyset \neq S \subseteq {}_{\wp}$. Note for any $v \in (\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$, the $U(\mathfrak{g}_J)$-submodule generated by v is finite dimensional (and we call such vectors $U(\mathfrak{g}_J)$-finite). However, if S is not contained in ${}_{\wp} \setminus J$, $I(\chi_S^c \delta_B^{-1})$ does not have non-zero $U(\mathfrak{g}_J)$-finite vectors, and thus $I(\chi_S^c \delta_B^{-1})$ can not be a subrepresentation of $(\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$. This, together with the assumption, implies that $I(\chi_S^c \delta_B^{-1})$ can not be a subrepresentation of $(\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$ for all $\emptyset \neq S \subseteq {}_{\wp}$. Thus the morphism (4.8) factors through the locally algebraic vectors, so $v \in J_B((\underline{k}_J, w)_{\mathrm{lalg}})^{T(F_\wp) = \chi, \mathcal{H}^p = \lambda}$, and z is classical.

Let us then prove (2). If z is not $_\wp \setminus J$-very classical, then there exists $\emptyset \neq S \subseteq {}_\wp \setminus J$, such that $I(\chi_S^c \delta_B^{-1}) \hookrightarrow (\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$. By applying the Jacquet–Emerton functor, one gets $\chi_S^c \hookrightarrow J_B((\underline{k}_J, w))^{\mathcal{H}^p = \lambda}$ (e.g. by [14, Thm. 4.3]), and thus $z_S^c \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$. Now suppose there exists $\emptyset \neq S \subseteq {}_\wp \setminus J$ such that $z_S^c \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, by Breuil's adjunction formula [14, Thm. 4.3] applied to a non-zero vector in $J_B((\underline{k}_J, w))^{T(F_\wp) = \chi_S^c, \mathcal{H}^p = \lambda}$, one gets a non-zero morphism

$$(4.9) \quad \mathcal{F}_{\overline{B}}^{\mathrm{GL}_2}((\mathrm{U}(\mathfrak{g}_{\Sigma_\wp}) \otimes_{\mathrm{U}(\overline{\mathfrak{b}}_\wp)} (-\mathrm{wt}(\chi_S^c)))^\vee, \psi_z \delta_B^{-1}) \longrightarrow (\underline{k}_J, w)^{\mathcal{H}^p = \lambda}.$$

Any irreducible constituent of the left object has the form $I(\chi_{S'}^c \delta_B^{-1})$ with $S \subseteq S' \subseteq {}_\wp$. On the other hand, by the same argument as in the proof of (1), one sees if S' is not contained in ${}_\wp \setminus J$, $I(\chi_{S'}^c \delta_B^{-1})$ can not be a subrepresentation of $(\underline{k}_J, w)$. So there exists $S' \subseteq {}_\wp \setminus J$ such that (4.9) induces an injection $I(\chi_{S'}^c \delta_B^{-1}) \hookrightarrow (\underline{k}_J, w)^{\mathcal{H}^p = \lambda}$, thus z is not ${}_\wp \setminus J$-very classical.

(3) follows by applying Breuil's adjunction formula [14, Thm. 4.3] to the $T(F_\wp)$-representation on the right side of (4.7), and by the same arguments as in the proof of (1). $\qquad \square$

Since $(\underline{k}_J, w)$ is contained in the unitary Banach $\mathrm{GL}_2(F_\wp)$-representation $\widetilde{H}^1(K^p, E)$, the following proposition follows easily from [12, Prop. 5.1]:

PROPOSITION 4.9. — *Let* $z = (\chi, \lambda)$ *be point in* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ *with* χ *locally algebraic and dominant, and suppose*

$$(4.10) \quad v_\wp(q\chi_1(\varpi)) < \inf_{\sigma \in \Sigma_\wp \setminus J} \{\mathrm{wt}(\chi)_{1,\sigma} - \mathrm{wt}(\chi)_{2,\sigma} + 1\}$$

then the point z *is* ${}_\wp \setminus J$*-very classical.*

A point $z = (\chi, \lambda)$ of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ is called *spherical* if χ is the product of an unramified character with an algebraic character (i.e. $\mathrm{wt}(\chi) \in \mathbb{Z}^{2|d|}$ and $\chi \delta_{\mathrm{wt}(\chi)}^{-1}$ is unramified). By the standard arguments as in [19, §6.4.5] (see also [19, Prop. 6.2.7]), one can deduce from Proposition 4.9 (and Proposition 4.6):

THEOREM 4.10.

(1) *The set of spherical points satisfying the assumption in Proposition 4.9 are Zariski dense in* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ *and accumulates over spherical points.*

(2) *The set of points satisfying the assumption in Proposition 4.9 accumulates over points with integer weights.*

By Chenevier's method [21, §4.4], one can prove

PROPOSITION 4.11. — *Let $z \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(E)$ be a $\wp \setminus J$-very classical point, then the weight map κ is étale at z; moreover, there exists an affinoid neighborhood U of z with $\kappa(U)$ affinoid open in $\mathcal{W}(\underline{k}_J)$ such that $\mathcal{O}(U) \cong \mathcal{O}(\kappa(U))$.*

Proof. — Indeed, by Proposition 4.6, Theorem 4.10, one can reduce to a similar situation as in the beginning of the proof of [21, Thm. 4.8]. Since z is $\wp \setminus J$-very classical, one has the bijection (4.7) (which is an analogue of [21, (4.20)], see also [24, Lem. 4]). The proposition then follows from the multiplicity one result for automorphic representations of $G(\mathbb{A})$, by the same argument as in the proof of [21, Thm. 4.8] (see also [24, §4.4.3] especially the arguments after [24, Lem. 4]).   □

4.2.4. Families of Galois representations

By Carayol's results [18], the theory of pseudo-characters and the density of classical points, we have

THEOREM 4.12. — *For a point $z = (\chi, \lambda)$ of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, there exists a unique continuous irreducible representation $\rho_z : \mathrm{Gal}_F \to \mathrm{GL}_2(k(z))$ which is unramified at places $\mathfrak{l} \notin S(K^p)$ satisfying $\rho_z(\mathrm{Frob}_{\mathfrak{l}}^{-2}) - \lambda(T_{\mathfrak{l}})\rho_z(\mathrm{Frob}_{\mathfrak{l}}^{-1}) + \lambda(S_{\mathfrak{l}}) = 0$, where $k(z)$ denotes the residue field at z.*

By the fact that $\rho_z|_{\mathrm{Gal}_{F_\wp}}$ is de Rham for classical points $z \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ (and of Hodge–Tate weights $(\frac{w - k_\sigma + 2}{2}, \frac{w + k_\sigma}{2})$ for $\sigma \in J$), Shah's results [42] and the density of classical points, one has

THEOREM 4.13. — *Let $z \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(\overline{E})$, the restriction $\rho_z|_{\mathrm{Gal}_{F_\wp}}$ is J-de Rham of Hodge–Tate weights $(\frac{w - k_\sigma + 2}{2}, \frac{w + k_\sigma}{2})$ for $\sigma \in J$.*

Proof. — The theorem follows from the same arguments of the proof of [24, Thm. 6] (see also [23, Prop. 6.2.40]) by replacing the global triangulation results by Shah's interpolation result [42, Thm. 2.19] in the last step (note by [2, Lem. 7.2.11], $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ is nested since it's finite over $\widehat{T}$).   □

PROPOSITION 4.14. — *For $z \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(\overline{E})$, there exists an open affinoid U of $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}, \mathrm{red}}$ and a continuous representation $\rho_U : \mathrm{Gal}_F \to \mathrm{GL}_2(\mathcal{O}(U))$ such that the specialization of ρ_U at any point $z' \in U(\overline{E})$ equals $\rho_{z'}$. Moreover, for $\sigma \in J$, $D_{\mathrm{dR}}(\rho_U)_\sigma := (B_{\mathrm{dR}, \sigma} \widehat{\otimes}_E \rho_U)^{\mathrm{Gal}_{F_\wp}}$ is a locally free $\mathcal{O}(U)$-module of rank 2.*

Proof. — The first part follows from [3, Lem. 5.5]; the second is hence from Theorem 4.13 and [42, Thm. 2.19] applied to ρ_U. □

By [38], $\rho_{z,\wp} := \rho_z|_{\mathrm{Gal}_{F_\wp}}$ is semi-stable (thus trianguline) for any spherical classical point z of $\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$. As in [24, Thm. 6], by global triangulation theory [31] [32] applied to $\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$ (note $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ is a closed rigid subspace of $\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$), we get

THEOREM 4.15. — *For any point* $z = (\chi = \chi_1 \otimes \chi_2, \lambda)$ *of* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, $\rho_{z,\wp}$ *is trianguline with a triangulation given by*

$$0 \to \mathcal{R}_{k(z)}(\delta_1) \to D_{\mathrm{rig}}(\rho_{z,\wp}) \to \mathcal{R}_{k(z)}(\delta_2) \to 0$$

with

$$\begin{cases} \delta_1 = \mathrm{unr}(q)\chi_1 \prod_{\sigma \in \Sigma_z} \sigma^{\mathrm{wt}(\chi)_{2,\sigma} - \mathrm{wt}(\chi)_{1,\sigma} - 1}, \\ \delta_{2,z} = \chi_2 \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in \Sigma_z} \sigma^{\mathrm{wt}(\chi)_{1,\sigma} - \mathrm{wt}(\chi)_{2,\sigma} + 1}, \end{cases}$$

where $\Sigma_z \subseteq C(\chi)$, $\mathcal{R}_{k(z)}$ *denotes the Robba ring* $B_{\mathrm{rig},F_\wp}^\dagger \otimes_{\mathbb{Q}_p} k(z)$, *and* $D_{\mathrm{rig}}(\rho_{z,\wp}) := (B_{\mathrm{rig}}^\dagger \otimes_{\mathbb{Q}_p} \rho_{z,\wp})^{\mathrm{Gal}_{F_\wp}}$ *is the* (φ, Γ)-*module (of rank 2) over* $\mathcal{R}_{k(z)}$ *associated to* $\rho_{z,\wp}$ *(we refer to [5] for* $B_{\mathrm{rig},F_\wp}^\dagger$, $B_{\mathrm{rig}}^\dagger$ *and* (φ, Γ)-*modules).*

COROLLARY 4.16. — *Let* $z = (\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(E)$ *and suppose*

(4.11) $$\mathrm{unr}(q)\chi_1\chi_2^{-1} \neq \prod_{\sigma \in \Sigma_\wp} \sigma^{n_\sigma} \text{ for all } (n_\sigma)_{\sigma \in \Sigma_\wp} \in \mathbb{Z}^d,$$

for $S \subseteq \Sigma_\wp \setminus J$, *if* z *admits an* S-*companion point then* $S \subseteq \Sigma_z$.

Proof. — Applying Proposition 4.15 to the point z_S^c, the corollary then follows from [33, Thm. 3.7]. □

One can moreover deduce from the proof of [31, Thm. 6.3.9] (see also [23, Prop. 6.2.49]):

PROPOSITION 4.17. — *Let* z *be a classical point of* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, U *be an affinoid neighborhood of* z, *suppose any point of* U *satisfies* (4.11), *then for any* $\sigma \in \Sigma_\wp$, $Z_{U,\sigma} := \{z' \in U(\overline{E}) \mid \sigma \in \Sigma_{z'}\}$ *is a Zariski-closed subset of* $U(\overline{E})$.

DEFINITION 4.18. — *Let* $z = (\chi, \lambda)$ *be a point of* $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$, *for* $S \subseteq \Sigma_\wp$, *we say* z *is non-*S-*critical if* (4.11) *is satisfied and* $\Sigma_z \cap S = \emptyset$.

COROLLARY 4.19. — *Let* $z = (\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(\overline{E})$ *with* χ *locally algebraic and* $C(\chi) = \Sigma_\wp$, *if* z *is non-*$\Sigma_\wp \setminus J$-*critical, then* z *is* $\Sigma_\wp \setminus J$-*very classical.*

Proof. — By Lemma 4.8(2), it's sufficient to show z does not have S-companion point for $\emptyset \neq S \subseteq \wp \setminus J$. But this follows from Corollary 4.16.   □

THEOREM 4.20. — *Let* $z = (\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(E)$ *be a non-$\wp \setminus J$-critical classical point, then the weight map κ is étale at z. Moreover, there exists an affinoid neighborhood U of z such that $W = \kappa(U)$ is an affinoid open in $\mathcal{W}(\underline{k}_J)$ and $\mathcal{O}(U) \cong \mathcal{O}(W)$.*

Proof. — The theorem follows from Proposition 4.11 combined with Corollary 4.19.   □

The following proposition, which follows from the same argument as in [24, Cor. 5], would be useful to apply the adjunction formula in families.

