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ON SOME GLOBAL SEMIANALYTIC SETS

by Abdelhafed ELKHADIRI (*)

Abstract. — We give some structures without quantifier elimination but in which the closure, and hence the interior and the boundary, of a quantifier free definable set is also a quantifier free definable set.

Résumé. — On donne quelques structures n'ayant pas l'élimination des quantificateurs, mais dans lesquelles l'adhérence, et donc l'intérieur et le bord, d'un ensemble défini sans quantificateur est encore un ensemble défini sans quantificateur.

1. Introduction

Recall that a subset of $\mathbb{R}^n$ is called *semi-algebraic* if it can be represented as a finite boolean combination of sets of the form $\{x \in \mathbb{R}^n \,/\, P(x) = 0\}$, $\{x \in \mathbb{R}^n \,/\, Q(x) > 0\}$, where $P(x), Q(x)$ are n variables polynomials with real coefficients. A fundamental result of Tarski says that the projection of a semi-algebraic set is also semi-algebraic. Immediate consequences are the facts that the closure, interior and boundary of a semi-algebraic set are semi-algebraic. The result of Tarski is know to logicians as quantifiers elimination for the structure $\overline{\mathbb{R}} = (\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the language of ordered rings. In this paper, we consider expressions, with unbounded variables, built from global real analytic functions in some algebra satisfying some conditions. We prove that the closure of a set defined by a quantifier free formula is also a set defined by quantifier free formula. In the local analytic situation, i.e. in the case of germs of real analytic functions, the result is proved by Łojasiewicz [4]: the closure of semi-analytic set is again semi-analytic. The case where the variables are bounded is proved by Gabrielov [3].

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We denote by $\mathcal{H}(\mathbb{R}^n)$ the algebra of real analytic functions on $\mathbb{R}^n$. Let $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ be a subalgebra, a subset of $\mathbb{R}^n$ is called $\mathcal{O}$-semi-analytic, if it can be represented as a finite boolean combination of sets of the form $\{x \in \mathbb{R}^n \,/\, f(x) = 0\}$, $\{x \in \mathbb{R}^n \,/\, g(x) > 0\}$, where f, g are in the algebra $\mathcal{O}(\mathbb{R}^n)$. We want to see, under what conditions on the algebra $\mathcal{O}(\mathbb{R}^n)$, when the closure, and hence the interior and the boundary of an $\mathcal{O}$-semi-analytic is also $\mathcal{O}$-semi-analytic.
At the first sight, if the structure $(\overline{\mathbb{R}}, \mathcal{O})$ admits quantifiers elimination, we are done. But by a result of Lou van den Dries [7], each $f \in \mathcal{O}(\mathbb{R}^n)$ must be semi-algebraic, i.e. the graph of f is a semi-algebraic set. Hence every $\mathcal{O}$-semi-analytic set, in this case, is a semi-algebraic set.
If we consider, for each $n \in \mathbb{N}$,

$$\mathcal{O}_e(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n, \exp x_1, \ldots, \exp x_n].$$

We know by a deep result of Wilkie [9], that the structure $(\overline{\mathbb{R}}, \mathcal{O}_e)$ is model-complete, that is, every definable set is a projection of a quantifier free definable set. As a consequence of Wilkie's result, we see that the closure of an $\mathcal{O}_e$-semi-analytic set is the projection of an $\mathcal{O}_e$-semi-analytic set. Using our result, we will see that this closure is in fact an $\mathcal{O}_e$-semi analytic set.

Let us consider a more general situation: for each $n, p \in \mathbb{N},\ n, p \geqslant 1$, let $F_1, \ldots, F_p : \mathbb{R}^n \to \mathbb{R}$ be analytic functions and suppose that there exist polynomials $P_{ij} \in \mathbb{R}[Y_1, \ldots, Y_{n+i}]$, for $i = 1 \ldots, p,\ j = 1, \ldots, n$ such that:

$$\frac{\partial F_i(x)}{\partial x_j} = P_{ij}(x, F_1(x), \ldots, F_i(x)), \quad \forall x \in \mathbb{R}^n.$$

The sequence $F_1, \ldots, F_p$ is called a Pfaffian chain on $\mathbb{R}^n$.
Denote, for each $n \in \mathbb{N}$,

$$\mathcal{O}_P(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n, F_1, \ldots, F_p],$$

where $F_1, \ldots, F_p$ is a Pfaffian chain on $\mathbb{R}^n$. It is not known yet if the structure $(\overline{\mathbb{R}}, \mathcal{O}_P)$ is model-complete, but our result shows again, in this case, that the closure of an $\mathcal{O}_P$-semi-analytic set is again an $\mathcal{O}_P$-semi-analytic set. This is a direct consequence of lemma 3.1 b) and corollary 10.2.

To summarize, we consider a collection $\mathcal{O}$ of real valued functions of real variables not necessarily all of the same number of variables. We consider the structure of ordered field $\overline{\mathbb{R}}$ augmented by the functions on $\mathcal{O}$. We denote by $L_{\mathcal{O}}$ the induced language. Consider a definable subset, X say, of $\mathbb{R}^n \times \mathbb{R}^m$. For each $b \in \mathbb{R}^m$, let X_b denote the fibre $\{a \in \mathbb{R}^n \,/\, (a, b) \in X\}$ and $\overline{X_b}$ its closure, with respect the Euclidean topology of $\mathbb{R}^n$. The main result of this paper can be stated as follow: With some natural assumption on $\mathcal{O}$, if X is a quantifier free definable set in the language $L_{\mathcal{O}}$, then the set $\bigcup_{b\in\mathbb{R}^m}\overline{X_b}$ is also a quantifier free definable set in the language $L_{\mathcal{O}}$.

The main idea of the proof can be summarized as follows:

Let $\mathcal{O}(\mathbb{R}^n)\subset\mathcal{H}(\mathbb{R}^n)$ be a subalgebra and consider the set:

$$A=\{x\in\mathbb{R}^n\,/\,\varphi_0(x)=0,\,\varphi_1(x)>0,\dots,\varphi_q(x)>0\},$$

where $\varphi_0,\dots,\varphi_q\in\mathcal{O}(\mathbb{R}^n)$.
We see that the closure of A, $\overline{A}$, is the set of all $a\in\mathbb{R}^n$ such that there exists a germ of real analytic curve

$$t\to\xi(t)=(\xi_1(t),\dots,\xi_n(t)),\ \xi(0)=a,$$

such that $\varphi_{0,a}(\xi(t))=0$, for t small, and, for each $i=1,\dots,q$, the first coefficient not zero of $\varphi_{i,a}(\xi(t))$ is positive, where $\varphi_{0,a},\dots,\varphi_{q,a}$ are the germs at a of the functions $\varphi_0,\dots,\varphi_q$ respectively.
If we suppose that the algebra $\mathcal{O}(\mathbb{R}^n)$ is $\mathbb{R}^n$–noetherian, see definition 2.1, then we can find an integer $N\in\mathbb{N}$, such that it is enough to find a polynomial curve

$$t\to\alpha(t)=(a_1+\alpha_1(t),\dots,a_n+\alpha_n(t)),\ \alpha_i(t)=\sum_{j=1}^{2N+1}C_{i,j}t^j,\ i=1,\dots,n,$$

with $\varphi_{0,a}(\alpha(t))=0$, and the first coefficient not zero of $\varphi_{i,a}(\alpha(t))$ is positive. So we see that the closure of A is the projection on $\mathbb{R}^n$ of a set $B\subset\mathbb{R}^n\times\mathbb{R}^{2N+1}$, where B is an $\mathcal{O}(\mathbb{R}^n)[t_1,\dots,t_{2N+1}]$- semianalytic set. We deduce the result by a generalized version of Tarski's theorem [4].

2. Definitions and recalls

2.1. $\mathbb{R}^n$–noetherian algebra

For each $n\in\mathbb{N}$, let $\mathcal{O}(\mathbb{R}^n)\subset\mathcal{H}(\mathbb{R}^n)$ be a subalgebra such that:

a) $\mathbb{R}[x_1,\dots,x_n]\subset\mathcal{O}(\mathbb{R}^n)$, where $\mathbb{R}[x_1,\dots,x_n]$ is the ring of polynomials.
b) $\mathcal{O}(\mathbb{R}^n)$ is closed upon taking derivation.

In this paper we call such algebras *analytic algebras.*
If $\mathcal{O}(\mathbb{R}^n)$ is an analytic algebra, we denote by $SM\mathcal{O}(\mathbb{R}^n)$ the maximal spectrum of $\mathcal{O}(\mathbb{R}^n)$. Since $\mathbb{R}[x_1,\dots,x_n]\subset\mathcal{O}(\mathbb{R}^n)$, we have an injection

$$\mathbb{R}^n\to SM\mathcal{O}(\mathbb{R}^n),$$

every $x\in\mathbb{R}^n$ is identified with the maximal ideal

$$\underline{m}_x=\{f\in\mathcal{O}(\mathbb{R}^n)\,/\,f(x)=0\}.$$

We denote by $\mathbb{R}^n(\mathcal{O})$ the topological space $\mathbb{R}^n$ with the induced topology of $SM\mathcal{O}(\mathbb{R}^n)$.

