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ON VARIETIES OF HILBERT TYPE

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Abstract. — A variety X over a field K is of Hilbert type if $X(K)$ is not thin. We prove that if $f\colon X \to S$ is a dominant morphism of K-varieties and both S and all fibers $f^{-1}(s)$, $s \in S(K)$, are of Hilbert type, then so is X. We apply this to answer a question of Serre on products of varieties and to generalize a result of Colliot-Thélène and Sansuc on algebraic groups.

Résumé. — Une variété X sur un corps K a la propriété de Hilbert si $X(K)$ n'est pas mince. Nous montrons que si $f\colon X \to S$ est un morphisme de K-variétés dominant et si S ainsi que toutes les fibres $f^{-1}(s)$ pour $s \in S(K)$ ont la propriété de Hilbert, alors X aussi. Ceci nous permet de répondre à une question de Serre concernant les produits de variétés, et de généraliser un résultat de Colliot-Thélène et Sansuc sur les groupes algébriques.

1. Introduction

In the terminology of thin sets (we recall this notion in Section 2), Hilbert's irreducibility theorem asserts that $\mathbb{A}^n_K(K)$ is not thin, for any number field K and any $n \geqslant 1$. As a natural generalization a K-variety X is called of Hilbert type if $X(K)$ is not thin. The importance of this definition stems from the observation of Colliot-Thélène and Sansuc [2] that the inverse Galois problem would be settled if every unirational variety over $\mathbb{Q}$ was of Hilbert type.

In this direction, Colliot-Thélène and Sansuc [2, Cor. 7.15] prove that any connected reductive algebraic group over a number field is of Hilbert type. This immediately raises the question whether the same holds for all linear algebraic groups (note that these are unirational). Another question, asked by Serre [19, p. 21], is whether a product of two varieties of Hilbert

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type is again of Hilbert type. The main result of this paper gives a sufficient condition for a variety to be of Hilbert type:

Theorem 1.1. — *Let K be a field and $f\colon X \to S$ a dominant morphism of K-varieties. Assume that the set of $s \in S(K)$ for which the fiber $f^{-1}(s)$ is a K-variety of Hilbert type is not thin. Then X is of Hilbert type.*

As an immediate consequence we get the following result for a family of varieties over a variety of Hilbert type:

Corollary 1.2. — *Let K be a field and $f\colon X \to S$ a dominant morphism of K-varieties. Assume that S is of Hilbert type and that for every $s \in S(K)$ the fiber $f^{-1}(s)$ is of Hilbert type. Then X is of Hilbert type.*

Using this result we resolve both questions discussed above affirmatively, see Corollary 3.4 and Proposition 4.2.

2. Background

Let K be a field. A *K-variety* is a separated scheme of finite type over K which is geometrically reduced and geometrically irreducible. Thus, a nonempty open subscheme of a K-variety is again a K-variety. If $f\colon X \to S$ is a morphism of K-varieties and $s \in S(K)$, then $f^{-1}(s) := X \times_S \operatorname{Spec}(\kappa(s))$, where $\kappa(s)$ is the residue field of s, denotes the scheme theoretic fiber of f at s. This fiber is a separated scheme of finite type over K, which needs not be reduced or connected in general. We identify the set $f^{-1}(s)(K)$ of K-rational points of the fiber with the set theoretic fiber $\{x \in X(K) \mid f(x) = s\}$.

Let X be a K-variety. A subset T of $X(K)$ is called *thin* if there exists a proper Zariski-closed subset C of X, a finite set I, and for each $i \in I$ a K-variety Y_i with $\dim(Y_i) = \dim(X)$ and a dominant separable morphism $p_i\colon Y_i \to X$ of degree $\geqslant 2$ (in particular, p_i is generically étale, *cf.* Lemma 3.3) such that

$$T \subseteq \bigcup_{i\in I} p_i(Y_i(K)) \cup C(K).$$

A K-variety X is *of Hilbert type* if $X(K)$ is not thin, *cf.* [19, Def. 3.1.2]. Note that X is of Hilbert type if and only if some (or every) open subscheme of X is of Hilbert type, *cf.* [19, p. 20]. A field K is *Hilbertian* if $\mathbb{A}^1_K$ is of Hilbert type. We note that if there exists a K-variety X of positive dimension such that X is of Hilbert type, then K is Hilbertian [5, Prop. 13.5.3].

All global fields and, more generally, all infinite fields that are finitely generated over their prime fields are Hilbertian [5, Thm. 13.4.2]. Many more fields are known to be Hilbertian, for example the maximal abelian Galois extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$, [5, Thm. 16.11.3]. On the other hand, local fields like $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are not Hilbertian [5, Ex. 15.5.5].

