ANNALES

DE

L'INSTITUT FOURIER

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On the torsion of the first direct image of a locally free sheaf

ON THE TORSION OF THE FIRST DIRECT IMAGE
OF A LOCALLY FREE SHEAF

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ABSTRACT. — Let $\pi\colon M \to B$ be a proper holomorphic submersion between complex manifolds and $\mathcal{E}$ a holomorphic bundle on M. We study and describe explicitly the torsion subsheaf $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ of the first direct image $R^1\pi_*(\mathcal{E})$ under the assumption $R^0\pi_*(\mathcal{E}) = 0$. We give two applications of our results. The first concerns the locus of points in the base of a generically versal family of complex surfaces where the family is non-versal. The second application is a vanishing result for $H^0(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$ in a concrete situation related to our program to prove existence of curves on class VII surfaces.

RÉSUMÉ. — Soit $\pi\colon M \to B$ un submersion propre entre variétés complexes, et soit $\mathcal{E}$ un fibré holomorphe sur M. Nous étudions et décrivons explicitement le sous-faisceau de torsion $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ de la première image directe $R^1\pi_*(\mathcal{E})$ en supposant que $R^0\pi_*(\mathcal{E}) = 0$. Nous discutons deux applications des résultats obtenus : la première concerne le lieu des points où une famille génériquement verselle de surfaces complexes est non-verselle. La deuxième application est un résultat d'annulation pour $H^0(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$ dans une situation concrète liée à notre programme pour démontrer l'existence des courbes sur les surfaces de la classe VII.

Introduction

Let B, M be complex manifolds, $\pi\colon M \to B$ a proper, surjective holomorphic submersion, and let $\mathcal{E}$ be a holomorphic bundle on M. Denote by $\mathcal{E}_x$ the restriction of $\mathcal{E}$ to the fiber M_x, regarded as a locally free sheaf on M_x. Grauert's locally freeness theorem states that a direct image $R^k\pi_*(\mathcal{E})$ is locally free when the map $x \mapsto h^k(\mathcal{E}_x)$ is constant on B. The starting

point of this article is the following natural questions: What can be said about the singularities of the sheaf $R^k\pi_*(\mathcal{E})$ when this condition is not satisfied? Can one describe explicitly the torsion subsheaf of $R^k\pi_*(\mathcal{E})$?

In this article we will deal with these questions for $k = 1$ under the assumption $R^0\pi_*(\mathcal{E}) = 0$. Our first result concerns the support of the torsion subsheaf $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ and states

THEOREM 1.3. — *Let B, M be complex manifolds, $\pi\colon M \to B$ a proper, surjective holomorphic submersion, and $\mathcal{E}$ a holomorphic bundle on M. If $R^0\pi_*(\mathcal{E}) = 0$ then the support $\mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$ coincides with the maximal pure 1-codimensional analytic subset of B which is contained in the Brill-Noether locus*

$$\mathrm{BN}_\pi(\mathcal{E}) := \big\{x \in B \mid h^0(\mathcal{E}_x) \neq 0\big\}.$$

In particular this support has pure codimension 1.

The statement yields strong a priori properties of the first direct image in a very general framework. This result has interesting consequences:

COROLLARY 1.4. — *If $\mathrm{BN}_\pi(\mathcal{E})$ has codimension $\geqslant 2$ at any point, then $R^1\pi_*(\mathcal{E})$ is torsion free. In particular the singularity set of this sheaf has codimension $\geqslant 2$ at any point.*

Equivalently,

COROLLARY 1.5. — *If $R^0\pi_*(\mathcal{E}) = 0$ and $\mathrm{Tors}(R^1\pi_*(\mathcal{E})) \neq 0$, then the Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ contains a non-empty effective divisor.*

Another result concerns the natural question: supposing that we are in the conditions of Corollary 1.4, how far is the torsion free sheaf $R^1\pi_*(\mathcal{E})$ from being a free $\mathcal{O}_B$-module? The proposition below shows that the Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ can be regarded as an obstruction to trivializing globally the torsion free sheaf $R^1\pi_*(\mathcal{E})$.

PROPOSITION 1.8. — *Let $\pi\colon M \to B$ be a proper, surjective holomorphic submersion with connected base B, connected surfaces as fibers, and $\mathcal{E}$ a holomorphic bundle on M and such that*

    (1) *The Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ has codimension $\geqslant 2$ at every point.*

    (2) *The map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant.*

Put $k := \mathrm{rk}(R^1\pi_*(\mathcal{E}))$ and assume that

$$s = (s_1, \ldots, s_k) \in H^0(B, R^1\pi_*(\mathcal{E}))^{\oplus k}$$

is a system of sections such that $s(x_0)$ is linearly independent in the fiber $(R^1\pi_*(\mathcal{E}))(x_0)$ for a point $x_0 \in B \smallsetminus \mathrm{BN}_\pi(\mathcal{E})$. Put

$$\mathcal{S} := \big\{ x \in B \smallsetminus \mathrm{BN}_\pi(\mathcal{E}) \mid s(x) \text{ is linearly dependent in } (R^1\pi_*(\mathcal{E}))(x) \big\}.$$

Then the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ is an effective divisor containing $\mathrm{BN}_\pi(\mathcal{E})$. In particular this divisor is non-empty if $\mathrm{BN}_\pi(\mathcal{E})$ is non-empty.

In sections 2 and 3 we give two explicit descriptions of the torsion subsheaf $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$. The first result concerns only the case $R^0\pi_*(\mathcal{E}) = 0$ and identifies this torsion sheaf with an inductive limit indexed by the ordered set $\mathcal{D}iv(B)$ of effective divisors of the base B.

The second statement is more general, more precise, and implies the first. On the other hand, whereas the second result uses formal algebraic homological techniques, the first one is proved with classical complex geometric methods. Therefore we believe that both approaches are interesting.

THEOREM 2.5. — *Suppose that π has connected fibers and $R^0\pi_*(\mathcal{E}) = 0$. Then there exists a canonical isomorphism*

$$\varinjlim_{\substack{D \in \mathcal{D}iv(B) \\ |D| \subset \mathrm{BN}_\pi(\mathcal{E})}} R_D^\pi = \varinjlim_{D \in \mathcal{D}iv(B)} R_D^\pi \xrightarrow{\simeq \ \psi} \mathrm{Tors}(R^1\pi_*(\mathcal{E})),$$

where $\mathcal{D}iv(B)$ denotes the ordered set of effective divisors on B and, for an effective divisor $D \subset B$, we put $\mathcal{D} := \pi^{-1}(D)$ and

$$R_D^\pi := \pi_*(\mathcal{E}_\mathcal{D})(D) = \pi_*(\mathcal{E}_\mathcal{D}(\mathcal{D})) \in \mathcal{C}oh(B).$$

Let $\pi \colon M \to B$ be a proper morphism of complex spaces and $\mathcal{E}$ a coherent sheaf on M which is flat over B. According to [1, Corollary 4.11 p. 133] there exists a coherent sheaf $\mathcal{T}_\mathcal{E}$ on B, unique up to isomorphism with the following property: for every coherent sheaf $\mathcal{A}$ defined on an open set $U \subset B$, there exists an isomorphism $\pi_*(\mathcal{E} \otimes \pi^*(\mathcal{A})) = \mathcal{H}om(\mathcal{T}_\mathcal{E}, \mathcal{A})$ which is functorial with respect to $\mathcal{A}$. With this notation we can state:

THEOREM 3.5. — *Let $\pi \colon M \to B$ be a proper morphism of complex spaces and $\mathcal{E}$ a coherent sheaf on M which is flat over B. Suppose that B is locally irreducible. Then*

    (i) *One has canonical isomorphisms*

$$\mathcal{E}xt^1(\mathcal{T}, \mathcal{O}_B) \xrightarrow{\cong} \mathrm{Tors}(R^1\pi_*(\mathcal{E})).$$

    (ii) *If $\pi_*(\mathcal{E}) = 0$ and B is smooth, then*
        (a) *$\mathcal{T}$ is a torsion sheaf,*

(b) *Denoting by $D_{\max}$ the maximal divisor contained in the complex subspace defined by $\mathrm{Ann}(\mathcal{T})$, one has canonical isomorphisms*

$$\mathcal{H}om(\mathcal{T}_{D_{\max}}, \mathcal{O}_{D_{\max}}) \xrightarrow{\simeq} \mathcal{H}om(\mathcal{T}, \mathcal{O}_{D_{\max}})$$
$$\xrightarrow{\simeq} \mathrm{Ext}^1(\mathcal{T}, \mathcal{O}_B) \xrightarrow{\simeq} \mathrm{Tors}(R^1\pi_*(\mathcal{E})).$$

These results have effective applications. For instance Corollary 1.5 can be viewed as a tool to prove existence of divisors on a given complex manifold. More precisely, let B be a complex surface (for instance a class VII surface) and $\mathcal{E}$ the universal bundle on $B \times Y$ associated with an embedding of a compact complex manifold Y in a moduli space of simple bundles on B [31]. We conclude that, if $\mathrm{Tors}(R^1\pi_*(\mathcal{E})) \neq 0$, then B has curves.

Proposition 1.8 has been used in [7] for studying deformations of class VII surfaces, and is applied taking for $\mathcal{E}$ the relative tangent bundle of the deformation. The author defines an explicit family of class VII surfaces with $b_2 = b$ parameterized by an open set $B \subset \mathbb{C}^{2b}$ which is generically versal. This family contains surfaces which admit (non-trivial) global tangent vector fields, so in this case the Brill-Noether locus is non-empty. Therefore Proposition 1.8 applies and gives a non-empty divisor in B containing the points which correspond to surfaces with global tangent vector fields and the points where the family is non-versal. An important problem in [7] is to determine this divisor explicitly.

Finally, Theorem 2.5 can be used to "compute" $H^0(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$ supposing that the restrictions of $\mathcal{E}$ to the "vertical" divisors $\mathcal{D} := \pi^{-1}(D)$ are known. Note that, under the assumption $R^0\pi_*(\mathcal{E}) = 0$, the Leray spectral sequence associated with the pair $(\mathcal{E}, \pi)$ induces a canonical isomorphism $H^1(\mathcal{E}) \xrightarrow{\simeq} H^0(R^1\pi_*(\mathcal{E}))$. Therefore $H^0(R^1\pi_*(\mathcal{E}))$ (in particular its subspace $H^0(\mathrm{Tors}(R^1\pi_*(\mathcal{E}))))$ is relevant for the computation of the cohomology space $H^1(\mathcal{E})$. We will discuss these applications in section 4.

1. First properties of $R^1\pi_*(\mathcal{E})$

In this section we prove the first general properties of the sheaf $R^1\pi_*(\mathcal{E})$ under the assumption $R^0\pi_*(\mathcal{E}) = 0$. As usually we will use the same notation for a holomorphic bundle and the associated locally free sheaf of local sections.

Let B, M be complex manifolds, $\pi \colon M \to B$ a proper, surjective holomorphic submersion, and let $\mathcal{E}$ be a holomorphic bundle on M. Let $U \subset B$ be an open set, $\varphi \in \mathcal{O}(U)$ a non-trivial holomorphic function, and $D := Z(\varphi)$

the associated effective divisor. Let $m_\varphi \colon R^1\pi_*(\mathcal{E})|_U \to R^1\pi_*(\mathcal{E})|_U$ be the morphism defined by multiplication with φ. By the definition of the $\mathcal{O}_B$-module structure on $R^1\pi_*(\mathcal{E})$, the morphism m_φ is just $R^1\pi_*(m_\Phi)$, where m_Φ is the morphism of sheaves $\mathcal{E}|_{\pi^{-1}(U)} \to \mathcal{E}|_{\pi^{-1}(U)}$ defined by multiplication with the function

$$\Phi := \pi^*(\varphi) = \varphi \circ \pi \in \mathcal{O}(\pi^{-1}(U)).$$

Tensorising by the locally free sheaf $\mathcal{E}$ the tautological exact sequence associated with the pull-back divisor $\mathcal{D} = Z(\Phi)$, we get the short exact sequence

$$0 \longrightarrow \mathcal{E}|_{\pi^{-1}(U)} \xrightarrow{\Phi\cdot} \mathcal{E}|_{\pi^{-1}(U)} \longrightarrow \mathcal{E}_\mathcal{D} \longrightarrow 0$$

which yields

(1.1) $\quad 0 \longrightarrow \pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \xrightarrow{\varphi\cdot} \pi_*(\mathcal{E}|_{\pi^{-1}(U)})$

$$\longrightarrow \pi_*(\mathcal{E}_\mathcal{D}) \to R^1\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \xrightarrow{R^1\pi_*(m_\Phi)} R^1\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \cdots$$

Denote by j and J the inclusions of D and $\mathcal{D}$ in B and M respectively. The sheaf $\mathcal{E}_\mathcal{D}$ can be written as $J_*(\mathcal{E}|_\mathcal{D})$ hence, since $\pi \circ J = j \circ (\pi|_\mathcal{D})$ we obtain

(1.2) $\qquad \pi_*(\mathcal{E}_\mathcal{D}) = \pi_*(J_*(\mathcal{E}|_\mathcal{D})) = (\pi \circ J)_*(\mathcal{E}|_\mathcal{D})$

(1.3) $\qquad\qquad = (j \circ \pi|_\mathcal{D})_*(\mathcal{E}|_\mathcal{D}) = j_*\left((\pi|_\mathcal{D})_*(\mathcal{E}|_\mathcal{D})\right).$

DEFINITION 1.1. — *The Brill-Noether locus of the pair $(\pi, \mathcal{E})$ is defined by*

$$\mathrm{BN}_\pi(\mathcal{E}) := \left\{ x \in B \mid h^0(\mathcal{E}_x) \neq 0 \right\} \subset B,$$

where, for $x \in B$ we denoted by $\mathcal{E}_x$ the restriction of $\mathcal{E}$ to the fiber $M_x := \pi^{-1}(x)$.

