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A SIMPLER PROOF OF TOROIDALIZATION OF MORPHISMS FROM 3-FOLDS TO SURFACES

by Steven Dale CUTKOSKY

Abstract. — We give a simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces, over an algebraically closed field of characteristic zero. A toroidalization is obtained by performing sequences of blow ups of nonsingular subvarieties above the domain and range, to make a morphism toroidal. The original proof of toroidalization of morphisms of 3-folds to surfaces is much more complicated.

Résumé. — On présente une démonstration plus simple et plus conceptuelle de la toroïdalisation des morphismes des variétés de dimension trois vers les surfaces, sur un corps algébriquement clos de caractéristique zéro. On obtient la toroïdalisation par une série d'éclatements de sous-variétés non singulières au-dessus de la source et de l'image, afin d'obtenir un morphisme torique. La démonstration originale de la toroïdalisation des morphismes des variétés de dimension trois vers les surfaces était beaucoup compliquée.

1. Introduction

Let $\mathfrak{k}$ be an algebraically closed field of characteristic zero. If X is a nonsingular variety, then the choice of a simple normal crossings divisor (SNC divisor) on X makes X into a toroidal variety.

Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular $\mathfrak{k}$-varieties, and there is a SNC divisor D_Y on Y such that $D_X = \Phi^{-1}(D_Y)$ is a SNC divisor on X. Then Φ is torodial (with respect to D_Y and D_X) if and only if $\Phi^*(\Omega^1_Y(\log D_Y))$ is a subbundle of $\Omega^1_X(\log D_X)$ (Lemma 1.5 [11]). A toroidal morphism can be expressed locally by monomials. All of the cases are written down for toroidal morphisms from a 3-fold to a surface in Lemma 19.3 [11].

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The toroidalization problem is to determine, given a dominant morphism $f : X \to Y$ of $\mathfrak{k}$-varieties, if there exists a commutative diagram

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\to} & Y_1 \\ \Phi\downarrow & & \downarrow\Psi \\ X & \stackrel{f}{\to} & Y \end{array}$$

such that Φ and Ψ are products of blow ups of nonsingular subvarieties, X_1 and Y_1 are nonsingular, and there exist SNC divisors D_{Y_1} on Y_1 and $D_{X_1} = f^*(D_{Y_1})$ on X_1 such that f_1 is toroidal (with respect to D_{X_1} and D_{Y_1}).

The toroidalization problem does not have a positive answer in positive characteristic p, even for maps of curves; $t = x^p + x^{p+1}$ gives a simple example.

In characteristic zero, the toroidalization problem has an affirmative answer if Y is a curve and X has arbitrary dimension; this is really embedded resolution of hypersurface singularities, so follows from resolution of singularities [24] (some of the simplified proofs are [5], [4] [12], [19] and [22]). Toroidalization is proven for morphisms from a 3-fold to a surface in [11] and for the case of a 3-fold to a 3-fold in [14]. Detailed history and references on the toroidalization problem are given in the introductions to [11] and [14].

We consider the problem of toroidalization as a resolution of singularities type problem. When the dimension of the base is larger than one, the problem shares many of the complexities of resolution of vector fields ([30], [6], [28]) and of resolution of singularities in positive characteristic (some references are [1], [2], [25], [7], [8], [9], [3], [15], [18], [23], [20], [21], [26], [27], [31]). In particular, natural invariants do not have a "hypersurface of maximal contact" and are sometimes not upper semicontinuous.

Toroidalization, locally along a fixed valuation, is proven in all dimensions and relative dimensions in [10] and [13].

The proof of toroidalization of a dominant morphism from a 3-fold to a surface given in [11] consists of 2 steps.

The first step is to prove "strong preparation". Suppose that X is a nonsingular variety, S is a nonsingular surface with a SNC divisor D_S, and $f : X \to S$ is a dominant morphism such that $D_X = f^{-1}(D_S)$ is a SNC divisor on X which contains the locus where f is not smooth. f is strongly prepared if $f^*(\Omega^2_S(\log D_S)) = \mathcal{I}\mathcal{M}$ where $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf, and $\mathcal{M}$ is a subbundle of $\Omega^2_X(\log D_X)$ (Lemma 1.7 [11]). A strongly prepared morphism has nice local forms which are close to being toroidal (page 7 of [11]).

Strong preparation is the construction of a commutative diagram

$$\begin{array}{ccc} X_1 & & \\ \downarrow & \searrow & \\ X & \xrightarrow{f} & S \end{array}$$

where S is a nonsingular surface with a SNC divisor D_S such that $D_X = f^*(D_S)$ is a SNC divisor on the nonsingular variety X which contains the locus where f is not smooth, the vertical arrow is a product of blow ups of nonsingular subvarieties so that $X_1 \to S$ is strongly prepared. Strong preparation of morphisms from 3-folds to surfaces is proven in Theorem 17.3 of [11].

The second step is to prove that a strongly prepared morphism from a 3-fold to a surface can be toroidalized. This is proven in Sections 18 and 19 of [11].

This second step is generalized in [16] to prove that a strongly prepared morphism from an n-fold to a surface can be toroidalized. Thus to prove toroidalization of a morphism from an n-fold to a surface, it suffices to proof strong preparation.

The proof of strong preparation in [11] is extremely complicated, and does not readily generalize to higher dimensions. The proof of this result occupies 170 pages of [11]. We mention that that the main invariant considered in this paper, ν, can be interpreted as the adopted order of Section 1.2 of [6] of the 2-form $du \wedge dv$.

In this paper, we give a significantly simpler and more conceptual proof of strong preparation of morphisms of 3-folds to surfaces. It is our hope that this proof can be extended to prove strong preparation for morphisms of n-folds to surfaces, for $n > 3$. The proof is built around a new upper semicontinuous invariant σ_D, whose value is a natural number or ∞. if $\sigma_D(p) = 0$ for all $p \in X$, then $X \to S$ is prepared (which is slightly stronger than being strongly prepared). A first step towards obtaining a reduction in σ_D is to make X 3-prepared, which is achieved in Section 3. This is a nicer local form, which is proved by making a local reduction to lower dimension. The proof proceeds by performing a toroidal morphism above X to obtain that X is 3-prepared at all points except for a finite number of 1-points. Then general curves through these points lying on D_X are blown up to achieve 3-preparation everywhere on X. if X is 3-prepared at a point p, then there exists an étale cover U_p of an affine neighborhood of p and a local toroidal structure $\overline{D}_p$ at p (which contains D_X) such that there exists a projective toroidal morphism $\Psi : U' \to U_p$ such that σ_D has dropped everywhere above p (Section 4). The final step of the proof is to make these local constructions algebraic, and to patch them. This is accomplished in Section 5. In Section 6 we state and prove strong preparation for morphisms of 3-folds to surfaces (Theorem 6.1) and toroidalization of morphisms from 3-folds to surfaces (Theorem 6.2). Important definitions along the way are:

prepared, Definition 2.4,

1-prepared, Definition 2.1,

2-prepared, after the proof of Proposition 2.7,

3-prepared, Definition 3.3.

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2. The invariant σ_D, 1-preparation and 2-preparation.

For the duration of the paper, $\mathfrak{k}$ will be an algebraically closed field of characteristic zero. A $\mathfrak{k}$-variety is an integral quasi projective $\mathfrak{k}$-scheme. We will write curve (over $\mathfrak{k}$) to mean a 1-dimensional $\mathfrak{k}$-variety, and similarly for surfaces and 3-folds. We will assume that varieties are quasi-projective. This is not really a restriction, by the fact that after a sequence of blow ups of nonsingular subvarieties, all varieties satisfy this condition. By a general point of a $\mathfrak{k}$-variety Z, we will mean a member of a nontrivial open subset of Z on which some specified good condition holds. When we say that "p is a point of X" or "$p \in X$" we will mean that p is a closed point, unless we indicate otherwise (for instance, by saying that "p is a generic point of a subvariety Y of X").

A reduced divisor D on a nonsingular variety Z of dimension n is a simple normal crossings divisor (SNC divisor) if all irreducible components of D are nonsingular, and if $p \in Z$, then there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ such that $x_1x_2\cdots x_r = 0$ is a local equation of D at p, where $r \leqslant n$ is the number of irreducible components of D containing p. Two nonsingular subvarieties X and Y intersect transversally at $p \in X \cap Y$ if there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ and subsets $I, J \subset \{1, \ldots, n\}$ such that $\mathcal{I}_{X,p} = (x_i \mid i \in I)$ and $\mathcal{I}_{Y,p} = (x_j \mid j \in J)$.

Definition 2.1. — *Let S be a nonsingular surface over $\mathfrak{k}$ with a reduced SNC divisor D_S. Suppose that X is a nonsingular 3-fold, and $f : X \to S$ is a dominant morphism. X is 1-prepared (with respect to f) if $D_X =$*

$f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth, and if C_1, C_2 are the two components of D_S whose intersection is nonempty, T_1 is a component of X dominating C_1 and T_2 is a component of D_X which dominates C_2, then T_1 and T_2 are disjoint.

The following lemma is an easy consequence of the main theorem on resolution of singularities.

Lemma 2.2. — *Suppose that $g : Y \to T$ is a dominant morphism of a 3-fold over $\mathfrak{k}$ to a surface over $\mathfrak{k}$ and D_T is a 1-cycle on T such that $g^{-1}(D_R)$ contains the locus where g is not smooth. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} Y_1 & \overset{g_1}{\to} & T_1 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ Y & \overset{g}{\to} & T \end{array}$$

such that the vertical arrows are products of blow ups of nonsingular subvarieties contained in the preimage of D_T, Y_1 and T_1 are nonsingular and $D_{T_1} = \pi_1^{-1}(D_T)$ is a SNC divisor on T_1 such that Y_1 is 1-prepared with respect to g_1.

For the duration of this paper, S will be a fixed nonsingular surface over $\mathfrak{k}$, with a (reduced) SNC divisor D_S. To simplify notation, we will often write D to denote D_X, if $f : X \to S$ is 1-prepared.

Suppose that X is 1-prepared with respect to $f : X \to S$. A nonsingular curve C of X which is contained in D_X makes SNCs with D_X if either C is a 2-curve, or C contains no 3-points, and if $q \in C$ is a 2-point, then there are regular parameters x, y, z in the local ring $\mathcal{O}_{X,q}$ such that $xy = 0$ is a local equation of D_X at q, and $x = z = 0$ are local equations of C at q.

A permissible blow up of X is the blow up $\pi_1 : X_1 \to X$ of a point of D_X or a nonsingular curve contained in D_X which makes SNCs with D_X. Then $D_{X_1} = \pi_1^{-1}(D_X)_{red} = (f \circ \pi_1)^{-1}(X_S)_{red}$ is a SNC divisor on X_1 and X_1 is 1-prepared with respect to $f \circ \pi_1$. A permissible curve is a curve which satisfies these conditions (so its blow up is permissible).

Assume that X is 1-prepared with respect to D. We will say that $p \in X$ is a n-point (for D) if p is on exactly n components of D. Suppose $q \in D_S$ and u, v are regular parameters in $\mathcal{O}_{S,q}$ such that either $u = 0$ is a local equation of D_S at q or $uv = 0$ is a local equation of D_S at q. u, v are called permissible parameters at q.

For $p \in f^{-1}(q)$, we have regular parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that

1) If p is a 1-point,

$$u = x^a, v = P(x) + x^b F \tag{2.1}$$

where $x=0$ is a local equation of D, $x \not| F$ and x^bF has no terms which are a power of x.

2) If p is a 2-point, after possibly interchanging u and v,

$$u=(x^ay^b)^l, v=P(x^ay^b)+x^cy^dF \tag{2.2}$$

where $xy=0$ is a local equation of D, $a,b>0$, $\gcd(a,b)=1$, $x,y \not| F$ and x^cy^dF has no terms which are a power of x^ay^b.

3) If p is a 3-point, after possibly interchanging u and v,

$$u=(x^ay^bz^c)^l, v=P(x^ay^bz^c)+x^dy^ez^fF \tag{2.3}$$

where $xyz=0$ is a local equation of D, $a,b,c>0$, $\gcd(a,b,c)=1$, $x,y,z \not| F$ and $x^dy^ez^fF$ has no terms which are a power of $x^ay^bz^c$.

regular parameters x,y,z in $\hat{\mathcal{O}}_{X,p}$ giving forms (2.1), (2.2) or (2.3) are called permissible parameters at p for u,v.

Suppose that X is 1-prepared. We define an ideal sheaf

$$\mathcal{I}= \text{ fitting ideal sheaf of the image of } f^*:\Omega^2_S\to\Omega^2_X(\log(D))$$

in $\mathcal{O}_X$. $\mathcal{I}=\mathcal{O}_X(-G)\overline{\mathcal{I}}$ where G is an effective divisor supported on D and $\overline{\mathcal{I}}$ has height $\geqslant 2$.

Suppose that $E_1,\ldots,E_n$ are the irreducible components of D. For $p\in X$, define

$$\sigma_D(p)=\mathrm{order}_{\mathcal{O}_{X,p}/(\sum_{p\in E_i}\mathcal{I}_{E_i,p})}\overline{\mathcal{I}_p}\left(\mathcal{O}_{X,p}/\sum_{p\in E_i}\mathcal{I}_{E_i,p}\right)\in\mathbb{N}\cup\{\infty\}.$$

Lemma 2.3. — *σ_D is upper semicontinuous in the Zariski topology of the scheme X.*

Proof. — For a fixed subset $J\subset\{1,2,\ldots,n\}$, we have that the function

$$\mathrm{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J}\mathcal{I}_{E_i,p})}\overline{\mathcal{I}_p}\left(\mathcal{O}_{X,p}/\sum_{i\in J}\mathcal{I}_{E_i,p}\right)$$

is upper semicontinuous, and if $J\subset J'\subset\{1,2,\ldots,n\}$. we have that

$$\mathrm{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J}\mathcal{I}_{E_i,p})}\overline{\mathcal{I}_p}\left(\mathcal{O}_{X,p}/\sum_{i\in J}\mathcal{I}_{E_i,p}\right)$$
$$\leqslant \mathrm{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J'}\mathcal{I}_{E_i,p})}\overline{\mathcal{I}_p}\left(\mathcal{O}_{X,p}/\sum_{i\in J'}\mathcal{I}_{E_i,p}\right).$$

□

Thus for $r \in \mathbb{N} \cup \{\infty\}$,

$$\mathrm{Sing}_r(X) = \{p \in X \mid \sigma_D(p) \geqslant r\}$$

is a closed subset of X, which is supported on D and has dimension $\leqslant 1$ if $r > 0$.

Definition 2.4. — *A point $p \in X$ is prepared if $\sigma_D(p) = 0$.*

We have that $\sigma_D(p) = 0$ if and only if $\overline{\mathcal{I}}_p = \mathcal{O}_{X,p}$. Further,

$$\mathrm{Sing}_1(X) = \{p \in X \mid \overline{\mathcal{I}}_p \neq \mathcal{O}_{X,p}\}.$$

If $p \in X$ is a 1-point with an expression (2.1) we have

$$(\overline{\mathcal{I}}_p + (x))\hat{\mathcal{O}}_{X,p} = (x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}). \tag{2.4}$$

If $p \in X$ is a 2-point with an expression (2.2) we have

$$(\overline{\mathcal{I}}_p + (x,y))\hat{\mathcal{O}}_{X,p} = (x, y, (ad-bc)F, \frac{\partial F}{\partial z}). \tag{2.5}$$

If $p \in X$ is a 3-point with an expression (2.3) we have

$$(\overline{\mathcal{I}}_p + (x,y,z))\hat{\mathcal{O}}_{X,p} = (x, y, z, (ae-bd)F, (af-cd)F, (bf-ce)F). \tag{2.6}$$

If $p \in X$ is a 1-point with an expression (2.1), then

$$\sigma_D(p) = \mathrm{ord}\ F(0,y,z) - 1.$$

We have $0 \leqslant \sigma_D(p) < \infty$ if p is a 1-point. If $p \in X$ is a 2-point, we have

$$\sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,z) = 0 \text{ (in this case, } ad-bc \neq 0) \\ \mathrm{ord}\ F(0,0,z) - 1 & \text{if } 1 \leqslant \text{ord } F(0,0,z) < \infty \\ \infty & \text{if ord } F(0,0,z) = \infty. \end{cases}$$

If $p \in X$ is a 3-point, let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

we have

$$\sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,0) = 0 \text{ (in this case, rank}(A) = 2) \\ \infty & \text{if ord } F(0,0,0) = \infty. \end{cases}$$

Lemma 2.5. — *Suppose that X is 1-prepared and $\pi_1 : X_1 \to X$ is a toroidal morphism with respect to D. Then X_1 is 1-prepared and $\sigma_D(p_1) \leqslant \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.*

Proof. — Suppose that $p \in X$ is a 2-point and $p_1 \in \pi_1^{-1}(p)$. Then there exist permissible parameters x, y, z at p giving an expression (2.2). In $\hat{\mathcal{O}}_{X_1,p_1}$, there are regular parameters x_1, y_1, z where

$$x = x_1^{a_{11}}(y_1+\alpha)^{a_{12}},\ y = x_1^{a_{21}}(y_1+\alpha)^{a_{22}} \tag{2.7}$$

with $\alpha \in \mathfrak{k}$ and $a_{11}a_{22} - a_{12}a_{22} = \pm 1$. If $\alpha = 0$, so that p_1 is a 2-point, then x_1, y_1, z are permissible parameters at p_1 and substitution of (2.7) into (2.2) gives an expression of the form (2.2) at p_1, showing that $\sigma_D(p_1) \leqslant \sigma_D(p)$. If $\alpha \neq 0 \in \mathfrak{k}$, so that p_1 is a 1-point, set $\lambda = \frac{aa_{12}+ba_{22}}{aa_{11}+ba_{21}}$ and $\overline{x}_1 = x_1(y_1+\alpha)^{\lambda}$. Then $\overline{x}_1, y_1, z$ are permissible parameters at p_1. Substitution into (2.2) leads to a form (2.1) with $\sigma_D(p_1) \leqslant \sigma_D(p)$.

If $p \in X$ is a 3-point and $\sigma_D(p) \neq \infty$, then $\sigma_D(p) = 0$ so that p is prepared. Thus there exist permissible parameters x, y, z at p giving an expression (2.3) with $F = 1$. Suppose that $p_1 \in \pi_1^{-1}(p)$. In $\hat{\mathcal{O}}_{X_1,p_1}$ there are regular parameters x_1, y_1, z_1 such that

$$\begin{aligned} x &= (x_1+\alpha)^{a_{11}}(y_1+\beta)^{a_{12}}(z_1+\gamma)^{a_{13}}\\ y &= (x_1+\alpha)^{a_{21}}(y_1+\beta)^{a_{22}}(z_1+\gamma)^{a_{23}}\\ z &= (x_1+\alpha)^{a_{31}}(y_1+\beta)^{a_{32}}(z_1+\gamma)^{a_{33}} \end{aligned} \tag{2.8}$$

where at least one of $\alpha, \beta, \gamma \in \mathfrak{k}$ is zero. Substituting into (2.3), we find permissible parameters at p_1 giving a prepared form. □

Suppose that X is 1-prepared with respect to $f : X \to S$. Define

$$\Gamma_D(X) = \max\{\sigma_D(p) \mid p \in X\}.$$

Lemma 2.6. — *Suppose that X is 1-prepared and C is a 2-curve of D and there exists $p \in C$ such that $\sigma_D(p) < \infty$. Then $\sigma_D(q) = 0$ at the generic point q of C.*

Proof. — If p is a 3-point then $\sigma_D(p) = 0$ and the lemma follows from upper semicontinuity of σ_D.

