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LANDAU-GINZBURG MODELS IN REAL MIRROR SYMMETRY

by Johannes WALCHER

ABSTRACT. — In recent years, mirror symmetry for open strings has exhibited some new connections between symplectic and enumerative geometry (A-model) and complex algebraic geometry (B-model) that in a sense lie between classical and homological mirror symmetry. I review the rôle played in this story by matrix factorizations and the Calabi-Yau/Landau-Ginzburg correspondence.

RÉSUMÉ. — Récemment, la symétrie miroir pour les cordes ouvertes a dévoilé de nouveaux liens entre la géométrie symplectique et énumérative (modèle A) et la géométrie algébrique complexe (modèle B) qui en un certain sens se situent entre la symétrie miroir classique et sa version homologique. On résume ici le rôle que jouent dans cette histoire les factorisations matricielles et la correspondance Calabi-Yau/Landau-Ginzburg.

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1. Introduction

We begin with an overview of the (pre-)history of the line of research reported here.

Mirror symmetry rose to prominence around 1990 after a computation by Candelas, de la Ossa, Green and Parkes [6]. These authors used the conjectural equivalence of $\mathcal{N} = 2$ superconformal field theories attached to mirror pairs (X, Y) of Calabi-Yau manifolds (as well as some other information about the use of these manifolds for compactifying string theory) to make a prediction about the number of rational curves on a generic

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quintic threefold. That prediction was expressed in terms of the periods of an assumed mirror manifold previously constructed by Greene and Plesser [13].

The mathematical theory relevant to the computations on the mirror manifold (“B-model”) was rapidly understood to be related to classical Hodge theory [27]. Based on the development of Gromov-Witten theory, the enumerative predictions (“A-model”) were verified over subsequent years, culminating in the proof of the now classical “mirror theorems” [23, 11, 25], which may be informally stated as

$$\text{Gromov-Witten theory of } X \xleftarrow{\text{solved by}} \text{Hodge theory of } Y \quad (1.1)$$

Somewhat more precisely, the right-hand side is only the genus 0 theory, and the right-hand side is the variation of Hodge structure associated with a *family* $\mathcal{Y} \rightarrow B$ of Calabi-Yau threefolds. Formulated in this way, the correspondence (1.1) sounds more asymmetrical than necessary, because the parameter space B appears to be absent on the left hand side. As a remedy, physicists have reasoned from (before) the beginning that the complete correspondence of superconformal field theories in fact implies an identification of B (and all geometric structure attached to it) with a “complexified Kähler moduli space”, suitably constructed from the genus 0 Gromov-Witten theory of X . The correspondence would become even more symmetrical by including the reverse correspondence in which the roles of X and Y are exchanged.

Part of the problem with the larger correspondence is that, even assuming it exists, Gromov-Witten theory controls only an infinitesimal neighborhood of a special point in B , (the so-called large complex structure limit), and the construction of B appears to convey no mathematically useable information away from that point. Recent developments, however, have begun to improve that situation. Specifically, mathematical theories have emerged that fit in with the physicists’ notion of a mirror correspondence away from the large complex structure limit.

That, of course, is one of the central themes of this workshop: the Calabi-Yau/Landau-Ginzburg correspondence. Originally proposed in [14], it was fully developed in [43] into a relationship between definitions of physical theories involving non-trivial geometries (on the Calabi-Yau side) and non-trivial interactions (on the Landau-Ginzburg side). The sense of this relationship is an analytical continuation of certain physical observables depending on a suitable set of complex parameters (or coupling constants).

In the simplest example, X is a hypersurface of degree d in projective space \mathbb{P}^{d-1} , defined as the vanishing locus of a homogeneous polynomial V of degree d . Let us write the correspondence as

$$\begin{array}{c} \text{non-linear sigma-model on } X = \{V = 0\} \subset \mathbb{P}^4 \\ \updownarrow \text{ related to} \\ \text{Landau-Ginzburg orbifold } V/\Gamma \end{array} \tag{1.2}$$

where “orbifold” refers to the version equivariant with respect to the group $\Gamma \cong \mathbb{Z}/d$ that acts multiplicatively on the d variables entering V .

It should be clear that the correspondence (1.2), although also coming from physics, is very different in character from (1.1): while the former exchanges symplectic and complex structures, the latter remains “on one side” of mirror symmetry. There are by now two well-developed, and interrelated ways, in which (1.2) has been given mathematical meaning. The first of those is homological in nature, and will play an important role momentarily. The second is based on the enumerative theory of singularities developed by Fan-Jarvis-Ruan [9], and can also be elevated to a mirror theorem [7].

$$\text{FJR theory of } V/\Gamma \xleftarrow{\text{solved by}} \text{Hodge theory of } Y \tag{1.3}$$

Putting together (1.1), (1.2), and (1.3) provides the A-model at least at two points (and their neighborhoods) in the parameter space B that is implicit on the right hand side.

