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Local compactness and cartesian products of quotient maps and $K$-spaces

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LOCAL COMPACTNESS
AND CARTESIAN
PRODUCTS OF QUOTIENT MAPS AND $k$-SPACES

by Ernest MICHAEL (1)

1. Introduction.

In 1948, J.H.C. Whitehead [8; Lemma 4] proved that, if $X$ is locally compact Hausdorff, then the Cartesian product \(^{(2)}\) $i_x \times g$ is a quotient map \(^{(3)}\) for every quotient map $g$. Using this result, D.E. Cohen proved in [1; 3.2] that, if $X$ is locally compact Hausdorff, then $X \times Y$ is a $k$-space \(^{(4)}\) for every $k$-space $Y$. The principal purpose of this note is to show that these results are the best possible, in the sense that, if a regular space $X$ is not locally compact, then the conclusions of both results are false. (That the conclusions are false without some restrictions on $X$ is well known; see, for instance, Bourbaki [2, p. 151, Exercise 6] and C.H. Dowker [4; p. 563]).

Our main results are formally stated and proved in sections 2 and 3, while section 4 contains analogous results for sequential spaces, and section 5 considers the special case where $X$ is metrizable.

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\(^{(2)}\) If $f_i: X_i \rightarrow Y_i$ ($i = 1, 2$), the product $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. We use $i_X$ to denote the identity map on $X$.

\(^{(3)}\) A map $f: X \rightarrow Y$ is a quotient map if a set $V \subseteq Y$ is open in $Y$ if and only if $f^{-1}(V)$ is open in $X$.

\(^{(4)}\) A topological space $X$ is a $k$-space if a subset $A$ of $X$ is closed whenever $A \cap K$ is closed in $K$ for every compact $K \subseteq X$. All locally compact spaces and all first-countable spaces are $k$-spaces.
I am grateful to S.P. Franklin and A.H. Stone for a valuable conversation over a Mexican dinner during an Arizona sandstorm.

2. Products of quotient maps.

**Theorem 2.1.** — The following properties of a regular (5) space $X$ are equivalent.

(a) $X$ is locally compact.

(b) $i_X \times g$ is a quotient map for every quotient map $g$.

(c) $i_X \times g$ is a quotient map for every closed compact-covering (6) map $g$ with domain and range paracompact $k$-spaces.

**Proof.** — The implication (a) $\rightarrow$ (b) is the theorem of J.H.C. Whitehead quoted in the introduction, and (b) $\rightarrow$ (c) is obvious because continuous closed maps are quotient maps. It remains to prove (c) $\rightarrow$ (a).

Suppose $X$ is not locally compact at some $x_0 \in X$. Let $\{U_\alpha\}_{\alpha \in A}$ be a local base at $x_0$. Then, for all $\alpha \in A$, the closure $\overline{U_\alpha}$ is not compact, and thus has a well ordered family $\{F_\lambda\}_{\lambda < \lambda(\alpha)}$ of non-empty closed subsets whose intersection is empty (7). We assume that the collection of all the well-ordered index sets $\Lambda_\alpha = \{\lambda : \lambda \leq \lambda(\alpha)\}$, with $\alpha \in A$, is disjoint. Topologize each $\Lambda_\alpha$ with the order topology, which makes it compact Hausdorff. Let $\Lambda$ denote the topological sum $\sum_{\alpha \in A} \Lambda_\alpha$, and let $Y$ be the space obtained from $\Lambda$ by identifying all the final points $\lambda(\alpha) \in \Lambda_\alpha$ to a single point $y_0 \in Y$. Clearly $\Lambda$ is a paracompact $k$-space, and it is easy to check directly that so is $Y$. Let $g : \Lambda \rightarrow Y$ be the quotient map. Clearly $g$ is closed, and $g$ is compact-covering because every compact subset of $Y$ is contained in the union of

(5) I am grateful K. A. Baker for informing me that, while our proof of (c) $\rightarrow$ (a) makes essential use of regularity, (b) $\rightarrow$ (a) can nevertheless be proved for all Hausdorff spaces $X$ by constructing a separate proof in case $X$ is not regular. I don’t know whether (c) $\rightarrow$ (a) remains true for all Hausdorff $X$.

(6) A continuous map $f : X \rightarrow Y$ is compact-covering if every compact subset of $Y$ is the image of some compact subset of $X$.

(7) This follows from [6; p. 163 H] and the fact that every simply ordered set has a cofinal well-ordered subset.
finitely many \( g(\Lambda_\alpha) \). It remains to show that \( h = i_X \times g \) is not a quotient map.

For each \( \alpha \in A \) and \( \lambda \in \Lambda_\alpha \), let \( E_\lambda = \bigcap_{\nu < \lambda} F_\nu \). Then \( E_{\lambda(\alpha)} = \emptyset \), and \( E_\lambda \supseteq F_\lambda \neq \emptyset \) if \( \lambda < \lambda(\alpha) \). For each \( \alpha \in A \), define \( S_\alpha \subseteq X \times \Lambda_\alpha \) by

\[
S_\alpha = \bigcup \{ E_\lambda \times \{ \lambda \} : \lambda \in \Lambda_\alpha \}.
\]

Then \( S_\alpha \) is clearly closed in \( X \times \Lambda_\alpha \). Define \( S \subseteq X \times Y \) by \( S = \bigcup_{\alpha \in A} h(S_\alpha) \).