Suppose that p is a 2-point. If $\sigma_D(p) = 0$ then the lemma follows from upper semicontinuity of σ_D, so suppose that $0 < \sigma_D(p) < \infty$. There exist permissible parameters x, y, z at p giving a form (2.2), such that x, y, z are uniformizing parameters on an étale cover U of an affine neighborhood of p. Thus for α in a Zariski open subset of $\mathfrak{k}$, $x, y, \overline{z} = z - \alpha$ are permissible parameters at a 2-point $\overline{p}$ of C. After possibly replacing U with a smaller neighborhood of p, we have

$$\frac{\partial F}{\partial z} = \frac{1}{x^c y^d}\frac{\partial v}{\partial z} \in \Gamma(U, \mathcal{O}_X)$$

and $\frac{\partial F}{\partial z}(0,0,z) \neq 0$. Thus there exists a 2-point $\overline{p} \in C$ with permissible parameters $x, y, \overline{z} = z - \alpha$ such that $\frac{\partial F}{\partial z}(0,0,\alpha) \neq 0$, and thus there is an expression (2.2) at $\overline{p}$

$$
\begin{aligned}
u &= (x^a y^b)^l \\
v &= P_1(x^a y^b) + x^c y^d F_1(x, y, \overline{z})
\end{aligned}
$$

with ord $F_1(0,0,\overline{z}) = 0$ or 1, so that $\sigma_D(\overline{p}) = 0$. By upper semicontinuity of σ_D, $\sigma_D(q) = 0$. ☐

Proposition 2.7. — *Suppose that X is 1-prepared with respect to $f : X \to S$. Then there exists a toroidal morphism $\pi_1 : X_1 \to X$ with respect to D, such that π_1 is a sequence of blow ups of 2-curves and 3-points, and*

1) *$\sigma_D(p) < \infty$ for all $p \in D_{X_1}$.*
2) *X_1 is prepared (with respect to $f_1 = f \circ \pi_1 : X_1 \to S$) at all 3-points and the generic point of all 2-curves of D_{X_1}.*

Proof. — By upper semicontinuity of σ_D, Lemma 2.6 and Lemma 2.5, we must show that if $p \in X$ is a 3-point with $\sigma_D(p) = \infty$ then there exists a toroidal morphism $\pi_1 : X_1 \to X$ such that $\sigma_D(p_1) = 0$ for all 3-points $p_1 \in \pi_1^{-1}(p)$ and if $p \in X$ is a 2-point with $\sigma_D(p) = \infty$ then there exists a toroidal morphism $\pi_1 : X_1 \to X$ such that $\sigma_D(p_1) < \infty$ for all 2-points $p_1 \in \pi_1^{-1}(p)$.

First suppose that p is a 3-point with $\sigma_D(p) = \infty$. Let x, y, z be permissible parameters at p giving a form (2.3). There exist regular parameters $\tilde{x}, \tilde{y}, \tilde{z}$ in $\mathcal{O}_{X,p}$ and unit series $\alpha, \beta, \gamma \in \hat{\mathcal{O}}_{X,p}$ such that $x = \alpha\tilde{x}$, $y = \beta\tilde{y}$, $z = \gamma\tilde{z}$. Write $F = \sum b_{ijk} x^i y^j z^k$ with $b_{ijk} \in \mathfrak{k}$. Let $I = (\tilde{x}^i \tilde{y}^j \tilde{z}^k \mid b_{ijk} \neq 0)$, an ideal in $\mathcal{O}_{X,p}$. Since $\tilde{x}\tilde{y}\tilde{z} = 0$ is a local equation of D at p, there exists a toroidal morphism $\pi_1 : X_1 \to X$ with respect to D such that $I\mathcal{O}_{X_1,p_1}$ is principal for all $p_1 \in \pi_1^{-1}(p)$. At a 3-point $p_1 \in \pi_1^{-1}(p)$, there exist permissible parameters x_1, y_1, z_1 such that

$$
\begin{aligned}
x &= x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}} \\
y &= x_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}} \\
z &= x_1^{a_{31}} y_1^{a_{32}} z_1^{a_{33}}
\end{aligned}
$$

with $\mathrm{Det}(a_{ij}) = \pm 1$. Substituting into (2.3), we obtain an expression (2.3) at p_1, where

$$
\begin{aligned}
u &= (x_1^{a_1} y_1^{b_1} z_1^{c_1})^l \\
v &= P_1(x_1^{a_1} y_1^{b_1} z_1^{c_1}) + x_1^{d_1} y_1^{e_1} z_1^{f_1} F_1
\end{aligned}
$$

where $P_1(x_1^{a_1}y_1^{b_1}z_1^{c_1}) = P(x^ay^bz^c)$ and

$$F(x,y,z) = x_1^{\overline{a}}y_1^{\overline{b}}z_1^{\overline{c}}F_1(x_1,y_1,z_1).$$

with $x_1^{\overline{a}}y_1^{\overline{b}}z_1^{\overline{c}}$ a generator of $I\hat{\mathcal{O}}_{X_1,p_1}$ and $F_1(0,0,0) \neq 0$. Thus $\sigma_D(p_1) = 0$.

Now suppose that p is a 2-point and $\sigma_D(p) = \infty$. There exist permissible parameters x, y, z at p giving a form (2.2). Write $F = \sum a_i(x,y)z^i$, with $a_i(x,y) \in \mathfrak{k}[[x,y]]$ for all i. We necessarily have that no $a_i(x,y)$ is a unit series.

Let I be the ideal $I = (a_i(x,y) \mid i \in \mathbb{N})$ in $\mathfrak{k}[[x,y]]$. There exists a sequence of blow ups of 2-curves $\pi_1 : X_1 \to X$ such that $\hat{\mathcal{O}}_{X_1,p_1}$ is principal at all 2-points $p_1 \in \pi_1^{-1}(p)$. There exist $x_1, y_1 \in \mathcal{O}_{X_1,p_1}$ so that x_1, y_1, z are permissible parameters at p_1, and

$$x = x_1^{a_{11}}y_1^{a_{12}},\ y = x_1^{a_{21}}y_1^{a_{22}}$$

with $a_{11}a_{22} - a_{12}a_{21} = \pm 1$. Let $x_1^{\overline{a}}y_1^{\overline{b}}$ be a generator of $I\mathcal{O}_{T_1,q_1}$. Then $F = x_1^{\overline{a}}y_1^{\overline{b}}F_1(x_1,y_1,z)$ where $F_1(0,0,z) \neq 0$, and we have an expression (2.2) at p_1, where

$$u = (x_1^{a_1}y_1^{b_1})^{l_1}$$
$$v = P_1(x_1^{a_1}y_1^{b_1}) + x_1^{d_1}y_1^{e_1}F_1$$

where $P_1(x_1^{a_1}y_1^{b_1}) = P(x^ay^b)$. Thus $\sigma_D(p_1) < \infty$ and $\sigma_D(q) < \infty$ if q is the generic point of the 2-curve of D_{X_1} containing p_1. $\square$

We will say that X is 2-prepared (with respect to $f : X \to S$) if it satisfies the conclusions of Proposition 2.7. We then have that $\Gamma_D(X) < \infty$.

If X is 2-prepared, we have that $\mathrm{Sing}_1(X)$ is a union of (closed) curves whose generic point is a 1-point and isolated 1-points and 2-points. Further, $\mathrm{Sing}_1(X)$ contains no 3-points.

3. 3-preparation

Lemma 3.1. — *Suppose that X is 2-prepared. Suppose that $p \in X$ is such that $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Then there exist permissible parameters x, y, z at p such that there exist $\tilde{x}, y \in \mathcal{O}_{X,p}$, an étale cover U of an affine neighborhood of p, such that $x, z \in \Gamma(U, \mathcal{O}_X)$ and x, y, z are uniformizing parameters on U, and $x = \gamma\tilde{x}$ for some unit series $\gamma \in \hat{\mathcal{O}}_{X,p}$. We have an expression (2.1) or (2.2), if p is respectively a 1-point or a 2-point, with*

$$F = \tau z^m + a_2(x,y)z^{m-2} + \cdots + a_{m-1}(x,y)z + a_m(x,y) \tag{3.1}$$

where $m \geqslant 2$ and $\tau \in \hat{\mathcal{O}}_{X_1,p} = \mathfrak{k}[[x,y,z]]$ is a unit, and $a_i(x,y) \neq 0$ for $i = m-1$ or $i = m$. Further, if p is a 1-point, then we can choose x, y, z so that $x = y = 0$ is a local equation of a generic curve through p on D.

For all but finitely many points p in the set of 1-points of X, there is an expression (3.1) where

(3.2)

a_i is either zero or has an expression $a_i = \overline{a}_i x^{r_i}$ where $\overline{a}_i$ is a unit and $r_i > 0$ for $2 \leqslant i \leqslant m$, and $a_m = 0$ or $a_m = x^{r_m}\overline{a}_m$ where $r_m > 0$ and ord$(\overline{a}_m(0,y)) = 1$.

Proof. — There exist regular parameters $\tilde{x}, y, \overline{z}$ in $\mathcal{O}_{X,p}$ and a unit $\gamma \in \hat{\mathcal{O}}_{X,p}$ such that $x = \gamma\tilde{x}, y, \overline{z}$ are permissible parameters at p, with $\mathrm{ord}(F(0,0,\overline{z})) = m$. Thus there exists an affine neighborhood Spec(A) of p such that $V = \mathrm{Spec}(R)$, where $R = A[\gamma^{\frac{1}{a}}]$ is an étale cover of Spec(A), $x, y, \overline{z}$ are uniformizing parameters on V, and $u, v \in \Gamma(V, \mathcal{O}_X)$. Differentiating with respect to the uniformizing parameters $x, y, \overline{z}$ in R, set

$$\tilde{z} = \frac{\partial^{m-1} F}{\partial \overline{z}^{m-1}} = \omega(\overline{z} - \varphi(x,y)) \tag{3.3}$$

where $\omega \in \hat{\mathcal{O}}_{X,p}$ is a unit series, and $\varphi(x,y) \in \mathfrak{k}[[x,y]]$ is a nonunit series, by the formal implicit function theorem. Set $z = \overline{z} - \varphi(x,y)$. Since R is normal, after possibly replacing Spec(A) with a smaller affine neighborhood of p,

$$\tilde{z} = \frac{1}{x^b}\frac{\partial^{m-1} v}{\partial \overline{z}^{m-1}} \in R.$$

By Weierstrass preparation for Henselian local rings (Proposition 6.1 [29]), $\varphi(x,y)$ is integral over the local ring $\mathfrak{k}[x,y]_{(x,y)}$. Thus after possibly replacing A with a smaller affine neighborhood of p, there exists an étale cover U of V such that $\varphi(x,y) \in \Gamma(U, \mathcal{O}_X)$, and thus $z \in \Gamma(U, \mathcal{O}_X)$.

Let $G(x,y,z) = F(x,y,\overline{z})$. We have that

$$G = G(x,y,0) + \frac{\partial G}{\partial z}(x,y,0)z + \cdots + \frac{1}{(m-1)!}\frac{\partial^{m-1} G}{\partial z^{m-1}}(x,y,0)z^{m-1} + \frac{1}{m!}\frac{\partial^m G}{\partial z^m}(x,y,0)z^m + \cdots$$

We have

$$\frac{\partial^{m-1} G}{\partial z^{m-1}}(x,y,0) = \frac{\partial^{m-1} F}{\partial \overline{z}^{m-1}}(x,y,\varphi(x,y)) = 0$$

and

$$\frac{\partial^m G}{\partial z^m}(x,y,0) = \frac{\partial^m F}{\partial \overline{z}^m}(x,y,\varphi(x,y))$$

is a unit in $\hat{\mathcal{O}}_{X,p}$. Thus we have the desired form (3.1), but we must still show that $a_m \neq 0$ or $a_{m-1} \neq 0$. If $a_i(x,y) = 0$ for $i = m$ and $i = m-1$, we have that $z^2 \mid F$ in $\hat{\mathcal{O}}_{X,p}$, since $m \geqslant 2$. This implies that the ideal of 2×2 minors

$$I_2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} \subset (z),$$

which implies that $z = 0$ is a component of D which is impossible. Thus either $a_{m-1} \neq 0$ or $a_m \neq 0$.

Suppose that C is a curve in $\mathrm{Sing}_1(X)$ (containing a 1-point) and $p \in C$ is a general point. Let $r = \sigma_D(p)$. Set $m = r + 1$. Let $x, y, \overline{z}$ be permissible parameters at p with $y, \overline{z} \in \mathcal{O}_{X,p}$, which are uniformizing parameters on an étale cover U of an affine neighborhood of p such that $x = \overline{z} = 0$ are local equations of C and we have a form (2.1) at p with

$$F = \tau \overline{z}^m + a_1(x, y)\overline{z}^{m-1} + \cdots + a_m(x, y). \tag{3.4}$$

For α in a Zariski open subset of $\mathfrak{k}$, $x, \overline{y} = y - \alpha, \overline{z}$ are permissible parameters at a point $q \in C \cap U$. For most points q on the curve $C \cap U$, we have that $a_i(x, y) = x^{r_i}\overline{a}_i(x, y)$ where $\overline{a}_i(x, y)$ is a unit or zero for $1 \leqslant i \leqslant m - 1$ in $\hat{\mathcal{O}}_{X,q}$. Since $\sigma_D(p) = r$ at this point, we have that $1 \leqslant r_i$ for all i. We further have that if $a_m \neq 0$, then $a_m = x^{r_m}a'$ where $a' = f(y) + x\Omega$ where $f(y)$ is non constant. Thus

$$0 \neq \frac{\partial a_m}{\partial y}(0, y) = \frac{\partial F}{\partial y}(0, y, 0).$$

After possibly replacing U with a smaller neighborhood of p, we have

$$\frac{\partial F}{\partial y} = \frac{1}{x^b}\frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).$$

Thus $\frac{\partial a_m}{\partial y}(0, \alpha) \neq 0$ for most $\alpha \in \mathfrak{k}$. Since $r > 0$, we have that $r_m > 0$, and thus $r_i > 0$ for all i in (3.4). We have

$$\frac{\partial^{m-1} F}{\partial \overline{z}^{m-1}} = \xi \overline{z} + a_1(x, y),$$

where ξ is a unit series. Comparing the above equation with (3.3), we observe that $\varphi(x, y)$ is a unit series in x and y times $a_1(x, y)$. Thus x divides $\varphi(x, y)$. Setting $z = \overline{z} - \varphi(x, y)$, we obtain an expression (3.1) such that x divides a_i for all i. Now argue as in the analysis of (3.4), after substituting $z = \overline{z} - \varphi(x, y)$, to conclude that there is an expression (3.1), where (3.2) holds at most points $q \in C \cap U$. Thus a form (3.1) and (3.2) holds at all but finitely many 1-points of X. $\square$

Lemma 3.2. — *Suppose that X is 2-prepared, C is a curve in $\mathrm{Sing}_1(X)$ containing a 1-point and p is a general point of C. Let $m = \sigma_D(p) + 1$. Suppose that $\tilde{x}, y \in \mathcal{O}_{X,p}$ are such that $\tilde{x} = 0$ is a local equation of D at p*

and the germ $\tilde{x} = y = 0$ intersects C transversally at p. Then there exists an étale cover U of an affine neighborhood of p and $z \in \Gamma(U, \mathcal{O}_X)$ such that $\tilde{x}, y, z$ give a form (3.1) at p.

Proof. — There exists $\overline{z} \in \mathcal{O}_{X,p}$ such that $\tilde{x}, y, \overline{z}$ are regular parameters in $\mathcal{O}_{X,p}$ and $x = \overline{z} = 0$ is a local equation of C at p. There exists a unit $\gamma \in \hat{\mathcal{O}}_{X,p}$ such that $x = \gamma\tilde{x}, y, \overline{z}$ are permissible parameters at p. We have an expression of the form (2.1),

$$u = x^a, v = P(x) + x^b F$$

at p. Write $F = f(y, \overline{z}) + x\Omega$ in $\hat{\mathcal{O}}_{X,p}$. Let I be the ideal in $\hat{\mathcal{O}}_{X,p}$ generated by x and

$$\{\frac{\partial^{i+j} f}{\partial y^i \partial \overline{z}^j} \mid 1 \leqslant i + j \leqslant m - 1\}.$$

The radical of I is the ideal $(x, \overline{z})$, as $x = \overline{z} = 0$ is a local equation of $\mathrm{Sing}_{m-1}(X)$ at p. Thus $\overline{z}$ divides $\frac{\partial^{i+j}}{\partial y^i \partial \overline{z}^j}$ for $1 \leqslant i + j \leqslant m - 1$ (with $m \geqslant 2$). Expanding

$$f = \sum_{i=0}^{\infty} b_i(y) \overline{z}^i$$

(where $b_0(0) = 0$) we see that $\frac{\partial b_0}{\partial y} = 0$ (so that $b_0(y) = 0$) and $b_i(y) = 0$ for $1 \leqslant i \leqslant m - 1$. Thus $\overline{z}^m$ divides $f(y, \overline{z})$. Since $\sigma_D(p) = m - 1$, we have that $f = \tau \overline{z}^m$ where τ is a unit series. Thus $x, y, \overline{z}$ gives a form (2.1) with $\mathrm{ord}(F(0, 0, \overline{z})) = m$. Now the proof of Lemma 3.1 gives the desired conclusion. □

Let $\omega(m, r_2, \ldots, r_{m-1})$ be a function which associates a positive integer to a positive integer m, natural numbers $r_2, \ldots, r_{m-2}$ and a positive integer r_{m-1}. We will give a precise form of ω after Theorem 4.1.

Definition 3.3. — *X is 3-prepared (with respect to $f : X \to S$) at a point $p \in D$ if $\sigma_D(p) = 0$ or if $\sigma_D(p) > 0$, f is 2-prepared with respect to D at p and there are permissible parameters x, y, z at p such that x, y, z are uniformizing parameters on an étale cover of an affine neighborhood of p and we have one of the following forms, with $m = \sigma_D(p) + 1$:*

1) *p is a 2-point, and we have an expression (2.2) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^{r_m} y^{s_m} \tag{3.5}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero), $r_i + s_i > 0$ whenever $\tau_i \neq 0$ and $(r_m + c)b - (s_m + d)a \neq 0$. Further, $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

2) *p is a 1-point, and we have an expression (2.1) with*

$$F=\tau_0 z^m+\tau_2 x^{r_2} z^{m-2}+\cdots+\tau_{m-1} x^{r_{m-1}} z+\tau_m x^{r_m} \tag{3.6}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leqslant i \leqslant m-1$, $\tau_m \in \hat{\mathcal{O}}_{X,p}$ and $\operatorname{ord}(\tau_m(0,y,0))=1$ (or $\tau_m=0$). Further, $r_i>0$ if $\tau_i \neq 0$, and $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

3) *p is a 1-point, and we have an expression (2.1) with*

$$F=\tau_0 z^m+\tau_2 x^{r_2} z^{m-2}+\cdots+\tau_{m-1} x^{r_{m-1}} z+x^t \Omega \tag{3.7}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leqslant i \leqslant m-1$, $\Omega \in \hat{\mathcal{O}}_{X,p}$, $\tau_{m-1} \neq 0$ and $t>\omega(m, r_2, \ldots, r_{m-1})$ (where we set $r_i=0$ if $\tau_i=0$). Further, $r_i>0$ if $\tau_i \neq 0$.

X is 3-prepared if X is 3-prepared for all $p \in X$.