LEMMA 1.2. — *Suppose that the divisor $Z(\varphi)$ is reduced, and that $\mathrm{BN}_\pi(\mathcal{E}) \cap Z(\varphi)$ has codimension $\geqslant 2$ at every point. Then*

$$\ker\left(m_\varphi \colon R^1\pi_*(\mathcal{E})|_U \to R^1\pi_*(\mathcal{E})|_U\right) = 0.$$

Proof. — Taken into account the exact sequence (1.1) and formula (1.2) it suffices to prove that $(\pi|_\mathcal{D})_*(\mathcal{E}|_\mathcal{D}) = 0$. Let $V \subset D := Z(\varphi)$ be an open set and W its pre-image in $\mathcal{D}$. One has

$$(\pi|_\mathcal{D})_*(\mathcal{E}|_\mathcal{D})(V) = H^0(W, \mathcal{E}|_\mathcal{D}).$$

Since W is reduced, the vanishing of a section $s \in H^0(W, \mathcal{E}|_\mathcal{D})$ can be tested pointwise. But the restriction of any such section to the dense set

$$W \smallsetminus (\pi|_\mathcal{D})^{-1}(\mathrm{BN}_\pi(\mathcal{E}))$$

vanishes obviously (because it vanishes fiberwise). This shows $H^0(W, \mathcal{E}|_{\mathcal{D}}) = 0$.
                                                                                            $\square$

Using Lemma 1.2 we can prove now our first result about the torsion of $R^1\pi_*(\mathcal{E})$:

THEOREM 1.3. — *Let B, M be complex manifolds, $\pi \colon M \to B$ a proper, surjective holomorphic submersion, and let $\mathcal{E}$ be a holomorphic bundle on M. If $R^0\pi_*(\mathcal{E}) = 0$ then the support $\mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$ coincides with the maximal pure 1-codimensional analytic subset $D_\pi(\mathcal{E}) \subset B$ contained in $\mathrm{BN}_\pi(\mathcal{E})$. In particular this support has pure codimension 1.*

Proof. — Let $x \in \mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$. Therefore there exists $u \in R^1\pi_*(\mathcal{E})_x \smallsetminus \{0\}$ and a germ $\varphi_x = (x, U \xrightarrow{\varphi} \mathbb{C}) \in \mathcal{O}_x \smallsetminus \{0\}$ such that $\varphi_x u = 0$. We may suppose that φ_x is irreducible in the local ring $\mathcal{O}_x$. Recall that the set of points at which a complex space is reduced is Zariski open. This is a consequence of Cartan's coherence theorem for the ideal sheaf of nilpotent elements (see [19, 47.II, E 47 c p. 182]). Therefore there exists an open neighborhood of x in $Z(\varphi)$ which is reduced. In other words, replacing U by a smaller open neighborhood of x in B if necessary, we may suppose that the effective divisor $D := Z(\varphi) \subset U$ is reduced[1].

Using Lemma 1.2 and taking into account that D is irreducible at x, we conclude that locally around x the divisor D is contained in $\mathrm{BN}_\pi(\mathcal{E})$. In other words, taking U sufficiently small we will have $D \subset \mathrm{BN}_\pi(\mathcal{E})$, hence $x \in D_\pi(\mathcal{E})$. This proves the inclusion $\mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E}))) \subset D_\pi(\mathcal{E})$. Conversely let $x \in D_\pi(\mathcal{E})$ and $\varphi_x = (x, U \xrightarrow{\varphi} \mathbb{C}) \in \mathcal{O}_x \smallsetminus \{0\}$ an irreducible germ defining an irreducible component of $D_\pi(\mathcal{E})$ at x. Taking U sufficiently small we may assume that $D := Z(\varphi)$ is reduced. Using again the exact sequence (1.1) we see that the sheaf $\pi_*(\mathcal{E}_\mathcal{D}) = j_*((\pi_{|\mathcal{D}})_*(\mathcal{E}|_{\mathcal{D}}))$ is mapped injectively into $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$. But, by Grauert's theorems, the sheaf $(\pi_{|\mathcal{D}})_*(\mathcal{E}|_{\mathcal{D}})$ has positive rank on D. Therefore $D \subset \mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$, which proves the inclusion $D_\pi(\mathcal{E}) \subset \mathrm{supp}(\mathrm{Tors}(R^1\pi_*(\mathcal{E})))$. $\square$

COROLLARY 1.4. — *Suppose that the Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ of the pair $(\pi, \mathcal{E})$ has codimension $\geqslant 2$ at every point. Then the first direct image $R^1\pi_*(\mathcal{E})$ is torsion free.*

We also state explicitly the following obvious reformulation of this Corollary 1.4, which can be regarded as a *criterion for existence of divisors* in the base of the fibration:

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[1] Note that, in complex analytic geometry, being irreducible at a point is not an open property, as one can see in the example of the Withney umbrella, which is defined by the equation $xy^2 = z^2$.

COROLLARY 1.5. — If $R^0\pi_*(\mathcal{E}) = 0$ and $\mathrm{Tors}(R^1\pi_*(\mathcal{E})) \neq 0$, then the Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ contains a non-empty effective divisor.

A similar statement is obtained if one replaces the condition $R^0\pi_*(\mathcal{E}) = 0$ with the condition "cohomologically flat in dimension 0" (see [1, p. 133–134]):

PROPOSITION 1.6. — Suppose that $B \ni x \mapsto h^0(\mathcal{E}_x) \in \mathbb{N}$ is constant. Then $R^1\pi_*(\mathcal{E})$ is torsion free.

Proof. — Since the map $B \ni x \mapsto h^0(\mathcal{E}_x) \in \mathbb{N}$ is constant, $\pi_*(\mathcal{E})$ is locally free and commutes with base change by Grauert's theorems (see [1, Theorems 4.10(d), 4.12]). Here we used the properness and the flatness of π (which implies the flatness of $\mathcal{E}$ over B). Using the base change property we see that the natural morphism

$$\pi_*(\mathcal{E}|_{\pi^{-1}(U)})|_D \longrightarrow (\pi|_{\mathcal{D}})_*(\mathcal{E}_{\mathcal{D}})$$

is an isomorphism. Applying the functor $(i_D)_*$ to the two sheaves, it follows that the natural morphism

$$\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \otimes \mathcal{O}_D \to \pi_*(\mathcal{E}_{\mathcal{D}})$$

is an isomorphism. Via this isomorphism the morphism $\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \to \pi_*(\mathcal{E}_{\mathcal{D}})$ in the exact sequence (1.1) corresponds to the canonical epimorphism

$$\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \to \pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \otimes \mathcal{O}_D$$

so is surjective. Therefore

$$\ker\left(m_\varphi \colon R^1\pi_*(\mathcal{E})|_U \to R^1\pi_*(\mathcal{E})|_U\right) = \{0\}$$

by the exact sequence (1.1).

An alternative proof can be obtained using the definition of cohomologically flatness in dimension 0. For any open set $U \subset B$ and any non-trivial holomorphic map $\varphi \in \mathcal{O}^*(U)$ we get a monomorphism $0 \to \mathcal{O}(U) \xrightarrow{\varphi} \mathcal{O}(U)$ so, by [1, Theorem 4.10 (a')] the induced morphism $m_\varphi \colon R^1\pi_*(\mathcal{E})|_U \to R^1\pi_*(\mathcal{E})|_U$ is injective. □

Corollary 1.4 and Proposition 1.6 give criteria which guarantee the first direct image $R^1\pi_*(\mathcal{E})$ being torsion free. The proposition below gives a criterium which guarantees this sheaf being a free $\mathcal{O}_X$-module.

PROPOSITION 1.7. — Let $\pi\colon M \to B$ be a proper, surjective holomorphic submersion with connected surfaces as fibers, and $\mathcal{E}$ a holomorphic bundle on M such that

(1) The Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E}) := \{x \in B| \ h^0(\mathcal{E}_x) \neq 0\}$ has codimension $\geqslant 2$ at every point.

(2) *The map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant.*

Put $k := \operatorname{rk}(R^1\pi_*(\mathcal{E}))$ and assume that $s = (s_1, \ldots, s_k) \in H^0(B, R^1\pi_*(\mathcal{E}))^{\oplus k}$ is a system of sections such that $s(x)$ is linearly independent in the fiber $(R^1\pi_*(\mathcal{E}))(x)$ for every $x \in B \setminus \operatorname{BN}_\pi(\mathcal{E})$. Then

   (1) *The morphism $\sigma\colon \mathcal{O}_B^{\oplus k} \to R^1\pi_*(\mathcal{E})$ defined by s is an isomorphism, in particular $R^1\pi_*(\mathcal{E})$ is a free $\mathcal{O}_X$-module and $s(x)$ is linearly independent in $(R^1\pi_*(\mathcal{E}))(x)$ for every $x \in B$.*
   (2) $\operatorname{BN}_\pi(\mathcal{E}) = \emptyset$.

Therefore, under the assumptions of the theorem, the first direct image $R^1\pi_*(\mathcal{E})$ is free if this sheaf admits a system s of $k := \operatorname{rk}(R^1\pi_*(\mathcal{E}))$ sections which are linearly independent on the open set $B \setminus \operatorname{BN}_\pi(\mathcal{E})$ (on which $R^1\pi_*(\mathcal{E})$ is locally free).

Proof. — Since the map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant, it follows by Grauert's theorems that $R^2\pi_*(\mathcal{E})$ is locally free and that $R^2\pi_*(\mathcal{E})$, $R^1\pi_*(\mathcal{E})$ commute with base changes ([1, Theorem 3.4 p. 116]). Therefore the canonical morphisms $(R^i\pi_*(\mathcal{E}))(x) \to H^i(\mathcal{E}_x)$ are isomorphisms for $i = 1, 2$, and for every $x \in B$.

By Riemann-Roch theorem and the second assumption it follows that the map $B \ni x \mapsto h^1(\mathcal{E}_x)$ is constant on $B \setminus \operatorname{BN}_\pi(\mathcal{E})$, hence the sheaf $\mathcal{F} := R^1\pi_*(\mathcal{E})$ (which is torsion free by Proposition 1.4) is locally free on this open subset. We know by hypothesis that the morphism $\sigma\colon \mathcal{O}_B^{\oplus k} \to \mathcal{F}$ defined by s is a bundle isomorphism on $U := B \setminus \operatorname{BN}_\pi(\mathcal{E})$. The canonical embedding $c\colon \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ of $\mathcal{F}$ in its bidual sheaf is also an isomorphism on U, so the composition $c \circ \sigma$ has this property too. The inverse $\tau := \{c \circ \sigma_{|U}\}^{-1}$ can be regarded as a section of $\operatorname{Hom}((\mathcal{F}^\vee)^\vee, \mathcal{O}_B^{\oplus k}) = \{\mathcal{F}^\vee\}^{\oplus k}$ defined on U. Since $\{\mathcal{F}^\vee\}^{\oplus k}$ is a reflexive sheaf and U is the complement of a Zariski closed subset of codimension $\geqslant 2$, it follows that τ extends to a global morphism $\tilde{\tau}\colon (\mathcal{F}^\vee)^\vee \to \mathcal{O}_B^{\oplus k}$. We have

$$\tilde{\tau} \circ (c \circ \sigma) = \operatorname{id}_{\mathcal{O}_B^{\oplus k}}, \quad (c \circ \sigma) \circ \tilde{\tau} = \operatorname{id}_{(\mathcal{F}^\vee)^\vee}$$

because these equalities hold on U. This shows that $c \circ \sigma$ is an isomorphism, in particular σ is a monomorphism and c is an epimorphism. Since c is also a monomorphism, it follows that c is an isomorphism, so $\sigma = c^{-1} \circ (c \circ \sigma)$ will also be an isomorphism.

For the second statement, recall that the canonical map $(R^1\pi_*(\mathcal{E}))(x) \to H^1(\mathcal{E}_x)$ is an isomorphism for every $x \in B$, hence the map $x \mapsto h^1(\mathcal{E}_x)$ is constant on B (and coincides with k). Using the second assumption of the hypothesis and the Riemann-Roch theorem, we conclude that the map $x \mapsto h^0(\mathcal{E}_x)$ is constant on B, hence it vanishes identically. $\square$

Proposition 1.7 leads naturally to the question: if one assumes that the system s is *generically* linearly independent and the Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ is non-empty, what can be said about the set of points where s is linearly dependent? The answer is:

PROPOSITION 1.8. — *Let $\pi\colon M \to B$ be a proper, surjective holomorphic submersion with connected base B, connected surfaces as fibers, and $\mathcal{E}$ a holomorphic bundle on M such that*

(1) *The Brill-Noether locus $\mathrm{BN}_\pi(\mathcal{E})$ has codimension $\geqslant 2$ at every point.*
(2) *The map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant.*

Put $k := \mathrm{rk}(R^1\pi_*(\mathcal{E}))$ and assume that

$$s = (s_1, \ldots, s_k) \in H^0(B, R^1\pi_*(\mathcal{E}))^{\oplus k}$$

is a system of sections such that $s(x_0)$ is linearly independent in the fiber $(R^1\pi_*(\mathcal{E}))(x_0)$ for a point $x_0 \in B \smallsetminus \mathrm{BN}_\pi(\mathcal{E})$. Put

$$\mathcal{S} := \big\{ x \in B \smallsetminus \mathrm{BN}_\pi(\mathcal{E}) \mid s(x) \text{ is linearly dependent in } (R^1\pi_*(\mathcal{E}))(x) \big\}.$$

Then the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ is an effective divisor containing $\mathrm{BN}_\pi(\mathcal{E})$. In particular this divisor is non-empty if $\mathrm{BN}_\pi(\mathcal{E})$ is non-empty.