PROPOSITION 4.21. — *Let* $z = (\chi, \lambda) \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ *be a non-$\wp \setminus J$-critical classical point, and suppose* $\mathrm{unr}(q^e)\psi_{\chi,1}(p)\psi_{\chi,2}^{-1}(p) \neq 1$ *where* $\psi_\chi := \chi\delta_{\mathrm{wt}(\chi)}^{-1}$, *then there exists an admissible open U of z in $\mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}$ such that any point of U is non-$\wp \setminus J$-critical.*

4.3. Local-global compatibility

Let $\rho : \mathrm{Gal}_F \to \mathrm{GL}_2(E)$ be a continuous representation such that

(1) $\rho_\wp := \rho|_{\mathrm{Gal}_{F_\wp}}$ is semi-stable non-crystalline of Hodge–Tate weights $\underline{h}_{\Sigma_\wp} = (\frac{w-k_\sigma+2}{2}, \frac{w+k_\sigma}{2})_{\sigma\in\Sigma_\wp}$ for $k_\sigma \in 2\mathbb{Z}_{\geqslant 1}$ and $w \in 2\mathbb{Z}$ with $\{\alpha, q\alpha\}$ the eigenvalues of φ^{do} on $D_{\mathrm{st}}(\rho_\wp)$, $S_c := S_c(\rho_\wp)$ (cf. the discussion before Corollary 2.3) the set of embeddings where $\rho_\wp$ is critical, $S_n := S_n(\rho_\wp) = \wp \setminus S_c$ and $\underline{\mathcal{L}}_{S_n} \in E^{|S_n|}$ the associated Fontaine–Mazur $\mathcal{L}$-invariants;

(2) $\mathrm{Hom}_{\mathrm{Gal}_F}(\rho, H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_p}, w))) \neq 0$ (in particular, ρ is associated to certain Hilbert eigenforms);

(3) ρ is absolutely irreducible modulo ϖ_E.

Note that, by the condition (2), ρ is unramified for places in $S(K^p)$. And by the Eichler–Shimura relations, one can associate to ρ a system of eigenvalues $\lambda_\rho : \mathcal{H}^p \to E$. Put $\widehat{\pi}(\rho) := \mathrm{Hom}_{\mathrm{Gal}_F}(\rho, \widetilde{H}^1_{\text{ét}}(K^p, E))$, which is an admissible unitary Banach representation of $\mathrm{GL}_2(F_\wp)$. One has

$$\widehat{\pi}(\rho) = \mathrm{Hom}_{\mathrm{Gal}_F}(\rho, \ ) = \mathrm{Hom}_{\mathrm{Gal}_F}(\rho, \ ^{\mathcal{H}^p=\lambda_\rho}).$$

The injection $H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_\wp}, w))_{\overline{\rho}} \hookrightarrow \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_p}, w))_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}$ induces an injection

$$H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_\wp}, w))_{\overline{\rho}} \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee \longrightarrow \widetilde{H}^1(K^p, E)_{\mathbb{Q}_p-\mathrm{an}},$$

thus the condition (2) implies in particular $\widehat{\pi}(\rho) \neq 0$.

By local-global compatibility in classical local Langlands correspondence (for $\ell = p$, cf. [38]) and the isomorphism in Proposition 4.2 (1), there exists an isomorphism of locally algebraic representations of $\mathrm{GL}_2(F_\wp)$:

$$(4.12) \qquad \mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp})^{\oplus r} \xrightarrow{\sim} \widehat{\pi}(\rho)_{\mathrm{lalg}},$$

with some $r \in \mathbb{Z}_{\geqslant 1}$ (note that $\mathrm{alg}(\underline{h}_{\Sigma_\wp}) \cong W(\underline{k}_{\Sigma_\wp}, w)^\vee$ and thus we have $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}) \cong \mathrm{St} \otimes_E \mathrm{unr}(\alpha) \circ \det \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee)$. The main result of this section is (see Section 3 for notations)

THEOREM 4.22.

(1) *Let* $\tau \in {}_\wp$, *then* $\tau \in S_c$ *if and only* $I_\tau^c(\alpha, \underline{h}_{\Sigma_\wp})$ *is a subrepresentation of* $\widehat{\pi}(\rho)$.

(2) *The natural restriction map*

$$(4.13) \quad \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}(\ (\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n}), \widehat{\pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}})$$
$$\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}(\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}), \widehat{\pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}})$$

is bijective. In particular, $(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n})$ *is a subrepresentation of* $\widehat{\pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$.

By the same argument as in the proof of [24, Cor. 6], we have

COROLLARY 4.23. — *Let* $\underline{\mathcal{L}}'_{S_n} \in E^d$, *then* $(\alpha, \underline{h}_{\Sigma_\wp}, \underline{\mathcal{L}}'_{S_n})$ *is a subrepresentation of* $\widehat{\pi}(\rho)$ *if and only if* $\underline{\mathcal{L}}'_{S_n} = \underline{\mathcal{L}}_{S_n}$.

Combining Corollary 4.23 and Theorem 4.22 (1), we see

COROLLARY 4.24. — *The local Galois representation* $\rho_\wp$ *can be determined by* $\widehat{\pi}(\rho)$.

Proof of Theorem 4.22. — First note that we only need (and do) prove the same result with $\widehat{\pi}(\rho)$ replaced by ${}^{\mathcal{H}^p=\lambda_\rho}$. For $S \subseteq {}_\wp, \sigma \in {}_\wp$, put $S^\sigma := S \backslash \{\sigma\}$. Note the injection $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}) \hookrightarrow (\underline{k}_\emptyset, w)^{\mathcal{H}^p=\lambda}$ gives a spherical classical point $z_\rho = (\chi_\rho, \lambda_\rho) \in \mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}(E)$ where $\chi_\rho := \chi(\alpha, \underline{h}_{\Sigma_\wp})\delta_B$; moreover, for any $S \subseteq {}_\wp, z_\rho \in \mathcal{E}(\underline{k}_J, w)_{\overline{\rho}}(E)$.

Let us first prove (1). Let $\tau \in {}_\wp$, and consider $(\underline{k}_{\Sigma_\wp^\tau}, w)$ and $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$. If $I_\tau^c(\alpha, \underline{h}_{\Sigma_\wp}) \hookrightarrow (\underline{k}_{\Sigma_\wp^\tau}, w)^{\mathcal{H}^p=\lambda_\rho}$ (which is equivalent to $I_\tau^c(\alpha, \underline{h}_{\Sigma_\wp}) \hookrightarrow {}^{\mathcal{H}^p=\lambda_\rho}$, since any latter morphism factors through $(\underline{k}_{\Sigma_\wp^\tau}, w)$ by Lemma 4.4) then $(z_\rho)_\tau^c = ((\chi_\rho)_\tau^c, \lambda_\rho) \in \mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$ (see the proof of Lemma 4.8 (2)). However, if $(z_\rho)_\tau^c \in \mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$, by Corollary 4.16, $\tau \in {}_{z_\rho} = S_c$, the "if" part follows.

Now suppose $\tau \in S_c = {}_{z_\rho}$, we first use Bergdall's method [4] to show the weight map $\kappa : \mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w) \to \mathcal{W}_1(\underline{k}_{\Sigma_\wp^\tau})_{\overline{\rho}}$ is not étale at z_ρ:

We only need to consider the case where $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$ is reduced at z_ρ since otherwise, κ is not étale at z_ρ (in fact, by the same argument as in [20, §3.8], one can probably prove that $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$ is reduced at z_ρ). Take U to be an irreducible a noid neighborhood of z_ρ in $\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$ small enough such that Proposition 4.14 holds. The composition $\mathcal{O}(U_{\mathrm{red}}) \to \mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}} \to \widehat{T}$ gives a continuous character $\widetilde{\delta} : T(L) \to \mathcal{O}(U_{\mathrm{red}})^\times$ (with $\widetilde{\delta}|_{Z_1} = \mathcal{N}^{-w}$). By [32, Prop. 4.3.5], there exists (shrinking U if necessary) an injection of $(\varphi,)$-modules over $\mathcal{R}_{\mathcal{O}(U_{\mathrm{red}})} := \mathcal{R}_{F_\wp} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{O}(U_{\mathrm{red}})$:

(4.14) $$\mathcal{R}_{\mathcal{O}(U_{\mathrm{red}})}(\widetilde{\delta}_1 \,\mathrm{unr}(q)) \hookrightarrow D_{\mathrm{rig}}(\rho_{U_{\mathrm{red}}}) \,;$$

moreover, the specialisation of the above morphism to any point in U is still injective. Let $t : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \to U_{\mathrm{red}}$ be an element in the tangent space of U_{red} at z_ρ, one deduces from (4.14) an injection of $(\varphi,)$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ (where the injectivity follows from the fact that (4.14) specializing to z_ρ is still injective)

(4.15) $$\mathcal{R}_{E[\epsilon]/\epsilon^2}(t^*\widetilde{\delta}_1 \,\mathrm{unr}(q)) \hookrightarrow D_{\mathrm{rig}}(t^*\rho_{U_{\mathrm{red}}}).$$

Note $t^*\widetilde{\delta} \equiv \chi_\rho \pmod{\epsilon}$ and $D_{\mathrm{rig}}(t^*\rho_{U_{\mathrm{red}}})$ is an extension of $D_{\mathrm{rig}}(\rho)$ by $D_{\mathrm{rig}}(\rho)$. Since $\widetilde{\delta}|_{Z_1} = \mathcal{N}^{-w}$, $\mathrm{wt}(t^*\widetilde{\delta}) = (\frac{k_\sigma - w - 2}{2} - a_\sigma \epsilon, \frac{2 - k_\sigma - w}{2} + a_\sigma \epsilon)_{\sigma \in \Sigma_\wp}$ for $(a_\sigma)_{\sigma \in \Sigma_\wp} \in E^d$ and thus the Sen weights of $D_{\mathrm{rig}}(t^*\rho_{U_{\mathrm{red}}})$ are given by $(\frac{-k_\sigma - w}{2} + a_\sigma \epsilon, \frac{k_\sigma - w - 2}{2} - a_\sigma \epsilon)_{\sigma \in \Sigma_\wp}$. The map (4.15) induces an injection

$$\mathcal{R}_{E[\epsilon]/\epsilon^2} \hookrightarrow D_{\mathrm{rig}}(t^*\rho_{U_{\mathrm{red}}}) \otimes_{E[\epsilon]/\epsilon^2} \mathcal{R}_{E[\epsilon]/\epsilon^2}((t^*\widetilde{\delta}_1 \,\mathrm{unr}(q))^{-1}) =: D,$$

where D is an extension of $D_{\mathrm{rig}}(\rho) \otimes_{\mathcal{R}_E} \mathcal{R}_E(\chi_{\rho,1}^{-1} \,\mathrm{unr}(q^{-1}))$ by itself and has Sen weights $((1 - k_\sigma) + 2a_\sigma \epsilon, 0)_{\sigma \in \Sigma_\wp}$. By the same argument of [14, Lem. 9.6] (replacing the functor $D_{\mathrm{cris}}(\cdot)$ by $D_{\mathrm{st}}(\cdot)$), one can show $1 - k_\tau$ is a constant Sen weight of D, and hence $a_\tau = 0$. Consequently, we see the composition $T_{U_{\mathrm{red}}, z_\rho} \to T_{\mathcal{W}_1, \kappa(z_\rho)} \to T_{\mathcal{W}_1(\underline{k}_\wp^\tau), \kappa(z_\rho)}$ is zero, where $T_{X,x}$ denotes the tangent space of X at x for a point x in a rigid analytic space X, and the first map denotes the tangent map induced by κ. Thus the map $T_{\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}, \mathrm{red}}, z_\rho} \to T_{\mathcal{W}_1(\underline{k}_\wp^\tau), \kappa(z_\rho)}$ is zero; however, since we assume $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$ to be reduced at z_ρ, we see the induced tangent map $T_{\mathcal{E}(\underline{k}_\wp^\tau, w)_{\overline{\rho}}, z_\rho} \to T_{\mathcal{W}_1(\underline{k}_\wp^\tau), \kappa(z_\rho)}$ factors though the above zero map and thus also equals zero, from which we see $\kappa : \mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}} \to \mathcal{W}_1(\underline{k}_{\Sigma_\wp^\tau})$ is not étale at z_ρ.

By Prop. 4.11, z_ρ is not $\tau_\wp$-very classical, and hence by definition, $I_\tau^c(\alpha, \underline{h}_{\Sigma_\wp}) \cong I((\chi_\rho)_\tau^c \delta_B^{-1})$ is a subrepresentation of $(\underline{k}_{\Sigma_\tau}, w)^{\mathcal{H}^p = \lambda_\rho}$, which concludes the proof of Theorem 4.22 (1).