Lemma 3.4. — *Suppose that X is 2-prepared with respect to $f: X \rightarrow S$. Then there exists a sequence of blow ups of 2-curves $\pi_1: X \rightarrow X_1$ such that X_1 is 3-prepared with respect to $f \circ \pi_1$, except possibly at a finite number of 1-points.*

Proof. — The conclusions follow from Lemmas 3.1, 2.6 and 2.5, and the method of analysis above 2-points of the proof of 2.7. □

Lemma 3.5. — *Suppose that $u, v \in \mathfrak{k}[[x, y]]$. Let $T_0=\operatorname{Spec}(\mathfrak{k}[[x, y]])$. Suppose that $u=x^a$ for some $a \in \mathbb{Z}_{+}$, or $u=(x^a y^b)^l$ where $\gcd(a, b)=1$ for some $a, b, l \in \mathbb{Z}_{+}$. Let $p \in T_0$ be the maximal ideal (x, y). Suppose that $v \in(x, y) \mathfrak{k}[[x, y]]$. Then either $v \in \mathfrak{k}[[x]]$ or there exists a sequence of blow ups of points $\lambda: T_1 \rightarrow T_0$ such that for all $p_1 \in \lambda^{-1}(p)$, we have regular parameters x_1, y_1 in $\hat{\mathcal{O}}_{T_1, p_1}$, regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\mathcal{O}_{T_1, p_1}$ and a unit $\gamma_1 \in \hat{\mathcal{O}}_{T_1, p_1}$ such that $x_1=\gamma_1 \tilde{x}_1$, and one of the following holds:*

1)
$$u=x_1^{a_1}, v=P(x_1)+x_1^b y_1^c$$
with $c>0$ or

2) *There exists a unit $\gamma_2 \in \hat{\mathcal{O}}_{T_1, p_1}$ such that $y_1=\gamma_2 \tilde{y}_1$ and*
$$u=(x_1^{a_1} y_1^{b_1})^{\ell_1}, v=P(x_1^{a_1} y_1^{b_1})+x_1^{c_1} y_1^{d_1}$$
with $\gcd(a_1, b_1)=1$ and $a_1 d_1-b_1 c_1 \neq 0$.

Proof. — Let
$$J=\operatorname{Det}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

First suppose that $J = 0$. Expand $v = \sum \gamma_{ij} x^i y^j$ with $\gamma_{ij} \in \mathfrak{k}$. If $u = x^a$, then $\sum j\gamma_{ij} x^i y^{j-1} = 0$ implies $\gamma_{ij} = 0$ if $j > 0$. Thus $v = P(x) \in \mathfrak{k}[[x]]$. If $u = (x^a y^b)^l$, then

$$0 = J = l x^{la-1} y^{lb-1} (\sum_{i,j} (ja - ib)\gamma_{ij} x^i y^j)$$

implies $\gamma_{ij} = 0$ if $ja - ib \neq 0$, which implies that $v \in \mathfrak{k}[[x^a y^b]]$.

Now suppose that $J \neq 0$. Let E be the divisor $uJ = 0$ on T_0. There exists a sequence of blow ups of points $\lambda : T_1 \to T_0$ such that $\lambda^{-1}(E)$ is a SNC divisor on T_1. Suppose that $p_1 \in \lambda^{-1}(p)$. There exist regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\hat{\mathcal{O}}_{T_1,p_1}$ such that if

$$J_1 = \mathrm{Det} \begin{pmatrix} \frac{\partial u}{\partial \tilde{x}_1} & \frac{\partial u}{\partial \tilde{y}_1} \\ \frac{\partial v}{\partial \tilde{x}_1} & \frac{\partial v}{\partial \tilde{y}_1} \end{pmatrix},$$

then

$$u = \tilde{x}_1^{a_1}, \ J_1 = \delta \tilde{x}_1^{b_1} \tilde{y}_1^{c_1} \tag{3.8}$$

where $a_1 > 0$ and δ is a unit in $\hat{\mathcal{O}}_{T_1,p_1}$, or

$$u = (\tilde{x}_1^{a_1} \tilde{y}_1^{b_1})^{l_1}, \ J_1 = \delta \tilde{x}_1^{c_1} \tilde{y}_1^{d_1} \tag{3.9}$$

where $a_1, b_1 > 0$, $\gcd(a_1, b_1) = 1$ and δ is a unit in $\hat{\mathcal{O}}_{T_1,p_1}$. Expand $v = \sum \gamma_{ij} \tilde{x}_1^i \tilde{y}_1^j$ with $\gamma_{ij} \in \mathfrak{k}$.

First suppose (3.8) holds. Then

$$a_1 x_1^{a_1 - 1} \left(\sum_{i,j} j\gamma_{ij} \tilde{x}_1^i \tilde{y}_1^{j-1} \right) = \delta \tilde{x}_1^{b_1} \tilde{y}_1^{c_1}.$$

Thus $v = P(\tilde{x}_1) + \varepsilon \tilde{x}_1^e \tilde{y}_1^f$ where $P(\tilde{x}_1) \in \mathfrak{k}[[\tilde{x}_1]]$, $e = b_1 - a_1 + a$, $f = c_1 + 1$ and ε is a unit series. Since $f > 0$, we can make a formal change of variables, multiplying $\tilde{x}_1$ by an appropriate unit series to get the form 1) of the conclusions of the lemma.

Now suppose that (3.9) holds. Then

$$\tilde{x}_1^{a_1 l_1 - 1} \tilde{y}_1^{b_1 l_1 - 1} \left(\sum_{ij} (a_1 l_1 j - b_1 l_1 i)\gamma_{ij} \tilde{x}_1^i \tilde{y}_1^j \right) = \delta \tilde{x}_1^{c_1} \tilde{y}_1^{d_1}.$$

Thus $v = P(\tilde{x}_1^{a_1} \tilde{y}_1^{b_1}) + \varepsilon \tilde{x}_1^e \tilde{y}_1^f$, where P is a series in $\tilde{x}_1^{a_1} \tilde{y}_1^{b_1}$, ε is a unit series, $e = c_1 + 1 - a_1 l_1$, $f = d_1 + 1 - b_1 l_1$. Since $a_1 l_1 f - b_1 l_1 e \neq 0$, we can make a formal change of variables to reach 2) of the conclusions of the lemma. $\square$

Lemma 3.6. — *Suppose that X is 2-prepared with respect to $f : X \to S$. Suppose that $p \in D$ is a 1-point with $m = \sigma_D(p) + 1 > 1$. Let u, v be permissible parameters for $f(p)$ and x, y, z be permissible parameters for D at p such that a form (3.1) holds at p. Let U be an étale cover of an affine neighborhood of p such that x, y, z are uniformizing parameters on U. Let C be the curve in U which has local equations $x = y = 0$ at p.*

Let $T_0 = Spec(\mathfrak{k}[x, y])$, $\Lambda_0 : U \to T_0$. Then there exists a sequence of quadratic transforms $T_1 \to T_0$ such that if $U_1 = U \times_{T_0} T_1$ and $\psi_1 : U_1 \to U$ is the induced sequence of blow ups of sections over C, $\Lambda_1 : U_1 \to T_1$ is the projection, then U_1 is 2-prepared with respect to $f \circ \psi_1$ at all $p_1 \in \psi_1^{-1}(p)$. Further, for every point $p_1 \in \psi_1^{-1}(p)$, there exist regular parameters x_1, y_1 in $\hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ such that x_1, y_1, z are permissible parameters at p_1, and there exist regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\mathcal{O}_{T_1,\Lambda_1(p_1)}$ such that if p_1 is a 1-point, $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ where $\alpha(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ is a unit series and $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)$ with $\beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$, and if p_1 is a 2-point, then $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ and $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)\tilde{y}_1$, where $\alpha(\tilde{x}_1, \tilde{y}_1), \beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ are unit series. We have one of the following forms:

1) *p_1 is a 2-point, and we have an expression (2.2) with*

(3.10)
$$F = \tau z^m + \overline{a}_2(x_1, y_1)x_1^{r_2}y_1^{s_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}}y_1^{s_{m-1}}z + \overline{a}_m x_1^{r_m}y_1^{s_m}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1, y_1) \in \mathfrak{k}[[x_1, y_1]]$ are units (or zero) for $2 \leqslant i \leqslant m-1$, $\overline{a}_m = 0$ or 1 and if $\overline{a}_m = 0$, then $\overline{a}_{m-1} \neq 0$. Further, $r_i + s_i > 0$ whenever $\overline{a}_i \neq 0$ and $a(r_m + c)b - (s_m + d)a \neq 0$.

2) *p_1 is a 1-point, and we have an expression (2.1) with*

$$F = \tau z^m + \overline{a}_2(x_1, y_1)x_1^{r_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}}z + x_1^{r_m}y_1 \tag{3.11}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1, y_1) \in \mathfrak{k}[[x_1, y_1]]$ are units (or zero) for $2 \leqslant i \leqslant m-1$. Further, $r_i > 0$ (whenever $\overline{a}_i \neq 0$).

3) *p_1 is a 1-point, and we have an expression (2.1) with*

$$F = \tau z^m + \overline{a}_2(x_1, y_1)x_1^{r_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}}z + x_1^t y_1\Omega \tag{3.12}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1, y_1) \in \mathfrak{k}[[x_1, y_1]]$ are units (or zero) for $2 \leqslant i \leqslant m-1$ and $r_i > 0$ whenever $\overline{a}_i \neq 0$. We also have $t > \omega(m, r_2, \ldots, r_{m-1})$. Further, $\overline{a}_{m-1} \neq 0$ and $\Omega \in \hat{\mathcal{O}}_{U_1,p_1}$.

Proof. — Let $\overline{p} = \Lambda_0(p)$. Let $T = \{i \mid a_i(x, y) \neq 0 \text{ and } 2 \leqslant i < m\}$. There exists a sequence of blow ups $\varphi_1 : T_1 \to T_0$ of points over $\overline{p}$ such that at all points $q \in \psi_1^{-1}(p)$, we have permissible parameters x_1, y_1, z such that

x_1, y_1 are regular parameters in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$ and we have that u is a monomial in x_1 and y_1 times a unit in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$, where $g = \prod_{i\in T} a_i(x,y)$.

Suppose that $a_m(x,y) \neq 0$. Let $\overline{v} = x^b a_m(x,y)$ if (2.1) holds and $\overline{v} = x^c y^d a_m(x,y)$ if (2.2) holds. We have $\overline{v} \notin \mathfrak{k}[[x]]$ (respectively $\overline{v} \notin \mathfrak{k}[[x^a y^b]]$). Then by Theorem 3.5 applied to $u, \overline{v}$, we have that there exists a further sequence of blow ups $\varphi_2 : T_2 \to T_1$ of points over $\overline{p}$ such that at all points $q \in (\psi_1 \circ \psi_2)^{-1}(p)$, we have permissible parameters x_2, y_2, z such that x_2, y_2 are regular parameters in $\hat{\mathcal{O}}_{T_2,\Lambda_2(q)}$ such that $u = 0$ is a SNC divisor and either

$$u = x_2^{\overline{a}}, \overline{v} = \overline{P}(x_2) + x_2^{\overline{b}} \overline{y}_2^{\overline{c}}$$

with $\overline{c} > 0$ or

$$u = (x_2^{\overline{a}} y_2^{\overline{b}})^t, \overline{v} = \overline{P}(x_2^{\overline{a}} y_2^{\overline{b}}) + x_2^{\overline{c}} \overline{y}_2^{\overline{d}}$$

where $\overline{a}\overline{d} - \overline{b}\overline{c} \neq 0$.

If q is a 2-point, we have thus achieved the conclusions of the lemma. Further, there are only finitely many 1-points q above p on U_2 where the conclusions of the lemma do not hold. At such a 1-point q, F has an expression

(3.13)
$$F = \tau z^m + \overline{a}_2(x_2,y_2) x_2^{r_2} y_2^{s_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_2,y_2) x_2^{r_{m-1}} y_2^{s_{m-1}} z + \overline{a}_m x_2^{r_m} y_2^{s_m}$$

where $\overline{a}_m = 0$ or 1, $\overline{a}_i$ are units (or zero) for $2 \leqslant i \leqslant m$.

Let

$$J = I_2 \begin{pmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} & \frac{\partial v}{\partial z} \end{pmatrix} = x^n \left(\frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial z}\right)$$

for some positive integer n. Since D contains the locus where f is not smooth, we have that the localization $J_{\mathfrak{p}} = (\hat{\mathcal{O}}_{U_2,q})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal (y_2, z_2) in $\hat{\mathcal{O}}_{U_2,q}$.

We compute

$$\frac{\partial F}{\partial z} = \overline{a}_{m-1} x_2^{r_{m-1}} y_2^{s_{m-1}} + \Lambda_1 z$$

and

$$\frac{\partial F}{\partial y_2} = s_m \overline{a}_m y_2^{s_m - 1} x_2^{r_m} + \Lambda_2 z$$

for some $\Lambda_1, \Lambda_2 \in \hat{\mathcal{O}}_{U_2,q}$, to see that either $\overline{a}_{m-1} \neq 0$ and $s_{m-1} = 0$, or $\overline{a}_m \neq 0$ and $s_m = 1$.

Let q be one of these points, and let $\varphi_3 : T_3 \to T_2$ be the blow up of $\Lambda_2(q)$. We then have that the conclusions of the lemma hold in the form (3.10) at the 2-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3 y_3$ and $y_2 = y_3$. At a 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3, y_2 = x_3(y_3 + \alpha)$ with $\alpha \neq 0$, we have that a form (3.11) holds.

Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$ and $y_2 = x_3y_3$. We continue to blow up, so that there is at most one point where the conclusions of the lemma do not hold. This point is a 1-point, which has permissible parameters x_3, y_3, z where $x_2 = x_3$ and $y_2 = x_3^n y_3$ where we can take n as large as we like. We thus have a form

$$u = x_3^a, v = P(x_3) + x_3^b F_3 \tag{3.14}$$

with $F_3 = \tau z^m + \overline{b}_2 x_3^{r_2} z^{m-2} + \cdots + \overline{b}_{m-1} x_3^{r_{m-2}} z + x_3^t \Omega$, where either $\overline{b}_i(x_3, y_3)$ is a unit or is zero, $\overline{b}_{m-1} \neq 0$, and $t > \omega(m, r_2, \ldots, r_{m-1})$ if $\overline{a}_{m-1} \neq 0$ and $s_{m-1} = 0$ which is of the form of (3.12), or we have a form (3.11) (after replacing y_3 with y_3 times a unit series in x_3 and y_3) if $\overline{a}_m \neq 0$ and $s_m = 1$. □

Lemma 3.7. — *Suppose that X is 2-prepared with respect to $f : X \to S$. Suppose that $p \in D$ is a 1-point with $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Let x, y, z be permissible parameters for D at p such that a form (3.1) holds at p.*

Let notation be as in Lemma 3.6. For $p_1 \in \psi_1^{-1}(p)$ let $\overline{r}(p_1) = m + 1 + r_m$, if a form (3.11) holds at p_1, and

$$\overline{r}(p_1) = \begin{cases} \max\{m + 1 + r_m, m + 1 + s_m\} & \text{if } \overline{a}_m = 1 \\ \max\{m + 1 + r_{m-1}, m + 1 + s_{m-1}\} & \text{if } \overline{a}_m = 0 \end{cases}$$

if a form (3.10)holds at p_1. Let $\overline{r}(p_1) = m + 1 + r_{m-1}$ if a form (3.12) holds at p_1.

Let $r' = \max\{\overline{r}(p_1) \mid p_1 \in \psi_1^{-1}(p)\}$. Let

$$r = r(p) = m + 1 + r'. \tag{3.15}$$

Suppose that $x^* \in \mathcal{O}_{X,p}$ is such that $x = \overline{\gamma} x^*$ for some unit $\overline{\gamma} \in \hat{\mathcal{O}}_{X,p}$ with $\overline{\gamma} \equiv 1 \bmod m_p^r \hat{\mathcal{O}}_{X,p}$.

Let V be an affine neighborhood of p such that $x^*, y \in \Gamma(V, \mathcal{O}_X)$, and let C^* be the curve in V which has local equations $x^* = y = 0$ at p.

Let $T_0^* = \mathrm{Spec}(\mathfrak{k}[x^*, y])$. Then there exists a sequence of blow ups of points $T_1^* \to T_0^*$ above (x^*, y) such that if $V_1 = V \times_{T_0^*} T_1^*$ and $\psi_1^* : V_1 \to V$ is the induced sequence of blow ups of sections over C^*, $\Lambda_1^* : V_1 \to T_1^*$ is the projection, then V_1 is 2-prepared at all $p_1^* \in (\psi_1^*)^{-1}(p)$. Further, for every point $p_1^* \in (\psi_1^*)^{-1}(p)$, there exist $\hat{x}_1, \overline{y}_1 \in \hat{\mathcal{O}}_{V_1, p_1^*}$ such that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^* and we have one of the following forms:

1) p_1^* *is a 2-point, and we have an expression (2.2) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} \overline{y}_1^{s_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} \overline{y}_1^{s_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \overline{y}_1^{s_m} \tag{3.16}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero) for* $0 \leqslant i \leqslant m-1$, $\overline{\tau}_m$ *is zero or 1,* $\overline{\tau}_{m-1} \neq 0$ *if* $\overline{\tau}_m = 0$, $r_i + s_i > 0$ *if* $\overline{\tau}_i \neq 0$, *and*

$$(r_m + c)b - (s_m + d)a \neq 0.$$

2) p_1^* *is a 1-point, and we have an expression (2.1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \tag{3.17}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero), and* $\mathrm{ord}(\overline{\tau}_m(0,\overline{y}_1,0)) = 1$. *Further,* $r_i > 0$ *if* $\overline{\tau}_i \neq 0$.

3) p_1^* *is a 1-point, and we have an expression (2.1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} z + x_1^t \overline{\Omega} \tag{3.18}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero),* $\overline{\Omega} \in \hat{\mathcal{O}}_{V_1,p_1^*}$, $\overline{\tau}_{m-1} \neq 0$ *and* $t > \omega(m, r_2, \ldots, r_{m-1})$. *Further,* $r_i > 0$ *if* $\overline{\tau}_i \neq 0$.

Proof. — The isomorphism $T_0^* \to T_0$ obtained by substitution of x^* for x and subsequent base change by the morphism $T_1 \to T_0$ of Lemma 3.6, induces a sequence of blow ups of points $T_1^* \to T_0^*$. The base change $\psi_1^* : V_1 = V \times_{T_0^*} T_1^* \to V \cong V \times_{T_0^*} T_0^*$ factors as a sequence of blow ups of sections over C^*. Let $\Lambda_1^* : V_1 \to T_1^*$ be the natural projection.

Let $p_1^* \in (\psi_1^*)^{-1}(p)$, and let $p_1 \in \psi_1^{-1}(p) \subset U_1$ be the corresponding point.

First suppose that p_1 has a form (3.11). With the notation of Lemma 3.6, we have polynomials φ, ψ such that

$$x = \varphi(\tilde{x}_1, \tilde{y}_1), y = \psi(\tilde{x}_1, \tilde{y}_1)$$

determines the birational extension $\mathcal{O}_{T_0,p_0} \to \mathcal{O}_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1, y_1 = \beta(\tilde{x}_1, \tilde{y}_1)$$

for some unit series α and series β. We further have expansions

$$a_i(x,y) = x_1^{r_i} \overline{a}_i(x_1, y_1)$$

for $2 \leqslant i \leqslant m-1$ where $\overline{a}_i(x_1, y_1)$ are unit series or zero, and

$$a_m(x,y) = x_1^{r_m} y_1.$$

We have $x=\overline{\gamma}x^*$ with $\overline{\gamma}\equiv 1 \bmod m_p^r\hat{\mathcal{O}}_{X,p}$. Set $y^*=y$. At p_1^*, we have regular parameters x_1^*, y_1^* in $\mathcal{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^*=\varphi(x_1^*,y_1^*), y^*=\psi(x_1^*,y_1^*),$$

and $x_1^*, y_1^*, \tilde{z}$ are regular parameters in $\mathcal{O}_{V_1,\overline{p}_1^*}$ (recall that $z=\sigma\tilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x}_1, \overline{y}_1, \in \hat{\mathcal{O}}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\overline{x}_1=\alpha(x_1^*,y_1^*)x_1^*, \overline{y}_1=\beta(x_1^*,y_1^*).$$

We have $u=x^a=x_1^{a_1}$ where $a_1=ad$ for some $d\in\mathbb{Z}_+$. Since $[\alpha(\tilde{x}_1,\tilde{y}_1)\tilde{x}_1]^d=x$, we have that $[\alpha(x_1^*,y_1^*)x_1^*]^d=x^*$. Set $\hat{x}_1=\overline{\gamma}^{\frac{1}{d}}\overline{x}_1=\overline{\gamma}^{\frac{1}{d}}\alpha(x_1^*,y_1^*)x_1^*$. We have that $\overline{\gamma}^{\frac{1}{d}}\alpha(x_1^*,y_1^*)$ is a unit in $\hat{\mathcal{O}}_{V_1,p_1^*}$, and $x=\hat{x}_1^d$. Thus $x_1=\hat{x}_1$ (with an appropriate choice of root $\overline{\gamma}^{\frac{1}{d}}$). We have $u=\hat{x}_1^{ad}$, so that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^*.