Proof. — Using the notations introduced in the proof of Proposition 1.7, regard the wedge product $\wedge\sigma := s_1 \wedge \cdots \wedge s_k$ as a section in $\wedge^k(\mathcal{F})$. Let

$$d\colon \wedge^k(\mathcal{F}) \to \big\{\wedge^k(\mathcal{F})^\vee\big\}^\vee = \det(\mathcal{F})$$

be the canonical embedding (see [20, Proposition 6.10 ch. V]) and D the vanishing divisor of the section $d \circ (\wedge\sigma)$ of this holomorphic line bundle. Note that $D \cap (B \smallsetminus \mathrm{BN}_\pi(\mathcal{E})) = \mathcal{S}$ which (taken into account that $\mathrm{BN}_\pi(\mathcal{E})$ has codimension $\geqslant 2$ at every point) implies $\overline{\mathcal{S}} = D$. In order to prove that $\mathrm{BN}_\pi(\mathcal{E}) \subset D$ we will show that $B \smallsetminus D \subset B \smallsetminus \mathrm{BN}_\pi(\mathcal{E})$. Let $x_0 \in B \smallsetminus D$ and let U be an open neighborhood of x_0 such that $U \cap D = \emptyset$. This implies $U \cap \mathcal{S} = \emptyset$, hence $s_{|U}$ satisfies the hypothesis of Proposition 1.7, which gives $\mathrm{BN}_\pi(\mathcal{E}) \cap U = \emptyset$, hence $x \notin \mathrm{BN}_\pi(\mathcal{E})$. $\qquad\square$

REMARK 1.9. — *Note that, in general, for a point $x_0 \in \mathrm{BN}_\pi(\mathcal{E})$, the system $s(x_0)$ can be linearly independent in $(R^1\pi_*(\mathcal{E}))(x_0)$ although, by Proposition 1.8, such a point belongs to the closure of the set $\mathcal{S}$ of points in $B \smallsetminus \mathrm{BN}_\pi(\mathcal{E})$ where $s(x)$ is linearly dependent. This shows that, in general, for a system of sections in a torsion free coherent sheaf, being fiberwise linearly dependent is not always a closed condition.*

Example. — Let $\mathcal{F}$ be the ideal sheaf $\mathcal{I}_0$ of the origin in $\mathbb{C}^2$, and let $s \in H^0(\mathcal{I}_0)$ be the section defined by the holomorphic function $(z_1, z_2) \mapsto z_1$. Then $s(0)$ is non-zero in the fiber $\mathcal{I}_0(0)$, although $s(z)$ vanishes in $\mathcal{I}_0(z)$ for every point $z \in \{0\} \times \mathbb{C}^*$.

In section 4 we will see that Proposition 1.8 has found interesting applications in studying families of class VII surfaces [7].

2. The sheaf $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ as inductive limit.

Let again B, M be complex manifolds, $\pi\colon M \to B$ a proper, surjective holomorphic submersion, and let $\mathcal{E}$ be a holomorphic bundle on M. For an effective divisor $D \subset B$ we put $\mathcal{D} := \pi^{-1}(D)$, which is an effective divisor of M. The restriction $\mathcal{E}_\mathcal{D}$ will be alternatively regarded either as a locally free sheaf on $\mathcal{D}$, or as a torsion coherent sheaf on M.

Recall that for a decomposition $D = D' + D''$ of an effective divisor D of B as sum of effective divisors we have an associated *decomposition exact sequence*

(2.1) $0 \to \mathcal{O}_{D'}(-D'') \to \mathcal{O}_D \to \mathcal{O}_{D''} \to 0$

induced by the restriction morphism $r_{DD''}\colon \mathcal{O}_D \to \mathcal{O}_{D''}$ and the obvious isomorphisms

$$\ker(r_{DD''}) = \mathcal{I}_{D''}\big/\mathcal{I}_D = \mathcal{I}_{D''}\big/\mathcal{I}_{D'}\mathcal{I}_{D''}$$
$$= \mathcal{O}(-D'')\big/\mathcal{I}_{D'}\mathcal{O}(-D'') = \mathcal{O}_{D'}(-D'').$$

LEMMA 2.1. — *Let $D = \sum_{i \in I} n_i D_i$ be an effective divisor of B decomposed as sum of irreducible components, such that $D_i \not\subset \mathrm{BN}_\pi(\mathcal{E})$ for every $i \in I$. Then $H^0(\mathcal{E}_\mathcal{D}) = 0$.*

Proof. — The statement is clear when D is irreducible, because a section in a holomorphic bundle over an irreducible space vanishes if it vanishes generically (as a map with values in the total space of the bundle). On the other hand if $D \not\subset \mathrm{BN}_\pi(\mathcal{E})$, any section of $\mathcal{E}_\mathcal{D}$ (regarded as holomorphic bundle over $\mathcal{D}$) will vanish on the non-empty Zariski open subset $\mathcal{D} \smallsetminus \pi^{-1}(\mathrm{BN}_\pi(\mathcal{E}))$ of $\mathcal{D}$.

For D reducible we use induction with respect to $n := \sum_{i \in I} n_i$. We choose $i_0 \in I$, we put $D'' := D - D_{i_0}$, and we use the short exact sequence of sheaves on M:

$$0 \to \mathcal{E}_{\mathcal{D}_{i_0}}(-\mathcal{D}'') \to \mathcal{E}_\mathcal{D} \to \mathcal{E}_{\mathcal{D}''} \to 0.$$

Noting that $\mathcal{E}(-\mathcal{D}'') = \mathcal{E} \otimes \pi^*(\mathcal{O}_B(-D''))$, we see that the two bundles have the same Brill-Noether locus over B, hence $H^0(\mathcal{E}_{\mathcal{D}_{i_0}}(-\mathcal{D}'')) = 0$, by the first step applied to the bundle $\mathcal{E}(-\mathcal{D}'')$ on M and the irreducible divisor D_{i_0}. On the other hand $H^0(\mathcal{E}_{\mathcal{D}''}) = 0$ by the induction assumption. $\qquad \square$

For an effective divisor $D \subset B$ we put

$$R_D^\pi := \pi_*(\mathcal{E}_D)(D) = \pi_*(\mathcal{E}_D(\mathcal{D})) \in \mathcal{C}oh(B).$$

DEFINITION 2.2. — *We denote by $\mathcal{D}iv(B)$ the small category associated with the ordered set of effective divisors of B. The morphisms in this category correspond to inclusions of effective divisors (regarded as complex subspaces of B).*

PROPOSITION 2.3. — *The assignment $D \mapsto R_D^\pi$ defines a functor*

$$R^\pi \colon \mathcal{D}iv(B) \to \mathcal{C}oh(B)$$

with the following properties.

(1) *The support of R_D^π is contained in D, in particular R_D^π is a torsion sheaf.*
(2) *For $D' \leqslant D$ the corresponding morphism $R_{D'D}^\pi \colon R_{D'}^\pi \to R_D^\pi$ is a sheaf monomorphism.*
(3) *R_D depends only on the part of D which is contained in $\mathrm{BN}_\pi(\mathcal{E})$, more precisely consider the decomposition $D = \sum_{i \in I} n_i D_i$ of D in irreducible components, let $I_0 \subset I$ the subset of indices i for which $D_i \subset \mathrm{BN}_\pi(\mathcal{E})$, and put $D^0 := \sum_{i \in I_0} n_i D_i$. Then $R_{D^0 D}^\pi$ is an isomorphism.*

Proof. — The first statement is obvious. For the second, putting $D'' := D - D'$ and tensorizing with $\mathcal{E}(\mathcal{D})$ the exact sequence

$$0 \to \mathcal{O}_{\mathcal{D}'}(-\mathcal{D}'') \to \mathcal{O}_{\mathcal{D}} \to \mathcal{O}_{\mathcal{D}''} \to 0,$$

associated with the decomposition $D = D' + D''$, we obtain the short exact sequence

$$0 \to \mathcal{E}_{\mathcal{D}'}(\mathcal{D}') \to \mathcal{E}_{\mathcal{D}}(\mathcal{D}) \to \mathcal{E}_{\mathcal{D}''}(\mathcal{D}) \to 0.$$

The injectivity of $R_{D'D}^\pi$ follows applying the left exact functor π_* to this short exact sequence.

To prove the third statement we put $D'' := D - D^0$, and we use the short exact sequence

$$0 \to \mathcal{E}_{\mathcal{D}}^0(\mathcal{D}^0) \to \mathcal{E}_{\mathcal{D}}(\mathcal{D}) \to \mathcal{E}_{\mathcal{D}''}(\mathcal{D}) \to 0$$

obtained similarly. On the other hand, using Lemma 2.1 we obtain

$$H^0(\pi^{-1}(U), \mathcal{E}_{\mathcal{D}''}(\mathcal{D})) = 0$$

for every open set $U \subset B$, because no irreducible component of $\mathcal{D}'' \cap U$ is contained in $\mathrm{BN}_\pi(\mathcal{E}(\mathcal{D}))$. □

COROLLARY 2.4. — *The natural morphism*

$$\varinjlim_{\substack{D \in \mathcal{D}iv(B) \\ |D| \subset \mathrm{BN}_\pi(\mathcal{E})}} R_D^\pi \to \varinjlim_{D \in \mathcal{D}iv(B)} R_D^\pi$$

is an isomorphism.

THEOREM 2.5. — *Let $\pi \colon M \to B$ be a proper holomorphic submersion with connected fibers, and $\mathcal{E}$ a locally free coherent sheaf on M such that $\pi_*(\mathcal{E}) = 0$. There exists a canonical isomorphism*

$$\varinjlim_{\substack{D \in \mathcal{D}iv(B) \\ |D| \subset \mathrm{BN}_\pi(\mathcal{E})}} R_D^\pi = \varinjlim_{D \in \mathcal{D}iv(B)} R_D^\pi \xrightarrow{\simeq \ \psi} \mathrm{Tors}(R^1\pi_*(\mathcal{E})).$$

In particular, the inductive limit on the left is a coherent sheaf supported on the divisorial part $D_\pi(\mathcal{E})$ of $\mathrm{BN}_\pi(\mathcal{E})$.

Proof. — For every $D \in \mathcal{D}iv(B)$ we define a morphism $\psi_D \colon R_D^\pi \to \mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ in the following way: the long exact sequence associated with the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(\mathcal{D}) \to \mathcal{E}_\mathcal{D}(\mathcal{D}) \to 0$$

and the left exact functor π_* begins with

$$0 \to \pi_*(\mathcal{E}_\mathcal{D}(\mathcal{D})) = R_D^\pi \xrightarrow{\partial_D^\pi} R^1\pi_*(\mathcal{E}) \xrightarrow{\alpha_D^\pi} R^1\pi_*(\mathcal{E}(\mathcal{D})) \to \cdots$$

Using the identification $R^1\pi_*(\mathcal{E}(\mathcal{D})) = R^1\pi_*(\mathcal{E})(D) = R^1\pi_*(\mathcal{E}) \otimes_{\mathcal{O}_B} \mathcal{O}_B(D)$ we see that α_D^π is given by multiplication with the canonical morphism $\sigma_D \colon \mathcal{O}_B \to \mathcal{O}_B(D)$. A local equation φ of D defined on an open set $U \subset B$ defines an isomorphism $\mathcal{O}_U(D \cap U) \simeq \mathcal{O}_U$, and via this isomorphism the restriction

$$\sigma_D^U \colon \mathcal{O}_U \to \mathcal{O}_U(D \cap U) = \mathcal{O}_U$$

of σ_D to U is given by multiplication with φ. This shows that ∂_D^π defines a monomorphism $\psi_D \colon R_D^\pi \to \mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ whose image is the annihilator (in $R^1\pi_*(\mathcal{E})$) of the ideal sheaf of D.

For D', $D \in \mathcal{D}iv(B)$ such that $D' \leqslant D$ we use the functoriality of the connecting operator ∂ with respect to morphisms of short exact sequences, and we get

$$\partial_D^\pi \circ R_{D'D}^\pi = \partial_{D'}^\pi,$$

i.e., $\psi_D \circ R^\pi_{D'D} = \psi_{D'}$. Using the universal property of the inductive limit we see that the system $(\psi_D)_{D \in \mathcal{D}iv(B)}$ induces a morphism

$$\psi \colon \varinjlim_{D \in \mathcal{D}iv(B)} R^\pi_D \to \mathrm{Tors}(R^1 \pi_*(\mathcal{E})),$$

which is injective, because all morphisms ψ_D are injective.

The surjectivity of ψ is more delicate: let $x \in B$ and χ an element in the stalk $R^1 \pi_*(\mathcal{E})_x$ which is a torsion element of this $\mathcal{O}_{B,x}$-module. Therefore there exists an open neighborhood U of x and a representative $h \in R^1 \pi_*(\mathcal{E})(U)$ of χ and $\varphi \in \mathcal{O}(U) \smallsetminus \{0\}$ such that $\varphi h = 0$ in $R^1 \pi_*(\mathcal{E})(U)$. Let $D_U := Z(\varphi) \in \mathcal{D}iv(U)$ be the effective divisor defined by φ and $\mathcal{D}_U = Z(\varphi \circ \pi_U)$ its pull-back to $M_U := \pi^{-1}(U)$ via $\pi_U := \pi_{|U}$. We have a short exact sequence

$$0 \to \mathcal{E}_{M_U} \xrightarrow{\varphi \circ \pi_U} \mathcal{E}_{M_U} \longrightarrow \mathcal{E}_{\mathcal{D}_U} = \mathcal{E}_{\mathcal{D}_U}(\mathcal{D}_U) \to 0.$$

Here we used the trivialization of $\mathcal{O}_{M_U}(\mathcal{D}_U)$ defined by $\varphi \circ \pi_U$. The long exact sequence associated with this exact sequence and the left exact functor $(\pi_U)_*$ contains the segment

$$0 \to (\pi_U)_*(\mathcal{E}_{\mathcal{D}_U}(\mathcal{D}_U)) = R^U_{D_U} \xrightarrow{\partial^U_{D_U}} R^1 \pi_*(\mathcal{E}_{M_U})$$

$$\xrightarrow{\alpha^U_{D_U}} R^1 \pi_*(\mathcal{E}_{M_U}(\mathcal{D}_U)) = R^1 \pi_*(\mathcal{E}_{M_U}),$$

in which $\alpha^U_{D_U}$ is given by multiplication with φ in the $\mathcal{O}_U$-module $R^1 \pi_*(\mathcal{E}_{M_U})$. Therefore, replacing U by a smaller open neighborhood of x if necessary, we conclude that h belongs to the image of $\partial^U_{D_U}$. The problem is that D_U will not necessarily extend to a divisor of B, so this does not prove the surjectivity of ψ yet.

