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ULTRARIGID TANGENTS OF SUB-RIEMANNIAN NILPOTENT GROUPS

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Abstract. — We show that the tangent cone at the identity is not a complete quasiconformal invariant for sub-Riemannian nilpotent groups. Namely, we show that there exists a nilpotent Lie group equipped with left invariant sub-Riemannian metric that is not locally quasiconformally equivalent to its tangent cone at the identity. In particular, such spaces are not locally bi-Lipschitz homeomorphic. The result is based on the study of Carnot groups that are rigid in the sense that their only quasiconformal maps are the translations and the dilations.

Résumé. — Nous montrons que pour les groupes nilpotents sous-riemanniens, le cône tangent en l'identité n'est pas un invariant quasi-conforme complet. À savoir, nous montrons qu'il existe un groupe de Lie nilpotent muni d'une métrique sous-riemannienne invariante à gauche qui n'est pas localement quasi-conformément équivalent à son cône tangent en l'identité. En particulier, ces espaces ne sont pas localement bi-Lipschitziens. Le résultat repose sur l'étude des groupes de Carnot qui sont rigides dans le sens que leurs seules applications quasi-conformes sont les translations et les dilatations.

1. Overture

1.1. Overview of the results

By means of a result [2] of Margulis and Mostow, if two equiregular sub-Riemannian manifolds are quasiconformally equivalent, then almost everywhere they have isomorphic tangent cones. In particular, the tangent cone is a quasiconformal invariant. Their work extends a result [7] of Pansu,

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for which two Carnot groups are quasiconformally equivalent only if they are isomorphic.

The main goal of this paper is to show that the converse of the theorem of Margulis and Mostow fails in a strong sense. We show the following statement.

Theorem 1.1. — *There exists a nilpotent Lie group equipped with left invariant sub-Riemannian metric that is not locally quasiconformally equivalent to its tangent cone.*

Note that the theorem above holds in particular for locally bi-Lipschitz maps. We recall that the tangent cone of an equiregular sub-Riemannian manifold is a Carnot group, and it coincides with the nilpotentisation of its sub-Riemannian structure [4]. In order to establish the result, we shall study groups with the property that whenever they are quasiconformally equivalent to some other group, they are in fact isomorphic to it, see Theorem 4.2. With this purpose in mind, we consider Carnot groups whose quasiconformal maps are only translations and dilations. We shall refer to groups with this property as *ultrarigid groups*. In order to show that some group is ultrarigid, we prove the following algebraic characterization.

Theorem 1.2. — *Let $\mathbb{G}$ be a Carnot group. Then the following are equivalent:*

[1.2.1] *For any $U \subset \mathbb{G}$ open and any quasiconformal embedding $f : U \to \mathbb{G}$, one has that f is the restriction of the composition of a left translation and a dilation;*

[1.2.2] *Every strata preserving automorphism of Lie$(\mathbb{G})$ is a dilation.*

The class of groups defined by [1.2.2] was considered by Pansu in [7]. He showed that there exist infinitely many 2-step Carnot groups with this property, although his proof does not provide explicit examples. We exhibit two examples of groups satisfying [1.2.2]. The first one is a 2-step stratified nilpotent Lie group, whereas the latter has step 3 and it is not in the class studied by Pansu. Finally, we point out that, in the case of 2-step Carnot groups, the nontrivial implication [1.2.2] $\Rightarrow$ [1.2.1] of Theorem 1.2 was proved by Capogna and Cowling, see [1].

1.2. State of the art

Given a metric space (X, d) and a base point $x \in X$, one can consider the *blow-down spaces* and the *blow-up spaces* of X at x. Namely, a pointed metric space $((Z,\rho),z)$ is a blow-up (resp. a blow-down) of X at x if there exists a sequence of positive real numbers λ_j with $\lambda_j \to \infty$ (resp. $\lambda_j \to 0$), as $j \to \infty$, such that $((X,\lambda_j d),x)$ Gromov-Hausdorff converges to $((Z,\rho),z)$. Such blow-down spaces and blow-up spaces are not unique and do not always exist. Whenever the limit exists for any sequence $\lambda_j \to \infty$ (resp. $\lambda_j \to 0$) and does not depend on the sequence, the blow-up (resp. blow-down) space is called the *tangent cone* (resp. *asymptotic cone*). In many situations a map between metric spaces induces a map between blow-down or blow-up spaces. A key fact is that the induced map has often more geometric structure than the initial map.

We recall two examples of blow-down and blow-up spaces, which are well known in sub-Riemannian geometry and in Geometric Group Theory, respectively. Let M, M' be two manifolds endowed with some sub-Riemannian distances induced by equiregular horizontal distributions. In such a setting, the blow-up spaces do not depend on the scaling sequences and are stratified nilpotent Lie groups, see [4]. Let $f : M \to M'$ be a quasiconformal homeomorphism. According to [2], for almost every $p \in M$, the map f blows up at p to a strata preserving group isomorphism between the blow-up space at p and the one at $f(p)$. Regarding the large scale geometry of groups, let Γ be a finitely generated nilpotent group. Endow Γ with any word distance induced by a finite generating set. The unfamiliar reader might just think that Γ is a connected, simply connected nilpotent Lie group endowed with a Riemannian left invariant distance. By [6], the blow-down space of Γ is unique and is a stratified nilpotent Lie group endowed with a left invariant Carnot-Carathéodory distance, induced by a norm on the first stratum. Likewise the general framework, any quasi-isometry blows down to a bi-Lipschitz homeomorphism of the blow-down spaces. Consequently, such blow-down spaces are isomorphic.

Once a map is given at the blow-up or at the blow-down level, it is then natural to ask if we can integrate back to a map between the initial spaces. Namely, if we are given two sub-Riemannian manifolds with isomorphic blow-up spaces at a point (resp. two finitely generated nilpotent groups with isomorphic blow-down spaces), to what extent we may conclude that the two manifolds are quasiconformally equivalent (resp. the nilpotent groups are quasi-isometric)?