To prove (2), we use the same arguments as in [24, §5.3]. The injectivity of (4.13) (with $\widehat{\pi}(\rho)$ replaced by $\cdot^{\mathcal{H}^p = \lambda_\rho}$) follows from the fact that z_ρ (as a classical point of $\mathcal{E}(\underline{k}_{S_c}, w)_{\overline{\rho}}$) does not have S-companion point for $\emptyset \neq S \subseteq S_n$. Indeed, if (4.13) is not injective, by results on the Jordan–Holder factors of $(\alpha, \underline{h}_{\Sigma_\wp})$ (e.g. see [12, Thm. 4.1]), we see either $F(\alpha, \underline{h}_{\Sigma_\wp})$ is a subrepresentation of $\cdot^{\mathcal{H}^p = \lambda}$, or there exists $\emptyset \neq S \subseteq S_n$ such that $I((\chi_\rho)_S^c \delta_B^{-1})$ is a subrepresentation of $\cdot^{\mathcal{H}^p = \lambda}$ which are both impossible since the locally algebraic representation $F(\alpha, \underline{h}_{\Sigma_\wp})$ can not be injected into $\cdot^{\mathcal{H}^p = \lambda}$ by (4.12), and z_ρ is non-S_n-critical hence S_n-very classical by Corollary 4.19.

Let $\underline{h}'_{S_n} := (\frac{2-k_\sigma}{2}, \frac{k_\sigma}{2})_{\sigma \in S_n} = \underline{h}_{S_n} - (\frac{w}{2}, \frac{w}{2})_{\sigma \in S_n}$, thus by definition we have $\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}) \cong \mathrm{St}(\alpha, \underline{h}'_{S_n}) \otimes_E W(\underline{k}_{S_c}, w)^\vee$,

$$(\alpha, \underline{h}_{\Sigma_\wp}) \cong (\alpha, \underline{h}'_{S_n}) \otimes_E W(\underline{k}_{S_c}, w)^\vee,$$

$\chi(\alpha, \underline{h}_{\Sigma_\wp}) = \chi(\alpha, \underline{h}'_{S_n}) \chi(\underline{k}_{S_c}, w)$, and $(\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n}) \cong (\alpha, \underline{h}'_{S_n}, \mathcal{L}_{S_n}) \otimes_E W(\underline{k}_{S_c}, w)^\vee$ (see Remark 3.1 (2)). By Lemma 4.4, to prove

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\big((\alpha, \underline{h}_{\Sigma_\wp}, \mathcal{L}_{S_n}), \cdot^{\mathcal{H}^p = \lambda_\rho}\big)$$
$$\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}(\mathrm{St}(\alpha, \underline{h}_{\Sigma_\wp}), \cdot^{\mathcal{H}^p = \lambda_\rho})$$

is surjective, it's sufficient to prove the restriction map

$$(4.16) \quad \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big((\alpha, \underline{h}'_{S_n}, \mathcal{L}_{S_n}), \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n - \mathrm{an}}^{\mathcal{H}^p = \lambda_\rho}\Big)$$
$$\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\mathrm{St}(\alpha, \underline{h}'_{S_n}), \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n - \mathrm{an}}^{\mathcal{H}^p = \lambda_\rho}\Big)$$

is surjective.

It's convenient to work with a "twist" of the eigenvariety $\mathcal{E}(\underline{k}_{S_c}, w)_{\overline{\rho}}$: as in Section 4.2.2, one can construct an eigenvariety $\mathcal{E}$ together with a coherent sheaf $\mathcal{M}$ from the essentially admissible representation of $T(F_\wp)$

$$(4.17) \qquad J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{\overline{\rho}, S_n - \mathrm{an}}^{Z_1}\Big),$$

such that

$$(\mathcal{E}, \mathcal{M}) \cong J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{\overline{\rho}, S_n - \mathrm{an}}^{Z_1}\Big)^\vee;$$

the natural morphism $\mathcal{E} \to \widehat{T}$ would factor through $\widehat{T}_{S_n}$ (since (4.17) is locally S_n-analytic); moreover, by the isomorphism (4.6), one has a commutative diagram

(4.18)

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{(\chi,\lambda)\mapsto(\chi\cdot\chi(\underline{k}_{S_c},w),\lambda)} & \mathcal{E}(\underline{k}_{S_c},w)_{\overline{\rho}} \\
\downarrow & & \downarrow \\
\widehat{T}_{S_n} & \xrightarrow{\chi\mapsto\chi\cdot\chi(\underline{k}_{S_c},w)} & \widehat{T}(\underline{k}_{S_c},w) \\
\downarrow & & \downarrow \\
\mathcal{W}_{1,S_n} & \xrightarrow{\chi\mapsto\chi\prod_{\sigma\in S_c}\sigma^{k_\sigma-2}} & \mathcal{W}_1(\underline{k}_J)
\end{array}
$$

where the upper and outer square are Cartesian. Moreover $\mathcal{M}$ equals the pull-back of $\mathcal{M}(\underline{k}_{S_c},w)$ via the top horizontal map. Denote by $z'_\rho = (\chi'_\rho, \lambda_\rho)$ the preimage of z_ρ in $\mathcal{E}$, where $\chi'_\rho = \chi(\alpha, \underline{h}'_{S_n})\delta_B$. There exists an admissible strictly quasi-Stein (cf. [25, Def. 2.1.17 (iv)]) open U of z_ρ in $\mathcal{E}(\underline{k}_{S_c},w)_{\overline{\rho}}$ satisfying

- any point of U is non-S_n-critical (Proposition 4.21),
- $(U, \mathcal{M}(\underline{k}_{S_c},w))$ is a torsion free $\mathcal{O}(\mathcal{W}_1(\underline{k}_{S_c}))$-module (Proposition 4.6).

Take $\mathcal{U}$ to be the preimage of U in $\mathcal{E}$, which satisfies hence

(1) for $z = (\chi, \lambda) \in \mathcal{U}$, $S \subseteq C(\chi) \cap S_n$, $z_S^c := (\chi_S^c, \lambda)$ does not lie in $\mathcal{E}$,
(2) $\mathcal{U}$ is strictly quasi-Stein,
(3) $(\mathcal{U}, \mathcal{M})$ is a torsion free $\mathcal{O}(\mathcal{W}_{1,S_n})$-module.

The natural restriction map (which has dense image) $(\mathcal{E}, \mathcal{M}) \to (\mathcal{U}, \mathcal{M})$ induces (by taking the dual with the strong topology)

$$(4.19) \qquad (\mathcal{U}, \mathcal{M})^\vee \lhook\joinrel\longrightarrow (\mathcal{E}, \mathcal{M})^\vee \cong J_B\left(\widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c},w))^{Z_1}_{\overline{\rho},S_n-\text{an}}\right),$$

and we have:

PROPOSITION 4.25. — *The map* (4.19) *induces a* $\mathrm{GL}_2(F_\wp)\times\mathcal{H}^p$-*invariant morphism*

$$(4.20) \qquad \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} (\mathcal{U}, \mathcal{M})^\vee \otimes_E \delta_B^{-1}\right)^{S_n-\text{an}}$$

$$\longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c},w))^{Z_1}_{\overline{\rho},S_n-\text{an}}.$$

Proof. — The proposition follows from the same argument for [24, Cor. 7] (by replacing "$\mathbb{Q}_p - \text{an}$" by "$S_n - \text{an}$"). Indeed, by assumption and [24, Lem. 14], $(\mathcal{U}, \mathcal{M})^\vee$ is an allowable locally S_n-analytic representation of

$T(F_\wp)$ equipped with a continuous action of $\mathcal{H}^p$ which commutes with $T(F_\wp)$. By the property (1) of $\mathcal{U}$, Lemma B.4 and Remark B.5 below, one can prove as in [24, Lem. 16] that (4.19) is balanced. Since $(\mathcal{U}, \mathcal{M})^\vee$ is $\mathcal{O}(\mathcal{W}_{1,S_n})$-torsion free, the proposition follows by Lemma B.6 below. $\square$

Return to the proof of Theorem 4.22 (2). Let $\tau \in S_n$, and $\mathcal{W}_{1,S_n}(\underline{k}_{S_n^\tau})$ denote the closed rigid subspace of $\mathcal{W}_{1,S_n}$ parametrizing characters moreover with fixed weights $k_\sigma - 2$ for $\sigma \in S_n^\tau$. Put $\mathcal{E}_\tau := \mathcal{E} \times_{\mathcal{W}_{1,S_n}} \mathcal{W}_{1,S_n}(\underline{k}_{S_n^\tau})$. Note $z'_\rho \in \mathcal{E}_\tau$ for all $\tau \in S_n$. Moreover, since $\mathcal{E}$ is étale over $\mathcal{W}_{1,S_n}$ at z'_ρ, $\mathcal{E}_\tau$ is étale over $\mathcal{W}_{1,S_n}(\underline{k}_{S_n^\tau})$ at z'_ρ. Let $t_\tau : \operatorname{Spec} E[\epsilon]/\epsilon^2 \to \mathcal{E}_\tau$ be a non-zero element in the tangent space of $\mathcal{E}_\tau$ at z'_ρ, the composition $t_\tau : \operatorname{Spec} E[\epsilon]/\epsilon^2 \to \mathcal{E}_\tau \to \widehat{T}$ thus gives a locally S_n-analytic character $\widetilde{\chi}'_{\rho,\sigma} : T(F_\wp) \to (E[\epsilon]/\epsilon^2)^\times$ satisfying that $\widetilde{\chi}'_{\rho,\tau} \equiv \chi'_\rho \pmod{\epsilon}$, $\widetilde{\chi}'_{\rho,\tau}|_{Z_1} = 1$, and $\widetilde{\chi}'_{\rho,\tau}(\chi'_\rho)^{-1}$ is locally τ-analytic.

Consider $(t_\tau^* \mathcal{M})^\vee$, which is a subrepresentation of $(\mathcal{U}, \mathcal{M})^\vee$ (since the restriction map $(\mathcal{U}, \mathcal{M}) \to t_\tau^* \mathcal{M}$ is surjective) of $T(F_\wp)$ equipped with a continuous action of $\mathcal{H}^p$. By the second part of Theorem 4.20 (note that we have a similar result for $(\mathcal{E}, \mathcal{W}_{1,S_n})$ thus for $(\mathcal{E}_\tau, \mathcal{W}_{1,S_n}(\underline{k}_{S_n^\tau}))$), we have

(1) there exists r such that $(t_\tau^* \mathcal{M})^\vee \cong (\widetilde{\chi}'_{\rho,\tau})^{\oplus r}$ as $T(F_\wp)$-representations,
(2) $(t_\tau^* \mathcal{M})^\vee$ is a generalized λ_ρ-eigenspace.