On the other hand we know by Proposition 2.3 that $R^U_{D_U} = R^U_{D^0_U}$, where the support of D^0_U is contained in the maximal 1-codimensional analytic subset $D_{\pi_U}(\mathcal{E}_{M_U})$ of $\mathrm{BN}_{\pi_U}(\mathcal{E}_{M_U})$. The point is that $\mathrm{BN}_{\pi_U}(\mathcal{E}_{M_U}) = \mathrm{BN}_\pi(\mathcal{E}_M) \cap U$, so the similar relation $D_{\pi_U}(\mathcal{E}_{M_U}) = D_\pi(\mathcal{E}_M) \cap U$ holds for the maximal 1-codimensional analytic subsets of these analytic sets. Therefore there exists a divisor D whose support is contained in $D_\pi(\mathcal{E}_M)$ such that $D \cap U \geqslant D^0_U$. Note that in general one cannot obtain equality, because different irreducible components of $D_{\pi_U}(\mathcal{E}_{M_U})$ (which can appear with different multiplicities in the decomposition of D^0_U) might belong to the same irreducible component of $D_\pi(\mathcal{E}_M)$.

This shows that h belongs to the image of ψ_D, which proves the surjectivity of the morphism ψ. $\qquad\square$

COROLLARY 2.6. — *In the conditions of Theorem 2.5, suppose that B is compact. Then one has a canonical isomorphism*

$$\varinjlim_{\substack{D \in \mathcal{D}iv(B) \\ |D| \subset BN_\pi(\mathcal{E})}} H^0(\mathcal{E}_{\mathcal{D}}(\mathcal{D})) \xrightarrow{\Psi} H^0(\mathrm{Tors}(R^1\pi_*(\mathcal{E}))).$$

Proof. — It suffices to recall that the functor Γ commutes with inductive limits of sheaves on compact spaces ([16, Theorem 3.10.1, p. 162]) and that

$$H^0(B, \pi_*(\mathcal{E}_{\mathcal{D}}(\mathcal{D}))) = H^0(M, \mathcal{E}_{\mathcal{D}}(\mathcal{D})). \qquad \square$$

3. $\mathrm{Tors}(R^1\pi_*(\mathcal{E}))$ as $\mathcal{E}xt^1$ and $\mathcal{H}om$

Let A be an integral domain. For an A-module N we denote by $\mathrm{Ann}(N) \subset A$ the annihilator ideal of N. For an element $\alpha \in A$ we denote by $\mathrm{Tors}_\alpha N \subset N$ the annihilator submodule of α in N. In other words

$$\mathrm{Tors}_\alpha N := \ker(\alpha\,\mathrm{id}_N).$$

We have obviously

$$(3.1) \qquad\qquad \mathrm{Tors}\,N = \varinjlim_{\alpha \in A^*} \mathrm{Tors}_\alpha N,$$

where the set $A^* := A \smallsetminus \{0\}$ is ordered with respect to divisiblity, and the inductive limit is constructed using the family of inclusions $\iota_{\alpha\beta}\colon \mathrm{Tors}_\alpha N \hookrightarrow \mathrm{Tors}_\beta N$ associated with pairs $\alpha|\beta$. For an element $\alpha \in A^*$ the natural morphism

$$\mathrm{Hom}(N, A)\big/_{\alpha\,\mathrm{Hom}(N, A)} \to \mathrm{Hom}\left(N, {}^A\big/_{\alpha A}\right).$$

is injective, hence the left hand module can be identified with a submodule of the right hand module. The multiplication with an element $\gamma \in A^*$ defines an injective morphism

$$m_\gamma\colon {}^A\big/_{\alpha A} \to {}^A\big/_{\alpha\gamma A}$$

so an injective morphism

$$\mu_\gamma\colon \mathrm{Hom}\left(N, {}^A\big/_{\alpha A}\right) \to \mathrm{Hom}\left(N, {}^A\big/_{\alpha\gamma A}\right)$$

defined by composition with m_γ. One has

$$\mu_\gamma^{-1}\left(\mathrm{Hom}(N, A)\big/_{\alpha\gamma\,\mathrm{Hom}(N, A)}\right) = \mathrm{Hom}(N, A)\big/_{\alpha\,\mathrm{Hom}(N, A)}$$

so μ_γ induces an injective morphism

$$\eta_\gamma \colon \frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha \operatorname{Hom}(N,A)}\right\}}$$
$$\longrightarrow \frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha\gamma A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha\gamma \operatorname{Hom}(N,A)}\right\}}.$$

If $\alpha|\beta$ we put $m_{\alpha\beta} := m_{\beta/\alpha}$, $\mu_{\alpha\beta} := \mu_{\beta/\alpha}$, $\eta_{\alpha\beta} := \eta_{\beta/\alpha}$.

PROPOSITION 3.1. — *Let N be an A-module.*

(i) *For any $\alpha \in A^*$ one has a canonical isomorphism*

$$\operatorname{Tors}_\alpha(\operatorname{Ext}^1(N,A)) = \frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha \operatorname{Hom}(N,A)}\right\}}.$$

(ii) *One has a canonical isomorphism*

$$\varinjlim_{\alpha \in A^*}\left\{\frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha \operatorname{Hom}(N,A)}\right\}}\right\}$$
$$\xrightarrow{\simeq} \operatorname{Tors}\operatorname{Ext}^1(N,A),$$

where the inductive limit is constructed using the family of morphisms $(\eta_{\alpha\beta})_{\alpha|\beta}$.

(iii) *If N and $\operatorname{Tors}(N)$ are finitely generated, then $\operatorname{Ann}(\operatorname{Ext}^1(N,A)) \neq 0$, in particular $\operatorname{Ext}^1(N,A)$ is a torsion module, the inductive limit in (ii) computes the whole $\operatorname{Ext}^1(N,A)$ and it stabilizes, i.e., there exists $\alpha_0 \in A^*$ such that the canonical morphisms*

$$\frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha_0 A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha_0 \operatorname{Hom}(N,A)}\right\}}$$
$$\longrightarrow \varinjlim_{\alpha \in A^*}\left\{\frac{\operatorname{Hom}\left(N, {}^A\!/_{\alpha A}\right)}{\left\{{}^{\operatorname{Hom}(N,A)}\!/_{\alpha \operatorname{Hom}(N,A)}\right\}}\right\}$$
$$\longrightarrow \operatorname{Ext}^1(N,A)$$

are isomorphisms.

Proof.

(i) The long exact sequence associated with the short exact sequence

$$0 \to A \xrightarrow{\alpha\cdot} A \to {}^A\!/_{\alpha A} \to 0$$

and the functor $\mathrm{Hom}(N, \cdot)$ reads

$$0 \to \mathrm{Hom}(N, A) \xrightarrow{\alpha\cdot} \mathrm{Hom}(N, A) \to \mathrm{Hom}(N, {}^A/_{\alpha A})$$
$$\to \mathrm{Ext}^1(N, A) \xrightarrow{\alpha\cdot} \mathrm{Ext}^1(N, A),$$

which gives a canonical isomorphism

$$\mathrm{Hom}\left(N, {}^A/_{\alpha A}\right) \Big/ \left\{ {}^{\mathrm{Hom}(N, A)}/_{\alpha\, \mathrm{Hom}(N, A)} \right\} \xrightarrow{\cong} \mathrm{Tors}_\alpha \mathrm{Ext}^1(N, A).$$

It remains to prove that via these identifications the morphism $\eta_{\alpha\beta}$ corresponds to the inclusion morphism

$$\iota_{\alpha\beta} \colon \mathrm{Tors}_\alpha \mathrm{Ext}^1(N, A) \to \mathrm{Tors}_\beta \mathrm{Ext}^1(N, A)$$

associated with a pair $\alpha|\beta$. Using the morphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha\cdot} & A & \longrightarrow & {}^A/_{\alpha A} & \longrightarrow & 0 \\
 & & \mathrm{id}_A \downarrow & & \gamma\cdot \downarrow & & \gamma\cdot \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\beta\cdot} & A & \longrightarrow & {}^A/_{\beta A} & \longrightarrow & 0
\end{array},
$$

we get the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}(N, {}^A/_{\alpha A}) & \longrightarrow & \mathrm{Ext}^1(N, A) & \xrightarrow{\alpha\cdot} & \mathrm{Ext}^1(N, A) \\
\eta_{\alpha\beta} \downarrow & & \mathrm{id} \downarrow & & \gamma\cdot \downarrow \\
\mathrm{Hom}(N, {}^A/_{\beta A}) & \longrightarrow & \mathrm{Ext}^1(N, A) & \xrightarrow{\beta\cdot} & \mathrm{Ext}^1(N, A)
\end{array},
$$

which proves the claim.

(ii) Follows directly from (i) taking into account (3.1).

(iii) Let N_0 be the quotient $N_0 := N/\mathrm{Tors}(N)$, which is torsion free. Since N_0 is a finitely generated torsion free module over the integral ring A, it can be embedded in a finitely generated free module F such that $\mathrm{Ann}(F/N_0) \neq 0$. We recall briefly the argument. Let K be the field of fractions of A and $\iota \colon N_0 \to V_0 := K \otimes_A N_0$ the canonical embedding. We identify N_0 with its image $\iota(N_0)$, and we fix a finite A-generating set G of N_0 (which will also be a K-generating set of V_0) and a system $(g_1, \ldots, g_n)$ of G which is a K-basis of V_0. Developing all generators $g \in G$ with respect to this basis, we see that there exists $\alpha \in A^*$ such that $N_0 \subset \frac{1}{\alpha}(Ag_1 \oplus \cdots \oplus Ag_n)$. The A-module $F := \frac{1}{\alpha}(Ag_1 \oplus \cdots \oplus Ag_n)$ is obviously free and $\alpha F \subset N_0$. Therefore $\alpha(F/N_0) = 0$, hence $\alpha \in \mathrm{Ann}(F/N_0)$.

The short exact sequence

$$0 \to N_0 \to F \to F/N_0 \to 0$$

yields an isomorphism $\operatorname{Ext}^1(N_0, A) \simeq \operatorname{Ext}^2((F/N_0), A)$. Using [4, Corollary to Proposition 6, p. AX 89, n°4], we obtain

$$\{\alpha\} \subset \operatorname{Ann}(F/N_0) \subset \operatorname{Ann}(\operatorname{Ext}^1(N_0, A)).$$

Therefore $\operatorname{Ann}(\operatorname{Ext}^1(N_0, A)) \neq 0$. On the other hand the exact sequence

$$0 \to \operatorname{Tors}(N) \to N \to N_0 \to 0,$$

yields the exact sequence

$$0 \to \operatorname{Ext}^1(N_0, A) \to \operatorname{Ext}^1(N, A) \to \operatorname{Ext}^1(\operatorname{Tors}(N), A),$$

which shows that

$$\operatorname{Ann}(\operatorname{Ext}^1(N_0, A)) \operatorname{Ann}(\operatorname{Ext}^1(\operatorname{Tors}(N), A)) \subset \operatorname{Ann}(\operatorname{Ext}^1(N, A)).$$

We know that $\operatorname{Ann}(\operatorname{Ext}^1(N_0, A)) \neq 0$ and, since $\operatorname{Tors}(N)$ is also assumed finitely generated, we have $\operatorname{Ann}(\operatorname{Tors}(N)) \neq 0$, which implies

$$\operatorname{Ann}(\operatorname{Ext}^1(\operatorname{Tors}(N), A)) \neq 0$$

(again by [4, Corollary to Proposition 6, p. AX 89, n°4]).

Therefore, the annihilator $\operatorname{Ann}(\operatorname{Ext}^1(N, A))$ contains a non-zero element α_0, hence

$$\operatorname{Ext}^1(N, A) = \operatorname{Tors}(\operatorname{Ext}^1(N, A)) = \operatorname{Tors}_{\alpha_0}(\operatorname{Ext}^1(N, A)),$$

which can be identified with $\operatorname{Hom}\left(N, {A}/{\alpha_0 A}\right)/\left\{{\operatorname{Hom}(N, A)}/{\alpha_0 \operatorname{Hom}(N, A)}\right\}$ by (i). $\qquad\square$

We will see that in the case when A is a unique factorization domain, N is a finitely generated torsion module, and $\operatorname{Ann}(N)$ is a finitely generated ideal, there is an explicit, canonical choice of an element $\alpha_0 \in A^*$ which computes $\operatorname{Ext}^1(N, A)$ as in Proposition 3.1 (iii).

Suppose that A is a unique factorization domain and that N is a finitely generated torsion module. This implies that the annihilator ideal $\operatorname{Ann}(N)$ does not vanish. Suppose also that $\operatorname{Ann}(N)$ is finitely generated. This condition holds automatically if we assume A to be Noetherian. We can define $a := \gcd(\operatorname{Ann}(N)) \in A^*$. In other words one has $a = \gcd(\varphi_1, \ldots, \varphi_k)$, where (φ_i) is a system of generators of $\operatorname{Ann}(N)$. For any $\beta \in A^*$ we put $(a, \beta) := \gcd(a, \beta)$.