If everything makes sense, of course, the (individual member of the family on the) right hand side would also be subject to the Calabi-Yau/Landau-Ginzburg correspondence. In fact the physics proof of mirror symmetry given in 2000 by K. Hori and C. Vafa [17] (see [28] for earlier work) is stated as an equivalence

$$\text{Sigma-model on } X \xleftarrow{\text{dual to}} \text{Landau-Ginzburg orbifold } W/\Delta \tag{1.4}$$

Note that (1.4) is a combination of mirror symmetry and Calabi-Yau/Landau-Ginzburg correspondence. (To depict the full situation would require a third dimension in which acts the Calabi-Yau/Landau-Ginzburg correspondence for Y .) In particular, it exchanges symplectic and complex structures. This should be strictly kept in mind in the examples, in which W also turns out to be a homogeneous polynomial of degree d . The only clear distinction between (1.2) and (1.4) then comes from the fact that the group $\Delta \cong \ker((\mathbb{Z}/d)^d \rightarrow \mathbb{Z}/d)$ is larger than $\Gamma \cong \mathbb{Z}/d$.

We now turn to the categorical framework. In 1994, M. Kontsevich proposed to understand the mathematical origin of mirror symmetry from an

underlying equivalence of triangulated categories [24],

$$\mathrm{Fuk}(X) \cong D^b(Y) \tag{1.5}$$

It has become popular to think of mirror symmetry in these terms, although it is perhaps not entirely clear how exactly one would recover (1.1) from (1.5). The two basic ideas to reconstruct the parameter space B is either as the deformation space of the category, or as the “space of stability conditions” (in the sense of Douglas-Bridgeland). For the purpose of this reconstruction, the proposal (1.5) suffers from the same shortcomings discussed below (1.1), that the left hand side is defined only at a particular point in parameter space.

In 2002, M. Kontsevich made a proposal for the category that would underly the correspondence (1.2) (in the B-model) and could also be used on one side of homological mirror symmetry: the category of matrix factorizations [5]. For our purposes,

$$D^b(Y) \cong \mathrm{MF}(W/\Delta) \tag{1.6}$$

This proposal was picked up by D. Orlov [31], and by two groups of physicist [21, 3]. Combining (1.6) with (1.5) would provide a homological version of (1.4).

I became interested in matrix factorizations around that time. Together with K. Hori, I wrote a few papers [19, 18, 20] exploring the possibility of using (1.6) for concrete computations in mirror symmetry with D-branes. (“D-brane” is the word physicists use in this context to refer to an object in one of the triangulated category. The intuition is that computations in Landau-Ginzburg models are much simpler than those in the derived category. But there is another side to that coin.) In particular, we proposed in [18] a mirror conjecture relating the set of 625 real quintics (as objects in the Fukaya category) to a set of 625 matrix factorizations of the mirror Landau-Ginzburg superpotential. This will be reviewed below.

The paper [39], which is of special relevance for this talk, contains three main results. (i) The detailed definition of the right hand side of (1.6), and especially a definition of the important \mathbb{Z} -grading. (ii) An index theorem for matrix factorizations in the framework of (1.6), including a formula for the Chern character. (iii) A discussion of the notion of stability for the category of matrix factorizations, including a formula for the central charge (an additive complex function whose argument is the slope).

The equivalence (1.6) was proven by Orlov [30]. An index theorem for matrix factorizations was also proven by van Straten [37], and more recently by Polishchuk-Vaintrob in [33]. The formula for the central charge

plays a rôle in work by Takahashi [36], and subsequent work. But the definition of a stability condition is not yet complete. For a relevant discussion of geometric invariant theory involving non-reductive group actions, see Doran-Kirwan [8].

The main evidence for the conjecture with Hori [18] emerged after my 2006 paper ref. [41]. My interest in the problem was initially revived after J. Solomon computed the number of real lines on the quintic, see ref. [35]. Along a heuristic route analogous to that originally employed by Candelas et al. [6], Solomon’s result enabled me to predict the number of real rational curves in all degrees. More to the point, the argument involves consideration of holomorphic discs ending on the real locus L inside a quintic X defined over the real numbers, and the full picture includes (1.5) and (1.6) in an essential way.

The enumerative predictions (A-model) were verified first: The joint work with Pandharipande and Solomon [32] uses (a suitable real version of) localization on the moduli space of stable maps, together with Givental equivariant mirror transform. The B-model explanation was found with Morrison [29]: this work shows that the relevant Hodge theoretic object is a normal function attached to an algebraic cycle $C \subset Y$ in the mirror family, by identifying the cycle dual to the real quintic, and calculating the associated normal function. Taking all this together, the results of ref. [41] can now be stated as a real mirror correspondence, generalizing (1.1) ⁽¹⁾

$$\begin{array}{ccc} \text{real Gromov-Witten} & \xleftrightarrow{\text{solved by}} & \text{normal function} \\ \text{theory of } (X, L) & & \text{attached to } (Y, C) \end{array} \quad (1.7)$$

One of the most interesting aspects of this correspondence is that the cycle C is actually *derived* from the matrix factorization that is conjecturally dual, via the equivalence $\text{Fuk}(X) \cong \text{MF}(W/\Delta)$, to the real quintic L (as an object in $\text{Fuk}(X)$). In that sense the theorem (1.7) provides evidence for the conjecture of ref. [18].

In the rest of this talk, I will describe the various ingredient in the open string mirror correspondence (1.7). In the final section, I will describe a few consequences that one might draw for the (possible) rôle of matrix factorizations in mirror symmetry.

Remark 1.1. — As in the review [40], I will here concentrate on progress made on open mirror symmetry on compact Calabi-Yau threefolds. For a review of the progress on non-compact manifolds, see [26]. Landau-Ginzburg

⁽¹⁾ Here, $L \subset X$ is the real quintic, and $C \subset Y$ is an algebraic cycle in the mirror family, see below for the details.

models and matrix factorizations also play a rôle in mirror symmetry for non-Calabi-Yau manifolds, some of which was discussed at other talks in this conference.

