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BROWNIAN MOTION ON TREEBOLIC SPACE: POSITIVE HARMONIC FUNCTIONS

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ABSTRACT. — This paper studies potential theory on treebolic space, that is, the horocyclic product of a regular tree and hyperbolic upper half plane. Relying on the analysis on strip complexes developed by the authors, a family of Laplacians with "vertical drift" parameters is considered. We investigate the positive harmonic functions associated with those Laplacians.

RÉSUMÉ. — Ce travail est dedié à une étude de la théorie du potentiel sur l'espace arbolique, i.e., le produit horcyclique d'un ârbre régulier avec le demi-plan hyperbolique supérieur. En se basant sur l'analyse sur les complexes à bandes Riemanniennes développée par les auteurs, on considère une famille de Laplaciens avec deux paramètres concernant la dérive verticale. On examine les fonctions harmoniques associées à ces Laplaciens.

1. Introduction

We recall the basic description of treebolic space. For many further details on the geometry, metric structure and isometry group, the reader is referred to [3].

We consider the homogeneous tree $\mathbb{T} = \mathbb{T}_{\mathsf{p}}$, drawn in such a way that each vertex v has one predecessor v^- and p successors. We consider $\mathbb{T}$ as a metric graph, where each edge is a copy of the unit interval $[0\,,1]$. The discrete, integer-valued graph metric $\mathsf{d}_{\mathbb{T}}$ on the vertex set (0-skeleton) $V(\mathbb{T})$ of $\mathbb{T}$ has an obvious "linear" extension to the entire metric graph. We can
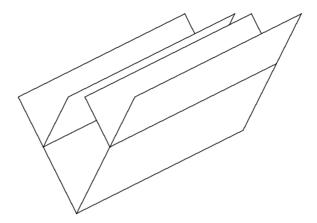
Figure 1. A compact piece of treebolic space, with $\mathsf{p} = 2$.[1]

partition the vertex set into countably many sets H_k, $k \in \mathbb{Z}$, such that each H_k is countably infinite, and every vertex $v \in H_k$ has its predecessor v^- in H_{k-1} and its successors in H_{k+1}. We write $[v^-, v]$ for the metric edge between those two vertices, parametrised by the unit interval. For any $w \in [v^-, v]$ we let $t = k - \mathsf{d}_\mathbb{T}(w, v)$ and set $\mathfrak{h}(w) = t$. In particular, $\mathfrak{h}(v) = k$ for $v \in H_k$. In general, we set $H_t = \{w \in \mathbb{T} : \mathfrak{h}(w) = t\}$. These sets are the *horocycles.* We choose a root vertex $o \in H_0$.

Second, we consider hyperbolic upper half space $\mathbb{H}$, and draw the horizontal lines $\mathsf{L}_k = \{x + \mathsf{i}\,\mathsf{q}^k : x \in \mathbb{R}\}$, thereby subdividing $\mathbb{H}$ into the strips $\mathsf{S}_k = \{x + \mathsf{i}\,y : x \in \mathbb{R}, \, \mathsf{q}^{k-1} \leqslant y \leqslant \mathsf{q}^k\}$, where $k \in \mathbb{Z}$. We sometimes write $\mathbb{H}(\mathsf{q})$ for the resulting strip complex, which is of course hyperbolic plane in every geometric aspect.

The reader is invited to have a look at the respective figures of $\mathbb{T}_\mathsf{p}$ and "sliced" hyperbolic plane $\mathbb{H}(\mathsf{q})$ in [3]. Treebolic space with parameters q and p is then

(1.1) $\mathsf{HT}(\mathsf{q}, \mathsf{p}) = \{\mathfrak{z} = (z, w) \in \mathbb{H} \times \mathbb{T}_\mathsf{p} : \mathfrak{h}(w) = \log_\mathsf{q}(\mathit{Im}\, z)\}.$

Thus, in treebolic space $\mathsf{HT}(\mathsf{q}, \mathsf{p})$, infinitely many copies of the strips S_k are glued together as follows: to each vertex v of $\mathbb{T}$ there corresponds the *bifurcation line*

$\mathsf{L}_v = \{(x + \mathsf{i}\,\mathsf{q}^k, v) : x \in \mathbb{R}\} = \mathsf{L}_k \times \{v\}, \quad \text{where} \quad \mathfrak{h}(v) = k.$

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[1] This figure also appears in [2].

Along this line, the strips

$$\mathsf{S}_v = \left\{ (x + \mathfrak{i}\, y, w) : x \in \mathbb{R}\,,\ y \in [\mathsf{q}^{k-1}\,,\,\mathsf{q}^k]\,,\ w \in [v^-, v]\,,\ \mathfrak{h}(w) = \log_{\mathsf{q}} y \right\}$$

and S_u, where $u \in V(\mathbb{T})$ with $u^- = v$, are glued together. The *origin* of our space is $\mathfrak{o} = (0, o)$.

In [3], we outlined several viewpoints why this space is interesting. It can be considered as a concrete example of a Riemannian complex in the spirit of Eells and Fuglede [15]. Treebolic space is a *strip complex* ("quantum complex") in the sense of [2], where a careful study of Laplace operators on such spaces is undertaken, taking into account the serious subtleties arising from the singularites of such a complex along its *bifurcation manifolds.* In our case, the latter are the lines L_v, $v \in V(\mathbb{T})$. The other interesting feature is that treebolic spaces are *horocyclic products* of $\mathbb{H}(\mathsf{q})$ and $\mathbb{T}_\mathsf{p}$, with a structure that shares many features with the *Diestel-Leader graphs* $\mathsf{DL}(\mathsf{p}, \mathsf{q})$ [13] as well as with the manifold (Lie group) $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$, the horocyclic product of two hyperbolic planes with curvatures $-\mathsf{p}^2$ and $-\mathsf{q}^2$, respectively, where $\mathsf{p}, \mathsf{q} > 0$. Compare with Woess [25], Brofferio and Woess [7], [8] and with Brofferio, Salvatori and Woess [6]; see also the survey by Woess [26]. The graph $\mathsf{DL}(\mathsf{p}, \mathsf{p})$ is a Cayley graph of the Lamplighter group $\mathbb{Z}_\mathsf{p} \wr \mathbb{Z}$. Similarly, the Baumslag-Solitar group $\mathsf{BS}(\mathsf{p}) = \langle a, b \mid ab = b^\mathsf{p} a \rangle$ is a prominent group that acts isometrically and with compact quotient on $\mathsf{HT}(\mathsf{p}, \mathsf{p})$.

In the present paper, we take up the thread from [3], where we have investigated the details of the metric structure, geometry and isometries before describing the spatial asymptotic behaviour of Brownian motion on treebolic space. Brownian motion is induced by a Laplace operator $\Delta_{\alpha, \beta}$, where the parameter α is the coefficient of a vertical drift term in the interior of each strip, while β is responsible for (again vertical) Kirchhoff type bifurcation conditions along the lines L_v. The serious task of rigorously constructing $\Delta_{\alpha, \beta}$ as an essentially self-adjoint diffusion operator was undertaken in the general setting of strip complexes in [2]. For the specific case of HT, it is explained in detail in [3]. We shall recall only its basic features, while we shall use freely the geometric details from [3].

Here, we face the rather difficult issue to describe, resp. determine the positive harmonic functions on $\mathsf{HT}(\mathsf{q}, \mathsf{p})$. In §2, we recall the basic features of treebolic space and the family of Laplacians with Kirchhoff conditions at the bifurcation lines. In §3, we start with a Poisson representation on "rectangular" sets which are compact. Then we obtain such a representation on simply connected sets which are unions of strips and a solution of the Dirichlet problem on those sets. (For the technical details of the main

results presented here, the reader is referred to the respective sections.) For the following, let $\overline{\Omega}$ be the union of all closed strips adjacent to L_o and Ω its interior, and let $\mu_{\mathfrak{w}}^{\Omega}$ be the exit distribution from Ω of Brownian motion starting at $\mathfrak{w} \in \Omega$.

THEOREMS 3.9 & 3.11.

(I) *Let h be continuous on $\overline{\Omega}$ and harmonic on Ω, and suppose that on that set, h grows at most exponentially with respect to the metric of* HT. *Then for every $\mathfrak{w} \in \Omega$,*

$$h(\mathfrak{w}) = \int_{\partial\Omega} h(\mathfrak{z}) \, d\mu_{\mathfrak{w}}^{\Omega}(\mathfrak{z}) \,.$$

(II) *Let f be continuous on $\partial\Omega$, and suppose that on that set, f grows at most exponentially with respect to the metric of* HT. *Then*

$$h(\mathfrak{w}) = \int_{\partial\Omega} f(\mathfrak{z}) \, d\mu_{\mathfrak{w}}^{\Omega}(\mathfrak{z}) \,, \quad \mathfrak{w} \in \Omega$$

defines an extension of f that is harmonic on Ω and continuous on $\overline{\Omega}$.

The difficulties arise from the singularities at the bifurcation lines plus the fact that Ω is unbounded in the horizontal direction. We obtain that the law $\mu = \mu_{\mathfrak{w}}^{\Omega}$ of the random walk induced by Brownian motion on the union LT of all bifurcation lines has a continuous density and exponential tails. The main result of §4 is as follows.

THEOREM 4.7. — *Restricting positive harmonic functions on* HT *to* LT *yields a one-to-one relation with the positive harmonic functions of the random walk driven by μ on* LT.

This is of interest, because the disconnected subspace LT $\subset$ HT and that random walk are invariant under the transitive action of the isometry group of HT. In §5, we use Martin boundary theory to prove the following decomposition.

THEOREM 5.2. — *Every positive harmonic function h on* HT *has the form*

$$h(\mathfrak{z}) = h^{\mathbb{H}}(z) + h^{\mathbb{T}}(w) \,, \quad \mathfrak{z} = (z, w) \in \mathsf{HT} \,,$$

where $h^{\mathbb{H}}$ is non-negative harmonic on $\mathbb{H}(\mathsf{q})$ and $h^{\mathbb{T}}$ is non-negative harmonic on $\mathbb{T}_{\mathsf{p}}$.

Given the drift parameters α and β, there is a formula for the *vertical drift* $\ell(\alpha, \beta) \in \mathbb{R}$, and $|\ell(\alpha, \beta)|$ is the rate of escape of our Brownian motion.

THEOREM 5.6. — *Suppose that* $\mathsf{p} \geqslant 2$. *Then the Laplacian* $\Delta_{\alpha,\beta}$ *on* HT *has the weak Liouville property (all bounded harmonic functions are constant) if and only if* $\ell(\alpha,\beta) = 0$.

Finally, in the situation when the projection of the Laplacian on the hyperbolic plane is smooth, we can describe all minimal harmonic functions explicitly.

The techniques that we employ here can be applied to obtain similar results on other types of strip complexes. The simplest one is where one replaces "sliced" hyperbolic by Euclidean plane. Very close to the present study is the case where one takes hyperbolic upper half space $\mathbb{H}_d$ in arbitrary dimension, subdivided by level sets (horospheres) of the Busemann function with respect to the boundary point ∞. Also of interest: to replace the tree by the 1-skeleton of a higher-dimensional $\widetilde{A}_d$-building, or just to take different types of level functions on a regular tree; compare e.g. with the space considered by Cuno and Sava-Huss [12].

2. Laplacians on treebolic space

We briefly recapitulate the most important features of HT, in particular, the construction of our family of Laplace operators $\Delta_{\alpha,\beta}$ on $\mathsf{HT}(\mathsf{q},\mathsf{p})$ with parameters $\alpha \in \mathbb{R}$ and $\beta > 0$. For more details, see [3]. The rigorous construction is carried out in [2].

For any function $f : \mathsf{HT} \to \mathbb{R}$, we write f_v for its restriction to S_v. For $(z,w) \in \mathsf{S}_v$, the element $w \in [v^-, v]$ is uniquely determined, so that we can omit w and write $f_v(z) = f(z,w)$: formally, f_v is defined on $\mathsf{S}_k \subset \mathbb{H}$, where $k = \mathfrak{h}(v)$.

DEFINITION 2.1. — *We let* $\mathcal{C}^{\infty}(\mathsf{HT})$ *be the space of those continuous functions* f *on* HT *such that, for each* $v \in V(\mathbb{T})$, *the restriction* f_v *on* S_v *has continous derivatives* $\partial_x^m \partial_y^n f_v(z)$ *of all orders in the interior* S_v^o *which satisfy, for all* $R > 0$,

$$\sup\big\{|\partial_x^m \partial_y^n f_v(z)| : |Re\, z| \leqslant R,\ \mathsf{q}^{\mathfrak{h}(v)-1} < Im\, z < \mathsf{q}^{\mathfrak{h}(v)}\big\} < \infty.$$

Thus, on each strip S_v, each partial derivative has a continuous extension $\partial_x^m \partial_y^n f_v(z)$ to the strip's boundary. However, except for $m = n = 0$, when $w^- = v$, we do in general *not* have that $\partial_x^m \partial_y^n f_w = \partial_x^m \partial_y^n f_v$ on $\mathsf{L}_v = \mathsf{S}_v \cap \mathsf{S}_w$. The hyperbolic gradient

$$\nabla f_v(z) = \big(y^2 \partial_x f_v(z)\,,\ y^2 \partial_y f_v(z)\big)$$

is defined in the interior of each strip. On any bifurcation line L_v, we have to distinguish between all the one-sided limits of the gradient, obtaining the family

$$\nabla f_v(z) \quad \text{and} \quad \nabla f_w(z) \quad \text{for all } w \in V(\mathbb{T}) \text{ with } w^- = v, \quad (z,v) \in \mathsf{L}_v.$$

For any open domain $\Theta \subset \mathsf{HT}$, we let let $\mathcal{C}_c^\infty(\Theta)$ be the space of all functions in $\mathcal{C}^\infty(\mathsf{HT})$ which have compact support contained in Θ. We let

$$(2.1) \qquad \mathsf{LT} = \bigcup_{v \in V(\mathbb{T})} \mathsf{L}_v \quad \text{and} \quad \mathsf{HT}^o = \bigcup_{v \in V(\mathbb{T})} \mathsf{S}_v^o = \mathsf{HT} \setminus \mathsf{LT}.$$

The *area element* of HT is $d\mathfrak{z} = y^{-2}dx\,dy$ for $\mathfrak{z} = (z,w) \in \mathsf{HT}^o$, where $z = x + \mathsf{i}\,y$ and dx, dy are Lebesgue measure: this is (a copy of) the hyperbolic upper half plane area element. The area of the lines L_v is 0. For $\alpha \in \mathbb{R}$, $\beta > 0$, we define the measure $\mathbf{m}_{\alpha,\beta}$ on HT by

$$(2.2) \qquad \begin{aligned} &d\mathbf{m}_{\alpha,\beta}(\mathfrak{z}) = \phi_{\alpha,\beta}(\mathfrak{z})\,d\mathfrak{z}, \quad \text{with} \\ &\phi_{\alpha,\beta}(\mathfrak{z}) = \beta^{\mathfrak{h}(v)}\,y^\alpha \quad \text{for} \quad \mathfrak{z} = (x + \mathsf{i}\,y, w) \in \mathsf{S}_v \setminus \mathsf{L}_{v^-}, \end{aligned}$$

where $v \in V(\mathbb{T})$, that is, $w \in (v^-, v]$ and $\log_{\mathsf{q}} y = \mathfrak{h}(w)$.

DEFINITION 2.2. — *For $f \in \mathcal{C}^\infty(\mathsf{HT})$ and $\mathfrak{z} = (x + \mathsf{i}\,y, w) \in \mathsf{HT}^o$,*

$$\Delta_{\alpha,\beta} f(\mathfrak{z}) = y^2(\partial_x^2 + \partial_y^2)f(\mathfrak{z}) + \alpha\,y\,\partial_y f(\mathfrak{z}).$$

Let $\mathcal{D}_{\alpha,\beta,c}^\infty$ be the space of all functions $f \in \mathcal{C}_c^\infty(\mathsf{HT})$ with the following properties.

   (i) *For any k, the k-th iterate $\Delta_{\alpha,\beta}^k f$, originally defined on HT^o, admits a continuous extension to all of HT (which then belongs to $\mathcal{C}_c^\infty(\mathsf{HT})$ and is also denoted $\Delta_{\alpha,\beta}^k f$).*

   (ii) *The function f, as well as each of its iterates $\Delta_{\alpha,\beta}^k f$, satisfies the bifurcation conditions*

$$(2.3) \qquad \partial_y f_v = \beta \sum_{w\,:\,w^- = v} \partial_y f_w \quad \text{on } L_v \text{ for each } v \in V(\mathbb{T}).$$

PROPOSITION 2.3 ([2]). — *The space $\mathcal{D}_{\alpha,\beta,c}^\infty$ is dense in the Hilbert space $\mathcal{L}^2(\mathsf{HT}, \mathbf{m}_{\alpha,\beta})$. The operator $(\Delta_{\alpha,\beta}, \mathcal{D}_{\alpha,\beta,c}^\infty)$ is essentially self-adjoint in $\mathcal{L}^2(\mathsf{HT}, \mathbf{m}_{\alpha,\beta})$.*

We also write $\big(\Delta_{\alpha,\beta}, \operatorname{Dom}(\Delta_{\alpha,\beta})\big)$ for its unique self-adjoint extension. Basic properties of this Laplacian and the associated heat semigroup are derived in [2]. In particular, there is the positive, continuous, symmetric heat kernel $\mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})$ on $(0, \infty) \times \mathsf{HT} \times \mathsf{HT}$ such that for all $f \in \mathcal{C}_c(\mathsf{HT})$,

$$e^{t\Delta_{\alpha,\beta}} f(\mathfrak{z}) = \int_{\mathsf{HT}} \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\,f(\mathfrak{z})\,d\mathbf{m}_{\alpha,\beta}(\mathfrak{z}).$$

$\Delta_{\alpha,\beta}$ is the infinitesimal generator of our Brownian motion $(X_t)_{t \geqslant 0}$ on HT. It has infinite life time and continuous sample paths. For every starting point $\mathfrak{w} \in$ HT, its distribution $\mathbb{P}_{\mathfrak{w}}^{\alpha,\beta}$ on $\Omega = \mathcal{C}([0,\infty] \to$ HT$)$ is determined by the one-dimensional distribution

$$\mathbb{P}_{\mathfrak{w}}^{\alpha,\beta}[X_t \in U] = \int_U \mathbf{h}_{\alpha,\beta}(t,\mathfrak{w},\mathfrak{z})\, d\mathbf{m}_{\alpha,\beta}(\mathfrak{z}) = \int_U \mathbf{p}_{\alpha,\beta}(t,\mathfrak{w},\mathfrak{z})\, d\mathfrak{z}\,,$$

where U is any Borel subset of HT and

$$(2.4) \qquad\qquad \mathbf{p}_{\alpha,\beta}(t,\mathfrak{w},\mathfrak{z}) = \mathbf{h}_{\alpha,\beta}(t,\mathfrak{w},\mathfrak{z})\, \phi_{\alpha,\beta}(\mathfrak{z})$$

with $\phi_{\alpha,\beta}$ as in (2.2).

