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RIEMANN SURFACES IN STEIN MANIFOLDS WITH THE DENSITY PROPERTY

by Rafael B. ANDRIST & Erlend Fornæss WOLD (*)

Abstract. — We show that any open Riemann surface can be properly immersed in any Stein manifold with the (Volume) Density property and of dimension at least 2. If the dimension is at least 3, we can actually choose this immersion to be an embedding. As an application, we show that Stein manifolds with the (Volume) Density property and of dimension at least 3, are characterized among all other complex manifolds by their semigroup of holomorphic endomorphisms.

Résumé. — Nous montrons que toute surface de Riemann ouverte peut être immergée proprement dans toute variété de Stein avec la propriété de densité (volumique) et de dimension au moins 2. Si la dimension est au moins 3, cette immersion est en fait un plongement propre. Les résultats obtenus sont appliqués pour montrer que toutes les variétés de Stein avec la propriété de densité (volumique) et de dimension au moins 3 sont caractérisées entre toutes les variétés complexes par leur demi-groupe d'endomorphismes holomorphes.

1. Introduction

An open Riemann surface always admits a proper holomorphic embedding into $\mathbb{C}^3$ and a proper holomorphic immersion into $\mathbb{C}^2$ (special case for dimension one of the Embedding Theorem for Stein manifolds by Remmert [26], Narasimhan [25] and Bishop [7]).

In this paper we generalize these results to embeddings and immersions of open Riemann surfaces into Stein manifolds with the density property or the volume density property. Our main result is the following:

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Theorem (5.1). — *Let X be a Stein manifold with the density property or with the volume density property and $\mathcal{R}$ an open Riemann surface.*

(a) *If $\dim X \geqslant 3$ then there is a proper holomorphic embedding $\mathcal{R} \hookrightarrow X$*
(b) *If $\dim X = 2$ then there is a proper holomorphic immersion $\mathcal{R} \to X$.*

Our main motivation is the recent work of the first author [6], who has shown that Stein manifolds are characterized by their endomorphism semigroups as long as they admit a proper holomorphic embedding of the complex line $\mathbb{C}$. As a corollary to our main theorem we thereby obtain:

Theorem (5.5). — *Let X and Y be complex manifolds and $\Phi : End(X) \to End(Y)$ an epimorphism of semigroups of holomorphic endomorphisms. If X is a Stein manifold with the density or volume density property and of dimension at least 3, then there exists a unique $\varphi : X \to Y$ which is is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ for all $f \in End(X)$.*

Secondly, our work generalizes recent work of Drinovec-Drnovšek and Forstnerič [9] who have proved that *bordered* Riemann surfaces immerse properly into Stein manifolds of dimension greater than one. Note that, due to hyperbolicity, the complex plane does not embed in all Stein manifolds, so some extra structure (e.g. the density property) is needed. One might ask whether our main theorem holds with X an Oka manifold [17] instead.

Thirdly, it was conjectured by Schoen and Yau [27] in 1997 that no proper harmonic map could exist from the unit disk onto $\mathbb{R}^2$. The conjecture was first disproved by Forstnerič and Globevnik [19] in 2001 using a proper holomorphic immersion of the unit disk into $\mathbb{C}^* \times \mathbb{C}^*$. More recently, the conjecture was disproved again by Alarcón and Galvéz [1] in another way. However, a much stronger result follows easily from our main theorem:

Theorem (5.6). — *Every open Riemann surface admits a proper harmonic map into $\mathbb{R}^2$.*

The same result was also recently obtained with different methods by Alarcón and López [2].

Finally, it is of general interest to find new methods to produce proper holomorphic maps from Riemann surfaces into complex manifolds, due to the long standing open problem whether any open Riemann surface admits a proper holomorphic embedding in $\mathbb{C}^2$.

2. The Density Property

The density property was introduced in Complex Geometry by Varolin [32], [31]. For a survey about the current state of research related to the density property and Andersén–Lempert theory, we refer to Kaliman and Kutzschebauch [24].

Definition 2.1. — *A complex manifold X has the* density property *if in the compact-open topology the Lie algebra $Lie_{hol}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $VF_{hol}(X)$ of all holomorphic vector fields on X.*

Definition 2.2. — *Let a complex manifold X be equipped with a holomorphic volume form ω (i.e. ω is a nowhere vanishing section of the canonical bundle). We say that X has the* volume density property *with respect to ω if in the compact-open topology the Lie algebra $Lie^{\omega}_{hol}(X)$ generated by completely integrable holomorphic vector fields ν such that $\nu(\omega) = 0$, is dense in the Lie algebra $VF^{\omega}_{hol}(X)$ of all holomorphic vector fields that annihilate ω.*

The following theorem is the central result of Andersén– Lempert theory (originating from works of Andersén and Lempert [4], [5]), and is given in the following form in [24] by Kaliman and Kutzschebauch, but essentially (for $\mathbb{C}^n$) proved already in [20] by Forstnerič and Rosay.

Theorem 2.3. — *Let X be a Stein manifold with the density (resp. volume density) property and let Ω be an open subset of X. In case of the volume density property further assume that $H^{n-1}(\Omega, \mathbb{C}) = 0$. Suppose that $\Phi : [0,1] \times \Omega \to X$ is a $\mathcal{C}^1$-smooth map such that*

1. *$\Phi_t : \Omega \to X$ is holomorphic and injective (and resp. volume preserving) for every $t \in [0,1]$*
2. *$\Phi_0 : \Omega \to X$ is the natural embedding of Ω into X*
3. *$\Phi_t(\Omega)$ is a Runge subset of X for every $t \in [0,1]$*

Then for each $\varepsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family $\alpha : [0,1] \to \mathrm{Aut}\,(X)$ of holomorphic (and resp. volume preserving) automorphisms of X such that

$$\alpha_0 = \mathrm{id} \text{ and } |\alpha_t - \Phi_t|_K < \varepsilon$$

for every $t \in [0,1]$.

