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A SPLITTING THEOREM FOR GOOD COMPLEXIFICATIONS

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Abstract. — The purpose of this paper is to produce restrictions on fundamental groups of manifolds admitting good complexifications by proving the following Cheeger-Gromoll type splitting theorem: Any closed manifold M admitting a good complexification has a finite-sheeted regular covering M_1 such that M_1 admits a fiber bundle structure with base $(S^1)^k$ and fiber N that admits a good complexification and also has zero virtual first Betti number. We give several applications to manifolds of dimension at most 5.

Résumé. — Le but de cet article est de montrer qu'il existe des restrictions aux groupes fondamentaux que peuvent avoir les variétés admettant une bonne complexification, en démontrant le théorème suivant de décomposition, de type Cheeger–Gromoll: Toute variété fermée M admettant une bonne complexification a un recouvrement fini M_1, possédant un structure de fibré de base $(S^1)^k$ et de fibre N ayant une bonne complexification et un premier nombre de Betti virtuel nul. On donne plusieurs applications de ce théorème aux variétés de dimension au plus 5.

1. Introduction

A good complexification of a closed smooth manifold M is defined to be a smooth affine algebraic variety U over the real numbers such that M is diffeomorphic to $U(\mathbb{R})$ and the inclusion $U(\mathbb{R}) \longrightarrow U(\mathbb{C})$ is a homotopy equivalence [26], [15]. A good complexification comes naturally equipped with a natural antiholomorphic involution A on $U(\mathbb{C})$ whose fixed point set

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is precisely the set of real points $U(\mathbb{R})$. Kulkarni [15] and Totaro [26] investigate the topology of good complexifications using characteristic classes and Euler characteristic. In this paper we prove a Cheeger-Gromoll type splitting theorem and initiate a systematic study of fundamental groups of good complexifications.

In [26, p. 69, 2nd para], Totaro asks the following question:

Question 1.1. — *If a closed smooth manifold M admits a good complexification, does M also admit a metric of non-negative curvature?*

The following Cheeger–Gromoll type splitting theorem is the main result proved here.

Theorem 1.2 (See Theorem 2.8). — *Let M be a closed manifold admitting a good complexification. Then M has a finite-sheeted regular covering M_1 satisfying the following:*

(1) *M_1 admits a fiber bundle structure with fiber N and base $(S^1)^d$. Here d denotes the (real) Albanese dimension of M_1.*
(2) *The first virtual Betti number $vb_1(N) = 0$,*
(3) *N admits a good complexification.*

Let us spell out the analogy with the Cheeger-Gromoll splitting theorem [19, Theorem 69, p. 288]:

A closed manifold of non-negative curvature has a finite sheeted cover of the form $N \times (S^1)^d$, where N is simply connected. Theorem 1.2 similarly furnishes a fibering over a torus $(S^1)^d$ with fiber N having $vb_1(N) = 0$.

Gromov [8] proves that if a closed smooth manifold M of dimension n admits a metric of non-negative curvature, then there is an upper bound, that depends only on n, on the sum of the Betti numbers of M. He further conjectures, [11, Section 7], that $b_i(M) \leqslant b_i((S^1)^n)$. Theorem 1.3 below furnishes positive evidence towards a combination of Question 1.1 with this conjecture of Gromov by giving an affirmative answer for the first Betti number of manifolds admitting good complexifications.

We shall say that a finitely presented group G is a *good complexification group* if G can be realized as the fundamental group of a closed smooth manifold admitting a good complexification (see also [2]). We deduce from Theorem 1.2 the following critical restriction on good complexification groups (See Theorem 3.1):

Theorem 1.3. — *Let G be a good complexification group. Then there exists a finite index subgroup G_1 of G such that two following statements hold:*

(1) *There is an exact sequence:*

$$1 \longrightarrow H \longrightarrow G_1 \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where k can be zero.

(2) *The above H is a finitely presented good complexification group with $vb_1(H) = 0$, where $vb_1(H)$ denotes the virtual first Betti number of H.*

Recall that for a group H, the *virtual first Betti number* $vb_1(H)$ is the supremum of first Betti numbers $b_1(H_1)$ as H_1 runs over finite index subgroups of H.

The following classes of groups are then ruled out as good complexification groups (See Section 3, especially Corollary 3.2):

(1) Groups with infinite vb_1, in particular large groups.
(2) Hyperbolic CAT(0) cubulated groups.
(3) Solvable groups that are not virtually abelian.
(4) 2- and 3-manifold groups that are not virtually abelian.
(5) any group admitting a surjection onto any of the above.

Question 1.1 has an affirmative answer for 2-manifolds; this is probably classical but follows also from [15, 26]. An affirmative answer to Question 1.1 for 3-manifolds was given in [5]. As a consequence of the above restrictions, Question 1.1 has an affirmative answer for 2 and 3-manifold (see Sections 4.1 and 4.2). Thus a new self-contained proof of the main Theorem of [5] on good complexifications is obtained. In Section 4 we give a number of applications to low-dimensional manifolds.

Theorem 1.4.

(1) *Question 1.1 has an affirmative answer for 2-manifolds.*
(2) *Question 1.1 has an affirmative answer for 3-manifolds* [5]*.*
(3) *Let M be a closed simply connected 4-manifold admitting a symplectic good complexification. Then M admits a metric of non-negative curvature.*
(4) *Let M be a closed 4-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2$ or 4. Moreover, the manifold M admits a finite-sheeted cover with a metric of non-negative curvature (i.e. Question 1.1 has an affirmative answer up to finite-sheeted covering).*

(5) *Let M be a closed 5-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2, 3$ or 5. Further, if $d = 2, 3$ or 5, then M admits a finite-sheeted cover M_1 homeomorphic to $S^2 \times T^3$ or $S^3 \times T^2$ or T^5. In particular, M_1 admits a metric of non-negative curvature.*

For an algebraic variety X, we shall denote the real (respectively, complex) points by $X_{\mathbb{R}}$ (respectively, $X_{\mathbb{C}}$).