For $2\leqslant i\leqslant m-1$, we have

$$a_i(x,y)=a_i(\overline{\gamma}x^*,y^*)\equiv a_i(x^*,y^*) \bmod m_p^r\hat{\mathcal{O}}_{V,p}$$

and

$$\begin{aligned}a_i(x^*,y^*)&=a_i(\varphi(x_1^*,y_1^*),\psi(x_1^*,y_1^*))\\&=\overline{x}_1^{r_i}\overline{a}_i(\overline{x}_1,\overline{y}_1)\\&\equiv x_1^{r_i}\overline{a}_i(x_1,\overline{y}_1) \bmod m_p^r\mathcal{O}_{V_1,p_1^*}.\end{aligned}$$

We further have

$$a_m(x^*,y^*)\equiv x_1^{r_m}\overline{y}_1 \bmod m_p^r\hat{\mathcal{O}}_{V_1,p_1^*}.$$

Thus we have expressions

(3.19)
$$\begin{aligned}&u=x_1^{da}\\&v=P(x_1^d)+x_1^{bd}P_1(x_1)+x_1^{bd}(\overline{\tau}z^m+x_1^{r_2}\overline{a}_2(x_1,\overline{y}_1)z^{m-2}+\cdots+x_1^{r_m}\overline{y}_1+h)\end{aligned}$$

where $\overline{\tau}\in\hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit series and

$$h\in m_p^r\hat{\mathcal{O}}_{V_1,p_1^*}\subset(x_1,z)^r.$$

Set $s=r-m$, and write

$$\begin{aligned}h=z^m\Lambda_0(x_1,\overline{y}_1,z)&+z^{m-1}x_1^{1+s}\Lambda_1(x_1,\overline{y}_1)+z^{m-2}x_1^{2+s}\Lambda_2(x_1,\overline{y}_1)\\&+\cdots+zx_1^{(m-1)+s}\Lambda_{m-1}(x_1,\overline{y}_1)+x_1^{m+s}\Lambda_m(x_1,\overline{y}_1)\end{aligned}$$

with $\Lambda_0\in m_{p_1^*}\hat{\mathcal{O}}_{V_1,p_1^*}$ and $\Lambda_i\in\mathfrak{k}[[x_1,\overline{y}_1]]$ for $1\leqslant i\leqslant m$.

Substituting into (3.19), we obtain an expression

$$
\begin{aligned}
u &= x_1^{da}\\
v &= P(x_1^d) + x_1^{bd}P_1(x_1)\\
&\qquad + x_1^{bd}(\overline{\tau}_0 z^m + x_1^{r_2}\overline{\tau}_2 z^{m-2} + \cdots + x_1^{r_{m-1}}\overline{\tau}_{m-1}z + x_1^{r_m}\overline{\tau}_m)
\end{aligned}
$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero), for $1 \leqslant i \leqslant m-1$ and $\overline{\tau}_m \in \mathfrak{k}[[x_1,\overline{y}_1]]$ with $\operatorname{ord}(\overline{\tau}_m(0,\overline{y}_1)) = 1$.

We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(x_1,\overline{y}_1)$ for $2 \leqslant i \leqslant m-1$, and

$$
\overline{\tau}_m = \overline{y}_1 + z^{m-1}x_1^{1+s-r_m}\Lambda_1(x_1,\overline{y}_1) + \cdots + x_1^{m+s-r_m}\Lambda_m(x_1,\overline{y}_1)).
$$

We thus have the desired form (3.17).

In the case when p_1 has a form (3.12), a similar argument to the analysis of (3.11) shows that p_1^* has a form (3.18).

Now suppose that p_1 has a form (3.10). We then have

$$
m_p\mathcal{O}_{U_1,p_1} \subset (x_1y_1,z)\mathcal{O}_{U_1,p_1}, \tag{3.20}
$$

unless there exist regular parameters $x_1', y_1' \in \mathcal{O}_{T_1,\Lambda_1(p_1)}$ such that x_1', y_1', z are regular parameters in $\mathcal{O}_{U_1,p_1}$ and

$$
x = x_1', y = (x_1')^n y_1' \tag{3.21}
$$

or

$$
x = x_1'(y_1')^n, y = y_1' \tag{3.22}
$$

for some $n \in \mathbb{N}$. If (3.21) or (3.22) holds, then $\hat{\mathcal{O}}_{V_1,p_1^*} = \hat{\mathcal{O}}_{U_1,p_1}$, and (taking $\hat{x}_1 = x_1$, $\overline{y}_1 = y_1$) we have that a form (3.16) holds at p_1^*. We may thus assume that (3.20) holds.

With the notation of Lemma 3.6, we have polynomials φ, ψ such that

$$
x = \varphi(\tilde{x}_1,\tilde{y}_1), y = \psi(\tilde{x}_1,\tilde{y}_1)
$$

determines the birational extension $\mathcal{O}_{T_0,p_0} \to \mathcal{O}_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$
x_1 = \alpha(\tilde{x}_1,\tilde{y}_1)\tilde{x}_1, y_1 = \beta(\tilde{x}_1,\tilde{y}_1)\tilde{y}_1
$$

for some unit series α and β. We further have expansions

$$
a_i(x,y) = x_1^{r_i}y_1^{s_i}\overline{a}_i(x_1,y_1)
$$

for $2 \leqslant i \leqslant m-1$ where $\overline{a}_i(x_1,y_1)$ are unit series or zero, and

$$
a_m(x,y) = x_1^{r_m}y_1^{s_m}\overline{a}_m,
$$

where $\overline{a}_m = 0$ or 1. We have $x = \overline{\gamma}x^*$ with $\overline{\gamma} \equiv 1 \bmod m_p^r\hat{\mathcal{O}}_{X,p}$. Set $y^* = y$. At p_1^*, we have regular parameters x_1^*, y_1^* in $\mathcal{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*),$$

and $x_1^*, y_1^*, \tilde{z}$ are regular parameters in $\mathcal{O}_{V_1,\overline{p}_1^*}$ (recall that $z = \sigma\tilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x}_1, \overline{y}_1, \in \hat{\mathcal{O}}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\overline{x}_1 = \alpha(x_1^*, y_1^*)x_1^*, \overline{y}_1 = \beta(x_1^*, y_1^*)y_1^*.$$

We calculate

$$u = x^a = (x_1^{a_1}y_1^{b_1})^{t_1} = [\alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1]^{a_1t_1}[\beta(\tilde{x}_1, \tilde{y}_1)\tilde{y}_1]^{b_1t_1}$$

which implies

$$(x^*)^a = [\alpha(x_1^*, y_1^*)x_1^*]^{a_1t_1}[\beta(x_1^*, y_1^*)y_1^*]^{b_1t_1} = \overline{x}_1^{a_1t_1}\overline{y}_1^{b_1t_1}.$$

Set $\hat{x}_1 = \overline{\gamma}^{\frac{a}{a_1t_1}}\overline{x}_1$ to get $u = (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1}$, so that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^*.

For $2 \leqslant i \leqslant m$, we have

$$a_i(x, y) = a_i(\overline{\gamma}x^*, y^*) \equiv a_i(x^*, y^*) \bmod m_p^r\hat{\mathcal{O}}_{V,p}$$

and

$$\begin{aligned} a_i(x^*, y^*) &= a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \\ &= \overline{x}_1^{r_i}\overline{y}_1^{s_i}\overline{a}_i(\overline{x}_1, \overline{y}_1) \\ &\equiv \hat{x}_1^{r_i}\overline{y}_1^{s_i}\overline{a}_i(\hat{x}_1, \overline{y}_1) \bmod m_p^r\mathcal{O}_{V_1,p_1^*}. \end{aligned}$$

Thus we have expressions

(3.23)

$$\begin{aligned} u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1} \\ v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}z^m \\ &\qquad + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{a}_2(\hat{x}_1, \overline{y}_1)z^{m-2} + \cdots + \hat{x}_1^{r_m}\overline{y}_1^{s_m}\overline{a}_m + h) \end{aligned}$$

where $\overline{\tau} \in \hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit series and

$$h \in m_p^r\hat{\mathcal{O}}_{V_1,p_1^*} \subset (\hat{x}_1\overline{y}_1, z)^r.$$

Set $s = r - m$, and write

$$\begin{aligned} h &= z^m\Lambda_0(x_1, \overline{y}_1, z) + z^{m-1}(\hat{x}_1\overline{y}_1)^{1+s}\Lambda_1(\hat{x}_1, \overline{y}_1) \\ &\quad + z^{m-2}(\hat{x}_1\overline{y}_1)^{2+s}\Lambda_2(\hat{x}_1, \overline{y}_1) + \cdots \\ &\quad + z(\hat{x}_1\overline{y}_1)^{(m-1)+s}\Lambda_{m-1}(\hat{x}_1, \overline{y}_1) + (\hat{x}_1\overline{y}_1)^{m+s}\Lambda_m(\hat{x}_1, \overline{y}_1) \end{aligned} \tag{3.24}$$

with $\Lambda_0 \in m_{p_1^*}\hat{\mathcal{O}}_{V_1,p_1^*}$ and $\Lambda_i \in \mathfrak{k}[[\hat{x}_1, \overline{y}_1]]$ for $1 \leqslant i \leqslant m$.

First suppose that $\overline{a}_m = 1$. Substituting into (3.23), we obtain an expression

$$
\begin{aligned}
u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1}\\
v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1})\\
&\qquad + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}_0 z^m + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{\tau}_2 z^{m-2} + \cdots + \hat{x}_1^{r_m}\overline{y}_1^{s_m}\overline{\tau}_m)
\end{aligned}
$$

where $\overline{\tau}_0, \overline{\tau}_m \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero) for $2 \leqslant i \leqslant m-1$.

We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(\hat{x}_1, \overline{y}_1)$ for $2 \leqslant i \leqslant m-1$, and

$$
\begin{aligned}
\overline{\tau}_m = \overline{a}_m + z^{m-1}\hat{x}_1^{1+s-r_m}\overline{y}_1^{1+s-s_m}\Lambda_1(\hat{x}_1, \overline{y}_1) + \cdots&\\
+ \hat{x}_1^{m+s-r_m}\overline{y}_1^{m+s-s_m}\Lambda_m(\hat{x}_1, \overline{y}_1).&
\end{aligned}
$$

We thus have the desired form (3.16).

Now suppose that $\overline{a}_m = 0$. Then $\overline{a}_{m-1} \neq 0$, and z divides h in (3.23), so that $\Lambda_m = 0$ in (3.24). Substituting into (3.23), we obtain an expression

$$
\begin{aligned}
u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1}\\
v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1})\\
&\qquad + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}_0 z^m + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{\tau}_2 z^{m-2} + \cdots + \hat{x}_1^{r_{m-1}}\overline{y}_1^{s_{m-1}}\overline{\tau}_{m-1} z)
\end{aligned}
$$

where $\overline{\tau}_0, \overline{\tau}_{m-1} \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero) for $2 \leqslant i \leqslant m-2$.

We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(\hat{x}_1, \overline{y}_1)$ for $2 \leqslant i \leqslant m-2$, and

$$
\begin{aligned}
\overline{\tau}_{m-1} = \overline{a}_{m-1} + z^{m-1}\hat{x}_1^{1+s-r_{m-1}}\overline{y}_1^{1+s-s_{m-1}}\Lambda_1(\hat{x}_1, \overline{y}_1) + \cdots&\\
+ \hat{x}_1^{m-1+s-r_{m-1}}\overline{y}_1^{m-1+s-s_{m-1}}\Lambda_{m-1}(\hat{x}_1, \overline{y}_1).&
\end{aligned}
$$

We thus have the form (3.16). □

Lemma 3.8. — *Suppose that X is 2-prepared. Suppose that $p \in X$ is a 1-point with $\sigma_D(p) > 0$ and E is the component of D containing p. Suppose that Y is a finite set of points in X (not containing p). Then there exists an affine neighborhood U of p in X such that*

1) *$Y \cap U = \emptyset$.*
2) *$[E - U \cap E] \cap Sing_1(X)$ is a finite set of points.*
3) *$U \cap D = U \cap E$ and there exists $\overline{x} \in \Gamma(U, \mathcal{O}_X)$ such that $\overline{x} = 0$ is a local equation of E in U.*
4) *There exists an étale map $\pi : U \to \mathbb{A}_k^3 = Spec(\mathfrak{k}[\overline{x}, \overline{y}, \overline{z}])$.*
5) *The Zariski closure C in X of the curve in U with local equations $\overline{x} = \overline{y} = 0$ satisfies the following:*

i) *C is a nonsingular curve through p.*
ii) *C contains no 3-points of D.*
iii) *C intersects 2-curves of D transversally at prepared points.*
iv) *$C \cap Sing_1(X) \cap (X - U) = \emptyset$.*
v) *$C \cap Y = \emptyset$.*
vi) *C intersects $Sing_1(X) - \{p\}$ transversally at general points of curves in $Sing_1(X)$.*
vii) *There exist permissible parameters x, y, z at p, with $\tilde{x} = \overline{x}, y = \overline{y}$, which satisfy the hypotheses of lemma 3.1.*

Proof. — Let H be an effective, very ample divisor on X such that H contains Y and $D - E$, but H does not contain p and does not contain any one dimensional components of $\mathrm{Sing}_1(X, D) \cap E$. There exists $n > 0$ such that $E + nH$ is ample, $\mathcal{O}_X(E + nH)$ is generated by global sections and a general member H' of the linear system $|E + nH|$ does not contain any one dimensional components of $\mathrm{Sing}_1(X, D) \cap E$, and does not contain p. $H + H'$ is ample, so $V = X - (H + H')$ is affine. Further, there exists $f \in \mathfrak{k}(X)$, the function field of X, such that $(f) = H' - (E + nH)$. Thus $\overline{x} = \frac{1}{f} \in \Gamma(V, \mathcal{O}_X)$ as X is normal and $\overline{x}$ has no poles on V. $\overline{x} = 0$ is a local equation of E on V. We have that V satisfies the conclusions 1), 2) and 3) of the lemma.

Let $R = \Gamma(V, \mathcal{O}_X)$. $R = \cup_{s=1}^{\infty} \Gamma(X, \mathcal{O}_X(s(H + H'))$ is a finitely generated $\mathfrak{k}$-algebra. Thus for $s \gg 0$, R is generated by $\Gamma(X, \mathcal{O}_X(s(H + H'))$ as a $\mathfrak{k}$-algebra.

From the exact sequences

$$0 \to \Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p) \to \Gamma(X, \mathcal{O}_X(s(H + H')) \to \mathcal{O}_{X,p}/m_p \cong k$$

and the fact that $1 \in \Gamma(X, \mathcal{O}_X(s(H + H'))$, we have that R is generated by $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_P)$ as a $\mathfrak{k}$-algebra for all $s \gg 0$.

For $s \gg 0$, and a general member σ of $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p)$ we have that the curve $\overline{C} = B \cdot E$, where B is the divisor $B = (\sigma) + s(H + H')$, satisfies the conclusions of 5) of the lemma; since each of the conditions 5i) through 5vii) is an open condition on $\Gamma(X, \mathcal{O}_X(s(H + H') \otimes \mathcal{I}_p))$, we need only establish that each condition holds on a nonempty subset. This follows from the fact that $H + H'$ is ample, Bertini's theorem applied to the base point free linear system $|\varphi^*(s(H + H')) - A|$, where $\varphi : W \to X$ is the blow up of p with exceptional divisor A, and the fact that

$$\varphi_*(\mathcal{O}_W(\varphi^*(s(H + H') - A)) = \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p.$$

For fixed $s \gg 0$, let $\overline{x}, \overline{y}_1, \ldots, \overline{y}_n$ be a $\mathfrak{k}$-basis of $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p)$, so that $R = \mathfrak{k}[\overline{x}, \overline{y}_1, \ldots, \overline{y}_n]$. We have shown that there exists a Zariski open set $\overline{Z}$ of k^n such that for $(b_1,\ldots,b_n)\in\overline{Z}$, the curve C in X which is the Zariski closure of the curve with local equation $\overline{x}=b_1\overline{y}_1+\cdots+b_n\overline{y}_n=0$ in V satisfies 5) of the conclusions of the lemma.

Let $C_1,\ldots,C_t$ be the curves in $\mathrm{Sing}_1(X)\cap V$, and let $p_i\in C_i$ be closed points such that $p,p_1,\ldots,p_t$ are distinct. Let Q_0 be the maximal ideal of p in R, and Q_i be the maximal ideal in R of p_i for $1\leqslant i\leqslant t$. We have that $\overline{x}$ is nonzero in Q_i/Q_i^2 for all i. For a matrix $A=(a_{ij})\in\mathfrak{k}^{2n}$, and $1\leqslant i\leqslant 2$, let

$$L_i^A(\overline{y}_1,\ldots,\overline{y}_n)=\sum_{j=1}^n a_{ij}\overline{y}_j.$$

There exist $\alpha_{jk}\in\mathfrak{k}$ such that $Q_k=(\overline{y}_1-\alpha_{1,k},\ldots,\overline{y}_n-\alpha_{n,k})$ for $0\leqslant k\leqslant t$. By our construction, we have $\alpha_{1,0}=\cdots=\alpha_{n,0}=0$. For each $0\leqslant k\leqslant t$, there exists a non empty Zariski open subset Z_k of k^{2n} such that

$$\overline{x},L_1^A(\overline{y}_1,\ldots,\overline{y}_n)-L_1^A(\alpha_{1,k},\ldots,\alpha_{n,k}),L_2^A(\overline{y}_1,\ldots,\overline{y}_n)-L_2^A(\alpha_{1,k},\ldots,\alpha_{n,k})$$

is a $\mathfrak{k}$-basis of Q_k/Q_{k+1}^2. Suppose $(a_{1,1},\ldots,a_{1,n})\in\overline{Z}$ and $A\in Z_0\cap\cdots\cap Z_t$.

We will show that $\overline{x},L_1^A,L_2^A$ are algebraically independent over $\mathfrak{k}$. Suppose not. Then there exists a nonzero polynomial $h\in\mathfrak{k}[t_1,t_2,t_3]$ such that $h(\overline{x},L_1^A,L_2^A)=0$. Write $h=H+h'$ where H is the leading form of h, and $h'=h-H$ is a polynomial of larger order than the degree r of H. Now $H(\overline{x},L_1^A,L_2^A)=-h'(\overline{x},L_1^A,L_2^A)$, so that $H(\overline{x},L_1^A,L_2^A)=0$ in Q_0^r/Q_0^{r+1}. Thus $H=0$, since R_{Q_0} is a regular local ring, which is a contradiction. Thus $\overline{x},L_1^A,L_2^A$ are algebraically independent. Without loss of generality, we may assume that $L_i^A=\overline{y}_i$ for $1\leqslant i\leqslant 2$.

Let $S=\mathfrak{k}[\overline{x},\overline{y}_1,\overline{y}_2]$, a polynomial ring in 3 variables over $\mathfrak{k}$. $S\to R$ is unramified at Q_i for $0\leqslant i\leqslant t$ since

$$(\overline{x},\overline{y}_1-\alpha_{1,i},\overline{y}_2-\alpha_{2,i})R_{Q_i}=Q_iR_{Q_i}$$

for $0\leqslant i\leqslant t$.