Remark 2.4. — In the case of the volume density property it is enough to assume that $H^{n-1}(\Omega', \mathbb{C}) = 0$ for all connected components Ω' of Ω

where Φ is not the identity map. The assumption is used to solve a certain differential equation which is trivially solvable on the components where Φ is the identity.

Among one of the many results following from this theorem, we only need the following, first given by Varolin [31]:

Proposition 2.5. — *Let X be a Stein manifold of dimension $n \geqslant 2$ with the density (resp. volume density) property, K be a compact in X, and $x, y \in X$ be two points outside the holomorphically convex hull of K. Suppose that $x_1, \dots, x_m \in K$. Then there exists a (resp. volume-preserving) holomorphic automorphism Ψ of X such that $\Psi(x_i) = x_i$ for every $i = 1, \dots, m$, $\Psi|K : K \to X$ is as close to the natural embedding as we wish, and $\Psi(y) = x$.*

Corollary 2.6. — *If X is a Stein manifold of dimension $n \geqslant 2$ with the density (resp. volume density) property, then its group of holomorphic automorphisms acts m-transitively for any $m \in \mathbb{N}$.*

Stein manifolds with the (volume) density property are *elliptic* in the sense of Gromov [22] and satisfy the so-called Oka–Grauert–Gromov principle.

Definition 2.7. — *Let X be a complex manifold. It is said to have the* Convex Approximation Property *(introduced in* [15, 16]*) if every holomorphic map of a compact convex set $K \subset \mathbb{C}^n$ in X can be approximated uniformly on K by entire holomorphic maps $\mathbb{C}^n \to X$.*

In more recent terminology introduced by Forstnerič [17], a manifold satisfying the Convex Approximation Property is called an *Oka manifold*. All elliptic Stein manifolds, in particular those with the (volume) density property, are Oka manifolds.

Oka manifolds satisfy the following (see Drinovec-Drnovšek and Forstnerič [10]):

Theorem 2.8. — *Let S be a Stein manifold and $D \subset\subset S$ a strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geqslant 2$) whose closure $\overline{D}$ is $\mathcal{O}(S)$-convex, and let Y be an Oka manifold. Let $r \in \{0, 1, \dots, \ell\}$ and let $f : S \to Y$ be a C^r-map which is holomorphic in D. Then f can be approximated in the $C^r(\overline{D}, Y)$-topology by holomorphic maps $S \to Y$ which are homotopic to f.*

3. Approximating in the non-critical case

In the following we will say that $\mathcal{R}$ is a *bordered* Riemann surface if $\mathcal{R}$ is a relatively compact open subset of an ambient Riemann surface $\mathcal{R}'$ such that $\mathcal{R}$ is smoothly bounded in $\mathcal{R}'$. The goal of this section is to prove the following approximation result.

PROPOSITION 3.1. — *Let $\mathcal{R}_0 \subset\subset \mathcal{R}_1 \subset\subset \mathcal{R}_2$ be bordered Riemann surfaces, let X be a Stein manifold with the density or volume density property, and let $K \subset X$ be a holomorphically convex compact set. Let $f : \overline{\mathcal{R}_1} \to X$ be a holomorphic immersion, and assume that $f(\overline{\mathcal{R}_1 \setminus \mathcal{R}_0}) \subset X \setminus K$. Let Γ be one of the boundary components of $\mathcal{R}_1$, and $\mathcal{A} \subset \mathcal{R}_2 \setminus \overline{\mathcal{R}_0}$ be an annulus containing Γ.*

Then for any compact subset L of $\mathcal{A}$ and any $\varepsilon > 0$ there exists a holomorphic immersion $g : \mathcal{R}_1 \cup L \to X$ such that the following holds:

(1) $\|g - f\|_{\mathcal{R}_0} < \varepsilon$, *and*
(2) $g((\mathcal{R}_1 \cup L) \setminus \mathcal{R}_0) \subset X \setminus K$.

This will in turn depend on the following result:

PROPOSITION 3.2. — *Let $\mathcal{R}_0 \subset\subset \mathcal{R}_1 \subset\subset \mathcal{R}_2$ be bordered Riemann surfaces, let X be a Stein manifold with the density or volume density property, and let $K \subset X$ be a holomorphically convex compact set. Let $f : \overline{\mathcal{R}_1} \to X$ be a holomorphic immersion, and assume that $f(\overline{\mathcal{R}_1 \setminus \mathcal{R}_0}) \subset X \setminus K$. Let Γ be one of the boundary components of $\mathcal{R}_1$, let $V \subset \mathcal{R}_2 \setminus \mathcal{R}_0$ be an open set in the domain of f, intersecting Γ, and assume that $f|_{\overline{V}}$ is an embedding, with $f(\overline{V})$ not intersecting $K \cup f(\overline{\mathcal{R}_0})$.*

Let $U \subset \mathcal{R}_2 \setminus \mathcal{R}_0$ be a simply connected open set, $U \cap \mathcal{R}_1 \subset V \cap \mathcal{R}_1$, $U \cap \mathcal{R}_1$ connected, and assume that we are given a $\mathcal{C}^\infty$-smooth isotopy $\varphi(\cdot, t) : U \to \mathcal{R}_2 \setminus \mathcal{R}_0$, $t \in [0, 1]$, of injective holomorphic maps, such that the following holds:

(1) $\varphi(\cdot, 0) = \mathrm{id}_U$,
(2) $\varphi(U, 1) \subseteq V$,
(3) $\varphi(U \setminus \mathcal{R}_1, t) \subseteq \mathcal{R}_2 \setminus \mathcal{R}_1$ *for all* $t \in [0, 1]$, *and*
(4) $\varphi(U \cap \mathcal{R}_1, t) \subseteq V \cap \mathcal{R}_1$ *for all* $t \in [0, 1]$.

