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Labeled Rauzy classes and framed translation surfaces

LABELED RAUZY CLASSES AND FRAMED TRANSLATION SURFACES

by Corentin BOISSY

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ABSTRACT. — In this paper, we compare two definitions of Rauzy classes. The first one was introduced by Rauzy and was in particular used by Veech to prove the ergodicity of the Teichmüller flow. The second one is more recent and uses a "labeling" of the underlying intervals, and was used in the proof of some recent major results about the Teichmüller flow.

The Rauzy diagrams obtained from the second definition are coverings of the initial ones. In this paper, we give a formula that gives the degree of this covering.

This formula is related to moduli spaces of *framed* translation surfaces, which correspond to surfaces where we label horizontal separatrices on the surface. We compute the number of connected component of these natural coverings of the moduli spaces of translation surfaces.

Delecroix has given recently a formula for the cardinality of the (reduced) Rauzy classes. Therefore, we also obtain formula for labeled Rauzy classes.

RÉSUMÉ. — Dans cet article, on compare deux définitions de classes de Rauzy. La première a été introduite par Rauzy et a été utilisée en particulier par Veech pour démontrer l'ergodicité du flot de Teichmüller. La seconde est plus récente et utilise un « étiquetage » des intervalles sous-jacents. Elle a été utilisée récemment dans les preuves de plusieurs résultats majeurs sur le flot de Teichmüller.

Les diagrammes de Rauzy obtenus avec la seconde définition sont des revêtements de ceux obtenus avec la première définition. On donne ici une formule donnant le degré de ce revêtement.

Cette formule est reliée à un espace des modules de surfaces de translations *marquées*, qui correspond à des surfaces de translations pour lesquelles on marque des séparatrices horizontales sur la surface. On calcule le nombre de composantes connexes de ces revêtements naturels de l'espace des modules des surfaces de translation.

Delecroix a donné récemment le cardinal des classes de Rauzy (réduites). On peut donc en déduire le cardinal des classes de Rauzy marquées.

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1. Introduction

Rauzy induction was introduced in [17] as a tool to study interval exchange maps. It is a renormalization process that associates to an interval exchange map, another one obtained as a first return map on a well chosen subinterval. After the major works of Veech [19] and Masur [16], the Rauzy induction became a powerful tool to study the Teichmüller geodesic flow.

A slightly different tool was used by Kerckhoff [12], Bufetov [7], and Marmi-Moussa-Yoccoz [15]. It is obtained after labeling the intervals, and keeping track of them during the renormalization process. This small change was a significative improvement, and was used in the recent past to prove other important results about the Teichmüller geodesic flow, for instance the simplicity of the Liapunov exponents (Avila-Viana [2]), or the exponential decay of correlations (Avila-Gouezel-Yoccoz [1]).

An interval exchange map is naturally decomposed into a continuous and a combinatorial datum. A *Rauzy* class is a minimal set of such combinatorial data invariant by the combinatorial Rauzy induction. We will speak of *reduced* or *labeled* Rauzy classes depending whether we use the definition of Rauzy, or the other one.

Let $k_1, \ldots, k_r$ be some pairewise distinct nonnegative integers and let $n_1, \ldots, n_r$ be positive integers such that $\sum_{i=1}^{r} n_i k_i = 2g - 2$. The stratum of the moduli space of Abelian differentials whose corresponding surfaces have precisely n_i singularities of degree k_i, for all $i \in \{1, \ldots, r\}$, is usually denoted as $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$. In this notation, we implicitly assume that $k_i \neq k_j$ for all $i \neq j$. A singularity of degree zero is by convention a regular marked point on the surface.

The Veech construction naturally associates to a Rauzy class a connected component $\mathcal{C}$ of a stratum $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$ of the moduli space of Abelian differentials, and an integer $k \in \{k_1, \ldots, k_r\}$ that corresponds to degree of the singularity attached on the left in the Veech construction. In the next statement, the pair $(\mathcal{C}, k)$ will be refered to the data associated to a Rauzy class by the Veech construction.

THEOREM 1.1. — *Let R_{lab} be a labeled Rauzy class and let R be the corresponding reduced one. Let $(\mathcal{C}, k)$ be data associated to R by the Veech construction, where $\mathcal{C}$ is a connected component of a stratum $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$ of the moduli space of Abelian differentials, and $k \in \{k_1, \ldots, k_r\}$. Let n be number of singularities of degree k for a surface in $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$. We have:*

$$\frac{|R_{lab}|}{|R|} = \frac{\Pi_{i=1}^{r} n_i!(k_i+1)^{n_i}}{n(k+1)} \varepsilon$$

where ε satisfies:

- *$\varepsilon = \frac{1}{g}$ if $\mathcal{C}$ consists only of hyperelliptic surfaces with two conical singularities of degree $g-1$ and possibly some regular marked points.*
- *$\varepsilon = \frac{1}{2}$ if $\mathcal{C}$ contains nonhyperelliptic surfaces and if there exists $k' \in \{k_1, \ldots, k_r\}$ which is odd.*
- *$\varepsilon = 1$ in all the other cases.*

Delecroix [9] has proven a formula for the cardinality of the reduced Rauzy classes. The previous theorem complete his result for the case of labeled Rauzy classes.

Our result is related to moduli spaces of framed translation surfaces. Informally, we will call a *frame* on a translation surface X a map F_X from a discrete alphabet $\mathcal{A}$, to a set $\mathcal{S}_X$ of discrete combinatorial data on the surface X such that the moduli space of framed translation surfaces is a covering of the corresponding moduli space of translation surface.

In our case, we are interested in the case where $\mathcal{S}_X$ is the set of *horizontal outgoing separatrices*. Several different kind of frames will appear in this context. The most important will be the space $\mathcal{C}(\mathcal{F}_{comp})$. It corresponds to the space where we label exactly one horizontal separatrix for each singularity (see Section 3.1 for a more precise definition), and the alphabet $\mathcal{A}$ is a disjoint union of subalphabets $\mathcal{A}_k$, such that the labels in $\mathcal{A}_k$ correspond to singularities of degree k.

Choosing α in some $\mathcal{A}_k$, one gets a natural covering $p_\alpha : \mathcal{C}(\mathcal{F}_{comp}) \to \mathcal{C}(\mathcal{F}_k)$, where $\mathcal{C}(\mathcal{F}_k)$ is a moduli space of framed translation surfaces that corresponds to translation surfaces with a single marked separatrix adjacent to a singularity of degree k.

Theorem 1.1 will be a consequence of the two following results, which give a geometrical interpretation of the formula.

PROPOSITION 1.2. — *Let R_{lab} be a labeled Rauzy class and let R be the corresponding reduced one. The ratio $|R_{lab}|/|R|$ is equal to the degree of the covering p_α, when restricted to a connected component of $\mathcal{C}(\mathcal{F}_{comp})$.*

THEOREM 1.3. — *Let $\mathcal{C}$ be a connected component of a stratum of the moduli space of Abelian differentials. The space $\mathcal{C}(\mathcal{F}_{comp})$ is not connected in general. More precisely, it has:*

- *g connected components if $\mathcal{C}$ is the hyperelliptic connected component of $\mathcal{H}(g-1, g-1)$, with possibly some regular marked points.*

- *Two connected components if it does not corresponds to a hyper-elliptic connected component and if there exists odd degree conical singularities.*
- *One connected component in all the other cases.*

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2. Background

2.1. Labeled and reduced interval exchange transformations

We give here the two definitions of interval exchange transformations that are used in the literature. In order to distinguish them, we will add the terms *reduced* and *labeled*.

The first one is due to Rauzy [17].

DEFINITION 2.1. — *Let $I \subset \mathbb{R}$ be an open interval and let us choose a finite subset $\Sigma = \{x_1, \ldots, x_{d-1}\}$ of I. The complement of Σ in I is a disjoint union of $d \geqslant 2$ open subintervals $\{I_j, \ j = 1, \ldots, d\}$. An reduced interval exchange transformation is a map T from $I \backslash \Sigma$ to I that permutes, by translation, the subintervals I_j. It is easy to see that $T(I \backslash \Sigma) = I \backslash \Sigma'$, where $\Sigma' \subset I$ is of the same cardinality as Σ, and that T is precisely determined by:*

- *A combinatorial datum: a permutation $\pi \in \Sigma_d$ which expresses that the interval number k, when counted from the left to the right, is sent to the place $\pi(k)$ by the map T.*
- *A continuous datum: a vector $\lambda \in \mathbb{R}^d$ with positive entries that corresponds to the lengths of the intervals.*

We will identify an interval exchange transformation with its parameters (π, λ).

