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## KRONECKER'S SOLUTION OF PELL'S EQUATION FOR CM FIELDS

by Riad MASRI

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ABSTRACT. — We generalize Kronecker's solution of Pell's equation to CM fields  $K$  whose Galois group over  $\mathbb{Q}$  is an elementary abelian 2-group. This is an identity which relates CM values of a certain Hilbert modular function to products of logarithms of fundamental units. When  $K$  is imaginary quadratic, these CM values are algebraic numbers related to elliptic units in the Hilbert class field of  $K$ . Assuming Schanuel's conjecture, we show that when  $K$  has degree greater than 2 over  $\mathbb{Q}$  these CM values are transcendental.

RÉSUMÉ. — Nous généralisons la solution de Kronecker des équations Pell aux corps  $K$  CM dont le groupe de Galois sur  $\mathbb{Q}$  est un 2-groupe abélien élémentaire. Il s'agit d'une formule qui relie les valeurs CM d'une certaine fonction modulaire de Hilbert aux produits de logarithmes des unités fondamentales. Lorsque  $K$  est quadratique imaginaire, ces valeurs CM sont des nombres algébriques reliés aux unités elliptiques des corps de classes de Hilbert de  $K$ . Sous l'hypothèse que la conjecture de Schanuel soit vraie, nous montrons que, lorsque  $K$  est de degré plus grand que 2 sur  $\mathbb{Q}$ , ces valeurs CM sont transcendentes.

### 1. Introduction and statement of results

The analytic construction of solutions of certain natural Diophantine equations is a problem of central importance in number theory. One of the most remarkable examples of this is Kronecker's "solution" of Pell's equation

$$(1.1) \quad x^2 - dy^2 = \pm 1.$$

The fundamental unit  $\varepsilon_d$  in the real quadratic field  $\mathbb{Q}(\sqrt{d})$  satisfies (1.1). Kronecker expressed  $\varepsilon_d$  in terms of values of the Dedekind eta function  $\eta(z)$  at CM points on the modular curve  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  (see the discussion below, and in particular, equation (1.5)).

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*Math. classification:* 11F41.

In this paper we will generalize Kronecker's solution of Pell's equation to CM fields  $K$  whose Galois group over  $\mathbb{Q}$  is an elementary abelian 2-group (see Theorem 1.3). This is an identity which relates values of a certain Hilbert modular function at CM points on a Hilbert modular variety to products of logarithms of fundamental units. When  $K$  is imaginary quadratic, these CM values are algebraic numbers which can be expressed as absolute values of Galois conjugates of elliptic units in the Hilbert class field of  $K$  (see [8, p. 103]). In contrast, when  $K$  has degree greater than 2 over  $\mathbb{Q}$  we will show, assuming Schanuel's conjecture, that these CM values are *transcendental* (see Theorem 1.6). This result is related to interesting recent work of Murty and Murty [6, 7] on transcendental values of class group  $L$ -functions for imaginary quadratic fields.

We begin by reviewing Kronecker's solution of Pell's equation. For a quadratic field  $\mathbb{Q}(\sqrt{\Delta})$  of discriminant  $\Delta$ , let  $\chi_\Delta$  be the Kronecker symbol,  $L(\chi_\Delta, s)$  be the Dirichlet  $L$ -function,  $h(\Delta)$  be the class number,  $\varepsilon_\Delta$  be the fundamental unit, and  $w_\Delta$  be the number of roots of unity. Let  $K = \mathbb{Q}(\sqrt{D})$  be an imaginary quadratic field of discriminant  $D < -4$  (so  $w_D = 2$ ). For an ideal class  $C$  of  $K$ , let  $\tau_{\mathfrak{a}} \in \mathbb{H}$  be the CM point of discriminant  $D$  on the modular curve  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  corresponding to  $[\mathfrak{a}] = C^{-1}$  (here  $\mathbb{H}$  is the complex upper half-plane). More precisely, if  $Q(X, Y) = N(\mathfrak{a})X^2 + bXY + cY^2$  is the reduced, primitive, integral binary quadratic form of discriminant  $b^2 - 4N(\mathfrak{a})c = D$  corresponding to the class  $C^{-1}$ , then

$$\tau_{\mathfrak{a}} = \frac{-b + \sqrt{D}}{2N(\mathfrak{a})}$$

is the unique root in  $\mathbb{H}$  of the dehomogenized form  $Q(X, 1)$  (here  $N(\mathfrak{a})$  is the norm of  $\mathfrak{a}$ ). Kronecker established the following "limit formula" for the constant term in the Laurent expansion of the partial Dedekind zeta function  $\zeta_K(s, C)$  at  $s = 1$ ,

$$(1.2) \quad \lim_{s \rightarrow 1} \left[ \zeta_K(s, C) - \frac{\pi}{\sqrt{|D|}} \frac{1}{s-1} \right] = \frac{\pi}{\sqrt{|D|}} (2\gamma - \log |D| - 2 \log g(\tau_{\mathfrak{a}})),$$

where  $\gamma$  is Euler's constant and  $g: \mathbb{H} \rightarrow \mathbb{R}^+$  is the  $\mathrm{SL}_2(\mathbb{Z})$ -invariant function

$$g(z) := \sqrt{(2/\sqrt{|D|}) \mathrm{Im}(z) |\eta(z)|^2},$$

where

$$\eta(z) = e(z/24) \prod_{n=1}^{\infty} (1 - e(nz)), \quad e(z) := e^{2\pi iz}$$

is Dedekind's weight  $1/2$  modular form for  $\mathrm{SL}_2(\mathbb{Z})$ .