LEMMA 3.2. — *Suppose that A is a unique factorization domain, N is a finitely generated torsion module, and $\mathrm{Ann}(N)$ is finitely generated. For any $\beta \in A^*$ one has*

$$\mathrm{Hom}(N, {}^A\!/_{\beta A}) = \mu_{(a,\beta)\beta}\left(\mathrm{Hom}(N, {}^A\!/_{(a,\beta)A})\right)$$

Proof. — The inclusion

$$\mu_{(a,\beta)\beta}\left(\mathrm{Hom}(N, {}^A\!/_{(a,\beta)A})\right) \subset \mathrm{Hom}(N, {}^A\!/_{\beta A})$$

is obvious and has been used before. For the other inclusion let $f \in \mathrm{Hom}(N, {}^A\!/_{\beta A})$. Denote by $\mathcal{J}$ the ideal of A defined as the pre-image in A of $f(N)$. One has

$$(3.2) \qquad\qquad \mathrm{Ann}(N)\mathcal{J} \subset \beta A \subset \mathcal{J},$$

where the first inclusion follows from $\mathrm{Ann}(N)f(N) = 0$. We claim that

$$(3.3) \qquad\qquad \mathcal{J} \subset (\beta/(a,\beta))A.$$

Indeed, for a prime element $p \in A$ let p^m be its maximal power which divides all generators φ_i of $\mathrm{Ann}(N)$. In other words, p^m divides φ_i for any $i \in \{1, \ldots, k\}$ and there exists $i_0 \in \{1, \ldots, k\}$ such that p^m is the maximal power of p which divides φ_{i_0}. Let p^n be the maximal power of p which divides β. Using (3.2) we get for every $v \in \mathcal{J}$ a relation of the type

$$v \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_k \end{pmatrix} = \beta \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}$$

with $\alpha_i \in A$. The i_0-th equality shows that whenever $n > m$ we must have $p^{n-m}|v$, hence the inclusion (3.3) is proved. We can complete now the chain of inclusions (3.2) to

$$(3.4) \qquad\qquad \mathrm{Ann}(N)\mathcal{J} \subset \beta A \subset \mathcal{J} \subset (\beta/(a,\beta))A.$$

Taking quotients with respect to the ideal βA of the last two A-modules we get

$$f(N) \subset {}^{(\beta/(a,\beta))A}\!/_{\beta A} = m_{(a,\beta)\beta}\left({}^A\!/_{(a,\beta)A}\right).$$

Since $m_{(a,\beta)\beta}$ is a monomorphism this proves that f can be written as $m_{(a,\beta)\beta} \circ g$ for a morphism $g \in \mathrm{Hom}(N, {}^A\!/_{(a,\beta)A})$, which completes the proof. $\qquad\square$

PROPOSITION 3.3. — *Suppose that A is a unique factorization domain, N is a finitely generated torsion module, and $\mathrm{Ann}(N)$ is finitely generated. Put $a := \gcd(\mathrm{Ann}(N))$.*

(i) *The natural morphism*

$$c\colon \operatorname{Hom}\left(N, {}^{A}\!/_{aA}\right) \to \varinjlim_{\alpha \in A^*} \operatorname{Hom}\left(N, {}^{A}\!/_{\alpha A}\right)$$

is an isomorphism.

(ii) *One has a canonical isomorphism*

$$\operatorname{Hom}\left(N, {}^{A}\!/_{aA}\right) \xrightarrow{\simeq} \operatorname{Ext}^1(N, A).$$

Proof. — Since the morphisms $\mu_{\alpha\beta}$ which intervene in the inductive limit are injective, it follows that c is injective, so it suffices to prove that c is surjective.

Let $\beta \in A^*$ and $f \in \operatorname{Hom}\left(N, {}^{A}\!/_{\beta A}\right)$. By Lemma 3.2 there exists

$$g \in \operatorname{Hom}(N, {}^{A}\!/_{(a, \beta)A})$$

such that $f = \mu_{(a,\beta)\beta}(g)$. Noting that $(a, \beta)|a$, we see that the three elements

$$g \in \operatorname{Hom}(N, {}^{A}\!/_{(a, \beta)A}),\ \mu_{(a,\beta)a}(g) \in \operatorname{Hom}\left(N, {}^{A}\!/_{aA}\right),\ f = \mu_{(a,\beta)\beta}(g)$$

define the same element in the inductive limit. Therefore $\mu_{(a,\beta)a}(g)$ is a pre-image (with respect to c) of the element defined by f in the inductive limit. $\qquad\square$

[1, Corollary 4.11 p. 133] states:

PROPOSITION 3.4. — *Let $\pi\colon M \to B$ be a proper morphism of complex spaces and $\mathcal{E}$ a coherent sheaf on M which is flat over B. Then there exists a coherent sheaf $\mathcal{T}$ on B (unique up to isomorphism) with the following property: for every coherent sheaf $\mathcal{A}$ on an open set $U \subset B$ there exists an isomorphism*

$$(\pi_U)_*(\mathcal{E}_U \otimes \pi_U^*(\mathcal{A})) \xrightarrow{\simeq} \mathcal{H}om(\mathcal{T}_U, \mathcal{A})$$

which is functorial with respect to $\mathcal{A}$.

In this statement $\mathcal{E}_U$, π_U denote the restrictions of $\mathcal{E}$ and π to $\pi^{-1}(U)$, and $\mathcal{T}_U$ the restriction of $\mathcal{T}$ to U.

We will see that the torsion sheaf $\operatorname{Tors}(R^1\pi_*(\mathcal{E}))$ in which we are interested has a simple description in terms of the associated sheaf $\mathcal{T}$. Before stating our result we recall that for any complex manifold X and coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ (which is non-trivial on any connected component) there exists a minimal locally principal ideal sheaf $\gcd(\mathcal{I})$ containing $\mathcal{I}$. At a point $x \in X$ the stalk $\gcd(\mathcal{I})_x$ is generated by $a := \gcd(\varphi^1, \ldots, \varphi^k)$, where $(\varphi^1, \ldots, \varphi^k)$ is a system of generators of the stalk $\mathcal{I}_x$. The definition

is coherent because, choosing holomorphic functions $\tilde{a}$, $\tilde{\varphi}^i$ representing the germs a, φ^i, we have $\tilde{a}_u = \gcd(\tilde{\varphi}_u^1, \ldots, \tilde{\varphi}_u^k)$ for every $u \in X$ which is sufficiently close to x (see [5, Theorem 2.11, p. 93]).

The effective divisor $D_{\max}(Z)$ corresponding to $\gcd(\mathcal{I})$ is the maximal effective divisor contained in the complex space Z defined by $\mathcal{I}$. This maximal divisor might be of course empty (when $\gcd(\mathcal{I}) = \mathcal{O}_X$).

I am grateful to Daniel Barlet for explaining me interesting geometric constructions of $D_{\max}(Z)$. For instance, this divisor can be obtained using the blow up $p \colon \hat{X} \to X$ of X along Z and defining $D_{\max}(Z)$ as image (as analytic cycle) of the divisor on $\hat{X}$ corresponding to the locally principal sheaf of ideals $p^*(\mathcal{I})/\operatorname{Tors} \subset \mathcal{O}_{\hat{X}}$.

THEOREM 3.5. — *Let $\pi \colon M \to B$ be a proper morphism of complex spaces and $\mathcal{E}$ a coherent sheaf on M which is flat over B. Suppose that B is locally irreducible, and let $\mathcal{T}$ be the associated coherent sheaf on B given by Proposition 3.4. Then*

(i) *One has canonical isomorphisms*
$$\mathcal{E}xt^1(\mathcal{T}, \mathcal{O}_B) \xrightarrow{\simeq} \operatorname{Tors}(R^1\pi_*(\mathcal{E})).$$

(ii) *If $\pi_*(\mathcal{E}) = 0$ and B is smooth, then*
   (a) *$\mathcal{T}$ is a torsion sheaf,*
   (b) *Denoting by $D_{\max}$ the maximal divisor contained in the complex subspace defined by $\operatorname{Ann}(\mathcal{T})$, one has canonical isomorphisms*
$$\mathcal{H}om(\mathcal{T}_{D_{\max}}, \mathcal{O}_{D_{\max}}) \xrightarrow{\simeq} \mathcal{H}om(\mathcal{T}, \mathcal{O}_{D_{\max}})$$
$$\xrightarrow{\simeq} \operatorname{Ext}^1(\mathcal{T}, \mathcal{O}_B) \xrightarrow{\simeq} \operatorname{Tors}(R^1\pi_*(\mathcal{E})).$$

Proof.

(i) Let $U \subset B$ be a connected open set and $\varphi \in \mathcal{O}(U) \smallsetminus \{0\}$. We put $\Phi := \varphi \circ \pi$, and we denote by $D_\varphi \subset U$, $D_\Phi \subset \pi^{-1}(U)$ the effective divisiors defined by φ and Φ respectively. Since $\mathcal{E}$ is flat over B, the exact sequence
$$0 \to \mathcal{O}_U \xrightarrow{\varphi\cdot} \mathcal{O}_U \to \mathcal{O}_{D_\varphi} \to 0$$
on B gives an exact sequence
$$0 \to \mathcal{E}_U \xrightarrow{\Phi\cdot} \mathcal{E}_U \to \mathcal{E}_{D_\Phi} \to 0$$
of coherent sheaves on M (see [1, p. 172]). The corresponding long exact sequence on U is
$$0 \to (\pi_U)_*(\mathcal{E}_U) \xrightarrow{\varphi\cdot} (\pi_U)_*(\mathcal{E}_U) \to (\pi_U)_*(\mathcal{E}_{D_\Phi})$$
$$\to R^1(\pi_U)_*(\mathcal{E}_U) \xrightarrow{m_\varphi} R^1(\pi_U)_*(\mathcal{E}_U),$$

which gives a canonical isomorphism

$$(3.5) \qquad \left.(\pi_U)_*(\mathcal{E}_{D_\Phi})\middle/{}^{(\pi_U)_*(\mathcal{E}_U)}\big/{}_{\varphi(\pi_U)_*(\mathcal{E}_U)}\right. \xrightarrow{\;\simeq I_\varphi\;} \ker(m_\varphi),$$

which composed with the identifications

$$(\pi_U)_*(\mathcal{E}_U) = \mathcal{H}om(\mathcal{T}_U, \mathcal{O}_U), \;\; (\pi_U)_*(\mathcal{E}_{D_\Phi}) = \mathcal{H}om(\mathcal{T}_U, \mathcal{O}_{D_\varphi}),$$

given by Proposition 3.4 gives a canonical isomorphism

$$(3.6) \qquad \left.\mathcal{H}om(\mathcal{T}_U, \mathcal{O}_{D_\varphi})\middle/{}^{\mathcal{H}om(\mathcal{T}_U, \mathcal{O}_U)}\big/{}_{\varphi\mathcal{H}om(\mathcal{T}_U, \mathcal{O}_U)}\right. \xrightarrow{\;\simeq J_\varphi\;} \ker(m_\varphi).$$

For a product $\psi = \varphi\xi$ we get a morphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_{\pi^{-1}(U)} & \xrightarrow{\Phi\cdot} & \mathcal{E}_{\pi^{-1}(U)} & \longrightarrow & \mathcal{E}_{D_\Phi} & \longrightarrow & 0 \\
 & & {\scriptstyle\text{id}}\downarrow & & {\scriptstyle\Xi\cdot}\downarrow & & {\scriptstyle\Xi\cdot}\downarrow & & \\
0 & \longrightarrow & \mathcal{E}_{\pi^{-1}(U)} & \xrightarrow{\Psi\cdot} & \mathcal{E}_{\pi^{-1}(U)} & \longrightarrow & \mathcal{E}_{D_\Psi} & \longrightarrow & 0
\end{array},
$$

which gives the commutative diagram

$$
\begin{array}{ccccc}
(\pi_U)_*(\mathcal{E}_{D_\Phi}) & \longrightarrow & R^1(\pi_U)_*(\mathcal{E}_U) & \xrightarrow{\;m_\varphi\;} & R^1(\pi_U)_*(\mathcal{E}_U) \\
{\scriptstyle\xi\cdot}\downarrow & & {\scriptstyle\text{id}}\downarrow & & {\scriptstyle m_\xi}\downarrow \\
(\pi_U)_*(\mathcal{E}_{D_\Psi}) & \longrightarrow & R^1(\pi_U)_*(\mathcal{E}_U) & \xrightarrow{\;m_\psi\;} & R^1(\pi_U)_*(\mathcal{E}_U)
\end{array}.
$$

This shows that, via the isomorphisms I_φ, I_ψ (and also J_φ, J_ψ), the obvious inclusion $\ker(m_\varphi) \subset \ker(m_\psi)$ corresponds to the morphism induced by multiplication with ξ.

For a fixed point $x \in B$ the same results will hold if we use stalks at x instead of sheaves on U and germs $\varphi, \psi, \xi \in \mathcal{O}^*_{B,x}$ instead of non-zero holomorphic functions. Therefore we obtain a canonical isomorphism

$$\varinjlim_{\varphi \in \mathcal{O}^*_{B,x}} \left.\mathcal{H}om(\mathcal{T}_x, \mathcal{O}_{D_\varphi,x})\middle/{}^{\mathcal{H}om(\mathcal{T}_x, \mathcal{O}_x)}\big/{}_{\varphi\mathcal{H}om(\mathcal{T}_x, \mathcal{O}_x)}\right. \xrightarrow{\;\simeq J^x\;} \mathrm{Tors}(R^1\pi_*(\mathcal{E}))_x.$$

The claim follows now by Proposition 3.1 (iii).

(ii) Suppose now that $\pi_*(\mathcal{E}) = 0$. Using Proposition 3.4 this gives $\mathcal{H}om(\mathcal{T}, \mathcal{O}_B) = 0$, hence $\mathcal{T}$ is a sheaf of rank 0, in other words it is a torsion sheaf. The claim follows by (i) and Proposition 3.3. $\qquad\square$

4. Applications and examples

4.1. Parametrizing moduli stacks of complex manifolds

Let S be a fixed compact, connected, oriented smooth manifold, and let $\mathcal{M}_S$ be the moduli set of biholomorphism classes of complex manifolds which are diffeomorphic to S. This moduli set has a natural topology obtained by identifying it with the quotient $\mathcal{J}/\operatorname{Diff}(M)$ of the space $\mathcal{J}$ of integrable complex structures on S by the group $\operatorname{Diff}(M)$ of diffeomorphisms of S. It is well known that, even in simple cases, this quotient topology is highly non-Hausdorff.

In general the moduli space $\mathcal{M}_S$ is not a complex space, but can be naturally regarded as a complex analytic stack, *i.e.,* a "category fibered in groupoids" over the category $\mathcal{C}$ of complex spaces (endowed with the usual Grothendieck topology) satisfying the two standard conditions: the isomorphisms form a sheaf and every descent datum is effective (see [13, Definition 4.3]). The "fiber" of the stack $\mathcal{M}_S$ over a complex space B is just the groupoid of flat holomorphic families $\mathcal{X} \to B$ whose fibers are complex manifolds diffeomorphic to S. Intuitively, such a family can be interpreted as a holomorphic morphism $B \to \mathcal{M}_S$. A detailed construction of the moduli stack $\mathcal{M}_S$ can be found in [23].

