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Shunsuke YAMANA

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SYMPLECTIC PERIODS OF THE CONTINUOUS SPECTRUM OF GL$(2n)$

by Shunsuke YAMANA

Abstract. — We provide a formula for the symplectic period of an Eisenstein series on GL$(2n)$ and determine when it is not identically zero.

Résumé. — On donne une formule pour la période symplectique d'une série d'Eisenstein pour le groupe GL$(2n)$ et on détermine sous quelles conditions celle-ci n'est pas identiquement nulle.

Introduction

After Jacquet and Rallis [6] initiated the study of global symplectic periods for automorphic representations of GL$(2n)$, Offen [10, 11] determined which automorphic representations in the discrete spectrum of GL$(2n)$ have a nonvanishing symplectic period. In this paper we generalize his result to the entire automorphic spectrum of GL$(2n)$.

We write G for the group GL$(2n)$ viewed as an algebraic group over a number field F with adele ring $\mathbb{A}$. Fix a skew symmetric matrix ϵ in $G(F)$ and let $H = H_\epsilon$ denote its symplectic group. For an automorphic form ϕ on $G(\mathbb{A})$, we define the symplectic period of ϕ by

$$P^H(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)dh.$$

The integral may not converge in general, but can be defined via regularization (see [10]). Let π be an irreducible subrepresentation of the space of automorphic forms on $G(\mathbb{A})$. We say that π is H-distinguished if there is $\phi \in \pi$ such that $P^H(\phi) \neq 0$. In the description below, we refer to the body of this paper for all unexplained notation.

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The theory of Eisenstein series provides a description of the continuous spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$ in terms of the discrete spectrum of Levi subgroups of G. Let $Q = LV$ be a standard parabolic subgroup of G of type $(k_1, \dots, k_r)$. Given a square-integrable automorphic form ψ on $V(\mathbb{A})Q(F)\backslash G(\mathbb{A})$ and $s \in \mathfrak{a}^*_{L,\mathbb{C}}$, the Eisenstein series

$$E(g, \psi, s) = \sum_{\gamma \in Q(F)\backslash G(F)} \psi(\gamma g) e^{\langle s, H_L(\gamma g)\rangle}$$

converges for $\Re s$ regular enough in the positive Weyl chamber. The Eisenstein series is meromorphic in the complex parameter s and is holomorphic near the imaginary axis $\sqrt{-1}\mathfrak{a}^*_L$.

Lemma 2.4 shows that if $s \in \sqrt{-1}\mathfrak{a}^*_L$, then $E(\psi, s)$ has a convergent integral over $H(F)\backslash H(\mathbb{A})$. Thus $P^H(E(\psi, s))$ is a meromorphic function on $\mathfrak{a}^*_{L,\mathbb{C}}$ which is holomorphic on $\sqrt{-1}\mathfrak{a}^*_L$. In Proposition 2.5 we will show that $P^H(E(\psi, s))$ is identically zero unless all k_i are even. In the latter case we derive a formula of $P^H(E(\psi, s))$ from a formula of the symplectic period of truncated cuspidal Eisenstein series obtained by Offen [10] and the description of the discrete spectrum proven by Moeglin and Waldspurger [8], using Cauchy's integral formula and Fubini's theorem. The idea of the proof is the same as that of Arthur [1]. This formula generalizes the formula that was proven by Jacquet and Rallis [6] and then extended by Offen [10].

To rewrite this formula in a form which is more suitable for our purpose, Section 3 extends the theory of the intertwining period to square-integrable, but not necessarily cuspidal, automorphic forms on $V(\mathbb{A})Q(F)\backslash G(\mathbb{A})$. Suppose that all k_i are even. We take a skew symmetric matrix y in $L(F)$ and $\eta \in G(F)$ so that $y = \eta\epsilon\, {}^t\eta$. Let $H_y = \eta H \eta^{-1}$ be the symplectic group of y. Put $L_y = H_y \cap L$. The period integral

$$P^{L_y}(\psi)(g) = \int_{L_y(F)\backslash L_y(\mathbb{A})} \psi(lg)\,dl$$

is convergent. We define the global intertwining period by the integral

$$J(w_{\theta L}, \psi, s) = \int_{\eta^{-1}L_y(\mathbb{A})\eta\backslash H(\mathbb{A})} P^{L_y}(\psi)(\eta h) e^{\langle s, H_L(\eta h)\rangle}\,dh,$$

which converges absolutely for $\Re s \in \mathfrak{a}^*_L$ sufficiently regular in the positive Weyl chamber. We will prove in Theorem 3.2 that for $s \in \mathfrak{a}^*_{L,\mathbb{C}}$ in general position

$$P^H(E(\psi, s)) = J(w_{\theta L}, \psi, s).$$

By the description of the discrete spectrum of $\mathrm{GL}(N)$ alluded to above, there is a bijection between irreducible automorphic representations in the discrete spectrum of L and pairs $(\mathbf{d}, \sigma)$, where $\mathbf{d} = (d_1, \dots, d_r)$, d_i is a factor of k_i for each i, $n_i = k_i/d_i$ and $\sigma = \otimes_{i=1}^r \sigma_i$ is an irreducible cuspidal automorphic representation of $\prod_{i=1}^r \mathrm{GL}(n_i, \mathbb{A})$. Let P_i be the standard parabolic subgroup of $\mathrm{GL}(k_i)$ of type $(n_i, \dots, n_i)$ and view $\prod_{i=1}^r P_i$ as the standard parabolic subgroup of L. Let π be the unique irreducible quotient of the induced representation of $L(\mathbb{A})$ obtained from the representation $\otimes_{i=1}^r (\sigma_i^{\otimes d_i} \otimes \rho_{P_i}^{1/n_i})$ of $\prod_{i=1}^r P_i(\mathbb{A})$, where ρ_{P_i} is the square root of the modulus function of $P_i(\mathbb{A})$. Then π occurs in the discrete spectrum of L.

For $s \in \mathfrak{a}_{L,\mathbb{C}}^*$ let $I(\pi, s)$ denote the automorphic representation of $G(\mathbb{A})$ induced from $\pi[s] := \pi \otimes e^{\langle s, H_L(\cdot)\rangle}$. Let $\psi \in I(\pi, 0)$. If not all d_i are even, then $J(w_{\theta L}, \psi, s)$ is identically zero as $P^{L_y}(\psi) = 0$ by the result of Offen [10]. When all d_i are even, we will show in Section 4 that $J(w_{\theta L}, \psi, s)$ is factorizable and expressed as a ratio of L-functions up to finitely many local factors at the ramified places and conclude that $J(w_{\theta L}, \psi, s)$ is not identically zero for a suitable choice of ψ by appealing to Offen's works [10, 11]. In particular, $I(\pi, s)$ is H-distinguished for generic values of the parameter s, if and only if there is a point $s \in \mathfrak{a}_{L,\mathbb{C}}^*$ such that $I(\pi, s)$ is H-distinguished, and if and only if all d_i are even. This concludes the project initiated by Jacquet and Rallis and then developed by Offen.

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Notation

Let F be a number field with adele ring $\mathbb{A}$. For any positive integer m we denote by G_m the group $\mathrm{GL}(m)$ viewed as an algebraic group over F. We fix a natural number n and denote $G = G_{2n}$. Let K be the standard maximal compact subgroup of $G(\mathbb{A})$ and $P_0 = M_0 U_0$ the Borel subgroup of G of upper triangular matrices in G, where M_0 is the subgroup of diagonal matrices and the unipotent radical U_0 of P_0 is the subgroup of unipotent upper triangular matrices. A parabolic subgroup of G is called standard if it contains P_0. A Levi subgroup of a standard parabolic subgroup of G is called standard if it contains M_0. By parabolic and Levi subgroups of G

we always mean standard parabolic and Levi subgroups of G. There is a bijection between standard Levi subgroups of G_m and ordered partitions of m.