The fact that the blow-down space is not a complete quasi-isometric invariants was proved by Shalom [8], using group cohomology. Namely, he shows that quasi-isometric nilpotent groups have same Betti numbers. Then he exhibits an example due to Benoist of two nilpotent groups with same blow-down space and different Betti numbers. We summarize the result of Shalom as the following statement.

Theorem 1.3 (Shalom, [8, page 152]). — *There exist two finitely generated nilpotent groups* Γ *and* Λ *that have the same blow-down space, but they are not quasi-isometric equivalent.*

Although blow-down spaces capture the asymptotic geometry, Shalom's result shows that they do not capture the whole large scale geometry. Similarly, for general sub-Riemannian manifolds, blow-up spaces capture only the infinitesimal geometry and not the local geometry. To see this, one can consider an example(1) of a sub-Riemannian manifold M whose blow-up space is not constant on a full measure set. Indeed, fix $p \in M$ and let $\mathbb{G}$ be the blow-up of M at p. We claim that no neighborhood of p is quasiconformally equivalent to an open set in $\mathbb{G}$. Since $\mathbb{G}$ is a cone, it is isometric to its blow-up space. Hence, the blow-up of M at p is isomorphic to the blow-up of $\mathbb{G}$ at the identity. Assume by contradiction that there exists a quasiconformal embedding $f : U \subset M \to \mathbb{G}$. Then by [2] almost every blow-up space in U needs to be isomorphic to $\mathbb{G}$. Since this is not the case, such a quasiconformal map does not exist.

We conclude that a necessary condition for a sub-Riemannian manifold M to be quasiconformally equivalent to a Carnot group $\mathbb{G}$ is that the blow-up space of M is $\mathbb{G}$ at almost every point. It is then natural to ask what happens when the manifold has the same blow-up space at every point. Here the work of Pansu, Margulis, and Mostow fails to give an answer. One needs to find a different strategy.

A natural example of a manifold with constant blow-up spaces is provided by a Lie group G endowed with a left invariant sub-Riemannian distance. In this case, the isometry group of G acts transitively on G. Therefore the blow-up space is the same at every point of G. In this article, we provide a nilpotent Lie group of dimension 16 that is not locally quasiconformal (and hence not locally bi-Lipschitz) equivalent to its blow-up. In particular, we have the following consequence.

Corollary 1.4. — *There exist two sub-Riemannian nilpotent Lie groups* H *and* G*, that have the same blow-up space at every point, but they are not (locally) quasiconformally equivalent.*

(1) The existence of sub-Riemannian manifolds whose blow-up space varies continuously was noticed by Pansu. An explicit 11-dimensional example has being given by Varchenko in [10].

We conclude our survey section by recalling some positive results: blow-up spaces of Riemannian manifolds and contact 3-manifolds do capture the local geometry. Indeed, every point p in a Riemannian n-manifold M has a neighborhood that is bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$, which is the blow-up of M at p. The same phenomenon appears for contact 3-manifolds. By Darboux's Theorem, every point in a contact manifold has a neighborhood that is contactomorphic to an open set of the standard contact structure. We can see this as a metric statement. Indeed, every contact manifold can be endowed with a sub-Riemannian structure, which is unique up to bi-Lipschitz equivalence. Now, Darboux's Theorem implies that every point p in a sub-Riemannian 3-manifold M has a neighborhood that is bi-Lipschitz equivalent to an open set in the sub-Riemannian standard contact structure. The latter is the sub-Riemannian Heisenberg group. Since the Heisenberg group is dilation invariant, we also have that the sub-Riemannian Heisenberg group is the blow-up at any point of any sub-Riemannian 3-manifold. We can therefore conclude that every nilpotent Lie group G of dimension 3 has the property that, when it is endowed with a left invariant sub-Riemannian metric, any element of G has a neighborhood that is bi-Lipschitz homeomorphic to an open set in the blow-up space of G.

We remark that in the setting of Riemannian groups or of 3-dimensional contact groups, the blow-up spaces may not preserve the algebraic structure of the original space. Examples in the Riemannian setting are easy to find, because there are diffeomorphic Lie groups that are not isomorphic. On the other hand, the sub-Riemannian roto-translation group is not isomorphic to its blow-up space, which is the Heisenberg group.

1.3. Structure of the paper

The article is organized as follows. In Section 2 we fix notation and state the results of the literature that are the building blocks of our work. In Section 3 we restate Theorem 1.2, which characterizes ultrarigidity in purely Lie theoretic terms. Then we give two examples of ultrarigid Carnot groups. In Section 4 we establish our main results. To begin we prove Theorem 4.2, which is a rigidity type theorem for sub-Riemannian nilpotent Lie groups with ultrarigid tangent cone. Secondly, we exhibit two example of a sub-Riemannian nilpotent Lie group with ultrarigid tangent cone. This together with Theorem 4.2 imply Theorem 1.1 and Corollary 1.4. In Section 5 we recall the definition of Tanaka prolongation of a stratified nilpotent Lie algebra and state a result of Tanaka. We then use this to prove Theorem 1.2.

2. Notation and preliminaries

2.1. Carnot Groups

Let $\mathbb{G}$ be a stratified nilpotent Lie group with identity $e_{\mathbb{G}}$ or e if no confusion arises. This means that its Lie algebra $\mathfrak{g}$ admits an s-step stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s,$$

where $[V_j, V_1] = V_{j+1}$, for $1 \leqslant j \leqslant s$, and with $V_s \neq \{0\}$ and $V_{s+1} = \{0\}$. To avoid degeneracies, we assume $\mathfrak{g}$ to have at least dimension two, which is reasonable for our purposes.