The second definition of interval exchange transformation was first introduced by Kerckhoff [12] and later formalized by Bufetov [7] and Marmi, Moussa & Yoccoz [15]. As we will see later, it simplifies the description of the Rauzy induction, but we get bigger Rauzy classes.

DEFINITION 2.2. — *A labeled interval exchange map is a reduced interval exchange map with a pair* (π_t, π_b) *of one-to-one maps from a finite alphabet* $\mathcal{A}$ *to* $\{1, \ldots, d\}$. *The interval number* k, *when counted from the left to the right, is denoted by* $I_{\pi_t^{-1}(k)}$. *Once the intervals are exchanged, the interval number* k *is* $I_{\pi_b^{-1}(k)}$.

In the previous definition, it is easy to see that the permutation corresponding to the underlying reduced interval exchange map is $\pi = \pi_b \circ \pi_t^{-1}$, and the continuous datum is a vector with positive entries $\lambda \in \mathbb{R}^{\mathcal{A}}$. We will identify a labeled interval exchange map with the pair $(\tilde{\pi}, \lambda)$, where $\tilde{\pi} = (\pi_t, \pi_b)$. We will call $\tilde{\pi}$ a *labeled permutation*.

We will usually represent a labeled permutation by a table:

$$\tilde{\pi} = \left(\begin{array}{cccc} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \ldots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \ldots & \pi_b^{-1}(d) \end{array} \right).$$

As we can see from this representation, we have "t" for top, "b" for bottom in the notation π_t, π_b.

A *renumbering* of a labeled permutation is the composition of (π_t, π_b) by a one-to-one map f from $\mathcal{A}$ to $\mathcal{A}'$. It just corresponds to changing the labels without changing the underlying permutation. From the previous definitions, it is clear that a reduced interval exchange transformation (*resp.* a permutation) is an equivalence class of labeled interval exchange maps (*resp.* labeled permutations) up to renumbering. We will sometime identify a permutation with its unique representative (π_t, π_b) with $\mathcal{A} = \{1, \ldots, d\}$ and $\pi_t = Id$.

2.2. Rauzy-Veech induction

Let T be a labeled or reduced interval exchange map. The Rauzy–Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [17, 15] for details).

We recall briefly the construction for the labeled case. Following the terminology which is introduced in [15] we define the *type* of T by t if $\lambda_{\pi_t^{-1}(d)} > \lambda_{\pi_b^{-1}(d)}$ and b if $\lambda_{\pi_t^{-1}(d)} < \lambda_{\pi_b^{-1}(d)}$. When T is of type t (respectively, b) we will say that the label $\pi_t^{-1}(d)$ (respectively, $\pi_b^{-1}(d)$) is the winner and that $\pi_b^{-1}(d)$ (respectively, $\pi_t^{-1}(d)$) is the looser. We define a subinterval J of I by

$$J = \left\{ \begin{array}{ll} I \backslash T(I_{\pi_b^{-1}(d)}) & \text{if } T \text{ is of type t;} \\ I \backslash I_{\pi_t^{-1}(d)} & \text{if } T \text{ is of type b.} \end{array} \right.$$

The image of T by the Rauzy-Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again a labeled interval exchange transformation, defined on d letters. The combinatorial datum of this new interval exchange transformation is very easy to calculate in terms of the one of T. Indeed, let $\alpha \in \mathcal{A}$ (resp. $\beta \in \mathcal{A}$) be the winner (resp. looser). Let $\lambda' \in \mathbb{R}^{\mathcal{A}}$ such that:

$$
\begin{aligned}
\lambda'_{\alpha} &= \lambda_{\alpha} - \lambda_{\beta} \\
\lambda'_{\nu} &= \lambda_{\nu} \qquad \text{for all } \nu \in \mathcal{A}\backslash\{\alpha\}
\end{aligned}
$$

Then, $\mathcal{R}(T) = (\mathcal{R}_{\varepsilon}(\pi), \lambda')$, where ε is the type of T, and $\mathcal{R}_t, \mathcal{R}_b$ are the following combinatorial maps:

(1) $\mathcal{R}_t$: let $k = \pi_b(\pi_t^{-1}(d))$ with $k \leqslant d - 1$. Then, $\mathcal{R}_t(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$ where $\pi_t = \pi'_t$ and

$$
\pi'^{-1}_b(j) = \begin{cases}
\pi_b^{-1}(j) & \text{if } j \leqslant k \\
\pi_b^{-1}(d) & \text{if } j = k + 1 \\
\pi_b^{-1}(j - 1) & \text{otherwise.}
\end{cases}
$$

(2) $\mathcal{R}_b$: let $k = \pi_t(\pi_b^{-1}(d))$ with $k \leqslant d - 1$. Then, $\mathcal{R}_b(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$ where $\pi_b = \pi'_b$ and

$$
\pi'^{-1}_t(j) = \begin{cases}
\pi_t^{-1}(j) & \text{if } j \leqslant k \\
\pi_t^{-1}(d) & \text{if } j = k + 1 \\
\pi_t^{-1}(j - 1) & \text{otherwise.}
\end{cases}
$$

The maps $\mathcal{R}_t, \mathcal{R}_b$ are called the combinatorial Rauzy moves. It is easy to define similar maps for reduced permutations. We first identify π with the corresponding (Id, π_b) and perform the Rauzy move. Then, if needed, we renumber the result so that it corresponds to a reduced permutation. We will still denote the corresponding maps by $\mathcal{R}_t, \mathcal{R}_b$, since it will always be clear, when the distinction is needed, whether or not the objects are labeled or reduced.

We define the Rauzy induction for reduced interval exchange maps by considering labeled interval exchange maps up to renumbering.

DEFINITION 2.3. — *A Rauzy class, usually denoted by R is a minimal set of labeled or reduced permutations invariant by the combinatorial Rauzy moves.*

A Rauzy diagram is a graph whose vertices are the elements of a Rauzy class and whose vertices are the combinatorial Rauzy maps.

A Rauzy class or Rauzy diagram will be called labeled or reduced depending on the corresponding permutations.

Examples 2.4.

- Let $\tau_n = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right)$. Rauzy proved that the cardinality of $R(\tau_n)$ is $2^{n-1} - 1$ for the reduced case, and one can prove "by hand" that it is the same for the labeled case.
- Let $\pi_n = \left(\begin{smallmatrix} 0 & 2 & 3 & \cdots & n-1 & 1 & n \\ n & n-1 & \cdots & 3 & 2 & 1 & 0 \end{smallmatrix}\right)$. Contrary to the previous case, the labeled and reduced diagrams are not isomorphic anymore. The structure of the labeled and reduced Rauzy diagrams is precisely described in [6]. It is in particular shown that the cardinality of the reduced diagram is $2^{n-1} - 1 + n$ and the cardinality of the labeled diagram is $(2^{n-1} - 1 + n)(n - 1)$.
- Consider $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 4 & 3 & 2 & 5 & 8 & 7 & 6 \end{smallmatrix}\right)$. The reduced Rauzy class is of size[1] 1255 and the labeled Rauzy class is of size 30120. The ratio is 24.

2.3. Translation surfaces and moduli space

A *translation surface* is a (real, compact, connected) genus g surface X with a translation atlas *i.e.* a triple $(X, \mathcal{U}, \Sigma)$ such that Σ is a finite subset of X (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $X \setminus \Sigma$ whose transition maps are translations. We will require that for each $s \in \Sigma$, there is a neighborhood of s isometric to a Euclidean cone. One can show that the holomorphic structure on $X \setminus \Sigma$ extends to X and that the holomorphic 1-form $\omega = dz_i$ extends to a holomorphic $1-$form on X where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *Abelian differential*.

For $g \geqslant 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X, ω) where X is a genus g (compact, connected) Riemann surface and ω non-zero holomorphic $1-$form defined on X. The term moduli space means that we identify the points (X, ω) and (X', ω') if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. The group $\mathrm{SL}_2(\mathbb{R})$ naturally acts on the moduli space of translation surfaces by post composition on the charts defining the translation structures.