Let  $D = D_1D_2$  be a nontrivial factorization of  $D$  into coprime fundamental discriminants  $D_1 > 0$  and  $D_2 < 0$ . Let  $\chi$  be the genus character of  $K$  corresponding to the decomposition  $D = D_1D_2$  and let

$$L_K(\chi, s) = \sum_{C \in \text{CL}(K)} \chi(C)\zeta_K(s, C)$$

be the  $L$ -function of  $\chi$  where  $\text{CL}(K)$  is the ideal class group of  $K$ . Kronecker established the factorization

$$(1.3) \quad L_K(\chi, s) = L(\chi_{D_1}, s)L(\chi_{D_2}, s).$$

By orthogonality of group characters, one obtains from (1.2) the formula

$$L_K(\chi, 1) = -\frac{2\pi}{\sqrt{|D|}} \sum_{C \in \text{CL}(K)} \chi(C) \log g(\tau_a).$$

On the other hand, by Dirichlet's class number formula for quadratic fields one has

$$(1.4) \quad L(\chi_\Delta, 1) = \begin{cases} \frac{2 \log(\varepsilon_\Delta)h(\Delta)}{\sqrt{\Delta}}, & \text{if } \Delta > 0, \\ \frac{2\pi h(\Delta)}{w_\Delta \sqrt{|\Delta|}}, & \text{if } \Delta < 0. \end{cases}$$

Equating both sides of Kronecker's factorization (1.3) at  $s = 1$  yields the beautiful identity

$$-\sum_{C \in \text{CL}(K)} \chi(C) \log g(\tau_a) = \frac{2h(D_1)h(D_2)}{w_{D_2}} \log(\varepsilon_{D_1}),$$

or equivalently

$$(1.5) \quad \prod_{C \in \text{CL}(K)} g(\tau_a)^{-\chi(C)} = \varepsilon_{D_1}^{2h(D_1)h(D_2)/w_{D_2}}.$$

The fundamental unit  $\varepsilon_{D_1}$  satisfies Pell's equation

$$x^2 - D_1y^2 = \pm 1,$$

thus one has a "solution" of this equation in terms of the CM values  $g(\tau_a)$ .

Recall that a *CM field* is a totally imaginary quadratic extension of a totally real number field. In order to generalize Kronecker's identity (1.5) to CM fields we proceed as follows. First, we evaluate the special value  $L_K(\chi, 1)$  where  $\chi$  is a nontrivial class group character of a CM field  $K$  (see Theorem 1.1). To do this we establish a suitable version of the Kronecker limit formula for CM fields, which relates the constant term in the Laurent expansion at  $s = 1$  of  $\zeta_K(s, C)$  to values of a Hilbert modular function at CM points on a Hilbert modular variety (see Theorem 4.1). Second, we

identify the CM fields which possess a genus character  $\chi$  whose  $L$ -function  $L_K(\chi, s)$  factors as a product of quadratic Dirichlet  $L$ -functions. These are the CM fields whose Galois group over  $\mathbb{Q}$  is an elementary abelian 2-group. Given such a factorization, we can evaluate  $L_K(\chi, 1)$  using Dirichlet's class number formula for quadratic fields. By equating the two different evaluations of  $L_K(\chi, 1)$  we will generalize (1.5).

Note that a limit formula for CM fields was established by Konno in [5]. See also the work of Asai [1], who calculated the constant term in the Laurent expansion at  $s = 1$  of the real-analytic Eisenstein series associated to any number field of class number 1. Our approach to the limit formula for CM fields differs from [5]. In particular, we proceed via the Fourier expansion of the Hilbert modular Eisenstein series, which enables us to use periods of this Eisenstein series to explicitly determine the CM zero-cycles along which we evaluate the Hilbert modular function.

In order to state our results we fix the following notation. Let  $F$  be a totally real number field of degree  $n$  over  $\mathbb{Q}$  with embeddings  $\sigma_1, \dots, \sigma_n$  and ring of integers  $\mathcal{O}_F$ . Let  $K$  be a CM extension of  $F$  with a CM type  $\Phi$ , and let

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} \in \mathbb{H}^n : [\mathfrak{a}] \in \text{CL}(K)\}$$

be the zero-cycle of CM points on the Hilbert modular variety  $X_F = \text{SL}_2(\mathcal{O}_F) \backslash \mathbb{H}^n$  (see Section 3). Let  $R_K$ ,  $w_K$  and  $d_K$  be the regulator, number of roots of unity, and absolute discriminant of  $K$ , respectively.

In the following theorem we give a formula for the special value  $L_K(\chi, 1)$ .

**THEOREM 1.1.** — *Let  $F$  be a totally real number field of degree  $n$  over  $\mathbb{Q}$  with narrow class number 1. Let  $K$  be a CM extension of  $F$  with a CM type  $\Phi$ . For each class  $C \in \text{CL}(K)$ , let  $z_{\mathfrak{a}}$  be the CM point in  $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$  corresponding to  $C^{-1}$ . Then for each nontrivial class group character  $\chi$  of  $K$ ,*

$$L_K(\chi, 1) = -\frac{2^{n+1}\pi^n R_K}{w_K \sqrt{d_K}} \sum_{C \in \text{CL}(K)} \chi(C) \log G(z_{\mathfrak{a}}),$$

where  $G: \mathbb{H}^n \rightarrow \mathbb{R}^+$  is the  $\text{SL}_2(\mathcal{O}_F)$ -invariant function

$$G(z) := \sqrt{\left(2^n d_F / \sqrt{d_K}\right) \prod_{i=1}^n \text{Im}(z_i) \cdot \phi(z)^2}, \quad z = (z_1, \dots, z_n) \in \mathbb{H}^n$$

and  $\phi(z)$  is the positive, real-analytic function generalizing  $|\eta(z)|$  defined by (1.6).

*Remark 1.2.* — The narrow class number 1 assumption in Theorem 1.1 can be removed by working adelicly. We have worked classically throughout the paper to emphasize the parallels with Kronecker's original work.

The function  $\phi(z)$  in Theorem 1.1 is defined by

$$(1.6) \quad \phi(z) := f(z)^{-\sqrt{d_F}/2\pi^n r_F},$$

where  $r_F$  is the residue of  $\zeta_F(2s - 1)$  at  $s = 1/2$  and

$$f(z) := \exp \left( \zeta_F(2) \prod_{i=1}^n y_i + \frac{\pi^n}{\sqrt{d_F}} \sum'_{\tilde{a} \in \mathcal{O}_F^*} \sum_{\substack{\tilde{a}=ab \\ a \in \mathcal{O}_F^* \\ b \in \mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(aby)}}{N_{F/\mathbb{Q}}((b))} e^{2\pi i T(afx)} \right),$$

where  $z = x + iy \in \mathbb{H}^n$ ,  $\mathcal{O}_F^*$  is the dual lattice,  $\mathcal{O}_F^\times$  is the unit group,

$$S(aby) = \sum_{i=1}^n |\sigma_i(ab)| y_i,$$

$$T(afx) = \sum_{i=1}^n \sigma_i(ab) x_i,$$

and the prime means the sum is over nonzero elements. In Proposition 4.3 we will show that  $\phi(z)$  transforms like

$$\phi(Mz) = \left| \prod_{i=1}^n (\sigma_i(\gamma) z_i + \sigma_i(\delta)) \right|^{\frac{1}{2}} \phi(z)$$

for  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathcal{O}_F)$ .