Let W be the closed locus in V where $V\to\mathrm{Spec}(S)$ is not étale. We have that $p,p_1,\ldots,p_t\notin W$, so there exists an ample effective divisor $\overline{H}$ on X such that $W\subset\overline{H}$ and $p,p_1,\ldots,p_t\notin\overline{H}$. Let $U=V-\overline{H}$. U is affine, and $U\to\mathrm{Spec}(S)\cong\mathbb{A}^3$ is étale, so satisfies 4) of the conclusions of the lemma. $\square$

Lemma 3.9. — *Suppose X is 2-prepared with respect to $f:X\to S$, $p\in D$ is a prepared point, and $\pi_1:X_1\to X$ is the blow up of p. Then all points of $\pi_1^{-1}(p)$ are prepared.*

Proof. — The conclusions follow from substitution of local equations of the blow up of a point into a prepared form (2.1), (2.2) or (2.3). $\square$

Lemma 3.10. — *Suppose that X is 2-prepared with respect to $f : X \to S$, and that C is a permissible curve for D, which is not a 2-curve. Suppose that $p \in C$ satisfies $\sigma_D(p) = 0$. Then there exist permissible parameters x, y, z at p such that one of the following forms hold:*

1) *p is a 1-point of D of the form of (2.1), $F = z$ and $x = y = 0$ are formal local equations of C at p.*
2) *p is a 1-point of D of the form of (2.1), $F = z$ and $x = z = 0$ are formal local equations of C at p.*
3) *p is a 1-point of D of the form of (2.1), $F = z$, $x = z + y^r\sigma(y) = 0$ are formal local equations of C at p, where $r > 1$ and σ is a unit series.*
4) *p is a 2-point of D of the form of (2.2), $F = z$, $x = z = 0$ are formal local equations of C at p.*
5) *p is a 2-point of D of the form of (2.2), $F = z$, $x = g(y, z) = 0$ are formal local equations of C at p, where $g(y, z)$ is not divisible by z.*
6) *p is a 2-point of D of the form of (2.2), $F = 1$ (so that $ad - bc \neq 0$) and $x = z = 0$ are formal local equations of C at p.*

Further, there are at most a finite number of 1-points on C satisfying condition 3) (and not satisfying condition 1) or 2)).

Proof. — Suppose that p is a 1-point. We have permissible parameters x, y, z at p such that a form (2.1) holds at p with $F = z$. There exists a series $g(y, z)$ such that $x = g = 0$ are formal local equations of C at p. By the formal implicit function theorem, we get one of the forms 1), 2) or 3). A similar argument shows that one of the forms 4), 5) or 6) must hold if p is a 2-point.

Now suppose that $p \in C$ is a 1-point, $\sigma_D(p) = 0$ and a form 3) holds at p. There exist permissible parameters x, y, z at p, with an expression (2.1), such that $x = z = 0$ are formal local equations of C at p and x, y, z are uniformizing parameters on an étale cover U of an neighborhood of p, where we can choose U so that

$$\frac{\partial F}{\partial y} = \frac{1}{x^b}\frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).$$

Since there is not a form 2) at p, we have that z does not divide $F(0, y, z)$, so that $F(0, y, 0) \neq 0$. Since F has no constant term, we have that $\frac{\partial F}{\partial y}(0, y, 0) \neq 0$. There exists a Zariski open subset of $\mathfrak{k}$ such that $\alpha \in \mathfrak{k}$ implies $x, y - \alpha, z$ are regular parameters at a point $q \in U$. There exists a Zariski open subset of $\mathfrak{k}$ of such α so that $\frac{\partial F}{\partial y}(0, \alpha, 0) \neq 0$. Thus $x, y - \alpha, z$ are permissible parameters at q giving a form 1) at $q \in C$. □

Lemma 3.11. — *Suppose that X is 2-prepared. Suppose that C is a permissible curve on X which is not a 2-curve and $p \in C$ satisfies $\sigma_D(p) = 0$. Further suppose that either a form 3) or 5) of the conclusions of Lemma 3.10 hold at p. Then there exists a sequence of blow ups of points $\pi_1 : X_1 \to X$ above p such that X_1 is 2-prepared and $\sigma_{D_1}(p_1) = 0$ for all $p_1 \in \pi_1^{-1}(p)$, and the strict transform of C on X_1 is permissible, and has the form 4) or 6) of Lemma 3.10 at the point above p.*

Proof. — If p is a 1-point, let $\pi' : X' \to X$ be the blow ups of p, and let C' be the strict transform of C on X'. Let p' be the point on C' above p. Then p' is a 2-point and $\sigma_D(p') = 0$. We may thus assume that p is a 2-point and a form 5) holds at p. For $r \in \mathbb{Z}_+$, let

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X$$

be the sequence of blow ups of the point p_i which is the intersection of the strict transform C_i of C on X_i with the preimage of p.

There exist permissible parameters x, y, z at p such that $x = z = 0$ are formal local equations of C at p, and a form (2.2) holds at p with $F = x\Omega + f(y, z)$. We have that ord $f(y, z) = 1$, ord $\Omega(0, y, z) \geqslant 1$, y does not divide $f(y, z)$ and z does not divide $f(y, z)$.

At p_r, we have permissible parameters x_r, y_r, z_r such that

$$x = x_r y_r^r,\ y = y_r,\ z = z_r y_r^r.$$

$x_r = z_r = 0$ are local equations of C_r at p_r. We have a form (2.2) at p_r with

$$\begin{aligned} u &= (x_r^a y_r^{ar+b})^l \\ v &= P(x_r^a y_r^{ar+b}) + x_r^c y_r^{cr+d+r} F' \end{aligned}$$

where

$$F' = x_r\Omega + \frac{f(y_r, z_r y_r^r)}{y_r^r},$$

if $\frac{f(y_{r-1}, z_{r-1} y_{r-1}^{r-1})}{y_{r-1}^{r-1}}$ is not a unit series. Thus for r sufficiently large, we have that F' is a unit, so that a form 6) holds at p_r. □

Lemma 3.12. — *Suppose that X is 2-prepared and that C is a permissible curve on X. Suppose that $q \in C$ is a point with $\sigma_D(q) = 0$ which has a form 1), 4) or 6) of Lemma 3.10. Let $\pi_1 : X_1 \to X$ be the blow up of C. Then X_1 is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$. Further, $\sigma_{D_1}(q_1) = 0$ for all $q_1 \in \pi_1^{-1}(q)$.*

Proof. — The conclusions follow from substitution of local equations of the blow up of C into the forms 1), 4) and 6) of Lemma 3.10. □

Proposition 3.13. — *Suppose that X is 2-prepared. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$, such that X_1 is 3-prepared. We further have that $\sigma_D(p_1) \leqslant \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.*

Proof. — Let T be the points $p \in X$ such that X is not 3-prepared at p. By Lemmas 3.4 and 2.5, after we perform a sequence of blow ups of 2-curves, we may assume that T is a finite set consisting of 1-points of D.

Suppose that $p \in T$. Let $T' = T \setminus \{p\}$. Let $U = \operatorname{Spec}(R)$ be the affine neighborhood of p in X and let C be the curve in X of the conclusions of Lemma 3.8 (with $Y = T'$), so that C has local equations $\overline{x} = \overline{y} = 0$ in U.

Let $\Sigma_1 = C \cap \operatorname{Sing}_1(X)$. $\Sigma_1 = \{p = p_0, \ldots, p_r\}$ is the union of $\{p\}$ and a finite set of general points of curves in $\operatorname{Sing}_1(X)$, which must be 1-points. We have that $\Sigma_1 \subset U$. Let

$$\Sigma_2 = \{q \in C \cap U \mid \sigma_D(q) = 0 \text{ and a form 2) of Lemma 3.10 holds at } q\}.$$

Σ_2 is a finite set by Lemma 3.10. Let $\Sigma_3 = C \setminus U$, a finite set of 1-points and 2-points which are prepared.

Set $U' = U \setminus \Sigma_2$. There exists a unit $\tau \in R$ and $a \in \mathbb{Z}_+$ such that $u = \tau\overline{x}^a$.

By 5 vi), 5 vii) of Lemma 3.8 and Lemma 3.2, there exist $z_i \in \hat{\mathcal{O}}_{X,p_i}$ such that for all $p_i \in \Sigma_1$, $x = \tau^{\frac{1}{a}}\overline{x}, \overline{y}, z_i$ are permissible parameters at p_i giving a form (3.1).

Let $t = \max\{r(p_i) \mid 0 \leqslant i \leqslant r\}$, where $r(p_i)$ are calculated from (3.15)) of Lemma 3.7. There exists $\lambda \in R$ such that $\lambda \equiv \tau^{-\frac{1}{a}} \bmod m_{p_i}^t\hat{\mathcal{O}}_{X,p_i}$ for $0 \leqslant i \leqslant r$. Let $x^* = \lambda^{-1}\overline{x}$, $\overline{\gamma} = \tau^{\frac{1}{a}}\lambda$. Then $x = \tau^{\frac{1}{a}}\overline{x} = \overline{\gamma}x^*$ with $\overline{\gamma} \equiv 1 \bmod m_{p_i}^t\hat{\mathcal{O}}_{X,p_i}$ for $0 \leqslant i \leqslant r$. Let $U' = U \setminus \Sigma_2$.

Let $T_0^* = \operatorname{Spec}(\mathfrak{k}[x^*, \overline{y}])$, and let $T_1^* \to T_0^*$ be a sequence of blow ups of points above $(x^*, \overline{y})$ such that the conclusions of Lemma 3.7 hold on $U_1' = U' \times_{T_0^*} T_1^*$ above all p_i with $0 \leqslant i \leqslant r$. The projection $\lambda_1 : U_1' \to U'$ is a sequence of blow ups of sections over C. λ_1 is permissible and $\lambda_1^{-1}(C \cap (U' \setminus \Sigma_1))$ is prepared by Lemma 3.12.

All points of $\Sigma_2 \cup \Sigma_3$ are prepared. Thus by Lemma 3.9, Lemmas 3.11 and Lemma 3.12, by interchanging some blowups of points above $\Sigma_2 \cup \Sigma_4$ between blow ups of sections over C, we may extend λ_1 to a sequence of permissible blow ups over X to obtain the desired sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that X_1 is 2-prepared. π_1 is an isomorphism over T', X_1 is 3-prepared over $\pi_1^{-1}(X_1 \setminus T')$, and $\sigma_D(p_1) \leqslant \sigma_D(p)$ for all $p \in X_1 \setminus T'$.

By induction on $|T|$, we may iterate this procedure a finite number of times to obtain the conclusions of Proposition 3.13. □

The following proposition is proven in a similar way.

PROPOSITION 3.14. — *Suppose that X is 1-prepared and D' is a union of irreducible components of D. Suppose that there exists a neighborhood V of D' such that V is 2-prepared and V is 3-prepared at all 2-points and 3-points of V.*

Let A be a finite set of 1-points of D', such that A is contained in $\mathrm{Sing}_1(X)$ and A contains the points where V is not 3-prepared, and let B be a finite set of 2-points of D'. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that

1) *X_1 is 3-prepared in a neighborhood of $\pi_1^{-1}(D')$.*
2) *π_1 is an isomorphism over $X_1 \setminus D'$.*
3) *π_1 is an isomorphism in a neighborhood of B.*
4) *π_1 is an isomorphism over generic points of 2-curves on D' and over 3-points of D'.*
5) *Points on the intersection of the strict transform of D' on X_1 with $\pi_1^{-1}(A)$ are 2-points of D_{X_1}.*
6) *$\sigma_D(p_1) \leqslant \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.*

4. Reduction of σ_D above a 3-prepared point.

THEOREM 4.1. — *Suppose that $p \in X$ is a 1-point such that X is 3-prepared at p, and $\sigma_D(p) > 0$. Let x, y, z be permissible parameters at p giving a form (3.6) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U. Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by z^m, x^{r_m} if $\tau_m \neq 0$, and by*

$$\{x^{r_i} z^{m-i} \mid 2 \leqslant i \leqslant m-1 \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U' \to U$ is a toroidal morphism with respect to $\overline{D}$ such that U' is nonsingular and $I\mathcal{O}_{U'}$ is locally principal. Then (after possibly replacing U with a smaller neighborhood of p) U' is 2-prepared and $\sigma_D(q) < \sigma_D(p)$ for all $q \in U'$.

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \to S$. We have that the curve C_i blown up in $U_{i+1} \to U_i$ is in $\mathrm{Sing}_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $\mathrm{Sing}_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. — Suppose that $\psi : U' \to U$ is toroidal for $\overline{D}$ and U' is nonsingular. Let $\overline{D}' = \psi^{-1}(\overline{D})$.

The set of 2-curves of $\overline{D}'$ is the disjoint union of the 2-curves of $D_{U'}$ and the 2-curve which is the intersection of the strict transform of the surface $z = 0$ on U' with $D_{U'}$. ψ factors as a sequence of blow ups of 2-curves of (the preimage of) $\overline{D}$. We will verify the following three statements, from which the conclusions of the theorem follow.

(4.1) If $q \in \psi^{-1}(p)$ and $I\mathcal{O}_{U',q}$ is principal, then $\sigma_D(q) < \sigma_D(p)$. In particular, $\sigma_D(q) < \sigma_D(p)$ if q is a 1-point of $\overline{D}'$.

(4.2)
If C' is a 2-curve of $D_{U'}$, then U' is prepared at $q = C' \cap \psi^{-1}(p)$
if and only if $\sigma_D(q) < \infty$
if and only if $I\mathcal{O}_{U',q}$ is principal
if and only if U' is prepared at all $q' \in C'$ in a neighborhood of q.

(4.3)
If C' is the 2-curve of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $\tilde{z} = 0$ in U', then $\sigma_D(q) \leqslant \sigma_D(p)$ if $q = C' \cap \psi^{-1}(p)$, and $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\overline{D}'$. Then $I\hat{\mathcal{O}}_{U',q}$ is principal. At q, we have permissible parameters x_1, y, z_1 defined by

$$x = x_1^{a_1}, z = x_1^{b_1}(z_1 + \alpha) \tag{4.4}$$

for some $a_1, b_1 \in \mathbb{Z}_+$ and $0 \neq \alpha \in \mathfrak{k}$. Substituting into (3.6), we have

$$u = x_1^{aa_1}, v = P(x_1^{a_1}) + x_1^{ba_1} G$$

where

$$G = \tau_0 x_1^{b_1 m}(z_1 + \alpha)^m + \tau_2 x_1^{a_1 r_2 + b_1(m-2)}(z_1 + \alpha)^{m-2} + \cdots \\ + \tau_{m-1} x_1^{a_1 r_{m-1} + b_1}(z_1 + \alpha) + \tau_m x_1^{a_1 r_m}.$$

Let x_1^s be a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let $G' = \frac{G}{x_1^s}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U',q}$, then G' has an expansion

$$G' = \tau'(z_1 + \alpha)^m + g_2(z_1 + \alpha)^{m-2} + \cdots + g_{m-1}(z_1 + \alpha) + g_m + x_1\Omega_1 + y\Omega_2$$

where $0 \neq \tau' = \tau(0,0,0) \in \mathfrak{k}$, $g_2, \ldots, g_m \in \mathfrak{k}$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. We have $\operatorname{ord}(G'(0,0,z_1)) \leqslant m - 1$. Setting $F' = G' - G'(x_1, 0, 0)$ and $P'(x_1) =$

$P(x_1^{a_1}) + x_1^{ba_1+b_1m}G'(x_1, 0, 0)$, we have an expression

$$u = x_1^{aa_1}, v = P'(x_1) + x_1^{ba_1+b_1m}F'$$

of the form of (2.1). Thus U' is 2-prepared at q with $\sigma_{D'}(q) < m - 1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U',q}$, but there exists some i with $2 \leqslant i \leqslant m-1$ such that $x^{r_i}z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1 + \alpha)^{m-h} + \cdots + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. As in the previous case, we have that U' is 2-prepared at q with $\sigma_D(q) < m - h - 1 < m - 1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U',q}$ and $x^{r_i}z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U',q}$ for $2 \leqslant i \leqslant m-1$. Then $x_1^{r_m}$ is a local generator of $I\mathcal{O}_{U',q}$, and we have an expression

$$G' = \Lambda + x_1\Omega_1,$$

where $\Lambda(x_1, y, z_1) = \tau_m(x_1^{a_1}, y, x_1^{b_1}(z_1 + \alpha))$ and $\Omega_1 \in \hat{\mathcal{O}}_{U',q}$. Then

$$\operatorname{ord} \Lambda(0, y, 0) = \operatorname{ord} \tau_m(0, y, 0) = 1,$$

and we have that U' is prepared at q.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point for $D_{U'}$. We have permissible parameters x_1, y, z_1 in $\hat{\mathcal{O}}_{U',q}$ such that

$$x = x_1^{a_1}z_1^{b_1}, z = x_1^{c_1}z_1^{d_1} \tag{4.5}$$

with $a_1, b_1 > 0$ and $a_1d_1 - b_1c_1 = \pm 1$. Substituting into (3.6), we have

$$u = x_1^{a_1a}z_1^{b_1a}, v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b}z_1^{b_1b}G$$

where

$$G = \tau_0x_1^{c_1m}z_1^{d_1m} + \tau_2x_1^{r_2a_1+c_1(m-2)}z_1^{r_2b_1+d_1(m-2)} + \cdots + \tau_{m-1}x_1^{a_1r_{m-1}+c_1}z_1^{b_1r_{m-1}+d_1} + \tau_mx_1^{a_1r_m}z_1^{b_1r_m}.$$

Let C' be the 2-curve of $D_{U'}$ containing q. Since $\operatorname{ord}(\tau_m(0, y, 0)) = 1$ (if $\tau_m \neq 0$) we see that the three statements $\sigma_D(q) < \infty$, $\sigma_D(q) = 0$ and $I\mathcal{O}_{U',q}$ is principal are equivalent. Further, we have that $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $I\mathcal{O}_{U',q}$ is principal and let $x_1^sz_1^t$ be a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let $G' = G/x_1^sz_1^t$. We have that

$$u = (x_1^{a_1}z_1^{b_1})^a, \; v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b+s}z_1^{bb_1+t}G'$$

has the form (2.2), since we have made a monomial substitution in x and z. If z^m or $x^{r_i}z^{m-i}$ for some $i<m$ is a local generator of $I\hat{\mathcal{O}}_{U',q}$, then G' is a unit in $\hat{\mathcal{O}}_{U',q}$. If none of z^m, $x^{r_i}z^{m-i}$ for $i<m$ are local generators of $I\hat{\mathcal{O}}_{U',q}$, then

$$G' = \Lambda + x_1\Omega_1 + z_1\Omega_2,$$

where

$$\Lambda(x_1,y_1,z_1) = \tau_m(x_1^{a_1}z_1^{b_1}, y, x_1^{c_1}z_1^{d_1})$$

and $\Omega_1,\Omega_2 \in \hat{\mathcal{O}}_{U',q}$. Thus

$$\operatorname{ord}\Lambda(0,y,0) = \operatorname{ord}\tau_m(0,y,0) = 1.$$

We thus have that U' is prepared at q.

The final case is when $q\in\psi^{-1}(p)$ is on the 2-curve C' of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $z=0$ in U'. Then there exist permissible parameters x_1,y,z_1 at q such that

$$x = x_1, z = x_1^{b_1}z_1 \tag{4.6}$$

for some $b_1\in\mathbb{Z}_+$. The equations $x_1=z_1=0$ are local equations of C' at q. Let

$$s = \min\{b_1m, r_i+b_1(m-i) \text{ with } \tau_i\neq 0 \text{ for } 2\leqslant i\leqslant m-1, r_m \text{ if } \tau_i\neq 0\}.$$

We have an expression of the form (2.1) at q,

$$\begin{array}{rcl} u &=& x_1^a\\ v &=& P(x_1^a) + x_1^{ab+s}G' \end{array}$$

with

$$G' = \tau_0x_1^{b_1m-s}z_1^m + \tau_2x_1^{r_2+b_1(m-2)-s}z_1^{m-2} + \cdots \\ + \tau_{m-1}x_1^{r_{m-1}+b_1-s}z_1 + \tau_mx_1^{r_m-s}.$$

We see that $\sigma_D(q)\leqslant\sigma_D(p)$ (with $\sigma_D(q)<\sigma_D(p)$ if $s=r_i+b_1(m-i)$ for some i with $2\leqslant i\leqslant m-1$ or $s=r_m$) and $\sigma_D(q')=\sigma_D(q)$ for q' in a neighborhood of q on C'.