Definition 2.1. — *An analytic algebra $\mathcal{O}(\mathbb{R}^n)$ is said to be $\mathbb{R}^n$-noetherian if $\mathbb{R}^n(\mathcal{O})$ is a noetherian subspace of $SM\mathcal{O}(\mathbb{R}^n)$, i.e. every decreasing sequence of closed sets in $\mathbb{R}^n(\mathcal{O})$ is stationary.*

Let us remark that any noetherian analytic algebra is $\mathbb{R}^n$-noetherian.

Here we will give an example of an $\mathbb{R}^n$-noetherian algebra, but we do not know if it is noetherian.

We consider the structure $\mathcal{R}_e=(\overline{\mathbb{R}},\mathcal{O}_e)$. We denote by $\mathcal{D}_e(\mathbb{R}^n)$ the algebra of all real analytic functions on $\mathbb{R}^n$ which are definable in $\mathcal{R}_e$.
If $f\in\mathcal{D}_e(\mathbb{R}^n)$, we put $V(f)=\{x\in\Omega\,/\,f(x)=0\}$. We denote by $RegV(f)$ the set of all $x\in V(f)$ such that, there is an open neighborhood U of x with $U\cap V(f)$ an embedded real analytic submanifold of $\mathbb{R}^n$. It is clear that $RegV(f)$ is dense in $V(f)$.

Lemma 2.2. — *Let $f_i:\mathbb{R}^n\to\mathbb{R}$, $i\in\mathbb{N}$, be a family of analytic functions each belonging to $\mathcal{D}_e(\mathbb{R}^n)$. Then there exists $N\in\mathbb{N}$ such that:*

$$\bigcap_{i\in\mathbb{N}}V(f_i)=\bigcap_{i\leqslant N}V(f_i).$$

Proof. — Let $f\in\mathcal{D}_e(\mathbb{R}^n)$, we know that the structure $\mathcal{R}_e$ admits analytic cell decomposition, see [1], and by 1.8 of [8], we deduced that $RegV(f)$ is a definable set and therefore has only a finite number of connected components.

For each $V(f_i)$ we associate a $n+1$-tuple $\nu_i=(\nu_{i,n},\nu_{i,n-1},\dots,\nu_{i,0})\in\mathbb{N}^{n+1}$, where $\nu_{i,k}$ is the number of connected components of $RegV(f_i)$ of dimension k. If we consider the lexicographic order on $\mathbb{N}^{n+1}$, we have

$$\nu_j<\nu_i\quad\text{if}\quad V(f_j)\subsetneq V(f_i). \tag{2.1}$$

To avoid trivialities, let us suppose that $\emptyset \neq V(f_0) \neq \mathbb{R}^n$. We have:

$$\bigcap_{i\in\mathbb{N}} V(f_i) \subset \ldots \subset V(f_0^2 + f_1^2 + f_2^2) \subset V(f_0^2 + f_1^2) \subset V(f_0).$$

By (2.1), there exists $N \in \mathbb{N}$ such that $\forall j \in \mathbb{N}$, $j \geqslant N$, we have:

$$V(f_0^2 + \ldots + f_N^2) = V(f_0^2 + \ldots + f_j^2),$$

hence

$$\bigcap_{i\in\mathbb{N}} V(f_i) = V(f_0^2 + \ldots + f_N^2) = \bigcap_{i\leqslant N} V(f_i),$$

which proves the lemma. □

Theorem 2.3. — *the algebra $\mathcal{D}_e(\mathbb{R}^n)$ is $\mathbb{R}^n$-noetherian.*

Proof. — Let $F \subset \mathbb{R}^n(\mathcal{D}_e)$ be a closed set. There exists an ideal $I \subset \mathcal{D}_e(\mathbb{R}^n)$ such that

$$F = V(I) := \{x \in \Omega \,/\, f(x) = 0, \, \forall f \in I\}.$$

First, we will show that $V(I)$ can be defined by one equation, i.e. there is $h \in I$ such that

$$V(I) = V(h) = \{x \in \Omega \,/\, h(x) = 0\}.$$

Let $g \in I - \{0\}$, we have $V(I) \subset V(g)$. We denote by $\Gamma_1, \ldots, \Gamma_s$ the connected components of $RegV(g)$.
We put

$$\mu_1 = max\{dim\Gamma_j \,/\, \Gamma_j \not\subseteq V(I)\}.$$

If $\Gamma_1, \ldots, \Gamma_\nu$ are the connected components of $RegV(g)$ such that

$$dim\Gamma_l = \mu_1 \text{ and } \Gamma_j \not\subseteq V(I), \, l = 1, \ldots, \nu.$$

For each $l = 1, \ldots, \nu$, there exists $h_l \in I$ such that $h_{l|\Gamma_j} \neq 0$. We put $h = \sum_{l=1}^{\nu} h_l^2 \in I$. We have $V(I) \subset V(\psi) \subsetneq V(g)$, where $\psi = f^2 + g^2$. If we put

$$\mu_2 = max\{dim\Gamma_k \,/\, \Gamma_k \, \text{connected component of} \, Reg\psi, \, \Gamma_k \not\subseteq V(I)\},$$

we have $\mu_2 < \mu_1$. By continuing this processus with ψ and so on, we see that, by lemma 2.2, there is $\varphi \in I$ such that $V(I) = V(\varphi)$.
Now let $(F_j)_{j\in J}$ be a decreasing sequence of closed sets of the topological space $\mathbb{R}^n(\mathcal{D}_e)$. For each $j \in J$, there exists $\varphi_j \in \mathcal{D}_e(\mathbb{R}^n)$ such that $F_j = V(\varphi_j)$, which proves the result by using lemma 2.2. □

2.2. Question

We don't know if the algebra $\mathcal{D}_e(\mathbb{R}^n)$ is Noetherian.

3. Somes properties of $\mathbb{R}^n$-noetherian algebras

Let $y = (y_1, \ldots, y_m)$, and let $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ be an analytic algebra. We identify $\mathbb{R}^n \times \mathbb{R}^m$ with a subset of $SM\mathcal{O}(\mathbb{R}^n)[y_1, \ldots, y_m]$ as follow: let $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, every $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$ is identified with the maximal ideal generated by $\underline{m}_x$ and $y_1 - \xi_1, \ldots, y_m - \xi_m$.

LEMMA 3.1. — [2] *Let $\mathcal{O}(\mathbb{R}^n)$ be an $\mathbb{R}^n$-noetherian algebra. Then:*

a) *$\mathcal{O}(\mathbb{R}^n)[y_1, \ldots, y_m]$ is $\mathbb{R}^n \times \mathbb{R}^m$-noetherian.*
b) *Let $F_1, \ldots, F_p$ be a Pfaffian chain on $\mathbb{R}^n$. Then the algebra $\mathbb{R}[x_1, \ldots, x_n, F_1, \ldots, F_p]$ is $\mathbb{R}^n$-noetherian.*

Let $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ be a subalgebra, if $I \subset \mathcal{O}(\mathbb{R}^n)$ is an ideal, we put:

$$V(I) = \{x \in \mathbb{R}^n \ / \ g(x) = 0, \forall g \in I \ \}.$$

If $I = (f)\mathcal{O}(\mathbb{R}^n)$, where $f \in \mathcal{O}(\mathbb{R}^n)$, we write $V(f)$ instead of $V((f)\mathcal{O}(\mathbb{R}^n))$. If $X \subset \mathbb{R}^n$, let

$$I(X) = \{f \in \mathcal{O}(\mathbb{R}^n) \, / \, f(x) = 0, \ \forall x \in X\}.$$

A closed set $X \subset \mathbb{R}^n(\mathcal{O})$ is irreducible if, and only if, $I(X)$ is a prime ideal.

Remark 3.2. — If $\mathcal{O}(\mathbb{R}^n)$ is an $\mathbb{R}^n$-noetherian algebra and $f \in \mathcal{O}(\mathbb{R}^n)$, $V(f)$ can have infinitely many connected components. For example the algebra $\mathbb{R}[x, \sin x, \cos x] \subset \mathcal{H}(\mathbb{R})$ is not only $\mathbb{R}$-noetherian but actually it is a Noetherian ring. But $V(\sin x)$ has infinitely many connected components.

4. Noetherian family

Let A be an algebra over $\mathbb{R}$, and let Γ be a subset of SMA. We assume that A and Γ satisfy the following conditions:

(a) the canonical mapping $\mathbb{R} \to \frac{A}{\gamma}$ is an isomorphism for all $\gamma \in \Gamma$,
(b) Γ is a noetherian subspace of SMA.