We will concentrate on the open moduli substack $\mathcal{M}_S^{\mathrm{reg}} \subset \mathcal{M}_S$ of complex manifolds X diffeomorphic to S and having $H^2(\Theta_X) = 0$. The versal deformation of a manifold X with $H^2(\Theta_X) = 0$ is smooth, so the complex analytic stack $\mathcal{M}_S^{\mathrm{reg}}$ is locally isomorphic to the quotient stack of a complex manifold by a groupoid complex space (see [18, sect. 3] for the analogous concept in the algebraic geometric framework). An interesting example of such an analytic stack (obtained as quotient of a smooth complex manifold) is described below; it can be interpreted as the moduli stack classifying a family of contracting germs, so a family of minimal class VII surfaces (see [27, p. 341, the example at the end of sect. 7]):

Example. — Consider the action of $\alpha\colon \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\alpha(\zeta, (z_1, z_2)) := (z_1, z_2 + z_1\zeta).$$

The map $p\colon \mathbb{C}^2/\alpha \to \mathbb{C}$ given by $[(z_1, z_2)] \to z_1$ is surjective, its fiber over any point $z \in \mathbb{C}^*$ is a point, whereas the fiber over 0 is the line $\{[(0, u)] \mid u \in \mathbb{C}\}$. The topological quotient $\mathbb{C}^2/\alpha$ can be intuitively thought of as a complex line with a 1-parameter family of mutually non-separable "origins" $0_u := [(0, u)]$. Note that the dimension of the formal Zariski tangent space of the quotient stack $\mathbb{C}^2/\alpha$ at $[(z_1, z_2)]$ is 1 for $z_1 \neq 0$ and

2 for $z_1 = 0$. Therefore, as happens with many moduli stacks of the form $\mathcal{M}_S^{\mathrm{reg}}$, the dimension of the Zariski tangent spaces is non-constant (it "jumps"). Consider the family of holomorphic maps $f_u \colon \mathbb{C} \to \mathbb{C}^2$ given by $f_u(z) = (z, u)$ and note that the images of the induced maps $g_u \colon \mathbb{C} \to \mathbb{C}^2/\alpha$ cover the quotient $\mathbb{C}^2/\alpha$. Any g_u is injective on whole $\mathbb{C}$ and étale on $\mathbb{C}^*$. The stack $\mathbb{C}^2/\alpha$ can be reconstructed using this family of "parametrizations" and studying how their images glue together in the stack. Indeed, in this way we see that $\mathbb{C}^2/\alpha$ can be obtained from a 1-parameter family $(\mathbb{C}_u)_{u \in \mathbb{C}}$ of copies of $\mathbb{C}$ by identifying all punctured lines $\mathbb{C}_u^*$ to $\mathbb{C}^*$ in the obvious way. This example shows that for understanding geometrically a moduli stack $\mathcal{M}_S^{\mathrm{reg}}$ is important to study the following

PROBLEM. — *Classify generically versal holomorphic families $\pi \colon \mathcal{X} \to B$ with smooth base B, whose fibers X_b are connected complex manifolds diffeomorphic to S and have $H^2(\Theta_{X_b}) = 0$. For such a family describe explicitly the analytic subset $B_{\mathrm{nv}} \subset B$ of points where π is non-versal.*

The notion of "versality" used here follows the terminology introduced in [2, section I.10], hence it corresponds to "semi-universality" in the terminology of [9], [3] and [15].

In order to explain in detail the condition *generically versal* used above, consider a proper holomorphic submersion (with connected base and connected fibers) $\pi \colon \mathcal{X} \to B$, denote by $\Theta_{\mathcal{X}}^v$ the vertical tangent subbundle of the tangent bundle $\Theta_{\mathcal{X}}$. As usually we identify holomorphic vector bundles with the associated locally free coherent sheaves. The long exact sequence associated with the short exact sequence

$$0 \to \Theta_{\mathcal{X}}^v \to \Theta_{\mathcal{X}} \xrightarrow{\pi_*} \pi^*(\Theta_B) \to 0,$$

of sheaves on $\mathcal{X}$ reads

$$\cdots \to R^0 \pi_* \pi^*(\Theta_B) = \Theta_B \xrightarrow{\delta} R^1 \pi_*(\Theta_{\mathcal{X}}^v) \to R^1 \pi_*(\Theta_{\mathcal{X}}) \to \cdots.$$

We denote by

$$\delta_b \colon (\Theta_B)_b \to R^1 \pi_*(\Theta_{\mathcal{X}}^v)_b, \quad \delta(b) \colon \Theta_B(b) \to R^1 \pi_*(\Theta_{\mathcal{X}}^v)(b)$$

the morphisms induced by δ between the stalks, respectively the fibers of the two sheaves at a point $b \in B$. The Kodaira-Spencer map $\rho_b \colon \Theta_B(b) \to H^1(X_b, \Theta_b)$ can be written as

$$\rho_b = c_b \circ \delta(b),$$

where c_b denotes the canonical map $R^1 \pi_*(\Theta_{\mathcal{X}}^v)(b) \to H^1(X_b, \Theta_{X_b})$ (see [1, p. 112]).

PROPOSITION 4.1. — *Let $\pi\colon \mathcal{X} \to B$ be a proper holomorphic submersion with connected base and connected fibers. Suppose that there exists $b_0 \in B$ such that ρ_{b_0} is an isomorphism and the map $b \mapsto h^1(\Theta_{X_b})$ is constant on an open neighborhood U of b_0. For a point $b \in B$ the following conditions are equivalent:*

(1) *ρ_b is bijective,*
(2) *ρ_b is surjective,*
(3) *$h^1(\Theta_{X_b}) = \dim(B)$ and δ_b is an isomorphism of $\mathcal{O}_{B,b}$-modules.*
(4) *$h^1(\Theta_{X_b}) = \dim(B)$ and δ_b is an epimorphism of $\mathcal{O}_{B,b}$-modules.*

Proof. — (1)$\Rightarrow$(2) is obvious. Put $n := \dim(B)$. Using the hypothesis and the semicontinuity theorem [1], [15], [10], it follows that n is the minimal value of this map on B, hence the set

$$B_0 := \{b \in B \mid h^1(\Theta_{X_b}) = n\}$$

(which obviously contains U) is a non-empty Zariski open set. Suppose that (2) holds. The surjectivity of ρ_b implies $h^1(\Theta_{X_b}) \leqslant n$, hence in fact $h^1(\Theta_{X_b}) = n$. Therefore $b \in B_0$ and ρ_b is bijective. On the other hand $\Theta_{\mathcal{X}}^v$ is cohomologically flat in dimension 1 over B_0, hence $R^1\pi_*(\Theta_{\mathcal{X}}^v)$ is locally trivial of rank n on B_0 and the canonical map c_b is an isomorphism for every $b \in B_0$ (see [1, p. 134]). This shows that, for $b \in B_0$, the bijectivity of ρ_b is equivalent to the bijectivity of $\delta(b)$. On the other hand, for $b \in B_0$ the $\mathcal{O}_{B,b}$-modules $(\Theta_B)_b$, $R^1\pi_*(\Theta_{\mathcal{X}}^v)_b$ are free of rank n, hence

$\delta(b)$ is bijective $\Leftrightarrow \det(\delta(b)) \neq 0 \Leftrightarrow \det(\delta_b) \in \mathcal{O}_{B,b}^* \Leftrightarrow \delta_b$ is invertible.

Therefore (2)$\Rightarrow$(3). Obviously (3)$\Rightarrow$(4). On the other hand

$$\delta_b \text{ is surjective } \Rightarrow \delta(b) \text{ is surjective },$$

which, for $b \in B_0$, implies $\delta(b)$ is bijective. As above this implies that δ_b is invertible. This shows (4)$\Rightarrow$(3). Finally, if (3) holds then $\delta(b)$ is an isomorphism which, for $b \in B_0$, implies that $\rho_b = c_b \circ \delta(b)$ is an isomorphism. $\square$

A holomorphic family which satisfies the assumptions of Proposition 4.1 will be called *generically versal*. The terminology is justified by the following corollary, which can be regarded as a special case of the classical Zariski openness of versality [3], [9], [15]. Note however that in the terminology of these authors the versality condition is weaker than ours, so the corollary below is not formally a particular case of this classical result.

COROLLARY 4.2. — *Let $\pi\colon \mathcal{X} \to B$ be a proper holomorphic submersion with connected base and connected fibers such that there exists $b_0 \in B$*

for which the Kodaira-Spencer map ρ_{b_0} is an isomorphism, and the map $b \mapsto h^1(\Theta_{X_b})$ is constant on an open neighborhood U of b_0. Then the set

$$B_{\mathrm{v}} := \big\{ b \in B \mid \pi \text{ is versal at } b \big\}$$

is Zariski open in B.

Proof. — First, using the definition of versality and Kodaira-Spencer completeness theorem [21] (which applies because B is smooth) we see that π is versal at b if and only if ρ_b is an isomorphism. With the notations introduced in the proof of Proposition 4.1 and using this remark we obtain

$$(4.1) \qquad B_{\mathrm{v}} = B_0 \smallsetminus \mathrm{supp}\left({R^1 \pi_*(\Theta_{\mathcal{X}}^{\mathrm{v}})} \big/ {\delta(\Theta_B)} \right),$$

which proves that B_{v} is Zariski open. $\qquad\qquad\square$

Using Proposition 1.8 we obtain interesting general properties of generically versal holomorphic families.

Proposition 4.3. — Let $\pi\colon \mathcal{X} \to B$ be a generically versal holomorphic submersion with connected surfaces as fibers such that the map $B \ni x \mapsto h^2(\Theta_{X_x})$ is constant on B. Then the analytic subset $B_{\mathrm{nv}} \subset B$ of points where π is non-versal is an effective divisor containing BN_π.

Proof. — Using Riemann-Roch theorem fiberwise and the hypothesis, we get $B_0 = B \smallsetminus \mathrm{BN}_\pi$. Taking into account formula (4.1) it follows $\mathrm{BN}_\pi \subset B_{\mathrm{nv}}$. It remains to prove that B_{nv} is pure 1-codimensional at any point.

If not, there would exist a regular point $x \in B_{\mathrm{nv}}$ such that $\mathrm{codim}_x(B_{\mathrm{nv}}) \geqslant 2$. Then $x \in \mathrm{BN}_\pi$, because the intersection $B_{\mathrm{nv}} \cap (B \smallsetminus \mathrm{BN}_\pi)$ is obviously an effective divisor in the open set $B \smallsetminus \mathrm{BN}_\pi$: it coincides with the divisor of points where the Kodaira-Spencer map (which on this set is just a morphism of holomorphic vector bundles of the same rank) is degenerate.

Since we supposed $\mathrm{codim}_x(B_{\mathrm{nv}}) \geqslant 2$ and we have proved $\mathrm{BN}_\pi \subset B_{\mathrm{nv}}$, we obtain $\mathrm{codim}_x(\mathrm{BN}_\pi) \geqslant 2$. Choose a local chart $h\colon U \xrightarrow{\simeq} V \subset \mathbb{C}^n$ around x. Applying Proposition 1.8 to the vertical tangent bundle $\Theta_{\mathcal{X}_U}^v$ and the sheaf morphism

$$\delta\colon \Theta_U = \mathcal{O}_U^{\oplus n} \to R^1(\pi_U)_*(\Theta_{\mathcal{X}_U}^v)$$

(where $\pi_U\colon \mathcal{X}_U \to U$ stands for the restriction of π to $\mathcal{X}_U := \pi^{-1}(U)$), it follows that B_{nv} is a divisor at x, which contradicts our assumption. $\quad\square$

Corollary 4.4. — With the notations and conditions of Proposition 4.3:

(1) For a point $x \in BN_\pi$ with $\mathrm{codim}_x(\mathrm{BN}_\pi) \geqslant 2$, the intersection

$$U \cap (B_{\mathrm{nv}} \smallsetminus BN_\pi)$$

is non-empty, for every open neighborhood U of x.

(2) *The map $B \to \mathcal{M}_S$ induced by π is non-injective around any point*

$$x \in B_{\mathrm{nv}} \setminus BN_\pi.$$

More precisely, let $x_0 \in B_{\mathrm{nv}} \setminus BN_\pi$, X_0 the corresponding fiber, $\mathfrak{u}_0$ its universal deformation (which exists because $h^0(\Theta_{X_0}) = 0$), and

$$f_\pi^{x_0} : (B, x_0) \to \mathfrak{u}_0$$

the morphism of germs induced by π. Then either the fiber of $f_\pi^{x_0}$ has strictly positive dimension, or $f_\pi^{x_0}$ is a finite morphism of degree $\geqslant 2$.

Proof. — The first statement is an obvious consequence of Proposition 4.3. For the second note first that $B_{\mathrm{nv}} \setminus BN_\pi$ is just the subset of points $x \in B \setminus BN_\pi$ where the differential of $f_\pi^{x_0}$ is not bijective. The fiber of $f_\pi^{x_0}$ is a germ (F, x_0) of a complex space, namely the germ in x_0 of the complex space defined (in a neighborhood of x_0) by the equation $f_\pi^{x_0}(x) = f_\pi^{x_0}(x_0)$. If $\dim_{x_0}(F) = 0$ then x_0 is an isolated point in the fiber F, hence $f_\pi^{x_0}$ is a finite morphism at x_0 (see [14, Lemma 3.2 p. 132]) and its local degree at x_0 is defined. This local degree coincides with $\dim_{\mathbb{C}}(\mathcal{O}_{F,x_0})$ ([11, p. 24]). Since the differential of $f_\pi^{x_0}$ is degenerate, F cannot be reduced at x_0, hence $\dim_{\mathbb{C}}(\mathcal{O}_{F,x_0}) \geqslant 2$.   □

Corollary 4.4 plays an important role in [7]. In this article the author constructs explicitly generically versal families of class VII surfaces parameterized by open sets $B \subset \mathbb{C}^n$. Applying Corollary 4.4 one can prove that the induced map $B \to \mathcal{M}$ (in the moduli space of class VII surfaces with fixed b_2) is non-injective, more precisely is non-injective on every open subset which intersects the set $B_{\mathrm{nv}} \setminus \mathrm{BN}_\pi$ (which is non-empty for these particular families). This conclusion is very interesting, because for two different points $x \neq y$ of B the corresponding fibers X_x, X_y "look" different, hence there is no obvious isomorphism $X_x \simeq X_y$. It turned out that for Dloussky's families of class VII surfaces, it's very difficult to determine explicitly the divisor $B_{\mathrm{nv}} \subset B$ and the locus of pairs $(x, y) \in B \times B$ for which $X_x \simeq X_y$.