The map (4.20) thus induces

$$\left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} (t_\tau^* \mathcal{M})^\vee \otimes_E \delta_B^{-1} \right)^{S_n - \operatorname{an}}$$

$$\hookrightarrow \left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} (\mathcal{U}, \mathcal{M})^\vee \otimes_E \delta_B^{-1} \right)^{S_n - \operatorname{an}}$$

$$\longrightarrow \widetilde{H}_{\text{ét}}^1 \left(K^p, W(\underline{k}_{S_c}, w) \right)_{\overline{\rho}, S_n - \operatorname{an}}^{Z_1}.$$

In particular, each vector not killed by ϵ in $(t_\tau^* \mathcal{M})^\vee$ induces a morphism

$$\left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} \widetilde{\chi}'_{\rho,\tau} \delta_B^{-1} \right)^{S_n - \operatorname{an}} \longrightarrow \widetilde{H}_{\text{ét}}^1 (K^p, W(\underline{k}_{S_c}, w))_{\overline{\rho}, S_n - \operatorname{an}}^{Z_1} [\mathcal{H}^p = \lambda_\rho].$$

Since $\widetilde{\chi}'_{\rho,\tau}$ is an extension of $\chi'_\rho = \chi(\rho, \underline{h}'_{S_n}) \delta_B$ by itself, one has an exact sequence

$$0 \longrightarrow \left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} \chi(\rho, \underline{h}'_{S_n}) \right)^{S_n - \operatorname{an}} \longrightarrow \left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} \widetilde{\chi}'_{\rho,\tau} \delta_B^{-1} \right)^{S_n - \operatorname{an}}$$

$$\xrightarrow{s} \left(\operatorname{Ind}_{\overline{B}(F_\wp)}^{\operatorname{GL}_2(F_\wp)} \chi(\rho, \underline{h}'_{S_n}) \right)^{S_n - \operatorname{an}} \longrightarrow 0.$$

Let $_\tau := s^{-1}(F(\alpha, \underline{h}'_{S_n}))/F(\alpha, \underline{h}'_{S_n})$. By the same argument as in [24, §5.3] (see in particular the arguments after [24, Lem. 12]), we can prove

the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\ _\tau, \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n-\mathrm{an}}^{\mathcal{H}^p=\lambda_\rho}\right)$$

$$\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\mathrm{St}(\alpha, \underline{h}'_{S_n}), \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n-\mathrm{an}}^{\mathcal{H}^p=\lambda_\rho}\right)$$

is surjective. However, by Proposition 4.26 below, one has $\ _\tau \cong (\alpha, \underline{h}'_{S_n}, \mathcal{L}_\tau)$. Thus for any $\tau \in S_n$, the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left((\alpha, \underline{h}'_{S_n}, \mathcal{L}_\tau), \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n-\mathrm{an}}^{\mathcal{H}^p=\lambda_\rho}\right)$$

$$\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\mathrm{St}(\alpha, \underline{h}'_{S_n}), \widetilde{H}^1_{\text{ét}}(K^p, W(\underline{k}_{S_c}, w))_{S_n-\mathrm{an}}^{\mathcal{H}^p=\lambda_\rho}\right)$$

is surjective. From which, together with Remark 3.1 (4), we see (4.16) is surjective. This concludes the proof of Theorem 4.22 (2) (assuming Proposition 4.26). $\hfill\square$

PROPOSITION 4.26. — *Keep the notation as in the proof of Theorem 4.22, there exists a locally τ-analytic character ψ_τ such that*

$$\widetilde{\chi}'_{\rho,\tau}(\chi'_\rho)^{-1} \cong \begin{pmatrix} 1 & \log_{\tau,-\mathcal{L}_\tau}(ad^{-1}) + \psi_\tau(ad) \\ 0 & 1 \end{pmatrix}$$

as (2-dimensional) representations of $T(F_\wp)$. Consequently (by Remark 3.1 (3)), $\ _\tau \cong (\alpha, \underline{h}'_{S_n}, \mathcal{L}_\tau)$.

The rest of the paper is devoted to the proof of Proposition 4.26. Note that the image $\mathcal{E}'_\tau$ of $\mathcal{E}_\tau$ in $\mathcal{E}(\underline{k}_{S_c}, w)_{\overline{\rho}}$ is an one-dimensional rigid space containing $\mathcal{E}(\underline{k}_{\Sigma^\tau_\wp}, w)_{\overline{\rho}}$ as closed subspace. Since both $\mathcal{E}'_\tau$ and $\mathcal{E}(\underline{k}_{\Sigma^\tau_\wp}, w)_{\overline{\rho}}$ are étale over $\mathcal{W}_1(\underline{k}_{\Sigma^\tau_\wp})$ at z_ρ, and have the same residue field E, we see they are locally isomorphic at z_ρ. In particular, the composition $\mathrm{Spec}\, E[\epsilon]/\epsilon^2 \xrightarrow{t_\tau} \mathcal{E}_\tau \to \mathcal{E}(\underline{k}_{S_c}, w)_{\overline{\rho}}$ gives a non-zero element in the tangent space of $\mathcal{E}(\underline{k}_{\Sigma^\tau_\wp}, w)_{\overline{\rho}}$ at z_ρ, still denoted by $t_\tau : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \to \mathcal{E}(\underline{k}_{\Sigma^\tau_\wp}, w)_{\overline{\rho}}$, moreover it's straightforward to see (e.g. by (4.18)) the character of $T(F_\wp)$ induced by this map is given by $\widetilde{\chi}_{\rho,\tau} := \widetilde{\chi}'_{\rho,\tau}\chi(\underline{k}_{S_c}, w)$. Note $\widetilde{\chi}_{\rho,\tau}\chi_\rho^{-1} = \widetilde{\chi}'_{\rho,\tau}(\chi'_\rho)^{-1}$. Since $\widetilde{\chi}_{\rho,\tau}\chi_\rho^{-1}$ is locally τ-analytic, there exist $\gamma, \eta \in E$, $\mu \in E^\times$ such that (cf. Section 1.3.1)

$$\widetilde{\chi}\chi_\rho^{-1} = (1 + \gamma\epsilon\psi_{\mathrm{ur}} + \mu\epsilon\psi_{\tau,p}) \otimes (1 + \eta\epsilon\psi_{\mathrm{ur}} - \mu\epsilon\psi_{\tau,p}).$$

It's sufficient to prove

$$(4.21) \qquad\qquad\qquad \gamma - \eta = -2\mathcal{L}_\tau\mu.$$

Indeed, if (4.21) holds, we get

$$\widetilde{\chi}_{\rho,\tau}\chi_\rho^{-1} \cong \left(1 + \mu\epsilon(-\mathcal{L}_\tau\psi_{\mathrm{ur}} + \psi_{\tau,p}) + \frac{(\gamma + \eta)\epsilon}{2}\psi_{\mathrm{ur}}\right)$$

$$\otimes \left(1 - \mu\epsilon(-\mathcal{L}_\tau\psi_{\mathrm{ur}} + \psi_{\tau,p}) + \frac{(\gamma + \eta)\epsilon}{2}\psi_{\mathrm{ur}}\right)$$

$$\cong (1 + \log_{\tau,-\mathcal{L}_\tau} \epsilon + \psi_\tau\epsilon) \otimes (1 - \log_{\tau,-\mathcal{L}_\tau} \epsilon + \psi_\tau\epsilon),$$

with $\psi_\tau = \frac{\gamma+\eta}{2\mu}\psi_{\mathrm{ur}}$, from which Proposition 4.26 follows.

We show (4.21). Let U be an affinoid neighborhood of z_ρ in $\mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$ small enough such that Proposition 4.14 applies, we have thus a continuous representation $\rho_U : \mathrm{Gal}_F \to \mathrm{GL}_2(\mathcal{O}(U_{\mathrm{red}}))$.

Non-critical Case. — Suppose $S_n = \wp$, i.e. z is non-$\wp$-critical. By Proposition 4.21, shrinking U, we can assume any point in U is non-$\wp$-critical. Let U_τ be the preimage of U in $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$ (via the natural closed embedding $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}} \hookrightarrow \mathcal{E}(\underline{k}_\emptyset, w)_{\overline{\rho}}$), since U_τ is étale over $\mathcal{W}_1(\underline{k}_{\Sigma_\wp^\tau})$ at z_ρ, shrinking U_τ, we can assume U_τ is a smooth curve. Let $\rho_{U_\tau} : \mathrm{Gal}_F \to \mathrm{GL}_2(\mathcal{O}(U_\tau))$ be the representation induced by ρ_U, $\chi_{U_\tau} : T(L) \to \mathcal{O}(U_\tau)^\times$ be the character induced by the natural morphism $U_\tau \to \widehat{T}(\underline{k}_{\Sigma_\wp^\tau}, w)$. Applying [31, Thm. 6.3.9] to $D_{\mathrm{rig}}(\rho_{U_\tau,\wp})$ with $\rho_{U_\tau,\wp} := \rho_{U_\tau}|_{\mathrm{Gal}_{F_\wp}}$ (see Theorem 4.15, note $_{z'} = \emptyset$ for all $z' \in U_\tau$ by the assumption on U), we get an exact sequence

$$0 \to \mathcal{R}_{\mathcal{O}(U_\tau)}(\mathrm{unr}(q)\chi_{U_\tau,1}) \to D_{\mathrm{rig}}(\rho_{U_\tau,\wp}) \to \mathcal{R}_{\mathcal{O}(U_\tau)}(\chi_{U_\tau,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1}) \to 0,$$

which induces (where $\widetilde{\rho}_{\tau,\wp} := t_\tau^*\rho_{U_\tau}|_\wp : \mathrm{Gal}_{F_\wp} \to \mathrm{GL}_2(E[\epsilon]/\epsilon^2)$)

$$0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(q)\widetilde{\chi}_{\rho,\tau,1}) \to D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\chi}_{\rho,\tau,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1}) \to 0.$$

Thus, (4.21) follows from Theorem 2.1.

Critical case. — Assume henceforth $S_c \neq \emptyset$. We shrink U such that the Proposition 4.17 applies, so $Z_{U,\sigma}$ (if non-empty) is a Zariski-closed subset in U for any $\sigma \in \wp$. We know $z \in Z_{U,\sigma}$ if and only if $\sigma \in S_c$. By shrinking U (as a neighborhood of z), one can assume $Z_{U,\sigma} = \emptyset$ for $\sigma \in S_n$. Let $\tau \in S_n$, U_τ be the preimage of U in $\mathcal{E}(\underline{k}_{\Sigma_\wp^\tau}, w)_{\overline{\rho}}$, and shrink U such that U_τ is a smooth curve. Let $Z_{U_\tau,\sigma}$ the preimage of $Z_{U,\sigma}$ in U_τ, which is a non-empty Zariski-closed subset for $\sigma \in S_c$, whose dimension is either 0 or 1 locally at z. Denote by S_0 (resp. S_1) the subset of S_c of embeddings σ such that $Z_{U_\tau,\sigma}$ is of dimension 0 (resp. of dimension 1) locally at z_τ. By shrinking U (and thus U_τ, note U_τ is smooth), one can assume $Z_{U_\tau,\sigma} = \{z_\tau\}$ for $\sigma \in S_0$

and $Z_{U_\tau,\sigma} = U_\tau(\overline{E})$ for $\sigma \in S_1$. We define $\rho_{U_\tau,\wp}$, χ_{U_τ}, $\widetilde{\rho}_{\tau,\wp}$ the same way as in the non-critical case.

Critical case (1). — Suppose $S_0 = \emptyset$. In this case, for any $z \in U_\tau$, $_z = S_c$. By applying [31, Thm. 6.3.9] to $D_{\mathrm{rig}}(\rho_{U_\tau,\wp})$, we get

$$0 \to \mathcal{R}_{\mathcal{O}(U_\tau)}\left(\mathrm{unr}(q)\chi_{U_\tau,1} \prod_{\sigma \in S_c} \sigma^{1-k_\sigma} \right)$$

$$\to D_{\mathrm{rig}}(\rho_{U_\tau,\wp}) \to \mathcal{R}_{\mathcal{O}(U_\tau)}\left(\chi_{U_\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_c} \sigma^{k_\sigma - 1} \right) \to 0 \,,$$

which induces

$$0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}\left(\mathrm{unr}(q)\widetilde{\chi}_{\rho,\tau,1} \prod_{\sigma \in S_c} \sigma^{1-k_\sigma} \right)$$

$$\to D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) \to \mathcal{R}_{E[\epsilon]/\epsilon^2}\left(\widetilde{\chi}_{\rho,\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_c} \sigma^{k_\sigma - 1} \right) \to 0 \,.$$

On the other hand, by Proposition 4.14, $\widetilde{\rho}_{\tau,\wp}$ is $_\wp^\tau$-de Rham. We can hence apply Theorem 2.1, and (4.21) follows.