4.1.

If $a \in A$ and $\gamma \in \Gamma$, we denote by $a(\gamma) \in \mathbb{R}$ the image of a under the map $A \to \frac{A}{\gamma} \to \mathbb{R}$.
Let $x = (x_1, \ldots, x_n)$, if $\omega = (\omega_1, \ldots, , \omega_n) \in \mathbb{N}^n$, we write $|\omega| = \omega_1 + \ldots + \omega_n$ and $x^\omega = x_1^{\omega_1} \ldots x_n^{\omega_n}$.
We denote by $A[[x]]$ the ring of formal power series with coefficients in A.
If $\gamma \in \Gamma$ and $f = \sum_\omega a_\omega x^\omega \in A[[x]]$, we put

$$f_\gamma = \sum_\omega a_\omega(\gamma) x^\omega \in \mathbb{R}[[x]].$$

Let $A_c[[x]] \subset A[[x]]$ be the subring of all $f \in A[[x]]$ such that, $\forall \gamma \in \Gamma$, $f_\gamma \in \mathbb{R}\{x\}$, where $\mathbb{R}\{x\}$ is the ring of convergent power series. Finally if $I \subset A_c[[x]]$ is an ideal, we denote by I_γ the ideal of $\mathbb{R}\{x\}$ generated by the set $\{f_\gamma, / f \in I\}$.

Let A be an algebra over the field of reals $\mathbb{R}$ and let Γ be a subset of SMA. We assume that A and Γ satisfy the conditions (a) and (b).

Definition 4.1. — *Let A and Γ as above, a family $\mathcal{I}$ of ideals of $\mathbb{R}\{x\}$ is called a noetherian family parameterized by (A, Γ), if there exists an ideal $I \subset A_c[[x]]$, such that $\mathcal{I} = (I_\gamma)_{\gamma \in \Gamma}$.*

Example 4.2. — Let $\mathcal{O}(\mathbb{R}^n)$ be a $\mathbb{R}^n$-noetherian algebra, then the couple $(\mathcal{O}(\mathbb{R}^n), \mathbb{R}^n(\mathcal{O}))$ satisfies the conditions (a) and (b).

i) Consider $\mathcal{O}(\mathbb{R}^n)$ an $\mathbb{R}^n$-noetherian algebra, and let $I \subset \mathcal{O}(\mathbb{R}^n)$ be an ideal. We denote by $I_x \subset \mathcal{H}_x$ the ideal generated by I in the ring of germs, at $x \in \mathbb{R}^n$, of real analytic functions, say $\mathcal{H}_x$. The coherent analytic sheaf $\mathcal{I} = (I_x)_{x \in \mathbb{R}^n}$ is a noetherian family parameterized by $(\mathcal{O}(\mathbb{R}^n), \mathbb{R}^n(\mathcal{O}))$.
ii) Let $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ be an $\mathbb{R}^n \times \mathbb{R}^m$-noetherian algebra, if $I \subset \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ is an ideal generated by (f_i); for each $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$, we denote by I_a^b the ideal in $\mathcal{H}_a$ generated by the germ, at $a \in \mathbb{R}^n$, of all functions $x \to f_i(x, b)$. The family $(I_a^b)_{(a,b) \in \mathbb{R}^n \times \mathbb{R}^m}$ is a noetherian family parameterized by $(\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m), \mathbb{R}^n \times \mathbb{R}^m(\mathcal{O}))$.
iii) Fix an integer $d \in \mathbb{N}^*$ and consider the family of ideals in $\mathbb{R}\{x\}$ such that, each ideal is generated by a finite number of polynomials of degree at most d. Then this family is a noetherian family.

4.2. Ring of generic formal series

The main tool to study the noetherian family is the ring of *generic formal power series* introduced in [2]. We will describe it briefly.
Let A and Γ as above. Let $X \subset \Gamma$ be a closed irreducible set. Let $I(X) = \{a \in A / \, a(\gamma) = 0, \, \forall \gamma \in X\}$. Put $\Lambda = \frac{A}{I(X)}$, since $I(X)$ is a prime ideal, Λ is a domain. If $\delta \in \Lambda$ let $V(\delta) = \{\gamma \in X / \, \delta(\gamma) = 0\}$. If $\delta, \delta' \in \Lambda - \{0\}$, the expression $\delta \geqslant \delta'$ will be used to denote the fact that δ' divides δ in the domain Λ. If $\delta \in \Lambda - \{0\}$, let $\Lambda_\delta = \{\frac{a}{\delta^m} / \, a \in \Lambda, \, m \in \mathbb{N}\} \subset [\Lambda]$, where $[\Lambda]$ is the field of fractions of Λ.

If $\delta \in \Lambda - \{0\}$, let $\Lambda_c\{\{x\}\}_\delta$ be the subring of $(\Lambda_\delta)_c[[x]]$ consisting of all formal series of the form $\sum_{\omega \in \mathbb{N}^n} \frac{a_\omega}{\delta^{\alpha|\omega|+\beta}} x^\omega$, where α, $\beta \in \mathbb{N}$ and $a_\omega \in \Lambda$, for each $\omega \in \mathbb{N}^n$. Let δ_1, $\delta_2 \in \Lambda$, then if $\delta = \delta_1\delta_2$, we have two injections:

$$\Lambda_c\{\{x\}\}_{\delta_1} \hookrightarrow \Lambda_c\{\{x\}\}_\delta$$

and

$$\Lambda_c\{\{x\}\}_{\delta_2} \hookrightarrow \Lambda_c\{\{x\}\}_\delta.$$

Let $\Lambda_c\{\{x\}\}$ denote the direct limit of the family of rings $(\Lambda_c\{\{x\}\}_\delta)_{\delta \in \Lambda - \{0\}}$. The Weierstrass preparation and division theorems are valid in $\Lambda_c\{\{x\}\}$, in particular $\Lambda_c\{\{x\}\}$ is a noetherian ring. $\Lambda_c\{\{x\}\}$ is a local ring, its maximal ideal is generated by $x_1, \dots, x_n$ and its residue field is canonically isomorphic to the field of fraction of Λ. Since $\{x_1, \dots, x_n\}$ is a system of parameters of the ring $\Lambda_c\{\{x\}\}$, the ring $\Lambda_c\{\{x\}\}$ is a regular ring of dimension n. Consequently, it is a unique factorization domain [2].
For the proof of our result we need some properties of ideals of the ring $\Lambda_c\{\{x\}\}$ see [2].

4.3. Flat properties

Let us review some flatness properties of modules over the ring $\Lambda_c\{\{x\}\}$ see [2]. The following proposition is proved in [2, 3.1]:

PROPOSITION 4.3. — *Let $I \subset \Lambda_c\{\{x\}\}_{\delta_1}$ be an ideal. There exists $\delta_2 \geqslant \delta_1$ such that, for each $\delta \geqslant \delta_2$, we have:*

$$I\Lambda_c\{\{x\}\} \cap \Lambda_c\{\{x\}\}_\delta = I\Lambda_c\{\{x\}\}_\delta.$$

4.3.1. Consequences

i) Let $I \subset \Lambda_c\{\{x\}\}$ be an ideal. Since $\Lambda_c\{\{x\}\}$ is a noetherian ring, I is finitely generated, let $\{f_1, \ldots, f_q\}$ be a system of generators of I. There exists $\delta \in \Lambda - \{0\}$, such that $f_j \in \Lambda_c\{\{x\}\}_\delta$, $\forall j = 1, \ldots, q$. We denote by I^δ the ideal of $\Lambda_c\{\{x\}\}_\delta$ generated by $f_1, \ldots, f_q$. If $I^{\delta_1} \subset \Lambda_c\{\{x\}\}_{\delta_1}$ and $I^{\delta_2} \subset \Lambda_c\{\{x\}\}_{\delta_2}$ are the ideals associated with two systems of generators of I, then there is $\delta \in \Lambda$, with $\delta \geqslant \delta_1,\ \delta \geqslant \delta_2$, such that $I^{\delta_1}\Lambda_c\{\{x\}\}_\delta = I^{\delta_2}\Lambda_c\{\{x\}\}_\delta$ [2].

ii) Let I, I' two ideals of the ring $\Lambda_c\{\{x\}\}$. Then there exists $\delta \in \Lambda - \{0\}$ such that $\Lambda_c\{\{x\}\}_\delta$ contains a system of generators of I and I', and $(I : I')^\delta = (I^\delta : I'^\delta)$, where $(I : I') = \{\lambda \in \Lambda_c\{\{x\}\} \,/\, \lambda I' \subset I\}$.

iii) If $\wp \subset \Lambda_c\{\{x\}\}$ is a prime ideal, let $\delta_1 \in \Lambda - \{0\}$ such that $\Lambda_c\{\{x\}\}_{\delta_1}$ contains a system of generators of $\wp$. There exists $\delta_2 \geqslant \delta_1$ such that, for each $\delta \geqslant \delta_2$, $\wp^\delta$ is a prime ideal of $\Lambda_c\{\{x\}\}_\delta$.

iv) Let $I \subset \Lambda_c\{\{x\}\}$ be an ideal and consider $\wp_1, \ldots, \wp_q$ the minimal prime ideals associated to I. We have then: $\sqrt{I} = \wp_1 \cap \ldots \cap \wp_q$. Let $\delta_1 \in \Lambda - \{0\}$ such that $\Lambda_c\{\{x\}\}_{\delta_1}$ contains a system of generators of each $\wp_i,\ 1 \leqslant i \leqslant q$. There exists $\delta_2 \geqslant \delta_1$ such that, for each $\delta \geqslant \delta_2$, we have: $(\sqrt{I})^\delta = \wp_1^\delta \cap \ldots \cap \wp_q^\delta$.