Definition 1.5. — *We say that an algebraic map $f : X \longrightarrow Y$ between good complexifications is a* good complexification map *if there is a commutative diagram*

$$\begin{array}{ccc} X_{\mathbb{R}} & \subset & X_{\mathbb{C}} \\ \downarrow f_{\mathbb{R}} & & \downarrow f_{\mathbb{C}} \\ Y_{\mathbb{R}} & \subset & Y_{\mathbb{C}} \end{array} \tag{1.1}$$

such that $f_{\mathbb{C}}$ is equivariant with respect to the antiholomorphic involutions A_X, A_Y, meaning $f_{\mathbb{C}} \circ A_X = A_Y \circ f_{\mathbb{C}}$.

2. Fibrations and the Albanese

We need to first introduce some terminology following [25]. Let $f : X \longrightarrow \mathbb{C}^*$ be an algebraic map of quasiprojective varieties. This f is said to be topologically trivial at $z \in \mathbb{C}^*$ if there is a neighborhood D of z such that the restriction of f from $f^{-1}(D)$ to D is a topologically trivial fibration. If z does not satisfy this property, then we say that z is an *atypical value* and that $f^{-1}(z)$ is an *atypical fiber*. Clearly, critical values (respectively, critical fibers) are atypical values (respectively, atypical fibers). The book [25] gives examples of atypical values that are not critical.

2.1. Fibration over $\mathbb{C}^*$

The punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the standard good complexification of the unit circle S^1 with the antiholomorphic involution given by $z \longrightarrow \frac{1}{\bar{z}}$. In this section we study good complexification maps $f_{\mathbb{C}} : X \longrightarrow \mathbb{C}^*$ restricting to $f_{\mathbb{R}} := f_{\mathbb{C}}|_{X_{\mathbb{R}}} : X_{\mathbb{R}} \longrightarrow S^1$ and prove that such maps have to be topologically trivial fibrations.

Theorem 2.1. — *Let X be a good complexification of M, and let $\mathbb{C}^*$ be the standard good complexification of S^1. Let $f : X \longrightarrow \mathbb{C}^*$ be a good complexification map between the good complexifications of M and S^1. The corresponding map between real loci*

$$f_{\mathbb{R}} := f|_M : M \longrightarrow S^1$$

is a smooth fiber bundle. If N denotes the fiber of $f_{\mathbb{R}}$, then N admits a good complexification.

Proof. — The proof will be divided into a number of steps. We fix some notation first. The map f will also be denoted by $f_{\mathbb{C}}$. The restriction of f to $f^{-1}(S^1) \subset X_{\mathbb{C}}$ will be denoted as $\widehat{f}$.

In what follows a *small perturbation* will refer to a small perturbation in the real algebraic category.

Step 1: $f_{\mathbb{R}}$ has no critical points after small perturbation. — First, note that since $b_1(M) = b_1(X) \neq 0$, it follows from [15, 26] that the Euler characteristic $\chi(M) = \chi(X) = 0$. Next, assume for the time being, that f can be perturbed to a function g with isolated quadratic (i.e. Morse-type) singularities in a small tubular neighborhood X_M of $M = X(\mathbb{R})$ such that the image of g is an annular neighborhood $A_{\mathbb{C}}$ of S^1. Let F denote the general fiber of g.

Then the Euler characteristics of F, X_M and $A_{\mathbb{C}}$ are related by the following formula due to Suzuki [24] (Suzuki emphasizes dimension 2, though the formula works in the general context of Stein manifolds and Stein morphisms):

$$\chi(X_M) = \chi(F) \cdot \chi(A_{\mathbb{C}}) + \sum_{x \in A_{\mathbb{C}}} (\chi(F_x) - \chi(F)),$$

where F_x denotes the fiber of g over x. Let c_x be the defect in the Euler characteristic given by $(\chi(F_x) - \chi(F))$. Since g is defined in the neighborhood of M, the only exceptional fibers of g are critical (i.e. atypical non-critical fibers do not exist). Let c_i denote the finitely many non-zero defects. Each isolated quadratic critical point contributes a cell of a fixed real dimension m (one more than the complex dimension of the fiber). Hence all the c_i's are either $+1$ or -1 depending on the parity of m. Since $\chi(X) = 0 = \chi(\mathbb{C}^*)$, it follows from the Euler characteristic formula above that g has no critical points.

It remains to show that f can indeed be perturbed to a function g with isolated quadratic (i.e. Morse-type) singularities locally. Since f is algebraic onto $\mathbb{C}^*$, it has finitely many critical and atypical values. Restricted to a small tubular neighborhood of M, f has only critical points. Let $A_{\mathbb{C}}$

denote a small annular neighborhood of S^1 containing the corresponding critical values. Any small perturbation of the annulus $A_{\mathbb{C}}$ is again an annulus (unlike $\mathbb{C}^*$, which can be perturbed slightly to $\mathbb{C}$). We note that f is an affine and hence a Stein morphism, and $A_{\mathbb{C}}$ is Stein. X_M is also Stein. Since f is a Stein morphism between Stein manifolds with only critical points, it can be perturbed slightly to a Stein morphism with isolated quadratic singularities, with all singular values in $A_{\mathbb{C}}$.

Step 2: Existence of critical points of $f_{\mathbb{C}}$ is stable under perturbation; hence $f_{\mathbb{R}}$ has no critical points. — Further, if $f_{\mathbb{C}}$ has critical points inside X_M, then any small perturbation of $f_{\mathbb{C}}$ must also have critical points. This is clear for isolated critical points. In general, the set of critical points $Z(f_{\mathbb{C}})$ is a variety and taking a local (algebraic) section σ of $f_{\mathbb{C}}$ with the image of σ lying in a neighborhood of $Z(f_{\mathbb{C}})$, we can ensure that the intersection $Z(f_{\mathbb{C}})\bigcap\sigma$ is an isolated point. Restricting $f_{\mathbb{C}}$ to $Z(f_{\mathbb{C}})\bigcap\sigma$ we observe that a small perturbation of $f_{\mathbb{C}}$ must have critical points in $Z(f_{\mathbb{C}})\bigcap\sigma$.

Since a small perturbation g of f has no critical points inside X_M, it follows that $f = f_{\mathbb{C}}$ has no critical points inside X_M. If $f_{\mathbb{R}}$ has critical points, so does $f_{\mathbb{C}}$ inside X_M. It follows that $f_{\mathbb{R}}$ has no critical points.

Step 3: $f_{\mathbb{R}}$ is a smooth fiber bundle map. — We get as a direct consequence of Step 2 and Ehresmann's fibration theorem that $f_{\mathbb{R}}$ is a smooth fiber bundle map.

