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A FINITE DIMENSIONAL APPROACH TO BRAMHAM'S APPROXIMATION THEOREM

by Patrice LE CALVEZ (*)

Abstract. — Using pseudoholomorphic curve techniques from symplectic geometry, Barney Bramham proved recently that every smooth irrational pseudo-rotation of the unit disk is the limit, for the C^0 topology, of a sequence of smooth periodic diffeomorphisms. We give here a finite dimensional proof of this result that works in the case where the pseudo-rotation is smoothly conjugate to a rotation on the boundary circle. The proof extends to C^1 pseudo rotations and is based on the dynamical study of the gradient flow associated to a generating family of functions given by Chaperon's broken geodesics method.

Résumé. — À l'aide de la théorie des courbes pseudo-holomorphes de la géométrie symplectique, Barney Bramham a récemment montré que toute pseudo-rotation irrationnelle lisse du disque unité est limite, pour la topologie C^0, d'une suite de difféomorphismes lisses périodiques. Nous donnons ici une preuve du résultat dans un cadre de dimension finie, valable quand la pseudo-rotation est différentiablement conjuguée à une rotation sur le bord du disque. La preuve, qui s'étend aux pseudo-rotations de classe C^1, est basée sur l'étude dynamique du flot de gradient associé à une famille génératrice de fonctions, obtenue par la méthode des géodésiques brisées de Chaperon.

1. Introduction

We will denote by $\mathbb{D}$ the closed unit disk of the Euclidean plane and by $\mathbb{S}$ the unit circle. An *irrational pseudo-rotation* is an area preserving homeomorphism f of $\mathbb{D}$ that fixes 0 and that does not possess any other periodic point. To such a homeomorphism is associated an irrational number $\overline{\alpha} \notin \mathbb{Q}/\mathbb{Z}$ characterized by the following: every point admits $\overline{\alpha}$ as a

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rotation number. To give a precise meaning to this sentence, choose a lift $\widetilde{f}$ of $f|_{\mathbb{D}\setminus\{0\}}$ to the universal covering space $\widetilde{\mathbb{D}} = \mathbb{R}\times(0,1]$. There exists $\alpha\in\mathbb{R}$ satisfying $\alpha+\mathbb{Z}=\overline{\alpha}$ such that for every compact set $\Xi\subset\mathbb{D}\setminus\{0\}$, and every $\varepsilon>0$, one can find $N\geqslant 1$ such that

$$n\geqslant N \text{ and } \widetilde{z}\in\pi^{-1}(\Xi)\cap\widetilde{f}^{-n}(\pi^{-1}(\Xi))\Rightarrow\left|\frac{p_1(\widetilde{f}^n(\widetilde{z}))-p_1(\widetilde{z})}{n}-\alpha\right|\leqslant\varepsilon,$$

where $\pi:(\theta,r)\mapsto(r\cos 2\pi\theta, r\sin 2\pi\theta)$ is the covering projection and $p_1:(\theta,r)\mapsto\theta$ the projection on the first factor. In particular the Poincaré rotation number of $f|_{\mathbb{S}}$ is $\overline{\alpha}$. In the case where f is a C^k diffeomorphism, $1\leqslant k\leqslant\infty$, we will say that f is a C^k irrational pseudo-rotation. Constructions of dynamically interesting irrational pseudo-rotations are based on the method of fast periodic approximations, starting from the seminal paper of Anosov and Katok [1] (see [9], [10], [11], [12] for further developments about this method, see [4], [3], [13] for other results on irrational pseudo-rotations).

Barney Bramham has recently proved the following (see [6]):

Theorem 1.1. — *Every C^∞ irrational pseudo-rotation f is the limit, for the C^0 topology, of a sequence of periodic C^∞ diffeomorphisms.*

The result is more precise. Let $(q_n)_{n\geqslant 0}$ be a sequence of positive integers such that the sequence $(q_n\overline{\alpha})_{n\geqslant 0}$ converges to 0 in $\mathbb{T}^1=\mathbb{R}/\mathbb{Z}$. One can construct a sequence of homeomorphisms $(f_n)_{n\geqslant 0}$ fixing 0 and satisfying $(f_n)^{q_n}=\mathrm{Id}$ that converges to f for the C^0 topology. Such a map f_n is C^0 conjugate to a rotation of rational angle (mod. π). Approximating the conjugacy by a C^∞ diffeomorphism permits to approximate f_n by a C^∞ diffeomorphism of the same period.

The proof of Theorem 1.1 uses pseudoholomorphic curve techniques from symplectic geometry. Trying to find a finite dimensional proof of this result is natural, as some results of symplectic geometry admit finite dimensional proofs by the use of generating families. A seminal example is Chaperon's proof of Conley–Zehnder's Theorem via the broken geodesics method (see [8]): if F is the time one map of a Hamiltonian flow on the torus $\mathbb{T}^{2r}=\mathbb{R}^{2r}/\mathbb{Z}^{2r}$, a function can be constructed on a space $\mathbb{T}^{2r}\times\mathbb{R}^{2n}$ whose critical points are in bijection with the contractible fixed points of F. Studying the dynamics of the gradient vector field ξ permits to minimize the number of critical points. Writing F as a composition of diffeomorphisms C^1 close to the identity is the way Chaperon constructs a generating family. Decomposing F in monotone twist maps alternatively positive or negative is another possible way. It is the fact that F is isotopic to the identity that is essential in the construction of the vector field ξ, but in the general case ξ has no reason to be a gradient vector field and its dynamics may be more complicated. Nevertheless, if $r = 1$ the vector field will satisfy some "canonical dissipative properties" and its dynamics can be surprisingly well understood (see [15] for the case where F is decomposed in monotone twist maps). Among the applications, one can note the following approximation result (see [16]): every minimal C^1 diffeomorphism F of $\mathbb{T}^2$ that is isotopic to the identity is a limit for the C^0 topology of a sequence of periodic diffeomorphisms. The proofs given in [6] and [16] share a thing in common: the construction of a foliation satisfying a certain "dynamically transverse property" on which a finite group acts, the approximating map being naturally related to this action. In [6] the foliation is defined on $\mathbb{R} \times \mathbb{D} \times \mathbb{T}^1$ and the leaves are either pseudoholomorphic cylinders or pseudoholomorphic half cylinders transverse to the boundary; in [16], the foliation is singular and naturally conjugate to the foliation by orbits of ξ on an invariant torus. Therefore it is natural to look for a proof of Bramham's theorem by a method close to the one given in [16]. The original proof of Theorem 1.1 is divided in two cases: the case where the restriction of f to $\mathbb{S}$ is smoothly conjugate to a rotation, and the case where it is not. We succeeded to treat the first case, with some improvements due to the fact that we work in the C^1 category but unfortunately could not get the general case. Therefore we will prove:

THEOREM 1.2. — *Every C^1 irrational pseudo-rotation f, whose restriction to $\mathbb{S}$ is C^1 conjugate to a rotation, is the limit, for the C^0 topology, of a sequence of periodic smooth diffeomorphisms.*

Observe that it is sufficient to prove Theorem 1.2 in the case where the restriction to $\mathbb{S}$ is a rotation. Indeed, every C^1 diffeomorphism of $\mathbb{S}$ can be extended to a C^1 area preserving diffeomorphism of $\mathbb{D}$ (see [5] for example). So, every C^1 irrational pseudo-rotation, whose restriction to $\mathbb{S}$ is C^1 conjugate to a rotation, is itself conjugate to a C^1 irrational pseudo-rotation, whose restriction to $\mathbb{S}$ is a rotation.

Let us explain the ideas of the proof. The first difficulty arises from the fact that f is defined on a surface with boundary. If one supposes that $f|_{\mathbb{S}}$ is a rotation, one can extend easily our map to the whole plane. Inside a small neighborhood of $\mathbb{D}$ we extend our map by an integrable polar twist map and outside by a rotation whose angle is irrational (mod. π) and close (but different) from $2\pi\alpha$. This implies that $\mathbb{S}$ is accumulated from outside by invariant circles $S_{p/q}$ on which the map is periodic with a rotation number p/q that is a convergent of α, where $\alpha+\mathbb{Z}=\overline{\alpha}$. Our extended map is piecewise C^1 and one can construct a generating family of functions that are C^1 with Lipschitz derivatives (see Section 2). We could have chosen to decompose f in monotone twist maps in order to apply directly the results of [14] and [15], we have preferred to use a decomposition in maps close to the identity like in [8] to underline the fact that the way we construct the generating family is not important. One knows that for every $q \geqslant 1$, the fixed points set of f^q corresponds to the singular points set of a gradient vector field ξ_q defined on a space E_q depending on q. In particular each circle $S_{p/q} \subset \mathbb{R}^2$ corresponds to a curve $\Sigma_{p/q} \subset E_q$ of singularities of ξ_q. In Section 2 we will recall the immediate properties of ξ_q, in particular its invariance by the natural action of $\mathbb{Z}/q\mathbb{Z}$ on E_q. A crucial point is the fact that ξ_q is A Lipschitz with a constant A that does not depend on q. An important consequence is the existence of a uniform inequality between the L^2 norm of an orbit (the square root of the energy) and its L^∞ norm. In Section 4 we give the proofs of Theorem 1.2. The fundamental result (Proposition 4.1) is the fact that $\Sigma_{p/q}$ bounds a disk $\Delta_{p/q} \subset E_q$ that contains the singular point corresponding to the fixed point 0 and that is invariant by the flow and by the $\mathbb{Z}/q\mathbb{Z}$ action. Moreover the dynamics on $\Delta_{p/q}$ is North-South and the non trivial orbits have the same energy. This energy can be explicitely computed and is small if p/q is a convergent of α. Consequently the vector field is uniformly small on $\Delta_{p/q}$. The approximation map will be related to $\xi_q|_{\Delta_{p/q}}$, as it is done in [16]. It must be noticed that the arguments of this section are nothing but the finite dimensional analogous of the arguments of [6]. The rest of the paper is devoted to the proof of Proposition 4.1. If the vector field would have been C^1, one could have used the results of [15] and the following remark: the set $\{0\} \cup S_{p/q}$ is a *maximal unlinked fixed point set* of f^q, which means that there exists an isotopy from identity to f^q that fixes every point of $\{0\} \cup S_{p/q}$ and there is no larger subset of the fixed point set of f^q that satisfies this property. The vector field being Lipschitz, one must adapt what is known in the C^1 case to this wider situation. In Section 5 we recall the existence of a *dominated structure* by presenting a canonical filtration on the product flow on $E_q \times E_q$, postponing the technical proofs to the appendix. In Section 6 we explain how $\Delta_{p/q}$ appears as an "invariant manifold" of this dominated structure. We have tried to write the article as self-contained as possible.

2. Extension and decomposition of a pseudo-rotation

Let f be an orientation preserving homeomorphism of the Euclidean plane. We will say that f is *untwisted* if the map

$$(x, y) \mapsto (p_1(f(x, y)), y)$$

is a homeomorphism, which means that there exist two continuous functions g, g' on $\mathbb{R}^2$ such that

$$f(x, y) = (X, Y) \Leftrightarrow \begin{cases} x = g(X, y), \\ Y = g'(X, y). \end{cases}$$

In this case, the maps $X \mapsto g(X, y)$ and $y \mapsto g'(X, y)$ are orientation preserving homeomorphisms of $\mathbb{R}$. If moreover, f is area preserving, the continuous form $xdy + YdX$ is exact: there exists a C^1 function $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$g = \frac{\partial h}{\partial y}, \quad g' = \frac{\partial h}{\partial X}.$$

The function h, defined up to an additive constant is a *generating function* of f.

We will be interested in untwisted homeomorphisms satisfying some Lipschitz conditions. Let f be an orientation preserving homeomorphism of the Euclidean plane and $K \geqslant 1$. We will say that f is a K *Lipschitz untwisted homeomorphism* if

(i) f is untwisted;
(ii) f is K bi-Lipschitz;
(iii) the maps $X \mapsto g(X, y)$ and $y \mapsto g'(X, y)$ are K bi-Lipschitz;
(iv) the maps $y \mapsto g(X, y)$ and $X \mapsto g'(X, y)$ are K Lipschitz.

Let f be a C^1 diffeomorphism of $\mathbb{R}^2$ and denote by

$$\mathrm{Jac}(f) = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix}$$

its Jacobian matrix. One can verify that f is a K Lipschitz untwisted homeomorphism if and only if the eigenvalues of the matrix $\mathrm{Jac}(f)^t \mathrm{Jac}(f)$ lie between K^{-2} and K^2 and if the following conditions are fulfilled

$$K^{-1} \leqslant \frac{\partial X}{\partial x} \leqslant K, \quad K^{-1} \leqslant \det(\mathrm{Jac}(f))^{-1} \frac{\partial X}{\partial x} \leqslant K$$

and

$$\left| \frac{\partial X}{\partial y} \right| \leqslant K \frac{\partial X}{\partial x}, \quad \left| \frac{\partial Y}{\partial x} \right| \leqslant K \frac{\partial X}{\partial x}.$$

In particular all these conditions are satisfied if $\mathrm{Jac}(f)$ is (uniformly) sufficiently close to the identity matrix.