Let $P = MU$ be a parabolic subgroup of G. We denote the lattice of rational characters of M by $X^*(M)$. For $\chi \in X^*(M)$ we define a homomorphism $|\chi| : M(\mathbb{A}) \to \mathbb{R}_+^\times$ by $|\chi|(m) = \prod_v |\chi_v(m_v)|_v$, where the product ranges over all places v of F, χ_v is the extension of χ to $M(F_v)$ and $|\cdot|_v$ is the standard absolute value on F_v. We form the real vector space $\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(X^*(M), \mathbb{R})$. We also have the dual vector space $\mathfrak{a}_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$, and its complexification $\mathfrak{a}_{M,\mathbb{C}}^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$. In case $M = M_0$ we write $\mathfrak{a}_0^* = \mathfrak{a}_{M_0}^*$. The canonical pairing on $\mathfrak{a}_0^* \times \mathfrak{a}_0$ is denoted by $\langle\ ,\ \rangle$, which induces a nondegenerate pairing on $\mathfrak{a}_M^* \times \mathfrak{a}_M$. A height function $H_M : G(\mathbb{A}) \to \mathfrak{a}_M$ is the left $U(\mathbb{A})$-invariant, right K-invariant function on $G(\mathbb{A})$ satisfying $e^{\langle \chi, H_M(m)\rangle} = |\chi|(m)$ for $m \in M(\mathbb{A})$ and $\chi \in X^*(M)$. Put $M(\mathbb{A})^1 = \{m \in M(\mathbb{A}) \mid H_M(m) = 0\}$. Let A_0 be the image of $\mathbb{R}_+^{2n}$ in $M_0(\mathbb{A})$ under the isomorphism $M_0 \simeq \mathbb{G}_m^{2n}$, where $\mathbb{R} \hookrightarrow F \otimes \mathbb{R}$ is given by $x \mapsto 1 \otimes x$. We can form the central subgroup T_M of M, the intersection A_M of A_0 with $T_M(\mathbb{A})$ and the discrete part $L^2_{\mathrm{disc}}(M(F)\backslash M(\mathbb{A})^1)$ of $L^2(M(F)\backslash M(\mathbb{A})^1)$. Note that $M(\mathbb{A}) = A_M \times M(\mathbb{A})^1$ and H_M induces an isomorphism $A_M \simeq \mathfrak{a}_M$. It is well-known that $L^2_{\mathrm{disc}}(M(F)\backslash M(\mathbb{A})^1)$ decomposes with multiplicity one. We denote by $\mathit{\Pi}_d(M)$ the set of irreducible subrepresentations of the representation of $M(\mathbb{A})^1$ on $L^2_{\mathrm{disc}}(M(F)\backslash M(\mathbb{A})^1)$, and by $\mathit{\Pi}_c(M)$ the set of irreducible cuspidal automorphic representations of $M(\mathbb{A})^1$. We view irreducible representations of $M(\mathbb{A})^1$ as representations of $M(\mathbb{A})$ by extending the action of $M(\mathbb{A})^1$ to $M(\mathbb{A})$ so that A_M acts trivially. If π is an irreducible representation of $M(\mathbb{A})$ and λ belongs to $\mathfrak{a}_{M,\mathbb{C}}^*$, then $\pi[\lambda](m) = \pi(m)e^{\langle \lambda, H_M(m)\rangle}$ is another irreducible representation of $M(\mathbb{A})$. The set of associated $\mathfrak{a}_{M,\mathbb{C}}^*$ orbits is in bijective correspondence under the restriction mapping from $M(\mathbb{A})$ to $M(\mathbb{A})^1$ with the set of irreducible unitary representations of $M(\mathbb{A})^1$.

Let $R^+(M_0, M)$ and Δ_0^M be the sets of positive roots and simple roots of M_0 in M, respectively. We write $\rho_0^M \in \mathfrak{a}_0^*$ for half the sum of elements in $R^+(M_0, M)$. More generally, for another Levi subgroup L of G with $M \subset L$, the parabolic subgroup $P \cap L$ of L determines the sets $R^+(T_M, L)$ and Δ_M^L. Namely, $R^+(T_M, L)$ is the set of elements in $X^*(T_M)$ obtained by decomposing the Lie algebra of $U \cap L$ under the adjoint action of T_M, and Δ_M^L the set of linear forms on $\mathfrak{a}_M$ obtained by restriction of elements in the complement of Δ_0^M in Δ_0^L. Let $(\mathfrak{a}_M^L)^*$ be the vector subspace of $\mathfrak{a}_M^*$ generated by Δ_M^L. Note that there is a canonical direct sum decomposition

$\mathfrak{a}_M^* = \mathfrak{a}_L^* \oplus (\mathfrak{a}_M^L)^*$. Let $\rho_M^L = \rho_P^L$ be the projection of ρ_0^L on $\mathfrak{a}_M^*$. When $L = G$, we will write ρ_P and Δ_M in place of ρ_P^G and Δ_M^G.

Let W^M denote the Weyl group of M. We write $W = W^G$. For standard Levi subgroups M, M' of G we write $W(M, M')$ for the set of elements $w \in W$ of minimal length in wW^M such that $wMw^{-1} = M'$. A parabolic subgroup $P' = M'U'$ is said to be associated to P if $W(M, M')$ is not empty. Set $W(M) = \bigcup_{M'} W(M, M')$. Explicitly, an element of $W(M)$ is represented by a unique permutation matrix that shuffles the diagonal blocks of M without causing any internal change within each block.

Let w_n be the $n \times n$ permutation matrix with unit anti-diagonal. Put

$$\epsilon = \epsilon_{2n} = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}.$$

We represent θ as the automorphism $\theta(g) = \epsilon\, {}^t g^{-1} \epsilon^{-1}$. The symmetric space attached to (G, θ) is the variety

$$\mathcal{C} = \{x \in G \mid x\theta(x) = 1\}.$$

The group G acts on $\mathcal{C}$ by the twisted conjugation $g \star x = gx\theta(g)^{-1}$. Since $\mathcal{C}$ is a translate by ϵ of the space of nondegenerate skew symmetric matrices of size $2n$, the space $\mathcal{C}$ is a single G-orbit. For $x \in \mathcal{C}$ and any subgroup Q of G we will denote the stabilizer of x in Q by Q_x. However, we will denote by H_x the group G_x and further by

$$H = Sp(n) = \{g \in G \mid g\epsilon\, {}^t g = \epsilon\}$$

the stabilizer in G of the identity. For a subgroup Q of G we will always denote $Q_H = Q \cap H$, which gives a bijection between θ-stable parabolic subgroups of G and parabolic subgroups of H. If $Q = LV$ is a θ-stable parabolic subgroup, then $Q_H = L_H V_H$ is a Levi decomposition for Q_H.

Note that θ stabilizes the Borel subgroup P_0 and hence defines involutions on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. Let $(\mathfrak{a}_0^*)_\theta^\pm$ denote the ± 1 eigenspaces of θ in $\mathfrak{a}_0^*$. We identify $(\mathfrak{a}_0^*)_\theta^+$ with $X^*((M_0)_H) \otimes_{\mathbb{Z}} \mathbb{R}$. For θ-stable Levi subgroups $M \subset L$ of G let $\Delta_{M_H}^{L_H}$ be the set of nontrivial projections of elements of Δ_M^L onto $(\mathfrak{a}_0^*)_\theta^+$. Then $\Delta_{M_H}^{L_H}$ is a basis of $((\mathfrak{a}_M^L)^*)_\theta^+$, and $\Delta_{(M_0)_H}^H$ forms a set of simple roots for H with respect to the Borel subgroup $(P_0)_H$ of H. We make similar definitions for the set of coroots and denote by $(\hat{\Delta}^\vee)_{M_H}^{L_H}$ the dual basis of $\Delta_{M_H}^{L_H}$ in $(\mathfrak{a}_M^L)_\theta^+$. There is a unique element $\rho_{P_H} \in (\mathfrak{a}_M^*)_\theta^+$ such that $\delta_{P_H}(m) = e^{\langle 2\rho_{P_H}, H_M(m)\rangle}$ for all $m \in M_H(\mathbb{A})$, where δ_{P_H} denotes the modulus function on $P_H(\mathbb{A})$. We fix Haar measures on various groups as in [10].