One can also see a translation surface obtained as a polygon (or a finite union of polygons) whose sides come by pairs, and for each pairs, the corresponding segments are parallel and of the same lengths. These parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context,

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[1] This can be computed for instance using Zorich's MATHEMATICA software, or using the SAGE package developed by Delecroix

two translation surfaces are identified in the moduli space of Abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and gluing and preserving the identifications. Also, the $SL_2(\mathbb{R})$ action in this representation is just the natural linear action on the polygons.

The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes; we will denote by $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X, ω) such that ω possesses exactly n_i zeroes on X with multiplicities k_i for all $i \in \{1, \ldots, r\}$, and no other zeroes. It is a well known part of the Teichmüller theory that these spaces are (Hausdorff) complex analytic, and in fact algebraic, spaces. These strata are non-connected in general but each stratum has at most three connected components (see [13] for a complete classification, or see Section 4).

Given a translation surface X, we will call *separatrix* an oriented half line (possibly finite) starting from a singularity of X. A horizontal separatrix l will be *outgoing* if it goes on the right in a translation chart, and *incoming* otherwise.

2.4. Suspension data

The next construction provides a link between interval exchange transformations and translation surfaces. A suspension datum for $T = (\pi, \lambda)$ is a collection of vectors $\{\tau_\alpha\}_{\alpha \in \mathcal{A}}$ such that

- $\forall 1 \leqslant k \leqslant d-1, \ \sum_{\pi_t(\alpha) \leqslant k} \tau_\alpha > 0$,
- $\forall 1 \leqslant k \leqslant d-1, \ \sum_{\pi_b(\alpha) \leqslant k} \tau_\alpha < 0$.

We will often use the notation $\zeta = (\lambda, \tau)$. To each suspension datum τ, we can associate a translation surface $(X, \omega) = X(\pi, \zeta)$ in the following way.

Consider the broken line L_t on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (in this order) for $j = 1, \ldots, d$ with starting point at the origin. Similarly, we consider the broken line L_b defined by concatenation of the vectors $\zeta_{\pi_b^{-1}(j)}$ (in this order) for $j = 1, \ldots, d$ with starting point at the origin. If the lines L_t and L_b have no intersections other than the endpoints, we can construct a translation surface X by identifying each side ζ_j on L_t with the side ζ_j on L_b by a translation. The resulting surface is a translation surface endowed with the form $\omega = dz$. Note that the lines L_t and L_b might have some other intersection points. But in this case,
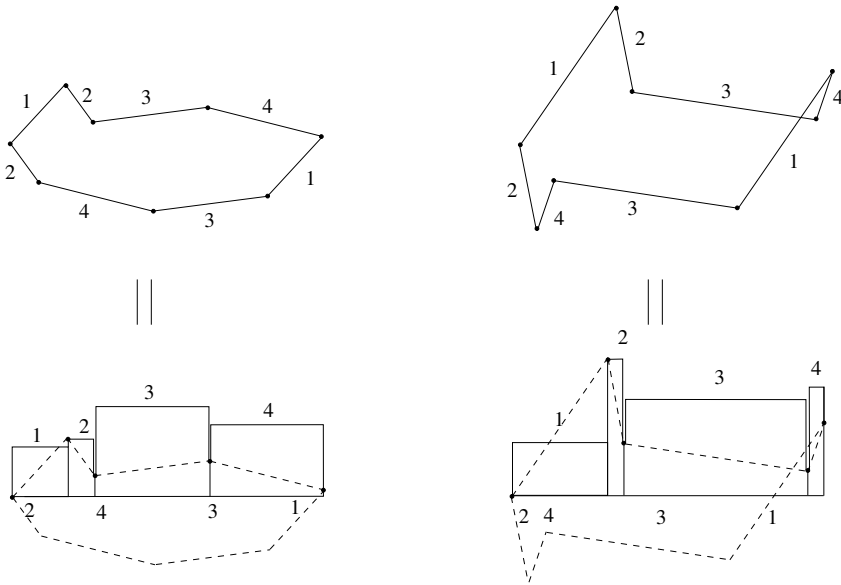
Figure 2.1. The zippered rectangle construction, for two examples of suspension data.

one can still define a translation surface by using the *zippered rectangle construction*, due to Veech ([19]). See for instance Figure 2.1.

Let $I \subset X$ be the horizontal interval defined by $I = (0, \sum_\alpha \lambda_\alpha) \times \{0\}$. The reduced interval exchange transformation T is precisely the one defined by the first return map to I of the vertical flow on X.

We can extend the Rauzy induction to suspension data in the following way: let τ be a suspension data over (π, λ), we define $\mathcal{R}(\pi, \lambda, \tau) = (\pi', \lambda', \tau')$ by:

- $\mathcal{R}(\pi, \lambda) = (\pi', \lambda')$
- $\tau'_\alpha = \tau_\alpha - \tau_\beta$, where α (resp. β) is the winner (resp. looser) for $T = (\pi, \lambda)$

This extension is known as the *Rauzy–Veech induction*, and is used as a discretization of the Teichmüller flow.

Remark 2.5. — By construction the two translation surfaces $X(\pi, \zeta)$ and $X(\pi', \zeta')$ define the same element in the moduli space.

Remark 2.6. — Note that λ, τ define natural local parameters for the stratum of the moduli space of Abelian differentials.

Remark 2.7. — The left end of the two lines in the previous construction L_t, L_b is a singularity, and the horizontal half line starting from this point to the right corresponds to a choice of a horizontal separatrix starting from this singularity, and it is easy to see that this combinatorial data is preserved under the Rauzy–Veech induction. Let us denote by $l(\pi, \zeta)$ this separatrix.

3. Rauzy diagrams and framed translation surfaces

3.1. Moduli space of framed translation surface.

A *frame* on a translation surface X a map F_X from a discrete alphabet $\mathcal{A}$, to a set $\mathcal{DC}_X$ of discrete combinatorial data on the surface X.

Let $\mathcal{C}$ be a connected component of a stratum of moduli space of translation surfaces, and a collection $\mathcal{F}$ of frames for translation surfaces in $\mathcal{C}$, with a fixed alphabet. One can define the corresponding moduli space of framed translation surfaces: two elements (X, F_X) and $(X', F'_{X'})$ are identified if there is a translation mapping $X \to X'$ which is consistent with the frames. Then, we will denote by $\mathcal{C}(\mathcal{F})$ the corresponding moduli space.

The sets $\mathcal{DC}_X$ can be many things: incoming or outgoing horizontal separatrices, $H_1(X, \mathbb{Z})$, etc... Here we will not study the precise conditions on the collections of frames so that $\mathcal{C}(\mathcal{F})$ is a "nice" space. We will just introduce three cases, for which $\mathcal{C}(\mathcal{F})$ are coverings of $\mathcal{C}$. We first ask that $\mathcal{DC}_X = \mathcal{S}_X$ is the set of *horizontal outgoing separatrices*, then we consider the three families of all frames that satisfy the following conditions respectively:

(1) $\mathcal{F}_k$: the set $\mathcal{A}$ is a singleton and the image of F_X is any separatrix adjacent to a degree k singularity.
(2) $\mathcal{F}_{sat}$: F_X is a one-to-one mapping from $\mathcal{A}$ to $\mathcal{S}_X$.
(3) $\mathcal{F}_{comp}$: $\mathcal{A} = \sqcup_k \mathcal{A}_k$, the map F_X is injective, and we require that for each k any singularity of S of degree k has a unique separatrix in $F(\mathcal{A}_k)$.

We will denote respectively by $\mathcal{C}(\mathcal{F}_k)$, $\mathcal{C}(\mathcal{F}_{sat})$, and $\mathcal{C}(\mathcal{F}_{comp})$, the corresponding moduli spaces, which are finite coverings of $\mathcal{C}$.

(1) The space $\mathcal{C}(\mathcal{F}_k)$ is the moduli space of pairs (X, l), where $X \in \mathcal{C}$ and l is a separatrix adjacent to a singularity of degree k in X.
(2) The space $\mathcal{C}(\mathcal{F}_{sat})$ corresponds to translation surfaces where we label each horizontal outgoing separatrix by an element in $\mathcal{A}$. It corresponds to a "saturated case".