3. Proof of Theorem 1.1

A key tool in the proof of Theorem 1.1 is the following consequence of Stein factorization.

Lemma 3.1. — *Let K be a field and $\psi\colon Y \to S$ a dominant morphism of normal K-varieties. Then there exists a nonempty open subscheme $U \subset S$, a K-variety T and a factorization*

$$\psi^{-1}(U) \xrightarrow{g} T \xrightarrow{r} U$$

of ψ such that the fibers of g are geometrically irreducible and r is finite and étale.

Proof. — See [13, Lemma 9]. □

Lemma 3.2. — *Let K be a field and $f\colon X \to S$ a dominant morphism of normal K-varieties. Assume that the set Σ of $s \in S(K)$ for which $f^{-1}(s)$ is a K-variety of Hilbert type is not thin. Let I be a finite set and let $p_i\colon Y_i \to X$, $i \in I$, be finite étale morphisms of degree $\geqslant 2$. Then $X(K) \not\subseteq \bigcup_{i\in I} p_i(Y_i(K))$.*

Proof. — For $i \in I$ consider the composite morphism $\psi_i := f \circ p_i\colon Y_i \to S$. By Lemma 3.1 there is a nonempty open subscheme U_i of S and a factorization

$$\psi_i^{-1}(U_i) \xrightarrow{g_i} T_i \xrightarrow{r_i} U_i$$

of ψ_i such that the morphism g_i has geometrically irreducible fibers, r_i is finite and étale, and such that T_i is a K-variety. We now replace successively S by $\bigcap_{i\in I} U_i$, X by $f^{-1}(S)$, T_i by $r_i^{-1}(S)$ and Y_i by $p_i^{-1}(X)$, to assume in addition that $r_i\colon T_i \to S$ is finite étale for every $i \in I$.

For $s \in S(K)$ denote by $X_s := f^{-1}(s)$ the fiber of f over s. Then X_s is a K-variety of Hilbert type for each $s \in \Sigma$. Furthermore we define $Y_{i,s} := \psi_i^{-1}(s)$ and let $p_{i,s}\colon Y_{i,s} \to X_s$ be the corresponding projection morphism. Then $p_{i,s}$ is a finite étale morphism of the same degree as p_i. In particular, the K-scheme $Y_{i,s}$ is geometrically reduced. For every $s \in S(K)$ and every $i \in I$ we have constructed a commutative diagram

$$\begin{array}{ccccc} Y_{i,s} & \longrightarrow & Y_i & \xrightarrow{g_i} & T_i \\ {\scriptstyle p_{i,s}}\downarrow & & {\scriptstyle p_i}\downarrow & \overset{\psi_i}{\searrow} & \downarrow{\scriptstyle r_i} \\ X_s & \longrightarrow & X & \xrightarrow{f} & S \end{array}$$

in which the left hand rectangle is cartesian. Set $J := \{i \in I : \deg(r_i) \geqslant 2\}$. Then $\bigcup_{i\in J} r_i(T_i(K)) \subseteq S(K)$ is thin, so by assumption there exists

$$s \in \Sigma \smallsetminus \bigcup_{i\in J} r_i(T_i(K)).$$

For $i \in J$ there is no K-rational point of T_i over s, hence $Y_{i,s}(K) = \emptyset$ for every $i \in J$. For $i \in I \smallsetminus J$, the finite étale morphism r_i is of degree 1, hence an isomorphism, and therefore $Y_{i,s}$ is geometrically irreducible. Thus, $Y_{i,s}$ is a K-variety. So since X_s is of Hilbert type, there exists $x \in X_s(K)$ such that $x \notin \bigcup_{i\in I\smallsetminus J} p_{i,s}(Y_{i,s}(K))$. Thus

$$x \notin \bigcup_{i\in J\smallsetminus I} p_{i,s}(Y_{i,s}(K)) = \bigcup_{i\in I} p_{i,s}(Y_{i,s}(K)),$$

hence $x \notin \bigcup_{i\in I} p_i(Y_i(K))$, as needed. □

The following fact is well-known, but for the sake of completeness we provide a proof:

Lemma 3.3. — *Let K be a field, let X, Y be K-varieties with $\dim(X) = \dim(Y)$, and let $p\colon Y \to X$ be a dominant separable morphism. Then there exists a nonempty open subscheme U of X such that the restriction of p to a morphism $p^{-1}(U) \to U$ is finite and étale.*