We have the projections $\pi^{\mathbb{H}} :$ HT $\to \mathbb{H}$, $\pi^{\mathbb{T}} :$ HT $\to \mathbb{T}$ and $\pi^{\mathbb{R}} :$ HT $\to \mathbb{R}$, where for $\mathfrak{z} = (z,w)$,

$$\pi^{\mathbb{H}}(\mathfrak{z}) = z\,, \quad \pi^{\mathbb{T}}(z,w) = w\,, \quad \text{and} \quad \pi^{\mathbb{R}}(\mathfrak{z}) = \log_{\mathsf{q}} Im(z) = \mathfrak{h}(w)\,.$$

On several occasions it will be useful to write

$$Re\,\mathfrak{z} = Re\,z \quad \text{for} \quad \mathfrak{z} = (z,w) \in \text{HT}\,.$$

The "sliced" hyperbolic $\mathbb{H}(\mathsf{q})$ plane is the treebolic space HT$(\mathsf{q},1)$: the tree is the bi-infinite line graph. In particular, we have the operator $\Delta_{\alpha,\beta}^{\mathbb{H}}$ on $\mathbb{H}$. We also have a Laplacian $\Delta_{\alpha,\beta}^{\mathbb{T}}$ on the metric tree, whose rigourous construction (in the same way as above) is considerably simpler. Note the different parametrisation in $\mathbb{T}$, where each edge $[v^-,v]$ corresponds to the real interval $[\mathfrak{h}(v)-1\,,\mathfrak{h}(v)]$. We write f_v for the restriction of $f : \mathbb{T} \to \mathbb{R}$ to $[v^-,v]$, which depends on one real variable. The analogue of the measure of (2.2) is

$$(2.5) \qquad \begin{aligned} & d\mathbf{m}_{\alpha,\beta}^{\mathbb{T}}(w) = \phi_{\alpha,\beta}^{\mathbb{T}}(w)\, dw\,, \quad \text{with} \\ & \phi_{\alpha,\beta}^{\mathbb{T}}(w) = \beta^{\mathfrak{h}(v)}\, \mathsf{q}^{(\alpha-1)\mathfrak{h}(w)} \log\mathsf{q} \quad \text{for} \quad w \in (v^-,v]\,, \end{aligned}$$

where $v \in V(\mathbb{T})$ and dw is the standard Lebesgue measure in each edge. The space $\mathcal{C}^\infty(\mathbb{T})$ is as in Definition 2.1, with the edges of $\mathbb{T}$ in the place of the strips. Definition 2.2 and the bifurcation condition (2.3) are replaced by the following: every $f \in \text{Dom}(\Delta_{\alpha,\beta}^{\mathbb{T}}) \cap \mathcal{C}^\infty(\mathbb{T})$ must satisfy for every $v \in V(\mathbb{T})$

$$(2.6) \qquad \begin{aligned} & f_v'(v) = \beta \sum_{w\,:\,w^-=v} f_w'(v) \quad \text{and} \\ & \Delta_{\alpha,\beta}^{\mathbb{T}} f = \frac{1}{(\log\mathsf{q})^2} f'' + \frac{\alpha-1}{\log\mathsf{q}} f' \quad \text{in the open edge } (v^-,v)\,. \end{aligned}$$

We also have this on the real line, by identifying $\mathbb{R}$ with the metric tree with vertex set $\mathbb{Z}$ and degree 2. The edges are the intervals $[k-1\,,k]$, $k \in \mathbb{Z}$,

and the Laplacian is $\Delta_{\alpha,\beta}^{\mathbb{R}}$. Its definition as a differential operator in each open interval $(k-1,\,k)$ is as in (2.6), and the bifurcation condition now is $f'(k-) = \beta\, f'(k+)$ for all $k \in \mathbb{Z}$. We have the following.

PROPOSITION 2.4. — Let (X_t) be Brownian motion on $\mathsf{HT}(\mathsf{q},\mathsf{p})$ with infinitesimal generator $\Delta_{\alpha,\beta}$.

    (a) $Z_t = \pi^{\mathbb{H}}(X_t)$ is the Markov process on $\mathbb{H}(\mathsf{q})$ with generator $\Delta_{\alpha,\beta\mathsf{p}}^{\mathbb{H}}$.

    (b) $W_t = \pi^{\mathbb{T}}(X_t)$ is the Markov process on $\mathbb{T}_{\mathsf{p}}$ with generator $\Delta_{\alpha,\beta}^{\mathbb{T}}$.

    (c) $Y_t = \pi^{\mathbb{R}}(X_t)$ is the Markov process on on $\mathbb{R}$ with generator $\Delta_{\alpha,\beta\mathsf{p}}^{\mathbb{R}}$.

We also need to recall further features of the geometry of HT, as well as of $\mathbb{T}$ and $\mathbb{H}$. For two points $w, w' \in \mathbb{T}$, their *confluent* $w \curlywedge w'$ is the unique element v on the geodesic path $\overline{w\,w'}$ where $\mathfrak{h}(v)$ is minimal. In the specific case when w and w' lie on a geodesic ray spanned by a sequence of vertices that are successive predecessors, the confluent is is one of w and w', while otherwise it is always a vertex. Analogously, for two points $z, z' \in \mathbb{H}$, we let $z \wedge z'$ be the point on the hyperbolic geodesic $\overline{z\,z'}$ where the imaginary part is maximal. Recall that hyperbolic geodesics lie on semi-circles which are orthogonal to the bottom boundary line $\mathbb{R}$, resp. on vertical straight lines. It is a straightforward (Euclidean) exercise to see that

$$(2.7) \qquad |Re\,z - Re\,z'| \leqslant 2\,Im\,z \wedge z'\,, \quad z, z' \in \mathbb{H}\,.$$

Returning to the tree, the *boundary* ∂T of $\mathbb{T}$ is its space of *ends*. Each geodesic ray in $\mathbb{T}$ gives rise to an end at infinity, and two rays have the same end if they coincide except for initial pieces of finite length. In our view on $\mathbb{T}$ as in [3, Figure 3], the tree has one end ϖ at the bottom, and all other ends at the top of the picture, forming the boundary part $\partial^*\mathbb{T}$. For any $w \in \mathbb{T}$ and $\xi \in \partial\mathbb{T}$, there is a unique geodesic ray $\overline{w\,\xi}$ starting at w and having ξ as its end. Analogously, for any two distinct $\xi, \eta \in \partial\mathbb{T}$, there is a unique bi-infinite geodesic $\overline{\eta\,\xi}$ such that if we split it at any point, one part is a ray going to ξ and the other a ray going to η. If both of them belong to $\partial^*\mathbb{T}$ then we can also define $\xi \curlywedge \eta$ as the element v on the geodesic where $\mathfrak{h}(v)$ is minimal; it is a vertex. If $\xi \in \partial^*\mathbb{T}$ then the "vertical" geodesic $\overline{\varpi\,\xi}$ is the side view of a copy of the hyperbolic plane sitting in HT, namely $\mathbb{H}_\xi = \{(z,w) \in \mathsf{HT} : w \in \overline{\varpi\,\xi}\}$.

The metric d_{HT} of HT is induced by the hyperbolic arc length inside each strip: let $\mathfrak{z}_1 = (z_1, w_1), \mathfrak{z}_2 = (z_2, w_2) \in \mathsf{HT}$. Let $v = w_1 \curlywedge w_2$. Then

$$(2.8) \qquad \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) = \begin{cases} \mathsf{d}_{\mathbb{H}}(z_1, z_2)\,, & \text{if } v \in \{w_1, w_2\} \\ \min\{\mathsf{d}_{\mathbb{H}}(z_1, z) + \mathsf{d}_{\mathbb{H}}(z, z_2) : z \in L_{\mathfrak{h}(v)}\}\,, \\ & \text{if } v \notin \{w_1, w_2\}\,. \end{cases}$$

In the first case, $\mathfrak{z}_1$ and $\mathfrak{z}_2$ belong to a common copy $\mathbb{H}_\xi$ of $\mathbb{H}$ in HT. In the second case, $v \in V(\mathbb{T})$, and there are $\xi_1, \xi_2 \in \partial^*\mathbb{T}$ such that $\xi_1 \curlywedge \xi_2 = v$ and both $\mathfrak{z}_i$ lie above L_v in $\mathbb{H}_{\xi_i}$, whence any geodesic from $\mathfrak{z}_1$ to $\mathfrak{z}_2$ must pass through a (unique!) point $\mathfrak{z} \in \mathsf{L}_v$. See [3, Figure 5].

Next, we recall the isometry group of $\mathsf{HT}(\mathsf{q},\mathsf{p})$. First, the locally compact group of affine transformations

$$(2.9) \qquad \mathsf{Aff}(\mathbb{H},\mathsf{q}) = \left\{ g = \begin{pmatrix} \mathsf{q}^n & b \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z},\ b \in \mathbb{R} \right\}, \quad gz = \mathsf{q}^n z + b$$

acts on $\mathbb{H}(\mathsf{q})$ by isometries and preserves the set of "slicing" lines L_k $(k \in \mathbb{Z})$. Left Haar measure dg and its modular function $\boldsymbol{\delta}_{\mathbb{H}} = \boldsymbol{\delta}_{\mathbb{H},\mathsf{q}}$ are

$$dg = \mathsf{q}^{-n}\, dn\, db \quad \text{and} \quad \boldsymbol{\delta}_{\mathbb{H}}(g) = \mathsf{q}^{-n}, \quad \text{if} \quad g = \begin{pmatrix} \mathsf{q}^n & b \\ 0 & 1 \end{pmatrix},$$

where dn is counting measure on $\mathbb{Z}$ and db is Lebesgue measure on $\mathbb{R}$.

Second, denote by $\mathsf{Aut}(\mathbb{T}_\mathsf{p})$ the full isometry group of $\mathbb{T}_\mathsf{p}$ and consider the *affine group* of $\mathbb{T}_\mathsf{p}$,

$$(2.10) \qquad \mathsf{Aff}(\mathbb{T}_\mathsf{p}) = \{ \gamma \in \mathsf{Aut}(\mathbb{T}_\mathsf{p}) : (\gamma v)^- = v^- \text{ for all } v \in V(\mathbb{T}_\mathsf{p}) \}.$$

The modular function $\boldsymbol{\delta}_{\mathbb{T}} = \boldsymbol{\delta}_{\mathbb{T}_\mathsf{p}}$ of $\mathsf{Aff}(\mathbb{T}_\mathsf{p})$ is given by

$$\boldsymbol{\delta}_{\mathbb{T}}(\gamma) = \mathsf{p}^{\Phi(\gamma)} \quad \text{where} \quad \Phi(\gamma) = \mathfrak{h}(\gamma w) - \mathfrak{h}(w), \quad \text{if} \quad \gamma \in \mathsf{Aff}(\mathbb{T}_\mathsf{p}),\ w \in \mathbb{T}.$$

The above mapping $\Phi : \mathsf{Aff}(\mathbb{T}) \to \mathbb{Z}$ is independent of $w \in \mathbb{T}$ and a homomorphism.

THEOREM 2.5 ([3]). — *The group*

$$\mathcal{A} = \mathcal{A}(\mathsf{q},\mathsf{p}) = \{ (g,\gamma) \in \mathsf{Aff}(\mathbb{H},\mathsf{q}) \times \mathsf{Aff}(\mathbb{T}_\mathsf{p}) : \log_\mathsf{q} \boldsymbol{\delta}_{\mathbb{H}}(g) + \log_\mathsf{p} \boldsymbol{\delta}_{\mathbb{T}}(\gamma) = 0 \}$$

acts by isometries $(g,\gamma)(z,w) = (gz,\gamma w)$ *on* $\mathsf{HT}(\mathsf{q},\mathsf{p})$*, with compact quotient isomorphic with the circle of length* $\log \mathsf{q}$*. It leaves the area element* $d\mathfrak{z}$ *of* HT *as well as the transition kernel (2.4) invariant:*

$$(2.11) \qquad\qquad \mathbf{p}_{\alpha,\beta}(t, \mathfrak{g}\mathfrak{w}, \mathfrak{g}\mathfrak{z}) = \mathbf{p}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})$$

for all $t > 0$*,* $\mathfrak{w}, \mathfrak{z} \in \mathsf{HT}$ *and* $\mathfrak{g} \in \mathcal{A}$*.*

Next, we recall from [3] the stopping times $\tau(n)$ $(n \in \mathbb{N}_0)$ of the successive visits of (Y_t) in $\mathbb{Z}$

$$(2.12) \qquad \tau(0) = 0, \quad \tau(n+1) = \inf\{ t > \tau(n) : Y_t \in \mathbb{Z} \setminus \{Y_{\tau(n)}\} \}.$$

Via Proposition 2.4, we can also interpret them in terms of Brownian motion on HT. If X_0 lies in S_v^o, then $\tau(1)$ is the instant when X_t first meets a point on $\mathsf{L}_v \cup \mathsf{L}_{v^-}$. If $X_{\tau(n)} \in \mathsf{L}_v$ for some $v \in V(\mathbb{T})$ (which holds for

all $n \geqslant 1$, and possibly also for $n = 0$), then $\tau(n+1)$ is the first instant $t > \tau(n)$ when X_t meets one of the bifurcation lines L_{v^-} or L_w with $w^- = v$.

The increments $\tau(n) - \tau(n-1)$, $n \geqslant 1$, are independent and almost surely finite. They are identically distributed for $n \geqslant 2$, and when $Y_0 \in \mathbb{Z}$ (equivalently, $Z_0 \in \mathsf{LT}$), then also $\tau(1)$ has the same distribution. The random variables

$$(2.13) \qquad \tau = \tau(2) - \tau(1) \quad \text{and} \quad Y = Y_{\tau(2)} - Y_{\tau(1)}$$

are independent,

$$(2.14) \quad \Pr[Y = 1] = \frac{\mathsf{a}}{\mathsf{a} + 1} \quad \text{and} \quad \Pr[Y = -1] = \frac{1}{\mathsf{a} + 1}, \quad \mathsf{a} = \beta \mathsf{p}\, \mathsf{q}^{\alpha - 1},$$

and τ has finite exponential moment $\mathsf{E}(e^{\lambda_0 \tau})$ for some $\lambda_0 > 0$.

For any open domain $\Theta \subset \mathsf{HT}$, we let

$$(2.15) \qquad\qquad \tau^\Theta = \inf\{t > 0 : X_t \in \mathsf{HT} \setminus \Theta\}$$

be the *first exit time* of (X_t) from Θ.

If $\tau = \tau^\Theta < \infty$ almost surely for the starting point $X_0 = \mathfrak{w} \in \Theta$, then we write $\mu_{\mathfrak{w}}^\Theta$ for the distribution of X_τ. In all cases that we shall consider, $\mu_{\mathfrak{w}}^\Theta$ is going to be a probability measure supported by $\partial\Theta$. We shall use analogous notation on $\mathbb{H}$, $\mathbb{T}$ and $\mathbb{R}$. We note that by group invariance of our Laplacian,

$$(2.16) \qquad \mu_{\mathfrak{w}}^\Theta(B) = \mu_{\mathfrak{g}\mathfrak{w}}^{\mathfrak{g}\Theta}(\mathfrak{g}B) \quad \text{for every } \mathfrak{g} \in \mathcal{A} \text{ and Borel set } B \subset \mathsf{HT}.$$

We conclude this section with the "official" definition of harmonic functions.

DEFINITION 2.6. — *Let $\Theta \subset \mathsf{HT}$ be open. A continuous function $f : \Theta \to \mathbb{R}$ is called* harmonic *on Θ if for every open, relatively compact set U with $\overline{U} \subset \Theta$,*

$$f(\mathfrak{z}) = \int f \, d\mu_{\mathfrak{z}}^U \quad \text{for all } \mathfrak{z} \in U.$$

As mentioned in [3], from a classical analytic viewpoint, "harmonic" should rather mean "annihilated by the Laplacian", but for general open domains in HT, the correct formulation in these terms is subtle in view of the relative location of the bifurcations. By [2, Theorem 5.9], if Θ is suitably "nice", then any harmonic function on Θ is anihilated by $\Delta_{\alpha,\beta}$. This is true in particular for the sets Ω_r that we are going to use in the next section; see Figure 2 below. More details will be stated and used later on; in particular, see Proposition 3.1 regarding globally harmonic functions on HT.