Critical case (2). — Suppose $S_0 \neq \emptyset$. By assumption, for $z \in U_\tau(\overline{E})$, $z \neq z_\rho$, $_z = S_1 \subsetneq S_c = S_0 \cup S_1$. By [31, Thm. 6.3.9] (see in particular [31, (6.3.14.1)]) applied to $D_{\mathrm{rig}}(\rho_{U_\tau,\wp})$, one gets an exact sequence

$$(4.22) \quad 0 \to \mathcal{R}_{\mathcal{O}(U_\tau)}\left(\mathrm{unr}(q)\chi_{U_\tau,1} \prod_{\sigma \in S_1} \sigma^{1-k_\sigma} \right)$$

$$\to D_{\mathrm{rig}}(\rho_{U,\wp}) \to \mathcal{R}_{\mathcal{O}(U_\tau)}\left(\chi_{U_\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_1} \sigma^{k_\sigma - 1} \right) \to Q \to 0$$

where Q is a finitely generated $\mathcal{R}_{\mathcal{O}(U_\tau)}$-module killed by certain powers of t ($\in \mathcal{R}_E$) and is supported at z_ρ. Tensoring (4.22) with $E[\epsilon]/\epsilon^2$ via t_τ, one gets exact sequences (see [31, Ex. 6.3.14])

$$(4.23) \quad D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) \xrightarrow{f} \mathcal{R}_{E[\epsilon]/\epsilon^2}\left(\widetilde{\chi}_{\rho,\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_1} \sigma^{k_\sigma - 1} \right)$$

$$\longrightarrow Q \otimes_{\mathcal{O}(U_\tau),t_\tau} E[\epsilon]/\epsilon^2 \longrightarrow 0 \,,$$

$$0 \longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}\left(\mathrm{unr}(q)\widetilde{\chi}_{\rho,\tau,1} \prod_{\sigma \in S_1} \sigma^{1-k_\sigma} \right) \longrightarrow \mathrm{Ker}(f) \,.$$

For simplicity, put

$$\widetilde{\delta} = \widetilde{\delta}_1 \otimes \widetilde{\delta}_2 := \left(\mathrm{unr}(q)\widetilde{\chi}_{\rho,\tau,1} \prod_{\sigma \in S_1} \sigma^{1-k_\sigma} \right) \otimes \left(\widetilde{\chi}_{\rho,\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_1} \sigma^{k_\sigma - 1} \right),$$

$$\delta = \delta_1 \otimes \delta_2 := \left(\mathrm{unr}(q)\chi_{\rho,1} \prod_{\sigma \in S_1} \sigma^{1-k_\sigma} \right) \otimes \left(\chi_{\rho,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_1} \sigma^{k_\sigma - 1} \right),$$

$$\widetilde{\delta}' = \widetilde{\delta}'_1 \otimes \widetilde{\delta}'_2 := \left(\mathrm{unr}(q)\widetilde{\chi}_{\rho,\tau,1} \prod_{\sigma \in S_c} \sigma^{1-k_\sigma} \right) \otimes \left(\widetilde{\chi}_{\rho,\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_c} \sigma^{k_\sigma - 1} \right),$$

$$\delta' = \delta'_1 \otimes \delta'_2 := \left(\mathrm{unr}(q)\chi_{\rho,1} \prod_{\sigma \in S_c} \sigma^{1-k_\sigma} \right) \otimes \left(\chi_{\rho,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1} \prod_{\sigma \in S_c} \sigma^{k_\sigma - 1} \right),$$

and note that δ' is the trianguline parameter of $\rho_\wp$.

We see $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ (cf. (4.23)) are $(\varphi, \ )$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ (i.e. $(\varphi, \ )$-modules over $\mathcal{R}_E$ equipped moreover an $E[\epsilon]/\epsilon^2$-action commuting with $\mathcal{R}_E$, note that such modules may not be free over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$). Denote by f_0 the map $D_{\mathrm{rig}}(\rho_\wp) \to \mathcal{R}_E(\delta_2)$ induced by (4.22) via the pull-back z_ρ^*, one has a commutative diagram (of $(\varphi, \ )$-modules over $\mathcal{R}_E$)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D_{\mathrm{rig}}(\rho_\wp) & \xrightarrow{\ \epsilon\ } & D_{\mathrm{rig}}(\widetilde{\rho}_\wp) & \longrightarrow & D_{\mathrm{rig}}(\rho_\wp) & \longrightarrow & 0 \\
 & & \ \downarrow{\scriptstyle f_0} & & \ \downarrow{\scriptstyle f} & & \ \downarrow{\scriptstyle f_0} & & \\
0 & \longrightarrow & \mathcal{R}_E(\delta_2) & \xrightarrow{\ \epsilon\ } & \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2) & \longrightarrow & \mathcal{R}_E(\delta_2) & \longrightarrow & 0
\end{array}
$$

which induces thus a long exact sequence

$$0 \to \mathrm{Ker}(f_0) \xrightarrow{\epsilon} \mathrm{Ker}(f) \xrightarrow{s} \mathrm{Ker}(f_0) \to Q \otimes_{\mathcal{O}(U_\tau), z_\rho} E$$
$$\xrightarrow{\epsilon} Q \otimes_{\mathcal{O}(U_\tau), t_\tau} E[\epsilon]/\epsilon^2 \to Q \otimes_{\mathcal{O}(U_\tau), z_\rho} E \to 0 \, .$$

By discussions in [31, Ex. 6.3.14], one has (where we refer to [31, Not. 6.2.7] for the t_σ's)

$$\mathrm{Ker}(f_0) \cong \mathcal{R}_E(\delta'_1) \, ,$$
$$\mathrm{Im}(f_0) \cong \mathcal{R}_E(\delta'_2) \, ,$$
$$Q \otimes_{\mathcal{O}(U_\tau), z_\rho} E \cong \mathcal{R}_E(\delta_1) / \left(\prod_{\sigma \in S_0} t_\sigma^{k_\sigma - 1} \right),$$

thus there exist $r_\sigma \in \mathbb{Z}$, $0 \leqslant r_\sigma \leqslant k_\sigma - 1$ for all $\sigma \in S_0$ such that $\mathrm{Im}(s) = \mathcal{R}_E(\delta''_1)$ where $\delta''_1 := \delta'_1 \prod_{\sigma \in S_0} \sigma^{r_\sigma}$. However, since $\mathrm{Ker}(f)$ is a saturated sub-$(\varphi, \ )$-module of $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$, and the latter has Sen weight of the form $(-\frac{k_\sigma + w}{2} + a_\sigma \epsilon, \frac{k_\sigma - w - 2}{2} + b_\sigma \epsilon)_{\sigma \in \Sigma_\wp}$, we see $r_\sigma = 0$ or $k_\sigma - 1$ for $\sigma \in S_0$.

One has a natural isomorphism (translating these in terms of E-B-pairs, one can check this isomorphism by the same argument as in the proof of Lemma 1.13)

$$\mathrm{Ext}^1(\mathcal{R}_E(\delta_1'), \mathcal{R}_E(\delta_1')) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}_E(\delta_1''), \mathcal{R}_E(\delta_1')).$$

We claim $[\mathrm{Ker}(f)]$ equals (up to scalars) the image of $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1')]$: Indeed one has isomorphisms

$$(4.24) \quad \mathrm{Ext}^1(\mathcal{R}_E(\delta_1'), \mathcal{R}_E(\delta_1')) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}_E(\delta_1''), \mathcal{R}_E(\delta_1'))$$

$$\xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1')) \cong \mathrm{Ext}^1(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)).$$

The composition i in (4.24) actually sends $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1')]$ to $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1)]$ (up to scalars), since both $i([\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1')])$ and $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1)]$ fit "$*$" in the following commutative diagram (with the maps on the left and right sides being the natural injections)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}_E(\delta_1) & \longrightarrow & * & \longrightarrow & \mathcal{R}_E(\delta_1) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_E(\delta_1') & \longrightarrow & \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1') & \longrightarrow & \mathcal{R}_E(\delta_1') & \longrightarrow & 0 \,;
\end{array}
$$

on the other hand, since $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1) \hookrightarrow \mathrm{Ker}(f)$, one sees the composition of the last two morphisms in (4.24) sends $[\mathrm{Ker}(f)]$ to $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1)]$ (up to scalars), the claim follows.

Similarly, $\mathrm{Im}(f)$ lies in an exact sequence of $(\varphi,\ )$-modules over $\mathcal{R}_E$:

$$0 \to \mathcal{R}_E(\delta_2'') \xrightarrow{\epsilon} \mathrm{Im}(f) \to \mathcal{R}_E(\delta_2') \to 0$$

with $\delta_2'' = \delta_2' \prod_{\sigma \in S_0} \sigma^{-r_\sigma}$, and the natural isomorphism

$$\mathrm{Ext}^1(\mathcal{R}_E(\delta_2'), \mathcal{R}_E(\delta_2')) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}_E(\delta_2'), \mathcal{R}_E(\delta_2''))$$

sends $[\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2')]$ to $\mathrm{Im}(f)$.

CLAIM. — *There exists a $(\varphi,\ )$-module D free of rank 2 over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ such that*

(1) *D lies in an exact sequence of $(\varphi,\ )$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$:*

$$0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1') \to D \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2') \to 0\,;$$

(2) *$D \equiv D_{\mathrm{rig}}(\rho_\wp) \pmod{\epsilon}$;*
(3) *D is S_c-de Rham.*

Assuming the claim, since $\widetilde{\delta}_1'(\widetilde{\delta}_2')^{-1} = \delta_1'(\delta_2')^{-1}(1 + (\gamma - \eta)\epsilon\psi_{\mathrm{ur}} + 2\mu\epsilon\psi_{\tau,p})$, one can deduce again from Theorem 2.1 that $\gamma - \eta = -2\mathcal{L}_\tau\mu$ (4.21). In the rest of this section, we "modify" $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$ to prove the claim.

The natural morphism of $(\varphi, \ )$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$: $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2) \hookrightarrow$ $\mathrm{Im}(f)$ induces a morphism

$$\mathrm{Ext}^1(\mathrm{Im}(f), \mathrm{Ker}(f)) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2), \mathrm{Ker}(f))$$

(here Ext^1 denotes the group of extensions of $(\varphi, \ )$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$). Denote by D' the image of $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$ via this morphism. In fact, D' is just the preimage of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2) \subset \mathrm{Im}(f)$ via the natural projection $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) \twoheadrightarrow$ $\mathrm{Im}(f)$. The natural morphism of $(\varphi, \ )$-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$: $\mathrm{Ker}(f) \hookrightarrow$ $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_1)$ induces a morphism

$$\mathrm{Ext}^1(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2), \mathrm{Ker}(f)) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2), \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_1)),$$

let D be the image of D' via this morphism. We check that D satisfies the properties (2) and (3) in the claim.

We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0
\end{array}
$$

which induces a long exact sequence

$$(4.25) \quad 0 \to \mathcal{R}_E(\delta'_1) \to D_{\mathrm{rig}}(\rho) \xrightarrow{r} \mathcal{R}(\delta''_2) \to \mathcal{R}_E(\delta''_1) \oplus \left(\mathcal{R}_E(\delta''_2)/\prod_{\sigma \in S_0} t_\sigma^{r_\sigma}\right)$$

$$\to D_{\mathrm{rig}}(\rho) \to \mathcal{R}_E(\delta'_2) \oplus \left(\mathcal{R}_E(\delta''_2)/\prod_{\sigma \in S_0} t_\sigma^{r_\sigma}\right) \to 0.$$

For a $(\varphi, \ )$-module D'' over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$, denote by $D''[\epsilon]$ the kernel of ϵ which is a saturated $(\varphi, \ )$-submodule (over $\mathcal{R}_E$) of D''. One sees the natural morphism $D' \hookrightarrow D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$ induces an isomorphism $D'[\epsilon] \cong D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon] \cong$ $D_{\mathrm{rig}}(\rho_\wp)$. Indeed, one gets an injection $D'[\epsilon] \hookrightarrow D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon]$, on the other hand, by (4.25), the image of $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon] \hookrightarrow D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}) \twoheadrightarrow \mathrm{Im}(f)$, which equals $\mathrm{Im}(r)$, is contained in $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}'_2)$, thus $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon] \subseteq D'$ so $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon] \subseteq D'[\epsilon]$, from which one gets the isomorphism. The $(\varphi, \ )$-module $D'/D'[\epsilon]$ sits in an exact sequence

$$0 \to \mathcal{R}_E(\delta''_1) \to D'/D'[\epsilon] \to \mathcal{R}_E(\delta'_2) \to 0$$

and is a submodule of $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})/D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})[\epsilon] \cong D_{\mathrm{rig}}(\rho_\wp)$. By the construction of D, one gets a natural morphism $D' \to D$ which induces an isomorphism $D_{\mathrm{rig}}(\rho_\wp) \cong D'[\epsilon] \xrightarrow{\sim} D[\epsilon] \cong D_{\mathrm{rig}}(\rho_\wp)$, and thus an injection

$D'/D'[\epsilon] \hookrightarrow D/D[\epsilon]$. One gets commutative diagrams

(4.26)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}_E(\delta_1'') & \longrightarrow & D'/D'[\epsilon] & \longrightarrow & \mathcal{R}_E(\delta_2') & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{R}_E(\delta_1') & \longrightarrow & D_* & \longrightarrow & \mathcal{R}_E(\delta_2') & \longrightarrow & 0
\end{array}
$$

for $D_* \in \{D/D[\epsilon], D_{\mathrm{rig}}(\rho_\wp)\}$ (for $D/D[\epsilon]$, this follows from the construction of D; for $D_{\mathrm{rig}}(\rho_\wp)$, this follows from the construction of D' discussed as above). So $D/D[\epsilon] \cong D_{\mathrm{rig}}(\rho_\wp)$ (which are both equal to the image of $D'/D'[\epsilon]$ via the natural morphism

$$
\mathrm{Ext}^1(\mathcal{R}_E(\delta_2'), \mathcal{R}_E(\delta_1'')) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\delta_2'), \mathcal{R}_E(\delta_1'))),
$$

the property (2) follows.