Then for any compact subset L of U and any $\varepsilon > 0$ there exists a holomorphic immersion $g : \mathcal{R}_1 \cup L \to X$ such that the following holds:

(1) $\|g - f\|_{\mathcal{R}_0} < \varepsilon$, *and*
(2) $g((\mathcal{R}_1 \cup L) \setminus \mathcal{R}_0) \subset X \setminus K$.

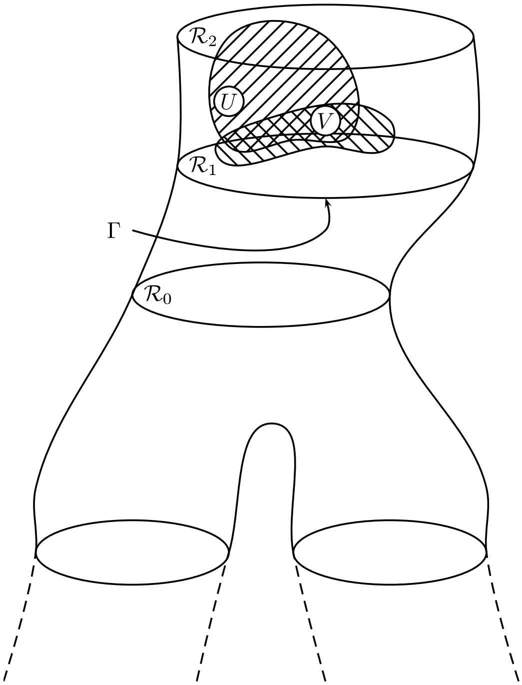


Figure 3.1. Setup for glueing a disk in Proposition 3.2

We cite Theorem 4.1 from Forstnerič [14] which will be needed in the proof:

Theorem 3.3. — *Let (A, B) be a Cartan pair in a complex manifold X. Given an open set $\tilde{C} \subseteq X$ containing $C := A \cap B$ there exist open sets $A' \supseteq A, B' \supseteq B, C' \supseteq C$ with $C' \subseteq A' \cap B' \subseteq \tilde{C}$, satisfying the following: For every injective holomorphic map $\gamma : \tilde{C} \to X$ which is sufficiently uniformly close to the identity on $\tilde{C}$ there exist injective holomorphic maps $\alpha : A' \to X, \beta : B' \to X$, uniformly close to the identity on their respective domains and satisfying*

$$\gamma = \beta \circ \alpha^{-1}$$

In analogy to the classical splitting for Cartan pairs, we call such A and B satisfying the assertions of the theorem also a *Cartan pair*.

We will also need the following:

Proposition 3.4 (Theorem 3.3.3 in [18]). — *Let Y be a Stein submanifold of a complex manifold X. Denote by $N_{Y/X}$ the normal bundle of Y*

in X. There exists a Stein neighborhood U of Y in X, biholomorphic to an open neighborhood Ω of the zero section in $N_{Y/X}$.

Corollary 3.5. — *Let X be a complex manifold of dimension n and let $f : \overline{\mathcal{R}} \to X$ be an immersion of a bordered Riemann surface $\mathcal{R}$. Then there exists a holomorphic immersion $F : \overline{\mathcal{R}} \times \mathbb{D}^{n-1} \to X$ such that $F|_{\mathcal{R}\times\{0\}} = f$.*

Proof. — Let $\tilde{X}$ be a complex manifold with an embedding $\tilde{f} : \overline{\mathcal{R}} \to \tilde{X}$ and an immersion $\rho : \tilde{X} \to X$ such that $f = \rho \circ \tilde{f}$. By Siu's theorem [30] we may assume that $\tilde{X}$ is Stein, and so the proposition applies to give an embedding $\tilde{F}$ into $\tilde{X}$. Note that any vector bundle over an open Riemann surface is trivial, and define $F = \rho \circ \tilde{F}$. □

To make use of the volume density property we need to extend isotopies defined on embedded Riemann surfaces to volume preserving isotopies defined on a full neighborhood of the embedded surface. Hence we need the following lemma

Lemma 3.6. — *Let $D \subset \mathbb{C}$ be a domain, let $\varphi_t : D \to D, t \in [0,1]$, be an isotopy of injective holomorphic maps, and let ω be a any volume form on $D \times \mu \cdot \mathbb{D}^{n-1}, \mu > 0$. Then for any $U \subset\subset D$ there exists $\mu_1 > 0$ and an isotopy $\phi_t : U \times \mu_1 \cdot \mathbb{D}^{n-1} \to D \times \mu \cdot \mathbb{D}^{n-1}$ of injective holomorphic maps such that ϕ_t extends φ_t and $\phi_t^*\omega = \omega$ for all t.*

Proof. — Write $\omega(z) = f(z) \cdot dz_1 \wedge \cdots \wedge dz_n$. We first change coordinates such that ω is the standard volume form on $\mathbb{C}^n$. For this we define $g(z) := 1/f(z)$, and then we find $h(z)$ such that $h(z_1, ..., z_{n-1}, 0) = 0$ and $\frac{\partial h}{\partial z_n}(z) = g(z)$. Defining $H(z) := (z_1, ..., z_{n-1}, h(z))$ we see that $H^*\omega = dz := dz_1 \wedge \cdots \wedge dz_n$, and H is biholomorphic near $z_n = 0$. Next define $\tilde{\phi}_t(z_1, ..., z_n) := (\varphi_t(z_1), z_2, ..., z_{n-1}, \frac{z_n}{\varphi_t'(z_1)})$. Then $\tilde{\phi}_t^* dz = dz$ for all t, and defining $\phi_t := H \circ \tilde{\phi}_t \circ H^{-1}$ we see that $\phi_t^*\omega = H_*\tilde{\phi}^* H^*\omega = \omega$. □