5. Generic flatness properties

In this section $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n)}{I(X)}$, $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ is a $\mathbb{R}^n$-noetherian algebra and $X \subset \mathbb{R}^n(\mathcal{O})$ is an irreducible closed set.
Let $g_1, \ldots, g_k \in \Lambda_c\{\{x\}\}_{\delta_1}$, $\delta_1 \in \Lambda - \{0\}$. We put $I = (g_1, \ldots, g_k)\Lambda_c\{\{x\}\}_{\delta_1}$ the ideal generated by $g_1, \ldots, g_k$.
Recall that, for all $\delta \geqslant \delta_1$, $\delta \in \Lambda - \{0\}$ and $a \in X - V(\delta) \subset SM\Lambda_\delta$, we have an homomorphism:

$$\Lambda_c\{\{x\}\}_\delta \to \mathbb{R}\{x\},$$

which associates to each $\sum_{\omega \in \mathbb{N}^n} \frac{a_\omega}{\delta^{\alpha|\omega|+\beta}} x^\omega \in \Lambda_c\{\{x\}\}_\delta$ the convergent series $\sum_{\omega \in \mathbb{N}^n} \frac{a_\omega(a)}{\delta^{\alpha|\omega|+\beta}(a)} x^\omega$, see 4.1. If $a \in X - V(\delta)$, I_a^δ is the ideal generated by the image of I^δ by this homomorphism.

Definition 5.1. — *We say that a property $(\mathcal{P})$ is generically satisfied by the family $(I_a^{\delta_1})_{a \in X - V(\delta_1)}$, if there exists $\delta_2 \geqslant \delta_1$ such that, $\forall \delta \geqslant \delta_2$, the property $(\mathcal{P})$ is satisfied by the ideal $I_a^\delta \subset \mathbb{R}\{x\}$ for all $a \in X - V(\delta)$.*

Let $I \subset \Lambda_c\{\{x\}\}$ be an ideal, and let $\wp_1, \dots, \wp_q$ the minimal prime ideals associated to I. We shall use the following proposition, see [2, 5.2.4 and 5.2.5] for the proof.

Proposition 5.2. — *Generically, we have:*

1- $ht(I_a^\delta) = ht(I)$,
2- $(\sqrt{I})_a^\delta = \sqrt{I_a^\delta}$,
3- $(\sqrt{I})_a^\delta = \wp_{1,a}^\delta \cap \dots \cap \wp_{q,a}^\delta$.

6. Bound of the multiplicity of a noetherian family

Let $\mathcal{O}(\mathbb{R}^n)$ be an $\mathbb{R}^n$-noetherian algebra and let $X \subset \mathbb{R}^n(\mathcal{O})$ be an irreducible closed subset in $\mathbb{R}^n(\mathcal{O})$. Put $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n)}{I(X)}$. If f_0 is a non zero element of $\Lambda_c\{\{x\}\}$, we denote by $\mu_0 = \mu(f_0)$ the degree of the initial form of f_0, μ_0 is called the multiplicity of f_0. After making a linear coordinate transformation:

$$\sigma_n(x_1, \dots, x_n) = (x_1 + a_{1,n}x_n, x_2 + a_{2,n}x_n, \dots, x_{n-1} + a_{n-1,n}x_n, x_n),$$

$f_0 \circ \sigma_n$ is regular of order μ_0 in x_n, that is $f_0 \circ \sigma_n(0, 0, \dots, x_n) = x_n^p g(x_n)$ with $g \in \Lambda_c\{\{x_n\}\}, g(0) \neq 0$. Hence $f_0 \circ \sigma_n$ is equivalent in the ring $\Lambda_c\{\{x\}\}$ with a distinguished polynomial in x_n of degree μ_0: $Q_0 \in \Lambda_c\{\{x'\}\}[x_n]$, where $x' = (x_1, \dots, x_{n-1})$. Let $f_1 \in \Lambda_c\{\{x'\}\}$ be the discriminant of $Red(Q_0)$, the reduced form of Q_0, $f_1 \neq 0$. We denote by μ_1 the degree of the initial form of f_1. As the first step, there exists a linear coordinate transformation:

$$\begin{aligned}&\sigma_{n-1}(x_1, \dots, x_{n-1})\\ &\quad = (x_1 + a_{1,n-1}x_{n-1}, x_2 + a_{2,n-1}x_{n-1}, \dots, x_{n-2} + a_{n-2,n-1}x_{n-1}, x_{n-1}),\end{aligned}$$

such that $f_1 \circ \sigma_{n-1}$ is regular of order μ_1 in x_{n-1}. Again $f_1 \circ \sigma_{n-1}$ is equivalent in the ring $\Lambda_c\{\{x'\}\}$ with a distinguished polynomial in x_{n-1} of degree μ_1: $Q_1 \in \Lambda_c\{\{(x_1, \dots, x_{n-2})\}\}[x_{n-1}]$. We denote by $f_2 \in \Lambda_c\{\{(x_1, \dots, x_{n-2})\}\}$ the discriminant of $Red(Q_1)$. By continuing this procedure, we get at the end, for each $j = 0, \dots, n-1$, an element $f_j \in \Lambda_c\{\{x_1, \dots, x_{n-j}\}\}$ and a linear transformation:

$$\sigma(x) = (x_1 + \sum_{j=2}^{n} a_{1,j}x_j, x_2 + \sum_{j=3}^{n} a_{2,j}x_j, \dots, x_{n-1} + a_{n,n-1}x_n, x_n) \ , \quad (6.1)$$

such that each $f_j \circ \sigma$ is regular of order μ_j in x_{n-j}, $j = 0, \dots, n-1$.

We put $\mu_{f_0} = \prod_{j=0}^{n-1} \mu_j$, this integer is called the integer associated to f_0.

Lemma 6.1. — *Let $f_0 \in \Lambda_c\{\{x\}\}$. After making a linear transformation of the form* (6.1)*, there exists an integer N and $\delta \in \Lambda - \{0\}$ such that, for each $\gamma \in SM\Lambda_\delta$, the integer associated to $f_{0,\gamma} \in \mathbb{R}\{x\}$ is less than N.*

Proof. — Let $f_0 \in \Lambda_c\{\{x\}\}$ and consider all $f_j \in \Lambda_c\{\{x_1, \dots, x_{n-j+1}\}\}$, $j = 0, \dots, n-1$, constructed above. There exists $\delta \in \Lambda - \{0\}$ such that each $f_j \in \Lambda_c\{\{x_1, \dots, x_{n-j+1}\}\}_\delta$. We can then see that, for all $\gamma \in SM\Lambda_\delta$, $\mu(f_j) = \mu(f_{j,\gamma})$, $\forall j = 0, \dots, n-1$. This proves the lemma. □

Proposition 6.2. — *Let $\mathcal{O}(\mathbb{R}^n)$ be an $\mathbb{R}^n$-noetherian algebra, and $f \in \mathcal{O}(\mathbb{R}^n)$. There exists $N \in \mathbb{N}$ such that, for all $a \in \mathbb{R}^n$, the integer associated to the germ of f at a, say f_a, is less than N ($\mathbb{R}\{x-a\} \simeq \mathbb{R}\{x\}$).*