    To show D is S_c-de Rham, one needs only to prove D' is S_c-de Rham since D' is a $(\varphi, \ )$-submodule of D with the same rank. Since D' is a $(\varphi, \ )$-submodule of $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$, by the equivalence of categories of B-pairs and $(\varphi, \ )$-modules ([6, Thm. 2.2.7]), one gets an injection $W(D') \hookrightarrow W(D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}))$ of E-B-pairs where $W(D'')$ denotes the associated B-pairs for a $(\varphi, \ )$-modules D''. Since D' and $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$ are both of rank 4 (over $\mathcal{R}_E$), one sees $W(D')_{\mathrm{dR}} \xrightarrow{\sim} W(D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp}))_{\mathrm{dR}}$. Since $D_{\mathrm{rig}}(\widetilde{\rho}_{\tau,\wp})$ is S_c-de Rham, so is D'. This finishes the proof of the claim and thus (4.21) in $S_0 \neq \emptyset$-case.

Appendix A. Partially de Rham trianguline representations

    In this appendix, we study some partially de Rham triangulable E-B-pairs, and show that partial non-criticalness implies partial de Rhamness for triangulable E-B-pairs. As an application, we get a partial de Rhamness result for finite slope overconvergent Hilbert modular forms.

    Let $F_\wp$ be a finite extension of $\mathbb{Q}_p$, $\ _\wp$ the set of embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$, $\mathrm{Gal}_{F_\wp} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$, E a finite extension of $\mathbb{Q}_p$ sufficiently large containing all the embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. Let χ be a continuous character of $F_\wp^\times$ over E, recall that we have defined the weights $(\mathrm{wt}(\chi)_\sigma)_{\sigma \in \Sigma_\wp} \in E^{|d|}$ of χ (cf. Section 2); in fact, $(-\mathrm{wt}(\chi)_\sigma)_{\sigma \in \Sigma_\wp}$ are equal to the *generalized Hodge–Tate weights* of the associated E-B-pair $B_E(\chi)$ (cf. [33, Def. 1.47]).

    LEMMA A.1. — *Let χ be a continuous character of $F_\wp^\times$ over E, for $\sigma \in \ _\wp$, $B_E(\chi)$ is σ-de Rham if and only if $\mathrm{wt}(\chi)_\sigma \in \mathbb{Z}$.*

Proof. — The "only if" part is clear. Suppose now $\mathrm{wt}(\chi)_\sigma \in \mathbb{Z}$, by multiplying χ by $\sigma^{-\mathrm{wt}(\chi)_\sigma}$ and then an unramified character of $F_\wp^\times$, one can assume that χ corresponds to a Galois character $\chi : \mathrm{Gal}_{F_\wp} \to E^\times$ and $\mathrm{wt}(\chi)_\sigma = 0$. In this case, by Sen's theory, one has $\mathbb{C}_{p,\sigma} \otimes_E \chi \cong \mathbb{C}_{p,\sigma}$ as $\mathrm{Gal}_{F_\wp}$-modules (since χ is of Hodge–Tate weight 0 at σ). Consider the exact sequence

$$0 \to (tB_{\mathrm{dR},\sigma}^+ \otimes_E \chi)^{\mathrm{Gal}_{F_\wp}} \to (B_{\mathrm{dR},\sigma}^+ \otimes_E \chi)^{\mathrm{Gal}_{F_\wp}}$$
$$\to (\mathbb{C}_{p,\sigma} \otimes_E \chi)^{\mathrm{Gal}_{F_\wp}} \to H^1(\mathrm{Gal}_{F_\wp}, tB_{\mathrm{dR},\sigma}^+ \otimes_E \chi),$$

it's su cient to prove $H^1(\mathrm{Gal}_{F_\wp}, tB_{\mathrm{dR},\sigma}^+ \otimes_E \chi) = 0$. For $i \in \mathbb{Z}_{>0}$, we claim $H^1(\mathrm{Gal}_{F_\wp}, t^{i+1}B_{\mathrm{dR},\sigma}^+ \otimes_E \chi) \to H^1(\mathrm{Gal}_{F_\wp}, t^i B_{\mathrm{dR},\sigma}^+ \otimes_E \chi)$ is an isomorphism: one has an exact sequence

$$(\mathbb{C}_{p,\sigma}(i) \otimes_E \chi)^{\mathrm{Gal}_{F_\wp}} \to H^1(\mathrm{Gal}_{F_\wp}, t^{i+1}B_{\mathrm{dR},\sigma}^+ \otimes_E \chi)$$
$$\to H^1(\mathrm{Gal}_{F_\wp}, t^i B_{\mathrm{dR},\sigma}^+ \otimes_E \chi) \to H^1(\mathrm{Gal}_{F_\wp}, \mathbb{C}_{p,\sigma}(i) \otimes_E \chi),$$

since $\mathbb{C}_{p,\sigma} \otimes_E \chi \cong \mathbb{C}_{p,\sigma}$, the first and fourth terms vanish when $i \geqslant 1$. We get thus an isomorphism $H^1(\mathrm{Gal}_{F_\wp}, tB_{\mathrm{dR},\sigma}^+ \otimes_E \chi) \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, t^n B_{\mathrm{dR},\sigma}^+ \otimes_E \chi)$ for $n \gg 0$, from which we deduce $H^1(\mathrm{Gal}_{F_\wp}, tB_{\mathrm{dR},\sigma}^+ \otimes_E \chi) = 0$. □

DEFINITION A.2 ([32, Def. 4.3.1]). — *Let W be a triangulable E-B-pair of rank r with a triangulation given by*

(A.1) $$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{r-1} \subsetneq W_r = W$$

with $W_{i+1}/W_i \cong B_E(\chi_i)$ for $0 \leqslant i \leqslant r-1$ where the χ_i's are continuous characters of $F_\wp^\times$ in $E^\times$. For $\sigma \in \ _\wp$, suppose $\mathrm{wt}(\chi_i)_\sigma \in \mathbb{Z}$ for all $0 \leqslant i \leqslant r-1$, W is called non σ-critical if (note the generalized Hodge–Tate weight of $B_E(\chi_i)$ at σ is $-\mathrm{wt}(\chi_i)_\sigma$)

$$\mathrm{wt}(\chi_1)_\sigma > \mathrm{wt}(\chi_2)_\sigma > \cdots > \mathrm{wt}(\chi_r)_\sigma;$$

for $\emptyset \neq J \subseteq \ _\wp$, suppose $\mathrm{wt}(\chi_i)_\sigma \in \mathbb{Z}$ for $0 \leqslant i \leqslant r-1$, $\sigma \in J$, then W is called non J-critical if W is non σ-critical for all $\sigma \in J$.

PROPOSITION A.3. — *Keep the notation in Definition A.2, let $\emptyset \neq J \subseteq \ _\wp$, suppose W is non J-critical, then W is J-de Rham.*

Proof. — It's su cient to prove if W is non-σ-critical, then W is σ-de Rham for $\sigma \in J$. Let $\sigma \in J$, we would use induction on $1 \leqslant i \leqslant r-1$: by Lemma A.1, W_1 is σ-de Rham; assume now W_i is σ-de Rham, we show W_{i+1} is also σ-de Rham. Note $[W_{i+1}] \in \mathrm{Ext}^1(W_i, B_E(\chi_{i+1}))$, let $W_i' := W_i \otimes B_E(\chi_{i+1}^{-1})$, $W_{i+1}' := W_{i+1} \otimes B_E(\chi_{i+1}^{-1})$, by Lemma A.1, W_{i+1} is σ-de Rham if and only if W_{i+1}' is σ-de Rham. One has $[W_{i+1}'] \in H^1(\mathrm{Gal}_{F_\wp}, W_i')$.

On the other hand, since $\mathrm{wt}(\chi_j)_\sigma > \mathrm{wt}(\chi_{i+1})_\sigma$ for $1 \leqslant j \leqslant i$, we see $H^0(\mathrm{Gal}_{F_\wp}, (W_i')^+_{\mathrm{dR},\sigma}) = 0$, thus by Lemma 1.11, $H^1_{g,\sigma}(\mathrm{Gal}_{F_\wp}, W_i') \xrightarrow{\sim} H^1(\mathrm{Gal}_{F_\wp}, W_i')$. So W_{i+1}' is σ-de Rham, and the proposition follows. $\square$

Example A.4. — Let $\chi_{\mathrm{LT}} : \mathrm{Gal}_{F_\wp} \to F_\wp^\times$ be a Lubin–Tate character, $\sigma : F_\wp \hookrightarrow E$, and consider $H^1(\mathrm{Gal}_{F_\wp}, \sigma \circ \chi_{\mathrm{LT}})$. By Proposition A.3, any element in $H^1(\mathrm{Gal}_{F_\wp}, \sigma \circ \chi_{\mathrm{LT}})$ is σ-de Rham, which generalizes the well-known fact that any extension of the trivial character by cyclotomic character is de Rham. In fact, suppose $F_\wp \neq \mathbb{Q}_p$, using (1.7), one can actually calculate: $\dim_E H^1_{g,J}(\mathrm{Gal}_{F_\wp}, \sigma \circ \chi_{\mathrm{LT}}) = d - |J \setminus \{\sigma\}|$.

Partially de Rham overconvergent Hilbert modular forms

Let F be a totally real number field of degree d_F, Σ_F the set of embeddings of F in $\overline{\mathbb{Q}}$, $w \in \mathbb{Z}$, and $k_\sigma \in \mathbb{Z}_{\geqslant 2}$, $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_F$. Let $\mathfrak{c}$ be a fractional ideal of F. Let h be an overconvergent Hilbert eigenform of weights $(\underline{k}, w)$ (where we adopt Carayol's convention of weights as in [18])), of tame level N ($N \geqslant 4$, $p \nmid N$), of polarization $\mathfrak{c}$, with Hecke eigenvalues in E (e.g. see [1, Def. 1.1], where E is big enough to contain all the embeddings of F in $\overline{\mathbb{Q}_p}$). For a place $\wp$ of F above p, let $a_\wp$ denote the $U_\wp$-eigenvalue of h, and suppose $a_\wp \neq 0$ for all $\wp | p$. Denote by $\rho_h : \mathrm{Gal}_F \to \mathrm{GL}_2(E)$ the associated (semi-simple) Galois representation (enlarge E if necessary) (e.g. see [1, Thm. 5.1]). For $\wp | p$, denote by $\rho_{h,\wp}$ the restriction of ρ_h to the decomposition group at $\wp$, which is thus a continuous representation of $\mathrm{Gal}_{F_\wp}$ over E, where $F_\wp$ denotes the completion of F at $\wp$. Let $v_\wp : \overline{\mathbb{Q}_p} \to \mathbb{Q} \cup \{+\infty\}$ be an additive valuation normalized by $v_\wp(F_\wp) = \mathbb{Z} \cup \{+\infty\}$. Denote by $\Sigma_\wp$ the set of embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. This section is devoted to prove

THEOREM A.5. — *With the above notation, and let $\emptyset \neq J \subseteq \Sigma_\wp$.*
  (1) *If $v_\wp(a_\wp) < \inf_{\sigma \in J}\{k_\sigma - 1\} + \sum_{\sigma \in \Sigma_\wp} \frac{w - k_\sigma + 2}{2}$, then $\rho_{h,\wp}$ is J-de Rham.*
  (2) *If $v_\wp(a_\wp) < \sum_{\sigma \in J}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_\wp} \frac{w - k_\sigma + 2}{2}$, then there exists $\sigma \in J$ such that $\rho_{h,\wp}$ is σ-de Rham.*

Remark A.6. — This theorem gives evidence for Breuil's conjectures in [10] (but in terms of Galois representations) (see in particular [10, Prop. 4.3]). When $J = \Sigma_\wp$ (and $F_\wp$ unramified), the part (1) follows directly from the known classicality result in [43].

One has as in Proposition 4.15

PROPOSITION A.7. — *For $\wp|p$, $\rho_{h,\wp}$ is trianguline with a triangulation given by*

$$0 \to B_E(\delta_1) \to W(\rho_{h,\wp}) \to B_E(\delta_2) \to 0 \,,$$

with

$$\begin{cases} \delta_1 = \mathrm{unr}_\wp(a_\wp) \prod_{\sigma \in \Sigma_\wp} \sigma^{-\frac{w-k_\sigma+2}{2}} \prod_{\sigma \in \Sigma_h} \sigma^{1-k_\sigma} \,, \\ \delta_2 = \mathrm{unr}_\wp(q_\wp b_\wp/a_\wp) \prod_{\sigma \in \Sigma_\wp} \sigma^{-\frac{w+k_\sigma}{2}} \prod_{\sigma \in \Sigma_h} \sigma^{k_\sigma-1} \,, \end{cases}$$

where $\mathrm{unr}_\wp(z)$ denotes the unramified character of $F_\wp^\times$ sending uniformizers to z, $q_\wp := p^{f_\wp}$ with $f_\wp$ the degree of the maximal unramified extension inside $F_\wp$ (thus $v_\wp(q_\wp) = d_\wp$, the degree of $F_\wp$ over $\mathbb{Q}_p$), and Σ_h is a certain subset of $\Sigma_\wp$.