Until the end of Section 5 we suppose given a C^1 pseudo-rotation of rotation number $\overline{\alpha}$ that coincides with a rotation on $\mathbb{S}$. We choose a real representant α of $\overline{\alpha}$ and an irrational number $\beta > \alpha$ such that $(\alpha, \beta) \cap \mathbb{Z} = \emptyset$. We extend our map to a homeomorphism f of the whole plane defined in polar coordinates as follows:

$$f(\theta, r) = \begin{cases} (\theta + 2\pi(\alpha + r - 1), r) & \text{if } r \in [1, 1 + \beta - \alpha], \\ (\theta + 2\pi\beta, r) & \text{if } r \geqslant 1 + \beta - \alpha. \end{cases}$$

We get a piecewise C^1 area preserving transformation that satisfies the following properties:

- 0 is the unique fixed point of f;
- there is no periodic point of period q if $(q\alpha, q\beta) \cap \mathbb{Z} = \emptyset$;
- if $(q\alpha, q\beta) \cap \mathbb{Z} \neq \emptyset$, the set of periodic points of period q can be written $\bigcup_{\alpha < p/q < \beta} S_{p/q}$, where $S_{p/q}$ is the circle of center 0 and radius $1 + p/q - \alpha$.

Proposition 2.1. — *For every $K > 1$, one can find a decomposition $f = f_m \circ \cdots \circ f_1$, where each f_i is an area preserving K Lipschitz untwisted homeomorphism that fixes 0 and induces a rotation on every circle of origin 0 and radius $r \geqslant 1$.*

Proof. — Denote by f' the plane homeomorphism defined in polar coordinates as follows:

$$f'(\theta, r) = \begin{cases} (\theta + 2\pi\alpha, r) & \text{if } r \in [0, 1], \\ (\theta + 2\pi(\alpha + r - 1), r) & \text{if } r \in [1, 1 + \beta - \alpha], \\ (\theta + 2\pi\beta, r) & \text{if } r \geqslant 1 + \beta - \alpha. \end{cases}$$

One gets $f' = (f'_{m'})^{m'}$, where

$$f'_{m'}(\theta, r) = \begin{cases} \theta + 2\pi\alpha/m', r) & \text{if } r \in [0, 1], \\ \theta + 2\pi(\alpha + r - 1)/m', r) & \text{if } r \in [1, 1 + \beta - \alpha], \\ (\theta + 2\pi\beta/m', r) & \text{if } r \geqslant 1 + \beta - \alpha. \end{cases}$$

If m' is large enough, $f'_{m'}$ is an area preserving K Lipschitz untwisted homeomorphism. Indeed it induces a diffeomorphism on each surface of equation

$$r \leqslant 1, \quad 1 \leqslant r \leqslant 1 + \beta - \alpha, \quad 1 + \beta - \alpha \leqslant r$$

and the Jacobian at every point is uniformly close to the identity.

One can write $f = f'' \circ f'$ where f'' coincides with the identity outside $\mathbb{D}$ and is an area preserving C^1 diffeomorphism, when restricted to $\mathbb{D}$. To get the proposition one can use the path-connectedness of the group $\mathrm{Diff}^1_{**}(\mathbb{D})$ of area preserving C^1 diffeomorphisms of $\mathbb{D}$ that fix 0 and every point of $\mathbb{S}$, when furnished with the C^1 topology(1). Indeed, for every neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^1_{**}(\mathbb{D})$, one can write $f''|_{\mathbb{D}} = f''_{m''} \circ \cdots \circ f''_1$ where $f''_i \in \mathcal{U}$. Choosing $\mathcal{U}$ sufficiently close to the identity and extending f''_i by the identity map outside $\mathbb{D}$, one gets a decomposition of f'' into area preserving K Lipschitz untwisted homeomorphisms. It remains to write $m = m' + m''$ and to define

$$f_i = \begin{cases} f'_{m'} & \text{if } i \leqslant m', \\ f''_{i-m'} & \text{if } m' < i \leqslant m. \end{cases}$$

□

Remark 2.2. — To each map f_i is naturally associated an isotopy $(f_{i,t})_{t\in[0,2]}$ starting from the identity: writing $f_i(x,y) = (X,Y)$, one sets

$$f_{i,t}(x,y) = \begin{cases} ((1-t)x + tX, y) & \text{if } t \in [0,1] \\ (X, (2-t)y + (t-1)Y) & \text{if } t \in [1,2] \end{cases}$$

One gets an isotopy $(f_t)_{t\in[0,2m]}$ joining the identity to f and fixing the origin by writing

$$f_t = f_{1+i,t-2i} \circ f_i \circ \cdots \circ f_1 \quad \text{if } \ t \in [2i, 2i+2].$$

This isotopy can be lifted to the universal cover $\mathbb{R} \times (0,+\infty)$ of $\mathbb{R}^2 \setminus \{0\}$ to an isotopy from the identity to a certain lift $\widetilde{f}$ of $f|_{\mathbb{R}^2\setminus\{0\}}$. The real rotation number (as explained in the introduction) defined by the restriction of $\widetilde{f}$ to the universal cover $\widetilde{\mathbb{D}} = \mathbb{R} \times (0,1)$ of $\mathbb{D} \setminus \{0\}$ is α.

Remark 2.3. — By choosing m' sufficiently large in the proof of Proposition 1, one can suppose that for every $i \in \{1,\dots,m\}$ and every $r \geqslant 1$, the rotation that coincides with f_i on the circle of origin 0 and radius r is a K Lipschitz untwisted homeomorphism. This fact will be used in Section 5.

3. The generating family and the gradient flow

We fix $K > 1$ and a decomposition $f = f_m \circ \cdots \circ f_1$ given by Proposition 2.1. We define two families $(g_i)_{1\leqslant i\leqslant m}$, $(g'_i)_{1\leqslant i\leqslant m}$ of continuous maps

(1) Having been unable to find a written proof of the path-connectedness of $\mathrm{Diff}^1_{**}(\mathbb{D})$ in the litterature, we have written one in the appendix (Lemma 7.1)

as follows

$$f_i(x,y)=(X,Y)\Leftrightarrow\begin{cases}x=g_i(X,y),\\ Y=g'_i(X,y),\end{cases}$$

and a family $(h_i)_{1\leqslant i\leqslant m}$ of C^1 maps, such that

$$g_i=\frac{\partial h_i}{\partial y},\quad g'_i=\frac{\partial h_i}{\partial X}.$$

We extend the families

$$(f_i)_{1\leqslant i\leqslant m},\quad (g_i)_{1\leqslant i\leqslant m},\quad (g'_i)_{1\leqslant i\leqslant m},\quad (h_i)_{1\leqslant i\leqslant m},$$

to m periodic families

$$(f_i)_{i\in\mathbb{Z}},\quad (g_i)_{i\in\mathbb{Z}},\quad (g'_i)_{i\in\mathbb{Z}},\quad (h_i)_{i\in\mathbb{Z}}.$$

We fix in this section an integer $q\geqslant 2$ such that $(q\alpha,q\beta)\cap\mathbb{Z}\neq\emptyset$. To lighten the notations, unlike in the introduction we do not refer to q while defining objects. We consider the finite dimensional vector space

$$E=\left\{\mathbf{z}=(z_i)_{i\in\mathbb{Z}}\in(\mathbb{R}^2)^{\mathbb{Z}}\;\middle|\;z_{i+mq}=z_i,\quad\text{for all }\ i\in\mathbb{Z}\right\},$$

furnished with the scalar product

$$\langle(z_i)_{i\in\mathbb{Z}},(z'_i)_{i\in\mathbb{Z}}\rangle=\sum_{0<i\leqslant mq}x_ix'_i+y_iy'_i,$$

where $z_i=(x_i,y_i)$ and $z'_i=(x'_i,y'_i)$. We denote by $\|\ \|$ the associated Euclidean norm and write

$$d(\mathbf{z},Z)=\inf_{\mathbf{z}'\in Z}\|\mathbf{z}-\mathbf{z}'\|$$

for the distance of a point $\mathbf{z}$ to a set $Z\subset E$.

We define on E a vector field $\xi=(\xi_i)_{i\in\mathbb{Z}}$ by writing

$$\xi_i(\mathbf{z})=\left(y_i-g'_{i-1}(x_i,y_{i-1}),\ x_i-g_i(x_{i+1},y_i)\right).$$

Note that ξ is invariant by the (q periodic) shift

$$\begin{aligned}\varphi:E&\to E\ ,\\ (z_i)_{i\in\mathbb{Z}}&\mapsto(z_{i+m})_{i\in\mathbb{Z}}.\end{aligned}$$

Indeed, φ being a linear map, one has

$$\begin{aligned}\xi(\varphi(\mathbf{z}))&=\left(y_{i+m}-g'_{i-1}(x_{i+m},y_{i+m-1}),\,x_{i+m}-g_i(x_{i+m+1},y_{i+m})\right)_{i\in\mathbb{Z}}\\ &=\left(y_{i+m}-g'_{i+m-1}(x_{i+m},y_{i+m-1}),\,x_{i+m}-g_{i+m}(x_{i+m+1},y_{i+m})\right)_{i\in\mathbb{Z}}\\ &=\varphi(\xi(\mathbf{z}))\\ &=D\varphi(\mathbf{z}).\xi(\mathbf{z}).\end{aligned}$$

In this section, we will state some easy facts about ξ.

LEMMA 3.1. — *The vector field is A Lipschitz, where $A = \sqrt{6K^2+3}$.*

Proof. — For every $\mathbf{z} = (z_i)_{i\in\mathbb{Z}}$ and $\mathbf{z}' = (z'_i)_{i\in\mathbb{Z}}$, one has

$$\begin{aligned}\big|(y_i - g'_{i-1}(x_i, y_{i-1})) - (y'_i - g'_{i-1}(x'_i, y'_{i-1}))\big| \\ \leqslant |y_i - y'_i| + K|x_i - x'_i| + K|y_{i-1} - y'_{i-1}|\end{aligned}$$

and

$$\begin{aligned}\big|(x_i - g_i(x_{i+1}, y_i)) - (x'_i - g_i(x'_{i+1}, y'_i))\big| \\ \leqslant |x_i - x'_i| + K|x_{i+1} - x'_{i+1}| + K|y_i - y'_i|.\end{aligned}$$

By Cauchy–Schwarz inequality, one knows that

$$(a+b+c)^2 \leqslant 3(a^2+b^2+c^2),$$

which implies that

$$\begin{aligned}\|\xi_i(\mathbf{z}) - \xi_i(\mathbf{z}')\|^2 \leqslant\ & 3(K^2+1)(|x_i - x'_i|^2 + |y_i - y'_i|^2) \\ & + 3K^2(|x_{i+1} - x'_{i+1}|^2 + |y_{i-1} - y'_{i-1}|^2)\end{aligned}$$

and that

$$\|\xi(\mathbf{z}) - \xi(\mathbf{z}')\|^2 \leqslant (6K^2+3)\|\mathbf{z} - \mathbf{z}'\|^2. \qquad \square$$

One deduces that the associated differential system

$$\begin{cases}\dot{x}_i = y_i - g'_{i-1}(x_i, y_{i-1}), \\ \dot{y}_i = x_i - g_i(x_{i+1}, y_i),\end{cases}$$

defines a flow on E. We will denote by $\mathbf{z}^t$ the image at time t of a point $\mathbf{z} \in E$ by this flow, and more generally by Z^t the image of a subset $Z \subset E$.

LEMMA 3.2. — *For every $(\mathbf{z}, \mathbf{z}') \in E^2$ and every $t \in \mathbb{R}$, one has*

$$e^{-A|t|}\|\mathbf{z} - \mathbf{z}'\| \leqslant \|\mathbf{z}^t - \mathbf{z}'^t\| \leqslant e^{A|t|}\|\mathbf{z} - \mathbf{z}'\|$$

and

$$e^{-A|t|}\|\xi(\mathbf{z})\| \leqslant \|\xi(\mathbf{z}^t)\| \leqslant e^{A|t|}\|\xi(\mathbf{z})\|.$$

Proof. — Let us begin with the first double inequality. For every $(\mathbf{z}, \mathbf{z}') \in E^2$ and $t \geqslant 0$, one has

$$\begin{aligned}\|\mathbf{z}^t - \mathbf{z}'^t\| &\leqslant \|\mathbf{z} - \mathbf{z}'\| + \|(\mathbf{z}^t - \mathbf{z}'^t) - (\mathbf{z} - \mathbf{z}')\| \\ &= \|\mathbf{z} - \mathbf{z}'\| + \left\|\int_0^t \xi(\mathbf{z}^s) - \xi(\mathbf{z}'^s)\,ds\right\| \\ &\leqslant \|\mathbf{z} - \mathbf{z}'\| + A\int_0^t \|\mathbf{z}^s - \mathbf{z}'^s\|\,ds\end{aligned}$$

which implies by Gronwall's Lemma that $\|\mathbf{z}^t - \mathbf{z}'^t\| \leqslant e^{At}\|\mathbf{z} - \mathbf{z}'\|$. The inequality on the right, for $t \leqslant 0$ can be proven similarly and the inequality on the left can be deduced immediately from the one on the right.