Given a point $p \in \mathbb{G}$ we denote by τ_p the left translation by p. An element X in the Lie algebra $\mathfrak{g}$ can be considered as a tangent vector at the identity. Such a vector induces the left invariant vector field given by $(\tau_p)_*|_e(X)$ at a point $p \in \mathbb{G}$. This vector field will still be denoted by X, unless confusion might arise. The set of all left invariant vector fields with the bracket operation is isomorphic to $\mathfrak{g}$ and it inherits the stratification of $\mathfrak{g}$. The sub-bundle $\mathcal{H} \subseteq T\mathbb{G}$ where $\mathcal{H}_p = (\tau_p)_*|_e(V_1)$ is called the *horizontal distribution*. A scalar product $\langle\, , \rangle$ on V_1 defines a left invariant scalar product on each $\mathcal{H}_p$ by setting

$$\langle v, w\rangle_p = \langle (\tau_{p^{-1}})_*|_p(v), (\tau_{p^{-1}})_*|_p(w)\rangle \tag{2.1}$$

for all $v, w \in \mathcal{H}_p$. The left invariant scalar product gives rise to a left invariant sub-Riemannian metric d on $\mathbb{G}$, the definition of which we shall give later in the more general setting of sub-Riemannian manifolds. We call $(\mathbb{G}, d)$ a *Carnot group*, which we simply denote by $\mathbb{G}$ if no ambiguity arises.

We denote by $\mathrm{Aut}_0(\mathfrak{g})$ the Lie group of strata preserving automorphisms of $\mathfrak{g}$. The Lie algebra of $\mathrm{Aut}_0(\mathfrak{g})$ is the space of strata preserving derivations of $\mathfrak{g}$, which we denote by $\mathfrak{g}_0$. In general, for any stratified nilpotent Lie algebra, there are distinguished elements of $\mathrm{Aut}_0(\mathfrak{g})$, which are called *dilations* (or better *algebra-dilations*). For each $\lambda \in \mathbb{R}$, the dilation δ_λ is defined linearly by setting $\delta_\lambda(X) := \lambda^j X$, for every $X \in V_j$ and every $j = 1, \ldots, s$. The subset $\{\delta_\lambda \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ is called the *algebra-dilation group*. The set of dilations with positive factor constitutes a one parameter subgroup of $\mathrm{Aut}_0(\mathfrak{g})$, whose Lie algebra is generated by the derivation $D \in \mathfrak{g}_0$ defined by $D(X) := jX$, for every $X \in V_j$ and every $j = 1, \ldots, s$. In particular, for every $t \in \mathbb{R}$, we have $\delta_{e^t} = e^{tD}$. Since any (algebra-)dilation δ_λ is an algebra homomorphism and the Lie group $\mathbb{G}$ is simply connected, the dilation induces a unique homomorphism on the group, which we still denote by δ_λ and call *group-dilation* or *dilation on the group level*. Since $\mathbb{G}$ is nilpotent and simply connected, the exponential map is a diffeomorphism. Thus, group-dilations δ_λ can be defined as $\delta_\lambda(p) = \exp \circ \delta_\lambda \circ \exp^{-1}(p)$, for all $\lambda \in \mathbb{R}$ and $p \in \mathbb{G}$.

The group generated by the left translations and the group-dilations will play an important role in our considerations. We refer to this group as the *translation-dilation group*, and denote it by

$$\mathrm{TD}(\mathbb{G}) := \{\tau_p \circ \delta_\lambda \ : \ p \in \mathbb{G}, \lambda \in \mathbb{R} \setminus \{0\}\}.$$

2.2. Sub-Riemannian manifolds

Throughout the paper, we shall write smooth when referring to C^∞ functions, maps or vector fields. A sub-Riemannian, or Carnot-Carathéodory manifold, is a triple $(M, \mathcal{H}, \mathrm{g})$, where M is a differentiable manifold, $\mathcal{H}$ is a bracket generating tangent sub-bundle of M, and g is a smooth section of the positive definite quadratic forms on $\mathcal{H}$.

Let m be $\dim \mathcal{H}_p$, which is independent on $p \in M$. Recall that being bracket generating means that, for every $p \in M$, there exists vector fields $X_1, \ldots, X_m$ in M, such that $\mathcal{H}_p = \mathrm{span}\{X_1(p), \ldots, X_m(p)\}$, and for some integer $s(p) \geqslant 1$,

$$\begin{aligned} T_pM = \mathrm{span}\big\{[X_{i_1}, [X_{i_2}, [\ldots, [X_{i_{k-1}}, X_{i_k}]\ldots]]](p) \ : \ k = 1, \ldots, s(p), \\ i_j \in \{1, \ldots, m\}, j = 1, \ldots, k\big\} \end{aligned}$$

The bundle $\mathcal{H}$ is called the *horizontal distribution*. The Carathéodory-Chow-Rashevsky Theorem shows that the bracket generating property implies that any two points in M can be joined by a *horizontal path*, *i.e.*, an absolutely continuous path whose tangents belong to the horizontal distribution. It follows that a sub-Riemannian manifold carries a natural metric, called the *sub-Riemannian* or *Carnot-Carathéodory metric*, defined by setting

$$d(p, q) := \inf \int_0^1 \sqrt{\mathrm{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt,$$

where the infimum is taken along all horizontal curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

For Carnot groups, the tensor g is given by the left invariant scalar product (2.1). Moreover, left translations are isometries, and d is homogeneous with respect to the group-dilations, that is, $d(\delta_\lambda(p), \delta_\lambda(q)) = |\lambda| d(p,q)$ for all $p, q \in \mathbb{G}$ and all $\lambda \in \mathbb{R}$.

The horizontal bundle induces a filtration of each T_pM as follows: Set $L_0 = \{0\}$, let L_1 denote the set of all smooth sections of $\mathcal{H}$ defined on a neighborhood of p, and by induction define $L_{i+1} = L_i + [L_1, L_i]$, for $i > 0$. It then follows that

$$L_0(p) \subset L_1(p) \subseteq \cdots \subseteq L_{s(p)}(p) = T_pM$$

and $[L_i, L_j](p) \subseteq L_{i+j}(p)$. If $V_i(p) := L_i(p)/L_{i-1}(p)$, then the nilpotentisation of T_pM is the vector space