3. Harmonic functions on rectangular sets

In this section, we want to derive a Poisson representation formula for harmonic functions on open domains $\Omega \subset \mathsf{HT}$ which have a rectangular shape. Prototypes are the following sets, where $v \in V(\mathbb{T})$ and $r > 0$.

(3.1)
$$\Omega_v = \{(z, w) \in \mathsf{HT} : w \in N(v)^o\} \subset \mathsf{HT}, \quad \text{and}$$
$$\Omega_{v,r} = \{\mathfrak{z} \in \Omega_v : |Re\,\mathfrak{z}| < r\},$$

where $N(v)$ is the "neighbourhood star" at v in $\mathbb{T}$. That is, $N(v)$ is the union of all edges ($\equiv$ intervals !) of $\mathbb{T}$ which have v as one endpoint. It is a compact metric subtree of $\mathbb{T}$, whose boundary $\partial N(v)$ consists of all neighbours of v in $V(\mathbb{T})$. We write $\partial^+ N(v) = \partial N(v) \setminus \{v^-\}$ (the forward neighbours of v).

Note that we can define the space $\mathcal{C}^\infty(\Omega)$ for $\Omega = \Omega_v$ or $\Omega_{v,r}$ in the same way as in Definition 2.1 by restricting to the (pieces of the) strips that make up Ω. We recall the following [2, Theorem 5.9]:

PROPOSITION 3.1. — *A function f is harmonic on $\Omega = \mathsf{HT}$, resp. $\Omega = \Omega_v$, resp. $\Omega = \Omega_{v,r}$ if and only if $f \in \mathcal{C}^\infty(\Omega)$, it satisfies the bifurcation conditions (2.3) at each bifurcation line inside Ω, and $y^2(\partial_x^2 + \partial_y^2)f + \alpha\, y\, \partial_y f = 0$ in each open strip in Ω.*

For deriving a Poisson representation for harmonic functions on Ω_v, it is sufficient to consider just Ω_o (by group invariance). We shall henceforth always reserve the letter Ω for this specific set. We follow classical reasoning, but here we have to be careful in view of the singularities of our domain along the bifurcation line L_o; the validity of each step has to be checked.

We first work with the bounded subsets $\Omega_r = \Omega_{o,r}$. Here, $r > 0$ will always be a real number, so that Ω_r should not be confounded with Ω_v for $v \in V(\mathbb{T})$. The boundary $\partial\Omega_r$ in HT consists of the *horizontal part* and the *vertical part*, given by

$$\partial^{\mathrm{hor}}\Omega_r = \{(z, v) \in \mathsf{HT} : v \in \partial N(o), |Re\,z| \leqslant r\} \quad \text{and}$$

(3.2)
$$\partial^{\mathrm{vert}}\Omega_r = N_r(o) \cup N_{-r}(o), \quad \text{where for any } r \in \mathbb{R},$$
$$N_r(o) = \{(z, w) \in \mathsf{HT} : w \in N(o), Re\,z = r\},$$

respectively. The horizontal part lies on the union of all bifurcation lines L_v, $v \in \partial N(o)$. The vertical part consists of two isometric copies of the compact metric tree $N(o)$ within HT, which delimit our set on the left and right hand sides, at which $Re\,z = -r$, resp. $Re\,z = r$. We call the elements of the finite set $\{(z, v) \in \mathsf{HT} : |Re\,z| = r, \, v \in V(N(o))\}$ the *corners* of Ω_r.
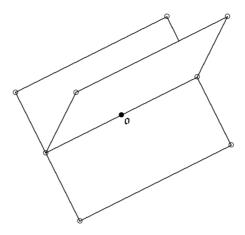
Figure 2. The compact set $\overline{\Omega}_r$. Its corners are encircled.

Our $\overline{\Omega}_r$ is the union of the finitely many closed "rectangles" $\overline{R}_{v,r}$, where

$$R_{v,r} = \{\mathfrak{z} \in \mathsf{S}_v^o : |Re\,\mathfrak{z}| < r\}.$$

The boundary of $R_{v,r}$ consists of two horizontal sides $L_{v,r}$ and $L_{v^-,r}$ and two vertical sides $J_{v,r}$ and $J_{v,-r}$, where $L_{v,r} = \{(z,v) \in \mathsf{L}_v : |z| \leqslant r\}$ and $J_{v,r} = \{(z,w) \in \mathsf{HT} : w \in [v^-,v], \, Re\,z = r\}$. Here, the vertex v ranges in $\{o\} \cup \partial^+ N(o)$, and the closed rectangles $\overline{R}_{v,r}$ and $\overline{R}_{o,r}$ meet at their lower, resp. upper horizontal sides, when $v^- = o$.

We want to write down Green's formulas for "nice" functions on $\overline{\Omega}_r$. We start by specifying those functions.

DEFINITION 3.2. — *Let $\mathcal{D}_{\alpha,\beta}^2(\overline{\Omega}_r)$ be the space of all continuous functions f on $\overline{\Omega}_r$ with the following properties.*

(i) *For $v = o$ as well as for any forward neighbour v of o in $V(\mathbb{T})$, the partial derivatives of f up to 2nd order exist and are bounded and continuous on $R_{v,r}$, and*

(ii) *they extend continuously from $R_{v,r}$ to $\partial R_{v,r}$, except possibly at the corners of Ω_r. (But on bifurcation lines, the extensions coming from different strips will in general not coincide.)*

(iii) *The first order derivatives satisfy the bifurcation condition (2.3) along the segment $L_{o,r}$.*

(iv) *$\Delta_{\alpha,\beta} f$ extends continuously to Ω_r across the bifurcation line.*

If $U \subset \Omega_r$ is a "nicely shaped" open subset of Ω_r, we define the space $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r \setminus U)$ analogously. Furthermore, we define $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega})$ as the space of all functions on $\overline{\Omega}$ whose restriction to $\overline{\Omega}_r$ is in $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r)$ for every $r > 0$.

Next, we need to specify the boundary (arc) measure induced by the measure $\mathbf{m} = \mathbf{m}_{\alpha,\beta}$ of (2.2). (We shall omit the paramters α, β in the index.) More precisely, we need one such arc measure $\mathbf{m}'_v = \mathbf{m}'_{\alpha,\beta;v}$ on each closed strip S_v, where $v \in V(\mathbb{T})$, given by

$$(3.3) \qquad d\mathbf{m}'_v(\mathfrak{z}) = \phi_{\alpha,\beta}(\mathfrak{z}) \sqrt{dx^2 + dy^2}/y \quad \text{for} \quad \mathfrak{z} = (x + \mathbf{i}\,y\,;v) \in \mathsf{S}_v\,,$$

where $\phi_{\alpha,\beta}(\mathfrak{z})$ is as in (2.2). Contrary to the definition (2.2) of $\mathbf{m}$, the lower boundary line L_{v-} of S_v is not excluded in this definition, so that on that line we have the two arc measures $\mathbf{m}'_v$ and $\mathbf{m}'_{v-}$ which differ by a factor of β. Their use depends on the side from which the line is approached. Now we can construct the boundary measure $\mathbf{m}^{\partial \Omega_r}$ and the boundary gradient of $f \in \mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r)$ as follows:

   (1) for each $v \in \partial^+ N(o)$, we let $\mathbf{m}^{\partial \Omega_r} = \mathbf{m}'_v$ and $\nabla^{\partial \Omega_r} f = \nabla f_v$ (the extension to $\partial \Omega_r$ within $\overline{\mathsf{S}}_v$) on the boundary parts $\mathsf{L}_{v,r}$ and on $J_{v,\pm r}$;

   (2) furthermore, $\mathbf{m}^{\partial \Omega_r} = \mathbf{m}'_o$ and $\nabla^{\partial \Omega_r} f = \nabla f_o$ (the extension to $\partial \Omega_r$ within $\overline{\mathsf{S}}_o$) on the boundary parts $\mathsf{L}_{o^-,r}$ and on $J_{o,\pm r}$.

In (1) and (2), the corners of Ω – a set with measure 0 – are excluded. In the following, the occuring boundaries are considered with positive orientation.

PROPOSITION 3.3. — For $f, h \in \mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r)$,

$$\int_{\Omega_r} \left(f\, \Delta_{\alpha,\beta} h + (\nabla f, \nabla h) \right) d\mathbf{m} = \int_{\partial \Omega_r} (\mathsf{n}, \nabla^{\partial \Omega_r} h)\, f\, d\mathbf{m}^{\partial \Omega_r} \quad \text{and}$$

$$\int_{\Omega_r} \left(f\, \Delta_{\alpha,\beta} h - h\, \Delta_{\alpha,\beta} f) \right) d\mathbf{m} = \int_{\partial \Omega_r} \left((\mathsf{n}, \nabla^{\partial \Omega_r} h)\, f - (\mathsf{n}, \nabla^{\partial \Omega_r} f)\, h \right) d\mathbf{m}^{\partial \Omega_r},$$

where $\mathsf{n}(\mathfrak{z}) = \mathsf{n}^{\partial \Omega_r}(\mathfrak{z})$ is the outward unit normal vector to $\partial \Omega_r$ at $\mathfrak{z} \in \partial \Omega_r$, defined except at the corners of Ω_r.

Proof. — We can write down the classical first Green formula as above on each $R = R_{v,r}$, with the boundary gradient $\nabla^{\partial R} f$:

$$\int_{R_{v,r}} \left(f\, \Delta_{\alpha,\beta} h + (\nabla f, \nabla h) \right) d\mathbf{m} = \int_{\partial R_{v,r}} (\mathsf{n}, \nabla^{\partial R} h)\, f\, d\mathbf{m}'_v\,.$$

Here, as above for Ω_r above, the boundary gradient $\nabla^{\partial R} f$ is the continuous extension of the gradient to $\partial R_{v,r}$, well defined except at the corners.

Indeed, we can first inscribe a slightly smaller rectangle whose closure is contained in $R_{v,r}$. We then have Green's first formula on that set, as stated in any calculus textbook. Then we can increase that inscribed rectangle and pass to the limit. The exchange of limit and integrals is legitimate because of the boundedness assumptions that appeared in the definition of the space $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega})$.

Now we take the sum of those integrals over all $R_{v,r}$ that make up Ω_r. The segment $L_{o,r}$ of the bifurcation line L_o lies in the interior of Ω. The bifurcation condition makes sure that the line integrals over $L_{o,r}$ coming from the different adjacent rectangles sum up to 0.

Green's second formula follows from the first one. $\square$

By [2, Theorems 5.9 and 5.19], the set Ω_r admits a Dirichlet Green kernel $\mathbf{g}^{\Omega_r}(\cdot, \mathfrak{w})$ which for $\mathfrak{w} \in \Omega_r$ belongs to $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r \setminus U)$ for any open neighbourhood U of its pole $\mathfrak{w}$. The fact that $\mathbf{g}^{\Omega_r}(\cdot, \mathfrak{w})$ is smooth up to the boundary segments of our Ω_r (with exception of the corners, where, however, the partial derivatives are bounded) follows from the general theory of elliptic 2nd order PDEs; see Evans [18, Theorem 6, p. 326].

We use it to define the *Poisson kernel*

$$(3.4) \qquad \Pi^{\Omega_r}(\mathfrak{w}, \mathfrak{z}) = -\left(\mathsf{n}^{\partial\Omega_r}(\mathfrak{z}), \, \nabla_{\mathfrak{z}}^{\partial\Omega_r} \mathbf{g}^{\Omega_r}(\mathfrak{z}, \mathfrak{w}) \right), \quad \mathfrak{w} \in \Omega_r, \; \mathfrak{z} \in \partial\Omega_r.$$

(The gradient is applied to the variable $\mathfrak{z}$ in the index, or rather to its hyperbolic part $z = x + \mathfrak{i}\, y$.) Note that the Poisson kernel is positive.

The argument used for the next lemma is classical, but it is important to check that each step works in our "non-classical" setting with the bifurcation lines and Kirchhoff condition.

LEMMA 3.4 (Poisson representation). — *Let $h \in \mathcal{D}^2_{\alpha,\beta}(\overline{\Omega}_r)$ be harmonic, that is, $\Delta_{\alpha,\beta} h = 0$ in Ω_r. Then for every $\mathfrak{w} \in \Omega_r$,*

$$h(\mathfrak{w}) = \int_{\partial\Omega_r} \Pi^{\Omega_r}(\mathfrak{w}, \mathfrak{z}) h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z}).$$

Proof. — Let φ be any non-negative function in $\mathcal{C}^\infty_c(\Omega_r)$ and let f be its Green potential on Ω_r, that is,

$$(3.5) \qquad\qquad f(\mathfrak{w}) = \int_{\Omega_r} \mathbf{g}^{\Omega_r}(\mathfrak{w}, \mathfrak{z}) \, \varphi(\mathfrak{z}) \, d\mathbf{m}(\mathfrak{z}).$$

Then f is a weak solution of the equation

$$\Delta_{\alpha,\beta} f = -\varphi \quad \text{in } \Omega_r.$$

Smoothness of the heat kernel $\mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})$ associated with $\Delta_{\alpha,\beta}$ (see [2, Theorem 5.23 and Appendix 1]) implies as in the classical theory of PDE

that f is a strong solution of the abvove Laplace equation, that is, $f \in D^2_{\alpha,\beta}(\overline{\Omega}_r)$ (indeed, it has all higher order derivatives). We have $f = 0$ on $\partial\Omega_r$.

Inserting f and h into Green's second identity of Proposition 3.3, we get with an application of Fubini's theorem

$$\int_{\Omega_r} h(\mathfrak{w})\,\varphi(\mathfrak{w})\,d\mathbf{m} = -\int_{\partial\Omega_r}\left(\mathsf{n}(\mathfrak{z}),\nabla_{\mathfrak{z}}^{\partial\Omega_r} f(\mathfrak{z})\right) h(\mathfrak{z})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})$$

$$= \int_{\partial\Omega_r}\int_{\Omega_r}\left(\mathsf{n}(\mathfrak{z}),\nabla_{\mathfrak{z}}^{\partial\Omega_r}\mathbf{g}^{\Omega_r}(\mathfrak{w},\mathfrak{z})\,\varphi(\mathfrak{z})\right) h(\mathfrak{z})\,d\mathbf{m}(\mathfrak{w})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})$$

$$= \int_{\Omega_r}\left(\int_{\partial\Omega_r}\Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z}) h(\mathfrak{z})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})\right)\varphi(\mathfrak{w})\,d\mathbf{m}(\mathfrak{w})\,.$$

Since this holds for any continuous function φ as specified above, the statement follows. $\qquad\square$

Let f be a continuous function on $\partial\Omega_r$. Then

$$h(\mathfrak{w}) = \int_{\partial\Omega_r} f\,d\mu_{\mathfrak{w}}^{\Omega_r} = \mathsf{E}_{\mathfrak{w}}\left(f(X_{\tau^{\Omega_r}})\right)$$

is the unique solution of the Dirichlet problem with boundary data f. By Lemma 3.4,

$$h(\mathfrak{w}) = \int_{\partial\Omega_r}\Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z}) f(\mathfrak{z})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})\,.$$

COROLLARY 3.5. — *For any $\mathfrak{w} \in \Omega_r$, the Poisson kernel $\Pi^{\Omega_r}(\mathfrak{w},\cdot)$ is the density of the hitting ($\equiv$ exit) distribution $\mu_{\mathfrak{w}}^{\Omega_r}$ with respect to the boundary measure $\mathbf{m}^{\partial\Omega_r}$.*

We want to have an analogous result for the non-compact set $\Omega = \Omega_o$. In spite of being "obvious", this requires further substantial work.

LEMMA 3.6. — *For any $\mathfrak{w} \in \Omega$ and any $r > |Re\,\mathfrak{w}|$,*

$$\mu_{\mathfrak{w}}^{\Omega_r}\left(\partial^{\mathrm{vert}}\Omega_r\right) \leqslant \mathsf{Pr}_{\mathfrak{w}}\left[\max\{|Re\,X_t| : t \leqslant \tau^{\Omega}\} > r\right] \leqslant 2\rho^{r-|Re\,\mathfrak{w}|-1}\,,$$

where $\rho < 1$.

Proof. — The first inequality is clear: if up to the exit time from the entire set Ω we have $|Re\,X_t| \leqslant r$ then X_t cannot exit Ω_r through its vertical boundary part. For the second inequality, the arguments are those used in the proof of [3, Proposition 4.18]. In particular, see [3, Figure 7]. The number ρ is as in that proposition. The result there is formulated in terms of the projected process on $\mathbb{H}(\mathsf{q})$, and we add some brief reminders in terms of HT itself.

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We "slice" Ω vertically by the sets $N_{\pm k}(o)$, $k \in \mathbb{N}$, and let $\sigma(k)$ be the exit time of our Brownian motion from Ω_k, while τ^Ω coincides with the σ of [3, (4.3)]: with respect to the projection $Y_t = \pi^{\mathbb{R}}(X_t)$ on the real line,

(3.6)
$$\tau^\Omega = \inf\{t > 0 : Y_t \notin [-1,1]\} = \inf\{t \geqslant 0 : Y_t = \pm 1\},$$
$$\text{where} \quad Y_0 \in [-1, 1],$$

which is a.s. finite and has an exponential moment. We also note that $\sigma(k) \leqslant \tau^\Omega$.