Proof. — By the theorem of generic flatness (*cf.* [11, 6.9.1]) there is a non-empty open subscheme V of X such that the restriction of p to a morphism $p^{-1}(V) \to V$ is flat (and in particular open). This restriction is quasi-finite by [12, 14.2.4], because the generic fiber of f is finite due to our assumption $\dim(X) = \dim(Y)$. By Zariski's main theorem there exists a K-variety $\overline{Y}$, an open immersion $i\colon p^{-1}(V) \to \overline{Y}$ and a finite morphism $f\colon \overline{Y} \to V$ such that $f \circ i = p$. The ramification locus $C \subset \overline{Y}$ of f is closed (*cf.* [8, I.3.3]), and $C \neq \overline{Y}$ because f is separable. Define $U := V \smallsetminus f((\overline{Y} \smallsetminus \mathrm{im}(i)) \cup C)$. Then U is open (*cf.* [6, 6.1.10]) and non-empty, and $f^{-1}(U) \subset \mathrm{im}(i) \smallsetminus C$. Hence the restriction of f to a morphism $f^{-1}(U) \to U$ is finite and étale, and the assertion follows from that. □

Proof of Theorem 1.1. — Let K be a field, and $f\colon X \to S$ a dominant morphism of K-varieties. Assume that the set Σ of those $s \in S(K)$ for which $f^{-1}(s)$ is of Hilbert type is not thin. Let $C \subseteq X$ be a proper Zariski-closed subset. Let I be a finite set and suppose that Y_i is a K-variety with $\dim(Y_i) = \dim(X)$ and $p_i\colon Y_i \to X$ is a dominant separable morphism of degree $\geqslant 2$, for every $i \in I$. We have to show that $X(K) \not\subseteq C(K) \cup \bigcup_{i\in I} p_i(Y_i(K))$.

By Lemma 3.3 and [11, 6.12.6, 6.13.5] there exists a normal nonempty open subscheme $X' \subset X \smallsetminus C$ such that the restriction of each p_i to a morphism $p_i^{-1}(X') \to X'$ is finite and étale. The image $f(X')$ contains a nonempty open subscheme S' of S (*cf.* [10, 1.8.4], [7, 9.2.2]). Furthermore, S' contains a nonempty normal open subscheme S''. Let us define $X'' := f^{-1}(S'')\cap X'$ and $Y_i'' := p_i^{-1}(X'')$. Then the restriction of f to a morphism $f''\colon X'' \to S''$ is a surjective morphism of normal K-varieties, $\Sigma \cap S''(K)$ is not thin, and $f''^{-1}(s)$ is of Hilbert type for every $s \in \Sigma \cap S''(K)$ because it is an open subscheme of $f^{-1}(s)$. The restriction p_i'' of p_i to a morphism $Y_i'' \to X''$ is finite and étale for every $i \in I$. By Lemma 3.2 applied to f'' and the p_i'' we have

$$\begin{aligned}\emptyset \neq X''(K) &\smallsetminus \bigcup_{i\in I} p_i''(Y_i''(K))\\ &= X''(K) \smallsetminus \bigcup_{i\in I} p_i(Y_i(K))\\ &\subseteq X(K) \smallsetminus \Big(C(K) \cup \bigcup_{i\in I} p_i(Y_i(K))\Big),\end{aligned}$$

so $X(K) \not\subseteq C(K) \cup \bigcup_{i\in I} p_i(Y_i(K))$, as needed. □

As an immediate consequence we get an affirmative solution of Serre's question mentioned in the introduction.

Corollary 3.4. — *Let K be a field. If X, Y are K-varieties of Hilbert type, then $X \times Y$ is of Hilbert type.*

Proof. — Denote by $f\colon X \times Y \to X$ the projection. Then $f^{-1}(x)$ is isomorphic to Y and hence of Hilbert type for every $x \in X(K)$. Thus Corollary 1.2 gives that $X \times Y$ is of Hilbert type. □

4. Algebraic groups of Hilbert type

By an *algebraic group* over a field K we shall mean a connected smooth group scheme over K. Recall that such an algebraic group is a K-variety, see [9, Exp $\mathrm{VI_A}$, 0.3, 2.1.2, 2.4]. If G is an algebraic group over K, then $G(K_s)$ is a $\mathrm{Gal}(K)$-group, where K_s denotes a separable closure of K and $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ is the absolute Galois group of K, and there is an associated Galois cohomology pointed set $H^1(K,G) = H^1(\mathrm{Gal}(K), G(K_s))$, which classifies isomorphism classes of $G(K_s)$-torsors, *cf.* [15, Prop. 1.2.3].

Proposition 4.1. — *Let K be a field and let*

$$1 \to N \to G \xrightarrow{p} Q \to 1$$

be a short exact sequence of algebraic groups over K. If $H^1(K,N) = 1$ and both N and Q are of Hilbert type, then G is of Hilbert type.