Proof. — Since $\mathbb{R}^n(\mathcal{O})$ is a noetherian topological space, it is sufficient to find for every irreducible closed set, say X, an integer N_X and a closed subset $F \subset X$, $F \neq X$, such that the integer associated to each f_a, $a \in X - F$, is less than N_X.
Let $X \subset \mathbb{R}^n(\mathcal{O})$ be an irreducible closed subset. We put $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n)}{I(X)}$, we have then a homomorphism of algebras:

$$\mathcal{O}(\mathbb{R}^n) \to \Lambda_c\{\{x\}\},$$

which associates to each $\varphi \in \mathcal{O}(\mathbb{R}^n)$ the element $\sum_{\omega \in \mathbb{N}^n} (\frac{D^\omega \varphi}{\omega!}) x^\omega$.
By lemma 6.1, there exists $\delta \in \mathcal{O}(\mathbb{R}^n) - I(X)$ and an integer N_δ such that, for every $a \in X - V(\delta)$, the integer associated to $\sum_{\omega \in \mathbb{N}^n} (\frac{D^\omega \varphi(a)}{\omega!}) x^\omega \in \mathbb{R}\{x\}$ is less then N_δ. We take $F = X \cap V(\delta) \subset X$, and the proposition follows. □

7. Curves in a germ of an analytic set

Definition 7.1. — *Let $f \in \mathbb{R}\{x\} - \{0\}$ and let $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ be a germ of an analytic curve. We say that the contact order of f with the curve α is less than p, if the multiplicity of the series $f \circ \alpha \in \mathbb{R}\{t\}$ is less than or equal to p.*

Proposition 7.2. — *Let $f \in \mathbb{R}\{x\} - \{0\}$, $f(0) = 0$, and let $N \in \mathbb{N}$ be the integer associated to f. Then for each connected component Γ of $\mathbb{R}^n - V(f)$, there exists a germ of an analytic curve $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ such that:*

1. *f has a contact of order less than N with ξ.*
2. *For each $t > 0$ and small, $\xi(t) \in \Gamma$.*

Proof. — We proceed by induction on n. For $n=1$, the result is clear. Suppose $n \geqslant 2$ and the result holds for $n-1$. We denote by $\mu_0 \in \mathbb{N}$ the multiplicity of f. After possibly a linear change of coordinates, we can suppose that f is a distinguished polynomial in x_n of degree μ_0. We consider $f_1 = Red(f)$ and let $\Delta \in \mathbb{R}\{(x_1,\dots,x_{n-1})\} - \{0\}$ be the discriminant of f_1 and p degree of f_1, $p \leqslant \mu_0$. Let Γ' be a connected component of $\mathbb{R}^{n-1} - V(\Delta)$. The equation $f_1(x', x_n) = 0$ admits, for each $x' \in \Gamma'$, a fixed number, say s, of real distinct roots: $r_1(x') < r_2(x') < \ldots < r_s(x')$, possibly $s=0$. We put $r_0(x') = -\infty, r_{s+1} = +\infty$.

Let Γ be a connected component of $\mathbb{R}^n - V(f)$. There exist Γ' a connected component of $\mathbb{R}^{n-1} - V(\Delta)$, $s \in \mathbb{N}$, such that, if for each $x' \in \Gamma'$, $r_1(x') < r_2(x') < \ldots < r_s(x')$ are the real distinct roots of the equation $f_1(x', x_n) = 0$, then Γ contains the germ at $0 \in \mathbb{R}^n$ of the set:

$$A_k = \{(x', x_n) \,/\, x' \in \Gamma',\ r_k(x') < x_n < r_{k+1}(x')\},$$

for some $0 \leqslant k \leqslant s$, recall that $r_0(x') = -\infty, r_{s+1} = +\infty$.
Let us first eliminate some marginal cases.
- If $s=0$ we take $t \to (0,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.
- If $s \geqslant 1$ and $k=0$ or $k=s$, we also take the curve $t \to (0,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.
- Suppose $k \neq 0$ and $k \neq s$. By the induction hypothesis, there exists an analytic curve $t \to \xi_1(t)$ in $\mathbb{R}^{n-1}$, $\xi_1(0) = 0$, such that, for every $t > 0$ in a neighborhood of $0 \in \mathbb{R}$, $\xi_1(t) \in \Gamma'$ and the contact of Δ with ξ_1 is less than N_1, where N_1 is the integer associated to Δ. If N is the integer associated to f, we have $N = N_1\mu_0$.
Let $\eta_1(t), \dots, \eta_{p-1}(t)$ be all the roots (possibly complex) of the function $y \to \frac{\partial f_1(\xi_1(t), y)}{\partial y}$. Then

$$\Delta(\xi_1(t)) = \prod_{j=1}^{p-1} f_1(\xi_1(t), \eta_j(t)). \tag{7.1}$$

By Rolle's theorem there exists at least one root $\eta_l(t) \in]r_k(t), r_{k+1}(t)[$, for $t>0$ small. By (7.1) we see that the order of $f_1(\xi(t), \eta_l(t))$ (as a Puiseux series) is less or equal than N_1.
We know that $\eta_l(t)$ is a Puiseux series with exponents $\frac{\nu}{q}$ for some $q \leqslant p-1$. Hence the function $\xi_n(t) = \eta_l(t^q)$ is analytic in a neighborhood of $0 \in \mathbb{R}$. Clearly the order of $f_1(\xi_1(t), \xi_n(t))$ is not greater than $(p-1)N_1 \leqslant \mu_0 N_1 = N$, which proves the result. □

8. $\mathcal{O}$-semianalytic set associated to an ideal

Let $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ be a subalgebra $\mathbb{R}^n$-noetherian. Let $X \subset \mathbb{R}^n(\mathcal{O})$ be a closed irreducible set. Put $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n)}{I(X)}$. For each prime ideal $\wp \subset \Lambda_c\{\{x\}\}$ and any finite family $\varphi_1, \dots, \varphi_q \in \Lambda_c\{\{x\}\}$ we shall associate a subset $A \subset \mathbb{R}^n \times \mathbb{R}^m$, for some $m \in \mathbb{N}$, such that A is an $\mathcal{O}(\mathbb{R}^n)[Y_1, \dots Y_m]$-semianalytic.

8.1. Properties of prime ideals of the ring $\Lambda_c\{\{x\}\}$

Let us recall a classical result on prime ideals of the ring $\mathbb{R}\{x\}$ and also valid, same proof, for prime ideals of $\Lambda_c\{\{x\}\}$ [2].
Let $x' = (x_1, \dots, x_{n-s})$ and let $\wp \subset \Lambda_c\{\{x\}\}$ be a prime ideal of height s. After a linear change of coordinates with coefficients in $\mathbb{Z}$, $\wp$ contains distinguished polynomials:

$$P_1 \in \Lambda_c\{\{x'\}\}[x_{n-s+1}]$$

$$P_2 \in \Lambda_c\{\{x'\}\}[x_{n-s+2}],\ P_3 \in \Lambda_c\{\{x'\}[x_{n-s+3}], \dots, P_s \in \Lambda_c\{\{x'\}\}[x_n]$$

P_1 is irreducible. Let $k = degP_1$ and $\Delta \in \Lambda_c\{\{x'\}\} - \{0\}$ its discriminant. There are polynomials $R_2, \dots, R_s \in \Lambda_c\{\{x'\}[x_{n-s+1}]$, with $degR_j < degP_1$, such that, $\wp$ contains the polynomials:

$$Q_2 := \Delta(x')x_{n-s+2} - R_2, \dots, Q_s := \Delta(x')x_n - R_s, \tag{8.1}$$

and finally

$$\wp \cap \Lambda_c\{\{x'\}\} = \{0\}.$$

The polynomials $P_i, i = 1, \dots, s$, $Q_j, j = 2, \dots, s$ do not, in general, generate the ideal $\wp$. We put

$$\nu = Sup\{\sum_{j=2}^{s}(degP_j - 1), degP_2, \dots, degP_s\}.$$

We denote by $(P_1, Q_2, \dots, Q_s)$ [resp. (Δ^ν)] the ideal of $\Lambda_c\{\{x\}\}$ generated by $P_1, Q_2, \dots, Q_s$ [resp. Δ^ν]. Finally set:

$$(P_1, Q_2, \dots, Q_s) : (\Delta^\nu) := \{f \in \Lambda_c\{\{x\}\} \,/\, \Delta^\nu f \in (P_1, Q_2, \dots, Q_s)\}.$$

Lemma 8.1. — *With the notations above, we have:*

$$\wp = (P_1, Q_2, \dots, Q_s) : (\Delta^\nu).$$

Proof. — Since $\wp$ is a prime ideal and $\Delta \notin \wp$, we see that

$$(P_1, Q_2, \dots, Q_s) : (\Delta^\nu) \subset \wp.$$

Let $f \in \wp$, we divide the function f successively by the polynomials $P_2, \dots, P_s$ and therefore we can assume that f is congruent modulo $(P_2, \dots, P_s)$ to a polynomial in $\Lambda_c\{\{x_1, \dots, x_{n-s+1}\}\}[x_{n-s+2}, \dots, x_n]$ of degree less than, or equal, to ν. Thus we can assume that

$$f \in \Lambda_c\{\{x_1, \dots, x_{n-s+1}\}\}[x_{n-s+2}, \dots, x_n], \quad d^0 f \leqslant \nu.$$