1. Residues of cuspidal Eisenstein series

This section explains how the general Eisenstein series is obtained as a residue of a cuspidal Eisenstein series. For two integers $a \leqslant b$ we denote the set $\{a, a+1, \dots, b\}$ by $[a, b]$. We understand that $[a, b] = \emptyset$ if $a > b$. For $\lambda \in \mathbb{C}$ we define the character ν^λ of $G_m(\mathbb{A})$ by $g \to |\det g|^\lambda$. Let $(m_1, \dots, m_t)$ be an ordered partition of $2n$ and $P = MU$ the standard parabolic subgroup of G of type $(m_1, \dots, m_t)$. For $\rho = \otimes_{i\in[1,t]}\rho_i \in \mathit{\Pi}_d(M)$ and $\lambda = (\lambda_1, \dots, \lambda_t) \in \mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathbb{C}^t$ we write $\rho[\lambda]$ for the pull-back of $\otimes_{i\in[1,t]}\nu^{\lambda_i}\rho_i$ to $P(\mathbb{A})$. We denote by $I(\rho, \lambda)$ the representation induced from $\rho[\lambda]$ to $G(\mathbb{A})$ using normalized parabolic induction. We will write $I(\rho)$ in place of $I(\rho, 0)$ and identify the spaces of the representations $I(\rho, \lambda)$ with the space $I(\rho)$ by restricting functions to K. The action is then given by

$$[I(\rho, \lambda)(g)\psi](y) = \psi(yg)e^{\langle \lambda, H_M(yg)\rangle - \langle \lambda, H_M(y)\rangle} \quad (\psi \in I(\rho),\ g, y \in G(\mathbb{A})).$$

For $\psi \in I(\rho)$ we define $\psi_\lambda \in I(\rho, \lambda)$ by

$$\psi_\lambda(g) = e^{\langle \lambda, H_M(g)\rangle}\psi(g).$$

We will identify $W(M)$ with the permutation group $\mathfrak{S}_t$ of $[1, t]$ in the following way. For $\tau \in \mathfrak{S}_t$ we define a permutation matrix $w_M(\tau) \in W(M)$ by $w_M(\tau) = (A_{ij})$, where A_{ij} is the $m_{\tau^{-1}(i)} \times m_j$ zero matrix unless $i = \tau(j)$, in which case $A_{ij} = \mathbf{1}_{m_j}$. Note that when $w = w_M(\tau)$,

$$w\mathrm{diag}(g_1, \dots, g_t)w^{-1} = \mathrm{diag}(g_{\tau^{-1}(1)}, \dots, g_{\tau^{-1}(t)}).$$

Thus wMw^{-1} is of type $(m_{\tau^{-1}(1)}, \dots, m_{\tau^{-1}(t)})$,

$$w\rho = \otimes_{i\in[1,t]}\rho_{\tau^{-1}(i)}, \qquad w\lambda = (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(t)}).$$

We form the Eisenstein series on $G(\mathbb{A})$ by

$$E(g, \psi, \lambda) = \sum_{\gamma\in P(F)\backslash G(F)} \psi_\lambda(\gamma g).$$

If $P' = M'U'$ is associated to P, then for $w \in W(M, M')$ the intertwining operator $M(w, \lambda)$ is defined by

$$M(w, \lambda)\psi(g) = e^{-\langle w\lambda, H_{M'}(g)\rangle} \int_{U_w(\mathbb{A})\backslash U'(\mathbb{A})} \psi(w^{-1}ug)e^{\langle \lambda, H_M(w^{-1}ug)\rangle} du,$$

where $U_w = U' \cap wUw^{-1}$. The series and the integral both converge absolutely if $\Re\lambda$ sufficiently regular in the positive Weyl chamber, and they possess meromorphic continuations to the space $\mathfrak{a}^*_{M,\mathbb{C}}$.

The classification of the discrete spectrum for $G_m(\mathbb{A})$ was established through a deep study by Moeglin and Waldspurger of residues of cuspidal Eisenstein series in [8]. The representations in $\mathit{\Pi}_d(G_m)$ are parametrized by pairs (d,σ_0) where d divides m and $\sigma_0 \in \Pi_c(G_{m/d})$. Given such a pair (d,σ_0), the representation $I(\sigma_0^{\otimes d},\Lambda_d)$ has the unique irreducible quotient which is denoted by $L(\sigma_0,\Lambda_d)$, where

$$\Lambda_d = \left(\frac{d-1}{2},\frac{d-3}{2},\dots,\frac{1-d}{2}\right).$$

The representation $L(\sigma_0,\Lambda_d)$ occurs in $L^2_{\mathrm{disc}}(G_m(F)\backslash G_m(\mathbb{A})^1)$ with multiplicity one. In particular, $L(\sigma_0,\Lambda_d) \in \Pi_c(G_m)$ if and only if $d=1$. For $\varphi \in I(\sigma_0^{\otimes d})$ the meromorphic function

$$E(\varphi,\lambda)\prod_{i=1}^{d-1}(\lambda_i-\lambda_{i+1}-1)$$

is holomorphic at $\lambda=\Lambda_d$. We define the multiresidue $E_{-1}(\varphi)$ of $E(\varphi,\lambda)$ at $\lambda=\Lambda_d$ to be its limit as $\lambda\to\Lambda_d$. The functions $E_{-1}(\varphi)$ are square integrable automorphic forms on $G_m(\mathbb{A})$, and $\varphi_{\Lambda_d}\mapsto E_{-1}(\varphi)$ defines an intertwining map from $I(\sigma_0^{\otimes d},\Lambda_d)$ onto $L(\sigma_0,\Lambda_d)$.

Let $k_i=d_in_i$ and let $(k_1,\dots,k_r)$ be a partition of $2n$. Take P to be the standard parabolic subgroup of G with Levi component

$$M=\underbrace{\mathrm{GL}(n_1)\times\cdots\times\mathrm{GL}(n_1)}_{d_1}\times\cdots\times\underbrace{\mathrm{GL}(n_r)\times\cdots\times\mathrm{GL}(n_r)}_{d_r}.$$

Let $Q=LV$ denote the standard parabolic subgroup of G of type $(k_1,\dots,k_r)$. Put

$$\sigma=\otimes_{i\in[1,r]}\sigma_i^{\otimes d_i}\in\Pi_c(M),\qquad \pi=\otimes_{i\in[1,r]}L(\sigma_i,\Lambda_{d_i})\in\Pi_d(L).$$

Put

$$\Delta_i=[d_i'+1,d_{i+1}'],\qquad \Delta_i'=[d_i'+1,d_{i+1}'-1],\qquad i\in[1,r],$$

where $d_i'=\sum_{j=1}^{i-1}d_j$ for $i\in[1,r+1]$. We put $|\mathbf{d}|=d_{r+1}'$ and set

$$\Lambda_{\mathbf{d}}=(\Lambda_{d_1},\Lambda_{d_2},\dots,\Lambda_{d_r})\in\mathfrak{a}_M^*\simeq\mathbb{R}^{|\mathbf{d}|}.$$

We define on $\mathfrak{a}_{M,\mathbb{C}}^*$ the linear functionals

$$R_j(\lambda)=\lambda_j-\lambda_{j+1},\qquad j\in[1,|\mathbf{d}|-1].$$

For $\varphi\in I(\sigma)$ let

$$E^Q(g,\varphi,\lambda)=\sum_{\gamma\in P(F)\backslash Q(F)}\varphi(\gamma g)e^{\langle\lambda,H_M(\gamma g)\rangle}$$

be an Eisenstein series induced from $P \cap L$ to L. The function $E^{Q}_{-1}(\varphi)$ is defined by

$$E^{Q}_{-1}(\varphi) = \lim_{\lambda \to \Lambda_{\mathbf{d}}} \left[E^{Q}(\varphi, \lambda) \prod_{i=1}^{r} \prod_{j \in \Delta'_i} (R_j(\lambda) - 1) \right].$$

The limit exists and $\varphi_{\Lambda_{\mathbf{d}}} \mapsto E^{Q}_{-1}(\varphi)$ defines a nonzero intertwining map

$$I(\sigma, \Lambda_{\mathbf{d}}) \to I(\pi).$$

As a representation of $G(\mathbb{A})$ induced from a unitary representation, $I(\pi)$ is known to be irreducible [2, 13]. For $s \in \mathfrak{a}^*_{L,\mathbb{C}}$ we study the Eisenstein series $E(E^{Q}_{-1}(\varphi), s)$. The series $E(E^{Q}_{-1}(\varphi), s)$ can be continued to a meromorphic function on the space $\mathfrak{a}^*_{L,\mathbb{C}}$ which is holomorphic on $\sqrt{-1}\mathfrak{a}^*_L$ (cf. [9]). It is important to note that

$$E(E^{Q}_{-1}(\varphi), s) = \lim_{\lambda \to \Lambda_{\mathbf{d}}} \left[E(\varphi, \lambda + s) \prod_{i=1}^{r} \prod_{j \in \Delta'_i} (R_j(\lambda) - 1) \right]. \tag{1.1}$$

For $w \in W(M)$ we define the multiresidue $M_{-1}(w, s)$ of the intertwining operator $M(w, \lambda)$ to be the limit

$$M_{-1}(w, s) = \lim_{\lambda \to \Lambda_{\mathbf{d}}} \left[M(w, \lambda + s) \prod_{i=1}^{r} \prod_{j \in \Delta'_i,\ w(j) > w(j+1)} (R_j(\lambda) - 1) \right].$$

The limit exists when $s \in \mathfrak{a}^*_{L,\mathbb{C}}$ is in a general position.