Writing this double inequality with $\mathbf{z}' = \mathbf{z}^s$, dividing by s and letting s tend to 0 permits to obtain the second double inequality. □

For every $i \in \mathbb{Z}$, define the maps

$$\begin{aligned} Q_i : E &\to \mathbb{R}^2, \\ \mathbf{z} &\mapsto (g_i(x_{i+1}, y_i), y_i), \end{aligned}$$

and

$$\begin{aligned} Q'_i : E &\to \mathbb{R}^2, \\ \mathbf{z} &\mapsto (x_i, g'_{i-1}(x_i, y_{i-1})). \end{aligned}$$

By definition of g_i and g'_i, one knows that

$$f_i((g_i(x_{i+1}, y_i), y_i) = (x_{i+1}, g'_i(x_{i+1}, y_i)),$$

which means that $f_i \circ Q_i(\mathbf{z}) = Q'_{i+1}(\mathbf{z})$. Observe also that

$$\xi_i = J \circ (Q'_i - Q_i),$$

where $J(x, y) = (-y, x)$. In particular $\mathbf{z}$ is a singularity of ξ if and only if $Q_i(\mathbf{z}) = Q'_i(\mathbf{z})$ for every $i \in \mathbb{Z}$. One deduces that Q_1 induces a bijection between the set of singularities of ξ and the set of fixed points of f^q. Indeed, if $\mathbf{z}$ is a singularity of ξ, we have $f_i \circ Q_i(\mathbf{z}) = Q_{i+1}(\mathbf{z})$ and consequently

$$f^q \circ Q_1(\mathbf{z}) = f_{mq} \circ \cdots \circ f_1 \circ Q_1(\mathbf{z}) = Q_{mq+1}(\mathbf{z}) = Q_1(\mathbf{z}).$$

Conversely, suppose that $f^q(z) = z$, and consider the sequence $\mathbf{z} = (z_i)_{i\in\mathbb{Z}}$, where $z_1 = z$ and $z_{i+1} = f_i(z_i)$. It belongs to E because $f^q(z) = z$ and is a singularity of ξ because $g_i(x_{i+1}, y_i) = x_i$ and $g'_i(x_{i+1}, y_i) = y_{i+1}$. Moreover we have $Q_1(\mathbf{z}) = (g_1(x_2, y_1), y_1) = (x_1, y_1) = z$. The set of singularities consists of the constant sequence $\mathbf{0} = (0)_{i\in\mathbb{Z}}$, whose image by Q_1 is the common fixed point 0 of all f_i, and of finitely many smooth closed curves $(\Sigma_p)_{p\in(q\alpha,q\beta)\cap\mathbb{Z}}$, each curve Σ_p being sent homeomorphically onto $S_{p/q}$ by Q_1 (and in fact by each Q_i or Q'_i). Observe that ξ is C^1 in a neighborhood of $\mathbf{0}$ and C^∞ in a neighborhood of Σ_p because each f_i is a C^1 diffeomorphism in a neighborhood of 0 and a C^∞ diffeomorphism in a neighborhood of $S_{p/q}$.

Observe that ξ is the gradient vector field of the function

$$\mathbf{h} : \mathbf{z} \mapsto \sum_{0<i\leqslant mq} x_i y_i - h_{i-1}(x_i, y_{i-1})$$

and that $\mathbf{h}$ is invariant by φ. The vector field ξ being a gradient vector field, one can define the *energy* of an orbit $(\mathbf{z}^t)_{t\in\mathbb{R}}$ to be

$$\int_{-\infty}^{+\infty} \|\xi(\mathbf{z}^t)\|^2\,dt = \lim_{t\to+\infty} \mathbf{h}(\mathbf{z}^t) - \lim_{t\to-\infty} \mathbf{h}(\mathbf{z}^t).$$

LEMMA 3.3. — *For every* $\mathbf{z}\in E$*, one has*

$$\|\xi(\mathbf{z})\|^2 \leqslant A\int_{-\infty}^{+\infty} \|\xi(\mathbf{z}^t)\|^2\,dt = A\left(\lim_{t\to+\infty} \mathbf{h}(\mathbf{z}^t) - \lim_{t\to-\infty} \mathbf{h}(\mathbf{z}^t)\right).$$

Proof. — It is an immediate consequence of the inequality

$$e^{-A|t|}\|\xi(\mathbf{z})\| \leqslant \|\xi(\mathbf{z}^t)\|$$

given by Lemma 3.2. □

LEMMA 3.4. — *For every* $\mathbf{z}\in\Sigma_p$*, one has*

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{0}) = \pi(p-q\alpha)\left(1+(p/q-\alpha)+\frac{(p/q-\alpha)^2}{3}\right).$$

Proof. — Recall that Q_1 sends Σ_p onto the circle $S_{p/q}$ and denote by $D_{p/q}\subset\mathbb{R}^2$ the disk bounded by $S_{p/q}$. The quantity $\mathbf{h}(\mathbf{z})-\mathbf{h}(\mathbf{0})$ is the difference of *action* between the two corresponding fixed points of f^q. It is equal to the opposite of the area displaced by an arc joining 0 to $Q_1(z)$ along an isotopy of $\mathbb{R}^2$ that fixes 0 and $Q_1(z)$. In our case, it is independent of $\mathbf{z}$, the set Σ_p being contained in the critical set of $\mathbf{h}$, and its dynamical meaning is the following: it is equal to the opposite of the rotation number of the Lebesgue measure in the annulus $D_{p/q}\setminus\{0\}$ defined by the map f^q for the lift to the universal covering space that fixes all the points of the boundary line. It is easy to compute. The rotation number of the Lebesgue measure restricted to $\mathbb{D}\setminus\{0\}$ is equal to $\pi(q\alpha-p)$, therefore:

$$\begin{aligned}\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{0}) &= \pi(p-q\alpha) - \int_1^{1+p/q-\alpha} 2\pi r((r+\alpha-1)q-p)\,dr\\ &= \pi(p-q\alpha)\left(1+(p/q-\alpha)+\frac{(p/q-\alpha)^2}{3}\right).\end{aligned}$$ □

4. The main proposition and its consequences

In this section we will give a proof of Theorem 1.2. It will follow from Proposition 4.1, whose proof is postponed to Section 6. The arguments are very close to the ones given by Bramham in [6]: we replace pseudoholomorphic curves by orbits of the gradient flow but the spirit of the proof is the same.

Proposition 4.1. — *The curve Σ_p bounds a topological disk $\Delta_p \subset E_q$ that satisfies the following:*

(i) *Δ_p contains the constant sequence $\mathbf{0}$;*
(ii) *Δ_p is invariant by φ;*
(iii) *each projection $\mathbf{z} \mapsto (x_i, y_{i-1})$, $i \in \mathbb{Z}$, is one to one on Δ_p;*
(iv) *each projection $\mathbf{z} \mapsto (x_i, y_i)$, $i \in \mathbb{Z}$, is one to one on Δ_p;*
(v) *Δ_p is invariant by the flow;*
(vi) *for every $\mathbf{z} \in \Delta_p \setminus (\{\mathbf{0}\} \cup \Sigma_p)$, one has $\lim_{t\to-\infty} \mathbf{z}^t = \mathbf{0}$ and $\lim_{t\to+\infty} d(\mathbf{z}^t, \Sigma_p) = 0$.*

Let us explain now why this proposition implies Theorem 1.2.

Proof of Theorem 1.2. — Let us begin by writing

$$C(p,q) = \pi(p - q\alpha)\left(1 + (p/q - \alpha) + \frac{(p/q-\alpha)^2}{3}\right).$$

The assertion (iii) tells us that the maps $Q_i|_{\Delta_p}$ and $Q'_i|_{\Delta_p}$, $i \in \mathbb{Z}$, induce homeomorphisms from Δ_p to $D_{p/q}$. One gets a family of homeomorphisms $(\widehat{f}_i)_{i\in\mathbb{Z}}$ of $D_{p/q}$ by writing:

$$\widehat{f}_i = (Q_{i+1}|_{\Delta_p}) \circ (Q_i|_{\Delta_p})^{-1}.$$

This family is m periodic because Δ_p is invariant by φ. Moreover $\widehat{f} = \widehat{f}_m \circ \cdots \circ \widehat{f}_1$ is q periodic because

$$\widehat{f} = (Q_{m+1}|_{\Delta_p}) \circ (Q_1|_{\Delta_p})^{-1} = (Q_1|_{\Delta_p}) \circ (\varphi|_{\Delta_q}) \circ (Q_1|_{\Delta_p})^{-1}.$$

Observe now that

$$f_i|_{D_{p/q}} = (Q'_{i+1}|_{\Delta_p}) \circ (Q_i|\Delta_p)^{-1}$$

and that

$$\widehat{f}_i - f_i|_{D_{p/q}} = J \circ \xi_{i+1} \circ (Q_i|\Delta_p)^{-1}.$$

By Lemma 3.3 and Lemma 3.4, one deduces that

$$\sup_{z\in D_{p/q}} |\widehat{f}_i(z) - f_i(z)| \leqslant A^{1/2} C(p,q)^{1/2}.$$

Observe also that $\widehat{f}_i$ fixes 0 and coincides with f_i on $S_{p/q}$.

On the disk $D_{p/q}$, one can write

$$\begin{aligned}\widehat{f} - f = {} & \widehat{f}_m \circ \widehat{f}_{m-1} \circ \cdots \circ \widehat{f}_1 - f_m \circ \widehat{f}_{m-1} \circ \cdots \circ \widehat{f}_1 \\ & + f_m \circ \widehat{f}_{m-1} \circ \cdots \circ \widehat{f}_1 - f_m \circ f_{m-1} \circ \cdots \circ \widehat{f}_1 \\ & + \cdots\cdots\cdots \\ & + f_m \circ \cdots \circ f_2 \circ \widehat{f}_1 - f_m \circ \cdots \circ f_2 \circ f_1.\end{aligned}$$

By definition of K Lipschitz untwisted homeomorphisms, one gets

$$\sup_{z\in D_{p/q}} |\widehat{f}(z)-f(z)| \leqslant (1+K+\cdots+K^{m-1})A^{1/2}C(p,q)^{1/2}.$$

Let us consider now the homothety H of ratio $1+p/q-\alpha$ and set $\check{f}=H^{-1}\circ\widehat{f}\circ H$. On the disk $\mathbb{D}$ one can write

$$\check{f}-f=(H^{-1}\circ\widehat{f}\circ H-H^{-1}\circ f\circ H)+(H^{-1}\circ f\circ H-f\circ H)+(f\circ H-f).$$

Using the fact that f is K^m Lipschitz, one gets

$$\sup_{z\in\mathbb{D}}|\check{f}(z)-f(z)| \leqslant (1+K+\cdots+K^{m-1})A^{1/2}C(p,q)^{1/2}+(1+K^m)(p/q-\alpha).$$

We will get the same upper bound for $\sup_{z\in\mathbb{D}}|\check{f}^{-1}(z)-f^{-1}(z)|$. One can choose p and q, with $p-q\alpha$ arbitrarily small, which means that the quantity on the right side itself can be chosen arbitrarily small. □

Remark 4.2. — Keeping the notation above, one gets

$$\begin{aligned}\mathrm{Id}-f^q \;=\; & \widehat{f}_{mq}\circ\widehat{f}_{mq-1}\circ\cdots\circ\widehat{f}_1 \;-\; f_m\circ\widehat{f}_{mq-1}\circ\cdots\circ\widehat{f}_1\\ &+\; f_{mq}\circ\widehat{f}_{mq-1}\circ\cdots\circ\widehat{f}_1 \;-\; f_{mq}\circ f_{mq-1}\circ\cdots\circ\widehat{f}_1\\ &+\;\cdots\cdots\cdots\\ &+\; f_{mq}\circ\cdots\circ f_2\circ\widehat{f}_1 \;-\; f_{mq}\circ\cdots\circ f_2\circ f_1.\end{aligned}$$

which implies that

$$\sup_{z\in D_{p/q}} |z-f^q(z)| \leqslant (1+K+\cdots+K^{mq-1})A^{1/2}C(p,q)^{1/2},$$

and that

$$\sup_{z\in\mathbb{D}}|z-f^q(z)| \leqslant (1+K+\cdots+K^{mq-1})A^{1/2}C(p,q)^{1/2}+(1+K^{mq})(p/q-\widetilde{\alpha}).$$

One gets a similar inequality for $\sup_{z\in\mathbb{D}}|z-f^{-q}(z)|$. Extending in a similar way our original diffeomorphism with the help of a negative polar twist map will give us a similar inequality for couples (p,q) such that $p/q<\alpha$. If α satisfies the following super Liouville condition: for every $\mu\in(0,1)$, there exists two sequences of integers $(q_n)_{n\geqslant 0}$ and $(p_n)_{n\geqslant 0}$, with $q_n>0$, such that $|q_n\alpha-p_n|\leqslant\mu^{q_n}$, then there exists a sequence $(r_n)_{n\geqslant 0}$ such that $(f^{r_n})_{n\geqslant 0}$ converges to the identity on $\mathbb{D}$ for the C^0 topology. One says that f is C^0 *rigid*. This is a C^1 version, but with the additional assumption of being C^1 conjugate to the rotation, of the following recent result of Bramham [7]:

THEOREM 4.3. — *Every C^∞ irrational pseudo rotation f of rotation number $\overline{\alpha}$ is C^0 rigid if it satisfies the following super Liouville condition:*

for every $\mu \in (0,1)$*, there exists a sequence of integers* $(q_n)_{n\geqslant 0}$ *such that* $d(q_n\overline{\alpha}, 0) \leqslant \mu^{q_n}$*.*

5. Canonical dominated structure for the gradient flow

In this section, we will do a deeper study of the vector field ξ. The fact that ξ is a gradient flow has no importance here. What is crucial is the fact that ξ is *tridiagonal and monotonically symmetric* (we will explain the meaning of this). We refer to [14] or [15] for detailed proofs. In what follows, the function sign assigns $+1$ to a positive number and -1 to a negative number.