$$\mathfrak{g}(p) = V_1(p) \oplus \cdots \oplus V_{s(p)}(p).$$

Since, for any $X \in L_i$ and $Y \in L_j$, one has that

$$[X + L_{i-1}, Y + L_{j-1}] = [X, Y] + L_{i+j-1},$$

the Lie bracket of vector fields induces a well defined bracket on $\mathfrak{g}(p)$ thus defining a stratified nilpotent Lie algebra of step $s(p)$. Since $\mathfrak{g}(p)$ is nilpotent, by the theory of nilpotent Lie groups, there exists a unique connected, simply connected Lie group $\mathbb{G}_p$ whose Lie algebra is $\mathfrak{g}(p)$. We might denote this group by $\exp(\mathfrak{g}(p))$. The group $\mathbb{G}_p$ together with the sub-Riemannian metric d_p induced by $\langle\, ,\, \rangle_p$, forms a Carnot group, which is called the *tangent cone* at p. Indeed, by a theorem of Mitchell [4], the pointed metric spaces $(M, \lambda d, p)$ Gromov-Hausdorff converge, as $\lambda \to \infty$, to the pointed metric space $(\mathbb{G}_p, d_p, e)$. In other words, any blow-up space of (M, d) at p is isometric to the Carnot group $\mathbb{G}_p$.

A sub-Riemannian manifold is called *equiregular*(2) if the functions $p \mapsto \dim V_i(p)$ are constant for all i. Note that in this case the function $p \mapsto s(p)$ is automatically constant. The important consequence of equiregularity is that the Hausdorff dimension of M, with respect to d, is the natural number $Q = \sum_{i=1}^{s(p)} i \dim \mathfrak{g}_i(p)$. Moreover, on any compact subset of M, the Q-dimensional Hausdorff measure is commensurate with any Lebesgue measure, see [4].

If the nilpotentisations are independent of p and thus isomorphic to a fixed Lie algebra $\mathfrak{g}$, then $(M, \mathcal{H})$ is said to be *strongly regular* and $\mathfrak{g}$ is called the *symbol algebra* of $(M, \mathcal{H})$. Clearly the tangent cones are all isomorphic to $\mathbb{G} = \exp(\mathfrak{g})$ in this case.

(2) In Mitchell's and Margulis-Mostow's works, one finds the term 'generic' instead of 'equiregular'.

2.3. Contact and quasiconformal maps

Let $(M,\mathcal{H})$ and $(M',\mathcal{H}')$ be manifolds with horizontal distributions. Let $U \subseteq M$ be an open set. A local C^1 diffeomorphism $f : U \to M'$ is called a *contact map* if $f_*(\mathcal{H}_p) = \mathcal{H}'_{f(p)}$. In particular, left translations and dilations are contact maps of a Carnot group to itself.

Let (M,d) and (M',d') be metric spaces and let $U \subseteq M$ be an open set. Let $f : U \to M'$ be a (topological) embedding. For $p \in M$ and for small $t \in \mathbb{R}$, we define the distortion function by

$$H_f(p,t) := \frac{\sup\{d'(f(p),f(q)) | d(p,q) \leqslant t\}}{\inf\{d'(f(p),f(q)) | d(p,q) \geqslant t\}}.$$

Definition 2.1. — *We say that f is K-*quasiconformal *for some $K \geqslant 1$ if*

$$\limsup_{t\to 0} H_f(p,t) \leqslant K,$$

for all $p \in M$.

In particular, left translations and dilations are 1-quasiconformal maps of a Carnot group to itself.

Quasiconformal maps between Carnot groups $(\mathbb{G},d)$ and $(\mathbb{G}',d')$ are Pansu differentiable almost everywhere with respect to any Haar measure, see [7, Théorème 2]. One such Haar measure is the Q-dimensional Hausdorff measure, where $Q = \sum_{i=1}^{s} i \dim V_i$ is the homogeneous dimension.

We recall that a continuous map $f : \mathbb{G} \to \mathbb{G}'$ is Pansu differentiable at $p \in \mathbb{G}$ if the limit

$$Df(p)(q) = \lim_{t\to 0^+} \delta_t^{-1} \circ \tau_{f(p)}^{-1} \circ f \circ \tau_p \circ \delta_t(q)$$

is uniform on compact sets and equals a homomorphism $Df(p) : \mathbb{G} \to \mathbb{G}'$. We call $Df(p)$ the *Pansu derivative* of f at p. The *Pansu differential* is the Lie algebra homomorphism $df(p) : \mathfrak{g} \to \mathfrak{g}'$ such that $Df(p) \circ \exp = \exp \circ\, df(p)$. Note that $Df(p)$ and $df(p)$ commute with dilating and so in particular, $df(p)$ is a strata preserving Lie algebra homomorphism.

The following results will be important for our purposes.

Theorem 2.2 (L. Capogna & M. Cowling, [1]). — *All 1-quasiconformal maps between Carnot groups are smooth.*

Furthermore, by [1, Corollary 7.2] we also have the following Lemma.

Lemma 2.3. — *If f is a quasiconformal embedding such that $df(p) = \delta_{\lambda(p)}$ for almost all points p in its domain of definition, then f is 1-quasiconformal and therefore smooth.*

2.4. Quasiconformal equivalence

The Pansu differential of a quasiconformal map is a graded group isomorphism. Consequently, we have the following fact.

Theorem 2.4 (P. Pansu, [7]). — *Two Carnot groups are quasiconformally equivalent if and only if they are isomorphic as groups.*

In particular, when f is an embedding of an open set $U \subset \mathbb{G}$ into $\mathbb{G}$ itself, we have that $df(p) \in \mathrm{Aut}_0(\mathfrak{g})$, for almost every $p \in U$. Furthermore, the set of smooth contact maps of $\mathbb{G}$ into itself coincides with the set of smooth Pansu differentiable maps of $\mathbb{G}$ to itself, see [11].

Theorem 2.4 was later generalized by Margulis and Mostow.

Theorem 2.5 (G. Margulis & G. Mostow, [2, 3]). — *Let $(M, \mathcal{H}, \mathrm{g})$ and $(M', \mathcal{H}', \mathrm{g}')$ be equiregular sub-Riemannian manifolds. Any quasiconformal embedding $f : U \subseteq M \to M'$ of an open set U of M induces, at almost every point $p \in U$, an isomorphism*

$$Df(p) : \mathbb{G}_p \to \mathbb{G}'_{f(p)},$$

between the tangent cones of M and M'.