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2. A-model

Our interest is concentrated on the quintic Calabi–Yau $X = \{V = 0\} \subset \mathbb{P}^4$, defined as the vanishing locus of a homogeneous degree 5 polynomial V in 5 complex variables x_1, \dots, x_5 . We assume that X is defined over the real numbers, which means that all coefficients of V are real (possibly up to some common phase). The real locus $\{x_i = \bar{x}_i\} \subset X$ is then a Lagrangian submanifold with respect to the standard symplectic structure, and after choosing a flat $U(1)$ connection, will define an object in the (derived) Fukaya category $\text{Fuk}(X)$.

Both the topological type and the homology class in $H_3(X; \mathbb{Z})$ of the real locus depend on the complex structure of X (the choice of (real) polynomial V). On the other hand, the Fukaya category is independent of the choice of V (real or not). The object in $\text{Fuk}(X)$ that we shall refer to as the real quintic is defined from the real locus L of X when V is the Fermat quintic $V = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$. It is not hard to see that topologically, $L \cong \mathbb{R}\mathbb{P}^3$. There are therefore two choices of flat bundles on L , and we will denote the corresponding objects of $\text{Fuk}(X)$ by L_+ and L_- , respectively. More precisely, since $\text{Fuk}(X)$ depends on the choice of a complexified Kähler structure on X , we define L_\pm for some choice of Kähler parameter t close to large volume $\text{Im}(t) \rightarrow \infty$, and then continue it under Kähler deformations. In fact, the rigorous definition of the Fukaya category is at present only known infinitesimally close to this large volume point [10]. However, $\text{Fuk}(X)$ does exist over the entire stringy Kähler moduli space of X , and at least some of the structure varies holomorphically. Our interest here is in the variation of the categorical structure associated with L_\pm over the entire stringy Kähler moduli space of X , identified via mirror symmetry with the complex structure moduli space of the mirror quintic, Y .

The Fermat quintic is invariant under more than one anti-holomorphic involution. If $\mathbb{Z}/5$ denotes the multiplicative group of fifth roots of unity,

we define for $\chi = (\chi_1, \dots, \chi_5) \in (\mathbb{Z}/5)^5$ an anti-holomorphic involution σ_χ of \mathbb{P}^4 by its action on homogeneous coordinates

$$\sigma_\chi : x_i \rightarrow \chi_i \bar{x}_i. \tag{2.1}$$

The Fermat quintic is invariant under any σ_χ . The involution and the fixed point locus only depend on the class of χ in $(\mathbb{Z}/5)^5/(\mathbb{Z}/5) \cong (\mathbb{Z}/5)^4$, and we obtain in this way $5^4 = 625$ (pairs of) objects $L_\pm^{[\chi]}$ in $\text{Fuk}(X)$.

We emphasize again that although we have defined the Lagrangians $L_\pm^{[\chi]}$ as fixed point sets of anti-holomorphic involutions of the Fermat quintic, we can think of the corresponding objects of $\text{Fuk}(X)$ without reference to the complex structure.

3. B-model

Let $V \in \mathbb{C}[x_1, x_2, \dots, x_5]$ be a polynomial. A *matrix factorization* of V is a \mathbb{Z}_2 -graded free $\mathbb{C}[x_1, \dots, x_5]$ -module M equipped with an odd endomorphism $Q : M \rightarrow M$ of square V ,

$$Q^2 = V \cdot \text{id}_M \tag{3.1}$$

In other words, Q is a “curved differential” when acting on M . Since the curvature is central and independent of M , Q^2 vanishes when acting on morphisms of free modules. The category $\text{MF}(V)$ is then the triangulated category of matrix factorizations with morphisms given by Q -closed morphisms of free modules, modulo Q -exact morphisms. Matrix factorizations are well-known objects since the mid '80's, see in particular [5], and it was proposed by Kontsevich that $\text{MF}(V)$ should be a good description of B-type D-branes in a Landau-Ginzburg model based on the worldsheet superpotential V [31, 21, 3]. To apply this to the case of interest, we need a little bit of extra structure.

When V is of degree 5, the so-called homological Calabi-Yau/Landau-Ginzburg correspondence [30] states that the derived category of coherent sheaves of the projective hypersurface $X = \{V = 0\} \subset \mathbb{P}^4$ is equivalent to the graded, equivariant category of matrix factorizations of the corresponding Landau-Ginzburg superpotential,

$$D^b(X) \cong \text{MF}(V/(\mathbb{Z}/5)) \tag{3.2}$$

where $\mathbb{Z}/5$ is the group of 5-th roots of unity acting diagonally on x_1, \dots, x_5 . To pass to the mirror quintic Y by the standard Greene-Plesser construction, we replace V with the one-parameter family of potentials W given

by

$$W = \frac{1}{5}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) - \psi x_1 x_2 x_3 x_4 x_5 \tag{3.3}$$

and enlarge the orbifold group to obtain (1.6)

$$D^b(Y) \cong \text{MF}(W/(\mathbb{Z}/5)^4) \tag{3.4}$$

where $(\mathbb{Z}/5)^4 = \text{Ker}((\mathbb{Z}/5)^5 \rightarrow \mathbb{Z}/5)$ is the subgroup of phase symmetries of W whose product is equal to 1.