Suppose that $I\mathcal{O}_{U',q}$ is principal. Then x^{r_m} generates $I\hat{\mathcal{O}}_{U',q}$. We have that $G' = x_1^{r_m}\Omega$ where $\Omega\in\hat{\mathcal{O}}_{U',q}$ satisfies $\operatorname{ord}\Omega(0,y,0)=1$. Thus U' is prepared at q. □

We will now construct the function $\omega(m,r_2,\ldots,r_{m-1})$ where $m>1$, $r_i\in\mathbb{N}$ for $2\leqslant i\leqslant m-1$ and $r_{m-1}>0$.

Let I be the ideal in the polynomial ring $\mathfrak{k}[x,z]$ generated by z^m and $x^{r_i}z^{m-i}$ for all i such that $2\leqslant i\leqslant m-1$ and $r_i>0$. Let $\mathfrak{m}=(x,z)$ be the maximal ideal of $k[x,z]$. Let $\Phi: V_1\to V=\operatorname{Spec}(\mathfrak{k}[x,z])$ be the toroidal morphism with respect to the divisor $xz = 0$ on V such that V_1 is the minimal nonsingular surface such that

1) $I\mathcal{O}_{V_1,q}$ is principal if $q \in \Phi^{-1}(\mathfrak{m})$ is not on the strict transform of $z = 0$.
2) If q is the intersection point of the strict transform of $z = 0$ and $\Phi^{-1}(\mathfrak{m})$, so that q has regular parameters x_1, z_1, with $x = x_1, z = x_1^b z_1$ for some $b \in \mathbb{Z}_+$, then $r_i + b_1(m - i) < b_1 m$ for some $2 \leqslant i \leqslant m - 1$ with $r_i > 0$.

Every $q \in \Phi^{-1}(\mathfrak{m})$ which is not on the strict transform of $z = 0$ has regular parameters x_1, z_1 at q which are related to x, z by one of the following expressions:

$$x = x_1^{a_1},\ z = x_1^{b_1}(z_1 + \alpha) \tag{4.7}$$

for some $0 \neq \alpha \in \mathfrak{k}$ and $a_1, b_1 > 0$, or

$$x = x_1^{a_1} z_1^{b_1},\ z = x_1^{c_1} z_1^{d_1} \tag{4.8}$$

with $a_1, b_1 > 0$ and $a_1 d_1 - b_1 c_1 = \pm 1$. There are only finitely many values of a_1, b_1 occurring in expressions (4.7), and a_1, b_1, c_1, d_1 occurring in expressions (4.8).

The point q on the intersection of the strict transform of $z = 0$ and $\Phi^{-1}(\mathfrak{m})$ has regular parameters x_1, z_1 defined by

$$x = x_1,\ z = x_1^{b_1} z_1 \tag{4.9}$$

for some $b_1 > 0$.

Now we define $\omega = \omega(m, r_2, \ldots, r_{m-1})$ to be a number such that

$$\omega > \max\{\frac{b_1}{a_1} m, r_i + \frac{b_1}{a_1}(m - i) \text{ for } 2 \leqslant i \leqslant m - 1 \text{ such that } r_i > 0\}.$$

For all expressions (4.7),

$$\omega > \max\{\frac{c_1}{a_1} m, \frac{d_1}{b_1} m, r_i + \frac{c_1}{a_1}(m - i), r_i + \frac{d_1}{b_1}(m - i) \\ \text{for } 2 \leqslant i \leqslant m - 1 \text{ such that } r_i > 0\}$$

for all expressions (4.8), and

$$\omega > \max\{b_1 m, r_i + b_1(m - i) \text{ for } 2 \leqslant i \leqslant m - 1 \text{ such that } r_i > 0\}$$

in (4.9).

Theorem 4.2. — *Suppose that $p \in \mathit{Sing}_1(X)$ is a 1-point and X is 3-prepared at p. Let x, y, z be permissible parameters at p giving a form (3.7) at p. Let U be an étale cover of an affine neighborhood of p in which*

x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U.

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \to S$. We have that the curve C_i blown up in $U_{i+1} \to U_i$ is in $\mathrm{Sing}_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $\mathrm{Sing}_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. — The proof is similar to that of Theorem 4.1, using the fact that $t > \omega(m, r_2, \ldots, r_{m-1})$ as defined above. □

Theorem 4.3. — *Suppose that $p \in X$ is a 2-point and X is 3-prepared at p with $\sigma_D(p) > 0$. Let x, y, z be permissible parameters at p giving a form (3.5) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters on U. Then $xyz = 0$ gives a toroidal structure $\overline{D}$ on U. Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by z^m, $x^{r_m}y^{s_m}$ if $\tau_m \neq 0$ and*

$$\{x^{r_i}y^{s_i}z^{m-i} \mid 2 \leqslant i \leqslant m-1 \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U_1 \to U$ is a toroidal morphism with respect to $\overline{D}$ such that U_1 is nonsingular and $I\mathcal{O}_{U_1}$ is locally principal. Then (after possibly replacing U with a smaller neighborhood of p) U_1 is 2-prepared for $U_1 \to S$, with $\sigma_D(q) < \sigma_D(p)$ for all $q \in U_1$.

Proof. — Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\psi^{-1}(\overline{D})$. Then q is also a 1-point for D_{U_1}. Since ψ is toroidal with respect to $\overline{D}$, there exist regular parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ in $\hat{\mathcal{O}}_{X_1,q}$ and a matrix $A = (a_{ij})$ with nonegative integers as coefficients such that $\mathrm{Det}\ A = \pm 1$, and we have an expression

$$\begin{aligned} x &= \hat{x}_1^{a_{11}}(\hat{y}_1 + \alpha)^{a_{12}}(\hat{z}_1 + \beta)^{a_{13}} \\ y &= \hat{x}_1^{a_{21}}(\hat{y}_1 + \alpha)^{a_{22}}(\hat{z}_1 + \beta)^{a_{23}} \\ z &= \hat{x}_1^{a_{31}}(\hat{y}_1 + \alpha)^{a_{32}}(\hat{z}_1 + \beta)^{a_{33}} \end{aligned} \tag{4.10}$$

with $a_{11}, a_{21}, a_{31} \neq 0$ and $0 \neq \alpha, \beta \in \mathfrak{k}$. Set

$$\overline{x}_1 = \hat{x}_1(\hat{y}_1 + \alpha)^{\frac{a_{12}}{a_{11}}}(\hat{z}_1 + \beta)^{\frac{a_{13}}{a_{11}}} \in \hat{\mathcal{O}}_{X_1,q}.$$

Substituting into (4.10), we have

$$\begin{aligned} x &= \overline{x}_1^{a_{11}} \\ y &= \overline{x}_1^{a_{21}}(\hat{y}_1 + \alpha)^{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}(\hat{z}_1 + \beta)^{a_{23} - \frac{a_{21}a_{13}}{a_{11}}} \\ z &= \overline{x}_1^{a_{31}}(\hat{y}_1 + \alpha)^{a_{32} - \frac{a_{31}a_{12}}{a_{11}}}(\hat{z}_1 + \beta)^{a_{33} - \frac{a_{31}a_{13}}{a_{11}}}. \end{aligned} \tag{4.11}$$

Let $B=(b_{ij})$ be the adjoint matrix of A. Let $\overline{\alpha}=\alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}}$, $\overline{\beta}=\alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}}$. Set

$$\overline{y}_1=\frac{y}{\overline{x}_1^{a_{21}}}-\overline{\alpha},\overline{z}_1=\frac{z}{\overline{x}_1^{a_{31}}}-\overline{\beta}.$$

We will show that $\overline{x}_1,\overline{y}_1,\overline{z}_1$ are regular parameters in $\hat{\mathcal{O}}_{X_1,q}$. We have that

$$\begin{aligned}
&(\hat{y}_1+\alpha)^{a_{22}-\frac{a_{21}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{23}-\frac{a_{21}a_{13}}{a_{11}}}\\
&\qquad=\overline{\alpha}+\frac{b_{33}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}-1}\beta^{-\frac{b_{23}}{a_{11}}}\hat{y}_1-\frac{b_{23}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}-1}\hat{z}_1+\cdots\\
&(\hat{y}_1+\alpha)^{a_{32}-\frac{a_{31}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{33}-\frac{a_{31}a_{13}}{a_{11}}}\\
&\qquad=\overline{\beta}-\frac{b_{32}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}-1}\beta^{\frac{b_{22}}{a_{11}}}\hat{y}_1+\frac{b_{22}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}-1}\hat{z}_1+\cdots
\end{aligned}$$

Let

$$C=\begin{pmatrix}\frac{b_{33}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}-1}\beta^{-\frac{b_{23}}{a_{11}}} & -\frac{b_{23}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}-1}\\ -\frac{b_{32}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}-1}\beta^{\frac{b_{22}}{a_{11}}} & \frac{b_{22}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}-1}\end{pmatrix}.$$

We must show that C has rank 2. C has the same rank as

$$\begin{pmatrix}b_{33}\beta & -b_{23}\alpha\\ b_{32}\beta & -b_{22}\alpha\end{pmatrix}=\begin{pmatrix}b_{33} & b_{23}\\ b_{32} & b_{22}\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & -\alpha\end{pmatrix}.$$

Since $\alpha,\beta\neq 0$, C has the same rank as

$$B'=\begin{pmatrix}b_{33} & b_{23}\\ b_{32} & b_{22}\end{pmatrix}.$$

Since B has rank 3,

$$\begin{pmatrix}b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$$

has rank 2. Since

$$\begin{pmatrix}b_{21}\\ b_{31}\end{pmatrix}=-\frac{a_{21}}{a_{11}}\begin{pmatrix}b_{22}\\ b_{32}\end{pmatrix}+\frac{a_{31}}{a_{11}}\begin{pmatrix}b_{23}\\ b_{33}\end{pmatrix},$$

we have that B' has rank 2, and hence C has rank 2. Thus $\overline{x}_1,\overline{y}_1,\overline{z}_1$ are regular parameters in $\hat{\mathcal{O}}_{X_1,q}$. We have

$$x=\overline{x}_1^{a_{11}},y=\overline{x}_1^{a_{21}}(\overline{y}_1+\overline{\alpha}),z=\overline{x}_1^{a_{31}}(\overline{z}_1+\overline{\beta}).$$

We have that $u=(x^ay^b)^\ell$. Let

$$t=-\frac{b}{a_{11}a+a_{21}b},$$

and set $\overline{x}_1 = x_1(y_1+\overline{\alpha})^t$. Define $\overline{y}_1 = y_1$, $\tilde{\alpha} = \overline{\alpha}$, $\tilde{\beta} = \overline{\alpha}^{ta_{31}}\overline{\beta}$ and $z_1 = (\overline{y}_1+\overline{\alpha})^{ta_{31}}(z_1+\overline{\beta}) - \tilde{\beta}$. Then x_1, y_1, z_1 are permissible parameters at q, with $u = x_1^{(aa_{11}+ba_{21})l}$,

$$x = x_1^{a_{11}}(y_1+\tilde{\alpha})^{ta_{11}}, y = x_1^{a_{21}}(y_1+\tilde{\alpha})^{ta_{21}+1}, z = x_1^{a_{31}}(z_1+\tilde{\beta}).$$

Thus we have shown that there exist (formal) permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_1}(y_1+\tilde{\alpha})^{\lambda_1}, y = x_1^{e_2}(y_1+\tilde{\alpha})^{\lambda_2}, z = x_1^{e_3}(z_1+\tilde{\beta})$$

where $e_1, e_2, e_3 \in \mathbb{Z}_+$, $\tilde{\alpha}, \tilde{\beta} \in \mathfrak{k}$ are nonzero, $\lambda_1, \lambda_2 \in \mathbb{Q}$ are both nonzero, and $u = x_1^{b_1 l}$, where $b_1 = ae_1 + be_2$, $a\lambda_1 + b\lambda_2 = 0$. We then have an expression

$$v = P(x_1^{ae_1+be_2}) + x_1^{ce_1+de_2}G,$$

where

$$\begin{aligned} G = (y_1+\tilde{\alpha})^{c\lambda_1+d\lambda_2}[&\tau_0 x_1^{e_3 m}(z_1+\tilde{\beta})^m \\ &+ \tau_2 x_1^{r_2e_1+s_2e_2+(m-2)e_3}(y_1+\tilde{\alpha})^{r_2\lambda_1+s_2\lambda_2}(z_1+\tilde{\beta})^{m-2} + \cdots \\ &\quad + \tau_{m-1}x_1^{r_{m-1}e_1+s_{m-1}e_2+e_3}(y_1+\tilde{\alpha})^{r_{m-1}\lambda_1+s_{m-1}\lambda_2}(z_1+\tilde{\beta}) \\ &\qquad + \tau_m x_1^{r_me_1+s_me_2}y_1^{r_m\lambda_1+s_m\lambda_2}]. \end{aligned}$$

Let $\tau' = \tau_0(0,0,0)$. Let x_1^s be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{F}{x_1^s}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, then G' has an expression

$$G' = \tau'\tilde{\alpha}^{\varphi}(z_1+\tilde{\beta})^m + g_2(z_1+\tilde{\beta})^{m-2} + \cdots + g_{m-1}(z+\tilde{\beta}) + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$, where $\varphi = c\lambda_1 + d\lambda_2$. Setting $F' = G' - G'(x_1,0,0)$, and $P'(x_1) = P(x_1^{ae_1+be_2}) + x_1^{ce_1+de_2+s}G'(x_1,0,0)$, we have that

$$u = x_1^{b_1 l}, v = P'(x_1) + x_1^{ce_1+de_2+s}F'$$

has the form (2.1) and $\sigma_D(q) \leqslant \operatorname{ord} F'(0,0,z_1) - 1 \leqslant m-2 < m-1 = \sigma_D(p)$ since $0 \neq \tilde{\beta}$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, but there exists some i with $2 \leqslant i \leqslant m-1$ such that $\tau_i x^{r_i}y^{s_i}z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\tilde{\beta})^{m-h} + \cdots + g_{m-1}(z_1+\tilde{\beta}) + g_m + x_1\Omega_1 + y_2\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$ As in the previous case, we have

$$\sigma_D(q) \leqslant m-h-1 < m-1 = \sigma_D(p).$$

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and $\tau_i x^{r_i} y^{s_i} z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$ for $2 \leqslant i \leqslant m$. Then $x^{r_s} y^{r_s}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and G' has an expression

$$G' = \tau'_m (y_1 + \tilde{\alpha})^{\varphi + r_m \lambda_1 + s_m \lambda_2} + x_1 \Omega$$

where $\tau'_m = \tau_m(0,0,0)$ for some $\Omega \in \hat{\mathcal{O}}_{U_1,q}$. Suppose, if possible, that $\varphi + r_m\lambda_1 + s_m\lambda_2 = 0$. Since $\varphi + r_m\lambda_1 + s_m\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2$, we then have that the nonzero vector (λ_1, λ_2) satisfies $a\lambda_1 + b\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2 = 0$. Thus the determinant $a(d + s_m) - b(c + r_m) = 0$, a contradiction to our assumption that F satisfies (2.2).

Now since $\varphi + r_m\lambda_1 + s_m\lambda_2 \neq 0$ and $\tilde{\alpha} \neq 0$, we have $1 = \operatorname{ord} G'(0, y_1, 0) < m$, so that $\sigma_D(q) = 0 < m - 1 = \sigma_D(p)$.

Suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(\overline{D})$. Then there exist (formal) permissible parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ at q such that

(4.12)

$$x = \hat{x}_1^{e_{11}} \hat{y}_1^{e_{12}} (\hat{z}_1 + \hat{\alpha})^{e_{13}}, y = \hat{x}_1^{e_{21}} \hat{y}_1^{e_{22}} (\hat{z}_1 + \hat{\alpha})^{e_{23}}, z = \hat{x}_1^{e_{31}} \hat{y}_1^{e_{32}} (\hat{z}_1 + \hat{\alpha})^{e_{33}}$$

where $e_{ij} \in \mathbb{N}$, with $\operatorname{Det}(e_{ij}) = \pm 1$, and $\hat{\alpha} \in \mathfrak{k}$ is nonzero. We further have

$$e_{11} + e_{12} > 0, e_{21} + e_{22} > 0 \text{ and } e_{31} + e_{32} > 0.$$

First suppose that $e_{11}e_{22} - e_{12}e_{21} \neq 0$. Then q is a 2-point of D_{U_1}.

There exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that upon setting

$$\hat{x}_1 = x_1(z_1 + \hat{\alpha})^{\lambda_1} \text{ and } \hat{y}_1 = y_1(z_1 + \hat{\alpha})^{\lambda_2},$$

we have

$$x = x_1^{e_{11}} y_1^{e_{12}}, y = x_1^{e_{21}} y_1^{e_{22}}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \hat{\alpha})^r,$$

where

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}.$$

By Cramer's rule,

$$r = \pm \frac{1}{e_{11}e_{22} - e_{12}e_{21}} \neq 0.$$

Now set $z_1 = (z_1 + \hat{\alpha})^r - \hat{\alpha}^r$ and $\alpha = \hat{\alpha}^r$ to obtain permissible parameters x_1, y_1, z_1 at q with

$$x = x_1^{e_{11}} y_1^{e_{12}}, y = x_1^{e_{21}} y_1^{e_{22}}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha).$$

We have an expression

$$u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^\ell = (x_1^{t_1} y_1^{t_2})^{\ell_1}$$

where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\gcd(t_1, t_2) = 1$.

We then have an expression

$$v = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + x_1^{ce_{11}+de_{21}}y_1^{ce_{12}+de_{22}}G,$$

where

$$\begin{aligned} G = [&\tau_0 x_1^{me_{31}}y_1^{me_{32}}(z_1+\alpha)^m \\ &+ \tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}}y_1^{r_2e_{12}+s_2e_{22}+(m-2)e_{32}}(z_1+\alpha)^{m-2}+\cdots \\ &\qquad + \tau_{m-1}x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1^{r_{m-1}e_{12}+s_{m-1}e_{22}+e_{32}}(z_1+\beta) \\ &\qquad\qquad + \tau_m x_1^{r_me_{11}+s_me_{21}}y_1^{r_me_{12}+s_me_{22}}]. \end{aligned}$$

Let $\tau' = \overline{\tau}_0(0,0,0)$. Let $x_1^s y_1^t$ be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{G}{x_1^s y_1^t}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, then G' has an expression

$$G' = \tau'(z_1+\alpha)^m + g_2(z_1+\alpha)^{m-2}+\cdots+g_{m-1}(z-\alpha)+g_m+x_1\Omega_1+y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ and $\Omega_1,\Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$. Let

$$(4.13) \qquad \overline{P}(x_1^{t_1}y_1^{t_2}) = \sum_{t_2i-t_1j=0} \frac{1}{i!j!}\frac{\partial(x_1^{ce_{11}+de_{21}}y_1^{ce_{12}+de_{22}}G)}{\partial x_1^i \partial y_1^j}(0,0,0)x_1^i y_1^j$$

and $F' = G' - \frac{\overline{P}(x_1^{t_1}y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}}$. Set $P'(x_1^{t_1}y_1^{t_2}) = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + \overline{P}(x_1^{t_1}y_1^{t_2})$. We have that

$$u = (x_1^{t_1}y_1^{t_2})^{\ell_1}, v = P'(x_1^{t_1}y_1^{t_2}) + x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}F'$$

has the form (2.2), and $\sigma_D(q) = \operatorname{ord} F'(0,0,z_1) - 1 \leqslant m-2 < m-1 = \sigma_D(p)$ since $0 \neq \alpha$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, but there exists some i with $2 \leqslant i \leqslant m-1$ such that $\tau_i x^{r_i}y^{s_i}z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\beta)^{m-h}+\cdots+g_m+x_1\Omega_1+y_2\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$ As in the previous case, we have $\sigma_D(q) \leqslant m-h-1 < m-1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and $\tau_i x^{r_i}y^{s_i}z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$ for $2 \leqslant i \leqslant m-1$. Then $x^{r_m}y^{r_m}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and then G' has an expression

$$G' = 1 + x_1\Omega_1 + y_1\Omega_2$$

for some $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$.