Lemma 3.7. — *Let X be a Stein manifold, let $K \subset X$ be a holomorphically convex compact set, and let $f : \overline{\mathbb{D}} \to X \setminus K$ be an embedding. Let $V' \subset X \setminus K$ be an open neighborhood of $f(\overline{\mathbb{D}})$ and assume given an isotopy of holomorphic injections $\phi_t : V' \to X \setminus K$. Then there exists an open neighborhood $V'' \subseteq V'$ of $f(\overline{\mathbb{D}})$ and an open neighborhood W of K such that $\Omega_t = \phi_t(V'') \cup W$ is a Runge domain in X for all $t \in [0,1]$.*

Proof. —

It is a well known fact that $K \cup \phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex for each fixed $t \in [0,1]$ (for lack of a reference we include an argument below). The result is then a consequence of Lemma 2.2 in [20] formulated for X instead of $\mathbb{C}^n$ (the proof in the case of a Stein manifold is identical).

We now show that $K\cup\phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex for each fixed $t\in[0,1]$. Let $r>1$ be chosen close enough to 1 such that $(\phi_t\circ f):\overline{\mathbb{D}}_r\to X\setminus K$ is an embedding, let $\Sigma=\phi_t(f(b\mathbb{D}))$, and $\Sigma'=\phi_t(f(b\mathbb{D}_r))$. We want to show that $\widehat{K\cup\Sigma}=K\cup\phi_t(f(\overline{\mathbb{D}}))$.

By Theorem 12.5. in [3] and the fact that X embeds properly in $\mathbb{C}^N$ for N sufficiently large, we have that $\widehat{K\cup\Sigma'}\setminus(K\cup\Sigma')$ (resp. $\widehat{K\cup\Sigma}\setminus(K\cup\Sigma)$) is a one-dimensional analytic subset of $X\setminus(K\cup\Sigma')$ (resp. $X\setminus(K\cup\Sigma)$. Note first that $\widehat{K\cup\Sigma}$ cannot contain a relatively open subset of $\phi_t(f(\mathbb{D}_r\setminus\overline{\mathbb{D}}))$. If it did, it would, by the identity principle for analytic sets, contain $\phi_t(f(\mathbb{D}_r\setminus\overline{\mathbb{D}}))$, and so $\widehat{K\cup\Sigma}\setminus K$ would be an analytic subset of $X\setminus K$. This is impossible since K is holomorphically convex. Since $\widehat{K\cup\phi_t(f(\overline{\mathbb{D}}))}\subset\widehat{K\cup\Sigma'}$ we get that

$$\overline{\widehat{K\cup\phi_t(f(\overline{\mathbb{D}}))}\setminus(K\cup\phi_t(f(\overline{\mathbb{D}})))}\cap(K\cup\phi_t(f(\overline{\mathbb{D}})))=K\cup A,$$

where A is a finite set of points. By Rossi's local maximum principle we have that $\widehat{K\cup\phi_t(f(\overline{\mathbb{D}}))}=(K\cup\phi_t(f(\overline{\mathbb{D}})))\cup\widehat{K\cup A}$ which implies that $K\cup\phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex. $\square$

Proof of Proposition 3.2. —
Since X is an Oka manifold we may assume, by approximation, that f is already defined on $\overline{\mathcal{R}_2}$; the task is is to find an approximation which achieves (2). Note that $K\cup f(\overline{\mathcal{R}_0})$ is holomorphically convex.

Define $A:=\overline{\mathcal{R}_1}$, and let $B\subset\mathcal{R}_2$ be a Stein compact such that the pair A, B is a Cartan pair as in Theorem 3.3, $A\cap B$ simply connected and contained in V, and $L\subset(A\cup B)^\circ$. We will approximate f on a certain thickening of $A\cup B$ in $\mathcal{R}_2\times\mathbb{C}^{n-1}$ which will allow us to exploit the density property of X.

Since f is an immersion, we have by Corollary 3.5 that f extends to an immersion

$$F:\mathcal{R}_2\times\mu\cdot\mathbb{D}^{n-1}\to X,$$

such that $F|_{\mathcal{R}_2\times\{0\}}=f$. Since $f|_{\overline{V}}$ is an embedding, we may assume that $F|_{\overline{V}\times\mu\mathbb{D}^{n-1}}$ is an embedding whose image does not intersect $K\cup f(\overline{\mathcal{R}_0})$.

Set $\tilde\omega:=F^*\omega$. By choosing μ_1 small enough we get from Lemma 3.6 that φ_t extends to an isotopy $\phi_t:U\times\mu_1\cdot\mathbb{D}^{n-1}\to(\mathcal{R}_2\setminus\mathcal{R}_0)\times\mu\cdot\mathbb{D}^{n-1}$ of the form

$$\phi_t(x,w)=(\varphi_t(x),\sigma_t(x,w)),\quad \sigma_t(x,0)=0,$$

and such that $\phi_t^*\tilde\omega=\tilde\omega$ for all $t\in[0,1]$.

Now the following is our strategy: note that there is an open neighborhood W of $C = A \cap B$, relatively compact in V, such that on the image $\Omega := F(W \times \mu_1 \cdot \mathbb{D}^{n-1})$ we have a well defined isotopy

$$\Phi_t := F \circ \phi_t \circ F^{-1} : \Omega \to X \setminus (K \cup f(\overline{\mathcal{R}_0})).$$

Choose W such that $H^{n-1}(W \times \mu_1 \cdot \mathbb{D}^{n-1}, \mathbb{C}) = 0$, and note that $\Phi_t^* \omega = \omega$ for all t. Note also that the composition $F_B := F \circ \phi_1$ is well defined near $B \times \mu_1 \cdot \mathbb{D}^{n-1}$. We will approximate Φ_1 well enough by an automorphism Λ of X, essentially fixing $K \cup f(\overline{\mathcal{R}_0})$, such that the map $\Lambda \circ F$ may be glued with minor perturbations to the map F_B.