To describe an object mirror to the real quintic, we begin with finding a matrix factorization of the one-parameter family of polynomials (3.3). If $S \cong \mathbb{C}^5$ is a 5-dimensional vector space, we can associate to its exterior algebra a $\mathbb{C}[x_1, \dots, x_5]$ -module $M = \wedge^* S \otimes \mathbb{C}[x_1, \dots, x_5]$. It naturally comes with the decomposition

$$M = M_0 + M_1 + M_2 + M_3 + M_4 + M_5, \quad \text{where } M_s = \wedge^s S \otimes \mathbb{C}[x_1, \dots, x_5], \tag{3.5}$$

and the \mathbb{Z}_2 -grading $(-1)^s$. Let η_i ($i = 1, \dots, 5$) be a basis of S and $\bar{\eta}_i$ the dual basis of S^* , both embedded in $\text{End}(M)$. We then define two families of matrix factorizations (M, Q_\pm) of W by

$$Q_\pm = \frac{1}{\sqrt{5}} \sum_{i=1}^5 (x_i^2 \eta_i + x_i^3 \bar{\eta}_i) \pm \sqrt{\psi} \prod_{i=1}^5 (\eta_i - x_i \bar{\eta}_i) \tag{3.6}$$

To check that $Q_\pm^2 = W \cdot \text{id}_M$, one uses that $\eta_i, \bar{\eta}_i$ satisfy the Clifford algebra

$$\{\eta_i, \bar{\eta}_j\} = \delta_{ij} \tag{3.7}$$

as well as the ensuing relations

$$\{(x_i^2 \eta_i + x_i^3 \bar{\eta}_i), (\eta_i - x_i \bar{\eta}_i)\} = 0 \quad \text{and} \quad (\eta_i - x_i \bar{\eta}_i)^2 = -x_i \tag{3.8}$$

The matrix factorization (3.6) is quasi-homogeneous (\mathbb{C}^* -gradable), but we will not need this data explicitly.

Now to specify objects in $\text{MF}(W/\Delta)$, where $\Delta = (\mathbb{Z}/5)^4$ for the mirror quintic, we have to equip M with a representation of Δ such that Q_\pm is equivariant with respect to the action of Δ on the x_i . Since Q_\pm is irreducible, this representation of Δ on M is determined up to a character of Δ by a representation on S , *i.e.*, an action on the η_i . For $\gamma \in \Delta$, we have $\gamma(x_i) = \gamma_i x_i$ for some fifth root of unity γ_i . We then set $\gamma(\eta_i) = \gamma_i^{-2} \eta_i$, making Q_\pm equivariant. As noted, this representation is unique up to an action on M_0 , *i.e.*, a character of Δ .

For the mirror quintic, $\Delta = \text{Ker}((\mathbb{Z}/5)^5 \rightarrow \mathbb{Z}/5)$, so $\Delta^* = (\mathbb{Z}/5)^5/\mathbb{Z}/5$, and we label the characters of Δ as $[\chi]$. The corresponding objects of

$\text{MF}(W/\Delta)$ constructed out of Q_{\pm} (3.6) are classified as $Q_{\pm}^{[\chi]} = (M, Q_{\pm}, \rho_{[\chi]})$, where $\rho_{[\chi]}$ is the representation on M we just described.

By using the explicit algorithm of [15], one may obtain representatives of the matrix factorizations $Q_{\pm}^{[\chi]}$ in $D^b(Y)$, which would be interesting to analyze further. A particularly nice one is the bundle

$$\text{Ker} \begin{pmatrix} M_0(2) & M_1(4) \\ \oplus M_2(1) & \xrightarrow{Q_{\pm}} \oplus M_3(3) \\ \oplus M_4(0) & \oplus M_5(2) \end{pmatrix} \tag{3.9}$$

where $M_s(k) = \wedge^s S \otimes \mathcal{O}_{\mathbb{P}^4}(k)$. See [29] for details.

4. Correspondence

CONJECTURE [18]. — *There is an equivalence of categories $\text{Fuk}(X) \cong \text{MF}(W/(\mathbb{Z}/5)^4)$ which identifies the 625 pairs of objects $L_{\pm}^{[\chi]}$ with the 625 pairs of equivariant matrix factorizations $Q_{\pm}^{[\chi]}$.*

Since (as follows from the calculation of the intersection indices below), the images of the 625 objects in K-theory generate the lattices of topological charges of the respective categories, it is natural to propose that, in an appropriate sense, *the $L_{\pm}^{[\chi]}$ generate $\text{Fuk}(X)$ and the $Q_{\pm}^{[\chi]}$ generate $\text{MF}(W/(\mathbb{Z}/5)^4)$.* In combination with the Conjecture (and the equivalence (1.6)), this would establish homological mirror symmetry for the pair (quintic, mirror quintic). Similar statements should also hold for other Calabi-Yau hypersurfaces or complete intersections, although the details would be different (possibly substantially so).

5. Evidence 1

The first main evidence for the Conjecture is the identity of intersection indices, originally due to [2].