Step 4: $f_{\mathbb{C}}$ is a smooth homotopy fibration. — In view of Steps 1, 2 and 3, we may, and we will, assume that $f_{\mathbb{R}}$ has no critical points. It remains to handle the atypical (including critical) values. We assume that the atypical values $z_1, \cdots, z_k$ of $f_{\mathbb{C}}$ are isolated points in $\mathbb{C}^*$. Let $D_1, \cdots, D_k$ be small (disjoint) analytic neighborhoods of $z_1, \cdots, z_k$ respectively. We join D_i by non-intersecting simple arcs α_i to S^1 and define

$$K := S^1\bigcup(\bigcup_{i=1}^{k}\alpha_i)\bigcup(\bigcup_{i=1}^{k}D_i)$$

and $W := f^{-1}(K)$. If some $z_i \in S^1$, we assume that α_i is the constant arc at z_i. Then $X_{\mathbb{C}}$ deformation retracts onto W, and hence W has the same homotopy type as $X_{\mathbb{R}}$ (since X is a good complexification).

The relative homotopy type of $(W, f^{-1}(S^1))$ is then given by the local topology changes at the critical fibers $f^{-1}(z_i)$ (cf. [16]). We shall now need to combine the above observation with the conclusion of Step 2.

Let $F_{\mathbb{R}}$ denote the fiber of $f_{\mathbb{R}} : X_{\mathbb{R}} \longrightarrow S^1$, and let $F_{\mathbb{C}}$ denote the general fiber of $f_{\mathbb{C}}$. Then $F_{\mathbb{C}}$ is also the general fiber of $\widehat{f}$. Pass to the cover $\widetilde{X_{\mathbb{C}}}$ of $X_{\mathbb{C}}$ corresponding to $\pi_1(F_{\mathbb{R}})$ (so $\widetilde{X_{\mathbb{C}}}$ is the fiber product of $X_{\mathbb{C}}$ with the universal cover of $\mathbb{C}^*$), and let

$$P : \widetilde{X_{\mathbb{C}}} \longrightarrow X_{\mathbb{C}}$$

denote the covering. Then $P^{-1}(X_{\mathbb{R}})$ is homeomorphic to $F_{\mathbb{R}} \times \mathbb{R}$. We note that $f_{\mathbb{C}}$ lifts to a map

$$\widetilde{f_{\mathbb{C}}} : \widetilde{X_{\mathbb{C}}} \longrightarrow \mathbb{C}\,,$$

where $\mathbb{C}$ is identified with the universal cover of $\mathbb{C}^*$, and $\widetilde{f_{\mathbb{C}}}$ restricts to a map $\widetilde{f_{\mathbb{R}}} : \widetilde{X_{\mathbb{R}}} \longrightarrow \mathbb{R}$, where $\widetilde{X_{\mathbb{R}}} = P^{-1}(X_{\mathbb{R}})$, and $\mathbb{R}$ is identified with the universal cover of S^1.

Fix fibers $F_{\mathbb{R}} \subset F_{\mathbb{C}} \subset \widetilde{X_{\mathbb{C}}}$ as above. Then the quotient space $\widetilde{X_{\mathbb{C}}}/F_{\mathbb{R}}$ is contractible by the definition of a good complexification. Further, $F_{\mathbb{C}}$ has the homotopy type of a finite CW complex H because it is a quasiprojective variety. Without loss of generality, we may assume that H is a compact subset of $F_{\mathbb{C}}$. Let F denote the (compact finite) quotient complex $H/F_{\mathbb{R}} \subset \widetilde{X_{\mathbb{C}}}/F_{\mathbb{R}}$. Each $z_i \in \mathbb{C}^*$ lifts to a countably infinite collection of atypical points in $\mathbb{C}$. Hence (in its cell complex structure) $\widetilde{X_{\mathbb{C}}}/F_{\mathbb{R}}$ has the same homotopy type as $H/F_{\mathbb{R}}$ with infinitely many cells attached, one for every lift of z_i, $i = 1, \cdots, k$. Since $\widetilde{X_{\mathbb{C}}}/F_{\mathbb{R}}$ is contractible, it follows that

(1) $k = 0$, and
(2) H/F_R is contractible, in particular, $F_{\mathbb{C}}$ has the same homotopy type as $F_{\mathbb{R}}$.

We summarize the conclusion of Step 4 as follows:

The map $f_{\mathbb{C}}$ is a smooth homotopy fibration over $\mathbb{C}^*$, where all fibers are smooth manifolds of the same homotopy type.

Step 5: The fiber of $f_{\mathbb{R}}$ admits a good complexification. — Let N denote a general fiber of $f_{\mathbb{R}}$. The antiholomorphic involution A preserves the fibers $f_{\mathbb{C}}^{-1}(p)$ for $p \in S^1$. Let $X_p = f_{\mathbb{C}}^{-1}(p)$ and $N_p = f_{\mathbb{R}}^{-1}(p)$. Then N_p is diffeomorphic to N for $p \in S^1$. It suffices to show that N_p is homotopy equivalent to X_p.

Now, by Step 4, the map $f_{\mathbb{C}}$ is a smooth homotopy fibration over $\mathbb{C}^*$, where all fibers are smooth manifolds of the same homotopy type as X_p. Hence we get the following commutative diagram of homotopy exact sequences; the vertical arrows in the diagram are maps induced by inclusion:

$$\begin{array}{ccccccccccc} \cdots \to & \pi_i(M) & \to & \pi_i(S^1) & \to & \pi_{i-1}(N) & \to & \pi_{i-1}(M) & \to & \pi_{i-1}(S^1) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \pi_i(X) & \to & \pi_i(\mathbb{C}^*) & \to & \pi_{i-1}(X_p) & \to & \pi_{i-1}(X) & \to & \pi_{i-1}(\mathbb{C}^*) & \to \cdots \end{array}$$

Since the homomorphism $\pi_i(S^1) \longrightarrow \pi_i(\mathbb{C}^*)$ is an isomorphism for all i (in fact they vanish for $i > 1$), and since the homomorphism $\pi_i(M) \longrightarrow \pi_i(X)$

is an isomorphism by the definition of a good complexification, it follows that the inclusion of N into X_p induces isomorphisms $\pi_i(N) \longrightarrow \pi_i(X_p)$ for all $i > 1$ (by the 5-Lemma, using the fact that higher homotopy groups are abelian). Note further that the homomorphism $\pi_1(N) \longrightarrow \pi_1(X_p)$ coincides with the identifications of the kernels of $\pi_1(M) \longrightarrow \pi_1(S^1)$ and $\pi_1(X) \longrightarrow \pi_1(\mathbb{C}^*)$. Also, $\pi_0(N) \longrightarrow \pi_0(X_p)$ is an isomorphism. Hence by the Whitehead Theorem, the inclusion of N into X_p is a homotopy equivalence. This concludes Step 6 and proves the Theorem. □

We point out a couple of consequences of the above proof of Theorem 2.1.