Let us consider the set

$$V = \left\{\mathbf{z} \in E \;\middle|\; x_i \neq 0 \text{ and } y_i \neq 0 \text{ for all } i \in \mathbb{Z}\right\}$$

and the function L on V defined by the formula

$$\begin{aligned} L(\mathbf{z}) &= \frac{1}{4}\sum_{0<i\leqslant mq} \operatorname{sign}(x_i)\left(\operatorname{sign}(y_i) - \operatorname{sign}(y_{i-1})\right), \\ &= \frac{1}{4}\sum_{0<i\leqslant mq} \operatorname{sign}(y_i)\left(\operatorname{sign}(x_i) - \operatorname{sign}(x_{i+1})\right). \end{aligned}$$

It extends continuously to the open set

$$W = \left\{\mathbf{z} \in E \;\middle|\; x_i = 0 \Rightarrow y_{i-1}y_i > 0,\ y_i = 0 \Rightarrow x_i x_{i+1} > 0\right\}.$$

Let us first explain the meaning of L. For every $z \in E$, one can define a loop $\gamma_{\mathbf{z}} : [0, 2mq] \to \mathbb{R}^2$ by writing:

$$\gamma_{\mathbf{z}}(t) = \begin{cases} ((1+2i-t)x_i + (t-2i)x_{i+1}, y_i) & \text{if } t \in [2i, 2i+1], \\ (x_{i+1}, (2+2i-t)y_i + (t-2i-1)y_{i+1}) & \text{if } t \in [2i+1, 2i+2]. \end{cases}$$

The fact that $\mathbf{z}$ belongs to W means that the image of this loop does not meet 0. Write $0x^+$, $0x^-$, $0y^+$, $0y^-$ for the half lines generated by the vectors $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$ respectively. The formulas given above tell us that

$$L(\mathbf{z}) = \frac{1}{2}(0x^+ \wedge \gamma_z + 0x^- \wedge \gamma_z) = \frac{1}{2}(0y^+ \wedge \gamma_z + 0y^- \wedge \gamma_z),$$

where $\wedge$ means the algebraic intersection number. The integer $L(z)$ is nothing but the indice of the loop $\gamma_{\mathbf{z}}$ relative to 0. In particular, L is integer valued and takes its values in $\{-[mq/2], \dots, [mq/2]\}$.

Let us state the fundamental result, whose proof is postponed to the appendix:

Proposition 5.1. — *If* $\mathbf{z}$, $\mathbf{z}$ *are two distinct points of* E *such that* $\mathbf{z}'-\mathbf{z}\notin W$, *then there exists* $\varepsilon>0$ *such that* ${\mathbf{z}'}^t-\mathbf{z}^t\in W$ *if* $0<|t|\leqslant\varepsilon$. *Moreover, for every* $t\in(0,\varepsilon]$, *one has*

$$L({\mathbf{z}'}^t-\mathbf{z}^t)=L({\mathbf{z}'}^\varepsilon-\mathbf{z}^\varepsilon)>L({\mathbf{z}'}^{-\varepsilon}-\mathbf{z}^{-\varepsilon})=L({\mathbf{z}'}^{-t}-\mathbf{z}^{-t}).$$

For every $p\in\{-[mq/2],\dots,[mq/2]\}$, let us write

$$W_p=\{\mathbf{z}\in W\mid L(\mathbf{z})=p\}$$

and define

$$W_p^+=\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{p'\geqslant p}W_{p'}\right)\right),\ W_p^-=\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{p'\leqslant p}W_{p'}\right)\right),$$

where Int and Cl mean the interior and the closure respectively. Similarly write

$$\begin{aligned}\mathcal{W}_p&=\{(\mathbf{z},\mathbf{z}')\in E\times E\mid \mathbf{z}'-\mathbf{z}\in W_p\},\\ \mathcal{W}_p^+&=\{(\mathbf{z},\mathbf{z}')\in E\times E\mid \mathbf{z}'-\mathbf{z}\in W_p^+\},\\ \mathcal{W}_p^-&=\{(\mathbf{z},\mathbf{z}')\in E\times E\mid \mathbf{z}'-\mathbf{z}\in W_p^-\}.\end{aligned}$$

Proposition 5.1 gives us a canonical filtration on the product flow defined on $E\times E\setminus\mathrm{diag}$, where $\mathrm{diag}=\{(\mathbf{z},\mathbf{z}')\in E\times E\mid\mathbf{z}=\mathbf{z}'\}$. Each set $\mathcal{W}_p^+$ is an attracting set of the product flow on $E\times E\setminus\mathrm{diag}$ and each set $\mathcal{W}_p^-$ a repulsing set. More precisely, if $(\mathbf{z},\mathbf{z}')\in\mathrm{Cl}(\mathcal{W}_p^+)\setminus\mathrm{diag}$, then $(\mathbf{z}^t,\mathbf{z}'^t)\in\mathcal{W}_p^+$ for every $t>0$; if $(\mathbf{z},\mathbf{z}')\in\mathrm{Cl}(\mathcal{W}_p^-)\setminus\mathrm{diag}$, then $(\mathbf{z}^t,\mathbf{z}'^t)\in\mathcal{W}_p^-$ for every $t<0$. Consequently, the boundary of $\mathcal{W}_p^+$ and $\mathcal{W}_p^-$ in $E\times E\setminus\mathrm{diag}$ are 1 codimensional topological submanifolds.

In particular, if $\mathbf{z}$ and $\mathbf{z}'$ are two singularities, then $\mathbf{z}'-\mathbf{z}\in W$ and $L(\mathbf{z}'-\mathbf{z})$ is well defined. Let us explain the meaning of this integer. Recall that to each map $f_i:(x,y)\to(X,Y)$ is naturally associated an isotopy $(f_{i,t})_{t\in[0,2]}$ starting from the identity defined as follows

$$f_{i,t}(x,y)=\begin{cases}((1-t)x+tX,y) & \text{if } t\in[0,1],\\ (X,(2-t)y+(t-1)Y) & \text{if } t\in[1,2],\end{cases}$$

and an isotopy $(f_t^{[q]})_{t\in[0,2mq]}$ joining the identity to f^q, where

$$f_t^{[q]}=f_{1+i,t-2i}\circ f_i\circ\cdots\circ f_1\quad\text{if } t\in[2i,2i+2].$$

The integer $L(\mathbf{z}'-\mathbf{z})$ is equal to the *linking number* of the two corresponding fixed points of f^q for this natural isotopy naturally defined by the decomposition of f, in other words to the topological degree of the map

$$t \mapsto \frac{f_t^{[q]}(\mathbf{z}') - f_t^{[q]}(\mathbf{z})}{|f_t^{[q]}(\mathbf{z}') - f_t^{[q]}(\mathbf{z})|}$$

from the "circle" $[0,2mq]|_{0=2mq}$ to the unit circle. In particular, if $p/q \in (\alpha,\beta)$, then $(\mathbf{0},\mathbf{z}) \in \mathcal{W}_p$ for every $z \in \Sigma_p$ and $(\mathbf{z},\mathbf{z}') \in \mathcal{W}_p$ for every $\mathbf{z}$ and $\mathbf{z}'$ in Σ_p.

The reason why Proposition 5.1 is true due to the fact that the vector field is tridiagonal and monotonically symmetric. Writing the coordinates in the following order

$$\dots, y_{i-1}, x_i, y_i, x_{i+1}, \dots$$

the corresponding coordinate of ξ depends only on this coordinate and its two neighbours. Moreover it depends monotonically on each of the neighbouring coordinates and for two neighbouring coordinates, the "cross monotonicities" are the same. In our example, $\dot{x}_i$ depends only on x_i, y_{i-1} and y_i, is a decreasing function of y_{i-1} and an increasing function of y_i whereas $\dot{y}_i$ depends only on y_i, x_i and x_{i+1}, is an increasing function of x_i and a decreasing function of x_{i+1}. To every tridiagonal and monotonically symmetric vector field is associated a natural function L satisfying Proposition 5.1. An important case is the linear case. Suppose that ξ_* is a linear tridiagonal and monotonically symmetric vector field on our space E. We obtain a dominated splitting (that has been known for a long time, see [17] for example): there exists a linear decomposition

$$E = \bigoplus_{p \in \{-[mq/2],\dots,[mq/2]\}} E_p$$

in invariant subspaces where

$$E_p \setminus \{0\} = \left\{\mathbf{z} \in E \,|\, e^{t\xi_*}(\mathbf{z}) \in W_p \text{ for all } t \in \mathbb{R}\right\},$$

and the real parts of the eigenvalues of $\xi_*|_{E_{*,p'}}$ are larger than the real parts of the eigenvalues of $\xi_*|_{E_p}$, if $p' < p$ (here $e^{t\xi_*}$ means the exponential of the linear map $t\xi_*$). Moreover, the spaces

$$E_p^+ = \bigoplus_{p' \geqslant p} E_{p'}, \quad E_p^- = \bigoplus_{p' \leqslant p} E_{p'}$$

satisfy

$$E_p^+ \setminus \{0\} = \left\{\mathbf{z} \in E \;\middle|\; e^{t\xi_*}(\mathbf{z}) \in W_p^+ \text{ for all } t \in \mathbb{R}\right\}$$

and

$$E_p^- \setminus \{0\} = \left\{\mathbf{z} \in E \;\middle|\; e^{t\xi_*}(\mathbf{z}) \in W_p^- \text{ for all } t \in \mathbb{R}\right\}.$$

In the case of a tridiagonal and monotonically symmetric C^1 vector field, with non zero cross derivatives, one gets such a decomposition of the tangent space at every singularity, for the linearized flow. The proof of Proposition 5.1, in the C^1 case is given in [14]. Starting with two distinct points $\mathbf{z}$ and $\mathbf{z}'$ such that $\mathbf{z}' - \mathbf{z} \notin W$, polynomial approximations obtained by successive integrations permit to determine the sign of the coordinates of $\mathbf{z}'^t - \mathbf{z}^t$, for small values of t. A more precise study, replacing polynomial approximations by explicit inequalities is given in [15, Lemma 2.5.1] and permits to extend Proposition 5.1 to the compactification of $E \times E \setminus \mathrm{diag}$ obtained by blowing up the diagonal and then to get a similar result for the linearized vector field on the tangent bundle. One proves in that way the existence of a global dynamically coherent dominated splitting (the definition is recalled in the next section). The proof of Proposition 5.1 that uses [15, Lemma 2.5.1] extends word to word to our Lipschitz case. However, Proposition 5.2 stated below, needed in the next section, necessitates an extension of this lemma to the case where vanishing conditions on the coordinates of $\mathbf{z}' - \mathbf{z}$, are replaced by smallness conditions. Its proof, postponed in the appendix, will be a direct consequence of Lemma 7.1, the main result of the appendix.

Let us define

$$E^i = \left\{ \mathbf{z} \in E \;\middle|\; i' \neq i \Rightarrow x_{i'} = 0 \;\text{ and }\; i' \neq i-1 \Rightarrow y_{i'} = 0 \right\}$$

and

$$E'^i = \left\{ \mathbf{z} \in E \;\middle|\; i' \neq i \Rightarrow x_{i'} = 0 \;\text{ and }\; i' \neq i \Rightarrow y_{i'} = 0 \right\}.$$

Write $\pi^i : E \to E^i$ and $\pi'^i : E \to E'^i$ for the orthogonal projections on E^i and E'^i respectively, write $\pi^{i\perp} : E \to E^{i\perp}$ and $\pi'^{i\perp} : E \to E'^{i\perp}$ for the orthogonal projections on $E^{i\perp}$ and $E'^{i\perp}$.

Proposition 5.2. — *For every $t > 0$, there exists a constant $N_t > 0$ such that for every pair of points $\mathbf{z}$, $\mathbf{z}'$ in E satisfying $\mathbf{z}'^s - \mathbf{z}^s \in W$ if $s \in [-t, t]$, and for every $i \in \mathbb{Z}$, one has*

$$\|\pi^{i\perp}(\mathbf{z}' - \mathbf{z})\| \leqslant N_t \, \|\pi^i(\mathbf{z}' - \mathbf{z})\|$$

and

$$\|\pi'^{i\perp}(\mathbf{z}' - \mathbf{z})\| \leqslant N_t \, \|\pi'^i(\mathbf{z}' - \mathbf{z})\|.$$

6. Proof of Proposition 4.1

The goal of this section is to prove Proposition 4.1, We will begin with a preliminary result (Proposition 6.1) that states the existence of a topological plane Π_p, containing $\mathbf{0}$ and Σ_p, invariant by φ, such that for every

$t \in \mathbb{R}$ and every $i \in \mathbb{Z}$, the projections $\mathbf{z} \mapsto (x_i, y_{i-1})$, and $\mathbf{z} \mapsto (x_i, y_i)$ send homeomorphically $(\Pi_p)^t$, the image of Π_p by the flow at time t, onto $\mathbb{R}^2$. This last fact will be a consequence of the inclusion

$$\big((\Pi_p)^t \times (\Pi_p)^t\big) \setminus \mathrm{diag} \subset \mathcal{W}_p.$$

The disk $\Delta_p \subset \Pi_p$ bounded by Σ_p satisfies the assertions (i) to (iv) of Proposition 4.1. In the second part of the section, we will prove that it is invariant by the flow (assertion **v)**) and that for every $\mathbf{z} \in \Delta_p \setminus (\{\mathbf{0}\} \cup \Sigma_p)$, one has $\lim_{t\to-\infty} \mathbf{z}^t = \mathbf{0}$ and $\lim_{t\to+\infty} d(\mathbf{z}^t, \Sigma_p) = 0$ (assertion (vi)).