Let us start with the geometric intersection index between⁽²⁾ $L^{[\chi]}$ and $L^{[\chi']}$. Because of the projective equivalence, we have to look at the intersection of the fixed point loci of σ_{χ} and $\sigma_{\omega\chi'}$ from (2.1) where ω runs over the 5 fifth roots of unity. It is not hard to see that topologically

$$\text{Fix}(\sigma_{\chi}) \cap \text{Fix}(\sigma_{\omega\chi'}) \cap X \cong \mathbb{R}\mathbb{P}^{d-2}, \quad \text{where } d = \#\{\chi'_i = \omega\chi_i\}. \tag{5.1}$$

⁽²⁾The intersection index, being topological, does not depend on the Wilson lines on the A-branes. For the B-branes, it is correspondingly independent of the sign of the square root in (3.6).

After making the intersection transverse by a small deformation in the normal direction, we obtain a vanishing contribution for $d = 0, 1, 3, 5$, and ± 1 for $d = 2, 4$, where the sign depends on the non-trivial phase differences $\chi_i^* \omega \chi'_i$. Explicitly, one finds

$$L^{[x]} \cap L^{[x']} = \sum_{\omega \in \mathbb{Z}/5} f_1(\chi'^* \omega \chi), \tag{5.2}$$

where

$$f_1(\chi) = \begin{cases} \prod_{i=1}^5 \operatorname{sgn}(\operatorname{Im}(\chi_i)), & \text{if } \#\{i, \chi_i = 1\} = 2, 4 \\ 0 & \text{else.} \end{cases} \tag{5.3}$$

To compute the intersection index between the matrix factorizations, we use the index theorem of [39]. It says in general

$$\begin{aligned} \chi \operatorname{Hom}((M, Q, \rho), (M', Q', \rho')) &:= \sum_i (-1)^i \dim \operatorname{Hom}^i((M, Q, \rho), (M', Q', \rho')) \\ &= \frac{1}{|\Delta|} \sum_{\gamma \in \Delta} \operatorname{Str}_{M'} \rho'(\gamma)^* \frac{1}{\prod_{i=1}^5 (1 - \gamma_i)} \operatorname{Str}_M \rho(\gamma), \end{aligned} \tag{5.4}$$

where γ_i are the eigenvalues of $\gamma \in \Delta$ acting on the x_i , and ρ, ρ' are the representations of Δ on M . For $M = M', Q = Q'$ and $\rho = \rho_{[x]}, \rho' = \rho_{[x']}$ described above, this evaluates to

$$-\frac{1}{5^4} \sum_{\gamma \in (\mathbb{Z}/5)^4} \chi(\gamma')^* \chi(\gamma) \prod_{i=1}^5 (\gamma_i + \gamma_i^2 - \gamma_i^3 - \gamma_i^4) = - \sum_{\omega \in \mathbb{Z}/5} f_2(\chi'^* \omega \chi), \tag{5.5}$$

where

$$f_2(\chi) = \begin{cases} \prod_{i=1}^5 \operatorname{sgn}(\operatorname{Im}(\chi_i)), & \text{if } \#\{i, \chi_i = 1\} = 0 \\ 0 & \text{else.} \end{cases} \tag{5.6}$$

We do not know any generally valid result from the representation theory of finite cyclic group which shows that (5.2) and (5.5) coincide. It is however not hard to check by hand or computer that for all χ ,

$$\sum_{\omega \in \mathbb{Z}/5} (f_1 + f_2)(\omega \chi) = 0. \tag{5.7}$$

Hence

$$L^{[x]} \cap L^{[x']} = \chi \operatorname{Hom}(Q^{[x]}, Q^{[x']}) \tag{5.8}$$

as claimed. A further computation shows that the rank of the 625×625 dimensional intersection matrix (5.2) is 204, which is equal to the rank of $H_3(X; \mathbb{Z})$, and the determinant is one. So the classes of the $L^{[x]}$ generate the homology, as claimed above.

6. Evidence 2

Consider the endomorphism algebra $\text{Hom}^*(Q, Q)$ of the matrix factorization $Q = Q_+$, as objects in $\text{MF}(W/(\mathbb{Z}/5)^4)$. This algebra is \mathbb{Z} -graded thanks to the homogeneity of W [39]. We also have $\text{Hom}^0(Q, Q) \cong \mathbb{C}$ since Q is irreducible, and this implies $\text{Hom}^3(Q, Q) \cong \mathbb{C}$ by Serre duality. Finally, it is shown in [18] that

$$\text{Hom}^1(Q, Q) = \text{Hom}^2(Q, Q) = \begin{cases} 0 & \psi \neq 0 \\ \mathbb{C} & \psi = 0 \end{cases} \tag{6.1}$$

The appearance of an additional cohomology element in $\text{Hom}^1(Q, Q)$ is another reflection of some results of [2] and was the initial motivation to investigate mirror symmetry for the real quintic.

To interpret (6.1) in the A-model, we recall that the morphism algebra of objects in the Fukaya category is defined using Lagrangian intersection Floer homology [10]. For the endomorphism algebra of a single Lagrangian, Floer homology is essentially a deformation of ordinary Morse homology by holomorphic disks.