One might wonder if the converse of the Margulis-Mostow Theorem holds true. Alas, we show that it is not the case. Indeed, we exhibit two sub-Riemannian manifolds that at every point have the same fixed Carnot group as tangent cone. Then we prove that they are not quasiconformally equivalent. In fact, we find such examples among the class of nilpotent Lie groups.

3. Ultrarigid groups

3.1. Definition of ultrarigidity

In this section we present a class of groups that we shall consider in proving the main theorem. Such groups have the property of having very few quasiconformal maps. Notice that left translations and group-dilations are always present. In fact, we are interested in the case when these are the only quasiconformal maps. Theorem 1.2 gives an algebraic characterization of such a situation and will be proved in Section 5. Because of Theorem 1.2, the notion of ultrarigid group may be defined in two equivalent manners.

Definition 3.1. — *A Carnot group* $\mathbb{G}$ *is said to be* ultrarigid *if one of the two equivalent properties [1.2.1] and [1.2.2] of Theorem 1.2 holds.*

Remark 3.2. — Property [1.2.2] has been considered by Pierre Pansu. He proved that there exist uncountably many groups with such property, see [7, Proposition 13.1]. This existence result of Pansu does not yield an example which suits our purpose. It is important to recall that Pansu mainly considered more general groups. Namely, we call *Pansu-rigid* those Carnot groups $\mathbb{G}$ for which any strata-preserving automorphism of Lie($\mathbb{G}$) is a similarity, *i.e.*, the composition of a dilation and an isometry. One of the main steps in proving Mostow rigidity for quaternionic hyperbolic spaces is to show that the boundary at infinity of such spaces are Pansu-rigid, see [7, Proposition 10.1]. Clearly, any ultrarigid group is Pansu-rigid.

Remark 3.3. — For Carnot groups of step 2, the nontrivial part of Theorem 1.2 has been proved by Capogna and Cowling, see [1, Corollary 7.4].

3.2. Examples of ultrarigid groups

In this section we present two examples of ultrarigid groups. The ultrarigidity can be verified by explicit computation of the strata preserving automorphisms using the MAPLE LieAlgebras package.

In Section 4.2 we shall need an example of an ultrarigid group whose structure can be deformed to a nonstratified nilpotent Lie group. The following Lie algebra determines an ultrarigid group having this flexibility.

Example 3.4. — Consider the sixteen dimensional Lie algebra with basis $\{e_i \mid i = 1, \dots, 16\}$ and bracket relations:

$$\begin{array}{lll}
[e_1,e_2] = e_{11}, & [e_1,e_3] = e_{13}, & [e_1,e_4] = e_{14}, \\
[e_1,e_5] = e_{15}, & [e_1,e_6] = e_{16}, & [e_2,e_3] = e_{13}, \\
[e_2,e_5] = e_{12}, & [e_2,e_6] = e_{14}, & [e_3,e_5] = e_{12}, \\
[e_3,e_6] = e_{13}, & [e_3,e_7] = e_{14}, & [e_4,e_5] = e_{12}, \\
[e_4,e_6] = e_{13}, & [e_4,e_8] = e_{14}, & [e_5,e_6] = e_{13}, \\
[e_5,e_8] = e_{12}, & [e_5,e_9] = e_{14}, & [e_6,e_8] = e_{12}, \\
[e_6,e_9] = e_{13}, & [e_6,e_{10}] = e_{14}, & [e_7,e_8] = e_{14}, \\
[e_7,e_9] = e_{12}, & [e_7,e_{10}] = e_{13}, & [e_8,e_9] = e_{13}. \\
[e_8,e_{10}] = e_{14}, & [e_9,e_{10}] = -e_{12} &
\end{array}$$

We note that this is a 2-step Carnot algebra with stratification $V_1 = \operatorname{span}\{e_1, \dots, e_{10}\}$ and $V_2 = \operatorname{span}\{e_{11}, \dots, e_{16}\}$. It can be deformed to a non-stratified nilpotent Lie algebra by adding the additional bracket $[e_1, e_{11}] = e_{14}$.

Example 3.5. — Our second example is the seventeen dimensional group corresponding to the Lie algebra obtained by extending the previous example by adding the additional bracket $[e_1, e_{11}] = e_{17}$. The strata are as follows: $V_1 = \operatorname{span}\{e_1, \dots, e_{10}\}$, $V_2 = \operatorname{span}\{e_{11}, \dots, e_{16}\}$ and $V_3 = \operatorname{span}\{e_{17}\}$. The significance of this example is that it shows that Theorem 1.2 does extend the result discussed in Remark 3.3.

4. A counterexample

4.1. A criterion for quasiconformal nonequivalence

In this section we show that sub-Riemannian nilpotent groups with same tangent cones need not to be quasiconformally equivalent. The main theorem of the section is Theorem 4.2, where ultrarigidity is assumed. Namely, we deal with a Carnot group $\mathbb{G}$ whose only quasiconformal maps are the elements of $\mathrm{TD}(\mathbb{G})$.

We start by showing that, in general, if a nilpotent subgroup H of $\mathrm{TD}(\mathbb{G})$ has codimension one, then H is in fact $\mathbb{G}$. To this end, let us study the group structure of $\mathrm{TD}(\mathbb{G})$. Composition of functions turns $\mathrm{TD}(\mathbb{G})$ into a Lie group that is isomorphic to a semidirect product $\mathbb{G} \rtimes \mathbb{R}$. Thus the Lie algebra of $\mathrm{TD}(\mathbb{G})$ is a semidirect product of $\mathfrak{g}$ with a one dimensional subgroup:

$$\operatorname{Lie}(\mathrm{TD}(\mathbb{G})) = \mathfrak{g} \rtimes \mathbb{R}.$$

Here the $\mathbb{R}$-factor is generated by the derivation D and the brackets in $\operatorname{Lie}(\mathrm{TD}(\mathbb{G}))$ are those of $\mathfrak{g}$ together with

$$[D, X] = D(X), \qquad \forall X \in \mathfrak{g}.$$

Lemma 4.1. — *Let $\mathbb{G} \neq \mathbb{R}$ be a Carnot group. Let $H < \mathrm{TD}(\mathbb{G})$ be a Lie subgroup of codimension 1 and assume that H is nilpotent. Then $H = \mathbb{G} \times \{0\}$.*