We now claim that after replacing G' with

$$F' = G' - \frac{\overline{P}(x_1^{t_1} y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s} y_1^{ce_{12}+de_{22}+t}},$$

where $\overline{P}$ is defined by (4.13), we have that $F'(0,0,0) \neq 0$. If this were not the case, we would have

$$\begin{aligned} 0 &= \mathrm{Det}\begin{pmatrix} (c+r_m)e_{11} + (d+s_m)e_{21} & (c+r_m)e_{12} + (d+s_m)e_{22} \\ ae_{11} + be_{21} & ae_{12} + be_{22} \end{pmatrix} \\ &= \mathrm{Det}\begin{pmatrix} c+r_m & d+s_m \\ a & b \end{pmatrix} \mathrm{Det}\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}. \end{aligned}$$

Since $e_{11}e_{22} - e_{21}e_{12} \neq 0$ (by our assumption), we get

$$0 = \mathrm{Det}\begin{pmatrix} c+r_m & d+s_m \\ a & b \end{pmatrix}$$

which is a contradiction to our assumption that F satisfies (2.2). Since $F'(0,0,0) \neq 0$, we have that $\sigma_D(q) = 0 < m-1 = \sigma_D(p)$.

Now suppose that q is a 2-point of $\psi^{-1}(\overline{D})$ with $e_{11}e_{22} - e_{21}e_{12} = 0$ in (4.12).

We make a substitution

$$\hat{x}_1 = x_1(z_1+\alpha)^{\varphi_1}, \hat{y}_1 = y_1(z_1+\alpha)^{\varphi_2}, \hat{z}_1 = z_1$$

where $\alpha = \hat{\alpha}$ and $\varphi_1, \varphi_2 \in \mathbb{Q}$ satisfy

$$\begin{aligned} 0 &= a(\varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}) + b(\varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}) \\ &= \varphi_1(ae_{11} + be_{21}) + \varphi_2(ae_{12} + be_{22}) + ae_{13} + be_{23}. \end{aligned}$$

We have $ae_{11} + be_{21} > 0$ and $ae_{12} + be_{22} > 0$ since $a, b > 0$ and by the condition satisfied by the e_{ij} stated after (4.12).

Let

$$\lambda_1 = \varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}, \lambda_2 = \varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}, \lambda_3 = \varphi_1 e_{31} + \varphi_2 e_{32} + e_{33}.$$

Then x_1, y_1, z_1 are permissible parameters at q such that

$$x = x_1^{e_{11}} y_1^{e_{12}} (z_1+\alpha)^{\lambda_1}, y = x_1^{e_{21}} y_1^{e_{22}} (z_1+\alpha)^{\lambda_2}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1+\alpha)^{\lambda_3} \tag{4.14}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}$, and $a\lambda_1 + b\lambda_2 = 0$.

Now suppose that $e_{11} > 0$ and $e_{12} > 0$, which is the case where q is a 2-point of D_{U_1}. Write

$$u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^\ell = (x_1^{t_1} y_1^{t_2})^{\ell_1}$$

where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\gcd(t_1, t_2) = 1$.

We then have an expression

$$v = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + x_1^{ce_{11}+de_{21}}y_1^{ce_{12}+de_{22}}G,$$

where

$$\begin{aligned} G =&(z_1+\alpha)^{c\lambda_1+d\lambda_2}[\tau_0 x_1^{me_{31}}y_1^{me_{32}}(z_1+\alpha)^{m\lambda_3} \\ &+\tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}}y_1^{r_2e_{12}+s_2e_{22}+(m-2)e_{32}}(z_1+\alpha)^{r_2\lambda_1+s_2\lambda_2+(m-2)\lambda_3}+\cdots \\ &+\tau_{m-1}x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1^{r_{m-1}e_{12}+s_{m-1}e_{22}+e_{32}}(z_1+\alpha)^{\lambda_1 r_{m-1}+\lambda_2 s_{m-1}+\lambda_3} \\ &+\tau_m x_1^{r_me_{11}+s_me_{21}}y_1^{r_me_{12}+s_me_{22}}(z_1+\alpha)^{r_m\lambda_1+s_m\lambda_2}]. \end{aligned}$$

Let $x_1^sy_1^t$ be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{F}{x_1^sy_1^t}$.

We will now establish that, with our assumptions, there is a unique element of the set S consisting of z^m, and

$$\{x^{r_i}y^{s_i}z^{m-i} \mid 2 \leqslant i \leqslant m \text{ and } \tau_i \neq 0\}$$

which is a generator of $I\hat{\mathcal{O}}_{U_1,q}$; that is, is equal to $x_1^sy_1^t$ times a unit in $\hat{\mathcal{O}}_{U_1,q}$. Let $r_0=0$ and $s_0=0$. Suppose that $x^{r_i}y^{r_i}z^{m-i}$ (with $0\leqslant i\leqslant m$) is a generator of $I\hat{\mathcal{O}}_{U_1,q}$. We have $x^{r_i}y^{s_i}z^{m-i} = x_1^sy_1^t(z_1+\alpha)^{\gamma_i}$ where

$$\begin{aligned} r_ie_{11}+s_ie_{21}+(m-i)e_{31} &= s \\ r_ie_{12}+s_ie_{22}+(m-i)e_{32} &= t \\ r_i\lambda_1+s_i\lambda_2+(m-i)\lambda_3 &= \gamma_i. \end{aligned}$$

Let

$$A = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}. \tag{4.15}$$

We have

$$A\begin{pmatrix} r_i \\ s_i \\ m-i \end{pmatrix} = \begin{pmatrix} s \\ t \\ \gamma_i \end{pmatrix}. \tag{4.16}$$

Let $\omega = \mathrm{Det}(A)$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix}\begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

implies $\omega = \mathrm{Det}(A) = \pm 1$.

By Cramer's rule, we have

$$\omega(m-i) = \mathrm{Det}\begin{pmatrix} e_{11} & e_{21} & s \\ e_{12} & e_{22} & t \\ \lambda_1 & \lambda_2 & \gamma_i \end{pmatrix}$$
$$= s\mathrm{Det}\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix} + \gamma_i\mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix}.$$

Since $e_{11}e_{21} - e_{12}e_{22} = 0$ by assumption, we have that

$$i = m - \frac{1}{\omega}\left(s\mathrm{Det}\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix}\right).$$

In particular, there is a unique element $x^{r_i}y^{r_i}z^{m-i} \in S$ which is a generator of $I\hat{\mathcal{O}}_{U_1,q}$. We have $x^{r_i}y^{s_i}z^{m-i} = x_1^s t_1^t (z_1+\alpha)^{\gamma_i}$.

We thus have that $G = x_1^s y_1^t[g(z_1+\alpha)^{\gamma_i + c\lambda_1 + d\lambda_2} + x_1\Omega_1 + y_1\Omega_2]$ for some $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$ and $0 \neq g \in \mathfrak{k}$.

We now establish that we cannot have that $\gamma_i + c\lambda_1 + d\lambda_2 = 0$ and $x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}$ is a power of $x_1^{t_1}y_1^{t_2}$. We will suppose that both of these conditions do hold, and derive a contradiction. Now we know that $x^a y^b = x_1^{ae_{11}+be_{21}}y_1^{ae_{12}+be_{22}}$ is a power of $x_1^{t_1}y_1^{t_2}$. By (4.15), (4.16) and our assumptions, we have that

$$A\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

and

$$A\begin{pmatrix} c+r_i \\ d+s_i \\ m-i \end{pmatrix}$$

are rational multiples of

$$\begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix}.$$

Since $\omega = \mathrm{Det}(A) \neq 0$, we have that $(c+r_i, d+s_i, m-i)$ is a rational multiple of $(a,b,0)$. Thus $x^c y^d x^{r_i} y^{s_i} z^{m-i}$ is a power of $x^a y^b$, a contradiction to our assumption that F satisfies (2.2).

Let

$$\overline{P}(x_1^{t_1}y_1^{t_2}) = \sum_{t_2 i - t_1 j = 0} \frac{1}{i!j!}\frac{\partial(x_1^{ce_{11}+de_{21}}y_1^{ce_{12}+de_{22}}G)}{\partial x_1^i \partial y_1^j}(0,0,0)x_1^i y_1^j,$$

and $F' = G' - \frac{\overline{P}(x_1^{t_1}y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}}$. Set

$P'(x_1^{t_1}y_1^{t_2}) = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + \overline{P}(x_1^{t_1}y_1^{t_2})$. We have that

$$u = (x_1^{t_1}y_1^{t_2})^{\ell_1}, v = P'(x_1^{t_1}y_1^{t_2}) + x_1^{ee_{11}+fe_{21}}y_1^{ce_{21}+de_{22}}F'$$

has the form (2.2) and $\sigma_D(q) = 0 \leqslant m-2 = \sigma_D(p)$.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(\overline{D})$, $e_{11}e_{22} - e_{12}e_{21} = 0$ in (4.12), and $e_{11} = 0$ or $e_{12} = 0$. Without loss of generality, we may assume that $e_{12} = 0$. q is a 1-point of D_{U_1}, and we have permissible parameters (4.14) at q. Since $\mathrm{Det}(e_{ij}) = \pm 1$, we have that $e_{32} = 1$, and $e_{11}e_{23} - e_{21}e_{13} = \pm 1$. Replacing y_1 with $y_1(z_1+\alpha)^{\lambda_3}$ in (4.14), we find permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_{11}}(z_1+\alpha)^{\lambda_1},\ y = x_1^{e_{21}}(z_1+\alpha)^{\lambda_2},\ z = x_1^{e_{31}}y_1, \tag{4.17}$$

with $e_{11}, e_{21} > 0$ and $a\lambda_1 + b\lambda_2 = 0$. We have

$$\begin{aligned} u &= x_1^{(ae_{11}+be_{21})l} = x_1^{l_1} \\ v &= P(x_1^{ae_{11}+be_{21}}) + x_1^{ce_{11}+de_{21}}G \end{aligned}$$

where

$$\begin{aligned} G = (z_1+\alpha)^{c\lambda_1+d\lambda_2}[&\tau_0 x_1^{me_{31}}y_1^m \\ &+ \tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}}y_1^{m-2}(z_1+\alpha)^{r_2\lambda_1+s_2\lambda_2} + \cdots \\ &+ \tau_{m-1}x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1(z_1+\alpha)^{r_{m-1}\lambda_1+s_{m-1}\lambda_2} \\ &+ \tau_m x_1^{r_me_{11}+s_me_{21}}(z_1+\alpha)^{r_m\lambda_1+s_m\lambda_2}]. \end{aligned}$$

Since $I\hat{\mathcal{O}}_{U_1,q}$ is principal and τ_m or $\tau_{m-1} \neq 0$, we have that $x_1^{r_me_{11}+s_me_{21}}$ is a generator of $I\hat{\mathcal{O}}_{U_1,q}$ if $\tau_m \neq 0$, and $x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1$ is a generator of $I\hat{\mathcal{O}}_{U_1,q}$ if $\tau_m = 0$ and $\tau_{m-1} \neq 0$.

First suppose that $\tau_m \neq 0$ so that

$$G = x_1^{r_me_{11}+s_me_{21}}[g_m(z_1+\alpha)^{(c+r_m)\lambda_1+(d+s_m)\lambda_2} + x_1\Omega + y_1\Omega_2]$$

with $0 \neq g_m \in \mathfrak{k}$, $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$. Since λ_1, λ_2 are not both zero, $a\lambda_1 + b\lambda_2 = 0$ and $a(d+s_m) - b(c+r_m) \neq 0$, we have that $(c+r_m)\lambda_1 + (d+s_m)\lambda_2 \neq 0$. Let $\overline{P}(x_1) = G(x_1, 0, 0)$, and $P'(x_1) = P(x_1^{ae_{11}+be_{21}}) + \overline{P}(x_1)$. Let

$$F' = \frac{1}{x_1^{ce_{11}+de_{21}}}(G - \overline{P}(x_1)).$$

Then

$$\begin{aligned} u &= x_1^{l_1} \\ v &= P'(x_1) + x_1^{ce_{11}+de_{21}}F' \end{aligned}$$

is of the form (2.1) with $\mathrm{ord}\, F'(0, y_1, z_1) = 1$. Thus $\sigma_D(q) = 0 < \sigma_D(p)$.

Now suppose that $\tau_m = 0$, so that

$$G = x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}[g_{m-1}y_1(z_1+\alpha)^{(c+r_{m-1})\lambda_1+(d+s_{m-1})\lambda_2} + x_1\Omega_1]$$

with $0 \neq g_{m-1} \in \mathfrak{k}$ and $\Omega_1 \in \hat{\mathcal{O}}_{U_1,q}$. Thus $\sigma_D(q) = 0 < \sigma_D(p)$.

The final case is when q is a 3-point for $\psi^{-1}(\overline{D})$, so that q is a 3-point or a 2-point of D_{U_1}. Then we have permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_{11}}y_1^{e_{12}}z_1^{e_{13}}, y = x_1^{e_{21}}y_1^{e_{22}}z_1^{e_{23}}, z = x_1^{e_{31}}y_1^{e_{32}}z_1^{e_{33}}$$

with $\omega = \mathrm{Det}(e_{ij}) = \pm 1$. Thus there is a unique element of the set S consisting of z^m and

$$\{x^{r_i}y^{s_i}z^{m-i} \mid 2 \leqslant i \leqslant m \text{ and } \overline{\tau}_i \neq 0\}$$

which is a generator $x_1^{s_1}y_1^{s_2}z_1^{s_3}$ of $I\hat{\mathcal{O}}_{U',q}$. Thus $\sigma_D(q) = 0$ if q is a 3-point of D_{U_1}. If q is a 2-point of D_{U_1}, we may assume that $e_{13} = e_{23} = 0$. Then $e_{33} = 1$. Since $\tau_m \neq 0$ or $\tau_{m-1} \neq 0$, we calculate that $\sigma_D(q) = 0$. □

5. Global reduction of σ_D

LEMMA 5.1. — *Suppose that X is 2-prepared and $p \in X$ is 3-prepared. Suppose that $r = \sigma_D(p) > 0$.*

a) *Suppose that p is a 1-point. Then there exists a unique curve C in $\mathrm{Sing}_1(X)$ containing p. The curve C is contained in $\mathrm{Sing}_r(X)$. If x, y, z are permissible parameters at p giving an expression (3.6) or (3.7) at p, then $x = z = 0$ are formal local equations of C at p.*

b) *Suppose that p is a 2-point and C is a curve in $\mathrm{Sing}_r(X)$ containing p. If x, y, z are permissible parameters at p giving an expression (3.5) at p, then $x = z = 0$ or $y = z = 0$ are formal local equations of C at p.*

Proof. — We first prove a). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining the reduced scheme $\mathrm{Sing}_1(X)$. Then $\mathcal{I}_p\hat{\mathcal{O}}_{X,p} = \sqrt{(x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})} = (x, z)$ is an ideal on U defining $\mathrm{Sing}_1(U)$. Thus the unique curve C in $\mathrm{Sing}_1(X)$ through p has (formal) local equations $x = z = 0$ at p. At points near p on C, a form (3.6) or (3.7) continues to hold with $m = r + 1$. Thus the curve is in $\mathrm{Sing}_r(X)$.

We now prove b). Suppose that $C \subset \mathrm{Sing}_r(X)$ is a curve containing p. By Theorem 4.3, there exists a toroidal morphism $\Psi : U_1 \to U$ where U is an étale cover of an affine neighborhood of p, and $\overline{D}$ is the local toroidal structure on U defined (formally at p) by $xyz = 0$, such that all points q

of U_1 satisfy $\sigma_D(q) < r$. Hence the strict transform on U_1 of the preimage of C on U must be empty. Since Ψ is toroidal for $\overline{D}$ and X is 3-prepared at p, C must have local equations $x = z = 0$ or $y = z = 0$ at p. □

Definition 5.2. — *Suppose that X is 3-prepared. We define a canonical sequence of blow ups over a curve in X, under the following conditions:*

1) *Suppose that C is a curve in X such that $t = \sigma_D(q) > 0$ at the generic point q of C, and all points of C are 1-points of D. Then we have that C is nonsingular and $\sigma_D(p) = t$ for all $p \in C$ by Lemma 5.1. By Lemma 5.1 and Theorem 4.1 or 4.2, there exists a unique minimal sequence of permissible blow ups of sections over C, $\pi_1 : X_1 \to X$, such that X_1 is 2-prepared and $\sigma_D(p) < t$ for all $p \in \pi_1^{-1}(C)$. We will call the morphism π_1 the canonical sequence of blow ups over C.*
2) *Suppose that C is a permissible curve in X which contains a 1-point such that $\sigma_D(p) = 0$ for all $p \in C$, and a condition 1), 4) or 6) of Lemma 3.10 holds at all $p \in C$. Let $\pi_1 : X_1 \to X$ be the blow up of C. Then by Lemma 3.12, X_1 is 3-prepared and $\sigma_D(p) = 0$ for $p \in \pi_1^{-1}(C)$. We will call the morphism π_1 the canonical blow up of C.*

Theorem 5.3. — *Suppose that X is 2-prepared. Then there exists a sequence of permissible blowups $\psi : Y \to X$ such that Y is prepared.*

Before proving this theorem, we introduce some notation, and give some idea of the main difficulty of the proof.

Suppose that $p \in X$ is a 2-point such that X is 3-prepared at p and $\sigma_D(p) = r > 0$. We can then define $(U_p, \overline{D}_p, I_p, \nu_p^1, \nu_p^2)$ as in Theorem 4.3, where ν_p^t are valuations on U_p which dominate the two curves C_1, C_2 which are the intersection of E_p with D_{U_p} on U_p (where $\overline{D}_p = D_{U_p} + E_p$), and which have the property that if $\pi : V \to U_p$ is a birational morphism, then the center $C(V, \nu_p^t)$ of ν_p^t on V is the unique curve on the strict transform of E_p on V which dominates C_t. We will call $(U_p, \overline{D}_p, I_p, \nu_p^1, \nu_p^2)$ a local resolver. We will think of U_p as a germ, so we will feel free to replace U_p with a smaller neighborhood of p whenever it is convenient.

If $\pi : Y \to X$ is a birational morphism, then we define $C(Y, \nu_p^i)$ to be the closed curve in Y which is the center of ν_p^t on Y. We define $\Lambda(Y, \nu_p^t)$ to be the point $C(Y, \nu_p^t) \cap \pi^{-1}(p)$. This defines a valuation which is composite with ν_p^t.

We define $W(Y, p)$ to be the germ in Y of the image of points in $\pi^{-1}(U_p) = Y \times_X U_p$ such that $I_p\mathcal{O}_Y \mid \pi^{-1}(U_p)$ is not invertible. $W(Y, p)$

is a subset of the union of the set of generic points of 2-curves for $\overline{D}_p$ in $Y \times_X U_p$, and the set of all points of $\pi^{-1}(p)$.

If $\pi : Y \to X$ is a morphism, define $\mathrm{Preimage}(Y, Z) = \pi^{-1}(Z)$ for Z a subset of X.

Suppose that $\pi : Y \to X$ is a composition of permissible blow ups which is toroidal for $\overline{D}_p$ above $\overline{Y} := \pi^{-1}(U_p)$. The blow up of a three point for $\overline{D}_p$ or of a 2-curve for $\overline{D}_p$ which π contracts to p extends readily to a permissible blow up of Y, as does a permissible blow up of a 2-curve of D. The only remaining case of the blow up of a 3-point or 2-curve of $\overline{D}_p$ on $\overline{Y}$ is the blow up of one of the two curves $C(Y, \nu_p^1)$ or $C(Y, \nu_p^2)$. Of course such a curve may only be permissible over U_p.