Our main result is the following theorem generalizing Kronecker's identity (1.5).

**THEOREM 1.3.** — *Let  $F$  be a totally real number field with narrow class number 1. Let  $K$  be a CM extension of  $F$  with  $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$  for some integer  $r \geq 2$ , and let  $E$  be an unramified quadratic extension of  $K$  with  $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$ . Let  $\chi$  be the genus character of  $K$  arising from the extension  $E/K$ . Let  $\Delta_i$  for  $1 \leq i \leq 2^r$  be the discriminants of the quadratic subfields  $\mathbb{Q}(\sqrt{\Delta_i})$  of  $E$  which are not contained in  $K$  and define  $S_R := \{\Delta_i : \Delta_i > 0\}$  and  $S_I := \{\Delta_i : \Delta_i < 0\}$ . Then*

$$\prod_{C \in \text{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)} = \exp \left( \frac{\alpha}{\prod_{i=1}^{2^r} \sqrt{|\Delta_i|}} \frac{\sqrt{d_K}}{R_F} \prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i}) \right),$$

where

$$\alpha := \frac{w_K \prod_{i=1}^{2^r} h(\Delta_i)}{\prod_{\Delta_i \in S_I} w_{\Delta_i}} \in \mathbb{Q}.$$

In the following theorem we give an explicit example of Theorem 1.3 for CM biquadratic fields.

**THEOREM 1.4.** — *Let  $F = \mathbb{Q}(\sqrt{p})$  where  $p \equiv 1 \pmod{4}$  is a prime such that  $F$  has narrow class number 1. Let  $D = D_1 D_2 < 0$  be a composite fundamental discriminant with  $D_1 > 0$  and  $D_2 < 0$  fundamental discriminants. Let  $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$  and  $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$ . Let  $\chi$  be the genus character of  $K$  arising from the extension  $E/K$ . Then*

$$\prod_{C \in \text{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)} = \exp \left( \beta \sqrt{d_K} \frac{\log(\varepsilon_{D_1}) \log(\varepsilon_{pD_1})}{\log(\varepsilon_p)} \right),$$

where

$$\beta := \frac{w_K h(D_1) h(D_2) h(pD_1) h(pD_2)}{pD_1 D_2 w_{D_2} w_{pD_2}} \in \mathbb{Q}.$$

Kronecker’s identity (1.5) implies that the product of CM values

$$\prod_{C \in \text{CL}(K)} g(\tau_{\mathfrak{a}})^{-\chi(C)}$$

is an algebraic number. This product is also related to elliptic units in the Hilbert class field  $H$  of  $K = \mathbb{Q}(\sqrt{D})$ . Namely, using quotients of powers of  $\eta(\tau_{\mathfrak{a}})$  and the theory of complex multiplication, one can construct a sequence  $\zeta_{\ell}$ ,  $\ell = 1, \dots, h(D) - 1$ , of independent units in  $H$  (see [8, p. 103]). If  $\sigma_k$  is the automorphism of  $H/K$  corresponding to the ideal class  $C_k$  under the isomorphism

$$\text{Gal}(H/K) \rightarrow \text{CL}(K),$$

one can show that

$$\frac{g(\tau_{\mathfrak{a}_k})}{g(\tau_{\mathfrak{a}_k \mathfrak{a}_{\ell}^{-1}})} = |\zeta_{\ell}^{(k)}|^{1/12h(D)}, \quad k, \ell = 1, \dots, h(D) - 1,$$

where  $\zeta_{\ell}^{(k)} := \sigma_k(\zeta_{\ell})$ . In particular, the quotients  $g(\tau_{\mathfrak{a}_k})/g(\tau_{\mathfrak{a}_k \mathfrak{a}_{\ell}^{-1}})$  are algebraic.

More generally, let  $H_K$  be the Hilbert class field of a CM field  $K$  as in Theorem 1.1 and let  $h_K$  be the class number of  $K$ . In light of the preceding facts, it is natural to ask whether the products of CM values

$$\prod_{C \in \text{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)}$$

are algebraic, and if so, whether they are related to analogs of elliptic units in  $H_K$ . We will show, assuming Schanuel's conjecture, that these products are *transcendental*.

Recall the following well-known conjecture of Schanuel from transcendental number theory (see e.g. [9, Conjecture 1.14]).

CONJECTURE 1.5 (Schanuel). — *Given complex numbers  $x_1, \dots, x_n$  that are linearly independent over  $\mathbb{Q}$ , the field*

$$\overline{\mathbb{Q}}(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n))$$

*has transcendence degree at least  $n$  over  $\overline{\mathbb{Q}}$ .*

We will prove the following theorem.

THEOREM 1.6. — *Let notation and assumptions be as in Theorem 1.3. Then assuming Schanuel's conjecture, the numbers*

$$\prod_{C \in \text{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)}$$

*are transcendental.*

Theorem 1.6 indicates that one cannot in general expect the quotients

$$\frac{G(z_{\mathfrak{a}_k})}{G(z_{\mathfrak{a}_k \mathfrak{a}_\ell^{-1}})}, \quad k, \ell = 1, \dots, h_K - 1,$$

to be related to analogs of elliptic units in  $H_K$ . For example, if we assume in Theorem 1.6 that  $K$  has class number 2, then Schanuel's conjecture implies that the quotients  $G(z_{\mathfrak{a}})/G(z_{\mathcal{O}_K})$  are transcendental. Note that there are more than 150 CM biquadratic fields with class number 2 (see [3]).

*Organization.* — The paper is organized as follows. In Section 2 we calculate the Laurent expansion at  $s = 1$  of the Hilbert modular Eisenstein series. In Section 3 we review some facts regarding CM zero-cycles on Hilbert modular varieties. Finally, in Sections 4, 5, 6, and 7, we prove Theorems 1.1, 1.3, 1.4, and 1.6, respectively.