Proof. — Denote by $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{g} \rtimes \mathbb{R}$ the Lie algebras of H, G, and $\mathrm{TD}(\mathbb{G})$, respectively. Thus we have $\dim \mathfrak{h} = \dim \mathfrak{g} = n$ and $\dim(\mathfrak{g} \rtimes \mathbb{R}) = n + 1$. Let V_1 be the first layer of $\mathfrak{g}$, for which we recall that $\dim V_1 \geqslant 2$. Hence we get

$$\dim(V_1 \cap \mathfrak{h}) = \dim V_1 + \dim \mathfrak{h} - \dim(V_1 + \mathfrak{h}) \geqslant 2 + n - (n + 1) = 1.$$

Thus there exists $X \in V_1 \cap \mathfrak{h}$ with $X \neq 0$.

We note that if $V_1 \subseteq \mathfrak{h}$, then $\mathfrak{g} \subseteq \mathfrak{h}$ and so $\mathbb{G} = H$. Now consider the case $V_1 \setminus (V_1 \cap \mathfrak{h}) \neq \emptyset$ and let Y be a nonzero element in $V_1 \setminus (V_1 \cap \mathfrak{h})$. Then, since $\mathfrak{h}$ has codimension 1 in $\mathfrak{g} \rtimes \mathbb{R}$, we have that $\mathfrak{g} \rtimes \mathbb{R} = \operatorname{span}\{\mathfrak{h}, Y\}$. Thus there exist $Z \in \mathfrak{h}$ and $\alpha \in \mathbb{R}$ such that $D = Z + \alpha Y$. Notice that $Z \neq 0$, otherwise $D = \alpha Y \in V_1$, which is not true. Therefore, the element $D - \alpha Y$ is in $\mathfrak{h} \setminus \{0\}$, and since D preserves strata, there exists $Z_m \in V_2 \oplus \cdots \oplus V_s$ such that

$$(\mathrm{ad}_{D-\alpha Y})^m (X) = X + Z_m, \qquad \forall m \in \mathbb{N}.$$

However, since $D - \alpha Y$ and X are both elements of the nilpotent algebra $\mathfrak{h}$, the iterated bracket $(\mathrm{ad}_{D-\alpha Y})^m (X)$ should be eventually 0, which contradicts the fact that $X \neq 0$. Thus we conclude that $V_1 \setminus (V_1 \cap \mathfrak{h}) \neq \emptyset$ cannot occur. □

Recall that, unless otherwise said, a Carnot group is always equipped with a left invariant sub-Riemannian distance with respect to the first stratum as horizontal distribution.

Theorem 4.2. — *Assume $\mathbb{G}$ is an ultrarigid Carnot group. Let H be a connected, simply connected nilpotent Lie group endowed with a left invariant sub-Riemannian distance. If there exist open sets $U \subset \mathbb{G}$, $U' \subset H$, and a quasiconformal homeomorphism between U and U', then H is isomorphic to $\mathbb{G}$.*

Proof. — Let $f : U \to U'$ be the quasiconformal homeomorphism. Composing with a suitable translation, we may assume that $f(e_{\mathbb{G}}) = e_H$. Then, for $h \in H$, we consider the quasiconformal map

$$f_h := f^{-1} \circ \tau_h \circ f : \tilde{U} \subseteq \mathbb{G} \to \mathbb{G},$$

which is well defined for h close enough to the identity in H and for some open set $\tilde{U} \subseteq \mathbb{G}$. By definition of ultrarigidity, we can assume f_h is in fact in $\mathrm{TD}(\mathbb{G})$.

We claim that the map $h \mapsto f_h$ is an injective local homomorphism of H into $\mathrm{TD}(\mathbb{G})$. Therefore, going to the Lie algebra level, we get a (globally defined) injective local homomorphism of $\mathfrak{h}$ into $\mathrm{Lie}(\mathrm{TD}(\mathbb{G}))$. Indeed, the map is a homomorphism, because

$$f_h \circ f_{h'} = f^{-1} \circ \tau_h \circ \tau_{h'} \circ f = f^{-1} \circ \tau_{hh'} \circ f = f_{hh'}, \qquad \forall h, h' \in H.$$

Regarding the injectivity, for $h \neq e_H$, we show that $f_h \neq \mathrm{id}$. Since $f(e_G) = e_H$,

$$f_h(e_G) = (f^{-1} \circ \tau_h \circ f)(e_G) = (f^{-1} \circ \tau_h)(e_H) = f^{-1}(h) \neq e_G.$$

Therefore $\mathfrak{h}$ is isomorphic to a subalgebra $\mathfrak{h}_0 < \mathrm{Lie}(\mathrm{TD})(\mathbb{G}))$. Since $\mathfrak{h}_0$ is nilpotent, by Lemma 4.1, we have that $\mathfrak{h}_0 = \mathfrak{g} \times \{0\}$. Thus $\mathfrak{h}$ is isomorphic to $\mathfrak{g}$ and therefore H is isomorphic to $\mathbb{G}$, since they are connected, simply connected nilpotent Lie groups. □

4.2. Example of a non-Carnot group with ultrarigid tangent

We present here an example of a sub-Riemannian nilpotent Lie group that demonstrates the validity of Theorem 1.1. Namely, we exhibit a nilpotent Lie group H whose tangent cone is the 16-dimensional group $\mathbb{G}$ as in Example 3.4 such that the pair $\mathbb{G}$, H satisfy the condition of Theorem 4.2. In turn, this implies Theorem 1.1 and Corollary 1.4.

The nilpotent group is the following. Take $\mathbb{G} = \exp(\mathfrak{g})$ where $\mathfrak{g}$ is Example 3.4, and let $H = \exp(\mathfrak{h})$ where $\mathfrak{h}$ is the 16-dimensional nilpotent Lie algebra with the same bracket relations as $\mathfrak{g}$ and the additional bracket $[e_1, e_{11}] = e_{14}$. Note that this additional bracket is of order 3 and so $\mathfrak{h}$ is not stratified.