The function $\mathfrak{z} \mapsto \mathsf{Pr}_{\mathfrak{z}}[X_{\sigma(1)} \in \partial^{\mathrm{hor}}\Omega_1]$ is strictly positive and weakly harmonic ($\equiv$ in the sense of distributions) on Ω_1, whence strongly harmonic by [2, Theorem 5.9], and consequently continuous. Along $N_0(o)$, it must attain its minimum in some $\mathfrak{z}_* \in N_0(o) \setminus \partial\Omega_1$. Then

$$\rho = 1 - \mathsf{Pr}_{\mathfrak{z}_*}[X_{\sigma(1)} \in \partial^{\mathrm{hor}}\Omega_1].$$

Now let us assume first that $\mathfrak{w} \in \Omega$ is such that $\operatorname{Re}\mathfrak{w} = 0$. If X_t starts at $\mathfrak{w}$ and $\max\{\operatorname{Re}X_t : t \leqslant \tau^\Omega\} > r$, then it must pass through all the vertical "barriers" $N_1(o), \cdots, N_n(o)$, where $n = \lfloor r \rfloor$. But by group-invariance (just using horizontal translations here), for any starting point in $N_{k-1}(o)$, the probability that X_t reaches $N_k(o) \setminus \partial\Omega$ before $\partial\Omega$ is bounded above by ρ. By a simple inductive argument as in in the proof of [3, Proposition 4.18], the probability to pass through all those $N_k(o)$ before exiting Ω is $\leqslant \rho^n \leqslant \rho^{r-1}$.

In the same way, $\mathsf{Pr}_{\mathfrak{w}}\left[\min\{\operatorname{Re}X_t : t \leqslant \tau^\Omega\} < -r\right] \leqslant \rho^{r-1}$. Thus, the stated upper bound of the second inequality holds when $\operatorname{Re}\mathfrak{w} = 0$.

If $\mathfrak{w} \in \Omega$ is arbitrary, then we can use group invariance and map $\mathfrak{w}$ to a point $(0, w_0)$ by a horizontal translation of HT. Then we must replace Ω_r by $\Omega_{r-|\operatorname{Re}\mathfrak{w}|}$ in the preceding arguments. $\qquad\square$

Since $\tau^\Omega = \sigma$ is a.s. finite, also the set Ω has a Dirichlet Green kernel $\mathbf{g}^\Omega(\cdot, \mathfrak{w})$ which for $\mathfrak{w} \in \Omega$ belongs to $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega} \setminus U)$ for every "nicely shaped" open neighbourhood U of its pole $\mathfrak{w}$. Here, the space $\mathcal{D}^2_{\alpha,\beta}(\overline{\Omega})$ is as in Definition (3.2), except that we do not have to refer to the corners, and boundedness of partial derivatives is local, i.e., it refers to arbitrary relatively compact subsets of any of the strips that make up Ω.

The following is another relative of [3, Proposition 4.18] and the above Lemma 3.6.

LEMMA 3.7. — *For any* $\mathfrak{w} \in \Omega$,

$$\mathbf{g}^\Omega(\mathfrak{z}, \mathfrak{w}) \to 0 \quad \text{as} \ \ d(\mathfrak{z}, \mathfrak{w}) \to \infty, \ \mathfrak{z} \in \Omega.$$

Proof. — We use classical *balayage*, see Dynkin [14, (13.82) in §13.25] and Blumenthal and Getoor [4, Theorem (1.16) on p. 261]. We have for all

$\mathfrak{z}, \mathfrak{w} \in \Omega$ and $r > |Re\,\mathfrak{w}|$

$$(3.7) \quad \mathbf{g}^\Omega(\mathfrak{w},\mathfrak{z}) - \mathbf{g}^{\Omega_r}(\mathfrak{w},\mathfrak{z}) = \int_{\partial\Omega_r} \mathbf{g}^\Omega(\cdot,\mathfrak{z})\,d\mu_{\mathfrak{w}}^{\Omega_r} = \int_{\partial^{\mathrm{vert}}\Omega_r} \mathbf{g}^\Omega(\cdot,\mathfrak{z})\,d\mu_{\mathfrak{w}}^{\Omega_r}\,,$$

where the last equality holds because $\mathbf{g}^\Omega(\cdot,\mathfrak{z}) = 0$ on $\partial^{\mathrm{hor}}\Omega_r$. Of course, we have $\mathbf{g}^{\Omega_r}(\mathfrak{w},\mathfrak{z}) = 0$ unless also $\mathfrak{z} \in \Omega_r$.

For fixed $\mathfrak{w}$, when $d(\mathfrak{z},\mathfrak{w})$ is sufficiently large, there is $r = r(\mathfrak{z})$ such that $\mathfrak{w} \in \Omega_r$ and $\mathfrak{z} \in \Omega \setminus \Omega_r$, and $r(\mathfrak{z}) \to \infty$ when $d(\mathfrak{z},\mathfrak{w}) \to \infty$. Then

$$\mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w}) = \mathbf{g}^\Omega(\mathfrak{w},\mathfrak{z}) \leqslant \mu_{\mathfrak{w}}^{\Omega_r}\big(\partial^{\mathrm{vert}}\Omega_r\big) \times \sup\{\mathbf{g}^\Omega(\mathfrak{u},\mathfrak{z}) : \mathfrak{u} \in \partial^{\mathrm{vert}}\Omega_r\}.$$

By Lemma 3.6, this tends to 0, as $r = r(\mathfrak{z}) \to \infty$. $\qquad\square$

We can now define the Poisson kernel $\Pi^\Omega(\mathfrak{w},\mathfrak{z})$ as in (3.4). It is positive and supported by $\partial\Omega$. We have $\Pi^\Omega(\mathfrak{w},\mathfrak{z}) = \partial_y \mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w})$ on L_{o^-} and $\Pi^\Omega(\mathfrak{w},\mathfrak{z}) = -\partial_y \mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w})$ on each L_v with $v^- = o$. (The partial derivative ∂_y refers to the y-coordinate of $\mathfrak{z} = x + \mathrm{i}\,y$.)

PROPOSITION 3.8.

$$\int_{\partial\Omega} \Pi^\Omega(\mathfrak{w},\mathfrak{z})\,d\mathbf{m}^{\partial\Omega} = 1\,.$$

Proof. — For $r > 0$, we use Green's second identity of Proposition 3.3 on Ω_r. We again let φ be any non-negative function in $\mathcal{C}_c^\infty(\Omega_r)$. We use the Green potential of φ as in (3.5), but this time with respect to the whole of Ω instead of Ω_r. Furthermore, we let $h \equiv 1$. We have $f = 0$ on $\partial\Omega$, in particular on $\partial^{\mathrm{hor}}\Omega_r$, and $\Delta_{\alpha,\beta} f = -\varphi$ on Ω. Thus, we get

$$\int_{\Omega_r} \varphi(\mathfrak{w})\,d\mathbf{m} = -\int_{\partial\Omega_r} \Big(\mathsf{n}^{\Omega_r}(\mathfrak{z})\,,\,\nabla_{\mathfrak{z}}^{\partial\Omega_r} f(\mathfrak{z})\Big)\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z}).$$

We fix $\mathfrak{w} \in \Omega_r$. Continuing as in the proof of Lemma 3.4, we see that

$$1 = -\int_{\partial\Omega_r} \Big(\mathsf{n}^{\Omega_r}(\mathfrak{z})\,,\,\nabla_{\mathfrak{z}}^{\partial\Omega_r} \mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w})\Big)\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})$$

$$= \int_{\partial^{\mathrm{hor}}\Omega_r} \Pi^\Omega(\mathfrak{w},\mathfrak{z})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})$$

$$+ \int_{N_{-r}} y^2 \partial_x \mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z}) + \int_{N_r} -y^2 \partial_x \mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w})\,d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})\,,$$

where $N_{-r} = N_{-r}(o)$ and $N_r = N_r(o)$ are the left and right hand vertical boundary parts of Ω_r.

Since we have fixed $\mathfrak{w}$, we can write $\mathbf{g}^\Omega(\mathfrak{z},\mathfrak{w}) = g(\mathfrak{z}) = g(x + \mathrm{i}\,y, w)$, when $\mathfrak{z} = (x + \mathrm{i}\,y, w)$ with $w \in T_v$. We observe that on N_r for arbitrary $r \in \mathbb{R}$,

$$d\mathbf{m}^{\Omega_r}(\mathfrak{z}) = \underbrace{\beta^{\lceil \log_q(y) \rceil} y^\alpha}_{=:\,\phi(y)} dy\,, \quad \text{where} \quad \mathfrak{z} = (r + \mathrm{i}\,y, w) \in N_r\,.$$

The derivative ∂_x refers to the x-coordinate of $\mathfrak{z}$. We can consider the last measure, as well as the next integral below, as a measure and integral on the neighbourhood star $N(o)$ in $\mathbb{T}$, because N_r is an isometric copy of that neighbourhood star.

Thus, for $r \in \mathbb{R}$, we can write

$$\int_{N_r} y^2 \partial_x \mathbf{g}^\Omega(\mathfrak{z}, \mathfrak{w})\, d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z}) = \int_{N(o)} y^2 \partial_x g(r + \mathrm{i}\, y, w)\, \phi(y)\, dy.$$

This notation is slightly inprecise; we keep in mind that in the integral over $N(o)$, one has to take the sum over such integrals, where each one ranges over one of the edges of $N(o)$.

We have obtained that for any $x \in \mathbb{R} \setminus \{0\}$,

$$1 - \Pi(|x|) = \int_{N(o)} y^2 \partial_x \Big(g(-|x| + \mathrm{i}\, y, w) - g(|x| + \mathrm{i}\, y, w)\Big) \phi(y)\, dy$$

$$\text{where} \quad \Pi(|x|) = \int_{\partial^{\mathrm{hor}}\Omega_{|x|}} \Pi^\Omega(\mathfrak{w}, \mathfrak{z})\, d\mathbf{m}^{\partial\Omega_{|x|}}(\mathfrak{z}).$$

Now we choose a fixed $\varepsilon > 0$ and let $r > \varepsilon$. We integrate both sides over x which varies in the interval $I = (r - \varepsilon, r + \varepsilon)$. Then we get, with an obvious exchange of the order of integration

$$2\varepsilon - \int_I \Pi(|x|)\, dx$$
$$= \int_{N(o)} \int_I y^2 \partial_x \Big(g(-|x| + \mathrm{i}\, y, w) - g(|x| + \mathrm{i}\, y, w)\Big) dx\, \phi(y)\, dy$$
$$= \int_{N(o)} \Big(g(-r - \varepsilon + \mathrm{i}\, y, w) - g(r + \varepsilon + \mathrm{i}\, y, w)$$
$$- g(-r + \varepsilon + \mathrm{i}\, y, w) + g(r - \varepsilon + \mathrm{i}\, y, w)\Big) \phi(y)\, dy$$

When $r \to \infty$, the last integral tends to to 0 by Lemma 3.7, while the one on the left hand side tends to $2\varepsilon \int_{\partial\Omega} \Pi^\Omega(\mathfrak{w}, \cdot)\, d\mathbf{m}^{\partial\Omega}$. Thus,

$$2\varepsilon = 2\varepsilon \int_{\partial\Omega} \Pi^\Omega(\mathfrak{w}, \mathfrak{z})\, d\mathbf{m}^\Omega(\mathfrak{z}),$$

which yields the proposed statement. $\qquad\square$

We are finally in the position to extend Lemma 3.4 to the unbounded set $\Omega = \Omega_o$ in HT. This will also prove that the exit measure μ from Ω has a continuous density with respect to Lebesgue measure on $\partial\Omega$, or equivalently, with respect to Haar measure on $\mathcal{A}$.

THEOREM 3.9 (Poisson representation). — *Let $h \in \mathcal{D}^2_{\alpha,\beta}(\overline{\Omega})$ be harmonic, that is, $\Delta_{\alpha,\beta}h = 0$ in Ω. Suppose that h satisfies the growth condition*

$$(3.8) \qquad\qquad |h(\mathfrak{z})| \leqslant C_1 \, e^{C_2 \, \mathsf{d}_{\mathsf{HT}}(\mathfrak{z},\mathfrak{o})}$$

on Ω $(C_1 , C_2 > 0)$. Then for every $\mathfrak{w} \in \Omega$,

$$h(\mathfrak{w}) = \int_{\partial\Omega} \Pi^{\Omega}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) \,.$$

In particular, the Poisson kernel $\Pi^{\Omega}(\mathfrak{w},\cdot)$ is the density of the exit distribution $\mu^{\Omega}_{\mathfrak{w}}$ with respect to the boundary measure $\mathbf{m}^{\partial\Omega}$, where $\mathfrak{w} \in \Omega$.

Proof. — We first assume that h is bounded. Let r be large enough such that $\mathfrak{w} \in \Omega_r$. Then by Lemma 3.4 and because $d\mu^{\Omega_r}_{\mathfrak{w}} = \Pi^{\Omega_r}(\mathfrak{w},\cdot) \, d\mathbf{m}^{\partial\Omega_r}$,

$$h(\mathfrak{w}) = \int_{\partial\Omega_r} \Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega_r}(\mathfrak{z})$$

$$= \int_{\partial^{\mathrm{hor}}\Omega_r} \Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) + \int_{\partial^{\mathrm{vert}}\Omega_r} h(\mathfrak{z}) d\mu^{\Omega_r}_{\mathfrak{w}}(\mathfrak{z}) \,.$$

When $r \to \infty$, the last integral along $\partial^{\mathrm{vert}}\Omega_r$ tends to 0 by Lemma 3.6 and boundedness of h.

$$\int_{\partial^{\mathrm{hor}}\Omega_r} \Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) \to \int_{\partial\Omega} \Pi^{\Omega}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) \,.$$

Taking normal derivatives in (3.7), we have for all $\mathfrak{w} \in \Omega_r$ and $\mathfrak{z} \in \partial\Omega$

$$\Pi^{\Omega}(\mathfrak{w},\mathfrak{z}) - \Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z}) = \int_{\partial^{\mathrm{vert}}\Omega_r} \Pi^{\Omega}(\cdot,\mathfrak{z}) \, d\mu^{\Omega_r}_{\mathfrak{w}} \geqslant 0$$

with $\Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z}) = 0$ when $\mathfrak{z} \in \partial\Omega \backslash \partial^{\mathrm{hor}}\Omega_r$. Therefore, using Proposition 3.8,

$$\left| \int_{\partial\Omega} \Pi^{\Omega}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) - \int_{\partial\Omega} \Pi^{\Omega_r}(\mathfrak{w},\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) \right|$$

$$= \left| \int_{\partial^{\mathrm{vert}}\Omega} \int_{\partial\Omega} \Pi^{\Omega}(\cdot,\mathfrak{z})h(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) \, d\mu^{\Omega_r}_{\mathfrak{w}} \right| \leqslant \|h\|_{\infty} \, \mu^{\Omega_r}_{\mathfrak{w}}\left(\partial^{\mathrm{vert}}\Omega_r\right) \,,$$

which tends to 0 as $r \to \infty$ by Lemma 3.6.

At this point, we have obtained the Poisson representation for all bounded harmonic functions. In the same way as we got Corollary 3.5, we get that

$$(3.9) \qquad\qquad \Pi^{\Omega}(\mathfrak{w},\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}) = d\mu^{\Omega}_{\mathfrak{w}}(\mathfrak{z}) \,,$$

as proposed.

Before we proceed, we make a simple observation on the metric. For any $\mathfrak{z} = (z, w) \in \mathsf{HT}$,

$$\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{o}) \leqslant 2 \log(|Re\,z| + 1) + \mathsf{d}_{\mathbb{T}}(w, o) \log \mathsf{q}.$$

Indeed, this is the triangle inequality: let $\mathfrak{z}' = (Re\,z + \mathfrak{i}, o)$ be the point on L_o with $Re\,\mathfrak{z}' = Re\,\mathfrak{z}$. Then $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{z}') = \mathsf{d}_{\mathbb{T}}(w, o) \log \mathsf{q}$, and $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}', \mathfrak{o}) = \mathsf{d}_{\mathbb{H}}(Re\,z + \mathfrak{i}, \mathfrak{i}) \leqslant 2 \log(|Re\,z| + 1)$, a simple computation with the hyperbolic metric. Thus,

$$(3.10) \qquad \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{o}) \leqslant 2 \log(|Re\,\mathfrak{z}| + 1) + D, \quad \text{for all } \mathfrak{z} \in \overline{\Omega},$$

where $D = \log \mathsf{q}$.

Now let h be an arbitrary harmonic function on Ω that satisfies (3.8). Suppose first that $Re\,\mathfrak{w} = 0$. Let $\sigma(0) = 0$, and as in the proof of Lemma 3.6, let $\sigma(k)$ be the exit time of Brownian motion from Ω_k, where $k \in \mathbb{N}$. Then harmonicity of h on Ω yields that $\big(h(X_{\sigma(k)})\big)_{k \geqslant 0}$ is a discrete-time martingale. Consider the random variable

$$(3.11) \qquad R_{\max} = \max\{|Re\,X_t| : t \leqslant \tau\}.$$

By Lemma 3.6, we have that

$$\mathsf{E}_{\mathfrak{z}}(R_{\max}^s) < \infty \quad \text{for every } s > 0.$$

Combining the growth condition (3.8) with (3.10), we get

$$|h(X_t)| \leqslant C_1 \, e^{C_2 D} \left(R_{\max} + 1\right)^{2C_2} \quad \text{for } 0 \leqslant t \leqslant \tau,$$

an integrable upper bound. In particular, our martingale converges almost surely.