Proof. — It suffices to show that $p^{-1}(x)$ is of Hilbert type for every $x \in Q(K)$, because then Corollary 1.2 implies the assertion. Let $x \in Q(K)$ and $F = p^{-1}(x)$. There is an exact sequence of $\mathrm{Gal}(K)$-groups

$$1 \to N(K_s) \to G(K_s) \to Q(K_s) \to 1,$$

where the right hand map is surjective, because for every point $x \in Q(K_s)$ the fiber over x is a non-empty K_s-variety and thus has a K_s-rational point. Since the $\mathrm{Gal}(K)$-set $F(K_s)$ is a coset of $N(K_s)$, it is a $N(K_s)$-torsor. Our hypothesis $H^1(K,N) = 1$ implies that $F(K_s)$ is isomorphic to the trivial $N(K_s)$-torsor $N(K_s)$. It follows that F is isomorphic to N as a K-variety, hence F is of Hilbert type. □

Using this, we generalize the result of Colliot-Thélène and Sansuc [2, Cor. 7.15] from reductive groups to arbitrary linear groups.

Theorem 4.2. — *Every linear algebraic group G over a perfect Hilbertian field K is of Hilbert type.*

Proof. — We denote by G_u the unipotent radical of G (*cf.* [14, Prop. XVII.1.2]). We have a short exact sequence of algebraic groups over K

$$(*) \qquad 1 \to G_u \to G \to Q \to 1$$

with Q reductive, *cf.* [14, Prop. XVII.2.2]. By [2, Cor. 7.15], Q is of Hilbert type. Since K is perfect, G_u is split, *i.e.* there exists a series of normal algebraic subgroups

$$1 = U_0 \subseteq \cdots \subseteq U_n = G_u$$

such that $U_{i+1}/U_i \cong \mathbb{G}_a$ for each i, *cf.* [1, 15.5(ii)]. The groups U_i are unipotent, hence $H^1(K,U_i) = 1$ by [18, Ch. III §2.1, Prop. 6], and $\mathbb{G}_a$ is of Hilbert type since K is Hilbertian. Thus, an inductive application of Proposition 4.1 implies that G_u is of Hilbert type. Finally we apply Proposition 4.1 to the exact sequence $(*)$ to conclude that G is of Hilbert type. □

Remark 4.3. — In the special case where K is a number field, Sansuc proved a much more precise result: It follows from [17, Cor. 3.5(ii)] that a linear algebraic group G over a number field satisfies the so-called *weak weak approximation* property [19, Def. 3.5.6], which, by a theorem of Colliot-Thélène and Ekedahl, in particular implies that G is of Hilbert type, *cf.* [19, Thm. 3.5.7].

Remark 4.4. — The special case of Theorem 4.2 where G is simply connected and K is finitely generated is also a consequence of a result of Corvaja, see [4, Cor. 1.7].

Remark 4.5. — We point out that Theorem 4.2 could be deduced also from Corollary 3.4 (instead of Corollary 1.2) via [16, Cor. 1] and the fact that a unipotent group over a perfect field is rational, *cf.* [9, XIV, 6.3].

As a consequence of Theorem 4.2, we get a more general statement for homogeneous spaces, which was pointed out to us by Borovoi:

Corollary 4.6. — *If G is a linear algebraic group over a perfect Hilbertian field K, and H is a connected algebraic subgroup of G, then the quotient G/H is of Hilbert type.*

Proof. — For the existence of the quotient $Q := G/H$ see for example [1, Ch. II Thm. 6.8]. If $\mathcal{H}$ denotes the generic fiber of $G \to Q$ and $\overline{F}$ is an algebraic closure of the function field $K(Q)$ of Q, then $\mathcal{H}_{\overline{F}} \cong H_{\overline{F}}$ by translation on G. Thus, $\mathcal{H}$ is geometrically irreducible since H is, so [2, Prop. 7.13] implies that Q is of Hilbert type. □

We also get a complete classification of the algebraic groups that are of Hilbert type over a number field:

Corollary 4.7. — *An algebraic group G over a number field K is of Hilbert type if and only if it is linear.*

Proof. — If G is linear, then it is of Hilbert type by Theorem 4.2. Conversely, assume that G is of Hilbert type. Chevalley's theorem [3, Thm. 1.1] gives a short exact sequence of algebraic groups over K,

$$1 \to H \to G \to A \to 1$$

with H linear and A an abelian variety. As in the proof of Corollary 4.6 we conclude that the generic fiber of $G \to A$ is geometrically irreducible, and therefore A is of Hilbert type. Since no nontrivial abelian variety over a number field is of Hilbert type, *cf.* [5, Remark 13.5.4], A is trivial and $G \cong H$ is linear. □

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