If X_i denotes the left invariant vector field corresponding to e_i, then the horizontal space $\mathcal{H} \subset TH$ is framed by $X_1, \ldots, X_{10}$. For a given point p, L_1 is the set of smooth sections of $\mathcal{H}$ defined on a neighbourhood of p, and $L_2 = L_1 + [L_1, L_1]$. It follows that $X_1 \in L_1$, $X_{11} \in L_2$ and $X_{14} \in L_2$, hence

$$[X_1, X_{11}] + L_2 = X_{14} + L_2 = 0 + L_2.$$

On the other hand, if $X, Y \in L_1$ and $[X, Y] = 0 + L_1$, then $[X, Y] = 0$ and so $\mathfrak{g}(p) = \mathfrak{g}$ for all $p \in H$.

5. Equivalence of definitions for ultrarigid groups

In this section we prove Theorem 1.2. Part of our proof uses a theorem of Tanaka, that provides a characterization of the space of contact maps on a Carnot group $\mathbb{G}$ at the infinitesimal level. In order to state Tanaka's theorem, it is convenient to change part of the notation. Throughout this section we shall denote by $\mathfrak{g}_{-i}$ the strata V_i of a nilpotent and stratified Lie algebra $\mathfrak{g}$, for every $i = 1, \ldots, s$.

5.1. Tanaka Prolongation

The *Tanaka prolongation* of $\mathfrak{g}$ is the graded Lie algebra $\mathrm{Prol}(\mathfrak{g})$ given by the direct sum

$$\mathrm{Prol}(\mathfrak{g}) := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k,$$

where $\mathfrak{g}_k = \{0\}$ for $k < -s$ and for each $k \geqslant 0$, $\mathfrak{g}_k$ is inductively defined by

$$\mathfrak{g}_k := \Big\{ u \in \bigoplus_{\ell<0} \mathfrak{g}_{\ell+k} \otimes \mathfrak{g}_\ell^* \mid u([X,Y]) = [u(X),Y] + [X,u(Y)] \Big\}.$$

Clearly, for $k = 0$ we get the strata preserving derivations. If $u \in \mathfrak{g}_k$, where $k \geqslant 0$, then the condition in the definition becomes the Jacobi identity upon setting $[u,X] = u(X)$ when $X \in \mathfrak{g}$. Furthermore, if $u \in \mathfrak{g}_k$ and $u' \in \mathfrak{g}_\ell$, where $k,\ell \geqslant 0$, then $[u,u'] \in \mathfrak{g}_{k+\ell}$ is defined inductively according to the Jacobi identity, that is

$$[u,u'](X) = [u,[u',X]] - [u',[u,X]].$$

In [9], Tanaka shows that $\mathrm{Prol}(\mathfrak{g})$ determines the structure of the contact vector fields on the group G. A *contact vector field* is defined as the infinitesimal generator of a local flow of contact maps, and the space of these vector fields forms a Lie algebra with the usual bracket of vector fields. We recall that D denotes the standard dilation defined in Section 2.1, and we rephrase the result of Tanaka with the following statement.

Theorem 5.1 (N. Tanaka, [9]). — *Let $U \subset \mathbb{G}$ be an open set. Denote by $\mathcal{C}(U)$ the Lie algebra of smooth contact vector fields on U. If* $\mathrm{Prol}(\mathfrak{g})$ *is finite dimensional, then there exists a Lie algebra isomorphism between* $\mathrm{Prol}(\mathfrak{g})$ *and* $\mathcal{C}(U)$. *In particular, if* $\mathrm{Prol}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}_0$ *and* $\mathfrak{g}_0 = \mathbb{R}D$, *one may choose this isomorphism to be the linear map ρ defined by the assignments*

$$\rho(D)\phi(p) = \frac{d}{dt}\phi(\exp(e^{-tD}\exp^{-1}(p)))|_{t=0}, \tag{5.1}$$

$$\rho(X)\phi(p) = \frac{d}{dt}\phi(\exp(-tX)p)|_{t=0}, \tag{5.2}$$

where $p \in U$, ϕ is a smooth function on U and X varies in $\mathfrak{g}$.

The interested reader can consult [9, 12] for a thorough overview, and [5] for a basic introduction.

Remark 5.2. — Since $\mathbb{G}$ is simply connected, then in the case $\mathrm{Prol}(\mathfrak{g})$ is finite dimensional, every $V \in \mathcal{C}(U)$ uniquely extends to an element of $\mathcal{C}(\mathbb{G})$, see [9, page 34].

Remark 5.3. — We notice that $\mathrm{Prol}(\mathfrak{g})$ is finite dimensional if and only if $\mathfrak{g}_k = \{0\}$ for some $k \geqslant 0$. In fact, we have that $\mathfrak{g}_k = \{0\}$ implies $\mathfrak{g}_{k+l} = \{0\}$ for every $l \geqslant 0$.

We note that if $\mathrm{Aut}_0(\mathfrak{g})$ consists only of dilations, then $\mathfrak{g}_0$ is exactly the span of D and thus one dimensional. The following lemma implies that in this case the only contact flows are dilations. We would like to thank M. Reimann for bringing this fact to our attention.

Lemma 5.4. — *Let $\mathfrak{g}$ be a nonabelian nilpotent and stratified Lie algebra such that $\mathfrak{g}_0$ is one dimensional. Then the prolongation of $\mathfrak{g}$ is $\mathfrak{g} \oplus \mathfrak{g}_0$.*

Proof. — We need to show that $\mathfrak{g}_1 = \{0\}$. Set $u \in \mathfrak{g}_1$. Then $u(\mathfrak{g}_j) \subset \mathfrak{g}_{j+1}$ for every $j = -s, \dots, -1$ and $u([X, Y]) = [u(X), Y] + [X, u(Y)]$ for all $X, Y \in \mathfrak{g}$. Since $\mathfrak{g}_0$ has dimension one, then $\mathfrak{g}_0 = \mathrm{span}\{D\}$. In particular, if $X \in \mathfrak{g}_{-1}$, then $u(X) = c(X)D$, where $c : \mathfrak{g}_{-1} \to \mathbb{R}$ is linear. In order to show that $u = 0$, it is enough to prove that $c(X) = 0$ for all $X \in \mathfrak{g}_{-1}$.