According to (8.1), it follows that $\Delta^\nu f$ is congruent modulo $(Q_2, \dots, Q_s)$ to $g \in \Lambda_c\{\{x_1, \dots, x_{n-s+1}\}\} \cap \wp$. Since $P_1 \in \wp$, we can see, by Weierstrass's Division Theorem in the ring $\Lambda_c\{\{x_1, \dots, x_{n-s+1}\}\}$, see 4.2, that g is divisible by P_1, which proves the lemma. □

Let $\delta \in \Lambda - \{0\}$ such that:

$$P_1 \in \Lambda_c\{\{x'\}\}_\delta[x_{n-s+j}] \text{ and } R_j \in \Lambda_c\{\{x'\}_\delta[x_{n-s+j}], \forall j = 2, \dots, s.$$

By 4.3.1, ii) we can choose $\delta \in \Lambda$, such that:

$$\wp^\delta = (P_1, Q_2, \dots, Q_s)^\delta : (\Delta^\nu)^\delta, \tag{8.2}$$

and by Proposition 5.2, $\forall \gamma \in SM\Lambda_\delta$, we have:

$$\wp^\delta_\gamma = (P_1, Q_2, \dots, Q_s)^\delta_\gamma : (\Delta^\nu)^\delta_\gamma. \tag{8.3}$$

and

$$ht(\wp^\delta_\gamma) = s.$$

For all $\gamma \in SM\Lambda_\delta$ there is a neighborhood W^γ of $0 \in \mathbb{R}^n$ in which Δ_γ, $P_{1,\gamma}$, $R_{j,\gamma}, j = 2, \dots, s$, and some system of generators of $\wp^\delta_\gamma$ are analytic. We put:

$$V_\Delta(\wp^\delta_\gamma) = V(\wp^\delta_\gamma) \cap \{x \in W^\gamma / \Delta_\gamma(x') \neq 0\},$$

By (8.3) we have:

$$\begin{aligned} V_\Delta(\wp^\delta_\gamma) = \{x \in W^\gamma / \Delta_\gamma(x') \neq 0, P_{1,\gamma}(x', x_{n-s+1}) = 0,& \\ \Delta_\gamma(x') x_{n-s+j} - R_{j,\gamma}(x', x_{n-s+j}) = 0, j = 2, \dots, s\}.& \end{aligned} \tag{8.4}$$

Theorem 8.2. — *Let $\wp \subset \Lambda_c\{\{x\}\}$ be a prime ideal as above of height s and let $\varphi_1, \dots, \varphi_l \in \Lambda_c\{\{x\}\}$. After performing a linear transformation of $\mathbb{R}^n$, there exist $\delta \in \Lambda - \{0\}$, $\psi \in \Lambda_c\{\{x'\}\}_\delta$ and an integer, N, such that $\Lambda_c\{\{x\}\}_\delta$ contains $\varphi_1, \dots, \varphi_l$, and a system of generators of $\wp$. And for each $\gamma \in SM\Lambda_\delta$ we have:*

i) $ht(\wp^\delta_\gamma + \psi_\gamma \mathbb{R}\{x\}) > ht(\wp^\delta_\gamma)$,

ii) *For each connected component, $\mathcal{C}$, of*

$$V(\wp^\delta_\gamma) \cap \{x / \psi_\gamma(x') \prod_{i=1}^{l} \varphi_{i,\gamma}(x) \neq 0\}$$

there exists a germ of an analytic curve $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ *such that, for all* t *sufficiently small,* $\xi(t) \in \mathcal{C}$ *and the multiplicities of* $\psi_\gamma(\xi(t)), \varphi_{1,\gamma}(\xi(t)), \ldots, \varphi_{l,\gamma}(\xi(t))$ *are less than* N*.*

iii) *If* $\eta : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ *is a germ of an analytic curve, we denote by* $\eta^* : \mathbb{R}\{x\} \to \mathbb{R}\{t\}$ *the induced homomorphism of rings. Suppose* $\eta^*(\wp_\gamma^\delta) \subset t^{2N+1}\mathbb{R}\{t\}$*, and the multiplicity of* $\psi_\gamma(\eta(t))$ *is less than* N*. Then there exists another germ of analytic curve* $\overline{\xi} : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ *with* $\overline{\xi}(t) \in V(\wp_\gamma^\delta)$*, for all* t *sufficiently small, and* $\overline{\xi}(t) - \eta(t) \in t^N(\mathbb{R}\{t\})^n$*.*

Proof. — We divide each φ_j, $j = 1, \ldots, l$, successively, by $P_2, \ldots, P_s$. Then there exists $\varphi'_j \in \Lambda_c\{\{x_1, \ldots, x_{n-s+1}\}\}[x_{n-s+2}, \ldots, x_n]$, $deg\varphi'_j \leqslant \nu$, such that:

$$\varphi_j = \varphi'_j\,,\ mod(P_2, \ldots, P_s).$$

By (8.1), we see that, for each $j=1, \ldots, l$, there exists $\psi_j \in \Lambda_c\{\{x_1, \ldots, x_{n-s+1}\}\}$ such that:

$$\Delta^{s\nu}\varphi'_j = \psi_j \quad mod(\wp). \tag{8.5}$$

After performing the division of each ψ_j, $j = 1, \ldots, l$, by P_1, we can suppose that $\psi_j \in \Lambda_c\{\{x_1, \ldots, x_{n-s}\}\}[x_{n-s+1}]$, $deg\psi_j < degP_1$.

For each $j = 1, \ldots, l$, let $\Delta_j \in \Lambda_c\{\{x'\}\}$ be the discriminant of the $Red\psi_j$ and put

$$\psi = \Delta \prod_{j=1}^{l} \Delta_j \in \Lambda_c\{\{x'\}\}.$$

There exist $\delta \in \Lambda - \{0\}$ such that:

$$P_1, \ldots, P_s, \varphi_1, \ldots, \varphi_l, \psi_1, \ldots, \psi_l, R_1, \ldots, R_s \in \Lambda_c\{\{x\}\}_\delta$$

By lemma 6.1, there exists $N_1 \in \mathbb{N}$ such that, for each $\gamma \in SM\Lambda_\delta$, the integer associated to:

$$P_{1,\gamma}, \ldots, P_{s,\gamma}, \varphi_{1,\gamma}, \ldots, \varphi_{l,\gamma}, \psi_{1,\gamma}, \ldots, \psi_{l,\gamma}, R_{1,\gamma}, \ldots, R_{s,\gamma}$$

is less than N_1.

We choose δ such that $\Lambda_c\{\{x\}\}_\delta$ contains a system of generators of the ideal $\wp$. Since $\psi \notin \wp$, we have $ht(\wp + \psi\Lambda_c\{\{x\}\}) > ht(\wp)$. By Proposition 5.2, we can also choose δ such that, for all $\gamma \in SM\Lambda_\delta$, $ht(\wp_\gamma^\delta + \psi_\gamma\mathbb{R}\{x\}) > ht(\wp_\gamma^\delta)$, this proves i).

To show ii) and iii) we can assume that $\wp = P_1\Lambda_c\{\{x\}\}$, this is based on (8.4). We can also suppose that $\varphi_j \in \Lambda_c\{\{x_1, \ldots, x_{n-s}\}\}_\delta[x_{n-s+1}]$, $j = 1, \ldots, l$.

Let $\gamma \in SM\Lambda_\delta$ and let $\mathcal{C}$ be a connected component of:

$$V(P_{1,\gamma}) \cap \{x \,/\, \psi_\gamma(x') \prod_{i=1}^{l} \varphi_{i,\gamma}(x) \neq 0\}.$$

There exists a connected component $\mathcal{C}'$ of $\{x'/\psi_\gamma(x') \neq 0\}$ such that the polynomial $P_{1,\gamma}$ has, for each $x' \in \mathcal{C}'$, a fixed number, say $p_\gamma \geqslant 1$, of real distinct roots $r_1(x') < \ldots < r_{p_\gamma}(x')$, and $\mathcal{C}$ countains the germ, at the origin in $\mathbb{R}^n$, of the set:

$$\{(x', x_{n-s+1}) \,/\, x' \in \mathcal{C}' \,\text{and}\, x_{n-s+1} = r_\mu(x')\},$$

where $1 \leqslant \mu \leqslant r_\gamma$.
By Proposition 7.2, there exists a germ of an analytic curve:

$$\xi_1 : (\mathbb{R}, 0) \to (\mathbb{R}^{n-1}, 0),$$

such that $\xi_1(t) \in \mathcal{C}'$ for all t in a neighborhood of $0 \in \mathbb{R}$, and the contact with ψ_γ is less then N_1. Put

$$\xi(t) = (\xi_1(t), r_\mu(\xi_1(t))).$$

We can see that $\xi(t) \in \mathcal{C}$ for all t in a neighborhood of $0 \in \mathbb{R}$. Let us look for the multiplicity of each $\varphi_j(\xi(t)), j = 1, \ldots, l$.
For each $j = 1, \ldots, l$, $Red(\varphi_j)$ admits on $\mathcal{C}'$ a constant number, say α_j, of distinct real roots (possibly $\alpha_j = 0$): $r_{j,1,\gamma} <, \ldots, < r_{j,\alpha_j,\gamma}$. We put $r_{j,0,\gamma}(x') = -\infty$ and $r_{j,\alpha_j+1,\gamma}(x') = +\infty$. By definition of N_1, for all $j = 1, \ldots, l$, the multiplicity of $\Delta_j(\xi_1(t))$ is less than N_1. Since $\mathcal{C}$ is situated between two sheets of the zeros of $Red(\varphi_j)$, we can see, like in the proof of the Proposition 7.2, that the multiplicity of $\varphi_j(\xi(t))$ is less than $N := N_1 deg(P_1)$. Hence we obtain ii).