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## 2. Laurent expansion of the Hilbert modular Eisenstein series

Let  $F$  be a totally real number field with class number 1. Let  $F$  have degree  $n$  over  $\mathbb{Q}$  with embeddings  $\sigma_1, \dots, \sigma_n$  and let

$$z = x + iy = (z_1, \dots, z_n) \in \mathbb{H}^n.$$

Let  $\mathcal{O}_F$  be the ring of integers of  $F$  and  $\mathrm{SL}_2(\mathcal{O}_F)$  be the Hilbert modular group. Then  $\mathrm{SL}_2(\mathcal{O}_F)$  acts componentwise on  $\mathbb{H}^n$  by linear fractional transformations,

$$Mz = (\sigma_1(M)z_1, \dots, \sigma_n(M)z_n), \quad M \in \mathrm{SL}_2(\mathcal{O}_F).$$

Let

$$N(y(z)) = \prod_{j=1}^n \mathrm{Im}(z_j) = \prod_{j=1}^n y_j$$

denote the product of the imaginary parts of the components of  $z \in \mathbb{H}^n$ . Define the real-analytic Hilbert modular Eisenstein series

$$\mathcal{E}(z, s) := \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathcal{O}_F)} N(y(Mz))^s, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1,$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \right\}.$$

Furthermore, let

$$N(a + bz) = \prod_{j=1}^n (\sigma_j(a) + \sigma_j(b)z_j)$$

for  $(a, b) \in \mathcal{O}_F \times \mathcal{O}_F$  and define the Eisenstein series

$$E(z, s) := \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F / \mathcal{O}_F^\times} \frac{N(y(z))^s}{|N(a + bz)|^{2s}}, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1,$$

where the sum is over a complete set of nonzero, nonassociated representatives of  $\mathcal{O}_F \times \mathcal{O}_F$  (recall that  $(a, b)$  and  $(a', b')$  are associated if there exists a unit  $\epsilon \in \mathcal{O}_F^\times$  such that  $(a, b) = (\epsilon a', \epsilon b')$ ). The two Eisenstein series are related by

$$(2.1) \quad E(z, s) = \zeta_F(2s)\mathcal{E}(z, s),$$

where  $\zeta_F(s)$  is the Dedekind zeta function of  $F$ .

The Eisenstein series  $E(z, s)$  has the Fourier expansion

$$\begin{aligned}
 (2.2) \quad E(z, s) &= N(y(z))^s \zeta_F(2s) + \frac{N(y(z))^{1-s}}{\sqrt{d_F}} \left[ \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \right]^n \zeta_F(2s - 1) \\
 &+ \frac{2^n N(y(z))^{\frac{1}{2}}}{\sqrt{d_F}} \left[ \frac{\pi^s}{\Gamma(s)} \right]^n \sum'_{\tilde{a} \in \mathcal{O}_F^*} \sum_{\substack{\tilde{a}=ab \\ a \in \mathcal{O}_F^* \\ b \in \mathcal{O}_F / \mathcal{O}_F^\times}} \times \\
 &\quad \left( \frac{N_{F/\mathbb{Q}}(a)}{N_{F/\mathbb{Q}}(b)} \right)^{s-\frac{1}{2}} e^{2\pi i T(ax)} \prod_{j=1}^n K_{s-\frac{1}{2}}(2\pi |\sigma_j(ab)| y_j) \\
 &=: A(s) + B(s) + C(s),
 \end{aligned}$$

where  $\mathcal{O}_F^*$  is the dual lattice,  $d_F$  is the absolute discriminant,  $T(ax) = \sum_{j=1}^n \sigma_j(a)x_j$  is the trace,  $K_s(v)$  is the usual  $K$ -Bessel function of order  $s$ , and  $A(s), B(s), C(s)$  are the three functions on the right hand side of (2.2), respectively.

The Fourier expansion provides a meromorphic continuation of  $E(z, s)$  to  $\mathbb{C}$  with a simple pole at  $s = 1$ . We now use this to compute the Laurent expansion at  $s = 1$ .

The Laurent expansion of  $A(s)$  at  $s = 1$  is

$$A(s) = N(y(z))\zeta_F(2) + O(s - 1).$$

Next, observe that

$$\begin{aligned}
 \frac{N(y(z))^{1-s}}{\sqrt{d_F}} &= \frac{1}{\sqrt{d_F}} - \frac{\log N(y(z))}{\sqrt{d_F}}(s - 1) + O(s - 1)^2, \\
 \left[ \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \right]^n &= \pi^n - 2n\pi^n \log(2)(s - 1) + O(s - 1)^2,
 \end{aligned}$$

and

$$\zeta_F(2s - 1) = \frac{r_F}{2(s - 1)} + A_F + O(s - 1).$$

After a calculation, we find that the Laurent expansion of  $B(s)$  at  $s = 1$  is

$$\begin{aligned}
 B(s) &= \frac{\pi^n r_F}{2\sqrt{d_F}} \frac{1}{(s - 1)} + \frac{\pi^n}{\sqrt{d_F}} A_F \\
 &\quad - \frac{\pi^n r_F}{2\sqrt{d_F}} [\log N(y(z)) + 2n \log(2)] + O(s - 1).
 \end{aligned}$$

Using

$$K_{1/2}(v) = \sqrt{\pi/2}ve^{-v}$$

we compute

$$\prod_{j=1}^n K_{1/2}(2\pi |\sigma_j(ab)| y_j) = \frac{N(y(z))^{-1/2}}{2^n} N_{F/\mathbb{Q}}((ab))^{-1/2} e^{-2\pi S(ab)},$$

where

$$S(ab) = \sum_{j=1}^n |\sigma_j(ab)| y_j.$$

Thus the Laurent expansion of  $C(s)$  at  $s = 1$  is

$$C(s) = \frac{\pi^n}{\sqrt{d_F}} \sum'_{\tilde{a} \in \mathcal{O}_F^*} \sum_{\substack{\tilde{a}=ab \\ a \in \mathcal{O}_F^* \\ b \in \mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(ab)}}{N_{F/\mathbb{Q}}((b))} e^{2\pi i T(abx)} + O(s - 1).$$

Putting things together, we find that the Laurent expansion of  $E(z, s)$  at  $s = 1$  is

$$(2.3) \quad E(z, s) = \frac{E_{-1}}{s - 1} + E_0(z) + O(s - 1),$$

where the residue

$$E_{-1} = \frac{\pi^n r_F}{2\sqrt{d_F}},$$

and

$$(2.4) \quad E_0(z) = \frac{\pi^n}{\sqrt{d_F}} A_F - E_{-1} 2n \log(2) + \log(N(y(z))^{-E_{-1}} f(z)),$$

where

$$\log f(z) = N(y(z))\zeta_F(2) + \frac{\pi^n}{\sqrt{d_F}} \sum'_{\tilde{a} \in \mathcal{O}_F^*} \sum_{\substack{\tilde{a}=ab \\ a \in \mathcal{O}_F^* \\ b \in \mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(ab)}}{N_{F/\mathbb{Q}}((b))} e^{2\pi i T(abx)}.$$