(3) The space $\mathcal{C}(\mathcal{F}_{comp})$ corresponds to translation surfaces where we label exactly one separatrix for each singularity, accordingly to the degree of the singularity.

As we will see, the spaces $\mathcal{C}(\mathcal{F}_k)$ (resp. $\mathcal{C}(\mathcal{F}_{sat})$) will appear naturally in the study of reduced (resp. labeled) Rauzy classes. We will then reduce the problem to the study of the space $\mathcal{C}(\mathcal{F}_{comp})$.

Note that the space $\mathcal{C}(\mathcal{F}_{comp})$ was also introduced recently by Marchese in [14] (Section 3.2).

3.2. Moduli space of reduced suspension data.

The set of suspension data associated to a labeled or reduced permutation is connected (in fact, convex). Hence, for a Rauzy class R, all flat surfaces obtained from the Veech construction are in the same connected component of a stratum of the moduli space of Abelian differentials. In fact, according to Remark 2.7 there is a natural map:

$$\Phi : \mathcal{H}_R = \left\{ \begin{array}{c} (\pi, \zeta), \pi \in R, \\ \zeta \text{ susp. dat. for } \pi \end{array} \right\} / \mathcal{R} \quad \to \quad \mathcal{C}(\mathcal{F}_k)$$

$$[(\pi, \zeta)] \quad \mapsto \quad \left(X(\pi, \zeta), l(\pi, \zeta) \right)$$

Example 3.1. — Let us consider the permutations given in Example 2.4. We have:

- For $\tau_n = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix} \right)$, the corresponding connected component is $\mathcal{H}^{hyp}(n-2)$ or $\mathcal{H}^{hyp}(\frac{n-1}{2}-1, \frac{n-1}{2}-1)$ depending on the parity of n.
- For $\pi_n = \left(\begin{smallmatrix} 0 & 2 & 3 & \cdots & n-1 & 1 & n \\ n & n-1 & \cdots & 3 & 2 & 1 & 0 \end{smallmatrix} \right)$, the corresponding connected component is $\mathcal{H}^{hyp}(0, n-2)$ or $\mathcal{H}^{hyp}(0, \frac{n-1}{2}-1, \frac{n-1}{2}-1)$, the singularity which is marked by the Veech construction being of degree 0.
- For $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 4 & 3 & 2 & 5 & 8 & 7 & 6 \end{smallmatrix} \right)$, the corresponding stratum is $\mathcal{H}(1^4) = \mathcal{H}(1, 1, 1, 1)$ and is connected.

When R is a Rauzy class of reduced permutations, then the following theorem was proven in [4].

THEOREM 3.2. — *The map Φ is a homeomorphism on its image. The complement of the image of Φ is contained in a codimension 2 subset of $\mathcal{C}(\mathcal{F}_k)$, which is connected.*

3.3. Moduli space of labeled suspension data

The previous section describes what represents "geometrically" a reduced Rauzy diagram (or more precisely, the corresponding moduli space of suspension data): a suspension data for a reduced permutation, modulo the Rauzy–Veech induction corresponds to a translation surface with a marked separatrix, *i.e.* an element of $\mathcal{F}_k(\mathcal{C})$.

In this section, we give an analogous description of a labeled Rauzy diagram. In this case, a suspension data for a labeled permutation, modulo the Rauzy–Veech induction corresponds to a translation surface where all the separatrices are labeled, *i.e* an element of $\mathcal{C}(\mathcal{F}_{sat})$. Then, we will show next that we can reduce the problem to studying the moduli space of translation surfaces with a single marked separatrix for each singularities.
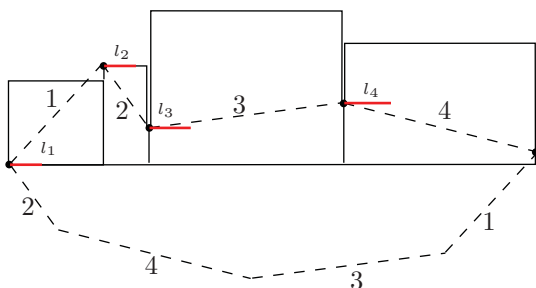

Figure 3.1. *A framing of a surface issued from the Veech construction. Here we have $l_i = F(i)$ for $i = 1, 2, 3, 4$ and $l_1 = l_2$.*

Let $\pi = (\pi_t, \pi_b)$ be a labeled permutation and let ζ be a suspension data for π. The zippered rectangle construction naturally defines a framed translation surface (see Figure 3.1) in the following way: for each rectangle in the Veech construction, the left vertical side contains a unique singularity. Hence, we can label the corresponding outgoing horizontal separatrix with the letter of the rectangle. Furthermore, two of these rectangles (corresponding to $\pi_t^{-1}(1)$ and $\pi_b^{-1}(1)$) intersect the corresponding singularity at a left corner (the bottom for one, the top for the other), and the corresponding horizontal outgoing separatrix is the same, so is labeled twice: once by the symbol $\pi_t^{-1}(1)$, and once by the symbol $\pi_b^{-1}(1)$. For all the other rectangles, the singularity on the left is in the interior of the left vertical side, hence, each corresponding separatrix is uniquely labeled. Therefore one gets a one-to-one map:

$$F : \mathcal{A}' \to \mathcal{S}_X$$

where $\mathcal{A}'$ is the quotient of $\mathcal{A}$ by the equivalence relation $\pi_t^{-1}(1) \sim \pi_b^{-1}(1)$.

The element $F(\{\pi_t^{-1}(1), \pi_b^{-1}(1)\})$ will be refered as the *doubly labeled* separatrix.

LEMMA 3.3. — *Let (π, ζ) as previously. The framed translation surface constructed as before from (π, ζ) and the one defined by $\mathcal{R}(\pi, \zeta)$ are the same element in $\mathcal{C}(\mathcal{F}_{sat})$.*

Proof. — This is an elementary check. $\qquad\square$

The following proposition transforms the initial combinatorial question into a topological one on the moduli space of Abelian differentials.

PROPOSITION 3.4. — *There is a natural one-to-one correspondence between labeled Rauzy classes and connected components of the moduli space of framed translation surfaces $\mathcal{C}(\mathcal{F}_{sat})$. The degree of the mapping from a labeled Rauzy diagram to the reduced one is then precisely the degree of the natural mapping from a connected component of $\mathcal{C}(\mathcal{F}_{sat})$ to $\mathcal{C}(\mathcal{F}_k)$.*

Proof. — Let R_{all} be the set of labeled permutations that corresponds to $\mathcal{C}(\mathcal{F}_k)$, and let:

$$\mathcal{H}_{R_{all}} = \{(\pi, \zeta), \pi \in R_{all}, \zeta \text{ suspension data for } \pi\}/\mathcal{R}.$$

By Lemma 3.3 there is a map $\Phi_{sat} : \mathcal{H}_{R_{all}} \to \mathcal{C}(\mathcal{F}_{sat})$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{H}_{R_{all}} & \xrightarrow{\Phi_{sat}} & \mathcal{C}(\mathcal{F}_{sat}) \\
\downarrow p_0 & & \downarrow p_1 \\
\mathcal{H}_R & \xrightarrow{\ \Phi\ } & \mathcal{C}(\mathcal{F}_k)
\end{array}
$$

Here p_0 is the canonical map that replace a labeled permutation by a reduced one, and p_1 is the map that "forget" all labels except for the doubly labeled separatrix. The maps Φ and Φ_{sat} are homeomorphisms on their images, and onto up to codimension 2 subsets (see [4], Section 3 for details.). Hence, $\mathcal{H}_{R_{all}}$ and $\mathcal{C}(\mathcal{F}_{sat})$ have the same number of connected components, and the degree of the maps p_1 and p_0, when restricted to a connected component are the same. But the degree of the map p_0 restricted to a connected component is precisely the degree of natural map from the labeled Rauzy diagram to the reduced one. $\qquad\square$

3.4. Moduli space of translation surfaces with frame.

There are obvious invariants for the connected components of the moduli space $\mathcal{C}(\mathcal{F}_{sat})$. Indeed, two elements of $\mathcal{C}(\mathcal{F}_{sat})$ that are in the same connected component must satisfy the following property:

- The labels that correspond to a given singularity on one surface must correspond to a same singularity on the other surface.
- The canonical cyclic order on the set of labels obtained by rotating clockwise around a singularity must be the same.