By Lemma 3.7 there exist a neighborhood Ω_1 of $K \cup f(\overline{\mathcal{R}_0})$ and a neighborhood $\Omega_2 \subset \Omega$ of $f(\overline{W})$ such that $\Phi_t(\Omega_1 \cup \Omega_2)$ is Runge for each t, where $\Phi_t|_{\Omega_1} \equiv \mathrm{id}$. Fix a $0 < \mu_2 < \mu_1$ such that $F(\overline{W} \times \mu_2 \cdot \mathbb{D}^{n-1}) \subset \Omega_2$. For any $0 < \delta < \mu_2$ let A_δ, B_δ denote the Cartan pairs $A \times \delta \cdot \overline{\mathbb{D}^{n-1}}$ and $B \times \delta \cdot \overline{\mathbb{D}^{n-1}}$, respectively. Let $\tilde{C}_\delta$ be a neighborhood of $C_\delta := A_\delta \cap B_\delta$ contained in $W \times \mu_2 \cdot \mathbb{D}^{n-1}$, and let C'_δ be the corresponding neighborhood of C_δ in Theorem 3.3.

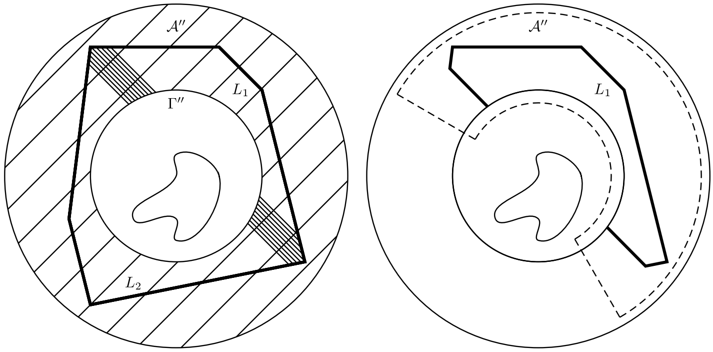
Now let Λ_j be a sequence of automorphisms of X converging uniformly to Φ_1 near $F(C'_\delta)$ and such that each Λ_j stays uniformly close to the identity near $K \cup f(\overline{\mathcal{R}_0})$. This is possible by the (volume) density property (Theorem 2.3 and the remark following it) of X and the choices made above.

Then $\gamma_j := F^{-1} \circ \Lambda_j^{-1} \circ F \circ \phi_1$ converges to the identity uniformly on C'_δ. Decompose $\gamma_j = \alpha_j \circ \beta_j^{-1}$ using Theorem 3.3. Define $G_j := \Lambda_j \circ F \circ \alpha_j$ on A'_δ and $G_j := F \circ \phi_1 \circ \beta_j$ on B'_δ. Now put $G := G_j$ for a large enough j and then $g := G|_{(A \cup B) \times \{0\}}$. □

Proof of Proposition 3.1. —

(1) The extension from f to g will be achieved in two steps, attaching in each step a simply connected domain in $\mathcal{R}_2$ using Proposition 3.2. Since f is defined on $\overline{\mathcal{R}_1} \subset \mathcal{R}_2$, it extends immersively to a neighborhood $\mathcal{R}'_1 \subset\subset \mathcal{R}_2$ such that f is injective on a neighborhood $V \subset \mathcal{R}_2$ of $b\mathcal{R}'_1$ which is generically the case.

(2) We embed the annulus $\mathcal{A}$ in $\mathbb{C}$ as a planar domain $\mathcal{A}'$. Now we can work entirely in $\mathbb{C}$ and identify all subsets of $\mathcal{A}$ with subsets of $\mathbb{C}$ in order to construct the sets U and isotopies φ needed for Proposition 3.2. The curve Γ is mapped to a smooth curve Γ' in the image $\mathcal{A}'$ of $\mathcal{A}$. Let D denote the bounded domain in $\mathbb{C}$ bordered by Γ'. Then D is homeomorphic to a disk and $\mathcal{A}' \setminus D$ is again an annular region, which we can identify with an annulus $\mathcal{A}'' =:$



(a) Overlapping (shaded) of L_1 and L_2 (b) Glueing the disk

Figure 3.2. Glueing an annulus to a bordered Riemann surface

$\ell\mathbb{D} \setminus \overline{\mathbb{D}}$, $\ell > 0$ via another uniformizing map. This map is a $\mathcal{C}^\infty$-smooth diffeomorphism up to the boundary and therefore extends to a small neighborhood. The set $L \cap \mathcal{A}''$ can be written as a union of two compact overlapping sets L_1 and L_2 which are both homotopic to disks inside in $\mathcal{A}''$, as depicted in figure 3.2(a).

Set $f_0 := f$. The immersion after the first extension to any compact $L_1 \subset U_1 \subset \mathcal{A}'$ will be denoted by f_1, and after the second extension to $L_2 \subset U_2 \subset \mathcal{A}'$ by $f_2 = g$.

(3) Define U_1 to be

$$U_1 := \left\{ r \cdot e^{i\theta} \in \mathbb{C} \,:\, (1-\delta) < r < \ell(1-\delta), \alpha < \theta < \beta \right\}$$

where $1 > \delta > 0$, and $\alpha, \beta \in \mathbb{R}$ are such that $L \subset (1-\delta)\ell\mathbb{D}$. The $\mathcal{C}^\infty$-smooth isotopy $\varphi_1(z,t),\ t \in [0,1]$, of holomorphic injections is given explicitly in equation (3.1) below:

$$\varphi_1(z,t) = \exp\left(\log(z) \cdot ((\gamma - 1) \cdot t + 1)\right) \tag{3.1}$$

with log defined on $\mathbb{C} \setminus \mathbb{R}^-$ and suitable angle $\gamma \in \mathbb{R}$. We now apply Proposition 3.2 to approximately extend f_0 to $L_1 \subset U_1$ using the isotopy φ_1. We denote the "extension" by f_1.