Let $\mathfrak{z}(\mathfrak{g})$ denote the centre of $\mathfrak{g}$, and let $Z \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{-k} \neq \emptyset$. Then for all $X \in \mathfrak{g}_{-1}$

$$0 = u([X, Z]) = [u(X), Z] + [X, u(Z)] = -c(X)kZ + [X, u(Z)],$$

whence

$$[X, u(Z)] = c(X)kZ.$$

By induction, it is easy to show that

$$[X, u^j(Z)] = c(X)\sum_{l=0}^{j-1}(k-l)u^{j-1}(Z) = c(X)\frac{j(2k-j+1)}{2}u^{j-1}(Z),$$

which in the case $j = k$ gives

$$[X, u^k(Z)] = c(X)\frac{k(k+1)}{2}u^{k-1}(Z). \tag{5.3}$$

Furthermore, iterating (5.3) gives

$$\mathrm{ad}_X^k u^k(Z) = A_k c(X)^k Z, \tag{5.4}$$

where A_k is a positive constant depending on k only. Notice that since $u^k(Z) \in \mathfrak{g}_0$, we have $u^{k-1}(Z) \in \mathfrak{g}_{-1}$, and it follows that

$$[X, u^k(Z)] = [X, u(u^{k-1}(Z))] = -c(u^{k-1}(Z))X,$$

for all $X \in \mathfrak{g}_{-1}$. We conclude that for $k \geqslant 2$, the left hand side of (5.4) is zero. It follows that since $\mathfrak{g}$ is nonabelian, then $c(X) = 0$ since we can set $k = s \geqslant 2$, and choose a nonzero $Z \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{-s}$. □

5.2. Proof of Theorem 1.2

[1.2.1] $\Rightarrow$ [1.2.2]. Every element $\alpha \in \mathrm{Aut}_0(\mathfrak{g})$ lifts to an automorphism of $\mathbb{G}$ which is also a contact map. Therefore by hypothesis this contact map is an element of $\mathrm{TD}(\mathbb{G})$, and since it is an automorphism it must be a group-dilation. We conclude that α is an algebra-dilation.

[1.2.2] $\Rightarrow$ [1.2.1]. Let $f : U \to \mathbb{G}$ be a quasiconformal embedding. Then f is Pansu differentiable at almost every $p \in U$. Therefore $df(p) = \delta_{\lambda(p)}$ for almost every $p \in U$ and by Lemma 2.3, f is 1-quasiconformal and smooth. In particular f is a smooth contact map.

After normalizing with left translations if necessary, we can assume that $e \in U$ and $f(e) = e$. Moreover $f_*W \in \mathcal{C}(f(U))$ for every $W \in \mathcal{C}(U)$. By Remark 5.2, f_* induces a Lie algebra isomorphism of $\mathcal{C}(U)$, which we also denote by f_*. It then follows that $\rho^{-1}f_*\rho$ is an automorphism of $\mathfrak{g} \oplus \mathfrak{g}_0$. This automorphism has some extra properties that we show in the following lemma.

Lemma 5.5. — *The automorphism $\alpha := \rho^{-1}f_*\rho$ preserves $\mathfrak{g}$ and $\mathfrak{g}_0$. Moreover, $\alpha|_{\mathfrak{g}} \in \mathrm{Aut}_0(\mathfrak{g})$.*

Proof. — By (5.1) we see that $\rho(D)(e) = 0$, whereas by (5.2) we have that $\rho(X)(e) \neq 0$ for every $X \in \mathfrak{g}$. Since $f(e) = e$, we conclude that $\mathfrak{g}_0$ is preserved. Since D is surjective, it follows that $[\mathfrak{g} \oplus \mathfrak{g}_0, \mathfrak{g} \oplus \mathfrak{g}_0] = \mathfrak{g}$ which implies $\alpha(\mathfrak{g}) = \mathfrak{g}$.

In order to show that $\alpha|_{\mathfrak{g}}$ preserves the strata, it is enough to prove that it preserves $\mathfrak{g}_{-1}$. Since f is contact, this is true if the equation

$$f_*\rho(X) = \rho(f_*|_e X), \tag{5.5}$$

holds for every $X \in \mathfrak{g}_{-1}$. To show (5.5), we first observe that the flow of $f_*\rho(X)$ through p is $f_t(p) = f(\exp(-tX)f^{-1}(p))$ and in particular $f_t(e) = f(\exp(-tX))$. Since $f(e) = e$, we have that $f_*|_e = df(e)$ on $\mathfrak{g}_{-1}$. By hypothesis, $df(e) = \delta_\lambda$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, and it follows that

$$\frac{d}{dt} f_t(e) = -f_*|_e(X) = -\lambda X.$$

Hence (5.5) is valid when evaluated at the identity. The equality at all points follows from the fact that both $f_*\rho(X)$ and $\rho(f_*|_e X)$ are right invariant vector fields, see (5.2). □

We now conclude the proof of Theorem 1.2. Since $\rho^{-1}f_*\rho|_{\mathfrak{g}} \in \mathrm{Aut}_0(\mathfrak{g})$, it follows that $\rho^{-1}f_*\rho|_{\mathfrak{g}} = \delta_\lambda$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. If $F(p) = \delta_{1/\lambda} \circ f$, then $F(e) = e$, and by (5.5), we see that $\rho^{-1}F_*\rho|_{\mathfrak{g}} = I$. In particular $F_*\rho(X) = \rho(X)$ for every $X \in \mathfrak{g}$. Thus, F_* preserves each right invariant vector field, and so F commutes with the left translations. Hence $F(pq) = pF(q)$, and putting $q = e$ shows that F is the identity. Therefore $f = \delta_\lambda$ and the proof of Theorem 1.2 is complete.

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