We put $N' = 2N+1$. Let $\eta : (\mathbb{R}, 0) \to (\mathbb{R}^s \times \mathbb{R}, 0)$ be a germ of an analytic curve, $\eta(t) = (\eta_1(t), \eta'(t)) \in \mathbb{R}^s \times \mathbb{R}$. We suppose that the multiplicity, say μ, of $P_{1,\gamma}((\eta_1(t), \eta'(t)))$ is greater than N' and the multiplicity of $\psi_\gamma(\eta_1(t))$ is less than N. We denote by μ_1 the multiplicity of $\frac{\partial P_{1,\gamma}(\eta_1(t),\eta'(t))}{\partial x_{n-s+1}}$, clearly $\mu_1 \leqslant N$.
We denote by τ an auxiliary variable and we put

$$f(t, \tau) := P_{1,\gamma}(\eta_1(t), \eta'(t) + \tau) \in \mathbb{R}\{t, \tau\}.$$

We consider the two ideals:

$$I = \tau^{\mu_1}\mathbb{R}\{\tau\}, \quad I' = \tau^{\mu - 2\mu_1}\mathbb{R}\{\tau\}.$$

Since the multiplicity of $f(t,0) = P_{1,\gamma}((\eta_1(t), \eta'(t))$ is μ, we have

$$f(t,0) \in I^2.I'.$$

We can then use the generalized implicit function theorem [5, III.3.2] to the function:

$$\tau \to f(t,\tau) := P_{1,\gamma}(\eta_1(t), \eta'(t) + \tau),$$

and the ideals:

$$I = \tau^{\mu_1}\mathbb{R}\{\tau\}, \quad \text{and} \quad I' = \tau^{\mu-2\mu_1}\mathbb{R}\{\tau\}.$$

By this theorem, there exists a germ of an analytic curve, $\tau \to \alpha(\tau)$, such that

$$\alpha(\tau) \in II', \quad \text{and} \quad P_{1,\gamma}(\eta_1(t), \eta'(t) + \alpha(\tau)) = 0$$

for all τ in a neighborhood of $0 \in \mathbb{R}$. Put:

$$\overline{\xi}(t) = (\eta_1(t), \eta'(t) + \alpha(t)).$$

Since $\alpha(t) \in II'$ and $\mu - \mu_1 \geqslant N$, we see that $\overline{\xi}(t) - \eta(t) \in t^N\mathbb{R}\{t\}^{s+1}$. □

9. The set associated to an ideal

Let us introduce some definitions and notations which we shall use. The reader is advised to recall the notation and definitions of 8.1.

For each $i = 1, \ldots, n$, we put:

$$\zeta_i(t) = \sum_{j=1}^{2N+1} Y_{i,j} t^j,$$

where $(Y_{i,j})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant 2N+1}$ are variables. We put:

$$\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t)).$$

In order to standardize the notation, we put $\varphi_0 = P_1$ and $\varphi_{l+1} = \psi$. For each $q = 0, 1, \ldots, l, l+1$, we have:

$$\varphi_q(\zeta(t)) = \sum_{m \in \mathbb{N}} L_{q,m} t^m,$$

with $L_{q,m} \in \Lambda_\delta[Y_{i,j}]$.

Put $\Gamma = X - V(\delta)$. For each $\mu \in \mathbb{N}$, and $q \in \{1, \ldots, l\}$, we put:

$$A_{q,\mu} = \{(x, c_{i,j}) \in \Gamma \times \mathbb{R}^{n(2N+1)} \,/\, L_{q,\nu}(x, c_{i,j}) = 0, \\ \forall \nu \leqslant \mu - 1,\, L_{q,\mu}(x, c_{i,j}) > 0\},$$

and if $\mu_1, \ldots, \mu_l \in \mathbb{N}$, we put:

$$A_{\mu_1 \ldots \mu_l} = \bigcap_{q=1}^{l} A_{q,\mu_i}$$

finely we put:

$$A_+ = \bigcup_{\mu_1,\ldots,\mu_l \leqslant N} A_{\mu_1 \ldots \mu_l}.$$

Let:

$$A_0 = \{(x, c_{i,j}) \in \Gamma \times \mathbb{R}^{n(2N+1)} \,/\, L_{0,\mu}(x, c_{i,j}) = 0, \ \ \forall \mu \in \{1, \ldots, 2N+1\}\},$$

and

$$A_{l+1} = \bigcup_{\mu=1}^{N} \{(x, c_{i,j}) \in \Gamma \times \mathbb{R}^{n(2N+1)} \,/\, L_{l+1,\mu}(x, c_{i,j}) \neq 0\}.$$

Definition 9.1. — *With the same notation as above, We put $A = A_+ \cap A_{l+1} \cap A_0 \subset (X - V(\delta)) \times \mathbb{R}^{n(2N+1)}$. The set A is called the set associated to the ideal $\wp$ and the family $\varphi_1, \ldots, \varphi_l$.*

Let us remark that the set $A \subset \mathbb{R}^n \times \mathbb{R}^{n(2N+1)}$ is $\mathcal{O}(\mathbb{R}^n)[Y_1, \ldots, Y_{n(2N+1)}]$-semi-analytic.
We put $\pi_{n,n(2N+1)} : \mathbb{R}^n \times \mathbb{R}^{n(2N+1)} \to \mathbb{R}^n$ the canonical projection.

Remark 9.2. — Since $A \subset \mathbb{R}^n \times \mathbb{R}^{n(2N+1)}$ is $\mathcal{O}(\mathbb{R}^n)[Y_1, \ldots, Y_{n(2N+1)}]$-semi-analytic i.e. semialgebraic with respect to the parameters of arcs, by using the generalized version of Tarski's theorem [4], see also [6, ch2, corollary 2.9], we see that $\pi_{n,n(2N+1)}(A) \subset \mathbb{R}^n$ is also $\mathcal{O}$-semianalytic.

10. The closure of an $\mathcal{O}(\mathbb{R}^n)$- semi-analytic set.

Let $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ be a subalgebra $\mathbb{R}^n \times \mathbb{R}^m$-noetherian. If $X \subset (\mathbb{R}^n \times \mathbb{R}^m)(\mathcal{O})$ is a closed irreducible set, we put $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)}{I(X)}$. We have then an homomorphism:

$$H : \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) \to \Lambda_c\{\{x\}\},$$

which associates to each $\varphi \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ the element $\sum_{\omega \in \mathbb{N}^n} \frac{D^{(\omega,0)}\varphi}{\omega!} x^\omega$.
For each $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\varphi \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$, we denote by φ_a^b the germ of the function $x \to \varphi(x, b)$ at $a \in \mathbb{R}^n$: $\sum_{\omega \in \mathbb{N}^n} \frac{D^{(\omega,0)}\varphi(a,b)}{\omega!} (x - a)^\omega$

Let $J \subset \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ be an ideal, the family $(J_a^b)_{(a,b) \in \mathbb{R}^n \times \mathbb{R}^m}$ is a noetherian family, see example 4.2, ii). We denote by $I \subset \Lambda_c\{\{x\}\}$ the ideal generated by the image of J by the homomorphism H. For each $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ we have then $J_a^b = I_{(a,b)}$. We keep the notations of section 8.