Let us show that \( h^{-1}(S) \) is closed in \( X \times \Lambda \), but that \( S \) is not closed in \( X \times Y \).

To see that \( h^{-1}(S) \) is closed in \( X \times \Lambda \), it suffices to check that \( h^{-1}(S) \cap (X \times \Lambda_\alpha) \) is closed in \( X \times \Lambda_\alpha \) for all \( \alpha \). But, since \( E_{\lambda(\alpha)} = \emptyset \) for all \( \alpha \),

\[
S_\alpha = h^{-1}(S) \cap (X \times \Lambda_\alpha) = S_\alpha,
\]

and \( S_\alpha \) is indeed closed in \( X \times \Lambda_\alpha \).

To see that \( S \) is not closed in \( X \times Y \), note first that \( (x_0, y_0) \notin S \). However, if \( U \times V \) is a neighborhood of \( (x_0, y_0) \) in \( X \times Y \), then \( \bar{U}_\beta \subseteq U \) for some \( \beta \in \Lambda \); if we pick \( \lambda \in g^{-1}(V) \cap \Lambda_\beta \) with \( \lambda \neq \lambda_\beta \), then

\[
\emptyset \neq h(E_\lambda \times \{ \lambda \}) \subseteq (U \times V) \cap S.
\]

Hence \( (x_0, y_0) \notin \bar{S} \), and that completes the proof.


**Theorem 3.1.** — The following properties of a regular \((5)\) space \( X \) are equivalent.

(a) \( X \) is locally compact.

(b) \( X \times Y \) is a k-space for every k-space \( Y \).

(c) \( X \times Y \) is a k-space for every paracompact k-space \( Y \).

**Proof.** — The implication \((a) \Rightarrow (b)\) is the result of D.E. Cohen quoted in the introduction, and \((b) \Rightarrow (c)\) is obvious. It remains to prove \((c) \Rightarrow (a)\).
Suppose $X$ is not locally compact. Then Theorem 2.1 implies that there exists a compact-covering map $g : A \to Y$, with $Y$ a paracompact $k$-space, such that $i_X \times g$ is not a quotient map. Since $g$ is compact-covering, so is $i_X \times g$. Now it is easy to show [7; Lemma 11.2] that any compact-covering map whose range is a Hausdorff $k$-space must be a quotient map. Since $i_X \times g$ is not a quotient map, it follows that $X \times Y$ is not a $k$-space. That completes the proof.

4. Two analogous results.

S. P. Franklin has pointed out that Theorems 2.1 and 3.1 have simple analogues in case the domain of $g$ in Theorem 2.1, or the space $Y$ in Theorem 3.1, are assumed to be sequential. Recall that a space $Y$ is called sequential [5] if a subset $A$ of $Y$ is closed whenever $A \cap S$ is closed in $S$ in for every convergent sequence (including the limit) $S$ in $Y$. Since such $S$ are compact, every sequential space is clearly a $k$-space. Moreover, quotients of sequential spaces are always sequential, and sequential spaces are precisely the quotients of (locally compact) metrizable spaces (see [5]).

For each infinite cardinal $m$, let $D_m$ denote the discrete space of cardinality $m$, let $Y_m$ be the quotient space obtained from $D_m \times [0, 1]$ by identifying all points in $D_m \times \{0\}$ (i.e. $Y_m$ is the cone over $D_m$), and let $g_m : D_m \times [0, 1] \to Y_m$ be the quotient map.

By the pointwise weight of a space $X$ we will mean the smallest cardinal $m$ such that each $x \in X$ has a neighborhood base of cardinality $\leq m$.

**Theorem 4.1.** — The following properties of a regular space $X$ are equivalent.

a) $X$ is locally countably compact.

b) $i_X \times g$ is a quotient map for every quotient map $g$ with sequential domain.

c) $i_X \times g_m$ is a quotient map, where $m$ is the pointwise weight of $X$. 

Proof. — (a) → (b). This proof goes just like J. H. C. Whitehead's proof [8; Lemma 4] that (a) → (b) in Theorem 2.1. In fact, Whitehead's proof is based on the fact that if \( U \) is an open subset of a product space \( E \times F \), and if \( C \subseteq F \) is compact, then \( \{ x \in E : \{ x \} \times C \subseteq U \} \) is open in \( E \). It is easy to check that, if \( E \) is sequential, this conclusion remains valid if \( C \) is only assumed to be countably compact.

(b) → (c) Obvious.

(c) → (a) Suppose \( X \) is not locally countably compact. Examining the proof of Theorem 2.1, one sees that then there are only \( m \) space \( \Lambda_x \), and each \( \Lambda_x \) can be chosen to be a convergent sequence or, if one prefers, a closed interval. In the latter case, the map \( g \) constructed in the proof of Theorem 2.1 is precisely \( g_m \). That completes the proof.

**Theorem 4.2.** — The following properties of a regular sequential space \( X \) are equivalent.

a) \( X \) is locally countably compact.

b) \( X \times Y \) is sequential for every sequential space \( Y \).

c) \( X \times Y_m \) is a k-space, where \( m \) is the pointwise weight of \( X \).

Proof. — (a) → (b). This follows immediately from T. K. Boehme [1; Theorem] and S. P. Franklin [5; Proposition 1.10].

(b) → (c). Obvious.

(c) → (a). This follows from 4.1 (c) → (a) in the same way that 3.1 (c) → (a) followed from 2.1 (c) → (a). That completes the proof.

**BIBLIOGRAPHIE**


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