2. The period of the residue

Let $\mathscr{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$. For $\phi \in \mathscr{A}(G)$ and a parabolic subgroup P of G we denote by $\mathcal{E}_P(\phi)$ the set of exponents of ϕ along P. Let $\mathscr{A}(G)'$ denote the space of automorphic forms on $G(\mathbb{A})$ whose exponents λ along P satisfy

$$\langle \lambda, \varpi^\vee \rangle \neq \langle 2\rho_{P_H} - \rho_P, \varpi^\vee \rangle$$

for all $\varpi^\vee \in (\hat{\Delta}^\vee)^H_{M_H}$ and all θ-stable standard parabolic subgroups $P = MU$ of G. Let $\mathbb{A}_{\mathrm{f}}$ be the finite part of $\mathbb{A}$. The procedure outlined in [4, 7], using a mixed truncation operator Λ^T_m to regularize the integral, can be applied to our case with little adjustment. We refer to [10] for the precise definition and necessary modifications. Here we recall the properties of the regularized period and its characterization:

Proposition 2.1 ([10]). — (1) *The regularized integral*

$$\phi \mapsto \int^*_{H(F)\backslash H(\mathbb{A})} \phi(h)dh$$

gives a right $H(\mathbb{A}_{\mathrm{f}})$-invariant functional on $\mathscr{A}(G)'$.

(2) *If $\phi \in \mathscr{A}(G)$ is integrable over the domain $H(F)\backslash H(\mathbb{A})$, then*

$$\int^*_{H(F)\backslash H(\mathbb{A})} \phi(h)dh = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)dh.$$

(3) *For any $\phi \in \mathscr{A}(G)$ the function $T \mapsto \int_{H(F)\backslash H(\mathbb{A})} \Lambda^T_m \phi(h)dh$, defined for $T \in (\mathfrak{a}_0)^+_\theta$ sufficiently regular in the positive Weyl chamber, is of the form $\sum_\lambda p_\lambda(T)e^{\langle \lambda, T\rangle}$, where λ may be taken from the set $\bigcup_{P_H}(\rho_P - 2\rho_{P_H} + \mathcal{E}_P(\phi))$ and $p_\lambda(T)$ are polynomials. Moreover, if $\phi \in \mathscr{A}(G)'$, then $p_0(T)$ is constant and is equal to $\int^*_{H(F)\backslash H(\mathbb{A})} \phi(h)dh$.*

Let $P = MU$ be a θ-stable standard parabolic subgroup of G and $\rho \in \mathit{\Pi}_c(M)$. We denote by v_{M_H} the volume of the parallelogram formed by $\Delta^\vee_{M_H}$. For $\psi \in I(\rho)$ we define $j(\psi)$ by

$$j(\psi) = \int_{K_H} \int_{M_H(F)\backslash M_H(\mathbb{A})^1} \psi(mk)dmdk.$$

Let us put

$$\mu = \rho_{P_0} - 2\rho_{(P_0)_H} = \left(-\frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \in (\mathfrak{a}^*_0)^+_\theta. \tag{2.1}$$

Theorem 2.2 (Offen [10]). — *Let $\rho \in \mathit{\Pi}_c(M)$ and $\psi \in I(\rho)$. Then*

$$\int_{H(F)\backslash H(\mathbb{A})} \Lambda^T_m E(h, \psi, \lambda)dh = \sum_w \frac{v_{M'_H} e^{\langle \mu + w\lambda, T\rangle}}{\prod_{\alpha \in \Delta^H_{M'_H}} \langle \mu + w\lambda, \alpha^\vee\rangle} j(M(w, \lambda)\psi),$$

where the sum is over all permutations $w \in W(M)$ such that the type of $M' = wMw^{-1}$ is of the form $(m_1, \dots, m_t, m_t, \dots, m_1)$.

Proof. — If $M' = wMw^{-1}$ is of type $(m_1, \dots, m_t, m_t, \dots, m_1)$, then

$$\rho_{P'} - 2\rho_{P'_H} = (\rho_{P_0} - \rho^{M'}_{P_0}) - 2(\rho_{(P_0)_H} - \rho^{M'_H}_{(P_0)_H}) = \rho_{P_0} - 2\rho_{(P_0)_H},$$

from which Theorem 2.2 is nothing but Theorem 7.8 of [10]. □

Let P, Q, σ and π be the same as in Section 1 in the rest of this section.

Lemma 2.3. — *Let $s \in \sqrt{-1}\mathfrak{a}^*_L$ and $\psi \in I(\pi)$. Then the real parts of cuspidal exponents of $E(\psi, s)$ are permutations of the sequence*

$$(-\Lambda_{d_1}; -\Lambda_{d_2}; \dots; -\Lambda_{d_r})$$

of length $|\mathbf{d}|$ in which the order of elements in the segment $-\Lambda_{d_i}$ is preserved for every $i \in [1, r]$.

Proof. — Let $\varphi \in I(\sigma)$. The Eisenstein series $E(\varphi, \lambda)$ is concentrated on parabolic subgroups associated to P and hence so is its residue $E(E^Q_{-1}(\varphi), s)$. If $P' = M'U'$ is associated to P, then the constant term of $E(\varphi, \lambda)$ relative to P' is given by

$$\sum_{w \in W(M,M')} M(w, \lambda)\varphi(g)e^{\langle w\lambda, H_{M'}(g)\rangle}.$$

Lemme on p. 650 of [8] and the adjoint formula of the intertwining operators show that

$$\lim_{\lambda \to \Lambda_{\mathbf{d}}} \left[M(w, \lambda + s)\varphi(g) \prod_{i=1}^{r} \prod_{j \in \Delta'_i} (R_j(\lambda) - 1) \right] = 0$$

unless w reverses the orders of elements in the segments $\Delta_1, \dots, \Delta_r$, which completes the proof by (1.1). □

Lemma 2.4. — *If $s \in \sqrt{-1}\mathfrak{a}^*_L$ and $\psi \in I(\pi)$, then the integral*

$$\int_{H(F)\backslash H(\mathbb{A})} E(h, \psi, s)dh$$

is absolutely convergent.

Proof. — It is explained in the proof of Proposition 1 of [3] how the convergence of the period of an automorphic form depends only on its cuspidal exponents. The symplectic period of an automorphic form ϕ on $G(\mathbb{A})$ converges if there is $\kappa \in \mathfrak{a}^*_0$ such that for each standard parabolic subgroup $P' = M'U'$ of G

$$\langle \rho_{P'} - 2\rho_{(P_0)_H} + \nu + \kappa^{M'}, \varpi^\vee \rangle < 0$$

for all $\varpi^\vee \in (\hat{\Delta}^\vee)^H_{(P_0)_H}$ and all the real parts of cuspidal exponents ν of ϕ along P', where $\kappa^{M'}$ is the projection of κ on $(\mathfrak{a}^{M'}_0)^*$. Put

$$e^+_i = (\underbrace{1, \dots, 1}_{i}, 0, \dots, 0), \qquad e^-_i = -(0, \dots, 0, \underbrace{1, \dots, 1}_{i}).$$

Lemma 2.3 shows that $\langle \nu, e^\pm_i \rangle \leqslant 0$ for all $i \in [1, 2n]$ and all the real parts of cuspidal exponents ν of $E(\psi, s)$. Since $\varpi^\vee \in (\hat{\Delta}^\vee)^H_{(P_0)_H}$ has the form $\frac{1}{2}(e^+_n + e^-_n)$ or $e^+_i + e^-_i$ for $i \in [1, n-1]$, we see that $\langle \nu, \varpi^\vee \rangle \leqslant 0$. Note that $\rho_{P_0} = \rho^{M'}_{P_0} + \rho_{P'}$. Thus $\kappa = \rho_{P_0}$ works in view of (2.1). □

For any permutation $\tau \in \mathfrak{S}_t$ we define $\kappa_\tau \in \mathfrak{S}_{2t}$ via

$$\kappa_\tau(2i-1) = \tau^{-1}(i), \qquad \kappa_\tau(2i) = 2t+1-\tau^{-1}(i), \qquad i \in [1,t].$$

Put $M^\tau = \kappa_\tau M \kappa_\tau^{-1}$. When $\tau = 1$, we denote $\kappa_{2t} = \kappa_\tau$ and $M^\dagger = M^\tau$.

Proposition 2.5. — *Let $\varphi \in I(\sigma)$ and $s \in \sqrt{-1}\mathfrak{a}_L^*$. Then*

$$\int_{H(F)\backslash H(\mathbb{A})} E(h, E_{-1}^Q(\varphi), s)dh = 0$$

unless all d_i are even. If all d_i are even, then for each $\tau \in \mathfrak{S}_t$,

$$\int_{H(F)\backslash H(\mathbb{A})} E(h, E_{-1}^Q(\varphi), s)dh = v_{M_H^\tau} j(M_{-1}(\kappa_\tau, s)\varphi).$$

In particular, the right hand side is independent of the choice of τ.