4.2. An explicit computation of $H^0(\mathrm{Tors}(R^1 p_*(\mathcal{E})))$

Let X be a minimal class VII surface with $b := b_2(X) > 0$. Recall from [33] that the intersection form

$$q_X \colon H^2(X, \mathbb{Z})/_{\mathrm{Tors}} \times H^2(X, \mathbb{Z})/_{\mathrm{Tors}} \to \mathbb{Z}$$

is negative definite and the lattice $H^2(X, \mathbb{Z})/\mathrm{Tors}$ admits a basis $(e_1, \ldots, e_b)$, unique up to order and called the Donaldson basis of $H^2(X, \mathbb{Z})/\mathrm{Tors}$, such that

$$e_i \cdot e_j = -\delta_{ij}, \ c_1(\mathcal{K}_X) = \sum_{i=1}^{b} e_i.$$

This existence theorem is obtained using Donaldson's first theorem and the fact that $c_1(\mathcal{K}_X)$ is a characteristic element for the intersection form on $H^2(X, \mathbb{Z})/\mathrm{Tors}$. For a subset $I \subset \{1, \ldots, b\}$ we put

$$e_I := \sum_{i \in I} e_i, \ \bar{I} := \{1, \ldots, b\} \smallsetminus I.$$

We recall that, by definition, a cycle of curves in a surface X is an effective divisor $D \subset X$ which is either an elliptic curve, or a rational curve with a simple singularity, or a sum $D = \sum_{s=1}^{k} D_s$ of $k \geqslant 2$ smooth rational curves D_s such that

$$D_1 D_2 = \cdots = D_{k-1} D_k = D_k D_1 = 1.$$

The GSS conjecture (which, if true, would complete the classification of class VII surfaces) states that any minimal class VII surface X with $b_2(X) > 0$ is a Kato surface (and has a global spherical shell). Any Kato surface has a cycle, so conjecturally any minimal class VII surface X with $b_2(X) > 0$ should have a cycle. A generic Kato surface (more precisely a Kato surface with non-trivial trace [6]) has a homologically trivial cycle of rational curves. Such a surface is called an Enoki surface and can be obtained as a compactification of an affine line bundle over an elliptic curve. Any minimal class VII surface which has a non-zero numerically trivial divisor (effective or not) is an Enoki surface [12].

There are two types of Enoki surfaces. An Enoki surface X of generic type has b irreducible curves $D_1, \ldots, D_b$, and these curves form a cycle of rational curves. More precisely supposing $b > 1$ each D_i is smooth rational and, with respect to a suitably ordered Donaldson basis $(e_1, \ldots, e_b)$, we have $c_1(\mathcal{O}(D_i)) = e_i - e_{i+1}$, the indices being considered modulo b. An Enoki surface of special type (also called a parabolic Inoue surface) has $b+1$ irreducible curves, namely b rational curves forming a cycle of the same type as in the generic case, and a smooth elliptic curve E with $c_1(E) = -\sum_i e_i$.

In our previous articles [30]–[33] we developed a program for proving the existence of curves on minimal class VII surfaces with $b_2 > 0$ using certain moduli spaces of polystable bundles on these surfaces: for a class VII surface X we consider a differentiable rank 2 bundle E on X with $c_2 = 0$ and $\det(E) = K_X$, where K_X denotes the underlying differentiable line bundle

of the canonical line bundle $\mathcal{K}_X$. The fundamental object intervening in our program is the moduli space

$$\mathcal{M}^{\text{pst}} := \mathcal{M}_{\mathcal{K}}^{\text{pst}}(E)$$

of polystable holomorphic structures on E which induce the canonical holomorphic structure $\mathcal{K}_X$ on K_X. This moduli space is alway compact but, for $b_2(X) > 0$, it is not a complex space. It can be written as the union

$$\mathcal{M}^{\text{pst}} = \mathcal{R} \cup \mathcal{M}^{\text{st}},$$

where $\mathcal{R}$ is a finite union of "circles of reductions" (*i.e.*, of split polystable bundles) and $\mathcal{M}^{\text{st}}$ is the moduli space of stable holomorphic structures on E which induce the canonical holomorphic structure $\mathcal{K}_X$ on K_X. The latter subspace is open in $\mathcal{M}^{\text{pst}}$ and has a natural complex space structure.

An important role in the proofs of the main results in [30], [33] (which concern the case $b_2(X) \in \{1, 2\}$) and in the program developed for the general case, is played by reduced, irreducible, compact complex subspaces $Y \subset \mathcal{M}^{\text{st}}$ which contain the point corresponding to the isomorphy class $[\mathcal{A}]$ of the "canonical extension" $\mathcal{A}$ of X [31], [32]. By definition $\mathcal{A}$ is the (essentially unique) non-trivial extension of the form

(4.2) $$0 \to \mathcal{K}_X \to \mathcal{A} \to \mathcal{O}_X \to 0.$$

This bundle is stable with respect to a suitable Gauduchon metric, unless X belongs to a very special class of Kato surfaces [31].

Supposing that a classifying bundle for the embedding $Y \hookrightarrow \mathcal{M}^{\text{st}}$ exists, and choosing a desingularization B of Y_{red}, we obtain:

- a compact complex manifold B,
- a rank 2 bundle $\mathcal{E}$ on $B \times X$,
- a point $a \in B$ such that the bundle $\mathcal{E}_a$ on X is isomorphic to the canonical extension $\mathcal{A}$.

We denote by π, p the projections of $B \times X$ on B and X respectively. For any bundle $\mathcal{E}$ on $B \times X$ and for every line bundle $\mathcal{T}$ on X we denote by ${}^{\mathcal{T}}\mathcal{E}$ the tensor product

$$ {}^{\mathcal{T}}\mathcal{E} := \mathcal{E} \otimes p^*(\mathcal{T}).$$

The goal of this section is the following vanishing result:

PROPOSITION 4.5. — *Let X be a minimal class VII surface with $b = b_2(X) > 0$ and $J \subset \{1, \ldots, b\}$ with $0 < |J| < b$. Let $\mathcal{E}$ be a rank 2 bundle on $B \times X$, where B is a connected compact complex manifold such that*

(a) *$p_*(\mathcal{E}) = 0$,*
(b) *There exists a point $a \in B$ such that $\mathcal{E}_a \simeq \mathcal{A}$.*

Then

(i) *For any* $[\mathcal{T}] \in \operatorname{Pic}^{-e_J}(X)$ *except for at most one point of* $\operatorname{Pic}^{-e_J}(X)$ *one has*

$$H^0(X, \operatorname{Tors}(R^1 p_*({}^{\mathcal{T}}\mathcal{E}))) = 0,$$

(ii) *For any point* $[\mathcal{T}] \in \operatorname{Pic}^{-e_J}(X)$ *with sufficiently negative degree*[2] *one has* $H^1({}^{\mathcal{T}}\mathcal{E}) = 0$.

Note that the condition $p_*(\mathcal{E}) = 0$ holds automatically when $a(B) = 0$ and the bundle $\mathcal{E}_z$ on X is non-filtrable for generic $z \in B$ (see in [29, section 4.3]). In [29] we use this vanishing result to prove a non-existence theorem for positive dimensional reduced, irreducible compact complex subspaces $Y \subset \mathcal{M}^{\mathrm{st}}$ containing the point $[\mathcal{A}]$ and an open neighborhood Y_0 of this point such that $Y_0 \smallsetminus \{[\mathcal{A}]\}$ consists only of non-filtrable bundles. The idea of the proof is to show that the vanishing of $H^1({}^{\mathcal{T}}\mathcal{E})$ leads to a contradiction, which gives $p_*(\mathcal{E}) \neq 0$, and using the results of [29, section 4.3], we obtain $a(B) > 0$. Therefore B (so also Y) is covered by divisors, so the result follows by induction with respect to $\dim(Y)$. This non-existence result is a generalization (for subspaces of arbitrary dimension) of [30, Corollary 5.3] which deals with the case $\dim(Y) = 1$ and plays a crucial role in proving the existence of a curve on class VII surfaces with $b_2 = 1$.

Proof.

(i) Recall that we denote by $\mathcal{D}iv(X)$ the set of *effective* divisors of X. Using Corollary 2.6 we see that in order to prove this proposition it suffices to show that for any effective divisor $D \in \mathcal{D}iv(X)$ we have

$$H^0({}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})) = 0,$$

where $\mathcal{D} := B \times D$, regarded as an effective divisor of $B \times X$. The projection $\pi_{\mathcal{D}} := \pi_{|\mathcal{D}} \colon \mathcal{D} \to B$ is a flat morphism and

$$\mathcal{{}^T}\mathcal{E}_{\mathcal{D}}(\mathcal{D}) = \mathcal{E}_{\mathcal{D}} \otimes p^*(\mathcal{T}(D))$$

is a locally free sheaf on $\mathcal{D}$, so is flat over B. Regarding ${}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})$ as a locally free sheaf on $\mathcal{D}$ and using the projection $\pi_{\mathcal{D}} \colon \mathcal{D} \to B$ we will prove that

Claim. — Under the assumptions of the theorem for any $[\mathcal{T}] \in \operatorname{Pic}^{-e_J}(X)$ except for at most one point of $\operatorname{Pic}^{-e_J}(X)$ it holds

$$H^0({}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})_a) = 0 \; \forall D \in \mathcal{D}iv(X).$$

By Lemma A.1 and Remark A.2 $H^0({}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})_a) = 0$ implies $(\pi_{\mathcal{D}})_*({}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D}))$ $= 0$, which obviously implies $H^0({}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})) = 0$. Therefore our claim completes the proof. Since ${}^{\mathcal{T}}\mathcal{E}_{\mathcal{D}}(\mathcal{D})_a = \mathcal{T} \otimes \mathcal{A}_D(D)$ our claim follows from Lemma 4.6 below.

(ii) The Leray spectral sequence associated with the projection p yields the exact sequence

$$0 \to H^1(p_*({}^{\mathcal{T}}\mathcal{E})) \to H^1({}^{\mathcal{T}}\mathcal{E}) \to H^0(R^1p_*({}^{\mathcal{T}}\mathcal{E})) \to H^2(p_*({}^{\mathcal{T}}\mathcal{E})) \to H^2({}^{\mathcal{T}}\mathcal{E}).$$

By the projection formula we have $p_*({}^{\mathcal{T}}\mathcal{E}) = p_*(\mathcal{E}) \otimes \mathcal{T}$, which vanishes by hypothesis. Therefore the canonical morphism $H^1({}^{\mathcal{T}}\mathcal{E}) \to H^0(R^1p_*({}^{\mathcal{T}}\mathcal{E}))$ is an isomorphism. By (i) we know that $H^0(\mathrm{Tors}(R^1p_*({}^{\mathcal{T}}\mathcal{E}))) = 0$ for any $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$ except for at most a point. It suffices to prove that for any $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$ with sufficiently negative degree one has $H^0({}^{\mathcal{T}}\mathcal{S}) = 0$, where ${}^{\mathcal{T}}\mathcal{S}$ denotes the quotient of $R^1p_*({}^{\mathcal{T}}\mathcal{E})$ by its torsion subsheaf. The projection formula gives a canonical isomorphism $R^1p_*({}^{\mathcal{T}}\mathcal{E}) = R^1p_*(\mathcal{E}) \otimes \mathcal{T}$ hence, since $\mathcal{T}$ is locally free, we get a canonical isomorphism

$${}^{\mathcal{T}}\mathcal{S} = \mathcal{S} \otimes \mathcal{T},$$

where $\mathcal{S}$ is the quotient of $R^1p_*(\mathcal{E})$ by its torsion subsheaf. Let $\mathcal{S} \hookrightarrow \mathcal{B}$ be the embedding of $\mathcal{S}$ in its bidual, which is a reflexive sheaf on a surface, hence is locally free. Fix a Hermitian metric h on $\mathcal{B}$, a Gauduchon metric g on X, and let $h_{\mathcal{T}}$ be a Hermite-Einstein Hermitian metric on $\mathcal{T}$. When the Einstein constant $c_{\mathcal{T}}$ of the Chern connection $A_{\mathcal{T},h_{\mathcal{T}}}$ is sufficiently negative, the Hermitian endomorphism $i\Lambda_g F_{h \otimes h_{\mathcal{T}}}$ of $\mathcal{B} \otimes \mathcal{T}$ becomes negative definite, hence $H^0(\mathcal{B} \otimes \mathcal{T}) = 0$ by a well-known vanishing theorem (see [20, Theorem 1.9 p. 52]). It suffices to recall that the Einstein constant of $A_{\mathcal{T},h_{\mathcal{T}}}$ is proportional with $\deg_g(\mathcal{T})$ (see in [22, Proposition 2.18]) and to note that $H^0(\mathcal{B} \otimes \mathcal{T}) = 0$ implies $H^0({}^{\mathcal{T}}\mathcal{S}) = 0$, because ${}^{\mathcal{T}}\mathcal{S}$ is a subsheaf of $\mathcal{B} \otimes \mathcal{T}$. $\quad\square$

LEMMA 4.6. — *Let X be a minimal class VII surface with $b = b_2(X) > 0$ and $J \subset \{1, \ldots, b\}$ with $0 < |J| < b$. Then for any $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$ except for at most one point of $\mathrm{Pic}^{-e_J}(X)$ it holds*

$$H^0(\mathcal{T} \otimes \mathcal{A}_D(D)) = 0 \ \forall D \in \mathcal{D}iv(X).$$

Proof. — Tensorizing with $\mathcal{T}(D)$ the exact sequence of *locally free* sheaves (4.2) and restricting it to D we obtain the exact sequence

(4.3) $$0 \to \mathcal{T} \otimes \mathcal{K}_D(D) \to \mathcal{T} \otimes \mathcal{A}_D(D) \to \mathcal{T}_D(D) \to 0,$$

hence we see that $H^0(\mathcal{T} \otimes \mathcal{A}_D(D)) = 0$ as soon as $H^0(\mathcal{T}_D(D)) = 0$ and $H^0(\mathcal{T} \otimes \mathcal{K}_D(D)) = 0$. We will see that the first space vanishes for every

$[\mathcal{T}] \in \text{Pic}^{-e_J}(X)$, whereas the second vanishes for every $[\mathcal{T}] \in \text{Pic}^{-e_J}(X)$ except for at most a point of $\text{Pic}^{-e_J}(X)$ (which is independent of D).