For example, consider the real quintic $L \cong \mathbb{RP}^3$. Think of \mathbb{RP}^3 as S^3/\mathbb{Z}_2 , and embed the S^3 in $\mathbb{R}^4 \ni (y_0, \dots, y_3)$ as $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$. A standard Morse function for \mathbb{RP}^3 in this presentation is given by $f = y_1^2 + 2y_2^2 + 3y_3^2$ restricted to the S^3 . This Morse function is self-indexing and has one critical point in each degree $i = 0, 1, 2, 3$. The Morse complex takes the form

$$C^0 \xrightarrow{0} C^1 \xrightarrow{\delta} C^2 \xrightarrow{0} C^3 \tag{6.2}$$

Working with integer coefficients, $C^i \cong \mathbb{Z}$ for all i , we have $\delta = 2$, and the complex (6.2) computes the well-known integral cohomology of \mathbb{RP}^3 .

To compute Floer homology of the real quintic, we have to deform (6.2) by holomorphic disks, *i.e.*, $\delta = 2 + \mathcal{O}(e^{-t/2})$. In the standard treatments, such as [10], this requires taking coefficients from a certain formal (Novikov) ring with uncertain convergence properties. In other words, Floer homology is at present only defined in an infinitesimal neighborhood of the large volume point in moduli space (which leads to the often heard remark that $\text{HF}^*(L, L)$ is isomorphic to $H^*(L)$). It is however natural to expect that we may in fact analytically continue (6.2) to the opposite end of moduli space, $\psi = 0$, where it is (conjecturally) identified with the deformation complex of Q . In particular, we conjecture $\delta(\psi = 0) = 0$ so as to reproduce (6.1).

7. “Proof”

The original construction of mirror symmetry by Greene and Plesser [13] exploited the fact that at $\psi = 0$, we may reduce the equivalence between the Fermat quintic $\{\sum x_i^5 = 0\}$ and the mirror quintic at $\psi = 0$, the LG orbifold $\sum x_i^5/(\mathbb{Z}/5)^4$, to the equivalence between “minimal models”

$$\text{Landau-Ginzburg model } x^5 \xleftarrow{\text{mirror to}} \text{Landau-Ginzburg orbifold } x^5/(\mathbb{Z}/5) \tag{7.1}$$

This equivalence (which, ultimately, is a special case of Arnold’s strange duality) has been explained at various levels in the literature, although perhaps not in sufficient detail to provide a proof of mirror symmetry along the following lines, initially envisioned by Greene and Plesser: One should first quotient by a diagonal $\mathbb{Z}/5$ action on the fifth tensor power of (7.1), to obtain

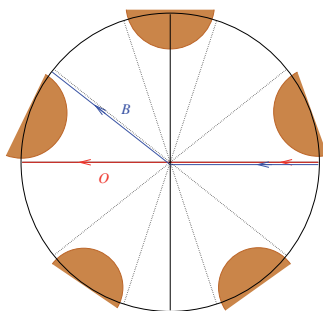
$$(\text{LG orbifold } \sum x_i^5)/(\mathbb{Z}/5) \xleftarrow{\text{mirror to}} (\text{LG orbifold } \sum x_i^5)/(\mathbb{Z}/5)^4 \tag{7.2}$$

Then one would study the deformation theory of the relevant symplectic/algebraic structures on the two sides, and show that they coincide. Particular finite deformations would implement the Calabi-Yau/Landau-Ginzburg correspondence, thereby connecting (7.2) to the various equivalences listed in the introduction.

The conjecture of ref. [18] was very much informed by the construction that we just sketched. At the homological level (in physicists’ language, at the level of D-branes), the correspondence (7.1) identifies the vanishing cycles of opening angle $4\pi/5$ depicted in the figure (and viewed as objects of the Fukaya category of the LG model x^5) with the building blocks of (3.6) at $\psi = 0$: these are simply the factorizations

$$x^5 = x^2 \cdot x^3 \tag{7.3}$$

equipped with a representation of $\mathbb{Z}/5$, as objects of $\text{MF}(x^5/(\mathbb{Z}/5))$ on the right hand side of (7.1). To give slightly more details, let us denote the factorization (7.3) with representation $m \in (\mathbb{Z}/5)^*$ by B_m , and the vanishing cycle of opening angle $4\pi/5$ and first leg oriented in the direction $2\pi/5 \cdot m$ by A_m . One may then check that $\text{Hom}(B_m, B_{m'}) \cong A_m \cap A_{m'}$,



where \cap' denotes the (asymmetrically transversalized) intersection in which the second cycle is slightly tilted into the shaded regions. Taking cones over these morphisms can then be seen to generate the entire categories: on the right hand side, factorizations $x^5 = x \cdot x^4$, and on the left hand side, cycles with opening angle $2\pi/5$.

To exploit (7.1) for the purposes of real mirror symmetry, *i.e.*, the conjecture of [18], one would like to identify the real locus⁽³⁾ of the Fermat quintic in the left hand side of (7.2). The naive guess that such a submanifold should come from the real slice in each of the building blocks in (7.1) cannot be quite correct, because that real slice is not on the list of admissible objects: as explained in [16] the image under x^5 should lie on the positive real axis.

The naive guess is not very far off the mark, however. First of all, among all possible vanishing cycles, those with opening angle $4\pi/5$ are the closest to the real slices. Actually, as explained in more detail in ref. [4], the real slices and the vanishing cycles have the same intersection number, in other words they are equivalent in homology. (This is indicated by the shaded regions in the figure, which are the pre-images of the upper half plane under $x \mapsto x^5$. The point is that the integrals of $\exp(ix^5)$ along contours which asymptote to these shaded regions are convergent.)