Hence, a connected component of $\mathcal{C}(\mathcal{F}_{sat})$ is clearly isomorphic to a connected component of $\mathcal{C}(\mathcal{F}_{comp})$.

Let $\alpha \in \mathcal{A}_k$ be a label associated to a degree k singularity. There is a natural covering p_α from $\mathcal{C}(\mathcal{F}_{comp})$ to $\mathcal{C}(\mathcal{F}_k)$ obtained by "forgetting" all the markings, except the one that corresponds to α. The following proposition summarizes the discussion of this section, and is equivalent to Proposition 1.2.

PROPOSITION 3.5. — *Let R_{lab} be a labeled Rauzy class and R be the corresponding reduced one. Let k be the degree of the marked singularity associated to R. The ratio $\frac{|R_{lab}|}{|R|}$ equals the degree of the canonical projection $p_\alpha : \mathcal{C}(\mathcal{F}_{comp}) \to \mathcal{C}(\mathcal{F}_k)$, restricted to a connected component of $\mathcal{C}(\mathcal{F}_{comp})$, where α is a label associated to a degree k singularity.*

Proof. — It was proven in [4] that $\mathcal{C}(\mathcal{F}_k)$ is connected. A connected component of $\mathcal{C}(\mathcal{F}_{comp})$ is naturally isomorphic to a connected component of $\mathcal{C}(\mathcal{F}_{sat})$. Then, we just apply Proposition 3.4.   □

4. Topological invariants for framed translation surfaces

From now on, a framed translation surface will be an element in $\mathcal{C}(\mathcal{F}_{comp})$.

As seen in Proposition 3.5, the formula given in Theorem 1.1 is related to the number of connected components of $\mathcal{C}(\mathcal{F}_{comp})$. Also, the degree of the covering $\mathcal{C}(\mathcal{F}_{comp}) \to \mathcal{C}$, restricted to a connected component of $\mathcal{C}(\mathcal{F}_{comp})$, is clearly $\frac{\Pi_{i=1}^{r} n_i!(k_i+1)^{n_i}}{c}$, where c is the number of connected component of $\mathcal{C}(\mathcal{F}_{comp})$, since $\Pi_{i=1}^{r} n_i!(k_i+1)^{n_i}$ is the number of possible frames $F \in \mathcal{F}_{comp}$ on a surface.

In this section, we give lower bounds on the number of connected components of $\mathcal{C}(\mathcal{F}_{comp})$. There are two cases.

- The "hyperelliptic case". If the corresponding surfaces are all hyperelliptic and have two singularities of degree $g - 1$, with possibly some added regular marked points, then $\mathcal{C}(\mathcal{F}_{comp})$ cannot be connected due to the extra symmetries of the underlying translation surfaces.

- The "odd singularity case". When there are odd degree singularities, we can define on $\mathcal{C}(\mathcal{F}_{comp})$ a topological invariant which generalizes the well known *spin structure invariant* for the moduli space of Abelian differentials, found by Kontsevich and Zorich.

Recall that a Riemann S surface is hyperelliptic if there exists an involution τ such that $S/\tau = \mathbb{CP}^1$. Since $\mathbb{CP}^1$ does not have any nontrivial Abelian differential, then for any translation surface (S, ω) such that S is hyperelliptic the corresponding involution τ satisfies $\tau^*\omega = -\omega$. In particular, this means that the translation surface (S, ω) have an isometric involution which reverses the vertical direction. Kontsevich and Zorich have shown that for each genus $g \geqslant 2$, there are exactly two strata that contain a connected component which consists only of hyperelliptic translation surfaces. These are the strata $\mathcal{H}(2g - 2)$ and $\mathcal{H}(g - 1, g - 1)$. Of course, for each stratum, one can also define new ones by adding regular marked points on the surfaces.

PROPOSITION 4.1. — *Assume that $\mathcal{C}$ consists only of hyperelliptic translation surfaces with two singularities of degree $g - 1$ and n_0 regular marked points. Then, $\mathcal{C}(\mathcal{F}_{comp})$ has at least g connected components.*

Proof. — Let $X \in \mathcal{C}(\mathcal{F}_{comp})$. We denote by l_0 and l_1 the marked separatrices associated to the degree $g - 1$ singularities, and we denote by P_i the singularity corresponding to l_i.

The hyperelliptic involution interchanges P_0 and P_1. Hence, there is a well defined (incoming) separatrix l_1' adjacent to P_1 which is the image of l_0. The angle θ between l_1' and l_1 is an odd multiple of π and is constant under continuous deformations of X inside the ambient stratum. Note that the value of θ does not depend on any choice. Hence the value of θ is an invariant of the connected component of $\mathcal{C}(\mathcal{F}_{comp})$. Since all the values $\pi, 3\pi, \ldots, (2g - 1)\pi$ are possible, we see that the number of connected components of $\mathcal{C}(\mathcal{F}_{comp})$ is at least g. $\qquad\square$

PROPOSITION 4.2. — *Assume that $\mathcal{C}$ consists of translation surfaces with at least one odd degree singularity. Then, $\mathcal{C}(\mathcal{F}_{comp})$ has at least 2 connected components.*

We postpone the proof of this proposition to the end of this section. We first define the "spin structure" invariant for $\mathcal{C}(\mathcal{F}_{comp})$.

Let X be a completely framed surface with at least one odd degree singularity. Note that the number of odd degree singularities of X is necessarily even, since the sum of the degree of the singularities must be equal to $2g - 2$ by the Riemann-Roch formula. For each singularity, we have given

a name $\alpha \in \mathcal{A}$ to a horizontal outgoing separatrix. Now let us fix a total order on the finite alphabet $\mathcal{A}$, so that the marked separatrices are naturally ordered. This order induces an oriented pairing of the separatrices corresponding to odd degree singularities.
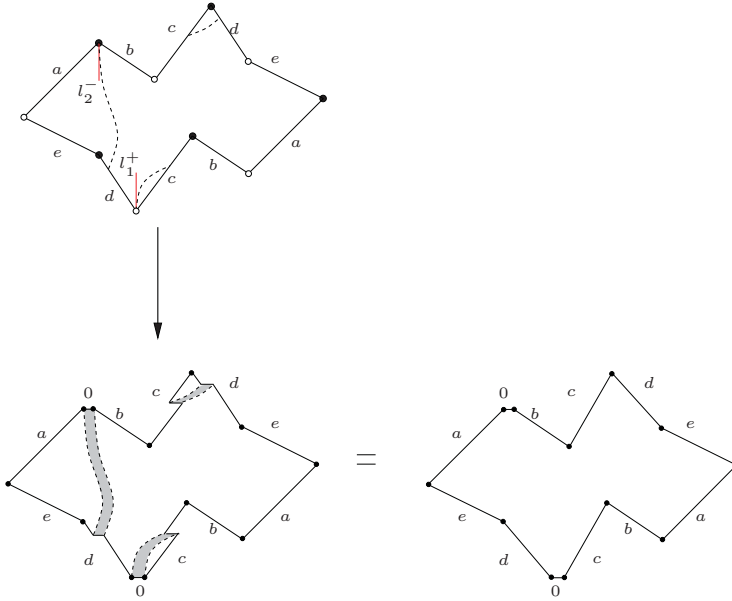

Figure 4.1. Building a surface with only even degree singularities.

Now let (l_1, l_2) be such a pair. We rotate the first separatrix clockwise by an angle $\pi/2$, and the second one counterclockwise by an angle $\pi/2$. We obtain pairs of vertical separatrices, the first one being on the positive direction, the second one on the negative direction. We denote by (l_1^+, l_2^-) this pair of positive/negative vertical separatrices, let us also denote by k_1, k_2 the degree of the corresponding zeroes. According to Hubbard–Masur [11], there exists a (smooth) path ν transverse to the horizontal foliation which starts being tangent to l_1^+ and ends being tangent to l_2^-. Now we consider the following surgery: we cut the surface X along the path and paste in a "curvilinear parallelogram" with two small horizontal sides and two opposite sides that are isomorphic to ν (see Figure 4.1). Then, gluing together the horizontal sides of the parallelogram, one obtains a translation surface

where the pair of singularities corresponding to l_1^+, l_2^- have become a singularity of degree $k_1 + k_2 + 2$, which is even. We will refer to this construction as the *parallelogram construction with parameters* (l_1, l_2).