By Step 3, the map $f_{\mathbb{R}}$ has no critical points, and by Step 6, f has no critical points in $f^{-1}(S^1)$. Further, by Step 6, the fiber $F_{\mathbb{C}}$ has the same homotopy type as $F_{\mathbb{R}}$. Thus we have the following:

Corollary 2.2. — *Let $f : X \longrightarrow \mathbb{C}^*$ be a good complexification map between good complexifications $X, \mathbb{C}^*$, and denote the antiholomorphic involution of X by A_X. Then*

- *$A_X(f^{-1}(p)) = f^{-1}(p)$ for all $p \in S^1$, and*
- *$A_X(f^{-1}(p))$ is a good complexification of $f^{-1}(p) \bigcap X_{\mathbb{R}}$.*

Proof. — The only point to note is that A_X preserves the fibers $f^{-1}(p)$. This in turn follows from the condition $f \circ A_X = A_X \circ f$ in the definition of a good complexification map. □

Combining Theorem 2.1 and Corollary 2.2, we obtain the following:

Corollary 2.3. — *Let M be a closed smooth manifold admitting a good complexification. Then there exists a good complexification map $f : X \longrightarrow \mathbb{C}^*$ between good complexifications X (of M) and $\mathbb{C}^*$ (of S^1) such that f is a smooth homotopy fibration. Further,*

(1) *M admits a smooth fibration over S^1 with fiber N such that $f_{\mathbb{R}}$ is a projection onto the base S^1, and*
(2) *X admits a homotopy fibration over $\mathbb{C}^*$ with generic fiber Y such that f is a projection onto $\mathbb{C}^*$ and Y is a good complexification of N.*

2.2. The Albanese

We refer the reader to [21], [14], [23] for quasi-Albanese maps (see also [7]), and to [18] for details on the semiabelian varieties (also called quasi-abelian varieties). For a smooth quasiprojective variety Z, the quasi-Albanese map

$$a_Z \,:\, Z \longrightarrow \mathrm{Alb}(Z) \tag{2.1}$$

is constructed by choosing a point $z_0 \in Z$ (the map on the degree zero part of $\mathrm{CH}_0(Z)$ does not depend on the choice of the point z_0). We will replace the quasi-Albanese $\mathrm{Alb}(Z)$ by a torsor for it so that the construction of the map from Z does not require choosing a base point. Introduce the following equivalence relation on $Z \times \mathrm{Alb}(Z)$:

$$(z_1\,,\alpha_1) \,\equiv\, (z_2\,,\alpha_2) \quad \text{if} \quad a_Z(z_1 - z_2) \,=\, \alpha_2 - \alpha_1\,;$$

as mentioned above, the map a_X on the degree zero part of $\mathrm{CH}_0(Z)$ does not depend on the choice of base point, so $a_Z(z_1 - z_2)$ does not depend on z_0. The corresponding quotient $\mathrm{Alb}(Z)^1$ of $Z \times \mathrm{Alb}(Z)$ is a torsor for $\mathrm{Alb}(Z)$. Now consider the map

$$Z \,\longrightarrow\, \mathrm{Alb}(Z)^1\,,\ x \,\longmapsto\, (z_0\,, a_Z(x))\,, \tag{2.2}$$

where a_Z is the map in (2.1) constructed using z_0. It is straight-forward check that this map does not depend on the choice of z_0. Henceforth, by the quasi-Albanese of Z we will mean $\mathrm{Alb}(Z)^1$ constructed above, and by quasi-Albanese map for Z we will mean the map in (2.2).

If Z is defined over $\mathbb{R}$, then both $\mathrm{Alb}(Z)^1$ and a_Z in (2.2) are also defined over $\mathbb{R}$.

LEMMA 2.4. — *Let $X_{\mathbb{C}}$ be a good complexification of $X_{\mathbb{R}}$. Then the quasi-Albanese of $X_{\mathbb{C}}$ is $(\mathbb{C}^*)^n$, where $n \,=\, b_1(X_{\mathbb{C}})$.*

Proof. — Let $W_{\mathbb{C}}$ denote the quasi-Albanese of $X_{\mathbb{C}}$, and let

$$\alpha \,:\, X_{\mathbb{C}} \longrightarrow W$$

be the Albanese map. The variety $W_{\mathbb{C}}$ being defined over $\mathbb{R}$, is equipped with an antiholomorphic involution A_a, and α intertwines the antiholomorphic involutions of $X_{\mathbb{C}}$ and W, or in other words, we have

$$A_a \circ \alpha \,=\, \alpha \circ A_X\,. \tag{2.3}$$

Let $W_{\mathbb{R}} \subset W_{\mathbb{C}}$ denote the fixed-point locus of the antiholomorphic involution. We have $\alpha(W_{\mathbb{C}}) \subset W_{\mathbb{R}}$.

Since the Albanese W_c is a semi-abelian variety, there is an exact sequence

$$1 \longrightarrow (\mathbb{C}^*)^m \longrightarrow W_{\mathbb{C}} \longrightarrow Q \longrightarrow 1\,,$$

where Q is an abelian variety.