Offen demonstrated the special case of this result for $r = 1$ in [10]. Though the proof holds almost verbatim for our general case, we reproduce it here. For $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ we write $W(M)_\lambda$ for the subset of $W(M)$ consisting of all elements w that satisfy the following conditions:

- the type of wMw^{-1} is of the form $(m_1, \dots, m_t, m_t, \dots, m_1)$;
- $\mu + w\lambda \in (\mathfrak{a}_0^*)_\theta^-$.

Lemma 2.6. — *Let $s = (s_1, \dots, s_r) \in \sqrt{-1}\mathfrak{a}_L^*$. Suppose that $s_1, \dots, s_r$ are distinct. Then $W(M)_{\Lambda_\mathbf{d}+s}$ is empty unless all d_i are even. If all d_i are even and if we put $t = |\mathbf{d}|/2$, then $\tau \mapsto \kappa_\tau$ is a bijection between $\mathfrak{S}_t$ and $W(M)_{\Lambda_\mathbf{d}+s}$.*

Proof. — Assume that $w \in W(M)_{\Lambda_\mathbf{d}+s}$. Note that for $x = (x_1, \dots, x_{2t}) \in \mathfrak{a}_{M',\mathbb{C}}^*$, where M' is of type $(m_1, \dots, m_t, m_t, \dots, m_1)$, $x \in (\mathfrak{a}_0^*)_\theta^-$ if and only if $x_j = x_{2t+1-j}$ for all $j \in [1,t]$. If we put $\lambda[y] = (\lambda_1 + y, \dots, \lambda_a + y)$ for $\lambda \in \mathbb{C}^a$ and $y \in \mathbb{C}$, then

$$\Lambda_\mathbf{d} + s = (\Lambda_{d_1}[s_1], \Lambda_{d_2}[s_2], \dots, \Lambda_{d_r}[s_r]).$$

By the assumption on s, for each j there is a segment Δ_i to which both $w^{-1}(j)$ and $w^{-1}(2t+1-j)$ belong. Therefore all d_i must be even for $W(M)_{\Lambda_\mathbf{d}+s}$ to be not empty. We can infer from (2.1) that

$$w^{-1}(2t+1-j) - w^{-1}(j) = 1, \qquad j \in [1,t]$$

Lemma 2.6 is now proven in exactly the same way as in the proof of Lemma 8.3 of [10]. □

We appeal to Lemma 8.1 of [10]. There is a minor error in that lemma. It is true not for a fixed T but as T varies.

Lemma 2.7 (cf. [10, Lemma 8.1]). — *Let V be a finite dimensional vector space over $\mathbb{C}$. Let*

$$f_\lambda(T)=\sum_{i=1}^{d}a_i(\lambda)e^{\langle b_i(\lambda),T\rangle},$$

where $T\in V$, a_i are meromorphic functions near a point $\lambda=\lambda_0\in V^*$ and b_i are linear endomorphisms of V^* such that $b_1(\lambda_0),\dots,b_d(\lambda_0)\in V^*$ are distinct. Assume that $\lim_{\lambda\to\lambda_0}f_\lambda(T)$ exists. Then a_i is holomorphic at λ_0 for all i and

$$\lim_{\lambda\to\lambda_0}f_\lambda(T)=\sum_{i=1}^{d}a_i(\lambda_0)e^{\langle b_i(\lambda_0),T\rangle}.$$

Now we are ready to prove Proposition 2.5. In view of Lemma 2.4 and Proposition 2.1(2) our task is to compute

$$\int^{*}_{H(F)\backslash H(\mathbb{A})}E(h,E^{Q}_{-1}(\varphi),s)dh. \tag{2.2}$$

We use Cauchy's integral formula to express the residue $E(E^{Q}_{-1}(\varphi),s)$ as a Cauchy integral of $E(\varphi,\lambda)$. The Cauchy integral can be interchanged with the truncation operator, and then Fubini's theorem allows us to exchange the Cauchy integral and the period integral. This argument is the same as that introduced by Arthur on pp. 47–48 of [1] (see also p. 293 of [10]). Therefore we deduce from (1.1) that

$$\begin{aligned}&\int_{H(F)\backslash H(\mathbb{A})}\Lambda^T_mE(h,E^{Q}_{-1}(\varphi),s)dh\\&\quad=\lim_{\lambda\to\Lambda_{\mathbf{d}}}\left[\int_{H(F)\backslash H(\mathbb{A})}\Lambda^T_mE(h,\varphi,\lambda+s)dh\prod_{i=1}^{r}\prod_{j\in\Delta'_i}(R_j(\lambda)-1)\right].\end{aligned}$$

This limit exists, and Theorem 2.2 combined with Proposition 2.1(3) and Lemma 2.7 shows that (2.2) is equal to

$$\lim_{\lambda\to\Lambda_{\mathbf{d}}}\sum_{w\in W(M)_{\Lambda_{\mathbf{d}}+s}}v_{M'_H}j(M(w,\lambda+s)\varphi)\frac{\prod_{i=1}^{r}\prod_{j\in\Delta'_i}(R_j(\lambda)-1)}{\prod_{\alpha\in\Delta^H_{M'_H}}\langle\mu+w(\lambda+s),\alpha^\vee\rangle}.$$

Since (2.2) can be viewed as a meromorphic function in s (cf. Proposition 12 of [4]), it suffices to prove Proposition 2.5 for s in a general position of $\sqrt{-1}\mathfrak{a}^*_L$. Thus we assume that $s_1,\dots,s_r$ are distinct and that $M_{-1}(\kappa_\tau,s)$ are holomorphic at s for all $\tau\in\mathfrak{S}_t$. Since the first part of Proposition 2.5 follows immediately from Lemma 2.6, we hereafter assume that all d_i are even. Since we know that the limit exists, we may compute it by computing a directional limit in a 'good' direction. Recall $t = |\mathbf{d}|/2$. For $w \in W(M)$ and $i \in [1, t-1]$ we define the functionals $L_{w,i}$ on $\mathfrak{a}^*_{M,\mathbb{C}}$ by

$$L_{w,i}(\lambda) = \lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)} + \lambda_{w^{-1}(2t-i)} - \lambda_{w^{-1}(2t+1-i)},$$

and we set $L_{w,t}(\lambda) = \lambda_{w^{-1}(t)} - \lambda_{w^{-1}(t+1)}$. Then

$$\{\langle w\lambda, \alpha^\vee\rangle \mid \alpha \in \Delta^H_{M'_H}\} = \{L_{w,i}(\lambda) \mid i \in [1, t]\}.$$

We fix $v_0 \in \mathfrak{a}^*_M$ so that $L_{\kappa_\tau,i}(v_0) \neq 0$ for all $i \in [1, t]$ and $\tau \in \mathfrak{S}_t$. Since $\langle \mu + \kappa_\tau(\Lambda_{\mathbf{d}} + s), \alpha^\vee\rangle = 0$ for all $\tau \in \mathfrak{S}_t$, (2.2) is equal to

$$\lim_{c\to 0} \sum_{\tau\in\mathfrak{S}_t} v_{M^\tau_H} c^{t-r} j(M(\kappa_\tau, \Lambda_{\mathbf{d}} + s + cv_0)\varphi) \frac{\prod_{i=1}^r \prod_{j\in\Delta'_i} R_j(v_0)}{\prod_{i=1}^t L_{\kappa_\tau,i}(v_0)}$$

$$= \sum_{\tau\in\mathfrak{S}_t} v_{M^\tau_H} j(M_{-1}(\kappa_\tau, s)\varphi) \frac{\prod_{j=1}^t R_{2j-1}(v_0)}{\prod_{i=1}^t L_{\kappa_\tau,i}(v_0)}$$

by Lemma 2.6. The remaining part of the proof continues as in p. 296 of [10]. Consequently, $v_{M^\tau_H} j(M_{-1}(\kappa_\tau, s)\varphi)$ is independent of τ. □

3. The intertwining periods

Let $P = MU$ be a parabolic subgroup of G. Offen provides a complete analysis of the double cosets $P\backslash G/H$ in [10, 11]. We recall the necessary definitions and results. Let w^M_0 denote the longest element of W^M. Put $w_0 = w^G_0$ and $w_{\theta M} = w^M_0 w^G_0$. Set

$$W(\theta) = \{w w_0 w^{-1} w_0 \mid w \in W\}.$$

We will identify $W^M\backslash W/W^{\theta M}$ with the set ${}_M W_{\theta M}$ of reduced representatives. We use the relative Bruhat decomposition to define a map $\iota_M : P\backslash\mathcal{C} \to {}_M W_{\theta M}$ by $\iota_M(P \star x) = \xi$, where $P\xi\theta(P) = Px\theta(P)$. Proposition 3.5 of [10] asserts that ι_M defines a bijection $P\backslash\mathcal{C} \simeq W(\theta) \cap {}_M W_{\theta M}$. For $\xi \in W(\theta) \cap {}_M W_{\theta M}$ we write $\mathcal{O}_\xi$ for the unique P-orbit in $\mathcal{C}$ that ι_M maps to ξ.