We now note that $\sigma(k) = \tau$ if (X_t) exits Ω_k at $\partial^{\mathrm{hor}}\Omega_k$ and that $\sigma(k) < \tau$ if (X_t) exits Ω_k at $\partial^{\mathrm{vert}}\Omega_k$. In particular, $\sigma(k) = \tau$ when $k \geqslant R_{\max}$. We can decompose

$$h(X_{\sigma(k)}) = h(X_\tau) \, \mathbf{1}_{[\sigma(k) = \tau]} + h(X_{\sigma(k)}) \, \mathbf{1}_{[\sigma(k) < \tau]},$$

and see that $h(X_{\sigma(k)}) \to h(X_\tau)$ almost surely. Using dominated convergence and the martingale property, we conclude that

$$f(\mathfrak{z}) = \mathsf{E}_{\mathfrak{z}}\big(h(X_\tau)\big),$$

which proves the required statement for the case when $Re\,\mathfrak{z} = 0$. For arbitrary $\mathfrak{z} \in \Omega$, it follows immediately from the horizontal translation invariance of our Laplacian. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

COROLLARY 3.10. — *For any $\mathfrak{w} \in \Omega$, the exit measure $\mu_{\mathfrak{w}}^{\Omega}$ has continuous, strictly positive density with respect to Lebesgue measure on the lines that make up $\partial\Omega$. It satisfies the moment condition*

$$\int_{\partial\Omega} \exp\left(\lambda \left| Re\, \mathfrak{z}\right|\right) d\mu_{\mathfrak{w}}^{\Omega}(\mathfrak{z}) < \infty$$

for any $\lambda < \log(1/\rho)$ with ρ as in Lemma 3.6. Thus, we have the double exponential moment condition,

$$\int_{\partial\Omega} \exp\left(\exp\left(\mathsf{d}_{\mathsf{HT}}(\mathfrak{o}, \mathfrak{z})/3\right)\right) d\mu_{\mathfrak{w}}^{\Omega}(\mathfrak{z}) < \infty.$$

Proof. — The fact that $\mathsf{supp}\, \mu_{\mathfrak{w}}^{\Omega} = \partial\Omega$ follows from (3.9), or equivalently from the observation made in the proof of the next corollary that all boundary points are regular: indeed, every regular point must belong to the support of the exit distribution.

The moment condition follows from Lemma 3.6: if (X_t) starts at $\mathfrak{w} \in \Omega_r$ and exits Ω at some $\mathfrak{z} \in \partial\Omega \setminus \partial\Omega_r$ then it must exit Ω_r through its vertical boundary, that is

$$\mu_{\mathfrak{w}}^{\Omega}(\partial\Omega \setminus \partial\Omega_r) \leqslant \mu_{\mathfrak{w}}^{\Omega_r}\left(\partial^{\mathrm{vert}}\Omega_r\right). \qquad \square$$

THEOREM 3.11 (Solution of the Dirichlet problem). — *Let $f \in \mathcal{C}(\partial\Omega)$ be such that the growth condition (3.8) holds for f on $\partial\Omega$. Then*

$$h(\mathfrak{w}) = \begin{cases} \displaystyle\int_{\partial\Omega} \Pi^{\Omega}(\mathfrak{w}, \mathfrak{z}) f(\mathfrak{z}) \, d\mathbf{m}^{\partial\Omega}(\mathfrak{z}), & \mathfrak{w} \in \Omega, \\ f(\mathfrak{w}), & \mathfrak{w} \in \partial\Omega \end{cases}$$

defines an extension of f that is harmonic on Ω and continuous on $\overline{\Omega}$. Furthermore, there is at most one such extension which satisfies (3.8) on the whole of Ω.

Proof. — A boundary point $\mathfrak{z}$ of any open domain $\Omega \subset \mathsf{HT}$ is regular for the Dirichlet problem for continuous functions on $\partial\Omega$ if and only if the exit distribution satisfies $\mathsf{Pr}_{\mathfrak{z}}[\tau^{\Omega} = 0] = 1$ (a general fact from Potential Theory).

Every boundary point of Ω is regular. This follows from the fact that τ^{Ω} (which was denoted σ above) is the same as the exit time of the projected process (Y_t) on $\mathbb{R}$ from the interval $[-1, 1]$, see (3.6). But the Dirichlet problem for the latter interval (with boundary values at ± 1) is obviously solvable, as verified by direct, elementary computations; see [3, Lemma 4.4 & proof].

Now, the function $h(\mathfrak{w})$ defined in the corollary is harmonic; its finiteness is guaranteed by (3.8) in combination with Corollary 3.10. By the same reason, it is continuous up to $\partial\Omega$, since every boundary point is regular, as we have just observed. Thus, we have a solution of the Dirichlet problem.

Since Ω is not bounded, uniqueness is not immediate. (In the compact case, uniqueness would follow from the maximum principle: a harmonic function assumes its maximum on the boundary.) However, if some extension satisfies (3.8) then we can apply Theorem 3.9 to see that it must coincide with the function defined in the present theorem. $\square$

Remark 3.12. — What we have stated and proved for Ω also works in precisely the same way, with some more notational efforts, for more general sets which are unions of finitely many strips. We state several facts that will be used in the sequel.

(a) As already mentioned, group invariance implies that Theorem 3.9, Theorem 3.11 and Corollary 3.10 are valid for any set Ω_v $(v \in V(\mathbb{T}))$ in the place of $\Omega = \Omega_o$.

(b) We can take a compact metric subtree T of $\mathbb{T}_{\mathsf{p}}$ which is *full* in the sense that for each of its vertices, either all, or else just one of its neigbours is in T. Then we can take the set

$$\overline{\Omega}_T = \{\mathfrak{z} = (z, w) \in \mathsf{HT} : w \in T\},$$

and Ω_T is its interior. By precisely the same methods as those used above, with only minor and straightforward notational adaptations, one shows that also in this case, the exit distribution from Ω_T (with respect to any starting point in that set) has a continuous density with respect to the Lebesgue measure on the union of the bifurcation lines that make up $\partial\Omega_T$. We also have the analogous Poisson representation and solution of the Dirichlet problem on Ω_T, as well as the exponential moment for the exit distribution, which is supported by the entire boundary of Ω_T.

(c) For any set $\Theta \subset \mathsf{HT}$ and $\varepsilon > 0$, we let

(3.12) $$\Theta^\varepsilon = \{\mathfrak{z} \in \Theta : \mathsf{d}(\mathfrak{z}, \partial\Theta) > \varepsilon\}.$$

For all results stated in (b), we can also replace Ω_T by Ω_T^ε, where $0 < \varepsilon < \log\mathsf{q}$: thus, $\partial\Omega_T^\varepsilon$ consists of horizontal lines in the interior of the boundary strips of Ω_T at (hyperbolic) distance ε from the boundary lines of Ω_T.

(d) The same holds for "sliced" hyperbolic plane, interpreted as $\mathsf{HT}(\mathsf{q},1)$. In this case, Ω_v should be replaced by the double strip $\widetilde{\Omega}_k = \pi^{\mathbb{H}}(\Omega_v)$

with $k = \mathfrak{h}(v)$, and more generally this holds for any union of finitely many strips which is connected in $\mathbb{H}$, and we may extend those strips (or truncate) above and below up to some intermediate boundary lines.

(e) In particular, all those results also hold for a single strip. Note that this is in reality a classical situation because before reaching the boundary of a strip, we are just observing ordinary hyperbolic Brownian motion with drift.

The reason why we did not state any more general results, e.g. for relatively compact domains that do not have rectangular shape, lies in the complication of how those sets may cross (several times from different sides) the bifurcation lines. For sets with sufficiently regular shape and piecewise smooth boundaries, such results can be stated and elaborated with considerable further notational efforts, while the techniques remain basically the same.

4. Positive harmonic functions and the induced random walk

We now finally turn our attention to positive harmonic functions on the whole of HT. Recall the stopping times $\tau(n)$ from (2.12). We assume that Brownian motion starts in some point of LT, so that all increments $\tau(n) - \tau(n-1)$, $n \geqslant 1$, are i.i.d. and have the same distribution as $\tau(1)$. Before coming to the core of this section, we start with a simple result concerning the induced random walk $(W_{\tau(n)})_{n \geqslant 0}$ on $\mathbb{T}$. An easy calculation [3, Cor. 4.9] shows that when it starts at a vertex of $\mathbb{T}$, then this is a nearest neighbour random walk on the vertex set of $\mathbb{T}$ with transition probabilities

$$(4.1) \qquad p_{\mathbb{T}}(v, v^-) = \frac{1}{1 + \mathsf{a}} \quad \text{and} \quad p_{\mathbb{T}}(v^-, v) = \frac{\mathsf{a}}{(1 + \mathsf{a})\mathsf{p}}, \quad v \in V(\mathbb{T})$$

with a as in (2.14). The one-step transition probabilities beween all other pairs of vertices are 0.

LEMMA 4.1.

(i) If a non-negative function f on $\mathbb{T}$ is $\Delta_{\alpha,\beta}^{\mathbb{T}}$-harmonic then its restriction to $V(\mathbb{T})$ is harmonic for the transition probabilites of $(W_{\tau(n)})$. That is, for any $v \in V(\mathbb{T})$

$$(4.2) \qquad f(v) = \sum_{u \in V(\mathbb{T}) : u \sim v} p_{\mathbb{T}}(v, u) \, f(u).$$

(ii) *Conversely, for any non-negative function f on $V(\mathbb{T})$ which satis-fies (4.2), its unique extension to a $\Delta_{\alpha,\beta}^{\mathbb{T}}$-harmonic function on $\mathbb{T}$ is given on each edge $[v^-, v]$ of $\mathbb{T}$ and $w \in [v^-, v]$ by*

$$f(w) = \lambda(w) \cdot f(v^-) + \big(1 - \lambda(w)\big) \cdot f(v) \quad \text{with}$$

(4.3)
$$\lambda(w) = \begin{cases} \dfrac{\mathsf{q}^{(\alpha-1)(\mathfrak{h}(v)-\mathfrak{h}(w))} - 1}{\mathsf{q}^{(\alpha-1)} - 1}, & \text{if } \alpha \neq 1, \\ \mathfrak{h}(v) - \mathfrak{h}(w), & \text{if } \alpha = 1, \end{cases}$$

where (recall) $[v^-, v]$ corresponds to the real interval $[\mathfrak{h}(v)-1, \mathfrak{h}(v)]$.

Proof. — (i) is clear from Definition 2.6 of harmonic functions.

(ii) Given f on $V(\mathbb{T})$ and any vertex, we see from (2.6) that the exten-sion to $\mathbb{T}$ within any edge $[v^-, v]$ must satisfy the boundary value problem $\frac{1}{(\log \mathsf{q})^2} f'' + \frac{\alpha-1}{\log \mathsf{q}} f' = 0$, with the given values $f(v^-)$ and $f(v)$ at the end-points of the interval corresponding to that edge. Straightforward compu-tation leads to the solution (4.3). It must satisfy the bifurcation condition of (2.6), which can also be verified directly. $\qquad\square$

After this warmup, our main focus is on the induced process $(X_{\tau(n)})_{n \geqslant 0}$ on LT. We want to relate its positive harmonic functions with the positive harmonic functions for our Laplacian $\Delta = \Delta_{\alpha,\beta}$ on HT. Here, a function $f : \mathsf{LT} \to \mathbb{R}$ is harmonic for that process if $f(\mathfrak{z}) = \mathsf{E}_{\mathfrak{z}}\big(f(X_{\tau(1)})\big)$ for any $\mathfrak{z} \in \mathsf{LT}$. In view of the previous section, this means that

(4.4) $$f(\mathfrak{z}) = \int_{\partial\Omega_v} f \, d\mu_{\mathfrak{z}}^{\Omega_v} \quad \text{for every } v \in V(\mathbb{T}) \text{ and } \mathfrak{z} \in \mathsf{L}_v.$$

The analogue of Lemma 4.1 is by no means as straightforward; the reason is that on HT, the strips are non-compact so that one can make long sideways detours in between two successive bifurcation lines.

Let us define the probability measure

(4.5) $$\mu = \mu_{\mathsf{o}}^{\Omega} \quad \text{on} \quad \partial\Omega = \bigcup_{v:v\sim o} \mathsf{L}_v.$$

Group invariance yields that for any $\mathfrak{z} = (z,v) \in \mathsf{LT}$ and $\mathfrak{g} \in \mathcal{A}$ with $\mathfrak{g}\mathsf{o} = \mathfrak{z}$, we have the convolution identity $\mu_{\mathfrak{z}}^{\Omega_v} = \delta_{\mathfrak{g}} * \mu$. Since $\mathcal{A}$ acts transitively on LT, the transition probabilities of the induced process are completely determined by μ, so that we can consider it as the random walk on LT with law μ. In view of this, we call the harmonic functions of (4.4) the *μ-harmonic functions* on LT. They are necessarily continuous, since μ has a continuous density.

We note here that the measure μ is invariant under the stabiliser $\mathcal{A}_\mathfrak{o}$ of our reference point $\mathfrak{o}$ in $\mathcal{A}$. For sliced hyperbolic plane, i.e., when $\mathsf{p} = 1$, the group is $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ and that stabiliser is trivial. When $\mathsf{p} \geqslant 2$, the stabiliser is the compact subgroup of $\mathcal{A}$ consisting of all $\mathfrak{g} = (id_\mathbb{H}, \gamma)$, where $\gamma \in \mathsf{Aff}(\mathbb{T})$ with $\gamma o = o$.

We also note the following fact that was anticipated in [3]: by Theorem 3.9, the measure μ has a continuous density with respect to Lebesgue measure on the union of the boundary lines L_v, $v \sim o$. Also, by Lemma 3.6 it has exponential moment with respect to the *Euclidean* distance on each of those lines.

For positive harmonic functions, the following uniform *Harnack inequality* is a consequence of [2, Theorem 4.2].

PROPOSITION 4.2. — *For every non-negative $\Delta_{\alpha,\beta}$-harmonic function h on (the whole of) HT and each $d > 0$,*

$$h(\mathfrak{z}') \leqslant C_d \, h(\mathfrak{z}), \quad \text{whenever} \quad \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{z}') \leqslant d,$$

where $C_d > 1$ is such that $C_d \to 1$ when $d \to 0$.

In particular, every non-negative harmonic function must be strictly positive in every point. The next lemma will be important for understanding the relation between Δ-harmonic functions on HT and μ-harmonic functions on LT. It makes use of the straightforward extensions of Theorem 3.9 and Theorem 3.11 to sets of the form Ω_T and Ω_T^ε for a full subtree T of $\mathbb{T}$, as clarified in Remark 3.12.

LEMMA 4.3. — *Let T be a full (compact) subtree of $\mathbb{T}$. For every $d > 0$ and $0 < \varepsilon < \log \mathsf{q}$ there is a constant $M_d = M_{d,T,\varepsilon} > 0$, with $M_d \to 1$ as $d \to 0$, such that for any positive harmonic function h on Ω_T,*

$$h(\mathfrak{z}') \leqslant M_d \, h(\mathfrak{z}), \quad \text{whenever} \quad \mathfrak{z}, \mathfrak{z}' \in \Omega_T^\varepsilon \quad \text{with} \quad |Re\,\mathfrak{z}' - Re\,\mathfrak{z}| \leqslant d.$$

In particular, in this situation, the exit distributions from the set Ω_T satisfy $\mu_{\mathfrak{z}'}^{\Omega_T} \leqslant M_d \, \mu_{\mathfrak{z}}^{\Omega_T}$.

Proof. — In analogy with the set $\Omega_{v,r}$ of (3.1), let us first take $\Omega_{T,r} = \{\mathfrak{z} \in \Omega_T : |Re\,\mathfrak{z}| < r\}$ and its subset $\Omega_{T,r}^\varepsilon$. The intrinsic metric of Ω_T is equivalent with the naturally defined Euclidean metric of that set (i.e., the metric has to be extended across bifurcation lines using straight geodesics). Then it follows once more from [2, Theorem 4.2] that every positive harmonic function on $\Omega_{T,r}$ satisfies a Harnack inequality as proposed for $\mathfrak{z}, \mathfrak{z}'$ in $\Omega_{T,r}^\varepsilon$. Note that the whole of Ω_T, resp. Ω_T^ε can be covered by translates of $\Omega_{T,r}$, resp. $\Omega_{T,r}^\varepsilon$. By group invariance of our Laplacian, that inequality holds on each translate with the same constants M_d. $\qquad\square$

For the next Harnack inequality, we need some of the features of the geometry of HT, $\mathbb{T}$ and $\mathbb{H}$, as outlined above after Proposition 2.4.

PROPOSITION 4.4. — *There are constants* $C_1, C_2 > 0$ *such that for every non-negative μ-harmonic function f on* LT *and all* $\mathfrak{z}, \mathfrak{z}' \in \mathsf{LT}$,

$$f(\mathfrak{z}') \leqslant C_1 \, e^{C_2 \, \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{z}')} \, f(\mathfrak{z}) \, .$$

Proof. — *Step 1.* Recall that Ω is the star of strips in LT with middle line L_o. Since μ has a continuous density, the assumption of μ-harmonicity yields that f is continuous, and it is $\mu_{\mathfrak{z}}^{\Omega}$-integrable on $\partial\Omega$ for every $\mathfrak{z} \in \Omega$. Therefore

$$h(\mathfrak{z}) = \int_{\partial\Omega} f \, d\mu_{\mathfrak{z}}^{\Omega}$$

defines a positive Δ-harmonic extension of f to the closure of Ω. Since f is μ-harmonic, h also coincides with f on the middle line L_o of Ω. The function h satisfies the Harnack inequality of Lemma 4.3. This yields the following for h and consequently for f.

If $\quad \mathfrak{z}, \mathfrak{z}' \in \mathsf{L}_0 \quad$ and $\quad |Re\,\mathfrak{z} - Re\,\mathfrak{z}'| \leqslant 2\mathsf{q}^2 \quad$ then $\quad f(\mathfrak{z}') \leqslant M \, f(\mathfrak{z}) \, ,$

where $M = M_{2\mathsf{q}^2}$. For any $v \in V(\mathbb{T})$, we can map Ω to the star of strips Ω_v with middle line L_v by a mapping $\mathfrak{g} = (g, \gamma) \in \mathcal{A}$ such that $g\mathfrak{z} = \mathsf{q}^k \, z$ with $k = \mathfrak{h}(v)$, and $\gamma \in \mathsf{Aff}(\mathbb{T})$. It preserves both Δ- and μ-harmonicity. That mapping dilates horizontal Euclidean distances by the factor of q^k. Therefore we also have the following statement for any $v \in V(\mathbb{T})$ with the same constant M.