### 3. CM zero-cycles on Hilbert modular varieties

In this section we review some facts we will need regarding CM zero-cycles on Hilbert modular varieties following Bruinier and Yang [2, Section 3]. See also the recent book of Howard and Yang [4]. Let  $F$  be a totally real number field of degree  $n$  over  $\mathbb{Q}$ . For  $S \subset F$ , let  $S^+$  be the

subset of  $S$  consisting of totally positive elements. For a fractional ideal  $\mathfrak{f}_0$  of  $F$ , let

$$\Gamma(\mathfrak{f}_0) = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{f}_0) \\ = \{M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(F) : \alpha, \delta \in \mathcal{O}_F, \beta \in \mathfrak{f}_0, \gamma \in \mathfrak{f}_0^{-1}\}.$$

Recall that  $\Gamma(\mathfrak{f}_0)$  acts on  $\mathbb{H}^n$  by

$$Mz = (\sigma_1(M)z_1, \dots, \sigma_n(M)z_n).$$

The quotient space

$$X(\mathfrak{f}_0) = \Gamma(\mathfrak{f}_0) \backslash \mathbb{H}^n$$

is the (open) Hilbert modular variety associated to  $\mathfrak{f}_0$ . The variety  $X(\mathfrak{f}_0)$  parameterizes isomorphism classes of triples  $(A, i, m)$  where  $(A, i)$  is an abelian variety with real multiplication  $i: \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$  and

$$m: (\mathfrak{M}_A, \mathfrak{M}_A^+) \rightarrow ((\partial_F \mathfrak{f}_0)^{-1}, (\partial_F \mathfrak{f}_0)^{-1,+})$$

is an  $\mathcal{O}_F$ -isomorphism from  $\mathfrak{M}_A$  to  $(\partial_F \mathfrak{f}_0)^{-1}$  which maps  $\mathfrak{M}_A^+$  to  $(\partial_F \mathfrak{f}_0)^{-1,+}$ . Here  $\mathfrak{M}_A$  is the polarization module of  $A$  and  $\mathfrak{M}_A^+$  is its positive cone.

Let  $K$  be a CM extension of  $F$  and  $\Phi = (\sigma_1, \dots, \sigma_n)$  be a CM type of  $K$ . A point  $z = (A, i, m) \in X(\mathfrak{f}_0)$  is a CM point of type  $(K, \Phi)$  if one of the following equivalent definitions holds:

- (1) As a point  $z \in \mathbb{H}^n$ , there is a point  $\tau \in K$  such that

$$\Phi(\tau) = (\sigma_1(\tau), \dots, \sigma_n(\tau)) = z$$

and

$$\Lambda_\tau = \mathfrak{f}_0 + \mathcal{O}_F \tau$$

is a fractional ideal of  $K$ .

- (2)  $(A, i')$  is a CM abelian variety of type  $(K, \Phi)$  with complex multiplication  $i': \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$  such that  $i = i'|_{\mathcal{O}_F}$ .

Fix  $\varepsilon_0 \in K^\times$  such that  $\bar{\varepsilon}_0 = -\varepsilon_0$  and  $\Phi(\varepsilon_0) = (\sigma_1(\varepsilon_0), \dots, \sigma_n(\varepsilon_0)) \in \mathbb{H}^n$ . Let  $\mathfrak{a}$  be a fractional ideal of  $K$  and  $\mathfrak{f}_\mathfrak{a} = \varepsilon_0 \partial_{K/F} \mathfrak{a} \bar{\mathfrak{a}} \cap F$ . By [2, Lemma 3.1], the CM abelian variety  $(A_\mathfrak{a} = \mathbb{C}^n / \Phi(\mathfrak{a}), i)$  defines a CM point on  $X(\mathfrak{f}_0)$  if there exists an  $r \in F^\times$  such that  $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$ . Thus any pair  $(\mathfrak{a}, r)$  with  $\mathfrak{a}$  a fractional ideal of  $K$  and  $r \in F^\times$  with  $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$  defines a CM point  $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$  (we refer the reader to [2] for a discussion of how the  $\mathcal{O}_F$ -isomorphism  $m$  depends on  $r$ ). Two such pairs  $(\mathfrak{a}_1, r_1)$  and  $(\mathfrak{a}_2, r_2)$  are equivalent if there exists an  $\alpha \in K^\times$  such that  $\mathfrak{a}_2 = \alpha \mathfrak{a}_1$  and  $r_2 = r_1 \alpha \bar{\alpha}$ . Write  $[\mathfrak{a}, r]$  for the class of  $(\mathfrak{a}, r)$  and identify it with its associated CM point  $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$ .

By [2, Lemma 3.2], given a CM point  $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$  there is a decomposition

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathfrak{f}_0 \beta$$

with  $z = \alpha/\beta \in K^\times \cap \mathbb{H}^n = \{z \in K^\times : \Phi(z) \in \mathbb{H}^n\}$ . Moreover,  $z$  represents the CM point  $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$ .

Let  $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$  be the set of CM points  $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$ , which we view as a CM zero-cycle in  $X(\mathfrak{f}_0)$ . Let

$$\mathcal{CM}(K, \Phi) = \sum_{[\mathfrak{f}_0] \in \text{CL}(F)^+} \mathcal{CM}(K, \Phi, \mathfrak{f}_0),$$

where  $\text{CL}(F)^+$  is the narrow ideal class group of  $F$ . The forgetful map

$$\begin{aligned} \mathcal{CM}(K, \Phi) &\rightarrow \text{CL}(K), \\ [\mathfrak{a}, r] &\mapsto [\mathfrak{a}] \end{aligned}$$

is surjective. Each fiber is indexed by  $\epsilon \in \mathcal{O}_F^{\times,+} / N_{K/F} \mathcal{O}_K^\times$ . Here  $\#(\mathcal{O}_F^{\times,+} / N_{K/F} \mathcal{O}_K^\times)$  equals 1 or 2; in particular, it equals 1 if  $\epsilon \in N_{K/F} \mathcal{O}_K^\times$ .