(4) Now consider $\mathcal{A}'' \setminus L_1$ which is again homeomorphic to an annulus and can be mapped to $\mathcal{A}''' =: \ell'\mathbb{D}\setminus\overline{\mathbb{D}}$, $\ell' > 0$ by another uniformizing map. Then we are back in the previous situation and define U_2 and φ_2 the same way. This leads to the desired approximation $g = f_2$. □

4. Approximating in the critical case

Definition 4.1. — *A strongly subharmonic exhaustion function ρ of a Riemann surface $\mathcal{R}$ is said to be a Morse function with* nice singularities, *if the set of critical values Σ is discrete, if for any value $c \in \Sigma$ the set $\rho^{-1}(c)$ is a single point, and any critical point ξ is either a local minimum, or there exist local coordinates $z = x + iy : U_\xi \to \mathbb{C}$ such that ρ is of the form*

$$\rho(z) = \rho(\xi) + x^2 - \mu \cdot y^2,$$

for some $\mu \in (0, 1)$.

Any open Riemann surface has a Morse exhaustion function with nice singularities.

The following proposition tells us that we can extend immersions also across nice critical points. The construction is standard, and we include only a brief sketch of the proof.

Proposition 4.2. — *Let $\mathcal{R}$ be an open connected Riemann surface, and let $\rho \in \mathcal{C}^\infty(\mathcal{R})$ be a Morse exhaustion function with nice singularities. Let $\xi \in \mathcal{R}$ be a critical point of ρ which is not a local minimum, and let $c = \rho(\xi)$. Then there exists a $\delta > 0$ such that the following holds: Let X be a Stein manifold with the density property or volume density property, let $K \subset X$ be a holomorphically convex compact set with $X \setminus K$ connected, let $f : \mathcal{R}_{c-\delta} \to X$ be a holomorphic immersion with $f(b\mathcal{R}_{c-\delta}) \in X \setminus K$, and let $\epsilon > 0$. Then there exists a holomorphic immersion $\tilde{f} : \mathcal{R}_{c+\delta} \to X$ such that*

1. $\|\tilde{f} - f\|_{\mathcal{R}_{c-\delta}} < \epsilon$
2. $\tilde{f}(\mathcal{R}_{c+\delta} \setminus \mathcal{R}_{c-\delta}) \subset X \setminus K$.

Proof. — We will describe how to cross the connected component of $\{\rho = c\}$ which contains the critical point ξ; we follow known arguments in the literature which can be found e.g. in [18, Sect. 3.8]. Crossing the other components is done by applying Proposition 3.1. Let $z : U_\xi \to \mathbb{C}$ be local coordinates with $z(\xi) = 0$, and such that, in local coordinates, $\rho(x, y) = x^2 - \mu \cdot y^2$, $0 < \mu < 1$ (for simplicity we assume $\rho(\xi) = 0$). Choosing δ small enough there are no critical values other than 0 in the interval $[-\delta, \delta]$, and if we let γ denote the arc $\gamma = \{iy : -\delta \leqslant y \leqslant \delta\}$, the set $A := \{\rho \leqslant -\delta\} \cup z^{-1}(\gamma)$ is holomorphically convex, and has a neighborhood basis of domains that non-critically extend to $\{\rho < \delta\}$. Now by the connectedness of $X \setminus K$ the map f extends smoothly to γ with the property that also $f(\gamma) \subset X \setminus K$, and by Megelyan's Theorem the extension may be approximated by holomorphic map $\tilde{f}$ on some neighborhood of A. In particular, $\tilde{f}$ is defined on some neighborhood U of A such that $\{\rho < \delta\}$ is a noncritical extension of U, and now Proposition 3.1 applies. □

5. Main theorem and Applications

THEOREM 5.1. — *Let X be a Stein manifold with the density property or with the volume density property and $\mathcal{R}$ an open Riemann surface.*

(a) *If $\dim X \geqslant 3$ then there is a proper holomorphic embedding $\mathcal{R} \hookrightarrow X$.*
(b) *If $\dim X = 2$ then there is a proper holomorphic immersion $\mathcal{R} \to X$.*

The main ingredients in the proof are Propositions 3.1 and 4.2 from the previous sections. They will be used in an inductive framework provided by Lemma 6.3 from Drinovec-Drnovšek and Forstnerič [9] which we cite here.

DEFINITION 5.2. — *Assume that X is a complex manifold, $\mathcal{R}$ is a relatively compact strongly pseudoconvex domain with $\mathcal{C}^2$ boundary in a Stein manifold S, and σ is a finite set of points in $\mathcal{R}$. A spray of maps of class $\mathcal{A}^2(\mathcal{R})$ with the exceptional set σ of order $k \in \mathbb{N}$ (and with values in X) is a map $f : \overline{\mathcal{R}} \times P \to X$, where P (the parameter set of the spray) is an open subset of a Euclidean space $\mathbb{C}^m$ containing the origin, such that the following hold:*

(1) *f is holomorphic on $\mathcal{R} \times P$ and of class $\mathcal{C}^2$ on $\overline{\mathcal{R}} \times P$*
(2) *the maps $f(\cdot, 0)$ and $f(\cdot, t)$ agree on σ up to order k for $t \in P$, and*
(3) *for every $z \in \overline{\mathcal{R}} \setminus \sigma$ and $t \in P$ the map*

$$\partial_t f(z,t) : T_t\mathbb{C}^m = \mathbb{C}^m \to T_{f(z,t)}X$$

is surjective (the domination property).