(4.6) If $\quad \mathfrak{z}, \mathfrak{z}' \in \mathsf{L}_v \quad$ and $\quad |Re\,\mathfrak{z} - Re\,\mathfrak{z}'| \leqslant 2\mathsf{q}^{k+2} \quad$ then $\quad f(\mathfrak{z}') \leqslant M \, f(\mathfrak{z}) \, .$

Step 2. We now use this for $V(\mathbb{T}) \ni v \sim o$ and the associated point $\mathfrak{v} = (\mathfrak{i}\,\mathsf{q}^{\mathfrak{h}(v)}, v) \in \mathsf{L}_v \subset \partial\Omega$, an endpoint of the "star" $N_0(o)$ as defined in (3.2). Let $I_v = \{\mathfrak{z} \in \mathsf{L}_v : |Re\,\mathfrak{z}| \leqslant 2\mathsf{q}^{2+\mathfrak{h}(v)}\} = \mathsf{L}_{v,r}$ with $r = 2\mathsf{q}^{2+\mathfrak{h}(v)}$. (Note that $\mathfrak{h}(v) = \pm 1$.) Combining the assumption of μ-harmonicity with (4.6),

$$f(\mathfrak{o}) = \int_{\partial\Omega} f \, d\mu \geqslant \int_{I^v} f \, d\mu \geqslant \mu(I_v) \, f(\mathfrak{v})/M.$$

By Corollary 3.10, $\mu(I_v) > 0$. Re-using (4.6) on each L_v,

(4.7)   if $\quad \mathfrak{z} \in \mathsf{L}_o \cup \bigcup_{v \sim o} \mathsf{L}_v \quad$ and $\quad |Re\,\mathfrak{z}| \leqslant 2\mathsf{q} \quad$ then $\quad f(\mathfrak{z}) \leqslant \overline{M} \, f(\mathfrak{i}) \, ,$

where $\overline{M} = M/\min\{\widetilde{\mu}(I_v) : v \sim o\}$.

Step 3. Now we let $u \in V(\mathbb{T})$ and $\mathfrak{z} \in \mathsf{L}_u$. Set $k = \mathfrak{h}(u)$, Then we can map $\mathfrak{i}$ to $\mathfrak{z}$ by a mapping $\mathfrak{g} = (g, \gamma) \in \mathcal{A}$, where $g = \left(\begin{smallmatrix} \mathsf{p}^k & Re\,\mathfrak{z} \\ 0 & 1 \end{smallmatrix} \right) \in \mathsf{Aff}(\mathbb{H}, \mathsf{q})$ and $\gamma \in \mathsf{Aff}(\mathbb{T})$ is such that $\gamma o = v$. Thus, (4.7) transforms into

(4.8)
$$\text{If} \quad \mathfrak{z}, \mathfrak{z}' \in \mathsf{L}_u \cup \bigcup_{v \sim u} \mathsf{L}_v \quad \text{and} \quad |Re\,\mathfrak{z} - Re\,\mathfrak{z}'| \leqslant 2\mathsf{q}^{k+1}$$
$$\text{then} \quad f(\mathfrak{z}') \leqslant \widetilde{M} f(\mathfrak{z}), \quad \text{where} \quad \widetilde{M} = 2\overline{M}.$$

Step 4. Now let $\mathfrak{z} = (z, v), \mathfrak{z}' = (z', v') \in \mathsf{LT}$, and consider the situation where $v \curlywedge v' = v$, that is, v' lies "above" v or is $= v$. Then we can choose some $\xi \in \partial^+ \mathbb{T}$ such that $v \in \overline{\varpi\,\xi}$ and $\mathfrak{z}, \mathfrak{z}' \in \mathbb{H}_\xi$, as outlined in the description of the metric (2.8) of HT. A geodesic arc $\overline{\mathfrak{z}\,\mathfrak{z}'}$ (depending on ξ) is then given by the (isometric) image in $\mathbb{H}_\xi$ of the geodesic arc $\overline{z\,z'}$ in $\mathbb{H}$.[2] The arc $\overline{\mathfrak{z}\,\mathfrak{z}'}$ has a highest point $\tilde{\mathfrak{z}} = (\tilde{z}, \tilde{w})$, where $\tilde{z} = z \wedge z'$. Our arc may cross more than one strip. It meets $\mathsf{LT} \cap \mathbb{H}_\xi$ in successive points $\mathfrak{z}_j = (z_j, v_j)$, $j = 0, \ldots, n$, where $\mathfrak{z}_0 = \mathfrak{z}$ and $\mathfrak{z}_n = \mathfrak{z}'$. All v_j are vertices on the ray $\overline{v\,\xi}$ in $\mathbb{T}$. Except possibly for one "top" sub-arc that contains $\tilde{\mathfrak{z}}$, any $\overline{\mathfrak{z}_{j-1}\,\mathfrak{z}_j}$ crosses one strip: in the initial, ascending part (if it is present), $v_j^- = v_{j-1}$ and that strip is S_{v_j}, while in the terminal, descending part (if present), $v_j = v_{j-1}^-$ and that strip is $S_{v_{j-1}}$. In both cases (i.e. except possibly for the top arc), $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_{j-1}, \mathfrak{z}_j) \geqslant \log \mathsf{q}$ (the distance between any two adjacent bifurcation lines), and we get

$$\mathsf{d}_{\mathsf{HT}}(z, z') \geqslant (n-1) \log \mathsf{q}.$$

For any $j \in \{1, \ldots, n\}$, let $k(j) = \mathfrak{h}(v_{j-1} \curlywedge v_j)$ be the level (horocycle number) of the lower boundary line of the strip in which the j-th sub-arc is contained. Therefore, the $\mathbb{H}$-coordinates of $\mathfrak{z}_{j-1}$ and $\mathfrak{z}_j$ satisfy $Im\,z_{j-1} \wedge z_j \leqslant \mathsf{q}^{k(j)+1}$. Thus by (2.7)

$$|Re\,\mathfrak{z}_j - Re\,\mathfrak{z}_{j+1}| = |Re\,z_j - Re\,z_{j+1}| \leqslant 2\mathsf{q}^{k(j)+1},$$

and (4.8) yields that $f(\mathfrak{z}_{j+1}) \leqslant \widetilde{M} f(\mathfrak{z}_j)$. Putting the sub-arcs together,

$$f(\mathfrak{z}') \leqslant \widetilde{M}^n f(\mathfrak{z}) \leqslant \exp\!\Big(C_2 \left(\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{z}') + 1 \right) \Big) f(\mathfrak{z}), \quad \text{where} \quad C_2 = \frac{\log \widetilde{M}}{\log \mathsf{q}}$$

---

[2] This explains that geodesics in HT are in general not unique, because there are different feasible choices for ξ that may lead to different geodesics, according to the mutual position of z and z' in $\mathbb{H}$.

Step 5. At last, let $\mathfrak{z} = (z, v), \mathfrak{z}' = (z', v') \in \mathsf{LT}$ be such that $v \curlywedge v' \notin \{v, v'\}$. Then (2.8) shows that any geodesic arc $\overline{\mathfrak{z}\mathfrak{z}'}$ in HT decomposes into two sub-arcs $\overline{\mathfrak{z}\mathfrak{z}''}$ and $\overline{\mathfrak{z}''\mathfrak{z}'}$, each of which is as in Step 4. Therefore

$$f(\mathfrak{z}') \leqslant \exp\Big(C_2 \big(\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}', \mathfrak{z}'') + 1\big)\Big) f(\mathfrak{z}'') \leqslant \exp\Big(C_2 \big(\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}', \mathfrak{z}) + 2\big)\Big) f(\mathfrak{z})$$

This concludes the proof. □

Now consider a set Ω_T, where T is a full subtree of $\mathbb{T}$, as defined in Remark 3.12(b). We note the following facts.

Remark 4.5. — If the starting point $\mathfrak{z}$ of (X_t) lies on a line L_v in the interior of Ω_T (that is, v is an interior vertex of T) then the exit distribution $\mu_{\mathfrak{z}}^{\Omega_T}$ of (X_t) from Ω_T is the same as the exit distribution of the random walk $(X_{\tau(n)})$ from Ω_T.

In view of Remark (3.12), it has the same properties as those of lemmas 3.6 (with different ρ) and 3.7, as well as Proposition 3.8 and corollaries 3.10 and 3.11.

Then we have the following (easier) analogue of Theorem 3.9.

PROPOSITION 4.6. — *Let f be a μ-harmonic function on $\overline{\Omega}_T$, that is, it satisfies (4.4) on every line $\mathsf{L}_v \subset \Omega_T$. Suppose furthermore that f satisfies the growth condition*

$$(4.9) \qquad\qquad |f(\mathfrak{z})| \leqslant C_1 \, e^{C_2 \, \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{o})}$$

on $\partial\Omega_T$, where $C_1, C_2 > 0$. Then for every $\mathfrak{z} \in \Omega_T \cap \mathsf{LT}$,

$$f(\mathfrak{z}) = \int_{\partial\Omega_T} f \, d\mu_{\mathfrak{z}}^{\Omega_T} \, .$$

Proof. — We first observe that (3.10) from the proof of Theorem 3.9 also holds on $\overline{\Omega}_T$ with constant $D = \max\{\mathsf{d}_{\mathbb{T}}(w, o) : w \in T\} \cdot \log \mathsf{q}$.

This time, let $\tau = \tau^{\Omega_T}$ be the exit time from Ω_T. Let the starting point be $\mathfrak{z} \in \Omega_T \cap \mathsf{LT}$. Consider the process $X_n^T = X_{\min\{\tau(n), \tau\}}$ on $\overline{\Omega}_T \cap \mathsf{LT}$. This is the induced random walk on LT stopped upon reaching $\partial\Omega_T$. Then $\big(f(X_n^T)\big)_{n \geqslant 0}$ is a martingale by μ-harmonicity of f. Note that $\tau = \tau(\boldsymbol{n})$ for some random $\boldsymbol{n}$. Thus, for non-random n tending to ∞,

$$f(X_n^T) = f(X_\tau) \, \mathbf{1}_{[\tau \leqslant \tau(n)]} + f(X_{\tau(n)}) \, \mathbf{1}_{[\tau > \tau(n)]} \to f(X_\tau) \quad \text{almost surely.}$$

As in (3.11), we consider the random variable $R_{\max} = \max\{|\mathrm{Re}\, X_t| : t \leqslant \tau\}$ (only that this time, τ is more general). By Lemma 3.6, resp. its variant for Ω_T, we have again that it has finite moments of all orders. Combining the growth condition (4.9) with (3.10), we get

$$|f(X_n^T)| \leqslant C_1 \, e^{C_2 D} \left(R_{\max} + 1\right)^{2C_2},$$

an integrable upper bound. Thus, by dominated convergence and the Martingale property, $f(\mathfrak{z}) = \mathsf{E}_{\mathfrak{z}}\big(f(X_\tau)\big)$, which is just another form of the proposed formula. □

Now we can formulate and prove the main result of this section, which is the analogue of Lemma 4.1 for HT.

THEOREM 4.7.

(i) If a non-negative function h on HT is $\Delta_{\alpha,\beta}$-harmonic then its restriction to LT is μ-harmonic.

(ii) Conversely, for any non-negative μ-harmonic function f on LT, its unique extension to a $\Delta_{\alpha,\beta}$-harmonic function on HT is given in each open strip S_v^o of HT by

$$(4.10) \qquad h(\mathfrak{z}) = \mathsf{E}_{\mathfrak{z}}\big(f(X_{\tau(1)})\big) = \int_{\partial\mathsf{S}_v} f \, d\mu_{\mathfrak{z}}^{\mathsf{S}_v^o},$$

where (recall) $\mu_{\mathfrak{z}}^{\mathsf{S}_v^o}$ is the exit distribution from S_v^o of (X_t) starting at $\mathfrak{z} \in \mathsf{S}_v^o$.

Proof. — (i) If h is non-negative and is $\Delta_{\alpha,\beta}$-harmonic on HT then it satisfies the Harnack inequality 4.2. Therefore, it also satisfies the growth condition (3.8), and Theorem 3.9 applies. That theorem provides (i) for points $\mathfrak{z} \in \mathsf{L}_o$. By transitivity of the action of the group $\mathcal{A}$ on LT as well as group invariance of the Laplacian, this holds along any line L_v.

(ii) We consider the following exhaustion of $\mathbb{T}$ by finite, full subtrees $T(k)$. Let o_k be the k-th predecessor of the root o of $\mathbb{T}$, that is, the element on $\overline{o\,\varpi}$ at distance k from o. Let $W(k)$ be the set of all vertices on the horocycle H_k at distance $2k-1$ from o_{k-1}. (That is, they have o_{k-1} as their $(2k-1)$-st predecessor.) Then $T(k)$ is the union of all geodesic segments $\overline{o_k\,w}$, where $w \in W(k)$. We consider $\Omega_{T(k)}$, with

$$\partial\Omega_{T(k)} = \mathsf{L}_{o_k} \cup \bigcup_{w \in W(k)} \mathsf{L}_w.$$

Now let f be a positive μ-harmonic function on LT. Then it is continuous (since μ has a continuous density by Theorem 3.9) and satisfies (4.9) by Proposition 4.4. In particular, we can apply Theorem 3.11, resp., its extended version according to Remark 3.12 to the restriction of f to $\partial\Omega_{T(k)}$, and

$$h_k(\mathfrak{z}) = \int_{\partial\Omega_{T(k)}} f \, d\mu_{\mathfrak{z}}^{\Omega_{T(k)}}, \quad \mathfrak{z} \in \Omega_{T(k)}$$

defines a harmonic function on $\Omega_{T(k)}$ which is continuous up to the boundary values of f. By Proposition 4.6, $h_k = f$ on every bifurcation line L_v with $v \in V\big(T(k)\big)$.

Now consider h_{k+1}: to this function, we can apply Lemma 4.3 on the set $\overline{\Omega}_{\mathbb{T}(k)} \subset \Omega^{\varepsilon}_{\mathbb{T}(k+1)}$: there is a constant $M(k)$ such that $h_{k+1}(\mathfrak{z}) \leqslant M(k)h_{k+1}(\tilde{\mathfrak{z}})$ for every $\mathfrak{z} \in \overline{\Omega}_{\mathbb{T}(k)}$, where $\tilde{\mathfrak{z}}$ is the element on L_o with $Re\,\tilde{\mathfrak{z}} = Re\,\mathfrak{z}$. But on L_o, the function h_{k+1} coincides with f, which satisfies the growth condition (4.9) by Proposition 4.6. Therefore also h_{k+1} satisfies (3.8) on the whole of $\overline{\Omega}_{\mathbb{T}(k)}$, with possibly modified constants. But now we can apply the uniqueness statement of Theorem 3.11, which yields that on $\overline{\Omega}_{\mathbb{T}(k)}$, the function h_{k+1} is the Poisson integral of f taken over $\partial\Omega_{\mathbb{T}(k)}$.

In other words, the restriction of h_{k+1} to $\overline{\Omega}_{\mathbb{T}(k)}$ is h_k. If $\mathfrak{z} \in \mathsf{HT}$ then there is k such that $\mathfrak{z} \in \Omega_{T(k)}$, and setting $h(\mathfrak{z}) = h_k(\mathfrak{z})$, we get a well-defined, positive Δ-harmonic function on HT that coincides with f on LT.

Uniqueness is straightforward: if $\tilde{h}$ is any other positive Δ-harmonic function that extends f, then by Proposition 4.2 it satisfies the growth condition (3.8) on the whole of HT. We can apply once more the uniqueness statement of Theorem 3.11 on any $\Omega_{T(k)}$ and get that $\tilde{h}$ concides with $h_k = h$ on the latter set, for any k.

The statement (ii) itself now follows a posteriori.   □

Remark 4.8. — The results of this section also apply to the Laplacian $\Delta^{\mathbb{H}}_{\alpha,\beta}$ on $\mathbb{H}(\mathsf{q}) = \mathsf{HT}(\mathsf{q},1)$, where the metric tree is $\mathbb{R}$ with vertex set $\mathbb{Z}$. In this case, LT is better denoted $\mathsf{L}\mathbb{H} = \bigcup_k \mathsf{L}_k \subset \mathbb{H}$, the union of all bifurcation lines in upper half plane. Note that $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ acts simply transitively on $\mathsf{L}\mathbb{H}$, so that this group can be identified with $\mathsf{L}\mathbb{H}$. The random walk $(Z_{\tau(n)})$ is governed the the probability measure $\tilde{\mu}$ on $\mathsf{L}_1 \cup \mathsf{L}_{-1}$ which is the exit distribution of (Z_t) from the double strip $\mathsf{S}_0 \cup \mathsf{S}_1$ when $Z_0 = \mathsf{i}$.

When we start with $\Delta^{\alpha,\beta}_{\mathsf{HT}}$ on $\mathsf{HT}(\mathsf{q},\mathsf{p})$, then in view of Proposition 2.4 we need to consider $\Delta^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$ here. In this situation, $\tilde{\mu}$ is the image of μ under the projection $\pi^{\mathbb{H}}$. What we have proved for μ on HT is also true for $\tilde{\mu}$ on $\mathbb{H}$. Of course, we shall call the associated harmonic functions on $\mathsf{L}\mathbb{H} \equiv \mathsf{Aff}(\mathbb{H},\mathsf{q})$ the $\tilde{\mu}$-harmonic functions.

5. Decomposition of positive harmonic functions

The aim of this section is to provide tools for giving a complete description of all positive Δ-harmonic functions on the whole of HT. For this

purpose, we shall need an understanding of the geometric boundary at infinity of HT. We quickly review the basic facts; a detailed description can be found in [3, Section 5].