1. $H^0(\mathcal{T}_D(D)) = 0$ for every $[\mathcal{T}] \in \text{Pic}^{-e_J}(X)$.

This follows by Lemma 4.7 (iii), (iv) below.

2. $H^0(\mathcal{T} \otimes \mathcal{K}_D(D)) = 0$ for every $[\mathcal{T}] \in \text{Pic}^{-e_J}(X)$ except for at most a point of $\text{Pic}^{-e_J}(X)$ (which is independent of $D \in \mathcal{D}iv(X)$).

The proof of 2. starts with the cohomology long exact sequence associated with the short exact sequence

$$0 \to \mathcal{T} \otimes \mathcal{K} \to \mathcal{T} \otimes \mathcal{K}(D) \to \mathcal{T} \otimes \mathcal{K}_D(D) \to 0,$$

which reads

$$(4.4) \quad 0 \to H^0(\mathcal{T} \otimes \mathcal{K}) \to H^0(\mathcal{T} \otimes \mathcal{K}(D))$$
$$\to H^0(\mathcal{T} \otimes \mathcal{K}_D(D)) \to H^1(\mathcal{T} \otimes \mathcal{K}) \to \cdots.$$

Note that $c_1(\mathcal{T} \otimes \mathcal{K}) = e_{\bar{J}}$, hence $H^0(\mathcal{T} \otimes \mathcal{K}) = 0$ by Lemma 4.7 (i) below. On the other hand $h^2(\mathcal{T} \otimes \mathcal{K}) = h^0(\mathcal{T}^\vee) = 0$, where the first equality follows by Serre duality, and the second by 4.7 (i) again. Using the Riemann-Roch theorem and taking into account that

$$\chi(\mathcal{T} \otimes \mathcal{K}) = \frac{1}{2} e_{\bar{J}} (e_{\bar{J}} - c_1(\mathcal{K})) = -\frac{1}{2} e_{\bar{J}} e_J = 0,$$

it follows $h^1(\mathcal{T} \otimes \mathcal{K}) = 0$. Therefore, using the exact sequence (4.4) we obtain

$$(4.5) \qquad h^0(\mathcal{T} \otimes \mathcal{K}_D(D)) = h^0(\mathcal{K} \otimes \mathcal{T}(D)).$$

Therefore it suffices to prove that for any $[\mathcal{T}] \in \text{Pic}^{-e_J}(X)$ except for at most one point of $\text{Pic}^{-e_J}(X)$ one has $h^0(\mathcal{K} \otimes \mathcal{T}(D)) = 0$ for any $D \in \mathcal{D}iv(X)$. If $h^0(\mathcal{K} \otimes \mathcal{T}(D)) > 0$, there exists an effective divisor $G \subset X$ such that

$$(4.6) \qquad \mathcal{K} \otimes \mathcal{T} \simeq \mathcal{O}(G - D).$$

This implies

$$(4.7) \qquad \sum_{i \in \bar{J}} e_i = c_1(\mathcal{O}(G - D)).$$

Suppose first that X is an Enoki surface. Taking into account the properties of Enoki surfaces explained above, we see that for any divisor A (effective or not) on an Enoki surface of generic type one has $c_1(A)c_1(\mathcal{K}) = 0$, whereas for any divisor A (effective or not) on an Enoki surface of special type one has $c_1(A)c_1(\mathcal{K}) \equiv 0 \mod b$. Taking $A = G - D$ this contradicts

formula (4.7). So for an Enoki surface one has $H^0(\mathcal{T} \otimes \mathcal{K}_D(D)) = 0$ for every $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$.

Suppose now that X is not an Enoki surface. The classes corresponding to the irreducible curves of X are linearly independent in the free $\mathbb{Z}$-module $H^2(X,\mathbb{Z})/\,\mathrm{Tors}$. Therefore there exists at most one divisor $A = A_+ - A_-$ (where $A_\pm$ are effective divisors with no common irreducible component) such that $\sum_{i \in \bar{J}} e_i = c_1(\mathcal{O}(A))$. This shows that the pairs of effective divisors (D, G) for which (4.7) holds are all given by

$$D = A_- + H, \ G = A_+ + H \text{ where } H \in \mathcal{D}iv(X).$$

But for any such pair one has $\mathcal{O}(G - D) \simeq \mathcal{O}(A)$, so $\mathcal{T}$ must be isomorphic with $\mathcal{K}^{-1}(A)$ by (4.6). $\square$

LEMMA 4.7. — *Let X be a minimal class VII surface with $b := b_2(X) > 0$ and $J \subset \{1, \ldots, b\}$. Then*

(i) *For every effective divisor $D \subset X$ one has*

$$\langle c_1(\mathcal{K}), [D] \rangle \geqslant 0.$$

(ii) *For every effective divisor $D \subset X$ one has*

$$\langle -e_J, D \rangle + D^2 \leqslant 0$$

with equality if and only if D is numerically trivial.

(iii) *If X is not an Enoki surface, then for every $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$ and for every effective divisor $D \subset X$ one has $H^0(\mathcal{T}_D(D)) = 0$.*

(iv) *If X is an Enoki surface and $0 < |J| < b$, then for every $[\mathcal{T}] \in \mathrm{Pic}^{-e_J}(X)$ and for every effective divisor $D \subset X$ one has $H^0(\mathcal{T}_D(D)) = 0$.*

Proof.

(i) This important inequality is due to Nakamura (see [26, Lemma 1.1.3]). It can be proved easily using the fact that, since X is minimal, the claimed inequality holds for any irreducible curve of X.

(ii) Decomposing $c_1(\mathcal{O}(D)) = \sum_{i=1}^{b} d_i e_i \mod \mathrm{Tors}$ (with $d_i \in \mathbb{Z}$) we have

$$\langle -e_J, D \rangle + D^2 = \sum_{j \in J} d_j - \sum_{i=1}^{b} d_i^2 = -\left(\sum_{j \in J} d_j(d_j - 1) + \sum_{i \in \bar{J}} d_i^2 \right) \leqslant 0,$$

with equality if and only if $d_j \in \{0, 1\}$ for $j \in J$ and $d_i = 0$ for $i \in \bar{J}$. Therefore one has equality if and only if there exists $J' \subset J$ such that $c_1(\mathcal{O}(D)) = e_{J'}$. By Lemma 4.7 (i) we must have $J' = \emptyset$, hence D is numerically trivial. Let $D = \sum_{s=1}^{k} n_s D_s$ (with $n_s > 0$) be the decomposition

of D in irreducible components. We will prove (iii), (iv) by induction with respect to $n := \sum_{s=1}^{k} n_s$. We have

$$(4.8) \qquad \sum_{s=1}^{k} n_s \langle c_1(\mathcal{T}(D)), D_s \rangle = \langle c_1(\mathcal{T}(D)), D \rangle$$
$$= \langle c_1(\mathcal{T}), D \rangle + D^2 = -\langle e_J, D \rangle + D^2.$$

(iii) If X is not an Enoki surface and D is not empty, then (ii) and (4.8) show that

$$\sum_{s=1}^{k} n_s \langle c_1(\mathcal{T}(D)), D_s \rangle < 0,$$

hence there exists $\sigma \in \{1, \ldots, k\}$ such that $\langle c_1(\mathcal{T}(D)), D_\sigma \rangle < 0$. Writing $D = D' + D_\sigma$ we obtain a short exact sequence

$$0 \to \mathcal{T}_{D'}(D') \to \mathcal{T}_D(D) \to \mathcal{T}_{D_\sigma}(D) \to 0$$

which gives the long exact sequence

$$0 \to H^0(\mathcal{T}_{D'}(D')) \to H^0(\mathcal{T}_D(D)) \to H^0(\mathcal{T}_{D_\sigma}(D)) \to \cdots .$$

But $H^0(\mathcal{T}_{D_\sigma}(D)) = 0$, because the degree of the restriction of the line bundle $\mathcal{T}(D)$ to the irreducible curve D_σ is negative. Therefore $H^0(\mathcal{T}_D(D)) = H^0(\mathcal{T}_{D'}(D'))$, and the sum of the coefficients of the decomposition of D' in irreducible components is $n' = n - 1$.

(iv) If X is an Enoki surface, the proof of (iii) applies except when D is numerically trivial, *i.e.*, when $D = nC$ where C is the cycle of rational curves of X. Write $C = \sum_{i=1}^{b} D_i$ where (for a suitably ordered Donaldson basis $(e_i)_{1 \leqslant i \leqslant b}$) $c_1(\mathcal{O}(D_i)) = e_i - e_{i+1}$, the indices being considered mod b. In this case we get

$$\langle c_1(\mathcal{T}(D)), D_i \rangle = \langle -e_J, D_i \rangle + \langle D, D_i \rangle = \langle -e_J, D_i \rangle.$$

Since we assumed $J \neq \emptyset$ and $J \neq \{1, \ldots, b\}$ there exists $\sigma \in \bar{J}$ such that $\sigma + 1 \in J$. Then $\langle c_1(\mathcal{T}(D)), D_\sigma \rangle = -1$, which shows that $H^0(\mathcal{T}_{D_\sigma}(D)) = 0$, hence the same argument by induction as in the proof of (iii) can also be used in the case when D is numerically trivial. $\qquad \square$

Appendix A.   Torsion free direct images

It is well known that "under suitable assumptions" the direct image of a locally free sheaf is torsion free. Since we need this result in a very general framework (when the total space is not supposed to be reduced) we state this result explicitly and give a complete proof.

LEMMA A.1. — *Let M, B be complex spaces with B locally irreducible[3], $\pi \colon M \to B$ a proper holomorphic map and $\mathcal{E}$ a coherent sheaf on M which is flat over B. Then*

(1) *$\pi_*(\mathcal{E})$ is a torsion-free coherent sheaf on B.*
(2) *Suppose B is connected and there exists $x_0 \in B$ such that $H^0(\mathcal{E}_{x_0}) = 0$. Then $\pi_*(\mathcal{E}) = 0$.*

Proof.

(1) Let $x \in B$ and let $\sigma \in \pi_*(\mathcal{E})_x$ be a torsion element of the $\mathcal{O}_{B,x}$-module $\pi_*(\mathcal{E})_x$. Therefore there exists $\psi \in \mathcal{O}_{B,x} \setminus \{0\}$ such that $\psi\sigma = 0$. The stalk $\pi_*(\mathcal{E})_x$ can be identified with the cohomology space $H^0(M_x, \mathcal{E})$ (see [16, Rem. 4.17.1, p. 202], [1, Lemma 1.3 p. 93]), *i.e.*, with the space of sections of the restriction $\mathcal{E}|_{M_x}$ in the sense of the theory of sheaves on topological spaces. Note that this restriction is *not* coherent on M_x. Using the natural $\mathcal{O}_B$-module structure of $\mathcal{E}$, the condition $\psi\sigma = 0$ becomes

$$(\text{A.1}) \qquad\qquad \psi\sigma_y = 0 \text{ in } \mathcal{E}_y \ \forall y \in M_x.$$

Since $\mathcal{O}_{B,x}$ is an integral domain by assumption (see [17, p. 8]) the multiplication with ψ defines a monomorphism of $\mathcal{O}_{B,x}$-modules

$$0 \to \mathcal{O}_{B,x} \xrightarrow{\psi} \mathcal{O}_{B,x},$$

hence, since $\mathcal{E}$ is flat over B, the induced morphism $\mathcal{E}_y \to \mathcal{E}_y$ is also injective for every $y \in M_x$. Therefore (A.1) implies $\sigma = 0$.

An alternative proof can be obtained using of [1, Corollary 4.11 p. 133] in the special case $\mathfrak{M} = \mathcal{O}_B$. It follows that there exists a coherent sheaf $\mathcal{T}$ on B and an isomorphism $\pi_*(\mathcal{E}) = \mathrm{Hom}(\mathcal{T}, \mathcal{O}_B)$. It suffices to note that $\mathrm{Hom}(\mathcal{T}, \mathcal{O}_B)$ is torsion free when the structure sheaf $\mathcal{O}_B$ is a sheaf of integral domains.

(2) We apply Grauert's Theorem ([1, Theorem 4.12 p. 134]). It follows that the map $x \mapsto H^0(\mathcal{E}_x)$ is upper semicontinuous on B. Since we assumed that this function vanishes at $x_0 \in B$, it follows that this function vanishes identically on a open neighborhood U of x_0. Since B is reduced, the second part of the quoted theorem shows that then the sheaf $\pi_*(\mathcal{E})$ is locally free on U. Therefore this sheaf vanishes on U which shows that its support is a proper analytic set of B hence, since B is locally irreducible, it is a torsion sheaf (see [28, Theorem 9.12 p. 60]). On the other hand by (1) this sheaf is torsion free. $\qquad\square$

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[3] Aqccording to the terminology of [17], a locally irreducible complex space is automatically reduced.

Note that in this lemma we don't have to assume the total space or the fibers to be reduced. For instance we have

REMARK A.2. — *Suppose that $\mathcal{E}$ is locally free on M and $p\colon M \to Y$ is a flat morphism (for instance a holomorphically locally trivial morphism). Then $\mathcal{E}$ is flat over Y, hence (supposing that Y is locally irreducible and p is proper) Lemma A.1 applies.*

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