Then, we apply this procedure on all the pairs of vertical separatrices that were defined previously. The resulting translation surface only has even singularities and is of genus at least 3, since the minimal genus case corresponds to starting from $\mathcal{H}(1,1)$ and ending in $\mathcal{H}(1+1+2)$.

Recall that the strata of the moduli space of Abelian differentials corresponding to only even degree singularities are not connected as soon as the genus is greater than or equal to 3, and are distinguished by the parity of spin structure. We will prove the following lemma:

LEMMA 4.3. — *The connected component of the resulting surface in the previous construction doesn't depend on the chosen paths.*

Proof. — Up to a small deformation of the surface X, one can assume that it is obtained by the Veech construction starting from a data (π, λ, τ). Then, for a pair l_1, l_2 of separatrices as previously, the surface obtained after the parallelogram construction with parameters l_1, l_2 also arises from the Veech construction, where the corresponding permutation is obtained from π by adding a new label on the top before the symbol corresponding to l_2 and the same label on the bottom before the label corresponding to l_1. For instance, in Figure 4.1, the labeled permutation $\left(\begin{smallmatrix} a & b & c & d & e \\ e & d & c & b & a \end{smallmatrix} \right)$ becomes $\left(\begin{smallmatrix} a & 0 & b & c & d & e \\ e & d & 0 & c & b & a \end{smallmatrix} \right)$, since l_1 corresponds to c and l_2 corresponds to b.

In particular, the permutation after removing all the odd degree singularities doesn't depend on the choices of the paths, but only on the order that we have chosen on $\mathcal{A}$. $\qquad\square$

Proof of Proposition 4.2. — We just need to show that the spin structure invariant defined before reaches all the possible values. First we recall Kontsevich–Zorich formula for the parity of spin structure for a translation surface X' of genus g' with only even degree singularities. Consider $a_1, b_1, \ldots, a_{g'}, b_{g'}$ a collection of closed paths that represent a symplectic basis of the homology $H_1(X'; \mathbb{Z})$, and such that each path does not pass through any singularity. We can assume that the a_i, b_i are parametrized by the arc length. For each a_i (resp. b_i), we define $\mathrm{ind}(a_i)$ (resp. $\mathrm{ind}(b_i)$) to be the index of the map $\mathbb{S}^1 \to \mathbb{S}^1$, $t \mapsto a_i'(t)$ (resp. $t \mapsto b_i'(t)$)). Then, the *parity of spin structure* of X' is defined by the following formula:

$$\sum_{i=1}^{g'} (\mathrm{ind}(a_i) + 1)(\mathrm{ind}(b_i) + 1) \mod 2$$

The result does not depend on the choice of the symplectic basis and is therefore an invariant of connected components of the strata of the moduli space of Abelian differentials (see [13]).

In the definition of the invariant for $\mathcal{C}(\mathcal{F}_{comp})$, we successively glue together some pairs of odd degree singularities. We can also glue all pairs except one and therefore we can assume that there is only one pair (P_1, P_2) of odd degree singularities, of degree k_1 and k_2 respectively, on the surface.

We present such surface X as coming from the Veech construction with parameters (π, λ, τ). Let g be the genus of this surface. As in Figure 4.1, we have a pair l_1^+, l_2^- of positive/negative vertical separatrix, we choose a path γ transverse to the horizontal foliation. There exists a collection of closed paths $a_1, b_1, \ldots, a_g, b_g$ that do not intersect γ and that represent a symplectic basis of the homology $H_1(X, \mathbb{Z})$. Let also a_0 be a small circle around the singularity P_1.

When doing the parallelogram construction with parameters l_1, l_2 using γ, the closed paths a_i, b_i persists and also the path a_0. Considering a path isometric to γ inside the parallelogram, one obtains a closed path b_0, that intersect a_0 only once, and that does not intersect a_i, b_i for all $i \in \{1, \ldots, g\}$. Hence, one gets symplectic basis on the homology of the newly built surface, that can be used to compute the corresponding parity of spin structure. Here, as we will see later, the only relevant data are the indices of a_0 and b_0. We clearly have:

- $\mathrm{ind}(a_0) = k_1 + 1 \mod 2 = 0 \mod 2$.
- $\mathrm{ind}(b_0) = 0$.

Now we start again from the surface X and replace the separatrix l_2 by the separatrix l_3, obtained by rotating l_2 by the angle 2π. Then, we do the parallelogram construction with parameters (l_1, l_3). We consider the following symplectic basis on the resulting flat surface:

- The path a_i, b_i, for $i \in \{1 \ldots g\}$ which persist under this construction.
- The path a_0, which also persists under this construction.
- A path b_0' obtained as in Figure 4.2

The indices of $a_0, a_1, \ldots, a_g$ and of $b_1, \ldots, b_g$ are the same as previously, but $\mathrm{ind}(b_0') = 1$.

Since, $\mathrm{ind}(a_0) + 1 = 1 \mod 2$, the surface obtained from (l_1, l_3) has a different parity of spin structure as the one obtained from (l_1, l_2). Hence, the two corresponding flat surfaces are in different connected components of the moduli space of Abelian differentials. This proves that $\mathcal{C}(\mathcal{F}_{comp})$ has at least 2 connected components. $\qquad\square$
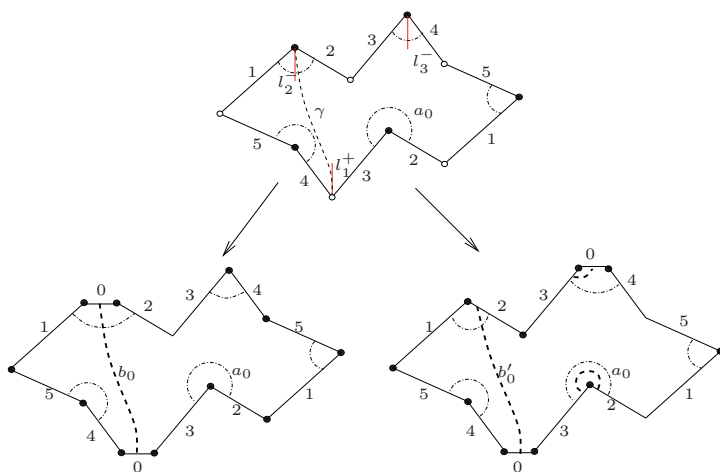
Figure 4.2. Building two surfaces with different spin structure

5. Number of connected components of $\mathcal{C}(\mathcal{F}_{comp})$

In the previous section, we have used topological invariants to find lower bounds on the number of connected components of $\mathcal{C}(\mathcal{F}_{comp})$. Here, we show that they are the exact values.

5.1. Three elementary surgeries.

Here we describe some elementary closed paths in $\mathcal{C}$ that lift to unclosed paths in $\mathcal{C}(\mathcal{F}_{comp})$.

Recall that a saddle connection γ joining two distinct singularities is *simple* if there exists no other saddle connection homologous to γ. In particular, it means that up to a small deformation of the surface in the ambient stratum, there is no other saddle connection in the surface parallel to γ. Then, deforming suitably the surface with the Teichmüller geodesic flow (see [3, 4] for instance), one gets a surface for which the saddle connection corresponding to γ is very short compared to the other ones. Then, one can show that such surface is obtained by the *breaking up a zero* surgery (see [10]). We give a short description of this surgery.