The map $f_0 = f(\cdot, 0)$ is called the core (or central) map of the spray f.

LEMMA 5.3. — *Let X be an irreducible complex space of dimension $n \geqslant 2$, and let $\tau : X \to \mathbb{R}$ be a smooth exhaustion function which is $(n-1)$-convex on $\{x \in X : \tau(x) > M_1\}$. Let $\mathcal{R}$ be a finite Riemann surface, let P be an open set in $\mathbb{C}^N$ containing 0, and let $M_2 > M_1$. Assume that $f : \overline{\mathcal{R}} \times P \to X$ is a spray of maps of class $\mathcal{A}^2(\mathcal{R})$ with the exceptional set $\sigma \subset \mathcal{R}$ of order $k \in \mathbb{N}$, and $U \subset \mathcal{R}$ is an open subset such that $f_0(z) \in \{x \in X_{\mathrm{reg}} : \tau(x) \in (M_1, M_2)\}$ for all $z \in \overline{\mathcal{R}} \setminus U$. Given $\varepsilon > 0$ and a number $M_3 > M_2$, there exist a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g : \overline{\mathcal{R}} \times P' \to X$ of class $\mathcal{A}^2(\mathcal{R})$, with exceptional set σ of order k, satisfying the following properties:*

(1) *$g_0(z) \in \{x \in X_{\mathrm{reg}} : \tau(x) \in (M_2, M_3)\}$ for $z \in b\mathcal{R}$,*
(2) *$g_0(z) \in \{x \in X : \tau(x) > M_1\}$ for $z \in \overline{\mathcal{R}} \setminus U$,*
(3) *$d(g_0(z), f_0(z)) < \varepsilon$ for $z \in U$, and*
(4) *f_0 and g_0 have the same k-jets at each of the points in σ.*

Moreover, g_0 can be chosen homotopic to f_0.

First we note that X in our case will be a Stein manifold and therefore τ can be taken to be a strongly plurisubharmonic exhaustion function, and we have $X = X_{\mathrm{reg}}$ as well. The existence of a metric follows in the general case from para-compactness, but in our case of Stein manifolds we can work with the restriction of an euclidean norm. We also cite from [11] their definition of spray of maps of class $\mathcal{A}^2(\mathcal{R})$:

In our case, S will be a fixed finite open Riemann surface and in any step $\mathcal{R}$ the sublevel set of a strongly plurisubharmonic exhaustion function of S. The core map of the spray will be the holomorphic immersion f of $\overline{\mathcal{R}}$ into the complex manifold X of dimension n, and we construct a spray as follows: Let $\mathcal{R}'$ be a Riemann surface with $\mathcal{R} \subset\subset \mathcal{R}'$. Since the tangent bundle of $\overline{\mathcal{R}'}$ is trivial, we may choose a non-vanishing holomorphic vector field V on $\overline{\mathcal{R}'}$, and we let φ_t denote its flow. Define a map $\tilde{f}$: $\overline{\mathcal{R}} \times \mathbb{D}^{n-1} \times \delta \cdot \mathbb{D} \to \mathcal{R}' \times \mathbb{D}^{n-1}$ by $(z, t_1, t_2) \mapsto (\varphi_{t_2}(z), t_1)$. Choose an immersion $F : \overline{\mathcal{R}'} \times \mathbb{D}^{n-1} \to X$ according to Corollary 3.5, and define $f := F \circ \tilde{f}$.

Proof of Theorem 5.1. —

1. Let $\mathcal{R}$ be an open connected Riemann surface. Since $\mathcal{R}$ is a Stein manifold we have that $\mathcal{R}$ admits a $\mathcal{C}^2$ strictly subharmonic exhaustion function $\rho : \mathcal{R} \to \mathbb{R}^+$. Since strict subharmonicity is stable under small $\mathcal{C}^2$-perturbations we may assume that ρ is a Morse function, meaning that all critical points of ρ are non-degenerate, and if ξ and ξ' are two critical points of ρ then $\rho(\xi) \neq \rho(\xi')$. By Lemma 2.5 in [23] we may further assume that any critical point ξ is either a local minimum, or there exist local coordinates $z = x + iy : U_\xi \to \mathbb{C}$ such that ρ is of the form
$$\rho(z) = \rho(\xi) + x^2 - \mu \cdot y^2,$$
for some $\mu \in (0, 1)$, i.e. that ρ has only nice singularities. In the following we denote by $\{\xi_k\}_{k \in I \subseteq \mathbb{N}}$ the critical points of ρ, and by $c_k := \rho(\xi_k)$ the corresponding critical values, where I is either $\mathbb{N}$ or a $I = [1, \ldots k_{\max}]$ for some $k_{\max} \in \mathbb{N}$. If there is only a finite number of critical points, we define inductively $c_{k+1} := c_k + 1$ for $k \geqslant k_{\max}$.

Let τ denote a strictly plurisubharmonic exhaustion function of the Stein manifold X. Choose a sequence of real $\varepsilon_k > 0$ such that $\sum_{k=1}^{\infty} \varepsilon_k < 1$. By

$$\mathcal{R}_\gamma := \{z \in \mathcal{R} \,:\, \rho(z) < \gamma\}, \quad \gamma \in \mathbb{R}$$

we denote the γ-sublevel set of ρ.

Let $\{K_j\}_{j\in\mathbb{N}}$ denote an exhaustion of the Stein manifold X with holomorphically convex compacts. Note that by a remark of Serre [29, p. 59], given as proposition with proof by Gilligan and Huckleberry [21, p. 186], the complements $X \setminus K_j$ are always connected, provided the Stein manifold X is at least of dimension 2. Therefore this assertion is always true when applying Proposition 4.2 in the following.