The antiholomorphic complex conjugation A_a on $W_{\mathbb{C}}$ descends to an antiholomorphic complex conjugation A_q on Q whose fixed point set has dimension $\dim_{\mathbb{C}}(Q)$ (note that the fixed point set is nonempty because the identity element is fixed). Hence $\alpha(X_{\mathbb{R}})$ has dimension at most $m+\dim_{\mathbb{C}}(Q)$ which is less than $\dim_{\mathbb{C}}(W_{\mathbb{C}})$ unless $\dim_{\mathbb{C}}(Q) = 0$. Since $\dim_{\mathbb{C}}(W_{\mathbb{C}}) = \dim_{\mathbb{R}}(W_{\mathbb{R}})$, it follows that $\dim_{\mathbb{C}}(Q) = 0$. Hence the quasi-Albanese of $X_{\mathbb{C}}$ is $(\mathbb{C}^*)^n$, where $n = b_1(X_{\mathbb{C}})$. □

Corollary 2.5. — *Let $X_{\mathbb{C}}$ be a good complexification of $X_{\mathbb{R}}$. Let α denote the Albanese map. Let*

$$\Pi : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$$

be a projection of the Albanese $W_{\mathbb{C}} = (\mathbb{C}^*)^n$ onto one of the factors. Then $\Pi \circ \alpha$ is a good complexification map.

Proof. — The quasi-Albanese of $X_{\mathbb{C}}$ is $(\mathbb{C}^*)^n$ by Lemma 2.4, and Π is clearly a good complexification map. Therefore, from (2.3) It follows that $\Pi \circ \alpha$ intertwines the antiholomorphic involution A_X on $X_{\mathbb{C}}$ and the standard antiholomorphic involution A_c on $\mathbb{C}^*$. Consequently, $\Pi \circ \alpha$ is a good complexification map. □

2.3. Albanese Fibrations

The next lemma is a generalization of Case 3 B of Proposition 4.3 of [5].

Lemma 2.6. — *Let $f : X \longrightarrow \mathbb{C}^*$ be a fibration that is a good complexification map between good complexifications $X, \mathbb{C}^*$ such that the general fiber is $(\mathbb{C}^*)^m$. Then f is a trivial fibration.*

Proof. — First note that X may be regarded as a principal $(\mathbb{C}^*)^m$–bundle over $\mathbb{C}^*$. Let V denote the vector bundle on $\mathbb{C}^*$ of rank m associated to the principal $(\mathbb{C}^*)^m$–bundle X for the natural action of $(\mathbb{C}^*)^m$ on $\mathbb{C}^{\oplus m}$. So V splits as a direct sum of m line bundles L_i, i.e. $V = \bigoplus_{i=1}^m L_i$. It suffices to show that L_i is a trivial line bundle or equivalently that the first Chern class $c_1(L_i) = 0$ for all i.

Take any algebraic line bundle L over $\mathbb{C}^*$. The line bundle L extends to an algebraic line bundle over the $\mathbb{P}^1$. To see this, take the image in $\mathbb{P}^1$ of any divisor in $\mathbb{C}^*$ representing L by the homomorphism defined by the inclusion map $\iota : \mathbb{C}^* \hookrightarrow \mathbb{P}^1$. Let $L' \longrightarrow \mathbb{P}^1$ be an extension of L. Therefore, $c_1(L) = \iota^* c_1(L')$. But

$$\iota^*(H^2(\mathbb{P}^1 \mathbb{Z})) = 0 .$$

Therefore, we conclude that $c_1(L) = 0$. □

Essentially the same proof works when the base $\mathbb{C}^*$ is replaced by $(\mathbb{C}^*)^n$. The only change is that we choose $(\mathbb{P}^1)^n$ to be the compactification of $(\mathbb{C}^*)^n$. This gives us the following:

Corollary 2.7. — *Let $f : X \longrightarrow (\mathbb{C}^*)^n$ be a fibration that is a good complexification map between good complexifications $X, (\mathbb{C}^*)^n$ such that the general fiber is $(\mathbb{C}^*)^m$. Then f is a trivial fibration.*

We shall need the notion of a relative Albanese in the following. Let $f : X \longrightarrow Y$ be a fibration. Then the relative Albanese is a fibration $f_\alpha : X_\alpha \longrightarrow Y$, where the fiber $f_\alpha^{-1}(p)$ is the Albanese of $f^{-1}(p)$ for $p \in Y$. Note that the relative Albanese is a bundle of torsors, with canonically defined fibers, and hence in our case, a principal $(\mathbb{C}^*)^k-$bundle for some integer $k \geqslant 0$ (multiplication is well-defined in local trivializations and the fact that the fibers are canonically defined shows that multiplication agrees on overlaps). The following is the main technical tool of this paper.

Theorem 2.8. — *Let M be a closed manifold admitting a good complexification. Then there exists a good complexification $X_\mathbb{C}$ of $M = X_\mathbb{R}$, such that the following holds:*
Let $W_\mathbb{C}$ be the quasi-Albanese of $X_\mathbb{C}$, and let $\alpha : X_\mathbb{C} \longrightarrow W_\mathbb{C}$ be the quasi-Albanese map. Then α is a homotopy smooth fibration with no critical points. Further, if F denote the general fiber, then

- *$b_1(F) = 0$, and*
- *F is a good complexification of the intersection $F \bigcap X_\mathbb{R}$.*

Proof. — If $W_\mathbb{C}$ is zero dimensional, then there is nothing to prove, so assume that $\dim W_\mathbb{C} \geqslant 1$.

We now proceed by induction on $b_1(X_\mathbb{C})$. By Lemma 2.4, we have $W_\mathbb{C} = (\mathbb{C}^*)^n$. Let $\Pi : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$ be a projection of the Albanese $W_\mathbb{C} = (\mathbb{C}^*)^n$ onto one of the factors. Then $\Pi \circ \alpha$ is a good complexification map by Corollary 2.5.

Hence, by Theorem 2.1, the composition is a homotopy smooth fibration with no critical points. Further, by Corollary 2.3, the fiber of $\Pi \circ \alpha$ over any point of S^1 is a good complexifications of the intersection of the fiber with $X_\mathbb{R}$.

Then $\Pi \circ \alpha : X_\mathbb{C} \longrightarrow \mathbb{C}^*$ induces a good complexification map with fiber F_z (say) over z. Let V_z denote the Albanese of the fiber $F_z = (\Pi \circ \alpha)^{-1}(z)$, and let

$$\widehat{\Pi \circ \alpha} : V \longrightarrow \mathbb{C}^*$$

be the bundle $\{V_z \mid z \in \mathbb{C}^*\}$. For any such fibration, we have $b_1(V_z) \geqslant (b_1(X_\mathbb{C}) - 1)$.