As explained in the previous section, a tridiagonal and monotonically symmetric C^1 vector field on E, with non zero cross derivatives, admits a dominated splitting, which means a decomposition of the tangent bundle

$$TE = \bigoplus_{p\in\{-[mq/2],\dots,[mq/2]\}} E_p(\mathbf{z}),$$

invariant by the linearized flow, with relative expanding properties, and this splitting is dynamically coherent, which means that every field

$$\bigoplus_{p_0\leqslant p\leqslant p_1} E_p(\mathbf{z}),$$

is integrable. This is a consequence of Proposition 5.1 stated in [15]. The coherency is obtained via graph transformations. One can integrate the fields

$$\bigoplus_{p_0\leqslant p} E_p(\mathbf{z}),\quad \bigoplus_{p\leqslant p_1} E_p(\mathbf{z}),$$

and then take the intersection of the integral manifolds. In particular a plane Π_p tangent to the bundle $E_p(\mathbf{z})$ is characterized by the property

$$\big((\Pi_p)^t \times (\Pi_p)^t\big) \setminus \mathrm{diag} \subset \mathcal{W}_p \ \text{ for all } \ t \in \mathbb{R}.$$

Here the vector field is no longer C^1, the decomposition

$$TE = \bigoplus_{p\in\{-[mq/2],\dots,[mq/2]\}} E_p(\mathbf{z})$$

does not exist but fortunately the graph transformation exists: there exist topological manifolds Γ_p^+ and Γ_p^- satisfying

$$\dim(\Gamma_p^+) + \dim(\Gamma_p^-) = mq + 2$$

and such that for every $t \in \mathbb{R}$, one has

$$\big((\Gamma_p^+)^t \times (\Gamma_p^+)^t\big) \setminus \mathrm{diag} \subset \mathcal{W}_p^+,\ \ \big((\Gamma_p^-)^t \times (\Gamma_p^-)^t\big) \setminus \mathrm{diag} \subset \mathcal{W}_p^-.$$

In case where the vector field is C^1, the manifolds Γ_p^+ and Γ_p^- are C^1 and the fact that $\Pi_p = \Gamma_p^+ \cap \Gamma_p^-$ is a C^1 plane follows almost immediately from the Implicit Function Theorem. In our situation, to prove that $\Pi_p = \Gamma_p^+ \cap \Gamma_p^-$ is a plane, we will need a Lipschitz Implicit Function Theorem which means that some Lipschitz conditions about the manifolds Γ_p^+, Γ_p^- and Π_p must be satisfied. This is the reason one needs Proposition 5.2.

PROPOSITION 6.1. — *For every $p \in (q\alpha, q\beta) \cap \mathbb{Z}$, there exists a set $\Pi_p \subset E$, the image of a proper topological embedding of $\mathbb{R}^2$, such that:*

(i) Π_p *contains* $\{\mathbf{0}\} \cup \Sigma_p$*;*
(ii) Π_p *is invariant by* φ*;*
(iii) $((\Pi_p)^t \times (\Pi_p)^t) \setminus \mathrm{diag} \subset \mathcal{W}_p$, *for every* $t \in \mathbb{R}$.

Proof. — As explained in Remark 2.3 at the end of Section 2, the rotation $f_{*,i}$ that coincides with f_i on the circle $S_{p/q}$ is an area preserving K Lipschitz untwisted homeomorphism. One gets a decomposition $f_* = f_{*,m} \circ \cdots \circ f_{*,1}$ of the rotation of angle $2\pi p/q$. To this decomposition is associated a linear vector field ξ_* on E which is the gradient of a quadratic form. Its kernel being homeomorphic to the fixed points set of $(f_*)^q$, is a plane. Since ξ_* coincides with ξ on $\{0\} \cup \Sigma_p$, its kernel contains this set: it is the plane generated by Σ_p. Applying what has been said in Section 5, one knows that there exists a linear (and orthogonal) decomposition

$$E = \bigoplus_{p' \in \{-[mq/2], \dots, [mq/2]\}} E_{p'},$$

where

$$E_{p'} \setminus \{0\} = \left\{ \mathbf{z} \in E \;\middle|\; e^{t\xi_*}(\mathbf{z}) \in W_{p'} \text{ for all } t \in \mathbb{R} \right\},$$

and that the spaces

$$E_{p'}^+ = \bigoplus_{p'' \geqslant p'} E_{p''}, \quad E_{p'}^- = \bigoplus_{p'' \leqslant p'} E_{p''},$$

satisfy

$$E_{p'}^+ \setminus \{0\} = \left\{ \mathbf{z} \in E \;\middle|\; e^{t\xi_*}(\mathbf{z}) \in W_{p'}^+ \text{ for all } t \in \mathbb{R} \right\}$$

and

$$E_{p'}^- \setminus \{0\} = \left\{ \mathbf{z} \in E \;\middle|\; e^{t\xi_*}(\mathbf{z}) \in W_{p'}^- \text{ for all } t \in \mathbb{R} \right\}.$$

Observe that E_p is the kernel of ξ_* and that E_p, E_p^+ and E_p^- are invariant by φ because ξ_* is invariant by φ. Write

$$\pi_p^+ : E \to E_p^+, \quad \pi_p^- : E \to E_p^-$$

for the orthogonal projections. Every vector "sufficiently close" to E_p^+ or E_p^- must belong to W_p^+ or W_p^- respectively. This implies that there exists a constant $M \geqslant 1$ such that:

$$(\mathbf{z}, \mathbf{z}') \in \mathcal{W}_p^+ \Rightarrow \|\pi_{p-1}^-(\mathbf{z}) - \pi_{p-1}^-(\mathbf{z}')\| \leqslant M \|\pi_p^+(\mathbf{z}) - \pi_p^+(\mathbf{z}')\|,$$

and

$$(\mathbf{z},\mathbf{z}')\in\mathcal{W}_p^-\Rightarrow\|\pi^+_{p+1}(\mathbf{z})-\pi^+_{p+1}(\mathbf{z}')\|\leqslant M\|\pi^-_p(\mathbf{z})-\pi^-_p(\mathbf{z}')\|.$$

Identifying the space $E=E_p^+\oplus E_{p-1}^-$ with the product $E_p^+\times E_{p-1}^-$, we write $\Gamma_\psi\subset E$ for the graph of a function $\psi:E_p^+\to E_{p-1}^-$. We define

$$\mathcal{G}_p^+=\left\{\psi:E_p^+\to E_{p-1}^-\;\middle|\;(\Gamma_\psi\times\Gamma_\psi)\setminus\mathrm{diag}\subset\mathcal{W}_p^+\right\}$$

and

$$\overline{\mathcal{G}}_p^+=\left\{\psi:E_p^+\to E_{p-1}^-\;\middle|\;(\Gamma_\psi\times\Gamma_\psi)\setminus\mathrm{diag}\subset\mathrm{Cl}(\mathcal{W}_p^+)\right\}.$$

Note that $\overline{\mathcal{G}}_p^+$ is closed for the compact-open topology, note also that every function in $\overline{\mathcal{G}}_p^+$ is M Lipschitz, by definition of the constant M.

Lemma 6.2. — *The vector field ξ induces a positive semi-flow $(t,\psi)\mapsto\psi^t$ on $\overline{\mathcal{G}}_p^+$, such that $\Gamma_{\psi^t}=(\Gamma_\psi)^t$. Moreover, one has $\psi^t\in\mathcal{G}_p^+$ for every $\psi\in\overline{\mathcal{G}}_p^+$ and every $t>0$.*

Proof. — By Proposition 5.1, for every $\psi\in\overline{\mathcal{G}}_p^+$ and every $t\geqslant 0$, one has

$$\left((\Gamma_\psi)^t\times(\Gamma_\psi)^t\right)\setminus\mathrm{diag}\subset\mathrm{Cl}(\mathcal{W}_p^+).$$

Consequently $(\Gamma_\psi)^t$ projects injectively into E_p^+. In fact it projects surjectively. Indeed, the map

$$\mathbf{z}\mapsto\pi_p((\mathbf{z}+\psi(\mathbf{z}))^t)$$

is more than an injective and continuous transformation of E_p^+. One deduces from Lemma 3.2 that it is Me^{At} bi-Lipschitz. In particular it is a homeomorphism of E_p^+. This means that $(\Gamma_\psi)^t$ is the graph of a continuous function $\psi^t\in\mathcal{G}_p^+$. The continuity of the map

$$(t,\psi)\mapsto\psi^t,$$

when $\overline{\mathcal{G}}_p^+$ is furnished with the compact-open topology, follows easily. The fact that ψ^t belongs to $\mathcal{G}_p^+$ for every $t>0$ is an immediate consequence of Proposition 5.1. □

Similarly, one can identify E with the product $E_p^-\times E_{p+1}^+$, write $\Gamma'_{\psi'}\subset E$ for the graph of a function $\psi':E_p^-\to E_{p+1}^+$ and define

$$\mathcal{G}_p^-=\left\{\psi':E_p^-\to E_{p+1}^+\;\middle|\;(\Gamma'_{\psi'}\times\Gamma'_{\psi'})\setminus\mathrm{diag}\subset\mathcal{W}_p^-\right\}$$

and

$$\overline{\mathcal{G}}_p^-=\left\{\psi:E_p^-\to E_{p+1}^+\;\middle|\;(\Gamma'_{\psi'}\times\Gamma'_{\psi'})\setminus\mathrm{diag}\subset\mathrm{Cl}(\mathcal{W}_p^-)\right\}.$$

We can define a negative semi-flow $(t,\psi')\mapsto\psi'^t$ on $\overline{\mathcal{G}}_p^-$ such that $\Gamma'_{\psi'^t}=(\Gamma'_{\psi'})^t$ and we have $\psi'^t\in\mathcal{G}_p^-$ for every $\psi'\in\overline{\mathcal{G}}_p^-$ and every $t<0$.

If $\mathbf{z} \in E^-_{p-1}$ denote by $\psi_{\mathbf{z}} \in \mathcal{G}^+_p$ the constant map equal to $\mathbf{z}$, and similarly if $\mathbf{z}' \in E^+_{p+1}$ denote by $\psi'_{\mathbf{z}'} \in \mathcal{G}^-_p$ the constant map equal to $\mathbf{z}'$. The graphs of these families of maps give us two transverse foliations. Let us study the time evolution of these foliations. For every $t > 0$, every $\mathbf{z} \in E^-_{p-1}$ and every $\mathbf{z}' \in E^+_{p+1}$, we define the set

$$\Pi_{\mathbf{z},\mathbf{z}',t} = \Gamma_{\psi^t_{\mathbf{z}}} \cap \Gamma'_{\psi'^{-t}_{\mathbf{z}'}}.$$

Lemma 6.3. — *Let us fix $i \in \mathbb{Z}$. The set $\Pi_{\mathbf{z},\mathbf{z}',t}$ is the graph of a map $\theta_{i,t} : E^i \to E^{i\perp}$ and the graph of a map $\theta'_{i,t} : E'^i \to E'^{i\perp}$. Moreover, $\theta_{i,t}$ and $\theta'_{i,t}$ are N_t Lipschitz, where N_t is defined by Proposition 5.2.*

Proof. — For every $s \in [-t, t]$, one has

$$(\Pi_{\mathbf{z},\mathbf{z}',t})^s \subset \Gamma_{\psi^{t+s}_{\mathbf{z}}} \cap \Gamma'_{\psi'^{-t+s}_{\mathbf{z}'}},$$

which implies that

$$((\Pi_{\mathbf{z},\mathbf{z}',t})^s \times (\Pi_{\mathbf{z},\mathbf{z}',t})^s) \setminus \mathrm{diag} \subset \mathcal{W}^+_p \cap \mathcal{W}^-_p = \mathcal{W}_p.$$

Therefore $\Pi_{\mathbf{z},\mathbf{z}',t}$ projects injectively on E^i and E'^i. We want to prove that it projects surjectively. Fix $\mathbf{z}_* \in E^i$ and look at the map

$$\begin{aligned} \Theta : E^{i\perp} &\to E^-_{p-1} \times E^+_{p+1}, \\ \mathbf{z} &\mapsto \left(\pi^-_{p-1}((\mathbf{z}_* + \mathbf{z})^{-t}), \pi^+_{p+1}((\mathbf{z}_* + \mathbf{z})^t)\right). \end{aligned}$$

Observe that

$$\mathbf{z}_* + \mathbf{z} \in \Pi_{\pi^-_{p-1}((\mathbf{z}_*+\mathbf{z})^{-t}),\pi^+_{p+1}((\mathbf{z}_*+\mathbf{z})^t),t},$$

which implies that Θ is injective. Let us prove that it is Me^{At} bi-Lipschitz if $E^-_{p-1} \times E^+_{p+1}$ is furnished with the supremum norm. The fact that it is e^{At} Lipschitz is an immediate consequence of Lemma 3.2. To prove that the inverse is Me^{At} Lipschitz, one can note that for every $\mathbf{z}$, $\mathbf{z}'$ in $E^{i\perp}$, one has $(\mathbf{z}^* + \mathbf{z}, \mathbf{z}^* + \mathbf{z}') \notin \mathcal{W}$ because $(\mathbf{z}^* + \mathbf{z}') - (\mathbf{z}^* + \mathbf{z})$ has two vanishing consecutive coordinates. So either

$$(\mathbf{z}^* + \mathbf{z}, \mathbf{z}^* + \mathbf{z}') \in \mathrm{Cl}(\mathcal{W}^-_{p-1})$$

or

$$(\mathbf{z}^* + \mathbf{z}, \mathbf{z}^* + \mathbf{z}') \in \mathrm{Cl}(\mathcal{W}^+_{p+1}).$$

One deduces that

$$\left((\mathbf{z}^* + \mathbf{z})^{-t}, (\mathbf{z}^* + \mathbf{z}')^{-t}\right) \in \mathcal{W}^-_{p-1}$$

or

$$\left((\mathbf{z}^* + \mathbf{z})^{t}, (\mathbf{z}^* + \mathbf{z}')^{t}\right) \in \mathcal{W}^+_{p+1},$$

which implies that

$$\|\pi^-_{p-1}((\mathbf{z}^*+\mathbf{z})^{-t})-\pi^-_{p-1}((\mathbf{z}^*+\mathbf{z}')^{-t})\|\geqslant M^{-1}e^{-At}\|\mathbf{z}-\mathbf{z}'\|$$

or

$$\|\pi^+_{p+1}((\mathbf{z}^*+\mathbf{z})^{-t})-\pi^+_{p+1}((\mathbf{z}^*+\mathbf{z}')^{-t})\|\geqslant M^{-1}e^{-At}\|\mathbf{z}-\mathbf{z}'\|.$$

We deduce from the fact that Θ is Me^{At} bi-Lipschitz that Θ is a homeomorphism, which implies that $\Pi_{\mathbf{z},\mathbf{z}',t}$ is the graph of a function $\theta_{i,t}:E^i\to E^{i\perp}$. By Proposition 5.2, the fact that $((\Pi_{\mathbf{z},\mathbf{z}',t})^s\times(\Pi_{\mathbf{z},\mathbf{z}',t})^s)\setminus\mathrm{diag}\subset W_p$, for every $s\in[-t,t]$, implies that this graph is N_t Lipschitz. We can define a similar map $\theta'_{i,t}:E'^i\to E'^{i\perp}$ and it will also be N_t Lipschitz. □

Let us finish the proof of Proposition 6.1. For every $t\geqslant 0$, define

$$\Pi_{p,t}=\Gamma_{\psi^t_{\mathbf{0}}}\cap\Gamma'_{\psi'^{-t}_{\mathbf{0}}}=\left(E^+_p\right)^t\cap\left(E^-_p\right)^{-t}.$$

Each graph $\Gamma_{\psi^t_{\mathbf{0}}}$ and $\Gamma'_{\psi^{-t}_{\mathbf{0}}}$ contains $\{\mathbf{0}\}\cup\Sigma_p$ because this is the case for E^+_p and E^-_p: one deduces that $\Pi_{p,t}$ contains $\{\mathbf{0}\}\cup\Sigma_p$.