THEOREM 10.1. — *Let $\varphi_1, \dots, \varphi_l \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ and let $\mathcal{F} = (J_a^b)_{(a,b)\in\mathbb{R}^n\times\mathbb{R}^m}$ be the family as above. We denote by $B \subset \mathbb{R}^n \times \mathbb{R}^m$ be the set of all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ such that there exists a germ of an analytic curve $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, a)$ with $\xi(t) \in V(J_a^b) \cap \{x \,/\, \varphi_{1,a}^b(x) > 0, \dots, \varphi_{l,a}^b(x) > 0\}$ for all $t > 0$, sufficiently small. Then B is an $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$-semi-analytic set.*

Proof. — Let $X \subset (\mathbb{R}^n \times \mathbb{R}^m)(\mathcal{O})$ be a closed irreducible set. Since $(\mathbb{R}^n \times \mathbb{R}^m)(\mathcal{O})$ is a noetherian topological space, it is enough to prove that there is a closed subset $F \subset X$, $F \neq X$, such that $B \cap (X - F)$ is an $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$-semi-analytic.
We put $\Lambda = \frac{\mathcal{O}(\mathbb{R}^n\times\mathbb{R}^m)}{I(X)}$ and $s = ht(I)$. By proposition 5.2, there exists $\delta \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) - I(X)$ such that, $\forall (a,b) \in X - V(\delta)$, $ht(I_{(a,b)}^\delta) = s$.
We proceed by induction on the height s. The case $s = n$ is clear, since in this case, we have $V(I_{(a,b)}) = \{(a,b)\}$ and then $B \cap X - V(\delta) = \{(a,b) \in X - V(\delta) / \varphi_1(a,b) > 0, \dots, \varphi_l(a,b) > 0\}$. We suppose that the result is true for all integers p such that $s < p \leqslant n$ and will show it for s. Let $\wp_1, \dots, \wp_q$ the minimal prime ideals associated to the ideal I. By 4.3.1, iv) and Proposition 5.2, there exists $\delta \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) - I(X)$, such that $\forall (a,b) \in X - V(\delta)$:

$$\sqrt{I_{(a,b)}^\delta} = \wp_{1,(a,b)}^\delta \cap \dots \cap \wp_{q,(a,b)}^\delta \quad \text{and} \quad ht(I_{(a,b)}^\delta) = s.$$

We have then, $\forall (a,b) \in X - V(\delta)$:

$$V(J_a^b) = V(\sqrt{I_{(a,b)}^\delta}) = V(\wp_{1,(a,b)}^\delta) \cup \dots \cup V(\wp_{q,(a,b)}^\delta).$$

Let $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, a)$ be a germ of an analytic curve such that $\xi(t) \in V(J_a^b)$ for all t small in a neighborhood of $0 \in \mathbb{R}$. Then there exists i, $1 \leqslant i \leqslant q$, such that $\xi(t) \in V(\wp_{i,(a,b)}^\delta)$ for all t small in a neighborhood of $0 \in \mathbb{R}$. If not, we can choose for each i, $1 \leqslant i \leqslant q$ an element $h_i \in \wp_{i,(a,b)}^\delta$ with $h_i(\xi(t)) \neq 0$. Put $h = h_1 \dots h_q$. Then $h \in \wp_{1,(a,b)}^\delta \cap \dots \cap \wp_{q,(a,b)}^\delta$, and $h(\xi(t)) \neq 0$, which is a contradiction. Therefore, we can suppose that there exist a prime ideal $\wp \in \Lambda_c\{\{x\}\}$, $\delta \in \Lambda - \{0\}$ such that, for each $(a,b) \in X - V(\delta)$, $\wp_{(a,b)}^\delta = \sqrt{J_a^b}$. We apply theorem 8.2 to the ideal $\wp$ and the image of $\varphi_1, \dots, \varphi_l$ by the homomorphism H, which we denote also by $\varphi_1, \dots, \varphi_l$.
let $\delta \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) - I(X)$, $\psi \in [\frac{\mathcal{O}(\mathbb{R}^n\times\mathbb{R}^m)}{I(X)}]_c\{\{x\}\}_\delta$ and $N \in \mathbb{N}$ given by theorem 8.2. We consider the set $A \subset X - V(\delta) \times \mathbb{R}^{n(2N+1)}$ associated to the ideal $\wp$ and $\varphi_1, \dots, \varphi_l$. We put $F = X \cap V(\delta)$. We shall prove that $(X - F) \cap B$ is an $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$-semi-analytic set.

Let Γ_1 [respectively Γ_2] the set of all $(a,b) \in X - F$ such that there exists a germ of an analytic curve, $\xi : (\mathbb{R},0) \to (\mathbb{R}^n, a)$, with $\xi(t) \in V(\wp^\delta_{(a,b)} + \psi^b_a \mathbb{R}\{x\}) \cap \{x/\varphi^b_{1,a}(x) > 0, \dots, \varphi^b_{l,a}(x) > 0\}$ [respectively $\xi(t) \in V(\wp^\delta_{a,b)}) \cap \{x/\psi^b_a(x) \neq 0, \varphi^b_{1,a}(x) > 0, \dots, \varphi^b_{l,a}(x) > 0\}$] for all $t > 0$ sufficiently small. We see that $(X - F) \cap B = \Gamma_1 \cup \Gamma_2$.

By the induction hypothesis, the set Γ_1 is $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$-semi-analytic. Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n(2N+1)} \to \mathbb{R}^n \times \mathbb{R}^m$ be the canonical projection. We shall prove that $\Gamma_2 = \pi(A)$. It is clear that $\Gamma_2 \subset \pi(A)$. Let $(a_1, \dots, a_n, b_1, \dots, b_m) = (a,b) \in \pi(A)$, then there exists $(c_{ij}) \in \mathbb{R}^{n(2N+1)}$ such that $(a, b, c_{ij}) \in A = A_+ \cap A_{l+1} \cap A_0$.
We put:

$$\gamma_i(t) = a_i + \sum_{j=1}^{2N+1} c_{ij} t^j \quad \text{and} \quad \%[\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)).$$

Since $(a, b, c_{ij}) \in A = A_+ \cap A_{l+1} \cap A_0 \subset A_{l+1} \cap A_0$, we have $P^b_{1a}(\gamma(t)) \in t^{2N+1}\mathbb{R}\{t\}$, and the multiplicity of $\psi^b_a(\gamma(t))$ is less than N. By property (iii) of theorem 8.2, we can find a germ of an analytic curve $\xi : (\mathbb{R},0) \to (\mathbb{R}^n, a)$, with $\xi(t) \in V(\wp^\delta_{(a,b)})$ for all t small, and $\xi(t) - \gamma(t) \in t^N \mathbb{R}\{t\}$. We have, for each $j = 1, \dots, l$,

$$\varphi^b_{ja}(\gamma(t)) = \sum_{m \geqslant \mu_i} L_{jm}(a, b, c_{c_{lk}}) t^m$$

and $L_{j\mu_i}(a, b, c_{lk}) > 0$.
Since $(a, b, c_{lk}) \in A_+ = \bigcup_{\mu_1, \dots, \mu_l \leqslant N} A_{\mu_1 \dots \mu_l}$ and $\xi(t) - \gamma(t) \in t^N \mathbb{R}\{t\}$ we deduce that, for each $j = 1, \dots, l$, $\varphi^b_{ja}(\gamma(t))$ and $\varphi^b_{ja}(\xi(t))$ have the same first non null term, hence the result. □

Corollary 10.2. — *Let $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ be a subalgebra $\mathbb{R}^n \times \mathbb{R}^m$-noetherian. Let $Y \subset \mathbb{R}^n \times \mathbb{R}^m$ be a $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$- semianalytic set. We denote by $\overline{Y}_{\mathbb{R}^m}$ the set $\bigcup_{b \in \mathbb{R}^m} \overline{Y_b}$, where $\overline{Y_b}$ is the closure, with respect of the euclidian topology, of the fiber $Y_b = Y \cap \mathbb{R}^n \times \{b\}$. Then $\overline{Y}_{\mathbb{R}^m}$ is a $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$- semianalytic set.*

Proof. — It is enough to prove the corollary in the case where the set Y is of the form:

$$Y = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m / \varphi_0(x,y) = 0, \varphi_1(x,y) > 0, \dots, \varphi_l(x,y) > 0\},$$

where $\varphi_0, \varphi_1, \dots, \varphi_l \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$.
We put $J = (\varphi_0)\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$ the ideal generated by the function φ_0. We denote by $B \subset \mathbb{R}^n \times \mathbb{R}^m$ the set of all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

there exists a germ of an analytic curve $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, a)$ with $\xi(t) \in V(\varphi^b_{0,a}) \cap \{x \,/\, \varphi^b_{1,a}(x) > 0, \ldots, \varphi^b_{l,a}(x) > 0\}$ for all $t > 0$, sufficiently small. By theorem 10.1, B is an $\mathcal{O}(\mathbb{R}^n \times \mathbb{R}^m)$-semi-analytic set. It's remain to show that $B = \overline{Y}_{\mathbb{R}^m}$.

If $(a, b) \in B$ then there exists a germ of an analytic curve $\xi : (\mathbb{R}, 0) \to (\mathbb{R}^n, a)$ as above. We see then that $(\xi(t), b) \in Y_b$, for all $t > 0$, hence $(a, b) \in \overline{Y_b}$. The other inclusion is a consequence of the curve selecting lemma. □

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