By Theorem 2.1, the above map $\widehat{\Pi \circ \alpha}$ defines a homotopy smooth fibration and the fibers V_z are of the form $(\mathbb{C}^*)^k$ by Lemma 2.4, where $k = b_1(V_z)$. Since V is a trivial bundle by Lemma 2.6, we have $(k+1) \leqslant b_1(X_{\mathbb{C}})$. It follows that $k = (b_1(X_{\mathbb{C}}) - 1)$.

By induction on the number of factors of $\mathbb{C}^*$ it follows that $F_z \longrightarrow V_z$ is a homotopy smooth fibration with no critical points. Hence α is a homotopy smooth fibration with no critical points.

Let F denote the general fiber of α. It remains to show that $b_1(F) = 0$. As above, pass to the relative Albanese bundle W_F of Albanese tori. Then W_F is a trivial smooth fibration over $W_{\mathbb{C}} = (\mathbb{C}^*)^n$ by Lemma 2.7 and hence has $b_1(W_F) > n$ unless $b_1(F) = 0$. Further, since W_F is a semi-abelian variety, we have $b_1(W_F) \leqslant n$. Therefore, it now follows that $b_1(F) = 0$.

That F is itself a good complexification now follows from Corollary 2.3 by induction the number of factors of $\mathbb{C}^*$. □

The virtual first Betti number $vb_1(M)$ is defined to be the supremum of the first Betti numbers of finite sheeted covers of M. It is known, [5, Lemma 4.1], that if M admits a good complexification and M_1 is a finite-sheeted cover of M, then M_1 admits a good complexification. We have the following immediate Corollary of Theorem 2.8:

Corollary 2.9. — *Suppose M is a closed n–dimensional manifold admitting a good complexification. Then $vb_1(M) \leqslant n$.*

Proof. — Let M_1 be a finite-sheeted cover of M. Then M_1 admits a good complexification [5, Lemma 4.1]. Let X be a good complexification of M_1. By Theorem 2.8 this X admits a smooth homotopy fibration with no critical points over the Albanese W of X. In particular, we have

$$n = \dim_{\mathbb{C}}(X) \geqslant b_1(X) = b_1(M_1)\,.$$

Since M_1 is arbitrary, the corollary follows. □

3. Good Complexification Groups

Theorem 3.1. — *Let G be a good complexification group. Then there exists a finite index subgroup G_1 of G such that*

(1) *There is an exact sequence:*

$$1 \longrightarrow H \longrightarrow G_1 \longrightarrow \mathbb{Z}^k \longrightarrow 1\,,$$

where k can be zero.

(2) *H is a finitely presented good complexification group with $vb_1(H) = 0$, where $vb_1(H)$ denotes the virtual first Betti number of H.*

Proof. — Let M be a closed smooth n–manifold admitting a good complexification with $\pi_1(M) = G$. By Corollary 2.9, there is a finite sheeted cover M_1 of M such that

$$vb_1(M) = b_1(M_1) = m \leqslant n\,.$$

Also by Theorem 2.8, M_1 fibers over $(S^1)^m$ with fiber a closed $(n-m)$–manifold N. Further N admits a good complexification by Theorem 2.8. Hence $vb_1(N) = 0$; indeed, if $vb_1(N) \neq 0$, by applying Theorem 2.8 again, $vb_1(M) = vb_1(M_1) > m$. Defining $H = \pi_1(N)$ and $G_1 = \pi_1(M_1)$, we obtain the following homotopy long exact sequence:

$$1 \longrightarrow HG_1 \longrightarrow \mathbb{Z}^{k'}\,.$$

Hence the theorem follows. □

Corollary 3.2. — *The following are not good complexification groups:*

1. (1) *Groups with infinite vb_1, in particular, large groups.*
2. (2) *Hyperbolic* CAT(0) *cubulated groups; in particular, infinite hyperbolic Coxeter groups.*
3. (3) *Solvable groups that are not virtually abelian.*
4. (4) *2-manifold groups that are not virtually abelian.*
5. (5) *3-manifold groups that are not virtually abelian.*

Proof.

(1) — Groups with infinite vb_1 cannot be good complexification groups by Theorem 3.1. Recall that G is large if G has a finite index subgroup that surjects onto the free group F_2 of two generators [5]. Clearly, large groups have infinite vb_1.

(2) — We refer the reader to [9, 6, 10] for generalities on hyperbolic and relatively hyperbolic groups, to [20, 27] for generalities on CAT(0) cube-complexes and their properties and to [12] for the theory of virtually special complexes.

A hyperbolic CAT(0) cubulated group G is virtually special and hence has infinite $vb_1(G)$ [1, 27]. We are thus reduced to Case (1) above.

The second statement follows from [12] where the authors show that infinite Coxeter groups are CAT(0) cubulated virtually special.

Non-elementary hyperbolic, or relatively hyperbolic, groups have asymptotic cones with cut points. On the other hand, a good complexification group with positive vb_1 must be virtually of the form $G_1 = \mathbb{Z}^k \times H$ by Theorem 3.1. Such a group has as asymptotic cone

$$AC(G) = AC(G_1) = \mathbb{R}^k \times AC(H),$$

where $AC(H)$ is an asymptotic cone of H, and $k \geqslant 1$. Hence $AC(G)$ can have a cut-point if and only if H is finite and $k = 1$, which means that G is virtually cyclic.

(3) — A solvable group G that is not virtually abelian admits, up to finite index, an exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where k is maximal, and $b_1(N) > 0$. This contradicts Theorem 3.1. Note that we avoided using [3] in this context.

(4) — Since 2-manifold groups G that are not virtually abelian have $vb_1(G) = \infty$, this follows from Case 1 above.

(5) — By work of a large number of people culminating in the work of Agol and Wise [1, 27] (see [4], especially Diagram 1 in page 36 for details), a 3-manifold group $G = \pi_1(M^3)$ is large unless G is virtually solvable, or equivalently, M^3 has geometry modelled on one of S^3, E^3, $S^2 \times \mathbb{R}$, Nil and Sol. Now, by Case 2 above, the group G must be virtually abelian. Hence M must have geometry modelled on one of S^3, E^3 and $S^2 \times \mathbb{R}$. □

Remark 3.3.

(1) Case 3 of Corollary 3.2 implies that any unipotent representation of a good complexification group is virtually abelian.
(2) It also follows from Theorem 3.1 that the commutator subgroup of a finite index subgroup of a good complexification group is finitely presented.
(3) Several cases of Corollary 3.2 above may be strengthened by saying that the groups listed there cannot even appear as quotients of good complexification groups. The proof involves applying Theorem 3.1 as done in Corollary 3.2.