Proof. — Consider the eigenvariety $\mathcal{E}$ constructed in [1, Thm. 5.1], one can associate to h a point z_h in $\mathcal{E}$. For classical Hilbert eigenforms, the result is known by Saito's results in [38] and Nakamura's results on triangulations of 2-dimensional semi-stable Galois representations (cf. [33, §4]). Since the classical points are Zariski-dense in $\mathcal{E}$ and accumulate over the point z_h (here one uses the classicality results, e.g. in [8]), the proposition follows from the global triangulation theory [31, Thm. 6.3.13] [32, Thm. 4.4.2]. □

Since $W(\rho_\wp)$ is étale (purely of slope zero), by Kedlaya's slope filtration theroy ([30, Thm. 1.7.1]), one has (see also [33, Lem. 3.1])

LEMMA A.8. — *Let $\varpi_\wp$ be a uniformizer of $F_\wp$, then $v_\wp(\delta_1(\varpi_\wp)) \geqslant 0$.*

Proof of Theorem A.5. — By the above lemma, $v_\wp(a_\wp) \geqslant \sum_{\sigma \in \Sigma_h}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_\wp} \frac{w-k_\sigma+2}{2}$. Thus for $\emptyset \neq J \subseteq \Sigma_\wp$, if $v_\wp(a_\wp) < \inf_{\sigma \in J}\{k_\sigma - 1\} + \sum_{\sigma \in \Sigma_\wp} \frac{w-k_\sigma+2}{2}$ $\left(\text{resp. } v_\wp(a_\wp) < \sum_{\sigma \in S}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_\wp} \frac{w-k_\sigma+2}{2} \right)$, then $J \cap \Sigma_h = \emptyset$ (resp. $J \nsubseteq \Sigma_h$) and thus $\rho_{h,\wp}$ is non-J-critical (resp. there exists $\sigma \in J$ such that $\rho_{h,\wp}$ is non-σ-critical) (note $\Sigma_{F_\wp} \setminus \Sigma_h$ is exactly the set of embeddings where $\rho_{h,\wp}$ is non-critical). The theorem then follows from Proposition A.3. □

We end this section by (conjecturally) constructing some partial de Rham families of Hilbert modular forms as closed subspaces of $\mathcal{E}$ ([1, Thm. 5.1]). For $\wp|p$, denote by $\mathcal{W}_\wp$ the rigid space over E parametrizing locally $\mathbb{Q}_p$-analytic characters of $\mathcal{O}_\wp^\times$. One has a natural morphism of rigid spaces $\mathcal{W}_\wp \to \mathbb{A}^{|\Sigma_\wp|}$, $\chi \mapsto (\mathrm{wt}(\chi)_\sigma)_{\sigma \in \Sigma_\wp}$. For $J \subseteq \Sigma_\wp$, $k_\sigma \in \mathbb{Z}$ for $\sigma \in J$, denote by $\mathcal{W}_\wp(\underline{k}_J)$ the preimage of the rigid subspace of $\mathbb{A}^{|\Sigma_\wp|}$ defined by fixing the σ-parameter to be k_σ for $\sigma \in J$. Let $\mathcal{W}_0$ denote the rigid space (over E) parametrizing locally $\mathbb{Q}_p$-analytic characters of $\mathbb{Z}_p^\times$. Recall (cf. [1, Thm. 5.1]), one has a natural morphism $\kappa : \mathcal{E} \to \prod_{\wp|p} \mathcal{W}_\wp \times \mathcal{W}_0$ (where

the right hand is denoted by $\mathcal{W}^G$ in loc. cit.), mapping each point of $\mathcal{E}$ (corresponding to overconvergent Hilbert eigenforms) to its weights.

Now fix $\wp|p$, $\emptyset \subsetneq J \subsetneq \Sigma_\wp$, $w \in \mathbb{Z}$, and $k_\sigma \in \mathbb{Z}_{\geqslant 2}$, $k_\sigma \equiv w$ (mod 2) for all $\sigma \in J$. Consider the closed subspace

$$\mathcal{W}_\wp(\underline{k}_J) \times \prod_{\substack{\wp'|p \\ \wp' \neq \wp}} \mathcal{W}_{\wp'} \longrightarrow \mathcal{W}_\wp \times \prod_{\substack{\wp'|p \\ \wp' \neq \wp}} \mathcal{W}_{\wp'} \longrightarrow \prod_{\wp'|p} \mathcal{W}_{\wp'} \times \mathcal{W}_0$$

where the last map is induced by the E-point $(x \mapsto x^w)$ of $\mathcal{W}_0$. Denote by $\mathcal{E}(\underline{k}_J, w)'$ the pull-back of $\mathcal{W}_\wp(\underline{k}_J) \times \prod_{\substack{\wp'|p \\ \wp' \neq \wp}} \mathcal{W}_{\wp'}$ via κ, which is a closed rigid subspace of $\mathcal{E}$ consisting of points with fixed weights k_σ for $\sigma \in J$ and w. Let $\mathcal{E}(\underline{k}_J, w)$ be the Zariski-closure of the classical points in $\mathcal{E}(\underline{k}_J, w)'$.

CONJECTURE A.9. — *Keep the above notation, let $z \in \mathcal{E}(\underline{k}_J, w)'(\overline{E})$, and suppose the associated Gal_F-representation ρ_z is absolutely irreducible. Then $z \in \mathcal{E}(\underline{k}_J, w)(\overline{E})$ if and only if $\rho_{z,\wp} := \rho_z|_{\mathrm{Gal}_{F_\wp}}$ is J-de Rham.*

Appendix B. Some locally analytic representation theory of $\mathrm{GL}_2(F_\wp)$

Recall some locally analytic representation theory of $\mathrm{GL}_2(F_\wp)$ used in the paper.

PROPOSITION B.1. — *Let V be a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E, $J \subseteq \Sigma_\wp$, and W be an irreducible algebraic locally J-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E. The composition*

(B.1) $$(V \otimes_E W^\vee)_{\Sigma_\wp \setminus J-\mathrm{an}} \otimes_E W \longrightarrow V \otimes_E W^\vee \otimes_E W \longrightarrow V$$

is injective, where $W^\vee$ denotes the dual representation of W.

Proof. — The proof is similar as that of [25, Prop. 4.2.4]. The case $J = \emptyset$ is trivial, and we suppose $J \neq \emptyset$. We equip $V \otimes_E W^\vee$ with a $\mathfrak{g}_{\Sigma_\wp} \times \mathfrak{g}_{\Sigma_\wp}$-action by $(X_1, X_2)(v \otimes w') = (X_1 v) \otimes w' + v \otimes (X_2 w')$. Denote by Δ the morphism $\mathfrak{g}_{\Sigma_\wp} \hookrightarrow \mathfrak{g}_{\Sigma_\wp} \times \mathfrak{g}_{\Sigma_\wp}$, $x \mapsto (x, x)$. We have thus

$$(V \otimes_E W^\vee)_{\Sigma_\wp \setminus J} \xrightarrow{\sim} (V \otimes_E W^\vee)^{\Delta(\mathfrak{g}_J)}.$$

Since $\mathrm{End}_{\mathfrak{g}_\wp}(W) \cong \mathrm{End}_{\mathfrak{g}_J}(W) \cong E$, by the double commutant theorem, the morphism of E-algebras

(B.2) $$\mathrm{U}(\mathfrak{g}_J) \longrightarrow \mathrm{End}_E(W)(\cong W \otimes_E W^\vee)$$

is surjective. We equip $\mathrm{U}(\mathfrak{g}_J)$ (resp. $W \otimes_E W^\vee$) with a $\mathfrak{g}_J \times \mathfrak{g}_J$-action by $(X_1, X_2)(X) = X_1 X - X X_2$ (resp. by $(X_1, X_2)(w \otimes w') = (X_1 w) \otimes w' +$

$w \otimes (X_2 w'))$. One can check the map (B.2) is $\mathfrak{g}_J \times \mathfrak{g}_J$-invariant. By taking the dual, we get a $\mathfrak{g}_J \times \mathfrak{g}_J$-invariant injection (recall for a Lie algebra $\mathfrak{h}$ over E, and a representation U of $\mathfrak{h}$ over E, $\mathrm{Hom}_E(U, E)$ is equipped with an $\mathfrak{h}$-action given by $(Xf)(u) = -f(Xu)$ for $X \in \mathfrak{h}$, $f \in \mathrm{Hom}_E(U, E)$ and $u \in U$)

$$\text{(B.3)} \qquad\qquad W^\vee \otimes_E W \hookrightarrow \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), E).$$

Moreover, one can check that the composition of (B.3) with the evaluation map $\mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), E) \to E$, $f \mapsto f(1)$ is equal to the natural morphism $W^\vee \otimes_E W \to E$, $w' \otimes w \mapsto w'(w)$.

Consider the following composition induced by (B.3)

$$\text{(B.4)} \quad V \otimes_E W^\vee \otimes_E W \hookrightarrow V \otimes_E \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), E) \hookrightarrow \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V),$$

which is in fact $\mathfrak{g}_{\Sigma_\wp} \times \mathfrak{g}_J \times \mathfrak{g}_J$-invariant, where the $\mathfrak{g}_{\Sigma_\wp} \times \mathfrak{g}_J \times \mathfrak{g}_J$-action on $V \otimes_E W^\vee \otimes_E W$ (resp. on $\mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)$) is given by $(X_1, X_2, X_3)(v \otimes w' \otimes w) = (X_1 v) \otimes w' \otimes w + v \otimes (X_2 w') \otimes w + v \otimes w' \otimes (X_3 w)$ (resp. by $((X_1, X_2, X_3)f)(X) = X_1(f(X)) - f(X_2 X) + f(XX_3)$). Moreover, the composition of (B.4) with the evaluation map

$$\text{(B.5)} \qquad\qquad \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V) \to V, \; f \mapsto f(1)$$

is equal to (B.1).

Denote by $\Delta_{12} : \mathfrak{g}_J \hookrightarrow \mathfrak{g}_{\Sigma_\wp} \times \mathfrak{g}_J \times \mathfrak{g}_J$, $X \mapsto (X, X, 0)$. Thus the left object of (B.1) is equal to $(V \otimes_E W^\vee \otimes_E W)^{\Delta_{12}(\mathfrak{g}_J)}$. We claim the map (B.5) induces an isomorphism $\mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)^{\Delta_{12}(\mathfrak{g}_J)} \xrightarrow{\sim} V$, from which the proposition follows. Indeed, the map $V \to \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)$, $v \mapsto [X \mapsto Xv]$ is obviously a section of (B.5), and one can check the image is contained in $\mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)^{\Delta_{12}(\mathfrak{g}_J)}$, and the induced map $V \to \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)^{\Delta_{12}(\mathfrak{g}_J)}$ is bijective (since any $f \in \mathrm{Hom}_E(\mathrm{U}(\mathfrak{g}_J), V)^{\Delta_{12}(\mathfrak{g}_J)}$ satisfies $f(XY) = Xf(Y)$ for $X \in \mathfrak{g}_J$, and $Y \in \mathrm{U}(\mathfrak{g}_J)$). $\qquad\square$

COROLLARY B.2. — *Keep the notation of Proposition B.1, suppose moreover V is admissible, then the representation $(V \otimes_E W^\vee)_{\Sigma_\wp \setminus J-\mathrm{an}} \otimes_E W$ is a closed subrepresentation of V.*

Proof. — Since V is admissible, so is $V \otimes_E W^\vee \otimes_E W$. Since $(V \otimes_E W^\vee)_{\Sigma_\wp \setminus J-\mathrm{an}} \otimes_E W$ is obviously a closed subrepresentation of $V \otimes_E W^\vee \otimes_E W$, by [40, Prop. 6.4], $(V \otimes_E W^\vee)_{\Sigma_\wp \setminus J-\mathrm{an}} \otimes_E W$ is also admissible. By loc. cit., in this case, the map (B.1) is strict and has closed image, which concludes the proof. $\qquad\square$

Let V be an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E, the associated Jacquet–Emerton module $J_B(V)$ is thus an

essentially admissible locally $\mathbb{Q}_p$-analytic representation of $T(F_\wp)$ (cf. [26, Thm. 0.5]). Let $U \in \mathrm{Rep}_{\mathrm{la},c}^z(T(F_\wp))$ (cf. [26, §3.1]), and suppose U is allowable (cf. [28, Def. 0.11]), recall first the following theorem of Emerton.