Assume now that  $F$  has narrow class number 1. Then

$$\mathcal{CM}(K, \Phi) = \mathcal{CM}(K, \Phi, \mathcal{O}_F),$$

and the forgetful map

$$\mathcal{CM}(K, \Phi) \rightarrow \text{CL}(K)$$

is injective (hence bijective) since  $N_{K/F} \mathcal{O}_K^\times = \mathcal{O}_F^\times$ . We will repeatedly use this bijection to identify the zero-cycle of CM points  $\mathcal{CM}(K, \Phi, \mathcal{O}_F) \subset X_F := X(\mathcal{O}_F)$  with the set

$$\{z_{\mathfrak{a}} \in K^\times \cap \mathbb{H}^n : [\mathfrak{a}] \in \text{CL}(K)\},$$

where  $z_{\mathfrak{a}}$  represents  $[\mathfrak{a}, r] \in X_F$  as above. The reader should keep in mind that the latter set depends on  $\Phi$ .

### 4. Proof of Theorem 1.1

We first establish the following version of the Kronecker limit formula for CM fields.

**THEOREM 4.1.** — *Let  $F$  be a totally real number field of degree  $n$  over  $\mathbb{Q}$  with narrow class number 1. Let  $K$  be a CM extension of  $F$  with a CM*

type  $\Phi$ . For each class  $C \in \text{CL}(K)$ , let  $z_{\mathfrak{a}}$  be the CM point in  $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$  corresponding to  $C^{-1}$ . Then we have

$$\begin{aligned} \lim_{s \rightarrow 1} \left[ \zeta_K(s, C) - \frac{(2\pi)^n R_K}{w_K \sqrt{d_K}} \frac{1}{s-1} \right] \\ = \frac{(2\pi)^n R_K}{w_K \sqrt{d_K}} \left( \frac{\pi^n A_F}{E_{-1} \sqrt{d_F}} + 2 \log(d_F) - \log(d_K) - 2 \log G(z_{\mathfrak{a}}) \right), \end{aligned}$$

where

$$(4.1) \quad G(z) := \sqrt{\left(2^n d_F / \sqrt{d_K}\right) N(y(z)) \cdot \phi(z)^2}$$

and

$$\phi(z) := f(z)^{-1/4E-1}.$$

*Proof.* — Fix a CM type  $\Phi$  for  $K$ . Let  $C \in \text{CL}(K)$ , and fix an integral ideal  $\mathfrak{a} \in C^{-1}$ . Then the partial Dedekind zeta function equals

$$\begin{aligned} \zeta_K(s, C) &= \sum'_{\mathfrak{b} \in C} N_{K/\mathbb{Q}}(\mathfrak{b})^{-s} \\ &= \sum'_{(\omega) \subset \mathfrak{a}} N_{K/\mathbb{Q}}(\mathfrak{a}^{-1}(\omega))^{-s} \\ &= N_{K/\mathbb{Q}}(\mathfrak{a})^s \sum'_{\omega \in \mathfrak{a}/\mathcal{O}_K^\times} N_{K/\mathbb{Q}}((\omega))^{-s}. \end{aligned}$$

Notice that

$$\sum'_{\omega \in \mathfrak{a}/\mathcal{O}_K^\times} N_{K/\mathbb{Q}}((\omega))^{-s} = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum'_{\omega \in \mathfrak{a}/\mathcal{O}_F^\times} N_{K/\mathbb{Q}}((\omega))^{-s}.$$

Thus we have

$$\zeta_K(s, C) = \frac{N_{K/\mathbb{Q}}(\mathfrak{a})^s}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum'_{\omega \in \mathfrak{a}/\mathcal{O}_F^\times} N_{K/\mathbb{Q}}((\omega))^{-s}.$$

By the facts in Section 3 there exists a decomposition

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathcal{O}_F \beta,$$

where  $z_a = \beta/\alpha \in K^\times \cap \mathbb{H}^n$  and  $z_a$  represents the CM point  $[a, r] \in X_F$  (here  $f_0 = \mathcal{O}_F$  since  $\#\text{CL}(F)^+ = 1$ ). Then

$$\begin{aligned} \sum'_{\omega \in \mathfrak{a}/\mathcal{O}_F^\times} N_{K/\mathbb{Q}}((\omega))^{-s} &= \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times} N_{K/\mathbb{Q}}((a\alpha + b\beta))^{-s} \\ &= N_{K/\mathbb{Q}}((\alpha))^{-s} \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times} N_{K/\mathbb{Q}}((a + bz_a)). \end{aligned}$$

By a calculation with the CM type  $\Phi$  we obtain

$$N_{K/\mathbb{Q}}((a + bz_a)) = |N(a + bz_a)|^2,$$

where we have identified  $z_a$  with  $\Phi(z_a) \in \mathbb{H}^n$ . Moreover, one has

$$N_{K/\mathbb{Q}}(\mathfrak{a}/(\alpha)) = N(y(z_a)) \frac{2^n d_F}{\sqrt{d_K}}.$$

By combining the preceding calculations, we obtain

$$\begin{aligned} \zeta_K(s, C) &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times} \frac{N(y(z_a))^s}{|N(a + bz_a)|^{2s}} \\ &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} E(z_a, s). \end{aligned}$$

Observe that

$$\left(\frac{2^n d_F}{\sqrt{d_K}}\right)^{s-1} = 1 + \log\left(\frac{2^n d_F}{\sqrt{d_K}}\right) (s - 1) + O(s - 1)^2.$$

Then after a calculation using the Laurent expansion

$$E(z_a, s) = \frac{E_{-1}}{s - 1} + E_0(z_a) + O(s - 1)$$

given by (2.3), we obtain the limit formula in the theorem. □

*Remark 4.2.* — If  $F = \mathbb{Q}$  in Theorem 4.1, we recover the Kronecker limit formula (1.2).

The function  $\phi(z)$  is positive and real-analytic. In the following proposition, we identify how  $\phi(z)$  transforms with respect to  $\text{SL}_2(\mathcal{O}_F)$  (see also [8, pp. 108-109]).