Breaking up a singularity. Let k_1, k_2 be the degree of the zeroes that are the endpoints of γ. We start from a zero P of degree $k_1 + k_2$. The

neighborhood $V_\varepsilon = \{x \in X, d(x, P) \leqslant \varepsilon\}$ of this conical singularity is obtained by considering $2(k_1 + k_2) + 2$ Euclidean half disks of radii r and gluing each half side of them to another one in a cyclic order. We can break the zero into two smaller one by changing continuously the way they are glued to each other as in Figure 5.1. Note that in this surgery, the metric is not modified outside V_ε. In particular, the boundary ∂V_ε is isometric to (a connected covering) of an Euclidean circle. Note that in this construction, we can "rotate" the two singularities by an angle θ by cutting the surface along ∂V_ε, rotating V_ε by an angle θ and regluing it.
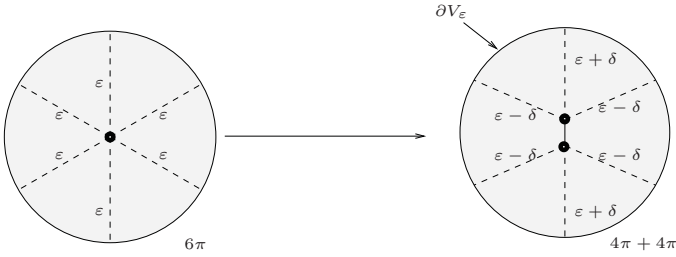

Figure 5.1. *Local surgery that break a zero of degree $k_1 + k_2$ into two zeroes of degree k_1 and k_2 respectively.*

MOVE 1. — *Let X be a framed surface and let P_1, P_2 be two distinct singularities of degree k, joined by a simple saddle connection γ. We deform slightly the surface so that no saddle connection is parallel to γ. Then, using the Teichmüller geodesic flow, we contract the saddle connection γ until it is very small compared to any other saddle connection. So the new surface X' is obtained by breaking a zero of degree $2k$ into two zeroes P_1' and P_2' of degree k. Now we continuously rotate these two zeroes by the angle $\theta = (2k + 1)\pi$. The resulting unframed surface is the same as X', but this procedure interchanges P_1' and P_2'. Then, we come back to the initial surface X, but the labeled zeroes P_1 and P_2 have been interchanged. The labels on the separatrices adjacent to the other singularities have not changed.*

The projection of this move in $\mathcal{C}$ is a closed path. This move in $\mathcal{C}(\mathcal{F}_{comp})$ interchanges P_1 and P_2, and fixes the separatrices adjacent to the other singularities.

The idea of this previous move is, as we will see, to authorize us to do any degree preserving permutation on the set of simply marked singularities. This explains the terms $n_i!$ in the formula of Theorem 1.1. Now the next

two moves will fix the labeled singularities and change the labeling on the outgoing separatrices.

MOVE 2. — *Let X be a framed surface and let P_1, P_2 be two distinct singularities of degree k_1 and k_2 respectively, joined by a simple saddle connection γ. Here we do not assume that $k_1 = k_2$. We perform the same as in Move 1, but we turn P_1' and P_2' by $(k_1 + k_2 + 1)2\pi$ instead.*

This move clearly corresponds to a closed path in $\mathcal{C}$. It also preserves P_1, P_2 pointwise. Let us look how changes the marked separatrices. For this, we can fix once and for all a marked separatrix for all singularities. Then, for a singularity of degree k, we can identify the set of corresponding horizontal separatrices to $\mathbb{Z}/(k+1)\mathbb{Z}$ by ordering them counterclockwise. We have the following lemma.

LEMMA 5.1. — *Let $l_1 \in \mathbb{Z}/(k_1+1)\mathbb{Z}$ and let $l_2 \in \mathbb{Z}/(k_2+1)\mathbb{Z}$ be the separatrices associated to P_1 and P_2. Then, Move 2 acts on the set of separatrices in the following way:*

- *l_1 becomes $l_1 - k_2 \mod k_1 + 1$*
- *l_2 becomes $l_2 - k_1 \mod k_2 + 1$*
- *All the other labeled separatrices remain unchanged*

Proof. — Note that it is enough to prove this lemma in the case when P_1, P_2 is obtained after breaking up a singularity. The last statement of the lemma is obvious by construction: we do not change the metric outside a small neighborhood of γ.

Now we look at the surgery, keeping track of the labeled separatrices. When turning continuously the set V_ε by an angle θ, one must simultaneously change the separatrices by an angle $-\theta$, so that they stay horizontal. At the end, they have moved by the angle $-(k_1 + k_2 + 1)2\pi$ each, so for $i = 1, 2$, l_i is replaced by $l_i - (k_1 + k_2 + 1)$ modulo $k_i + 1$, which gives the result. □

Note that this move is especially useful when $k_1 = k_2$. Then, the corresponding transformation is $(l_1, l_2) \mapsto (l_1 + 1, l_2 + 1)$.

Before describing the last move, we first describe a surgery which is analogous to the one presented in [13].

Bubbling r handles: We start from a singularity of degree $p \in \{0, 1\}$. Let us consider a small polygonal line L with no self intersection starting from the singularity. Let r be the number of segments $s_1, \ldots, s_r$ of L. We consider r parallelograms, each one having a pair of sides parallel to

one of the s_i. Then, we cut the surface along each s_i and paste in the corresponding parallelogram, and we glue by translation each remaining opposite sides of each parallelogram. We obtain a translation surface of genus $g(X)+r$, and the degree p singularity have been replaced by a degree $p+2r$ singularity. Note that this surgery can be performed without changing the metric outside a small neighborhood of the singularity of degree p. Note that we can "rotate" the construction in the following way: the surgery is performed inside a ε-neighborhood V_ε of the initial singularity of degree p. The boundary ∂V_ε remains a metric covering of a euclidean circle after bubbling the handles. Now we can cut the surface X along this circle and reglue it after a rotation by θ.

Now we can describe the last move.

MOVE 3. — *Assume that the translation surface X was obtained after bubbling r handles and let P be the corresponding singularity. We continuously rotate the construction as explained previously, by a angle of $(p+1)2\pi$.*

As before, the underlying surface in $\mathcal{C}$ is the same after Move 3 and any separatrix that does not correspond to the singularity P remains unchanged.

LEMMA 5.2. — *Let $l \in \mathbb{Z}/(p+2r+1)\mathbb{Z}$ be the separatrix corresponding to P. Then, Move 3 changes l in the following way:*

- *If $p=0$, l is replaced by $l-1$.*
- *If $p=1$, l is replaced by $l-2$.*

Proof. — It is easy to see that, as in the case of Move 2, a marked separatrix attached to the singularity is changed by the transformation $l \mapsto l-(p+1) \mod p+2r+1$, and the separatrices associated to the other singularities remain unchanged. □

In particular if $p=0$, we reach all possible separatrices adjacent P in this way. If $p=1$, then $p+2r+1$ is even, and we reach only half of the separatrices adjacent to P in this way.

5.2. Generating the monodromy group.

PROPOSITION 5.3. — *Assume that $\mathcal{C}$ contains nonhyperelliptic surfaces. Then, the following holds:*

- *The set $\mathcal{C}(\mathcal{F}_{comp})$ is connected if all the singularities have even degree*

- *The set $\mathcal{C}(\mathcal{F}_{comp})$ has two connected components otherwise.*

We start with the following lemma.

LEMMA 5.4. — *Let $\mathcal{C}$ be a connected component of $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$. Choose an ordering on the set with multiplicities $\{k_1, \ldots, k_1, \ldots, k_r, \ldots, k_r\}$. There exists $X \in \mathcal{C}$ and a polygonal line in X that consists of simple saddle connections and that joins all the singularities of X in that order.*

Proof. — The proof is the same as the proof of Proposition 4.1 in [4]. □

Proof of Proposition 5.3. — We first assume that there exist odd degree singularities in the underlying stratum. Since is not a hyperelliptic stratum by hypothesis, it is connected (see [13]). We write this stratum as $\mathcal{H}(k_1^{n_1}, \ldots, k_s^{n_s}, (2k_1')^{\beta_1}, \ldots, (2k_{s'}')^{\beta_{s'}})$, with $s + s' = r$, and $k_1, \ldots, k_s$ are odd.

Now we start from a surface in $\mathcal{H}(k_1^{n_1-1}, 1, \ldots, k_s^{n_s-1}, 1)$ (i.e. we don't take the even degree singularities and we replace one singularity of each odd degree by a singularity of degree one). From the previous lemma, we can assume that there is a polygonal path of simple saddle connections which has no self intersection and that joins successively all the singularities in the following order:

- first the singularities of degree k_1,
- then, a singularity of degree 1,
- then, the singularities of degree k_2,
- then, a singularity of degree 1,
- and so on ...

Now for each singularity of degree 1 that ends a group of singularities of degree k_i, we bubble $(k_i - 1)/2$ handles as in section 5.1. This replace the singularity of degree 1 by a singularity of degree k_i. Note that the polygonal path of simple saddle connections persists under this surgery. We will denote by γ this polygonal line.