The boundary (space of ends) $\partial\mathbb{T}$ was described in the lines after (2.7). The end compactification is $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial T$. The topology is such that a sequence (y_n) converges to a point y in the metric tree $\mathbb{T}$ if it converges with respect to the original metric, while it converges to an end $\xi \in \partial T$ if the geodesics $\overline{o\,z_n}$ and $\overline{o,\xi}$ share initial pieces whose graph length tends to infinity. The boundary of $\mathbb{H}(\mathsf{q})$, which is just hyperbolic half plane metrically, is $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The topology of $\widehat{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ is the classical hyperbolic compactification; it is maybe better understood when one considers the Poincaré disk model instead of upper half plane, where $\widehat{\mathbb{H}}$ is the Euclidean closure (the closed unit disk), and ∞ corresponds to the north pole, while the remaining part of the unit circle corresponds to the lower boundary line $\mathbb{R}$ of upper half plane.

Now $\mathsf{HT}(\mathsf{q}, \mathsf{p})$ is a subspace of the direct product $\mathbb{H}(\mathsf{q}) \times \mathbb{T}_\mathsf{p}$, so that the most natural geometric compactification $\widehat{\mathsf{HT}}$ of HT is its closure in $\widehat{\mathbb{H}} \times \widehat{\mathbb{T}}$. As mentioned, a detailed description of the resulting boundary $\partial\mathsf{HT}$ and of the different ways of convergence to the boundary can be found in [3].

Transience of (X_t) implies that the *Green kernel*

$$(5.1) \quad \mathbf{g}_{\alpha,\beta}(\mathfrak{w}, \mathfrak{z}) = \mathbf{g}(\mathfrak{w}, \mathfrak{z}) = \int_0^\infty \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\,dt \quad (\mathfrak{w}, \mathfrak{z} \in \mathsf{HT}\,, \; \mathfrak{w} \neq \mathfrak{z})$$

is strictly positive and finite, compare with [2]. The function $\mathfrak{w} \mapsto \mathbf{g}(\mathfrak{w}, \mathfrak{z})$ is harmonic on $\mathsf{HT} \setminus \{\mathfrak{z}\}$. Along with Proposition 4.2, the next uniform Harnack inequalities for the Green kernel are again a consequence of [2, Theorem 4.2].

PROPOSITION 5.1. — *With the same constants C_d as in Proposition 4.2 (in particular, $C_d \to 1$ when $d \to 0$)*

$$\frac{\mathbf{g}(\mathfrak{w}, \mathfrak{z}')}{\mathbf{g}(\mathfrak{w}, \mathfrak{z})} \leqslant C_d \quad and \quad \frac{\mathbf{g}(\mathfrak{z}', \mathfrak{w})}{\mathbf{g}(\mathfrak{z}, \mathfrak{w})} \leqslant C_d\,,$$

whenever $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{z}') \leqslant d$ and $\min\{\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{w}), \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}', \mathfrak{w})\} \geqslant 10(d+1)$.

Positive harmonic functions can be described via the *Martin boundary*. The regularity properties of our Green kernel allow us to use the following approach, which is a special case of the potential theory (more precisely, Brelot theory) outlined in the book of Constantinescu and Cornea [11, Chapter 11]. See also Brelot [5], and in the Markov chain setting, compare with Revuz [24, Chapter 7].

We fix the "origin" $\mathfrak{o} = (\mathfrak{i}, o)$ of HT as the reference point. A function $f \in \mathcal{H}^+$ is called *minimal* if $f(\mathfrak{o}) = 1$ and whenever $f_1 \in \mathcal{H}^+$ is such that $f \geqslant f_1$ then f_1/f is constant. The minimal harmonic functions are the extremal points of the convex set $\{f \in \mathsf{HT}^+ : f(\mathfrak{o}) = 1\}$, which is a base of the convex cone HT^+ of all positive harmonic functions. Every positive harmonic function is a convex combination (with respect to a Borel measure) of minimal ones. To make this more precise, define the *Martin kernel* by

$$(5.2) \qquad k(\mathfrak{w}, \mathfrak{z}) = \frac{\mathsf{g}(\mathfrak{w}, \mathfrak{z})}{\mathsf{g}(\mathfrak{o}, \mathfrak{z})}$$

Then the Martin compactification is the unique (up to homeomorphism) minimal compactification of the underlying space (i.e., HT) such that all functions $k(\mathfrak{w}, \cdot)$, $\mathfrak{w} \in \mathsf{HT}$, extend continuously. The extended kernel is also denoted by $k(\cdot, \cdot)$. The *Martin boundary* $\mathcal{M} = \mathcal{M}(\Delta)$ is the set of ideal boundary points in this compactification (i.e., the new points added "at infinity"). It is compact and metrizable.

From the general theory [11], it is known that every minimal harmonic function arises as a boundary function $k(\cdot, \boldsymbol{\xi})$, where $\boldsymbol{\xi} \in \mathcal{M}$. The (Borel) set $\mathcal{M}_{\min}$ of all $\boldsymbol{\xi} \in \mathcal{M}$ for which $k(\cdot, \boldsymbol{\xi})$ is minimal is called the *minimal Martin boundary*. The *Poisson-Martin* representation theorem says that for every $h \in \mathcal{H}^+$ there is a unique Borel measure ν^h on $\mathcal{M}$ such that $\nu(\mathcal{M} \setminus \mathcal{M}_{\min}) = 0$ and

$$(5.3) \qquad h = \int_{\mathcal{M}} k(\cdot, \boldsymbol{\xi}) \, d\nu^h(\boldsymbol{\xi}) \,.$$

If $h^{\mathbb{H}}$ is an $\Delta^{\mathbb{H}}$-harmonic function on $\mathbb{H}$ then by Proposition 2.4, the function $h^{\mathbb{H}} \circ \pi^{\mathbb{H}}$ is Δ-harmonic on HT, and the analogous property holds when we lift a $\Delta^{\mathbb{T}}$-harmonic function $h^{\mathbb{T}}$ from $\mathbb{T}$ to HT. Combining both cases, the function $h(\mathfrak{z}) = h^{\mathbb{H}}(z) + h^{\mathbb{T}}(w)$ (where $\mathfrak{z} = (z, w) \in \mathsf{HT}$) is also Δ-harmonic on HT.

The first main purpose of this section and the machinery set up so far is to prove the following.

THEOREM 5.2. — *Every positive $\Delta_{\alpha,\beta}$-harmonic h on HT has the form*

$$h(\mathfrak{z}) = h^{\mathbb{H}}(z) + h^{\mathbb{T}}(w), \quad \mathfrak{z} = (z, w) \in \mathsf{HT} \,,$$

where $h^{\mathbb{H}}$ is non-negative $\Delta^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$-harmonic on $\mathbb{H}$ and $h^{\mathbb{T}}$ is non-negative $\Delta^{\mathbb{T}}_{\alpha,\beta}$-harmonic on $\mathbb{T}$.

For the proof, we shall use the crucial fact that $\Delta_{\alpha,\beta}$ is invariant under the group $\mathcal{A} = \mathcal{A}(\mathsf{q}, \mathsf{p})$ defined in Theorem 2.5, that is, $\mathsf{t}_{\mathfrak{g}}(\Delta_{\alpha,\beta} f) = \Delta_{\alpha,\beta}(\mathsf{t}_{\mathfrak{g}} f)$

for all $f \in \mathrm{Dom}(\Delta_{\alpha,\beta})$ and all $\mathfrak{g} \in \mathcal{A}$, where

(5.4) $$\mathsf{t}_{\mathfrak{g}} f(\mathfrak{z}) = f(\mathfrak{g}\mathfrak{z}).$$

Let
$$G(\mathfrak{w}, \mathfrak{z}) = \mathbf{g}(\mathfrak{w}, \mathfrak{z})\phi_{\alpha,\beta}(\mathfrak{z}),$$
where $\phi_{\alpha,\beta}$ is as in (2.2). We have $k(\mathfrak{w}, \mathfrak{z}) = G(\mathfrak{w}, \mathfrak{z})/G(\mathfrak{o}, \mathfrak{z})$, and it follows from (2.11) that

$$G(\mathfrak{g}\mathfrak{w}, \mathfrak{g}\mathfrak{z}) = G(\mathfrak{w}, \mathfrak{z}) \quad \text{for all} \quad \mathfrak{w}, \mathfrak{z} \in \mathsf{HT} \ (\mathfrak{w} \neq \mathfrak{z}) \text{ and } \mathfrak{g} \in \mathcal{A}.$$

We shall use specific subgroups of $\mathcal{A}$. First, in the tree, consider once more the k-th predecessor o_k of the root vertex. Thus, $o_k \in H_{-k}$. Now define

$$\mathcal{B}(\mathbb{T}) = \{\mathfrak{g}_{\psi} = (id_{\mathbb{H}}, \psi) : \psi \in \mathsf{Aff}(\mathbb{T}), \ \psi o_k = o_k \text{ for some } k \in \mathbb{N}\}.$$

This is the embedding into $\mathcal{A}$ of the *horocyclic subgroup* of $\mathsf{Aff}(\mathbb{T})$, which consists of all isometries of the tree that leave each horocycle invariant. The latter is the analogue of the group of all "horizontal" translations in $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$, that is, of all mappings $g_b = (1, b)$, where $b \in \mathbb{R}$, acting on $\mathbb{H}$ by $z \mapsto z + b$. Thus, we define

$$\mathcal{B}(\mathbb{H}) = \{\mathfrak{g}_b = (g_b, id_{\mathbb{T}}) : b \in \mathbb{R}\},$$

which is of course isomorphic with the additive group $\mathbb{R}$ acting on $\mathbb{H}$.

THEOREM 5.3. — *If h is a minimal $\Delta_{\alpha,\beta}$-harmonic function on HT, then (at least) one of the following holds.*
  (a) *There is a minimal $\Delta_{\alpha,\beta\mathsf{p}}^{\mathbb{H}}$-harmonic function $h^{\mathbb{H}}$ on $\mathbb{H}$ such that $h(z, w) = h^{\mathbb{H}}(z)$ for all $\mathfrak{z} = (z, w) \in \mathsf{HT}$, or*
  (b) *there is a minimal $\Delta_{\alpha,\beta}^{\mathbb{T}}$-harmonic function $h^{\mathbb{T}}$ on $\mathbb{T}$ such that $h(z, w) = h^{\mathbb{T}}(w)$ for all $\mathfrak{z} = (z, w) \in \mathsf{HT}$.*

Proof. — We know that $h = k(\cdot, \boldsymbol{\xi})$ for some $\boldsymbol{\xi} \in \mathcal{M}$, the Martin boundary of HT with respect to $\Delta_{\alpha,\beta}$. There must be a sequence of points $\mathfrak{z}_n = (z_n = x_n + \mathrm{i}\,y_n, w_n) \in \mathsf{HT}$ such that $\mathsf{d}_{\mathsf{HT}}(\mathfrak{o}, \mathfrak{z}_n) \to \infty$ and

$$h(\mathfrak{z}) = \lim_{n \to \infty} k(\mathfrak{z}, \mathfrak{z}_n) \quad \text{for every} \quad \mathfrak{z} \in \mathsf{HT}.$$

Using compactness, we may suppose that $(\mathfrak{z}_n)$ converges to a boundary point in $\widehat{\mathsf{HT}}$. Then we either have that $w_n \to \varpi$ in $\widehat{\mathbb{T}}$, or else y_n is bounded below by a positive constant and $z_n \to \infty \in \partial\mathbb{H}$ in the topology of $\widehat{\mathbb{H}}$, that is, $y_n \to +\infty$ or $y_n \to y_0 \in \mathbb{R}$ and $|x_n| \to +\infty$.

Case 1. $w_n \to \varpi$. Let $\mathfrak{z} = (z, w)$ and $\mathfrak{z}' = (z, w') \in \mathsf{HT}$ be two distinct points with $\pi_{\mathbb{H}}(\mathfrak{z}) = \pi_{\mathbb{H}}(\mathfrak{z}') = z$, so that $\mathfrak{h}(w) = \mathfrak{h}(w')$ in $\mathbb{T}$. Consider the confluent $v = w \curlywedge w' \in V(\mathbb{T})$. Since $w_n \to \varpi$, we must have for all but

finitely many n that $w_n \in \mathbb{T} \setminus \mathbb{T}_v$, where $\mathbb{T}_v$ is the subtree $\mathbb{T}_v$ of $\mathbb{T}$ rooted at v (i.e., consisting of all $v' \in \mathbb{T}$ with $v' \curlywedge v = v$). There is an isometry of $\mathbb{T}_v$ that exchanges w and w' (and fixes v). It extends to an element ψ in the horocyclic subgroup of $\mathsf{Aff}(\mathbb{T})$, that fixes every point in $\mathbb{T} \setminus \mathbb{T}_v$. In particular, $\psi w_n = w_n$ for all but finitely many n. Therefore

$$G(\mathfrak{z}', \mathfrak{z}_n) = G(\mathfrak{g}_\psi \mathfrak{z}, \mathfrak{g}_\psi \mathfrak{z}_n) = G(\mathfrak{z}, \mathfrak{z}_n), \quad \text{whence} \quad k(\mathfrak{z}', \mathfrak{z}_n) = k(\mathfrak{z}, \mathfrak{z}_n)$$

for all but finitely many n. Thus, as $n \to \infty$, we find $h(\mathfrak{z}') = h(\mathfrak{z})$ for all $\mathfrak{z}, \mathfrak{z}'$ with $\pi_{\mathbb{H}}(\mathfrak{z}) = \pi_{\mathbb{H}}(\mathfrak{z}')$. This means precisely that there is a function $f^{\mathbb{H}}$ on $\mathbb{H}$ such that $h(z, w) = h^{\mathbb{H}}(z)$ for all $\mathfrak{z} = (z, w) \in \mathsf{HT}$. It is now straightforward that $h^{\mathbb{H}}$ must be minimal $\Delta^{\mathbb{H}}_{\alpha, \beta \mathsf{p}}$-harmonic.

Case 2. $z_n \to \infty \in \partial \mathbb{H}$ and $\inf y_n = y_0 > 0$. Let $b \in \mathbb{R}$. By (2.8) and standard properties of the hyperbolic metric,

$$\begin{aligned} \mathsf{d}_{\mathsf{HT}}(\mathfrak{g}_b \mathfrak{z}_n, \mathfrak{z}_n) &= \mathsf{d}_{\mathbb{H}}\big((b + x_n) + \mathrm{i}\, y_n, x_n + \mathrm{i}\, y_n\big) \\ &= \mathsf{d}_{\mathbb{H}}(b + \mathrm{i}\, y_n, \mathrm{i}\, y_n) \leqslant \mathsf{d}_{\mathbb{H}}(b + \mathrm{i}\, y_0, \mathrm{i}\, y_0) =: d_b\,. \end{aligned}$$

Let C_{d_b} be the corresponding Harnack constant in Proposition 5.1. Then, using that $G(\cdot, \cdot)$ is $\mathcal{A}$-invariant and that $\phi_{\alpha, \beta}(\mathfrak{g}_b \mathfrak{z}_n) = \phi_{\alpha, \beta}(\mathfrak{z}_n)$

$$\begin{aligned} k(\mathfrak{g}_b \mathfrak{z}, \mathfrak{z}_n) &= \frac{G(\mathfrak{g}_b \mathfrak{z}, \mathfrak{z}_n)}{G(\mathfrak{g}_b \mathfrak{z}, \mathfrak{g}_b \mathfrak{z}_n)} \, \frac{G(\mathfrak{g}_b \mathfrak{z}, \mathfrak{g}_b \mathfrak{z}_n)}{G(o, \mathfrak{z}_n)} \\ &= \frac{\mathsf{g}(\mathfrak{g}_b \mathfrak{z}, \mathfrak{z}_n)}{\mathsf{g}(\mathfrak{g}_b \mathfrak{z}, \mathfrak{g}_b \mathfrak{z}_n)} \, k(\mathfrak{z}, \mathfrak{z}_n) \leqslant C_{d_b} \, k(\mathfrak{z}, \mathfrak{z}_n). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\mathfrak{t}_b h(\mathfrak{z}) := h(\mathfrak{g}_b \mathfrak{z}) \leqslant C_{d_b} \, h(\mathfrak{z}) \quad \text{for all } \mathfrak{z} \in \mathsf{HT}\,.$$

Now, along with h, also $\mathfrak{t}_b h$ is positive harmonic, and minimality of h implies that the function $\mathfrak{t}_b h / h$ is constant. On the other hand, write $\mathfrak{z} = (x + \mathrm{i}\, y, w)$. If we let $y \to +\infty$ (and simultaneously $\mathfrak{h}(w) = \log_{\mathsf{q}} y \to +\infty$) then $d(\mathfrak{g}_b \mathfrak{z}, \mathfrak{z}) = \mathsf{d}_{\mathbb{H}}(b + \mathrm{i}\, y, \mathrm{i}\, y) \to 0$. Consequently, Proposition 4.2(a) implies that $h(\mathfrak{g}_b \mathfrak{z})/f(\mathfrak{z}) \to 1$. Therefore $\mathfrak{t}_b f = f$. This holds for every $b \in \mathbb{R}$. As in Case 1, this is equivalent with the fact that there is a function $h^{\mathbb{T}}$ on $\mathbb{T}$ such that $h(z, w) = h^{\mathbb{T}}(w)$ for all $\mathfrak{z} = (z, w) \in \mathsf{HT}$. It is again straightforward that $h^{\mathbb{T}}$ must be minimal $\Delta^{\mathbb{T}}_{\alpha, \beta}$-harmonic. $\qquad\square$

Remark 5.4. — If both cases (a) and (b) of the last theorem occur simultaneously, then this means that there is a minimal $\Delta^{\mathbb{R}}_{\alpha, \beta \mathsf{p}}$-harmonic function $h^{\mathbb{R}}$ on $\mathbb{R}$ such that $h(z, w) = h^{\mathbb{R}}\big(\mathfrak{h}(w)\big)$ for all $\mathfrak{z} = (z, w) \in \mathsf{HT}$. In the above proof, this happens when $\mathfrak{z}_n \to (\infty, \varpi)$.