PROPOSITION 4.3. — For all  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathcal{O}_F)$ , we have

$$\phi(Mz) = |N(\gamma z + \delta)|^{\frac{1}{2}} \phi(z).$$

*Proof.* — From the relation (2.1) we see that  $E(z, s)$  has weight 0 with respect to  $\mathrm{SL}_2(\mathcal{O}_F)$ . Then the Laurent expansion (2.3) implies that  $E_0(Mz) = E_0(z)$ , which by (2.4) implies that

$$\log f(Mz) = \log f(z) + E_{-1} \log \left( \frac{N(\mathrm{Im}(Mz))}{N(\mathrm{Im}(z))} \right).$$

A straightforward calculation shows that

$$\frac{N(\mathrm{Im}(Mz))}{N(\mathrm{Im}(z))} = |N(\gamma z + \delta)|^{-2},$$

and thus

$$f(Mz) = |N(\gamma z + \delta)|^{-2E_{-1}} f(z).$$

The result now follows from the definition of  $\phi(z)$ . □

*Remark 4.4.* — By Proposition 4.3, the function  $G: \mathbb{H}^n \rightarrow \mathbb{R}^+$  defined by (4.1) has weight 0 with respect to  $\mathrm{SL}_2(\mathcal{O}_F)$  and thus is well-defined on CM points.

We can now deduce Theorem 1.1.

*Proof of Theorem 1.1.* — For a class group character  $\chi$  of  $K$ , let

$$L_K(\chi, s) = \sum_{C \in \mathrm{CL}(K)} \chi(C) \zeta_K(s, C)$$

be its associated  $L$ -function. By orthogonality for group characters, if  $\chi$  is nontrivial we have

$$\sum_{C \in \mathrm{CL}(K)} \chi(C) = 0.$$

The theorem now follows from Theorem 4.1. □

### 5. Proof of Theorem 1.3

Let  $K$  be a CM field with  $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$  for some integer  $r \geq 2$ , and let  $E$  be an unramified quadratic extension of  $K$  with  $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$ . Then the zeta function  $\zeta_E(s)$  (resp.  $\zeta_K(s)$ ) factors as  $\zeta(s)$  times the product of the quadratic Dirichlet  $L$ -functions associated to the quadratic subfields of  $E$  (resp.  $K$ ). Note that there are  $2^r - 1$  quadratic subfields of  $K$ ,  $2^{r+1} - 1$  quadratic subfields of  $E$ , and  $2^r$  quadratic subfields of  $E$  that are not contained in  $K$ . By class field theory, the unramified extension



$E/K$  gives rise to a real class group character  $\chi$  of  $K$  (a genus character) whose  $L$ -function factors as

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$

Then by the preceding facts we obtain the factorization

$$L_K(\chi, s) = \prod_{i=1}^{2^r} L(\chi_{\Delta_i}, s),$$

where  $\chi_{\Delta_i}$  for  $1 \leq i \leq 2^r$  are the Kronecker symbols associated to the quadratic subfields  $\mathbb{Q}(\sqrt{\Delta_i})$  of  $E$  which are not contained in  $K$ .

Divide the discriminants  $\Delta_i$  into two disjoint sets,  $S_R := \{\Delta_i : \Delta_i > 0\}$  and  $S_I := \{\Delta_i : \Delta_i < 0\}$ . Then we obtain the following formula for  $L_K(\chi, 1)$  using Dirichlet’s class number formula (1.4) for quadratic fields,

$$(5.1) \quad L_K(\chi, 1) = \frac{2^{2^r} \pi^{\#S_I} \prod_{i=1}^{2^r} h(\Delta_i) \prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i})}{\prod_{i=1}^{2^r} \sqrt{|\Delta_i|} \prod_{\Delta_i \in S_I} w_{\Delta_i}}.$$

On the other hand, by Theorem 1.1 we have

$$(5.2) \quad L_K(\chi, 1) = \frac{2^{n+1} \pi^n R_K}{w_K \sqrt{d_K}} \left( - \sum_{C \in \text{CL}(K)} \chi(C) \log G(z_a) \right).$$

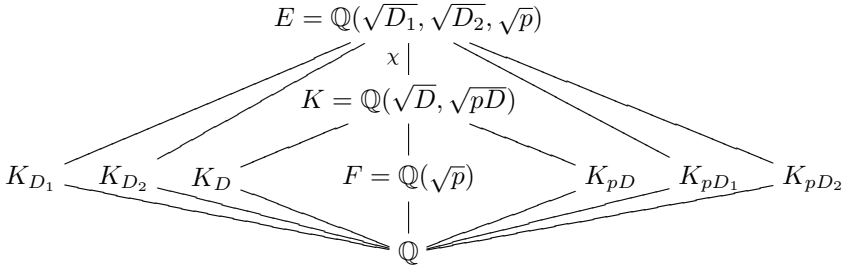
Observe that  $\#S_R = \#S_I = 2^{r-1} = [F : \mathbb{Q}] = n$ , and the regulators of  $K$  and  $F$  satisfy the relation

$$R_K = 2^{n-1} R_F$$

(see [10, p. 41]). The theorem now follows by equating (5.1) and (5.2) and simplifying the resulting expression. □

### 6. Proof Theorem 1.4

Let  $F = \mathbb{Q}(\sqrt{p})$  where  $p \equiv 1 \pmod{4}$  is a prime such that  $F$  has narrow class number 1. Let  $D = D_1 D_2 < 0$  be a composite fundamental discriminant with  $D_1 > 0$  and  $D_2 < 0$  fundamental discriminants. Let  $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$ , which is a CM biquadratic extension of  $\mathbb{Q}$  with  $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ . Let  $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$ , which is an unramified quadratic extension of  $K$  with  $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$ . Let  $\chi$  be the genus character of  $K$  arising from the extension  $E/K$ , and let  $K_\Delta$  denote  $\mathbb{Q}(\sqrt{\Delta})$  for a fundamental discriminant  $\Delta$ . Then we have the following diagram:



We have

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$

Then the factorizations

$$\begin{aligned} \zeta_E(s) &= \zeta(s)L(\chi_p, s)L(\chi_D, s)L(\chi_{pD}, s)L(\chi_{D_1}, s) \\ &\quad \times L(\chi_{D_2}, s)L(\chi_{pD_1}, s)L(\chi_{pD_2}, s) \end{aligned}$$

and

$$\zeta_K(s) = \zeta(s)L(\chi_p, s)L(\chi_D, s)L(\chi_{pD}, s)$$

yield

$$L_K(\chi, s) = L(\chi_{D_1}, s)L(\chi_{D_2}, s)L(\chi_{pD_1}, s)L(\chi_{pD_2}, s).$$