Each graph $\Gamma_{\psi^t_{\mathbf{0}}}$ and $\Gamma'_{\psi^{-t}_{\mathbf{0}}}$ is invariant by φ because this is the case for E^+_p and E^-_p and because ξ is invariant by φ: one deduces that $\Pi_{p,t}$ is invariant by φ.

For every $t\geqslant 1$ and every $i\in\mathbb{Z}$, one may write $\Pi_{p,t}$ as the graph of a function $\theta_{i,t}:E^i\to E^{i\perp}$ and as the graph of a function $\theta_{i,t}:E'^i\to E'^{i\perp}$ and all these functions are N_t Lipschitz.

Using Ascoli's Theorem, one may find for every $i\in\mathbb{Z}$ a map $\theta_i:E^i\to E^{i\perp}$, a map $\theta'_i:E'^i\to E'^{i\perp}$, both N_t Lipschitz for every $t>0$, and a sequence $(t_n)_{n\geqslant 0}$ satisfying $\lim_{n\to+\infty}t_n=+\infty$, such that each sequence $(\theta_{i,t_n})_{n\geqslant 0}$ converges to θ_i and each sequence $(\theta'_{i,t_n})_{n\geqslant 0}$ converges to θ'_i for the compact-open topology. The graphs of θ_i and θ'_i are equal and independant of i. The topological plane Π_p obtained in that way is invariant by φ, contains $\{\mathbf{0}\}\cup\Gamma_p$ and satisfies

$$\left((\Pi_p)^t\times(\Pi^t_p)\right)\setminus\mathrm{diag}\subset\mathcal{W}_p$$

for every $t\in\mathbb{R}$. □

Proof of Proposition 4.1. — Let Π_p be a plane given by Proposition 6.1. The curve Σ_p bounds a disk $\Delta_p\subset\Pi_p$. This disk is invariant by φ because it is the case for Π_p and Σ_p, so the assertion (i) is satisfied. The assertions (iii) and (iv) being true on Π_p are of course true on Δ_p. The map Q_1 sends homeomorphically Π_p onto $\mathbb{R}^2$ and satisfies $Q_1(\mathbf{0})=0$. Moreover its sends Σ_p onto $S_{p/q}$, which implies that it sends Δ_p onto $D_{p/q}$. Consequently one has $\mathbf{0}\in\Delta_p$, which means that (i) is true.

Let us prove now that for every $\mathbf{z} \in \Delta_p \setminus (\{\mathbf{0}\} \cup \Sigma_p)$, one has

$$\lim_{t\to-\infty} \mathbf{z}^t = \mathbf{0}, \quad \lim_{t\to+\infty} d(\mathbf{z}^t, \Sigma_p) = 0.$$

Observe that Σ_p bounds the disk $(\Delta_p)^t \subset (\Pi_p)^t$, which is also sent homeomorphically onto $D_{p/q}$ by each Q_i and Q'_i. Consequently, the orbit of every point $\mathbf{z} \in \Delta_p \setminus (\{0\} \cup \Sigma_q)$ is bounded. The flow of ξ being a gradient flow, the sets $\alpha(\mathbf{z})$ and $\omega(\mathbf{z})$ are not empty, either reduced to $\mathbf{0}$ or included in one of the curves of singularities. The only possible circle is Σ_p, because $(\mathbf{0}, \mathbf{z}^t) \in W_p$ for every $t \in \mathbb{R}$ and $(\mathbf{0}, \mathbf{z}') \in W_{p'}$ for every $\mathbf{z}' \in \Sigma_{p'}$. Using the fact that $\mathbf{h}(\mathbf{z}') > \mathbf{h}(\mathbf{0})$ if $\mathbf{z}' \in \Sigma_p$, one deduces that (vi) is satisfied.

It remains to prove (v), which means that Δ_p is invariant by the flow. It is sufficient to prove that the disk Δ_p is independent of the plane Π_p given by Proposition 6.1. Indeed $(\Pi_p)^t$ also satisfies the properties stated in Proposition 6.1, and Σ_p bounds the disk $(\Delta_p)^t \subset (\Pi_p)^t$. We will give a proof by contradiction, supposing that Π_p and Π'_p are two planes given by Proposition 6.1, such that Σ_p bounds two distinct disks $\Delta_p \subset \Pi_p$ and $\Delta'_p \subset \Pi'_p$.

Suppose that $\mathbf{z} \in \Delta_p \setminus \Delta'_p$. The map Q_1 induces homeomorphisms from Δ_p and Δ'_p onto $D_{p/q}$, so there exists $\mathbf{z}' \in \Delta'_p$ such that $Q_1(\mathbf{z}') = Q_1(\mathbf{z})$, which implies that $(\mathbf{z}, \mathbf{z}') \notin \mathcal{W}$. Therefore, one has $(\mathbf{z}, \mathbf{z}') \in \mathrm{Cl}(\mathcal{W}^+_{p+1})$ or $(\mathbf{z}, \mathbf{z}') \in \mathrm{Cl}(\mathcal{W}^-_{p-1})$. In the first case, there exist $p' > p$ and $T > 0$ such that $(\mathbf{z}^t, \mathbf{z}'^t) \in \mathcal{W}_{p'}$ for every $t > T$. In the second case, there exist $p' < p$ and $T < 0$ such that $(\mathbf{z}^t, \mathbf{z}'^t) \in \mathcal{W}_{p'}$ for every $t < T$. Let us consider the second case. The fact that $\lim_{t\to-\infty} \mathbf{z}^t = \mathbf{0}$ and that $(\mathbf{0}, \mathbf{z}^t) \in \mathcal{W}_p$ implies that the line generated by $\mathbf{z}^t$ is approaching the space $E_p(\mathbf{0})$ when t tends to $-\infty$. We have a similar result for $\mathbf{z}'^t$. The fact that $(\mathbf{z}^{t+s}, \mathbf{z}'^{t+s}) \in \mathcal{W}_{p'}$ for every $s \in (-\infty, -t+T)$ implies that the line generated by $\mathbf{z}^t - \mathbf{z}'^t$ is approaching the space $E_{p'}(\mathbf{0})$ when t tends to $-\infty$.

As we know that the eigenvalues of $D\xi(\mathbf{0})|_{E_{p'}(\mathbf{0})}$ are smaller than the eigenvalues of $D\xi(\mathbf{0})|_{E_p(\mathbf{0})}$, we deduce that there exist $C > 0$, $C' > 0$ and $\mu > \mu'$ such that, for $-t$ large enough, one has

$$\|\mathbf{z}^t\| \leqslant Ce^{\mu t}, \quad \|\mathbf{z}'^t\| \leqslant Ce^{\mu t}, \quad \|\mathbf{z}^t - \mathbf{z}'^t\| \geqslant C'e^{\mu' t},$$

which is impossible.

Let us consider the first case. The fact that $\Sigma_p \times \Sigma_p \setminus \mathrm{diag}$ is included in $\mathcal{W}_p$ implies that 0 is an eigenvalue of $D\xi(\mathbf{z}'')|_{E_p(\mathbf{z}'')}$ for every $\mathbf{z}'' \in \Sigma_q$. Consequently, there exists a uniform upper bound $\mu' > 0$ of the spectrum of $D\xi(\mathbf{z}'')|_{E_{p'}(\mathbf{z}'')}$. The fact that $\omega(\mathbf{z})$ and $\omega(\mathbf{z}')$ are included in Σ_p and that $(\mathbf{z}^t, \mathbf{z}'^t) \in \mathcal{W}_{p'}$ for every $t > T$ implies that $\lim_{t\to+\infty} \|\mathbf{z}^t - \mathbf{z}'^t\| = 0$. Moreover if t is large, then for every $s \in (-T - t, +\infty)$ one has $(\mathbf{z}^{t+s}, \mathbf{z}'^{t+s}) \in$

$\mathcal{W}_{p'}$. This implies that if t is large and $\mathbf{z}'^t$ is close to a point $\mathbf{z}'' \in \Sigma_p$, then it is also the case for $\mathbf{z}^t$ and the line generated by $\mathbf{z}^t - \mathbf{z}'^t$ is close to $E_{p'}(\mathbf{z}'')$. Consequently, if t is large, then

$$\|\mathbf{z}^{t+1} - \mathbf{z}'^{t+1}\| \geqslant \|\mathbf{z}^t - \mathbf{z}'^t\|,$$

which is impossible. □

Remark 6.4. — The uniqueness property that has been stated in the previous proof permits us to give a construction of Δ_p. Let $\Pi_{p,t} = \left(E_p^+\right)^t \cap \left(E_p^-\right)^{-t}$ be the plane defined in the proof of Proposition 6.1 and $\Delta_{p,t} \subset \Pi_{p,t}$ the disk bounded by Σ_t, then one has

$$\Delta_p = \lim_{t \to +\infty} \Delta_{p,t},$$

for each natural topology (induced by the Hausdorff distance, associated to the C^0 topology on maps $\theta_i : E^i \to E^{i\perp}$ or on maps $\theta'_i : E'^i \to E'^{i\perp}$). The uniqueness property, and consequently its invariance by the flow is a consequence of the fact that $\{0\} \cup S_{p/q}$ is a maximal unlinked fixed point set of f^q. The fact that ξ is a gradient flow was not essential in the proof, nevertheless the proof is easier in this case (see [15, Proposition 5.2.1]).

7. Appendix

The goal of the appendix is to give a proof of Proposition 5.2. This proposition will result from the technical Lemma 7.1. A particular case of this lemma, that we will explain in detail, implies Proposition 5.1. The proof of Lemma 7.1, in this particular case, is nothing but the proof of [15, Lemma 2.5.1]. We will look at a wider situation than the one studied in the present paper by looking at a general tridiagonal and monotonically symmetric Lipschitz vector field.

We fix an integer $r \geqslant 2$ and consider the finite dimensional vector space

$$F = \left\{\mathbf{x} = (x_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \;\middle|\; x_{i+r} = x_i, \text{ for all } i \in \mathbb{Z}\right\}.$$

Unlike in Section 3, we furnish F with the sup norm $\|\ \|$ where $\|\mathbf{x}\| = \max_{i\in\mathbb{Z}} |x_i|$.

We consider on F a tridiagonal and monotonically symmetric Lipschitz vector field $\zeta = (\zeta_i)_{i\in\mathbb{Z}}$. More precisely we suppose that

$$\zeta_i(\mathbf{x}) = \zeta_i(x_{i-1}, x_i, x_{i+1}),$$

where

— the map $x \mapsto \zeta_i(x_{i-1}, x, x_{i+1})$ is K Lipschitz;

— the maps $x \mapsto \zeta_i(x, x_i, x_{i+1})$ and $x \mapsto \zeta_i(x_{i-1}, x_i, x)$ are K bi-Lipschitz homeomorphisms;
— if $x \mapsto \zeta_i(x_{i-1}, x_i, x)$ is increasing, then $x \mapsto \zeta_{i+1}(x, x_{i+1}, x_{i+2})$ is increasing and we set $\sigma_i = 1$;
— if $x \mapsto \zeta_i(x_{i-1}, x_i, x)$ is decreasing, then $x \mapsto \zeta_{i+1}(x, x_{i+1}, x_{i+2})$ is decreasing and we set $\sigma_i = -1$.