Proof of Theorem 5.2. — Let $\mathcal{M}_{\min}^{(1)}$ and $\mathcal{M}_{\min}^{(2)}$ be the sets of all minimal harmonic functions on HT which are as in Theorem 5.3(a) and (b), respectiveley. Since the topology of the Martin boundary is the one of uniform convergence on compact sets, both are easily seen to be Borel sets in $\mathcal{M}$. Also, $\mathcal{M}_{\min} \setminus \mathcal{M}_{\min}^{(1)} \subset \mathcal{M}_{\min}^{(2)}$. If f is any positive harmonic function on HT with integral representation (5.3), then we can set

$$f^{\mathbb{H}} = \int_{\mathcal{M}_{\min}^{(1)}} k(\cdot, \boldsymbol{\xi})\, d\nu^h(\boldsymbol{\xi}). \quad \text{and} \quad f^{\mathbb{T}} = \int_{\mathcal{M}\setminus\mathcal{M}_{\min}^{(1)}} k(\cdot, \boldsymbol{\xi})\, d\nu^h(\boldsymbol{\xi}).$$

Then $f^{\mathbb{H}}(z, w)$ depends only on z, while $f^{\mathbb{T}}(z, w)$ depends only on w, and f is their sum, as proposed. $\qquad\square$

Remark 5.5. — In view of Theorem 4.7 and Remark 4.8, as well as Lemma 4.1 (which also applies to the metric tree $\mathbb{R}$ with vertex set $\mathbb{Z}$), we have the following: a function on any of our spaces HT, $\mathbb{H}$, $\mathbb{T}$, or $\mathbb{R}$ is minimal harmonic for the respective one of our Laplacians if and only its restriction to LT, $\mathsf{L}\mathbb{H}$, $V(\mathbb{T})$, or $\mathbb{Z}$ (respectively) is minimal harmonic for the respective induced random walk.

As a first application of Theorem 5.2, we can clarify when the weak Liouville property holds, that is, when all bounded harmonic functions are constant. For the following, we recall from [3, Thm. 5.1] that

$$\lim_{t\to\infty} \frac{1}{t}\, \mathsf{d}_{\mathsf{HT}}(X_t, X_0) = |\ell(\alpha, \beta)| \text{ almost surely, where } \ell(\alpha, \beta) = \frac{\log \mathsf{q}}{\mathsf{E}(\tau)}\frac{\mathsf{a} - 1}{\mathsf{a} + 1},$$

with τ and a given by (2.13) and (2.14), respectively. In particular, $\ell(\alpha, \beta) = 0$ if and only if $\beta \mathsf{p}\, \mathsf{q}^{\alpha-1} = 1$.

THEOREM 5.6. — *Suppose that* $\mathsf{p} \geqslant 2$. *Then the Laplacian* $\Delta_{\alpha,\beta}$ *on* HT *has the weak Liouville property if and only if* $\ell(\alpha, \beta) = 0$.

This follows from Theorem 5.2 in combination with the following.

PROPOSITION 5.7.
  (a) $\Delta_{\alpha,\beta}^{\mathbb{T}}$ *has the weak Liouville property on* $\mathbb{T}$ *if and only if* $\ell(\alpha, \beta) \leqslant 0$.
  (b) $\Delta_{\alpha,\beta\mathsf{p}}^{\mathbb{H}}$ *has the weak Liouville property on* $\mathbb{H}$ *if and only if* $\ell(\alpha, \beta) \geqslant 0$.

(Note the striking difference in the range of validity of the weak Liouville property on $\mathsf{HT}(\mathsf{q}, \mathsf{p})$ with $\mathsf{p} \geqslant 2$ as compared to $\mathbb{H}(\mathsf{q}) = \mathsf{HT}(\mathsf{q}, 1)$! Also, compare those results with Karlsson and Ledrappier [23], [22].)

Proof. — We note that the weak Liouville property is the same as triviality of the Poisson boundary, or equivalently, that the constant function $\mathbf{1}$ is minimal harmonic.

(a) It follows from [10] that the weak Liouville property holds for the random walk $(W_{\tau(n)})$ on $\mathbb{T}$ if and only if $\ell(\alpha, \beta) \leqslant 0$. Via Lemma 4.1, this transfers to $\Delta^{\mathbb{T}}_{\alpha,\beta}$. Statement (a) also follows from Proposition 5.8 below.

(b) Among the different possible approaches, we adapt the method of Theorem 5.3. Since $\mathbb{H}(\mathsf{q}) = \mathsf{HT}(\mathsf{q}, 1)$, everything that we have stated and proved for harmonic functions and the Green kernel on HT also applies to $\mathbb{H}$. (The decomposition result for positive harmonic functions becomes of course trivial.) In particular, we have the Green kernel $G^{\mathbb{H}}(w, z) = \phi(\alpha, \beta\mathsf{p})(z)\,\mathsf{g}^{\mathbb{H}}(w, z)$, where $\phi(\alpha, \beta\mathsf{p})(x+\mathsf{i}\,y) = (\beta\mathsf{p})^{\lfloor \log_\mathsf{q} y \rfloor} y^\alpha$. We also have the associated Martin kernel $k^{\mathbb{H}}(w, z)$.

A basic theorem in Martin boundary theory says that $k^{\mathbb{H}}(\cdot, Z_t)$ converges almost surely to a minimal harmonic function. We now show that when $\ell(\alpha, \beta) \geqslant 0$ then with positive probability, there is a random sequence $(Z_{t(n)})$ such that $k^{\mathbb{H}}(\cdot, Z_{t(n)}) \to \mathbf{1}$. Thus, the constant function $\mathbf{1}$ on $\mathbb{H}$ must be minimal harmonic.

To show what we claimed, in the case when $\ell(\alpha, \beta) > 0$ then the random walk $(Y_{\tau(n)})$ has positive drift, so that it tends to ∞ almost surely. When $\ell(\alpha, \beta) = 0$, it is recurrent: with probability 1, it visits every point in $\mathbb{Z}$. Thus it is unbounded and has a random subsequence that tends to ∞. In both cases, we have a random sequence $t(n)$ such that $Im\, Z_{t(n)} \to \infty$ almost surely.

Let us abbreviate $z_n = Z_{t(n)}$, so that $k^{\mathbb{H}}(\cdot, z_n)$ tends to a minimal harmonic function h, and $Im\, z_n \to \infty$. We can use exactly the same method as in Case 2 of the proof of Theorem 5.3 and get that $h(z + b) = h(z)$ for every $z \in \mathbb{H}$ and every $b \in \mathbb{R}$. Thus, h only depends on $Re\, z$, which means that it arises from lifting a minimal $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$-harmonic function $\widetilde{h}$ to $\mathbb{H}$ via $h(z) = \widetilde{h}(\log_\mathsf{q} Im\, z)$. We can apply Lemma 4.1 to the specific case where the tree is just $\mathbb{R}$, with vertex set $\mathbb{Z}$: we see that the minimal $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$ harmonic functions arise by interpolating the minimal harmonic functions for the random walk $Y_{\tau(n)}$ on $\mathbb{Z}$. The latter minimal harmonic functions are easily computed and are as follows.

When $\ell(\alpha, \beta) = 0$ then that random walk is recurrent, so that all its positive harmonic functions are constant. Thus, $h = \mathbf{1}$, as required.

When $\ell(\alpha, \beta) > 0$, the random walk has precisely two minimal harmonic functions, namely the constant function $\mathbf{1}$ and the function $\widehat{f}(k) = \mathsf{a}^{-k}$ (a well known exercise). We have to interpolate $\widehat{f}$ according to the variant of Lemma 4.1 where $\mathsf{p} = 1$ and $\mathbb{T} = \mathbb{R}$ with vertex set $\mathbb{Z}$. This yields the unique $\Delta^{\mathbb{R}}_{\alpha,\mathsf{p}\beta}$-harmonic extension $\widehat{h}$. Now we have $h(z) = \widehat{h}(\log_\mathsf{q} Im\, z)$.

Thus, with probability one, $k^{\mathbb{H}}(\cdot, Z_t)$ converges either to the constant $\mathbf{1}$ or to the latter function h. Suppose that it converges to h with probability 1. But then, the Poisson boundary has to consist of only one point, and in this case, every bounded harmonic function ought to be a multiple of h - a contradiction since h is unbounded. Thus, with positive probability, the limit is the function $\mathbf{1}$ which thus must be minimal harmonic on $\mathbb{H}$. (A posteriori, the limit function must be $\mathbf{1}$ almost surely.) $\qquad\square$

Theorem 5.2 tells us that for describing all positive harmonic functions on HT, we need to know all such functions on $\mathbb{T}$ and on $\mathbb{H}$. Here, "describing" means that we need to determine all minimal harmonic functions on each of those two spaces. This task is very easy on $\mathbb{T}$. In view of Remark 5.5, we first consider the induced random walk on $V(\mathbb{T})$. The minimal harmonic functions for any transient nearest neighbour random walk on (the vertex set of) a tree are completely understood since the seminal article of Cartier [9]. For $(W_{\tau(n)})$ on $V(\mathbb{T})$, we have the following explicit formulas, taken from [25, p. 424–425].

PROPOSITION 5.8. — *The minimal Martin boundary of the random walk on $V(\mathbb{T})$ coincides with the whole Martin boundary, which is $\partial\mathbb{T}$. For the transition probabilities of (4.1), and with* a *as in (2.14), the Martin kernels on $V(\mathbb{T})$ are given as follows, where we set*

$$\mathsf{b} = \max\{\mathsf{a}, 1\} \quad and \quad \mathsf{c} = \min\{\mathsf{a}, 1/\mathsf{a}\}/\mathsf{p}.$$

For the boundary point ϖ,

$$k^{\mathbb{T}}(v, \varpi) = \mathsf{b}^{-\mathfrak{h}(v)}, \quad v \in V(\mathbb{T}).$$

For $\xi \in \partial^*\mathbb{T}$,

$$k^{\mathbb{T}}(v, \xi) = \mathsf{b}^{-\mathfrak{h}(v)} \, \mathsf{c}^{\mathfrak{h}(o\wedge\xi)-\mathfrak{h}(v\wedge\xi)}, \quad v \in V(\mathbb{T}).$$

The minimal harmonic functions on the metric tree $\mathbb{T}$ are the respective extensions of these functions according to (4.3), which are also denoted $k^{\mathbb{T}}(w, \xi)$, where $w \in \mathbb{T}$ and $\xi \in \partial\mathbb{T}$.

We see indeed (as already mentioned) that the constant function $\mathbf{1}$ on $\mathbb{T}$ is minimal harmonic if and only if $\mathsf{a} \leqslant 1$, that is $\ell(\alpha, \beta) \leqslant 0$.

Describing the minimal harmonic functions on $\mathbb{H}$ is in general more complicated and less explicit. There is one exception which we describe next. Namely, if $\beta\mathsf{p} = 1$ for the Laplacian on HT, then its projection $\Delta^{\mathbb{H}}_{\alpha,\beta\mathsf{p}} = \Delta^{\mathbb{H}}_{\alpha}$ is nothing but the ordinary hyperbolic Laplacian on upper half plane $\mathbb{H}$ with vertical drift parameter α, as in the second line of Definition 3.2. The

bifurcation lines disappear in its analysis, and the minimal harmonic functions on $\mathbb{H}$ are well known; see e.g. Helgason [21], or many other sources.

PROPOSITION 5.9. — *The minimal Martin boundary of* $\Delta_\alpha^{\mathbb{H}} = \Delta_{\alpha,1}^{\mathbb{H}}$ *on* $\mathbb{H}$ *coincides with the whole Martin boundary, which is* $\partial\mathbb{H}$. *The associated Martin kernels on* $\mathbb{H}$ *are given by the following extended Poisson kernels. For the boundary point* ∞,

$$P_\alpha\big((x + \mathrm{i}\,y), \infty\big) = y^{\max\{1-\alpha,0\}}.$$

For any boundary point $\zeta \in \mathbb{R}$,

$$P_\alpha\big((x + \mathrm{i}\,y), \zeta\big) = y^{\max\{1-\alpha,0\}} \left(\frac{\zeta^2 + 1}{(\zeta - x)^2 + y^2} \right)^{\max\{\frac{\alpha}{2}, 1-\frac{\alpha}{2}\}}.$$

At last, we can deduce the following result.

THEOREM 5.10. — *Consider* $\mathsf{HT}(\mathsf{q}, \mathsf{p})$ *with* $\mathsf{p} \geqslant 2$ *and the Laplacian* $\Delta_{\alpha,\beta}$ *with* $\beta = 1/\mathsf{p}$, *so that* $\mathsf{a} = \mathsf{q}^{\alpha-1}$.

    (I) *If* $\alpha \neq 1$ *then the minimal harmonic functions on* HT *are parametrised by* $\mathbb{R} \cup \partial^*\mathbb{T}$ *and are given by*

$$\mathfrak{z} = (z, w) \mapsto k^{\mathbb{T}}(w, \xi), \ \xi \in \partial^*\mathbb{T}, \quad \text{and} \quad \mathfrak{z} = (z, w) \mapsto P_\alpha(z, \zeta), \ \zeta \in \mathbb{R}.$$

    (II) *If* $\alpha = 1$ *then the minimal harmonic functions are as in* (I) *plus, in addition, the constant function* $\mathbf{1}$.

Proof. — We first show that for $\zeta \in \mathbb{R}$, the function $\mathfrak{z} = (z, w) \mapsto P_\alpha(z, \zeta)$ is minimal harmonic. Suppose that $P_\alpha(z, \zeta) \geqslant h(z, w)$ for all $(z, w) = \mathfrak{z} \in \mathsf{HT}$, where h is non-negative harmonic on HT. We decompose $h(z, w) = h^{\mathbb{H}}(z) + h^{\mathbb{T}}(w)$ according to Theorem 5.2. By minimality of $P_\alpha(\cdot, \zeta)$ on $\mathbb{H}$, we have $h^{\mathbb{H}} = c \cdot P_\alpha(\cdot, \zeta)$ for some $c \in [0, 1]$. If $c = 1$ then we are done. If $c < 1$,

$$P_\alpha(z, \zeta) \geqslant \frac{1}{1-c} h^{\mathbb{T}}(w) = \int_{\partial\mathbb{T}} k^{\mathbb{T}}(w, \xi) \, d\nu(\xi) \quad \text{for all} \ (z, w) \in \mathsf{HT},$$

where ν is a Borel measure on $\partial\mathbb{T}$. We set $w = o$ and $z = x + \mathrm{i}$, where $x \in \mathbb{R}$. Then $P_\alpha(x + \mathrm{i}, \zeta) \geqslant \nu(\partial T)$ for all x. When $x \to \infty$, the Poisson kernel tends to 0, so that the measure ν vanishes. Thus, $h^{\mathbb{T}} \equiv 0$ and h is a multiple of our Poisson kernel.

Next, we need that for $\xi \in \partial^*T$, the function $\mathfrak{z} = (z, w) \mapsto k^{\mathbb{T}}(w, \xi)$ is minimal harmonic. This is completely analogous to the above, exchanging the roles of $\mathbb{T}$ and $\mathbb{H}$.

When $\alpha = 1$, we know from Theorem 5.6 that the constant function $\mathbf{1}$ is also minimal harmonic. When $\alpha \neq 1$, we still need to show that the

two functions $(z, w) \mapsto k^{\mathbb{T}}(w, \varpi)$ and $(z, w) \mapsto P_\alpha(z, \infty)$ are *not* minimal harmonic. These are the functions $\mathbf{1}$ and $(x + \mathrm{i}\, y, w) \mapsto y^{1-\alpha}$. The first one is not minimal, because there are non-constant bounded harmonic functions. The second one is not minimal because when $\alpha > 1$ it is already non-minimal on $\mathbb{H}$, while when $\alpha < 1$ is is already non-minimal on $\mathbb{T}$. □

OUTLOOK. — For a similar description of all minimal harmonic functions in the case when $\bar{\beta} = \beta \mathsf{p} \neq 1$, one needs to study the boundary theory of "sliced" Laplacians $\Delta^{\mathbb{H}}(\alpha, \bar{\beta})$. A natural approach is to apply Theorem 4.7 to that case. In view of Remark 4.8, this means that one wants to determine the minimal harmonic functions, or even the entire Martin compactification, for the random walk on $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ with law $\widetilde{\mu}$. Since $\widetilde{\mu}$ has super-exponential moments and continuous density with respect to Haar measure on that group, the guiding results are those of Elie [16], [17].

One expects that the minimal harmonic functions in the case $\ell(\alpha, \beta) \leqslant 0$ are given by an exponential depending only on the height (imaginary part) plus the Radon-Nikodym derivatives of translates of the unique $\widetilde{\mu}$-invariant measure on the real line. When $\ell(\alpha, \beta) > 0$ one should first invert the drift by conjugating with a suitable exponential, and then apply the negative-drift case.

However, at a closer look, one finds that Elie's methods do not apply directly to our disconnected subgroup of the affine group. A second possible approach is to clarify how the methods of Ancona [1] apply to $\Delta^{\mathbb{H}}(\alpha, \bar{\beta})$ in the "coercive" case $\bar{\beta}\, \mathsf{q}^{\alpha-1} \neq 1$, while in the case $\bar{\beta}\, \mathsf{q}^{\alpha-1} = 1$ one might attempt to adapt the methods of Gouëzel [19], [20] concerning Martin boundary of hyperbolic groups to our non-discrete situation. Thus, additional work is required – this will be done separately.

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