The set of admissible twisted involutions is defined by

$$\mathfrak{I}_M(\theta) = \{\xi \in {}_M W_{\theta M} \mid w_0\xi w_0 = \xi^{-1},\ \xi\theta(M)\xi^{-1} = M\} \subset W(\theta M, M).$$

If $\xi \in \mathfrak{I}_M(\theta)$, then $\xi\theta$ acts as an involution on $\mathfrak{a}^*_M$, and $(\mathfrak{a}^*_M)^\pm_{\xi\theta}$ denotes the ± 1 eigenspaces of $\xi\theta$ in $\mathfrak{a}^*_M$. For $\xi \in \mathfrak{I}_M(\theta)$ we put

$$\Phi_\xi = \{\beta \in R^+(T_M, G) \mid \xi\theta\beta < 0\},$$
$$\Psi_\xi = \{\beta \in R^+(T_M, G) \mid \xi\theta\beta = \beta\},$$
$$\Psi^0_\xi = \{\beta \in R^+(T_M, G) \mid \xi\theta\beta = \pm\beta\}.$$

For $\xi \in \mathfrak{I}_M(\theta)$ and $\xi' \in \mathfrak{I}_{M'}(\theta)$ we set

$$W(\xi,\xi') = \{w \in W(M,M') \mid w\xi = \xi' w_0 w w_0,\ w\beta > 0 \text{ for } \beta \in \Psi_\xi\},$$
$$W^0(\xi,\xi') = \{w \in W(M,M') \mid w\xi = \xi' w_0 w w_0,\ w\beta > 0 \text{ for } \beta \in \Psi_\xi^0\}.$$

Let $(m_1,\dots,m_t)$ be the type of M. An M-admissible involution of $[1,t]$ is a permutation $\tau \in \mathfrak{S}_t$ which satisfies $\tau^2 = 1$ and $m_i = m_{\tau^{-1}(i)}$ for $i \in [1,t]$ and such that m_i is even whenever $\tau(i) = i$. We associate to $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$ an M-admissible involution τ_ξ of $[1,t]$ via

$$w_0^M \xi w_0 = w_M(\tau_\xi).$$

The map $\xi \mapsto \tau_\xi$ is a bijection between $\mathfrak{I}_M(\theta) \cap W(\theta)$ and the set of all M-admissible involutions of $[1,t]$. We put $S_\xi = \{i \in [1,t] \mid \tau_\xi(i) = i\}$.

Let $\rho \in \Pi_d(M)$. Let $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$ and choose $x \in \mathcal{O}_\xi \cap M\xi$. We define $\rho_\xi \in (\mathfrak{a}_M^*)_{\xi\theta}^+$ by requiring $\delta_{P_x}(m) = e^{\langle 2\rho_\xi, H_M(m)\rangle}$ for all $m \in M_x(\mathbb{A})$, where δ_{P_x} is the modulus function of $P_x(\mathbb{A})$ and $(\mathfrak{a}_M^*)_{\xi\theta}^+$ is identified with $X^*(M_x)\otimes_{\mathbb{Z}}\mathbb{R}$. Since $\mathcal{O}_\xi \cap M\xi$ is a unique M orbit by [10, Proposition 3.6(2)], ρ_ξ is independent of the choice of x. There is $m \in M$ such that mM_xm^{-1} is the subgroup of M consisting of matrices of the form $\mathrm{diag}[a_1,\dots,a_t]$, where $a_i = {}^t a_j^{-1} \in G_{m_i}$ whenever $\tau_\xi(i) = j \neq i$, and $a_i \in Sp(m_i/2)$ whenever $\tau_\xi(i) = i$. Lemma 2.4 shows that for any $\psi \in I(\rho)$ the period integral

$$P^{M_x}(\psi)(g) = \int_{M_x(F)\backslash M_x(\mathbb{A})^1} \psi(mg)dm$$

is well-defined.

We choose η so that $x = \eta \star \mathbf{1}_{2n}$. The intertwining period is defined by

$$J(\xi,\psi,\lambda) = \int_{\eta^{-1}P_x(\mathbb{A})\eta\backslash H(\mathbb{A})} P^{M_x}(\psi)(\eta h)e^{\langle\lambda, H_M(\eta h)\rangle}dh \tag{3.1}$$

for λ in some open set of $2\rho_\xi - \rho_P + (\mathfrak{a}_{M,\mathbb{C}}^*)_{\xi\theta}^-$. The integral makes sense and depends neither on the choice of x nor on η.

Proposition 3.1. — *Notation being as above, if γ is a sufficiently large real number, then the integral (3.1) converges absolutely when $\Re\lambda - 2\rho_\xi + \rho_P$ belongs to*

$$\mathcal{D}_{\xi,M} = \{\Lambda \in (\mathfrak{a}_M^*)_{\xi\theta}^- \mid \langle\Lambda, \beta^\vee\rangle > \gamma \textit{ for all } \beta \in \Phi_\xi\}.$$

Proof. — For each i there is a pair (d_i,σ_i) such that $\rho_i \simeq L(\sigma_i, \Lambda_{d_i})$, where d_i divides m_i and $\sigma_i \in \Pi_c(G_{m_i/d_i})$. We can take $x \in \mathcal{O}_\xi \cap M_0\xi$ to define $J(\xi,\psi,\lambda)$. Let $P' = M'U'$ be the parabolic subgroup contained in P which corresponds to the partition obtained from $(m_1,\dots,m_t)$ by replacing the entry m_i by $\left(\frac{m_i}{d_i},\dots,\frac{m_i}{d_i}\right)$ for $i \in S_\xi$. Let $\rho' = \otimes_{i\in[1,t]}\rho_i'$ be a representation of $M'(\mathbb{A})$, where $\rho'_i = \rho_i$ if $i \notin S_\xi$, and $\rho'_i = \sigma_i^{\otimes d_i}$ if $i \in S_\xi$. Define $\lambda' = (\lambda'_1, \dots, \lambda'_t) \in \mathfrak{a}^*_{M'}$ by $\lambda'_i = 0$ if $i \notin S_\xi$, and $\lambda'_i = \kappa_{d_i}\Lambda_{d_i}$ if $i \in S_\xi$. Since Proposition 2.5 shows that $P^{M_x}(\psi)$ is identically zero unless all d_i are even, we may suppose that all d_i are even. Applying Proposition 2.5 to ρ_i for each $i \in S_\xi$, we see that there is an element $\psi' \in I(\rho')$ satisfying

$$P^{M_x}(\psi)(g) = \int_{P'_x(\mathbb{A})\backslash P_x(\mathbb{A})} P^{M'_x}(\psi')(pg)e^{\langle \lambda', H_{M'}(pg)\rangle}dp,$$

and hence

$$J(\xi, \psi, \lambda) = J(\xi_{M'}, \psi', \lambda + \lambda'),$$

where we view $\xi_{M'} = \xi$ as an element of $\mathfrak{I}_{M'}(\theta)$. One can readily check that $\mathcal{D}_{\xi,M} \subset \mathcal{D}_{\xi_{M'},M'}$ and

$$\lambda + \lambda' + \rho_{P'} - 2\rho_{\xi_{M'}} \in (\mathfrak{a}^*_{M',\mathbb{C}})^-_{\xi_{M'}\theta},$$

which reduces the statement to the case where S_ξ is empty.