THEOREM B.3 ([28, Thm. 0.13]). — *Keep the above notation and hypothesis, suppose moreover V is very strongly admissible (cf. [28, Def. 0.12]), then one has a natural bijection*

$$\mathrm{Hom}_{T(F_\wp)}(U \otimes_E \delta_B, J_B(V))^{\mathrm{bal}} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(I_{\overline{B}(L)}^{\mathrm{GL}_2(L)}(U), V\right),$$

where "bal" denotes the balanced maps (cf. [28, Def. 0.8]) and we refer to [28, §2.8] for the definition of $I_{\overline{B}(L)}^{\mathrm{GL}_2(L)}(U)$ (see also the paragraphs which follow).

Recall the definition of balanced maps (in $\mathrm{GL}_2(F_\wp)$-case). Let $N(F_\wp)$ be the nilpotent radical of $B(F_\wp)$, and let $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), E)$ denote the affine E-algebra of the algebraic group $\mathrm{Res}_{\mathbb{Q}_p}^{F_\wp} N \times_{\mathbb{Q}_p} E \cong \mathrm{Res}_{\mathbb{Q}_p}^{F_\wp} \mathbb{G}_a \times_{\mathbb{Q}_p} E$, thus $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), E) \cong \otimes_{\sigma \in \Sigma_\wp} E[\sigma(z)] =: E[\underline{z}_{\Sigma_\wp}]$. Let $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), U) := \mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), E) \otimes_E U \cong E[\underline{z}_{\Sigma_\wp}] \otimes_E U =: U[\underline{z}_{\Sigma_\wp}]$, which can be naturally equipped with a $\mathfrak{g}_{\Sigma_\wp}$-action (cf. [28, §2.5]) such that for $\underline{m}_{\Sigma_\wp} := (m_\sigma)_{\sigma \in \Sigma_\wp} \in \mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$, $z^{\underline{m}_\wp} := \prod_{\sigma \in \Sigma_\wp} \sigma(z)^{m_\sigma} \in \mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), E)$, and $u \in U$,

- $Z_\sigma \cdot (uz^{\underline{m}_\wp}) = m_\sigma(d_\sigma - a_\sigma)uz^{\underline{m}_\wp} + (Z_\sigma \cdot u)z^{\underline{m}_\wp}$, for $Z_\sigma = \begin{pmatrix} a_\sigma & 0 \\ 0 & d_\sigma \end{pmatrix} \in \mathfrak{t}_\sigma \subset \mathfrak{g}_\sigma$,

- $X_{+,\sigma} \cdot (uz^{\underline{m}_\wp}) = \begin{cases} 0 & \text{if } m_\sigma = 0 \\ m_\sigma uz^{\underline{m}_\wp - 1_\sigma} & \text{otherwise} \end{cases}$ where $1_\sigma \in \mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$ with $(1_\sigma)_{\sigma'} = \begin{cases} 1 & \sigma' = \sigma \\ 0 & \sigma' \neq \sigma \end{cases}$, and $X_{+,\sigma} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\sigma$,

- $X_{-,\sigma} \cdot (uz^{\underline{m}_\wp}) = (h_\sigma \cdot u)z^{\underline{m}_\wp + 1_\sigma} - m_\sigma uz^{\underline{m}_\wp + 1_\sigma}$ with $X_{-,\sigma} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}_\sigma$, $h_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}_\sigma$.

The embedding $U \hookrightarrow U[\underline{z}_{\Sigma_{F_\wp}}]$ (which can be easily checked to be $\mathfrak{b}_{\Sigma_\wp}$-invariant, where $\mathfrak{b}_{\Sigma_\wp}$ acts on U via $\mathfrak{b}_{\Sigma_\wp} \twoheadrightarrow \mathfrak{t}_{\Sigma_\wp}$, and the $\mathfrak{t}_{\Sigma_\wp}$-action on U is induced by the $T(F_\wp)$-action) thus induces

(B.6) $U(\mathfrak{g}_{\Sigma_\wp}) \otimes_{U(\mathfrak{b}_{\Sigma_\wp})} U \longrightarrow \mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), U) \cong U[\underline{z}_{\Sigma_\wp}]$.

Let $\bar{\mathfrak{n}}$ denote the Lie algebra of the nilpotent radical of $\overline{B}(F_\wp)$, since $U(\mathfrak{g}_{\Sigma_\wp}) \cong U(\bar{\mathfrak{n}}_{\Sigma_\wp}) \otimes_E U(\mathfrak{b}_{\Sigma_\wp})$, $U(\mathfrak{g}_{\Sigma_\wp}) \otimes_{U(\mathfrak{b}_{\Sigma_\wp})} U \cong U(\bar{\mathfrak{n}}_{\Sigma_\wp}) \otimes_E U$. One

gets thus a map

$$U(\bar{\mathfrak{n}}_{\Sigma_\wp}) \otimes_E U \longrightarrow \mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), U) \cong U[\underline{z}_{\Sigma_\wp}]$$

which is in fact given by

(B.7) $\left(\displaystyle\prod_{\sigma \in \Sigma_\wp} X_{-,\sigma}^{m_\sigma}\right) \otimes u \mapsto \left(\left(\displaystyle\prod_{\sigma \in \Sigma_\wp} \prod_{j=0}^{m_\sigma-1} (h_\sigma - j)\right) \cdot u\right) z^{\underline{m}}_{\ \wp}$

for all $u \in U$, $\underline{m}_{\Sigma_\wp} \in \mathbb{Z}_{\geq 0}^{|\Sigma_\wp|}$, where we let $\prod_{j=0}^{m_\sigma-1} (h_\sigma - j)$ be 1 when $m_\sigma = 0$.

Let $f : U \otimes_E \delta_B \to J_B(V)$ be a morphism of locally $\mathbb{Q}_p$-analytic $T(F_\wp)$-representations. Fix a compact open subgroup N_0 of $N(F_\wp)$, and consider the composition $\iota(f) : U \otimes_E \delta_B \to J_B(V) \to V$ (where the last map is the canonical lifting with respect to N_0, cf. [26, (3.4.8)]), which is $\mathfrak{b}_{\Sigma_\wp}$-invariant (where $\mathfrak{b}_{\Sigma_\wp}$ acts on $U \otimes_E \delta_B$ via $\mathfrak{b}_{\Sigma_\wp} \twoheadrightarrow \mathfrak{t}_{\Sigma_\wp}$), and induces thus a $U(\mathfrak{g}_{\Sigma_\wp})$-invariant map (where the first isomorphism follows from the fact that δ_B is smooth):

(B.8) $U(\mathfrak{g}_{\Sigma_\wp}) \otimes_{U(\mathfrak{b}_{\ \wp})} U \cong U(\mathfrak{g}_{\Sigma_\wp}) \otimes_{U(\mathfrak{b}_{\ \wp})} (U \otimes_E \delta_B) \longrightarrow V.$

Recall the map f is called *balanced* if the kernel of (B.6) is contained in the kernel of (B.8).

Let $J \subseteq {}_{\wp}$, and suppose moreover U and V are locally J-analytic. Let $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), E) := \otimes_{\sigma \in J} E[\sigma(z)] =: E[\underline{z}_J]$, and $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U) := \mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), E) \otimes_E U \cong U[\underline{z}_J]$. Since U is locally J-analytic, one can check that $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U)$ is a $U(\mathfrak{g}_{\Sigma_\wp})$-submodule of $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), U)$, and the action of $U(\mathfrak{g}_{\Sigma_\wp})$ on $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U)$ factors through $U(\mathfrak{g}_J)$. Moreover, one can check (e.g. by (B.7)) that the map (B.6) factors through

(B.9) $U(\mathfrak{g}_J) \otimes_{U(\mathfrak{b}_J)} U \longrightarrow \mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U).$

Identifying $U(\mathfrak{g}_J) \otimes_{U(\mathfrak{b}_J)} U$ with $U(\bar{\mathfrak{n}}_J) \otimes_E U$, $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U)$ with $E[\underline{z}_J]$, this map is in fact equal to

(B.10) $U(\bar{\mathfrak{n}}_J) \otimes_E U \longrightarrow E[\underline{z}_J]$,

$$\left(\prod_{\sigma \in J} X_{-,\sigma}^{m_\sigma}\right) \otimes u \mapsto \left(\left(\prod_{\sigma \in J} \prod_{j=0}^{m_\sigma-1} (h_\sigma - j)\right) \cdot u\right) z^{\underline{m}_J},$$

for all $u \in U$, $\underline{m}_J \in \mathbb{Z}_{\geq 0}^{|J|}$, where we let $\prod_{j=0}^{m_\sigma-1} (h_\sigma - j)$ be 1 when $m_\sigma = 0$. Let f be the morphism as above, and consider the $\mathfrak{b}_{\Sigma_\wp}$-invariant map $\iota(f) : U \otimes_E \delta_B \to V$. Since both U and V are locally J-analytic, the action of $\mathfrak{b}_{\Sigma_\wp}$

on $U \otimes_E \delta_B$ and V factors through $\mathfrak{b}_J$. Thus $\iota(f)$ induces a $\mathsf{U}(\mathfrak{g}_J)$-invariant map

(B.11) $\mathsf{U}(\mathfrak{g}_J) \otimes_{\mathsf{U}(\mathfrak{b}_J)} U \cong \mathsf{U}(\mathfrak{g}_J) \otimes_{\mathsf{U}(\mathfrak{b}_J)} (U \otimes_E \delta_B) \longrightarrow V.$

Since V is locally J-analytic, the morphism (B.8) in fact factors through (B.11). Thus we have

LEMMA B.4. — *Keep the above notation and hypothesis (in particular U and V are locally J-analytic), a morphism $f : U \otimes_E \delta_B \to J_B(V)$ is balanced if and only if the kernel of (B.9) is contained in the kernel of (B.11).*

Remark B.5. — Keep the above notation and hypothesis, by (B.10), we see f is balanced if and only if for any $(m_\sigma)_{\sigma \in J} \in \mathbb{Z}_{\geqslant 0}^{|J|}$, if $u \in U$ killed by $\prod_{\sigma \in J} \prod_{j=0}^{m_\sigma - 1} (h_\sigma - j)$, then $(\prod_{\sigma \in J} X_{-,\sigma}^{m_\sigma}) \cdot (\iota(f)(u)) = 0 \in V$.

Let $\mathfrak{t}'_J$ be the Lie subalgebra of $\mathfrak{t}_J$ generated by $\{h_\sigma\}_{\sigma \in J}$.

LEMMA B.6. — *Keep the above notation and hypothesis, and suppose the strong dual U'_b is a torsion free $\mathsf{U}(\mathfrak{t}'_J)$-module, then $I_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U) \cong (\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U)^{J-\mathrm{an}}$. In particular, in this case we have (by Theorem B.3)*

$\mathrm{Hom}_{T(F_\wp)}(U \otimes_E \delta_B, J_B(V))^{\mathrm{bal}} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)} \left(\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U \right)^{J-\mathrm{an}}, V \right).$

Proof. — Since U'_b is a torsion free $\mathsf{U}(\mathfrak{t}'_J)$-module, for any $0 \neq X \in \mathsf{U}(\mathfrak{t}'_J)$, the morphism $U \to U$, $u \mapsto Xu$ is surjective. Consequently, we see that (B.10) (and hence (B.9)) is surjective. The image of (B.6) is thus equal to $\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U)$. As in [28, §2.4], for a local closed subrepresentation (see loc. cit. for the definition) X of $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U)^{\mathbb{Q}_p-\mathrm{an}}$, denote by X_e the stalk of X at neutral element $e \in \mathrm{GL}_2(F_\wp)/\overline{B}(F_\wp)$. One has the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{J-\mathrm{pol}}(N(F_\wp), U) & \longrightarrow & \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U \right)^{J-\mathrm{an}}_e \\
\downarrow & & \downarrow \\
\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N(F_\wp), U) & \longrightarrow & \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U \right)^{\mathbb{Q}_p-\mathrm{an}}_e
\end{array}
$$

where the horizontal maps are injections with dense image as in [28, (2.5.20)], and the vertical maps are natural embeddings. By [28, Prop. 2.8.10], $I_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U)_e$ is the closure of the image of (B.6) in

$(\text{Ind}_{\overline{B}(F_\wp)}^{\text{GL}_2(F_\wp)} U)_e^{\mathbb{Q}_p-\text{an}}$, which thus coincides with $(\text{Ind}_{\overline{B}(F_\wp)}^{\text{GL}_2(F_\wp)} U)_e^{J-\text{an}}$, from which we deduce that $I_{\overline{B}(F_\wp)}^{\text{GL}_2(F_\wp)}(U) \cong (\text{Ind}_{\overline{B}(F_\wp)}^{\text{GL}_2(F_\wp)} U)^{J-\text{an}}$. $\qquad\square$

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