Moreover, the actual real slices play a central role in the context of “orientifolds”. Orientifolds are an important string theory construction that at the homological level amount simply to equipping the D-brane category (in either symplectic or algebraic incarnation) with a contravariant involution. It can be seen that on the symplectic side (in either Calabi-Yau or Landau-Ginzburg realization) the involution of the category comes with an anti-holomorphic involution of the underlying geometry. In the Calabi-Yau picture, the real locus then plays two distinguished roles—one as “orientifold plane”, and one as a D-brane (an object of the underlying category). When passing to the Landau-Ginzburg phase, the two roles would seem to be taken by somewhat different players (the O and B in the figure). The deviation should be viewed as a consequence of the different corrections from holomorphic maps: D-branes see holomorphic discs ending on them, and orientifold planes holomorphic crosscaps (maps $\mathbb{P}^1 \rightarrow X$ that are equivariant with respect to the antipodal map on \mathbb{P}^1 and anti-holomorphic involution on X). The Landau-Ginzburg analogue of these holomorphic maps in the framework of Fan-Jarvis-Ruan theory [9] should also have a bearing on this problem.

(3) with respect to one of the 625 possible involutions

8. Main evidence

The main evidence for the above conjecture is the enumeration of holomorphic disks using mirror symmetry [41, 32, 29].

We introduce the generating function of open Gromov-Witten invariants for the pair $(X, L) = (\text{quintic}, \text{real quintic})$ defined in [35].

$$T_A(q) = \frac{\log q}{2} + \frac{1}{4} + \sum_{d \text{ odd}} \tilde{n}_d q^{d/2} \tag{8.1}$$

where d indexes the degree of the holomorphic disk. One may compute the \tilde{n}_d similarly to the ordinary ($g = 0$) Gromov-Witten invariants by localization on the moduli space of maps from the disk to \mathbb{P}^4 . We then pull back $T_A(q)$ under the standard mirror map

$$q = q(z) = \exp(\varpi_1(z)/\varpi_0(z)) \tag{8.2}$$

where ϖ_0 and ϖ_1 are the power series and first logarithmic around $z = 0$ solutions of the Picard-Fuchs differential operator

$$\mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad \theta = \frac{d}{dz}, \quad z = (5\psi)^{-5} \tag{8.3}$$

Namely, we define

$$T_B(z) = \varpi_0(z)T_A(q(z)) \tag{8.4}$$

Then the main result of [32] is the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}T_B(z) = \frac{15}{8}\sqrt{z} \tag{8.5}$$

As shown in [29], one can identify $T_B(z)$ as a particular *truncated normal function*⁽⁴⁾ associated with the algebraic cycle $C_+ - C_-$, where

$$C_{\pm} = \{x_1 + x_2 = 0, x_3 + x_4 = 0, x_5^2 \pm \sqrt{5\psi}x_1x_3 = 0\} \subset \{W = 0\} \tag{8.6}$$

are two families of curves in the mirror quintic. Namely, we have

$$\mathcal{T}_B = \mathcal{T}_B(z) = \int_{\mathcal{S}} \hat{\Omega} \tag{8.7}$$

where \mathcal{S} is a particular three-chain bounding $C_+ - C_-$, and $\hat{\Omega}$ is a particular choice of holomorphic three-form on Y , defined as Poincaré residue by the formula

$$\hat{\Omega} = \left(\frac{5}{2\pi i}\right)^3 \psi \text{Res}_{W=0} \frac{\alpha}{W} \tag{8.8}$$

⁽⁴⁾ We recall the definition in the next section.

where α is the four-form on projective space

$$\alpha = \sum_i (-1)^{i-1} x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_5 \tag{8.9}$$

The choice of holomorphic three-form in (8.8) is precisely the one for which the Picard-Fuchs operator of the mirror quintic takes the form (8.3).

So finally, the equivalence (8.4) is evidence for the Conjecture because the cycles C_{\pm} provide representatives of the second algebraic Chern class of our matrix factorizations Q_{\pm} . In general, Grothendieck’s theory of Chern classes provides a map

$$c_i^{\text{alg}} : D^b(Y) \rightarrow \text{CH}^i(Y) \tag{8.10}$$

from the derived category of coherent sheaves to the Chow groups of algebraic cycles modulo rational equivalence. Composing this map on one side with the equivalence $D^b(Y) \cong \text{MF}(W/(\mathbb{Z}/5)^4)$, and on the other with the Abel-Jacobi map, we obtain the construction of a truncated normal function starting from a virtual matrix factorization of zero topological charge. In particular [29]

$$c_2^{\text{alg}}(Q_+) - c_2^{\text{alg}}(Q_-) = [C_+ - C_-] \in \text{CH}^2(Y) \tag{8.11}$$

9. Consequences, Infinitesimal invariant

What are the lessons of all of this for the geometry and physics of Landau-Ginzburg models?

First of all, there are a few loose ends to tie up in what is known already. For example, what is the Landau-Ginzburg version of the Chow group of algebraic cycles that we used at the end of the previous section? (This could merely involve writing out the abstract K-theoretic definition, see *e.g.*, [34, 1]. But for calculations, a more hands-on approach would be desirable.)

Secondly, it seems quite unfortunate that one has to go through a fairly long chain of correspondences, and a somewhat delicate geometric computation to arrive at the rather simple inhomogeneity in (8.5). In [38], I show that in fact there does exist a shortcut, along the following lines.