We write τ_ξ as a product of disjoint reflections

$$\tau_\xi = (i_1, j_1)\cdots(i_{t/2}, j_{t/2}).$$

There is no harm in assuming that $\rho_{i_k} \simeq \rho_{j_k}$ for $k \in [1, t/2]$. We write an Iwasawa decomposition of $g \in G(\mathbb{A})$ with respect to $P(\mathbb{A})$ as $g = p(g)k(g)$. Taking into account a canonical identification of ρ_{i_k} with its contragredient, we see that

$$e^{-\langle \rho_P, H_M(g)\rangle}P^{M_x}(\psi)(g) = \int_{M_x(F)\backslash M_x(\mathbb{A})^1} \rho(mp(g))\psi(k(g))dm$$

is a matrix coefficient of the unitary representation $\rho_{i_1} \otimes \cdots \otimes \rho_{i_{t/2}}$, so that

$$\sup_{g \in G(\mathbb{A})} |e^{-\langle \rho_P, H_M(g)\rangle}P^{M_x}(\psi)(g)| < \infty.$$

Proposition 4.3 of [10] now completes our proof. ☐

Theorem 3.2. — *Let $\psi \in I(\pi)$ and $s \in \sqrt{-1}\mathfrak{a}^*_L$. Then*

$$\int_{H(F)\backslash H(\mathbb{A})} E(h, \psi, s)dh = 0$$

unless all d_i are even. If all d_i are even, then

$$\int_{H(F)\backslash H(\mathbb{A})} E(h, \psi, s)dh = J(w_{\theta L}, \psi, s).$$

Proof. — We may assume that all d_i are even. Put

$$x = \mathrm{diag}[\epsilon_{k_1}, \dots, \epsilon_{k_r}]\epsilon \in \mathcal{C}, \qquad \kappa_{\mathbf{d}} = \mathrm{diag}[\kappa_{d_1}, \dots, \kappa_{d_r}] \in W(M, M).$$

Note that $\iota_L(x) = w_{\theta L} \in \mathfrak{I}_L(\theta) \cap W(\theta)$. The intertwining operator $M_{-1}(\kappa_{\mathbf{d}}) = M_{-1}(\kappa_{\mathbf{d}}, s)$ is independent of s. We define $w' \in W(M)$ by

$\kappa_{|\mathbf{d}|} = w'\kappa_{\mathbf{d}}$. If L' is the Levi subgroup of G of type $\left(\frac{k_1}{2}, \frac{k_1}{2}, \dots, \frac{k_r}{2}, \frac{k_r}{2}\right)$, then w' is given by $w' = w_{L'}(\kappa_{2r})$. When we view $w_{\theta L}$ as an element of $\mathfrak{I}_M(\theta)$, we will rewrite it as ξ_M. One can check that $w' \in W^0(\xi_M, \mathbf{1}_{2n})$. By the functional equation stated in Theorem 7.7 of [10]

$$j(M(w',\lambda)\phi) = J(\mathbf{1}_{2n}, M(w',\lambda)\phi, w'\lambda) = J(\xi_M, \phi, \lambda)$$

for all $\phi \in I(\sigma)$. By rewriting the formula of Proposition 2.5, we get

$$\begin{aligned}\int^*_{H(F)\backslash H(\mathbb{A})} E(h, E^Q_{-1}(\varphi), s)dh &= v_{M_H^\dagger} j(M(w', \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s)M_{-1}(\kappa_{\mathbf{d}})\varphi) \\ &= v_{M_H^\dagger} J(\xi_M, M_{-1}(\kappa_{\mathbf{d}})\varphi, \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s).\end{aligned}$$

Note that

$$x \in \mathcal{C} \cap M_0\xi_M, \quad Q_x = L_x, \quad \rho_{w_{\theta L}} = \rho_{Q_x} = 0, \quad (\mathfrak{a}^*_{L,\mathbb{C}})^-_{w_{\theta L}\theta} = \mathfrak{a}^*_{L,\mathbb{C}}.$$

Applying Proposition 2.5 to $E^Q_{-1}(\varphi)$ with L and L_x in place of G and H, we get

$$\begin{aligned}&v_{M_H^\dagger}\int_{P_x(\mathbb{A})\backslash Q_x(\mathbb{A})} P^{M_x}(M_{-1}(\kappa_{\mathbf{d}})\varphi)(qg)e^{\langle \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}}, H_M(qg)\rangle}dq \\ &= \int_{L_x(F)\backslash L_x(\mathbb{A})} E^Q_{-1}(\varphi)(hg)dh = P^{L_x}(E^Q_{-1}(\varphi))(g).\end{aligned}$$

Since

$$\langle s, H_M(qg)\rangle = \langle s, H_L(qg)\rangle = \langle s, H_L(g)\rangle$$

for $s \in \mathfrak{a}^*_{L,\mathbb{C}}$, $q \in Q_x(\mathbb{A})$ and $g \in G(\mathbb{A})$, we finally obtain

$$\int^*_{H(F)\backslash H(\mathbb{A})} E(h, E^Q_{-1}(\varphi), s)dh = J(w_{\theta L}, E^Q_{-1}(\varphi), s) \tag{3.2}$$

for s in some open set of $\mathfrak{a}^*_{L,\mathbb{C}}$. Since the left hand side is meromorphically continued to $\mathfrak{a}^*_{L,\mathbb{C}}$ and holomorphic on $\sqrt{-1}\mathfrak{a}^*_L$ by Lemma 2.4 (cf. Proposition 12 of [4]), so is $J(w_{\theta L}, E^Q_{-1}(\varphi), s)$. The stated identity is obtained by evaluating at $s \in \sqrt{-1}\mathfrak{a}^*_L$. ☐

We are going to prove the following result in the next section.

Proposition 3.3. — *Notation being as in Theorem 3.2, we assume that all d_i are even. Then there is $\psi \in I(\pi)$ such that the function $s \mapsto \int_{H(F)\backslash H(\mathbb{A})} E(h, \psi, s)dh$ is not identically zero.*

We set forth the following conjecture:

Conjecture 3.4. — *Let $\sigma \in \Pi_c(M)$ and $\pi \in \Pi_d(L)$ be as in Section 1. If all d_i are even, then $I(\pi, \lambda)$ is distinguished by H for each $\lambda \in \sqrt{-1}\mathfrak{a}^*_L$.*

The following theorem generalizes Theorem 7.7 of [10].

Theorem 3.5. — *Let $\rho \in \Pi_d(M)$, $\psi \in I(\rho)$ and $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$.*

1. *$J(\xi, \psi, \lambda)$ extends to a meromorphic function on the space $2\rho_\xi - \rho_P + (\mathfrak{a}^*_{M,\mathbb{C}})^-_{\xi\theta}$.*
2. *For $\xi' \in \mathfrak{I}_{M'}(\theta)$ and $w \in W(\xi, \xi')$*

$$J(\xi', M(w, \lambda)\psi, w\lambda) = J(\xi, \psi, \lambda).$$

Proof. — We can deduce the theorem from (3.2) and [10, Lemma 4.4] by the same technique as in [4, 7]. The detail is left to the reader. ☐

4. The local intertwining periods

Let $P = MU$ be a parabolic subgroup of G of type $(m_1, \ldots, m_t)$, $\rho \in \Pi_d(M)$ and $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$. Choose $x \in \mathcal{O}_\xi \cap M\xi$ and η so that $x = \eta \star \mathbf{1}_{2n}$. We assume that $\tau_\xi(i) \neq i$ for all $i \in [1, t]$. We may suppose that $\rho_i \simeq \rho_{\tau_\xi(i)}$ for all $i \in [1, t]$ as $P^{M_x}(\psi)$ is identically zero for all $\psi \in I(\rho)$ otherwise. Then the period integral P^{M_x} gives rise to the unique (up to a scalar) $M_x(\mathbb{A})$-invariant form l_{M_x} on ρ. We fix an identification of ρ with a restricted tensor product $\otimes_v \rho_v$. This identification presupposes the choice of K_v-fixed vectors in the space of ρ_v for almost all v. The invariant form l_{M_x} on ρ decomposes into local invariant forms $l_{M_x,v}$ on ρ_v. The local intertwining period is defined by

$$J_v(\xi, \psi_v, \lambda) = \int_{\eta^{-1} P_x(F_v)\eta \backslash H(F_v)} l_{M_x,v}(\psi_v(\eta h)) e^{\langle \lambda, H_M(\eta h)\rangle} dh$$

for $\psi_v \in I(\rho_v)$ and for λ in some open set of $2\rho_\xi - \rho_P + (\mathfrak{a}^*_{M,\mathbb{C}})^-_{\xi\theta}$. Then we have the factorization

$$J(\xi, \psi, \lambda) = \prod_v J_v(\xi, \psi_v, \lambda),$$

provided that $\psi = \otimes_v \psi_v$ is factorizable.

We switch to a local setting and drop the index v from our notation. Thus $F = F_v$ is a local field of characteristic zero. When X is an algebraic group over F, we will write $X = X(F)$ for simplicity. The length function $\ell_M : W(M) \to \mathbb{Z}_{\geqslant 0}$ is defined in [9] by

$$\ell_M(w) = \#\{\alpha \in R^+(T_M, G) \mid w\alpha < 0\}.$$

For any Levi subgroup M and $\alpha \in \Delta_M$ there is an element $s_\alpha \in W(M)$ characterized by the property that $\ell_M(s_\alpha) = 1$ and $s_\alpha \alpha < 0$.

Lemma 4.1. — *Let* $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$, $\xi' \in \mathfrak{I}_{M'}(\theta) \cap W(\theta)$, *and* $\alpha \in \Delta_M$. *Assume that* $s_\alpha \in W(\xi, \xi')$.