Now for each $i \in \{1, \ldots, s'\}$, we consider a polygonal line γ_i joining β_i regular points, such that the paths $\gamma, (\gamma_i)_i$ have no intersection points. Then, for each vertex, we bubble k_i' handles. We obtain s' chains of simple saddle connections that join each collection of singularities of degree $2k_i'$. The resulting surface is therefore in $\mathcal{H}(k_1^{n_1}, \ldots, k_s^{n_s}, (2k_1')^{\beta_1}, \ldots, (2k_{s'}')^{\beta_{s'}})$, which is the stratum that we study.

Now using Move 1, we see that for any polygonal line of simple saddle connections joining singularities with the same degree, we can perform any transposition of two consecutive singularities. Hence we can arbitrarily permute the singularities sharing the same degree.

Using Move 3, we see that we can reach any choice of separatrices for the even degree singularities.

Now we consider the chain γ of simple saddle connections joining all the separatrices of odd degree, that was constructed before. The first n_1 vertices of γ makes a chain of singularities of degree k_1 . Let us name the singularities $P_{1,1}, \ldots, P_{1,n_1}$ according to the order given by the polygonal path γ. If $n_1 > 1$, we reach any choice of separatrices for $P_{1,1}, \ldots, P_{1,n_1-1}$ by applying successively Move 2 on the pairs $(P_{1,i}, P_{1,i+1})$ for $i \in \{1, \ldots, n_1 - 1\}$, in order to choose arbitrarily a labeled separatrix of $P_{1,i}$. Note that once this is done for some i, the next moves don't change the marked separatrix corresponding to $P_{1,i}$. Then, for the singularity P_{1,n_1}, we use Move 3 to rotate the corresponding separatrix by any even number (recall that the set of outgoing separatrices corresponding to a singularity of degree k is naturally identified with $\mathbb{Z}/(k+1)\mathbb{Z}$). If P_{1,n_1} is not the end of γ, i.e. the polygonal line γ continues to some other (odd) degree singularity $P_{2,1}$, then Move 2 on the pair $P_{1,n_1}, P_{2,1}$ will act on the marked separatrix of P_{1,n_1} as $l_1 \mapsto l - k_2$. Hence, it will be changed by a *odd* number, so in combination with Move 3, we obtain all possible choices.

If we iterate this procedure until the last singularity of degree k_s, we see that we can reach any choice of separatrices for the singularities of odd degree, except the last one of the chain where we obtain only half of the possibilities. This proves the proposition in the case when there exist odd singularities.

If there does not exists any singularity of odd degree, the procedure described above works (with $\gamma = \emptyset$) as soon as we can find a surface like above in the connected component that we study. But in this case, the corresponding stratum of translation surfaces is not connected.

Consider a translation surface obtained from a torus with the "bubbling r handles" construction. We can easily show that in this case, each singularity contribute to zero to the spin structure. See Figure 5.2. Hence the resulting parity of spin structure is the same as for the flat torus, which is odd.

If there exists a singularity of degree $k \geqslant 4$, It is easy to see that one can slightly change the construction to make this singularity contribute to 1 to the spin structure, and obtain a surface with even spin structure (see Figure 5.3).

The last remaining case to see is when all the singularities are of degree 2, and the parity of spin structure is even. If there are exactly 2 singularities, then the connected component is $\mathcal{H}^{even}(2,2) = \mathcal{H}^{hyp}(2,2)$, which is a hyperelliptic case. If there are at least 3 singularities, we can find a surface
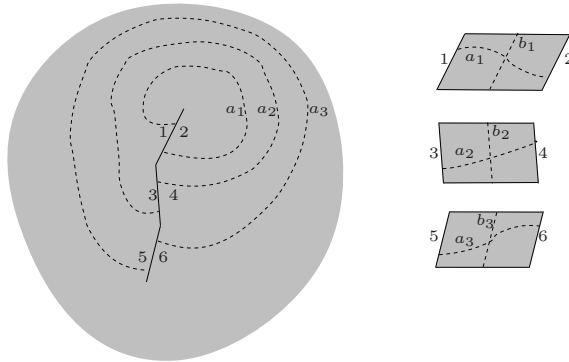
Figure 5.2. The "bubbling r-handles" standard construction. We have $\mathrm{ind}(a_i) = 1$, so the collection $(a_i, b_i)_{i \in \{1 \ldots \beta\}}$ contributes to 0 for the spin structure.
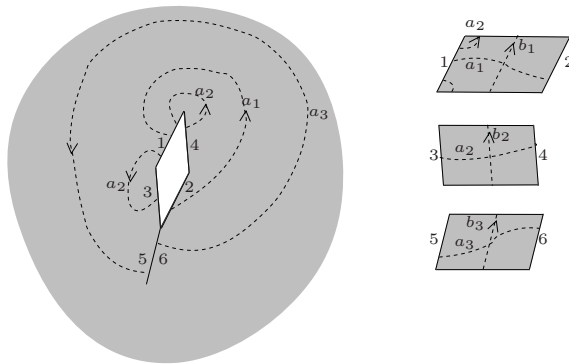

Figure 5.3. A slight change in the r-handle construction, for $r \geqslant 2$ changes the spin structure, since in this case, $\mathrm{ind}(a_2) = 0 \mod 2$.

with a chain of simple saddle connections joining all the singularities, and such that the last element of the chain is obtained by the "bubbling a handle" construction. Then, combining Move 2 along the chain, and Move 3 at the end of the chain, we obtain that $\mathcal{C}(\mathcal{F}_{comp})$ is connected. This concludes the proof of the proposition. $\square$

Proof of Theorem 1.3. — The nonhyperelliptic case is given by Proposition 5.3. For the hyperelliptic case, the lower bound on the number of

connected components is given by Proposition 4.1, and the upper bound is easy and left to the reader.   □

Proof of Theorem 1.1. — Recall that we denote by $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$ the ambient stratum of the moduli space of Abelian differentials. The degree of the covering $\mathcal{C}(\mathcal{F}_{comp}) \to \mathcal{C}$, restricted to a connected component of $\mathcal{C}(\mathcal{F}_{comp})$, is clearly $\frac{\Pi_{i=1}^r n_i!(k_i+1)^{\alpha_i}}{c}$, where c is the number of connected component of $\mathcal{C}(\mathcal{F}_{comp})$ and is given by Theorem 1.3.

Let k be the degree of the marked singularity associated the Rauzy class R, and let n be the number of singularities of degree k. The set $\mathcal{C}_k$ is connected and the degree of the projection $\mathcal{C}_k \to \mathcal{C}$ is $n(k+1)$. Hence we have:

$$\frac{|R_{lab}|}{|R|} = \frac{\Pi_{i=1}^r n_i!(k_i+1)^{\alpha_i}}{c.(k+1).n}$$

which gives Theorem 1.1.   □

Appendix A. Generalizations of Rauzy classes

Half-translation surfaces are a natural generalization of translation surfaces. They are surfaces with an atlas such that the changes of coordinates are not only translations, but can also be half-turns. They correspond to Riemann surfaces with *quadratic differentials*.

Danthony and Nogueira have generalized interval exchange transformations and Rauzy induction to describe first return maps of nonoriented measured foliations on transverse segment. One gets *linear involutions*, see [8].

The relation between quadratic differentials and linear involutions was described by the author and Lanneau in [5].

One can wonder if there is a analogous result for Rauzy classes appearing in this context. There doesn't seem to be a natural relation between "labeled generalized permutations" and framed half-translation surfaces. In particular, there is no "quadratic" equivalent of Lemma 3.3.

Numerical experiments with SAGE [18] on a significant number of Rauzy classes suggest that the ratio between a labeled Rauzy class and its corresponding reduced one is always either $n!$ or $\frac{n!}{2}$, where n is the number of underlying intervals, which is generally much more than for the Abelian case. In particular, using Rauzy induction and labelling the intervals, these numerical experiments suggests that one obtain either any renumbering of the intervals, or any even renumbering of the intervals depending on the stratum.

One can also look at *extended Rauzy classes* for Abelian differentials, where we add Rauzy moves that correspond to cutting on the left of the interval (see [13], appendix A.2). Again in this case, the geometrical interpretation of the ratio is unclear and similar numerical experiments suggests that the ratio is also either $n!$ or $n!/2$ depending on the stratum.

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