Here again, we write $\mathbf{x}^t$ for the image at time t of a point $\mathbf{x}$ by the flow. We will prove the following:

Lemma 7.1. — *For every integer $l \geqslant 2$ and every real number $\mu \in (0, 1]$, there exists:*

— *a sequence of polynomials $(P_k)_{k\geqslant 0}$ in $\mathbb{R}[X_1, X_2]$ depending on K, l and μ, such that $P_k(0, X_2) = a_k X_2^k$ with $a_k > 0$ if $k \leqslant [l/2] + 1$;*
— *a sequence of polynomials $(Q_k)_{k\geqslant 0}$ in $\mathbb{R}[X_1, X_2]$ depending on K, l, and μ, such that $Q_k(0, X_2) = b_k X_2^k$ with $b_k > 0$;*
— *real numbers $\varepsilon_0 > 0$ and $t_0 > 0$,*

which verify the following:

For every $\varepsilon \in [0, \varepsilon_0]$ and for every $\mathbf{x}$, $\mathbf{x}'$ such that

$$\begin{aligned} |x_i - x'_i| &\geqslant \mu \|\mathbf{x} - \mathbf{x}'\|, \\ |x_{i+1} - x'_{i+1}| &\leqslant \varepsilon \|\mathbf{x} - \mathbf{x}'\|, \\ &\cdots\cdots\cdots \\ |x_{i+l} - x'_{i+l}| &\leqslant \varepsilon \|\mathbf{x} - \mathbf{x}'\|, \\ |x_{i+l+1} - x'_{i+l+1}| &\geqslant \mu \|\mathbf{x} - \mathbf{x}'\|, \end{aligned}$$

the following inequalities are satisfied for every $t \in [-t_0, t_0] \setminus \{0\}$:

$$\begin{aligned} \sigma'_{i+k} \operatorname{sign}(t)^k (x^t_{i+k} - {x'}^t_{i+k}) &\geqslant P_k(\varepsilon, |t|)\, \|\mathbf{x} - \mathbf{x}'\|, \\ |x^t_{i+k+1} - {x'}^t_{i+k+1}| &\leqslant Q_k(\varepsilon, |t|)\, \|\mathbf{x} - \mathbf{x}'\|, \\ &\cdots\cdots\cdots \\ |x^t_{i+l-k} - {x'}^t_{i+l-k}| &\leqslant Q_k(\varepsilon, |t|)\, \|\mathbf{x} - \mathbf{x}'\|, \\ \sigma'_{i+l+1-k} \operatorname{sign}(t)^k (x^t_{i+l+1-k} - {x'}^t_{i+l+1-k}) &\geqslant P_k(\varepsilon, |t|)\, \|\mathbf{x} - \mathbf{x}'\|, \end{aligned}$$

where

$$\sigma'_{i+k} = \operatorname{sign}(x_i - x'_i)\, \sigma_i \ldots \sigma_{i+k-1}$$

and

$$\sigma'_{i+l+1-k} = \operatorname{sign}(x_{i+l+1} - x'_{i+l+1})\, \sigma_{i+l+1-k} \ldots \sigma_{i+l}.$$

Moreover if $l = 2m - 1$ *is odd and* $\sigma'_{i+m-1}\sigma_{i+m-1} = \sigma'_{i+m+1}\sigma_{i+m}$*, then*

$$\sigma'_{i+m}\ \mathrm{sign}(t)^m (x^t_{i+m} - x'^t_{i+m}) \geqslant 2P_m(\varepsilon, |t|)\, \|\mathbf{x} - \mathbf{x}'\|,$$

where

$$\sigma'_{i+m} = \sigma'_{i+m-1}\sigma_{i+m-1} = \sigma'_{i+m+1}\sigma_{i+m}.$$

Proof. — By replacing the vector field ξ with $-\xi$, one changes every σ_i into $-\sigma_i$. Thus it is sufficient to prove the lemma for $t > 0$. Let us begin by defining two sequences $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$ by the induction equations

$$a_{k+1} = k^{-1}\left(K^{-1}a_k - 2Kb_k\right),\ b_{k+1} = 3Kk^{-1}b_k$$

and the initial conditions

$$a_0 = \mu/2, \quad b_0 = \delta,$$

where $\delta > 0$ is chosen sufficiently small to ensure that a_k is positive if $k \leqslant [l/2] + 1$. The number δ depends on K, l and μ. Suppose moreover than $\delta \leqslant \mu$ and write

$$\varepsilon_0 = \frac{\delta}{2}, \quad t_0 = \frac{\log(\delta/2)}{3K}.$$

Let us continue by defining two sequences of polynomials $(P_k)_{k\geqslant 0}$, $(Q_k)_{k\geqslant 0}$ in $\mathbb{R}[X_1, X_2]$ by the recursive equations

$$\frac{\partial}{\partial X_2}P_{k+1} = K^{-1}P_k - 2KQ_k, \qquad P_{k+1}(X_1, 0) = -X_1,$$
$$\frac{\partial}{\partial X_2}Q_{k+1} = 3KQ_k(X_1, X_2), \qquad Q_{k+1}(X_1, 0) = X_1,$$

and the initial conditions

$$P_0 = \mu/2, \quad Q_0 = \delta.$$

Observe that

$$P_k(0, X_2) = a_k X_2^k, \quad Q_k(0, X_2) = b_k X_2^k.$$

Note also that $Q_k(x_1, x_2) \geqslant 0$ if $x_1 \geqslant 0$ and $x_2 \geqslant 0$.

We will prove by induction on $k \in \{0, \ldots, [l/2]\}$ that for every $t \in (0, t_0]$, one has

$$\sigma'_{i+k}(x^t_{i+k} - x'^t_{i+k}) \geqslant P_k(\varepsilon, t)\, \|\mathbf{x} - \mathbf{x}'\|$$
$$\sigma'_{i+l+1-k}(x^t_{i+l+1-k} - x'^t_{i+l+1-k}) \geqslant P_k(\varepsilon, t)\, \|\mathbf{x} - \mathbf{x}'\|$$

and

$$|x^t_j - x'^t_j| \leqslant Q_k(\varepsilon, t)\, \|\mathbf{x} - \mathbf{x}'\|$$

if $i + k < j < i + l + 1 - k$.

By hypothesis, the vector field is $3K$ Lipschitz. Using Gronwall's Lemma, like in Lemma 3.2, one deduces that

$$e^{-3Kt}\|\mathbf{x}^t - \mathbf{x}'^t\| \leqslant \|\mathbf{x}^t - \mathbf{x}'^t\| \leqslant e^{3Kt}\|\mathbf{x} - \mathbf{x}'\|.$$

Therefore, for every $i \in \mathbb{Z}$, one has

$$|x_i - x'_i| \leqslant e^{3Kt}\|\mathbf{x} - \mathbf{x}'\|,$$

which implies

$$|\dot{x}_i - \dot{x}'_i| \leqslant 3Ke^{3Kt}\|\mathbf{x} - \mathbf{x}'\|,$$

and so

$$|(x^t_i - x'^t_i) - (x_i - x'_i)| \leqslant e^{3Kt}\|\mathbf{x} - \mathbf{x}'\|.$$

By definition of ε_0 and t_0 one deduces that the induction hypothesis is satisfied for $k = 0$.

Suppose that the induction hypothesis has been proven until k, where $k < [l/2]$ and let us prove it for $k+1$. One has

$$\begin{aligned}
&\dot{x}_{i+k+1} - \dot{x}'_{i+k+1}\\
&\quad= \zeta_{i+k+1}(x_{i+k}, x_{i+k+1}, x_{i+k+2}) - \zeta_{i+k+1}(x'_{i+k}, x_{i+k+1}, x_{i+k+2})\\
&\qquad+ \zeta_{i+k+1}(x'_{i+k}, x_{i+k+1}, x_{i+k+2}) - \zeta_{i+k+1}(x'_{i+k}, x'_{i+k+1}, x_{i+k+2})\\
&\qquad+ \zeta_{i+k+1}(x'_{i+k}, x'_{i+k+1}, x_{i+k+2}) - \zeta_{i+k+1}(x'_{i+k}, x'_{i+k+1}, x'_{i+k+2}).
\end{aligned}$$

By definition of σ_{i+k+1}, σ'_{i+k+1} and by hypothesis, one deduces that

$$\sigma'_{i+k+1}(\dot{x}^t_{i+k+1} - \dot{x}'^t_{i+k+1}) \geqslant \left(K^{-1}P_k(\varepsilon,t) - 2KQ_k(\varepsilon,t)\right)\|\mathbf{x} - \mathbf{x}'\|.$$

Consequently, one has

$$\begin{aligned}
&\sigma'_{i+k+1}((x^t_{i+k+1} - x'^t_{i+k+1}) - (x_{i+k+1} - \dot{x}'_{i+k+1}))\\
&\qquad\qquad\geqslant \left(\int_0^t (K^{-1}P_k(\varepsilon,u) - 2KQ_k(\varepsilon,u))\,du\right)\|\mathbf{x} - \mathbf{x}'\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sigma'_{i+k+1}(x^t_{i+k+1} - x'^t_{i+k+1})\\
&\qquad\qquad\geqslant \left(\int_0^t K^{-1}P_k(\varepsilon,u) - 2KQ_k(\varepsilon,u)\,du - \varepsilon\right)\|\mathbf{x} - \mathbf{x}'\|\\
&\qquad\qquad= P_{k+1}(\varepsilon,t)\|\mathbf{x} - \mathbf{x}'\|.
\end{aligned}$$

The same proof gives us

$$\sigma_{i+l-k}(x^t_{i+l-k} - x'^t_{i+l-k}) \geqslant P_{k+1}(\varepsilon,t)\,\|\mathbf{x} - \mathbf{x}'\|.$$

Now, let us fix $j \in \{k+2, \dots, l-k-1\}$. One has

$$|\dot{x}_j^t - \dot{x}'^t_j| \leqslant 3KQ_k(\varepsilon, t)\|\mathbf{x} - \mathbf{x}'\|$$

and so

$$|(x_j^t - x'^t_j) - (x_i - x'_i))| \leqslant \left(\int_0^t 3KQ_k(\varepsilon, u)\, du\right) \|\mathbf{x} - \mathbf{x}'\|$$

which implies

$$|x_j^t - x'^t_j| \leqslant \left(\int_0^t 3KQ_k(\varepsilon, u)\, du + \varepsilon\right) \|\mathbf{x} - \mathbf{x}'\| = Q_{k+1}(\varepsilon, t)\|\mathbf{x} - \mathbf{x}'\|$$

It remains to study the case where $l = 2m-1$ is odd and $\sigma'_{i+m-1}\sigma_{i+m-1} = \sigma'_{i+m+1}\sigma_{i+m}$. One has

$$\sigma'_{i+m}(\dot{x}^t_{i+m} - \dot{x}'^t_{i+m}) \geqslant \left(2K^{-1}P_{m-1}(\varepsilon, t) - KQ_{m-1}(\varepsilon, t)\right) \|\mathbf{x} - \mathbf{x}'\|$$

which implies

$$\begin{aligned}
&\sigma'_{i+m}(x^t_{i+m} - x'^t_{i+m}) \\
&\qquad \geqslant \left(\int_0^t 2K^{-1}P_{m-1}(\varepsilon, u) - KQ_{m-1}(\varepsilon, u)\, du - \varepsilon\right) \|\mathbf{x} - \mathbf{x}'\| \\
&\qquad \geqslant \left(\int_0^t 2K^{-1}P_{m-1}(\varepsilon, u) - 4KQ_{m-1}(\varepsilon, u)\, du - 2\varepsilon\right) \|\mathbf{x} - \mathbf{x}'\| \\
&\qquad = 2P_m(\varepsilon, t)\|\mathbf{x} - \mathbf{x}'\|
\end{aligned}$$

□

Let us explain why Proposition 5.1 can be deduced from Lemma 7.1 in the case where $\varepsilon = 0$. The function defined by the formula

$$\mathcal{L}(\mathbf{x}) = \sum_{0<i\leqslant r} \sigma_i \operatorname{sign}(x_i) \operatorname{sign}(x_{i+1})$$

extends naturally to the open set

$$U = \{\mathbf{x} \in F \mid x_i = 0 \Rightarrow \sigma_{i-1}\sigma_i\, x_{i-1}x_{i+1} < 0\}.$$

Supposing $\varepsilon = 0$, Lemma 7.1 permits to determine the sign of every $x^t_j - x'^t_j$, $j \in \{i, \dots, i+l+1\}$, $|t| \in (0, t_0]$, with the exception of $x^t_{i+m} - x'^t_{i+m}$, in the case where $j = 2m-1$ is odd and $\sigma'_{i+m-1}\sigma_{i+m-1} \neq \sigma'_{i+m+1}\sigma_{i+m}$. One has

$$\operatorname{sign}(x^t_{i+k} - x'^t_{i+k}) = \operatorname{sign}(t)^k\, \sigma'_{i+k},$$
$$\operatorname{sign}(x^t_{i+l+1-k} - x'^t_{i+l+1-k}) = \operatorname{sign}(t)^k\, \sigma'_{i+l+1-k}.$$

Recall that $\sigma'_{i+k} = \sigma_{i+k-1}\sigma'_{i+k-1}$ and $\sigma'_{i+l+1-k} = \sigma_{i+l+1-k}\sigma'_{i+l+2-k}$, if $1 \leqslant k \leqslant [l/2]$, which implies that

$$\operatorname{sign}(x_j^t - {x'}_j^t)\operatorname{sign}(x_{j+1}^t - {x'}_{j+1}^t) = \operatorname{sign}(t)\,\sigma_j\,\sigma'_j\sigma'_{j+1} = \operatorname{sign}(t),$$

if $j \in \{i, \dots, i+[l/2]-1, i+[l/2]+1, \dots, i+l\}$.