By Dirichlet's class number formula (1.4) for quadratic fields, we have

$$\begin{aligned} (6.1) \quad L_K(\chi, 1) &= \frac{2 \log(\varepsilon_{D_1})h(D_1)}{\sqrt{D_1}} \frac{2\pi h(D_2)}{w_{D_2} \sqrt{|D_2|}} \frac{2 \log(\varepsilon_{pD_1})h(pD_1)}{\sqrt{pD_1}} \\ &\quad \times \frac{2\pi h(pD_2)}{w_{pD_2} \sqrt{|pD_2|}}. \end{aligned}$$

On the other hand, by Theorem 1.1 we have

$$(6.2) \quad L_K(\chi, 1) = \frac{16\pi^2 \log(\varepsilon_p)}{w_K \sqrt{d_K}} \left( - \sum_{C \in \text{CL}(K)} \chi(C) \log G(z_a) \right),$$

where we used  $R_K = 2 \log(\varepsilon_p)$  (see [10, Proposition 4.16]). The theorem now follows by equating (6.1) and (6.2) and simplifying the resulting expression. □

### 7. Proof of Theorem 1.6

Assume first that  $r = 2$ . Then  $K \cong (\mathbb{Z}/2\mathbb{Z})^2$ ,  $E \cong (\mathbb{Z}/2\mathbb{Z})^3$ , and the maximal totally real subfield  $F$  of  $K$  is real quadratic. Let  $\mathbb{Q}(\sqrt{D_1})$  and  $\mathbb{Q}(\sqrt{D_2})$  be the real quadratic subfields of  $E$  which are not contained in  $K$ , and let  $F = \mathbb{Q}(\sqrt{D_3})$ . Then because  $R_K = 2 \log(\varepsilon_{D_3})$ , it suffices to show that  $A := \exp(B)$  is transcendental, where

$$B := Q_1 \sqrt{Q_2} \frac{\log(\varepsilon_{D_1}) \log(\varepsilon_{D_2})}{\log(\varepsilon_{D_3})}$$

for rational numbers  $Q_1, Q_2 \in \mathbb{Q}$ .

Let  $x_1 := \log(\varepsilon_{D_1}), x_2 := \log(\varepsilon_{D_2})$  and  $x_3 := \log(\varepsilon_{D_3})$ . Then

$$\overline{\mathbb{Q}}(x_1, x_2, x_3, \exp(x_1), \exp(x_2), \exp(x_3)) = \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3})).$$

Because  $\varepsilon_{D_1}, \varepsilon_{D_2}$  and  $\varepsilon_{D_3}$  are multiplicatively independent,  $x_1, x_2$  and  $x_3$  are linearly independent over  $\mathbb{Q}$ . Then by Schanuel’s conjecture (see Conjecture 1.5), the field

$$\overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}))$$

has transcendence degree at least 3 over  $\overline{\mathbb{Q}}$ , and hence exactly 3 as it is generated by 3 elements. In particular,  $x_1, x_2$  and  $x_3$  are algebraically independent over  $\overline{\mathbb{Q}}$ .

We claim that because  $x_1, x_2$  and  $x_3$  are algebraically independent over  $\overline{\mathbb{Q}}$ , the numbers  $x_1, x_2, x_3$  and  $x_4 := B$  are linearly independent over  $\mathbb{Q}$ . To see this, suppose to the contrary that there exist rational numbers  $\alpha_i \in \mathbb{Q}$ , not all zero, such that

$$(7.1) \quad \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 B = 0.$$

Define the polynomial

$$q(t_1, t_2, t_3) := \alpha_1 t_1 t_3 + \alpha_2 t_2 t_3 + \alpha_3 t_3^2 + \alpha_4 Q_1 \sqrt{Q_2} t_1 t_2.$$

Then (7.1) implies that  $q(x_1, x_2, x_3) = 0$ , which contradicts the algebraic independence of  $x_1, x_2$  and  $x_3$  over  $\overline{\mathbb{Q}}$ . Thus  $x_1, x_2, x_3$  and  $x_4$  are linearly independent over  $\mathbb{Q}$ . By another application of Schanuel’s conjecture, the field

$$\begin{aligned} \overline{\mathbb{Q}}(x_1, x_2, x_3, x_4, \exp(x_1), \exp(x_2), \exp(x_3), \exp(x_4)) \\ = \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), B, A) \\ = \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), A) \end{aligned}$$

has transcendence degree at least 4 over  $\overline{\mathbb{Q}}$ , hence  $A$  must be transcendental. This completes the proof when  $r = 2$ .

Next assume that  $r \geq 2$ . Then  $K \cong (\mathbb{Z}/2\mathbb{Z})^r$ ,  $E \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$ , and  $F \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$ . The rank of the unit group  $\mathcal{O}_F^\times$  is  $n - 1$ , where  $n = [F : \mathbb{Q}]$ , and recall that the regulators of  $K$  and  $F$  satisfy the relation

$$R_K = 2^{n-1} R_F.$$

Let  $\varepsilon_1, \dots, \varepsilon_{n-1}$  be fundamental units for the  $n - 1$  real quadratic subfields of  $F$ . These units form a set of multiplicatively independent units in  $F$  which are a basis for  $\mathcal{O}_F^\times / \{\pm 1\}$ , and thus

$$R_F = |\det(\log |\sigma_i(\varepsilon_j)|)_{1 \leq i, j \leq n-1}|$$

where the  $\sigma_i$  run through any  $n - 1$  embeddings of  $F$ . The conjugate of a unit in a real quadratic field is, up to a sign, its inverse. Thus for  $\sigma \in \text{Gal}(F/\mathbb{Q})$ , either  $\sigma(\varepsilon_j) = \varepsilon_j$  or  $\sigma(\varepsilon_j) = \pm \varepsilon_j^{-1}$ . It follows that the regulator  $R_F$  is a positive integer multiple of the product  $\log(\varepsilon_1) \cdots \log(\varepsilon_{n-1})$ . Therefore it suffices to show that  $\exp(C)$  is transcendental, where

$$C := Q_3 \sqrt{Q_4} \frac{\prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i})}{\log(\varepsilon_1) \cdots \log(\varepsilon_{n-1})}$$

for rational numbers  $Q_3, Q_4 \in \mathbb{Q}$ . Because the units  $\{\varepsilon_1, \dots, \varepsilon_{n-1}\} \cup \{\varepsilon_{\Delta_i} : \Delta_i \in S_R\}$  are multiplicatively independent, a straightforward modification of the argument for  $r = 2$  shows that  $\exp(C)$  is transcendental.  $\square$

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