2. For each $k \geqslant 2$ we do the following: if ξ_k is a local minimum we put $\delta_k := \frac{1}{2}\min\{c_k - c_{k-1}, c_{k+1} - c_k\}$, and otherwise we choose a small δ_k according to Proposition 4.2. such that $\overline{\mathcal{R}_{c_k+\delta_k}} \setminus \mathcal{R}_{c_k-\delta_k}$ contains no other critical point of ρ than ξ_k. In the case of finitely many critical points we put $\delta_k = 0$ for $k > k_{\max}$.
3. Choose an initial embedding $f_2 : \overline{\mathcal{R}_{c_2-\delta_2}} \to X$. This is trivial since $\mathcal{R}_{c_2-\delta_2}$ is a disk. We will now describe an inductive procedure how to construct immersions (resp. embeddings) f_k of $\overline{\mathcal{R}_{c_k-\delta_k}}$ into X. Assume that we have constructed immersions (resp. embeddings) $f_k : \overline{\mathcal{R}_{c_k-\delta_k}} \to X$ and real numbers r_k for $k = 2, ..., N$, $r_k \geqslant r_{k-1} + 1$, and assume that $f_k(\overline{\mathcal{R}_{c_k-\delta_k}} \setminus \mathcal{R}_{c_{k-1}}) \subset X \setminus K_{r_k}$, $\|f_k - f_{k-1}\|_{\overline{\mathcal{R}_{c_{k-1}-\delta_{k-1}}}} < \epsilon_{k-1}$. We will now describe the inductive step how to construct f_{N+1}.
   (a) Choose $r_{N+1} \geqslant r_N + 1$ such that $X \setminus K_{r_{N+1}}$ is connected. We may assume that $f_N(b\mathcal{R}_{c_N-\delta_N}) \subset X \setminus K_{r_{N+1}}$: since f_N lives on a neighborhood of $\overline{\mathcal{R}_{c_N-\delta_N}}$, we may thicken f_N as described above, and the boundary may be pushed away using Lemma 5.3.
   (b) In the case of finitely many critical points, if $N > k_{\max}$, we may reach the next level set by attaching finitely many annuli to $\mathcal{R}_{c_N}$, hence the approximation is furnished by Proposition 3.1.
   (c) If ξ_N is a local minimum, start by extending the immersion (resp. embedding) to the component of $\overline{\mathcal{R}_{c_N+\delta_N}}$ that contains ξ_N; this is trivial because this component is a disk. Make sure that the image lies in $X \setminus K_{r_{N+1}}$. Since we may now reach

$\mathcal{R}_{c_{N+1}-\delta_{N+1}}$ by attaching a finite number of annuli, the approximation f_{N+1} is furnished by Proposition 3.1. If $\dim(X) \geqslant 3$ the separation of points is not a problem.

(d) If ξ_N is not a local minimum, we may choose an initial approximation $\tilde{f}_{N+1} : \overline{\mathcal{R}_{c_N+\delta_N}} \to X$ furnished by Proposition 4.2. Now $\mathcal{R}_{c_{N+1}-\delta_{N+1}}$ may be reached by attaching a finite number of annuli, and the approximation f_{N+1} is furnished by Proposition 3.1.

(e) It is now clear that the limit $f := \lim_{j\to\infty} f_j$ is well defined on $\mathcal{R}$, and gives us the desired immersion (resp. embedding) into X. For the embedding, the sequence ϵ_j should be modified along the way to avoid self intersections in the limit. □

The main result in [6] by the first author characterizes certain Stein manifolds by their endomorphism semigroup and gives an application of our theorem using a properly embedded complex line in a Stein manifold:

Theorem 5.4. — *Let X and Y be complex manifolds and $\Phi : \mathrm{End}(X) \to \mathrm{End}(Y)$ an isomorphism of semigroups of holomorphic endomorphisms. Then there exists a unique $\varphi : X \to Y$ which is is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ if the following criteria are fulfilled:*

(1) *X is a Stein manifold, and*
(2) *X admits a proper holomorphic embedding $i : \mathbb{C} \hookrightarrow X$.*

If the automorphism group of X acts (weakly) double-transitive, it is sufficient for Φ to be an epimorphism.

From Theorem 5.1 and the preceeding result and noting that a Stein manifold with the (volume) density property has a double-transitive action by Propostion 2.5 resp. its Corollary 2.6, we immediately get the following result:

Theorem 5.5. — *Let X and Y be complex manifolds and $\Phi : \mathrm{End}(X) \to \mathrm{End}(Y)$ an epimorphism of semigroups of holomorphic endomorphisms. If X is a Stein manifold with the density- or volume density property and of dimension at least 3, then there exists a unique $\varphi : X \to Y$ which is is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$.*

A conjecture by Schoen and Yau [27] claimed that no proper harmonic map could exist from the unit disk onto $\mathbb{R}^2$. The conjecture was first disproved by Forstnerič and Globevnik [19] in 2001 and again more recently disproved by Alarcón and Galvéz [1], but a much stronger result follows easily from our main theorem:

Theorem 5.6. — *Every open Riemann surface admits a proper harmonic map into* $\mathbb{R}^2$.

Proof. — The Stein manifold $\mathbb{C}^* \times \mathbb{C}^*$ has the volume density property (with standard volume form $\frac{dz}{z} \wedge \frac{dw}{w}$), see [32]. According to Theorem 5.1 there exists a proper holomorphic immersion $(f_1, f_2) : \mathcal{R} \to \mathbb{C}^* \times \mathbb{C}^*$. The map $(\log|f_1|, \log|f_2|) : \mathcal{R} \to \mathbb{R}^2$ is harmonic and still proper. □

Theorem 5.6 was also recently obtained with different methods by Alarcón and López [2].

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