4. Examples and Applications

In this Section, we apply Theorem 2.8 to furnish conclusions for low-dimensional examples.

4.1. 2 Manifolds

Lemma 4.1. — *Question 1.1 has an affirmative answer for 2-manifolds.*

Proof. — By Case 4 of Corollary 3.2, the only 2-manifolds that can possibly admit good complexifications are those finite covered by the sphere S^2 or the torus $S^1 \times S^1$. That all such manifolds do admit metrics of non-negative curvature is classical. Hence Question 1.1 has an affirmative answer for 2-manifolds. □

4.2. 3 Manifolds

Theorem 4.2 ([5]). — *Question 1.1 has an affirmative answer for 3-manifolds.*

Proof. — By Case 5 of Corollary 3.2, the only 3-manifolds that can possibly admit good complexifications are those finite covered by S^3, $S^2 \times S^1$ or $S^1 \times S^1 \times S^1$. That all such manifolds do admit metrics of non-negative curvature is a consequence of the Geometrization Theorem of Perelman. □

4.3. Simply Connected 4-manifolds

A stronger notion of good complexification appears implicitly in the work of McLean [17]. There, the author observes first that any smooth complex affine variety X has a natural structure of a symplectic manifold (up to symplectomorphisms) when regarded as a real smooth manifold. The cotangent bundle T^*M of a closed manifold M carries a natural symplectic structure (it is called the Liouville symplectic form). Corollary 1.3 of [17] implies that if X is *symplectomorphic* to T^*M for some closed manifold M (in particular, M admits a good complexification in the sense of [26] (recalled in Section 1)), then M is rationally elliptic. We say that M admits a *symplectic good complexification* if T^*M with its natural symplectic structure is symplectomorphic to a smooth complex affine variety X with its natural underlying symplectic structure.

Proposition 4.3. — *Let M be a closed simply connected 4-manifold admitting a symplectic good complexification. Then M admits a metric of non-negative curvature (i.e. Question 1.1 has an affirmative answer).*

Proof. — From [17, Corollary 1.3], the manifold M is rationally elliptic. Hence $b_2(M) \leqslant 2$. By the classification of simply connected 4-manifolds, M is homeomorphic to one of the following:

$$S^4,\ \mathbb{C}P^2,\ \mathbb{C}P^2\#\mathbb{C}P^2,\ \mathbb{C}P^2\#\overline{\mathbb{C}P^2},\ S^2 \times S^2 .$$

That each of these admits a metric of non-negative curvature follows from the examples constructed by Totaro in [26]. □

4.4. 4 manifolds with fundamental group $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$

We shall need the following theorem.

THEOREM 4.4 (Smale, Hatcher).

(1) $\mathrm{Diff}(S^2)$ *is homotopy equivalent to* $\mathrm{SO}(3)$ [22].
(2) $\mathrm{Diff}(S^3)$ *is homotopy equivalent to* $\mathrm{SO}(4)$ [13].

COROLLARY 4.5.

(a) *An S^2 bundle over T^2 is trivial after (at most) passing to a double cover of T^2.*
(b) *An S^2 bundle over T^3 is trivial after (at most) passing to a finite-sheeted cover of T^3.*
(c) *An S^3 bundle over T^2 is trivial after (at most) passing to a double cover of T^2.*
(d) *An S^3 bundle over T^3 is trivial after (at most) passing to a finite-sheeted cover of T^3.*

Proof.
(a) — The collection of distinct S^2-bundles over T^2 is in bijective correspondence with homotopy classes of maps $[T^2, \mathrm{BDiff}(S^2)]$. By the first statement of Theorem 4.4 this is equivalent to $[T^2, \mathrm{BSO}(3)]$. By the homotopy long exact sequence, $\pi_n(\mathrm{BSO}(3)) = \pi_{n-1}(SO(3))$. Since $\mathrm{BSO}(3)$ is simply connected, it follows that (after equipping T^2 with the standard CW complex structure consisting of one 0-cell, two 1-cells and one 2-cell) any map from T^2 to $\mathrm{BSO}(3)$ induces a map from S^2 to $\mathrm{BSO}(3)$, where S^2 is the quotient of T^2 obtained by collapsing its one skeleton to a point. Hence $[T^2, \mathrm{BDiff}(S^2)] = \pi_2(\mathrm{BSO}(3)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. It follows that after passing to a double cover of T^2 if necessary, the pullback S^2 bundle becomes trivial.

(b) — Repeating the argument for (a) above, we find that any S^2 bundle over T^3 can be made trivial over the 2-skeleton after passing to a finite-sheeted cover. Hence any map from T^3 to $\mathrm{BSO}(3)$ induces a map from S^3

to BSO(3), where S^3 is the quotient of T^3 obtained by collapsing its two skeleton to a point. Hence $[T^3, \mathrm{BDiff}(S^2)] = \pi_3(\mathrm{BSO}(3)) = \pi_2(\mathrm{SO}(3)) = 0$. It follows that after passing to a finite-sheeted cover of T^3 if necessary, the pullback S^2 bundle becomes trivial.

(c) — We repeat the argument in (a), replacing BSO(3) with BSO(4) (and using the second statement of Theorem 4.4) to obtain $[T^2, \mathrm{BDiff}(S^3)] = \pi_2(\mathrm{BSO}(4)) = \pi_1(\mathrm{SO}(4)) = \mathbb{Z}/2\mathbb{Z}$. Here the last statement follows from the fact that $SO(4)$ is diffeomorphic to $SO(3) \times S^3$. The rest of the argument is identical to that in (a).

(d) — We repeat the argument in (b) to obtain $[T^3, \mathrm{BDiff}(S^3)] = \pi_3(\mathrm{BSO}(4)) = \pi_2(\mathrm{SO}(4)) = 0$. The Corollary follows. □

Proposition 4.6. — *Let M be a closed 4-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2$ or 4. Further, the manifold M admits a finite-sheeted cover with a metric of non-negative curvature (i.e. Question 1.1 has an affirmative answer up to finite-sheeted covering).*

Proof. — By Theorem 2.8, the 4-manifold M admits a fiber bundle structure over $(S^1)^d$, where $1 \leqslant d \leqslant 4$ (by Corollary 2.9). Let N denote the fiber. Then $b_1(N) = 0$ (by Theorem 2.8 again). Hence N cannot be one-dimensional, and hence $d = 1, 2$ or 4.