Recall that the variation of Hodge structure associated with the family of mirror quintics $Y \rightarrow B$ is concerned with the variation of the *Hodge decomposition*

$$H^3(Y; \mathbb{C}) = H^{3,0}(Y) \oplus H^{2,1}(Y) \oplus H^{1,2}(Y) \oplus H^{0,3}(Y) \tag{9.1}$$

of the third cohomology group $H^3(Y; \mathbb{C}) \cong H^3(Y; \mathbb{Z}) \otimes \mathbb{C}$ with $z = (5\psi)^{-5} \in B$. After forming the *Hodge filtration*

$$F^p H^3(Y) = \bigoplus_{p' \geq p} H^{p', 3-p'}(Y) \tag{9.2}$$

we may write the important condition of Griffiths transversality of the VHS as

$$\nabla F^p H^3(Y) \subset F^{p-1} H^3(Y) \otimes \Omega_B \tag{9.3}$$

where ∇ is the flat (“Gauss-Manin”) connection originating from the local triviality of $H^3(Y; \mathbb{Z})$ over the moduli space B .

Moreover, we have the *Griffiths intermediate Jacobian fibration* which is the fibration $J^3(Y) \rightarrow B$ of complex tori

$$J^3(Y) = \frac{H^3(Y)}{F^2 H^3(Y) \oplus H^3(Y; \mathbb{Z})} \cong (F^2 H^3(Y))^* / H_3(Y; \mathbb{Z}) \tag{9.4}$$

Then, a *Poincaré normal function* of the variation of Hodge structure is a holomorphic section ν of $J^3(Y)$ satisfying *Griffiths transversality for normal functions*

$$\nabla \tilde{\nu} \in F^1 H^3(Y) \otimes \Omega_B \tag{9.5}$$

where $\tilde{\nu}$ is an arbitrary lift of ν from $J^3(Y)$ to $H^3(Y)$ (the condition (9.5) does not depend on the lift).

Finally, we need the notion of the *infinitesimal invariant of a normal function* (see [12] for details). In our context (co-dimension two cycles on Calabi-Yau threefolds), this invariant can be viewed as an analogue of the well-known Griffiths-Yukawa coupling. We recall that this coupling, let us denote it as κ , is a section of $(F^3 H^3)^{-2} \otimes \text{Sym}^3 \Omega_B \rightarrow B$ arising from the third iterate of the differential period mapping,

$$H^1(TY) \rightarrow \bigoplus \text{Hom}(H^{p,q}(Y), H^{p-1,q+1}(Y)). \tag{9.6}$$

and captures the invariant information about the infinitesimal variation of Hodge structure.

Given a normal function, its infinitesimal invariant can be obtained *by choosing a lift $\tilde{\nu}$ of ν from $J^3(Y)$ to $H^3(Y)$, contracting $\nabla \tilde{\nu}$ with $F^2 H^3(Y) \otimes \Omega_B$, and using (9.5) to obtain a section δ of $(F^3 H^3)^{-1} \otimes \text{Sym}^2 \Omega_B \rightarrow B$. This invariant captures the infinitesimal variation of mixed Hodge structure associated with the normal function, and therefore contains the same information as the inhomogeneous Picard-Fuchs equation. Note however that in contrast to (9.5), this definition of δ does depend on the lift.* [The

usual definition of the infinitesimal invariant takes the class of δ in H^1 of the Koszul complex

$$F^2H^3 \rightarrow F^1H^3 \otimes \Omega_B \rightarrow F^0H^3 \otimes \Omega_B^2, \tag{9.7}$$

which is independent of the lift.]

We are now nearing the punchline of the shortcut to (8.5). By contraction with your choice Ω of section of F^3H^3 , the Griffiths-Yukawa coupling is usually written as

$$\kappa = \int_Y \Omega \wedge \nabla^3 \Omega \tag{9.8}$$

which has a ‘‘Landau-Ginzburg’’ equivalent as a residue (written here up to normalization)

$$\kappa_{ijk} = \text{Res} \frac{\partial_i W \partial_j W \partial_k W}{(dW)^5}. \tag{9.9}$$

To write a similar formula for the infinitesimal invariant, we have to specify a lift of the normal function. I have studied two distinguished lifts. The first is the so-called *real lift* [42], which exploits the (non-holomorphic) splitting (9.1) of the Hodge filtration, and in which the infinitesimal invariant satisfies the interesting equation

$$\bar{\partial}_k \delta_{ij} = -\kappa_{ij}^{\bar{l}} \bar{\delta}_{l\bar{k}} \tag{9.10}$$

The other possibility is what I want to call the ‘‘Landau-Ginzburg lift’’, in which we have a formula similar to (9.9)

$$\delta_{ij} = \text{Res} \frac{\text{Str}(\partial_i W \partial_j Q (dQ)^5)}{(dW)^5} \tag{9.11}$$

in terms of the residue for matrix factorizations introduced in [22].

To finish up, I think it is fair to say that the Landau-Ginzburg formulation of D-brane categories has served very well to develop intuition, but that several quantitative details are still better understood in the geometric description. This is true in A- and B-model, and among physicists and mathematicians. All can benefit from filling in the details of the correspondence.

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Johannes WALCHER
McGill University,
Montréal,
Canada
CERN Physics Department, Theory Division
Geneva, Switzerland
johannes.walcher@mcgill.ca