In the case where t is positive, the value of

$$\sum_{i \leqslant j \leqslant i+l} \sigma_j \operatorname{sign}(x_j^t - {x'}_j^t)\operatorname{sign}(x_{j+1}^t - {x'}_{j+1}^t)$$

is

$$\begin{cases} l + \sigma_{i+m}\,\sigma'_{i+m}\sigma'_{i+m+1} & \text{if } l = 2m \text{ is even,} \\ l+1 & \text{if } l = 2m-1 \text{ is odd and } \\ & \qquad \sigma'_{i+m-1}\sigma_{i+m-1} = \sigma'_{i+m+1}\sigma_{i+m}, \\ l-1 & \text{if } l = 2m-1 \text{ is odd and } \\ & \qquad \sigma'_{i+m-1}\sigma_{i+m-1} \neq \sigma'_{i+m+1}\sigma_{i+m}. \end{cases}$$

In the case where t is negative, it is

$$\begin{cases} -l + \sigma_{i+m}\,\sigma'_{i+m}\sigma'_{i+m+1} & \text{if } l = 2m \text{ is even,} \\ -l-1 & \text{if } l = 2m-1 \text{ is odd and } \\ & \qquad \sigma'_{i+m-1}\sigma_{i+m-1} = \sigma'_{i+m+1}\sigma_{i+m}, \\ -l+1 & \text{if } l = 2m-1 \text{ is odd and } \\ & \qquad \sigma'_{i+m-1}\sigma_{i+m-1} \neq \sigma'_{i+m+1}\sigma_{i+m}. \end{cases}$$

The difference between the two values is a positive multiple of 4, except in the case where $l = 1$ and $\sigma'_i\sigma_i \neq \sigma'_{i+2}\sigma_{i+1}$, which means that

$$\sigma_i \operatorname{sign}(x_i - x'_i) \neq \sigma_{i+1}\operatorname{sign}(x_{i+2} - x'_{i+2}).$$

Consequently, if $\mathbf{x}$, $\mathbf{x}'$ are two distinct points of F such that $\mathbf{x}' - \mathbf{x} \notin U$, there exists $t_0 > 0$ such that ${\mathbf{x}'}^t - \mathbf{x}^t \in U$ if $0 < |t| \leqslant t_0$. Moreover, for $t \in (0, t_0])$ one has

$$\mathcal{L}({\mathbf{x}'}^t - \mathbf{x}^t) = \mathcal{L}({\mathbf{x}'}^{t_0} - \mathbf{x}^{t_0}) > \mathcal{L}({\mathbf{x}'}^{-t_0} - \mathbf{x}^{-t_0}) = \mathcal{L}({\mathbf{x}'}^{-t} - \mathbf{x}^{-t})\ .$$

The proof given above in the case where $\varepsilon = 0$ is nothing but the proof of [15, Lemma 2.5.1]. It implies Proposition 5.1 when applied to the vector field ξ because $L = \mathcal{L}/4$.

Lemma 7.1 tells us more. For every $l \geqslant 2$ and μ, there exists t_0 such that for every $t \in (0, t_0]$, there exists ϵ such that $P_k(\varepsilon, t) > 0$ for every

$k \leqslant [l/2]+1$. Indeed one has $P_k(0,t)=a_k t^k>0$. Consequently, if $\mathbf{x}$ and $\mathbf{x}'$ satisfies

$$\begin{aligned} |x_i - x'_i| &\geqslant \mu\|\mathbf{x}-\mathbf{x}'\| \\ |x_{i+1} - x'_{i+1}| &\leqslant \varepsilon\|\mathbf{x}-\mathbf{x}'\| \\ &\cdots\cdots\cdots \\ |x_{i+l} - x'_{i+l}| &\leqslant \varepsilon\|\mathbf{x}-\mathbf{x}'\| \\ |x_{i+l+1} - x'_{i+l+1}| &\geqslant \mu\|\mathbf{x}-\mathbf{x}'\| \end{aligned}$$

one can determine the signs of $x_j^t - x'^t_j$, $j \in \{i,\dots,i+l+1\}$, with the exception of $x_{i+m}^t - x'^t_{i+m}$, in the case where $j=2m-1$ is odd and $\sigma'_{i+m-1}\sigma_{i+m-1} \neq \sigma'_{i+m+1}\sigma_{i+m}$. Like in the case where $\varepsilon = 0$, the value of

$$\sum_{i\leqslant j\leqslant i+l} \sigma_j \operatorname{sign}(x_j^t - x'^t_j)\operatorname{sign}(x_{j+1}^t - x'^t_{j+1})$$

is larger at time t than at time $-t$ except if $l=1$ and $\sigma_i \operatorname{sign}(x_i - x'_i) \neq \sigma_{i+1}\operatorname{sign}(x_{i+2}-x'_{i+2})$.

As a consequence, if one defines, for $t>0$, the sets

$$\mathcal{X}_t = \left\{(\mathbf{x},\mathbf{x}') \in F^2 \;\middle|\; |s| \leqslant t \Rightarrow \mathbf{x}^s - \mathbf{x}'^s \in U\right\}$$

and

$$\mathcal{Y}_t = \left\{\frac{\mathbf{x}-\mathbf{x}'}{\|\mathbf{x}-\mathbf{x}'\|} \;\middle|\; (\mathbf{x},\mathbf{x}') \in \mathcal{X}_t\right\},$$

one deduces that the closure of $\mathcal{Y}_t$ is included in U. This implies Proposition 5.2.

Let us conclude this appendix with the following result:

Lemma 7.2. — *The space* $\mathrm{Diff}^1_{**}(\mathbb{D})$ *of area preserving* C^1 *diffeomorphisms of* $\mathbb{D}$ *that fix* 0 *and every point of* $\mathbf{S}$ *is path-connected when furnished with the* C^1 *topology.*

Proof. — Let us begin by proving that the space $\mathrm{Diff}^1_{*}(\mathbb{D})$ of area preserving C^1 diffeomorphisms of $\mathbb{D}$ that fix every point of $\mathbf{S}$ is path-connected when furnished with the C^1 topology. Let us consider the symplectic polar system of coordinates $(\theta,\lambda) \in \mathbb{T}^1 \times (0,+\infty)$ defined on on $\mathbb{R}^2 \setminus \{0\}$ by

$$(x,y) = (\sqrt{\lambda}\cos 2\pi\theta, \sqrt{\lambda}\sin 2\pi\theta).$$

Every $\Phi \in \mathrm{Diff}^1_{*}(\mathbb{D})$ induces a C^1 area preserving diffeomorphism, defined in a neighborhood of $\mathbb{T}^1\times\{1\}$ in $\mathbb{T}^1\times(0,1]$ that fixes every point of $\mathbb{T}^1\times\{1\}$. The image of a circle $\lambda=\lambda_0$ is a graph $\lambda=\psi(\theta)$ if λ_0 is close to 1. This implies that Φ is defined by a generating function in a neighborhood of

$\mathbb{T}^1 \times \{1\}$: if one writes $\Phi(\theta,\lambda) = (\Theta,\Lambda)$, one knows that (Θ,λ) defines a system of coordinates in a neighborhood of $\mathbb{T}^1 \times \{1\}$ in $\mathbb{T}^1 \times (0,1]$ and that $(\theta-\Theta)d\lambda + (\Lambda-\lambda)d\Theta$ is a C^1 exact form. There exists a C^2 function S defined on an annulus $\mathbb{T}^1 \times [1-\eta_0, 1\}$ such that

$$\theta - \Theta = \frac{\partial S}{\partial \lambda}, \;\; \Lambda - \lambda = \frac{\partial S}{\partial \Theta}$$

and such that

$$\frac{\partial^2 S}{\partial\Theta\partial\lambda} > -1.$$

The circle $\lambda = 1$ being invariant and contained in the fixed points set of Φ, one has

$$\frac{\partial S}{\partial \Theta}(\Theta, 1] = \frac{\partial S}{\partial \lambda}(\Theta,1) = 0$$

which implies

$$\frac{\partial^2 S}{\partial\Theta\partial\lambda}(\Theta,1) = 0.$$

Let $\nu : [0,+\infty) \to [0,1]$ be a C^2 function such that

$$\nu(t) = \begin{cases} 1 & \text{if } t \in [0,1/3], \\ 0 & \text{if } t \geqslant 2/3. \end{cases}$$

For every $\eta \in (0,\eta_0]$ define

$$S_\eta : (\Theta,\lambda) \mapsto \nu\left(\frac{1-\lambda}{\eta}\right) S(\Theta,\lambda).$$

One has

$$\frac{\partial^2 S_\eta}{\partial\Theta\partial\lambda} = \nu\left(\frac{1-\lambda}{\eta}\right)\frac{\partial^2 S}{\partial\Theta\partial\lambda} - \frac{1}{\eta}\nu'\left(\frac{1-\lambda}{\eta}\right)\frac{\partial S}{\partial\Theta}.$$

The fact that $\dfrac{\partial S}{\partial\Theta}(\Theta,1) = \dfrac{\partial^2 S}{\partial\Theta\partial\lambda}(\Theta,1) = 0$ implies that the quantity $\dfrac{1}{\eta}\dfrac{\partial S}{\partial\Theta}$ tends to zero uniformly on the annulus of equation $1-\eta \leqslant \lambda \leqslant 1$, when η tends to 0. As a consequence, if η is sufficiently small, one has

$$\frac{\partial^2 S}{\partial\Theta\partial\lambda} > -1$$

on $\mathbb{T}^1 \times [1-\eta_0, 1]$. The equations

$$\theta - \Theta = s\frac{\partial S_\eta}{\partial\lambda}, \;\; \Lambda - \lambda = s\frac{\partial S_\eta}{\partial\Theta}$$

define a family $(\Phi_{\eta,s})_{s\in[0,1]}$ of area preserving C^1 diffeomorphisms on the annulus of equation $1-\eta_0 \leqslant \lambda \leqslant 1$ that fix all points in a neighborhood of the circle $\lambda = 1-\eta_0$. Extending these maps by the identity on $\mathbb{D}$, one gets a continous arc in $\mathrm{Diff}^1_*(\mathbb{D})$ that joins the identity to a diffeomorphism $\Phi_{\eta,1}$

that coincides with Φ in a neighborhood of $\mathbb{S}$. One can write $\Phi = \Phi' \circ \Phi_{\eta,1}$, where Φ' coincides with the identity in a neighborhood of $\mathbb{S}$. It remains to prove that Φ' can be joined to the identity in $\mathrm{Diff}^1_*(\mathbb{D})$. A classical result of Zehnder [18], whose proof uses generating functions, states that every C^1 symplectic diffeomorphism on a compact symplectic manifold is the limit, for the C^1 topology of a sequence of C^∞ symplectic diffeomorphisms. If $\mathcal{U}$ is a neighborhood of the identity in $\mathrm{Diff}^1_*(\mathbb{D})$, Zehnder's proof permits us to write $\Phi' = \Phi'' \circ \Phi'''$, where Φ'' belongs to $\mathcal{U}$ and Φ''' is an area preserving C^∞ diffeomorphism of $\mathbb{D}$, both of them coinciding with the identity in a neighborhood of $\mathbb{S}$. If $\mathcal{U}$ is sufficiently small, Φ'' will be defined by a generating function: there exist a real number $\eta_1 > 0$ and a C^2 function h defined on the disk $X^2 + y^2 \leqslant 1$ such that

$$\Phi''(x,y) = (X,Y) \Leftrightarrow \begin{cases} x = \partial h/\partial y(X,y) \\ Y = \partial h/\partial X(X,y), \end{cases}$$

where

$$\frac{\partial^2 h}{\partial X \partial y} > 0$$

and

$$h(X,y) = Xy \ \text{ if } \ 1-\eta_1 \leqslant X^2 + y^2 \leqslant 1.$$

Writing $h_s(X,y) = sh(X,y) + (1-s)Xy$, for $s \in [0,1]$, the equations

$$\Phi''_s(x,y) = (X,Y) \Leftrightarrow \begin{cases} x = \partial h_s/\partial y(X,y) \\ Y = \partial h_s/\partial X(X,y). \end{cases}$$

define a continous arc in $\mathrm{Diff}^1_*(\mathbb{D})$ that joins the identity to Φ''. The fact that Φ''' can be joined to the identity comes from the following classical result whose proof uses a similar result, due to Smale, in the non area preserving case and the classical Moser's Lemma on volume forms (see [2] for example): the space of area preserving C^∞ diffeomorphisms of $\mathbb{D}$, that coincides with the identity on a neighborhood of $\mathbb{S}$ is path-connected when furnished with the C^∞ topology.

Let us prove now that $\mathrm{Diff}^1_{**}(\mathbb{D})$ is path-connected. It is sufficient to prove that for every compact set Ξ in the interior of $\mathbb{D}$, there exists a continuous family $(\Psi^z)_{z\in\Xi}$ in $\mathrm{Diff}^1_*(\mathbb{D})$ such that $\Psi^z(z) = 0$. Indeed, every $\Phi \in \mathrm{Diff}^1_{**}(\mathbb{D})$ can be joined to the identity by a path $s \mapsto \Phi_s$ in $\mathrm{Diff}^1_*(\mathbb{D})$. If Ξ is chosen to contain the image of the path $s \mapsto \Phi_s(0)$, then the path $s \mapsto \Psi^{\Phi_s(0)} \circ \Phi_s$ joins Φ to the identity in $\mathrm{Diff}^1_{**}(\mathbb{D})$.

For every $r \in (0,1)$ one can find a C^2 function $\nu_r : [0,1] \to \mathbb{R}$ such that

$$\nu(t) = \begin{cases} 1 & \text{if } t \in [0,r], \\ 0 & \text{if } t \geqslant 1. \end{cases}$$

Denote by $(\Psi_1^s)_{s\in\mathbb{R}}$ and $(\Psi_2^s)_{s\in\mathbb{R}}$ the Hamiltonian flows (for the usual symplectic form $dx \wedge dy$) induced by the function

$$H_1 : (x,y) \mapsto -y\nu_r(x^2+y^2), \;\; H_1 : (x,y) \mapsto x\nu_r(x^2+y^2)$$

respectively, and for every $z = (x,y) \in \mathbb{R}^2$ set $\Psi^z = \Psi_1^x \circ \Psi_2^y$. Observe that if $x^2+y^2 \leqslant r$, then $\Psi^z(z) = 0$. $\square$

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