(1) $\ell_{\theta M'}(\xi') = \ell_{\theta M}(\xi)$, $\ell_{\theta M}(\xi) + 2$ *or* $\ell_{\theta M}(\xi) - 2$ *according as* $\xi\theta\alpha = \pm\alpha$, $\alpha \neq \xi\theta\alpha > 0$ *or* $-\alpha \neq \xi\theta\alpha < 0$.

(2) *Assume that* S_ξ *is empty and* $-\alpha \neq \xi\theta(\alpha) < 0$. *Let* ρ *be an irreducible unitary representation of* M. *Let* $\psi \in I(\rho)$ *and* $\lambda \in 2\rho_\xi - \rho_P + (\mathfrak{a}^*_{M,\mathbb{C}})^-_{\xi\theta}$. *If the double integral defining* $J(\xi, \psi, \lambda)$*converges absolutely, then*

$$J(\xi', M(s_\alpha, \lambda)\psi, s_\alpha\lambda) = J(\xi, \psi, \lambda).$$

Proof. — The proof of (1) is the same as that of Lemma 3.2.1 of [7]. Since $w_{M'}(\tau_{\xi'}) = s_\alpha w_M(\tau_\xi) s_\alpha^{-1}$, if S_ξ is empty, then $S_{\xi'}$ is empty. The proof of (2) mimics the argument of Proposition 10.1.1 of [7] by utilizing Lemma 3.8 of [10]. □

By the same reasoning as [4, 7, 10] we can use Lemma 4.1 to deduce convergence and meromorphic continuation of $J(\xi, \psi, \lambda)$ from those of the intertwining operators. As far as the convergence is concerned, we may replace ρ by the trivial representation. We will not repeat the proof.

Proposition 4.2. — *Let* ρ *be an irreducible unitary representation of* M, $\psi \in I(\rho)$ *and* ξ *an element of* $\mathfrak{I}_M(\theta) \cap W(\theta)$ *such that* S_ξ *is empty.*

(1) $J(\xi, \psi, \lambda)$ *converges absolutely when* $\Re\lambda - 2\rho_\xi + \rho_P \in \mathcal{D}_{\xi,M}$.

(2) $J(\xi, \psi, \lambda)$ *is continued meromorphically to* $2\rho_\xi - \rho_P + (\mathfrak{a}^*_{M,\mathbb{C}})^-_{\xi\theta}$.

Let $(d_1n_1, \dots, d_rn_r)$ be a partition of $2n$, $P = MU$ the parabolic subgroup of G of type $(n_1, \dots, n_1, \dots, n_r, \dots, n_r)$, and $\sigma = \otimes_{i\in[1,r]}\sigma_i^{\otimes d_i}$ an irreducible unitary generic representation of M. Suppose that all $d_i = 2t_i$ are even. The local L factors $L(s, \sigma_i \times \sigma_j^\vee)$ are defined by Jacquet, Piatetski-Shapiro and Shalika [5] in the nonarchimedean case. Since σ is unitary and generic, the factors $L(s, \sigma_i \times \sigma_j^\vee)$ are holomorphic in $\Re s \geqslant 1$.

We use the notation defined in the proof of Theorem 3.2. Take a decomposition $w' = s_{\alpha_\ell} \cdots s_{\alpha_1}$, where $\ell = \ell_M(w')$, $\alpha_i \in \Delta_{M_i}$, and $M_1 = M$, $M_{i+1} = s_{\alpha_i} M_i s_{\alpha_i}^{-1}$ for $i \in [1, \ell]$. Since $\ell_{\theta M}(\xi_M) = 2\ell_M(w')$, Lemma 4.1(1) shows that $-\alpha_i \neq \xi_i\theta\alpha_i < 0$, where $\xi_1 = \xi_M$, $\xi_{i+1} = s_{\alpha_i}\xi_i(w_0 s_{\alpha_i} w_0)^{-1}$ for $i \in [1, \ell]$. Applying Lemma 4.1(2) successively, we get

$$J(\xi_M, \phi, \lambda) = J(\mathbf{1}_{2n}, M(w', \lambda)\phi, w'\lambda), \qquad \phi \in I(\sigma).$$

Observe that $M(\kappa_{\mathbf{d}}) = M(\kappa_{\mathbf{d}}, \Lambda_{\mathbf{d}} + s)$ is independent of s. Note that when d is even,

$$\{(j, k) \mid 1 \leqslant j < k \leqslant d,\ \kappa_d(j) > \kappa_d(k)\} = \{(2j, k) \mid 1 \leqslant 2j < k \leqslant d\}.$$

If F is a nonarchimedean field, σ is unramified and $\phi \in I(\sigma)$ is K-invariant such that $\phi(e) = 1$, then by the Gindikin-Karpelevich formula

$$\begin{aligned} M(\kappa_{\mathbf{d}})\phi &= \phi \prod_{i=1}^{r} \prod_{j=1}^{t_i-1} \prod_{k=2j+1}^{2t_i} \frac{L(k-2j, \sigma_i \otimes \sigma_i^\vee)}{L(k-2j+1, \sigma_i \otimes \sigma_i^\vee)} \\ &= \phi \prod_{i=1}^{r} \frac{L(1, \sigma_i \otimes \sigma_i^\vee)^{t_i-1}}{\prod_{j=1}^{t_i-1} L(2j+1, \sigma_i \otimes \sigma_i^\vee)} \end{aligned}$$

and $M(w', \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s)\phi(e)$ is equal to

$$\begin{aligned} &\prod_{1 \leqslant i < j \leqslant r} \prod_{k=1}^{t_i} \prod_{l=1}^{2t_j} \frac{L(l-2k+s_i-s_j+t_i-t_j, \sigma_i \otimes \sigma_j^\vee)}{L(l-2k+s_i-s_j+t_i-t_j+1, \sigma_i \otimes \sigma_j^\vee)} \\ &= \prod_{1 \leqslant i < j \leqslant r} \prod_{k=1}^{t_i} \frac{L(s_i-s_j+2k-t_i-t_j-1, \sigma_i \otimes \sigma_j^\vee)}{L(s_i-s_j+t_i+t_j+1-2k, \sigma_i \otimes \sigma_j^\vee)}. \end{aligned}$$

Proposition 4.3. — *Notation being as above, $J(\xi_M, M(\kappa_{\mathbf{d}})\varphi, \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s)$ is not identically zero as a meromorphic function and as φ varies.*

Proof. — For $g \in G$ and $\varphi \in I(\sigma)$ we put

$$j_{L_x}(\varphi)(g) = \int_{P_x \backslash L_x} l_{M_x}(M(\kappa_{\mathbf{d}})\varphi(qg)) e^{\langle \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}}, H_M(qg)\rangle} dq$$

as in [12]. Theorem 5 of [11] tells us that j_{L_x} is not identically zero on $I(\sigma)$. Note that

$$J(\xi_M, M(\kappa_{\mathbf{d}})\varphi, \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s) = \int_{Q_x \backslash H_x} j_{L_x}(\varphi)(h\eta) e^{\langle s, H_L(h\eta)\rangle} dh.$$

Since $\dim Q + \dim H_x - \dim L_x = \dim G$, we see that QH_x is an open set in G. Thus this integral can be taken to be nonzero by choosing φ to be supported in a small neighborhood inside $Q\backslash QH_x\eta$. □

Back to the global setup, we are now ready to prove Proposition 3.3. Let S be a finite set of places of F which contains all the archimedean places and such that for all $v \notin S$, σ_v is unramified and φ_v is K_v-invariant. Then

$$\begin{aligned} \int^{*}_{H(F)\backslash H(\mathbb{A})} E(h, E^Q_{-1}(\varphi), s) dh &= \prod_{i=1}^{r} \frac{\mathrm{Res}_{s=1} L^S(s, \sigma_i \otimes \sigma_i^\vee)^{t_i-1}}{\prod_{j=1}^{t_i-1} L^S(2j+1, \sigma_i \otimes \sigma_i^\vee)} \\ &\times \prod_{1 \leqslant i < j \leqslant r} \prod_{k=1}^{t_i} \frac{L^S(s_i-s_j+2k-t_i-t_j-1, \sigma_i \otimes \sigma_j^\vee)}{L^S(s_i-s_j+t_i+t_j+1-2k, \sigma_i \otimes \sigma_j^\vee)} \\ &\times \prod_{v \in S} J_v(\xi_M, M(\kappa_{\mathbf{d}})\varphi_v, \kappa_{\mathbf{d}}\Lambda_{\mathbf{d}} + s). \end{aligned}$$

Proposition 4.3 now completes the proof of Proposition 3.3.

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Shunsuke YAMANA
Graduate School of Mathematics,
Kyushu University,
744, Motooka, Nishi-ku, Fukuoka,
819-0395, Japan
yamana@math.kyushu-u.ac.jp