If $d = 4$, then M is diffeomorphic to $(S^1)^4$.

If $d = 2$, then M fibers over $(S^1)^2$ with fiber N a 2-manifold admitting a good complexification and having $b_1(N) = 0$. Hence M is an S^2-bundle over T^2 and is finitely covered by $S^2 \times T^2$. Clearly, $S^2 \times T^2$ admits a metric of non-negative curvature.

If $d = 1$, then M fibers over S^1 with fiber N a 3-manifold admitting a good complexification and having $b_1(N) = 0$. By Theorem 4.2, N must be finitely covered by $(S^1)^3$ or $S^1 \times S^2$ or S^3. Hence after passing to a finite-sheeted cover if necessary, M is a fiber bundle over B with fiber N_1, where B, N_1 satisfy one of the following (Corollary 4.5):

(1) $B = (S^1)^4$, and N_1 is a point. In this case, M admits a finite-sheeted cover diffeomorphic to $(S^1)^4$.
(2) $B = (S^1)^2$, and $N_1 = S^2$. In this case, M admits a finite-sheeted cover diffeomorphic to $(S^1)^2 \times S^2$ as for the case $d = 2$ above.
(3) $B = S^1$, and $N_1 = S^3$. In this case, M admits a finite-sheeted cover diffeomorphic to $S^1 \times S^3$.

The Theorem follows. □

4.5. 5 Manifolds

The argument for Proposition 4.6 coupled with Corollary 4.5 furnishes the following fact for 5-manifolds as well.

Proposition 4.7. — *Let M be a closed 5-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2, 3$ or 5. Further, if $d = 2, 3$ or 5, then M admits a finite-sheeted cover M_1 homeomorphic to $S^2 \times T^3$ or $S^3 \times T^2$ or T^5. In particular, M_1 admits a metric of non-negative curvature.*

We do not repeat the proof of Proposition 4.6 here but observe that (after passing to a finite-sheeted cover), $d = 2$ corresponds to an S^3-bundle over T^2, $d = 3$ corresponds to an S^2-bundle over T^3, $d = 5$ corresponds to T^5.

4.6. Branched Covers

Proposition 4.8. — *Let $f : X \longrightarrow Y$ be a good complexification map between good complexifications X, Y such that f is a branched cover. Then f is actually étale.*

Proof. — The branch locus of $f_{\mathbb{C}}$ has to be of complex codimension one in $Y_{\mathbb{C}}$. Hence the branch locus $B_{\mathbb{R}} \subset Y_{\mathbb{R}}$ of $f_{\mathbb{R}}$ is of real codimension one. If $B_{\mathbb{R}} \neq \emptyset$ (or equivalently, if $f_{\mathbb{R}}$ is not étale) then $X_{\mathbb{R}}$ cannot be a manifold because the local model of $X_{\mathbb{R}}$ around a preimage of a point in $B_{\mathbb{R}}$ will consist of more than two half-spaces glued along a codimension one hyperplane. It follows that $B_{\mathbb{R}} = \emptyset$ and hence $f_{\mathbb{R}}$ is étale.

It remains to show that $f_{\mathbb{C}}$ is étale. First, since $f_{\mathbb{R}}$ is étale, and $Y_{\mathbb{R}} \subset Y_{\mathbb{C}}$ is a homotopy equivalence, we can construct a new good complexification $X^1_{\mathbb{C}}$ of $X_{\mathbb{R}}$ by simply taking $X^1_{\mathbb{C}}$ to be the cover of $Y_{\mathbb{C}}$ corresponding to $f_{\mathbb{R},*}(\pi_1(X_{\mathbb{R}}))$. Hence by lifting the map $f_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_C$ to $X^1_{\mathbb{C}}$, we have an algebraic homotopy equivalence h between $X_{\mathbb{C}}$ and $X^1_{\mathbb{C}}$ which is the identity on $X_{\mathbb{R}}$. Since $X_{\mathbb{R}}$ is Zariski-dense in $X_{\mathbb{C}}$, h must be the identity map. i.e. $X_{\mathbb{C}} = X^1_{\mathbb{C}}$. □

4.7. Questions

In order to answer Question 1.1 affirmatively, it suffices by Theorems 2.8, 3.1, to answer the following questions affirmatively:

QUESTION 4.9.

(a) *Suppose that the virtual first Betti number of a closed manifold M is zero: $vb_1(M) = 0$. If M admits a good complexification, then does M admit a metric of non-negative curvature?*
(b) *If M is a fiber bundle over $(S^1)^d$ admitting a good complexification, is the bundle trivial (at least up to finite sheeted covers)?*

We divide Question 4.9 (a) into two further questions:

QUESTION 4.10.

(a) *Suppose that G is a good complexification group with $vb_1(G) = 0$, is G finite?*
(b) *If a closed manifold M with finite fundamental group admits a good complexification, then does M admit a metric, at least virtually, of non-negative curvature?*

Remark 4.11. — An affirmative answer to both parts of Questions 4.9 and 4.10 exist if and only if Question 1.1 has an affirmative answer.

To see this, note that if M admits a metric of non-negative curvature, then the universal cover splits (by the Cheeger-Gromoll splitting Theorem [19, Theorem 69, p. 288]) as a metric product $N \times \mathbb{R}^k$ and hence $G = \pi_1(M)$ is virtually $\mathbb{Z}^k$.

Conversely, let $G = \pi_1(M)$ such that M admits a good complexification. By Theorem 3.1, G is virtually of the form $H \rtimes \mathbb{Z}^k$, where $H = \pi_1(N)$ where N admits a good complexification and $vb_1(N) = 0$. By a positive answer to Question 4.10(a), H must be finite. Hence N admits a finite simply connected cover N_1, which in turn admits a good complexification [5, Lemma 4.1]. Hence N_1 admits a metric of non-negative curvature by a positive answer to Question 4.10(b). Hence $N_1 \times (S^1)^k$ admits a metric of non-negative curvature. Consequently, by a positive answer to Question 4.9(b), M admits a finite sheeted cover with a metric of non-negative curvature.

Since every finite group is a subgroup of some $\mathrm{SU}(n)$, every finite group is a good complexification group [26], and also the fundamental group of a closed manifold of non-negative curvature.

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