We can principalize I_p above U_p by the following algorithm: First perform any sequence $\overline{Y} \to U_p$ consisting of blow ups of 3-points of $\overline{D}_p$ and 2-curves of $\overline{D}_p$, with the restriction that the map is an isomorphism over the generic points of $C(U_p, \nu_p^1)$ and $C(U_p, \nu_p^2)$. Now construct $\overline{Y}_1 \to \overline{Y}$ be blowing up $C(\overline{Y}, \nu_p^t)$ for some t, such that $I_p\mathcal{O}_{\overline{Y},\eta}$ is not principal, where η is the generic point of $C(\overline{Y}, \nu_p^t)$. Then once again perform any sequence of blow ups $\overline{Y}_2 \to \overline{Y}_1$ consisting of blow ups of 3-points of $\overline{D}_p$ and 2-curves of $\overline{D}_p$, with the restriction that the map is an isomorphism over the generic points of $C(\overline{Y}_1, \nu_p^1)$ and $C(\overline{Y}_2, \nu_p^2)$. Now we define $\overline{Y}_3 \to \overline{Y}_2$ to be the blow up of $C(\overline{Y}_2, \nu_p^t)$ for some t, such that $I_p\mathcal{O}_{\overline{Y}_2,\xi}$ is not principal, where ξ is the generic point of $C(\overline{Y}_2, \nu_p^t)$. A chain of blowups of this type will eventually produce a $\overline{Y}_n$ such that $I_p\mathcal{O}_{\overline{Y}_n,\eta_t}$ is principal, where η_t is the generic point of $C(\overline{Y}_n, \nu_p^t)$ for $t = 1, 2$. If this has been accomplished, then we may perform a final sequence of blowups $\overline{Y}_{n+1} \to \overline{Y}_n$, consisting of blow ups of 3-points of $\overline{D}_p$ and 2-curves of $\overline{D}_p$, with the restriction that the map is an isomorphism over the generic points of $C(\overline{Y}_1, \nu_p^1)$ and $C(\overline{Y}_2, \nu_p^2)$, such that $I_p\mathcal{O}_{\overline{Y}_{n+1}}$ is locally principal. We thus have that $\sigma_D(q) < r$ for all points $q \in \overline{Y}_{n+1}$ (by Theorem 4.3).

The essential difficulty in extending this local argument to a proof of Theorem 5.3 is to extend the local blow ups of $C(\overline{Y}_i, \nu_p^t)$ to permissible global blow ups above X, which do not interfere with the the local resolution procedures above other points of X.

We will construct sequences

$$Y_n \to Y_{n-1} \to \cdots \to Y_0 = X \tag{5.1}$$

where each Y_i has an associated finite set $S(Y_i)$, which we will often abbreviate as $S(i)$. We require that $S(0) = \emptyset$, and that

$S(i)$ is contained in the disjoint union of the Y_j with $j < i$.

Each morphism $Y_{i+1} \to Y_i$ is a permissible blow up, or the identity map with $Y_{i+1} = Y_i$ and $S(i+1) = S(i) \cup \{p\}$ for some $p \in Y_i$, which is a 2-point for D with $\sigma_D(p) > 0$, such that Y_i is 3-prepared at p, and we introduce a local resolver $(U_p, \overline{D}_p, \nu_p^1, \nu_p^2)$ at p, or $Y_{i+1} = Y_i$ and $S(i+1)$ is a subset of $S(i)$. We require that $S(i)$ be contained in the disjoint union of the Y_j with $j < i$, and $p \in S(i) \cap Y_j$ implies p is a 3-prepared 2-point in $Y_j \setminus \left(\cup_{p' \in S(j)} W(Y_j, p')\right)$, with $\sigma_D(p) > 0$, and there is a given local resolver $(U_p, \overline{D}_p, \nu_p^1, \nu_p^2)$ in Y_j for p. Let $W(Y_i) = \cup_{p' \in S(i)} W(Y_i, p')$. We will often write $W(i) = W(Y_i)$. We require that each morphism $Y_{i+1} \to Y_i$ be an *admissible blow up*, which we define to be a permissible blow up such that for all $p \in S(i)$, $Y_{i+1} \to Y_i$ is toroidal for $\overline{D}_p$ above a neighborhood of $W(Y_i, p)$.

A sequence (5.1) will be called an admissible sequence. In the first approximation, $S(Y_i)$ may be seen as the set of "bad points" $p \in Y_j$ (for $j < i$) with "bad preimages" in Y_i. Their preimages are not fully 3-prepared, or contain singular points or $I_p \mathcal{O}_{Y_i}$ is not invertible. By performing a succession of admissible sequences, we want to obtain that $S(Y_n) = \emptyset$.

Define

$$\sigma(Y_i) := \max\{\{\sigma_D(p) \mid p \in Y_i \setminus W(i)\} \cup \{\sigma_D(q) \mid q \in S(i)\}\}.$$

Definition 5.4. — *Suppose that $Y_{i_0} \to X$ is an admissible sequence, and C is a curve in D_{Y_i} which contains a 1-point of D. Let η be the generic point of C. C is called a good curve if one of the following conditions hold:*

1. *If $\sigma_D(\eta) = 0$, then $\sigma_D(p) = 0$ for all $p \in C \setminus W(i_0)$ and $p \in C \cap W(i_0)$ implies $p = \Lambda(Y_{i_0}, \nu_b^t)$ and $C = C(Y_{i_0}, \nu_b^t)$ for some $b \in S(i_0)$ and t.*
2. *If $\sigma_D(\eta) > 0$, then $C \setminus W(i_0)$ is a set of 3-prepared 1-points and $p \in C \cap W(i_0)$ implies $p = \Lambda(Y_{i_0}, \nu_b^t)$, $C = C(Y_{i_0}, \nu_b^t)$ for some $b \in S(i_0)$ and t (in particular, p is a 2-point of D).*

We will be particularly concerned with sequences (5.1) which admit expressions

$$Y = Y_n = Y_{i_s} \to \cdots \to Y_{i_2} \to Y_{i_1} \to Y_0 = X \tag{5.2}$$

where each $Y_{i_{j+1}} \to Y_{i_j}$ is the sequence

$$Y_{i_{j+1}} \to Y_{i_{j+1}-1} \to \cdots \to Y_{i_j+1} \to Y_{i_j},$$

such that each of the $Y_{i_{j+1}} \to Y_{i_j}$ in (5.2) is one of the following, called an admissible transformation:

1. The blow up of a prepared point of D, and $S(i_{j+1}) = S(i_j)$.
2. The blow up of a 3-point or a 2-curve of D, and $S(i_{j+1}) = S(i_j)$.

3. The blow up of a 3-point or 2-curve for $\overline{D}_p$, contained in $W(Y_{i_j}, p)$ (with $p \in S(i_j) \cap Y_k$), which contracts to p under $Y_{i_j} \to Y_k$ and $S(i_{j+1}) = S(i_j)$.
4. $Y_{i_{j+1}} = Y_{i_j}$ and $S(i_{j+1}) = S(i_j) \cup \{p\}$ for some $p \in Y_{i_j} \setminus W(i_j)$, which is a 2-point for D such that Y_{i_j} is 3-prepared at p, and $\sigma_D(p) > 0$, and we introduce a local resolver $(U_p, \overline{D}_p, \nu_p^1, \nu_p^2)$ at p.
5. The sequence of permissible blow ups of Proposition 3.14, applied to a union of irreducible components E of D such that all 2 and 3 points for D in a neighborhood of E are 3-prepared, and $W(i) \cap E$ contains only a finite set of 2-points (which we take to be the set B of Proposition 3.14), over which $Y_{i_{j+1}} \to Y_{i_j}$ is an isomorphism. The effect of this transformation is to make all points in a neighborhood of Preimage$(Y_{i_{j+1}}, E)$ 3-prepared. We have $S(i_{j+1}) = S(i_j)$.
6. The "canonical sequence of blow ups" above a good curve C in $D_{Y_{i_j}}$ (This transformation will be defined after Lemma 5.10). We will generally have $S(i_{j+1}) \setminus S(i_j) \neq \emptyset$.
7. $Y_{i_{j+1}} = Y_{i_j}$ and $S(i_{j+1}) = S(i_j) \setminus \{p \in S(i_j) \mid W(Y_{i_j}, p) = \emptyset\}$.

Lemma 5.5. — *Suppose that (5.2) is an admissible sequence consisting entirely of admissible transformations of types 1 - 5 and 7. Then for $0 \leqslant j \leqslant n$ in (5.2), the following conditions (5.3) - (5.7) hold:*

(5.3)
The closed sets $W(Y_{i_j}, p) \cap$ Preimage(Y_{i_j}, p) are pairwise disjoint for $p \in S(i_j)$.

(5.4) *All points of $Y_{i_j} \setminus W(i_j)$ are 2-prepared.*

(5.5)
For $p \in S(i_j)$, $Y_{i_{j+1}} \to Y_{i_j}$ is toroidal for $\overline{D}_p$ above a neighborhood of $W(Y_{i_j}, p)$.

$$\sigma(Y_{i_{j+1}}) \leqslant \sigma(Y_{i_j}) \tag{5.6}$$

(5.7)
Suppose that $r = \sigma(Y_{i_j})$ and $\sigma(Y_{i_j} \setminus W(i_j)) < r$.
Then $\sigma(Y_{i_{j+1}} \setminus W(i_{j+1})) < r$ and if $p \in S(i_{j+1}) \setminus S(i_j)$, then $\sigma_D(p) < r$.

Proof. — For admissible transformations of types 1 - 5 (5.3) - (5.6) hold since $D \subset \overline{D}_p$ for all $p \in S(i_j)$, and by Lemma 3.9 (for transformations of type 1), Lemma 2.5 (for transformations of type 2), Theorem 4.3 (for transformations of type 3) and Propositions 3.14 (for transformations of type 5).

(5.7) holds for all admissible transformations of types 1 - 5, since for $p\in S(i_j)$, $0<\sigma_D(p)\leqslant r$. Thus by Theorem 4.3, we have

$$\sigma_D(q)<r \text{ if } q\in \mathrm{Preimage}(Y_{i_{j+1}},W(Y_{i_j},p)),$$

since $Y_{i_{j+1}}\to Y_{i_j}$ is toroidal for $\overline{D}_p$ above a neighborhood of $W(Y_{i_j},p)$ and $I_p\mathcal{O}_{Y_{i_{j+1}},q}$ is invertible. □

Remark 5.6. — (5.3) tells us that if $p\in Y_{i_j}\cap W(i_j)$ is a (closed) point, then there is a unique $q\in S(i_j)$ such that $p\in W(Y_{i_j},q)$. This observation is important in the structure of the proof of Theorem 5.3.

LEMMA 5.7. — *Suppose that $Y_{i_0}\to X$ is an admissible sequence, and $r=\sigma(Y_{i_0})>0$. Then there exists an admissible sequence $Y_{i_j}\to Y_{i_0}$, consisting of admissible transformations of types 2,3, 4 and 5, such that all points of $\mathrm{Sing}_r(Y_{i_j}\setminus W(i_j))$ are 3-prepared 1-points.*

Proof. — We will prove the lemma by constructing an admissible sequence

$$Y_{i_n}\to Y_{i_{n-1}}\to\cdots\to Y_{i_1}\to Y_{i_0}$$

where each $Y_{i_{j+1}}\to Y_{i_j}$ is an admissible transformation of type 2 or 3 for $j\leqslant n-3$, $Y_{i_{n-1}}\to Y_{i_{n-2}}$ is an admissible transformation of type 5 (so that $S(i_j)=S(i_0)$ for $j\leqslant n-1$) and $Y_{i_n}\to Y_{i_{n-1}}$ is a transformation of type 4, and for all j,

$$\text{(5.8)}\quad \begin{array}{l}\text{If } F \text{ is a component of } D_{Y_{i_j}} \text{ such that } F\subset \mathrm{Preimage}(Y_{i_j},S(i_0))\\ \text{then } F\cap \mathrm{Sing}_r(Y_{i_j}\setminus W(i_j))=\emptyset.\end{array}$$

Let $Y_{i_1}\to Y_{i_0}$ be a sequence of permissible blow ups of 2-curves of D such that if $p\in S(i_0)$, then for $j=1$, we have that

$$\text{(5.9)}\quad W(Y_{i_j},p)\subset C(Y_{i_j},\nu_p^1)\cup C(Y_{i_j},\nu_p^2)\cup \mathrm{Preimage}(Y_j,p).$$

We have that (5.8) holds for $j=1$ (by Theorem 4.3 and since $\sigma(Y_{i_0})=r$).

Let $Y_{i_2}\to Y_{i_1}$ be a sequence of blow ups of 2-curves and 3-points of D, such that for $j=2$, we have that

(5.10)

If E_1 and E_2 are distinct components of Y_{i_j} such that E_1 contains a curve $C(Y_{i_j},\nu_p^s)$ and E_2 contains a curve $C(Y_{i_j},\nu_q^t)$for some $p,q\in S(i_0)$ and s,t, then $E_1\cap E_2=\emptyset$

and

$$\text{(5.11)}\quad \begin{array}{l}\text{If } E \text{ is a component of } D_{Y_{i_j}}\text{, and } p\in S(i_0),\ t \text{ are such that}\\ \Lambda(Y_{i_j},\nu_p^t)\in E \text{ but } C(Y_{i_j},\nu_p^t)\not\subset E\text{, then } E \text{ contracts to } p.\end{array}$$

Suppose that $p \in S(i_0)$, and E is a component of $D_{Y_{i_2}}$ which contains $C(Y_{i_2}, \nu_p^1)$ for some t (E can contain at most one of these two curves). Let η_t be the generic point of $C(Y_{i_2}, \nu_p^t)$.

Since (5.9) holds for $j = 2$, $W(Y_{i_2}, p)$ intersects E in a union of 2-curves and 3-points for $\overline{D}_p$ which contract to p, as well as (possibly) the point η_t.

Let $\gamma_{p,t}$ be the 2-curve for $\overline{D}_p$ in E which contains the point $\Lambda(Y_{i_2}, \nu_p^t)$ (and is not equal to $C(Y_{i_2}, \nu_p^t)$). Let

$$Z = W(Y_{i_2}, p) \cap E \setminus \{\gamma_{p,t}, \eta_t\}. \tag{5.12}$$

If $W(Y_{i_2}, p) \cap E \subset Z \cup \{\eta_t\}$, then let $Y_{i_3} = Y_{i_2}$. Otherwise, the 2-curve $\gamma_{p,t}$ for $\overline{D}_p$ is in $W(Y_{i_2}, p) \cap E$. In this case we let $Y_{i_3} \to Y_{i_2}$ be a sequence of blow ups of 2-curves for $\overline{D}_p$, which are sections over $\gamma_{p,t}$, and lie in the strict transform of E. Under each such blow up, the strict transform of E maps isomorphically to E, so we may in fact identify E with its strict transform and $\gamma_{p,t}$ with its section. After enough such blow ups, on the strict transform E_3 of E (which is isomorphic to E), we have that $\gamma_{p,t}$ is not contained in $W(Y_{i_3}, p) \cap E_3$.

Let $G = W(Y_{i_3}, p) \cap E_3 \setminus \left(C(Y_{i_3}, \nu_p^1) \cup C(Y_{i_3}, \nu_p^2)\right)$. G is a closed subset of E_3 which is disjoint from $C(Y_{i_3}, \nu_p^1) \cup C(Y_{i_3}, \nu_p^2)$. Thus there exists an open neighborhood V of G in $\mathrm{Preimage}(Y_{i_3}, U_p)$ which is disjoint from $C(Y_{i_3}, \nu_p^1) \cup C(Y_{i_3}, \nu_p^2)$. There exists a sequence of blow ups of 3-points and 2-curves for $\overline{D}_p$ (which contract to p) $V_1 \to V$ such that $I_p|\mathcal{O}_{V_1}$ is locally principal. $V_1 \to V$ extends to an admissible sequence of transformations of type 3, $Y_{i_4} \to Y_{i_3}$, such that the strict transform E_4 of E on Y_{i_4} satisfies

$$W(Y_{i_4}, p) \cap E_4 \subset C(Y_{i_4}, \nu_p^1) \cup C(Y_{i_4}, \nu_p^2).$$

Repeat this last step (the construction of $Y_{i_4} \to Y_{i_2}$) for all $p \in S(i_0)$ and components E of $D_{Y_{i_4}}$ which contain $C(Y_{i_4}, \nu_p^t)$ for some $p \in S(i_0)$ and t, remembering that (5.10) holds, to obtain $Y_{i_5} \to Y_{i_4}$ where (5.8), (5.9), (5.10) and (5.11) continue to hold for $j = 5$, and we also have that for $j = 5$,

(5.13)

If $p \in S(i_0)$ and E is a component of $D_{Y_{i_j}}$ such that $C(Y_{i_j}, \nu_p^t) \subset E$ for some t, then $W(Y_{i_j}, p) \cap E \subset C(Y_{i_j}, \nu_p^1) \cup C(Y_{i_j}, \nu_p^2)$.

Suppose that E is a component of $D_{Y_{i_5}}$ such that $E \cap \mathrm{Sing}_r(Y_{i_5} \setminus W(i_5)) \neq \emptyset$, and $p \in S(i_0)$. If $C(Y_{i_5}, \nu_p^t) \subset E$ for some t, then $E \cap W(Y_{i_5}, p) \subset \{\eta_t\}$, where η_t is the generic point of $C(Y_{i_5}, \nu_p^t)$ (by (5.13)). If $C(Y_{i_5}, \nu_p^t) \not\subset E$ for $t = 1, 2$, then $\Lambda(Y_{i_5}, \nu_p^1), \Lambda(Y_{i_5}, \nu_p^2) \notin E$ by (5.11) and (5.8), and thus by (5.13), we have that $E \cap W(i_5, p) \cap \left(C(Y_{i_5}, \nu_p^1) \cup C(Y_{i_5}, \nu_p^2)\right) = \emptyset$.

Thus we can construct an allowable sequence of transformations of type 3, $Y_{i_6} \to Y_{i_5}$, so that if E_6 is the strict transform on Y_{i_6} of a component E of $D_{Y_{i_5}}$ such that $E \cap \mathrm{Sing}_r(Y_{i_6} \setminus W(i_6)) \neq \emptyset$, then

$$(5.14)\quad W(Y_{i_6}) \cap E \subset \{\eta_{p,t} \mid \eta_{p_t} \text{ is the generic point of a curve } C(Y_{i_6}, \nu_p^t) \text{ which lies on } E\}.$$

By (5.8), we have that all exceptional components F of $Y_{i_6} \to Y_{i_5}$ satisfy $F \cap \mathrm{Sing}_r(Y_{i_6} \setminus W(i_6)) = \emptyset$. Thus all components E of Y_{i_6} which satisfy $E \cap \mathrm{Sing}_r(Y_{i_6} \setminus W(i_6)) \neq \emptyset$ must satisfy (5.14). Thus for $j = 6$, we have that

(5.15)

$$\begin{gathered}\text{If } E \text{ is a component of } D_{Y_{i_j}} \text{ such that } E \cap \mathrm{Sing}_r(Y_{i_j} \setminus W(i_j)) \neq \emptyset, \text{ then}\\ W(Y_{i_j}) \cap E \subset \{\eta_{p,t} \mid \eta_{p_t} \text{ is the generic point of a curve } C(Y_{i_j}, \nu_p^t) \text{ which lies on } E\}.\end{gathered}$$

By Lemmas 2.5 and 3.4, there exists a further sequence $Y_{i_7} \to Y_{i_6}$ of blow ups of 3-points and 2-curves of D, such that $Y_{i_7} \setminus W(i_7)$ is 3-prepared, except possibly at a finite number of 1-points. The conditions of equations (5.8), (5.9), (5.10), (5.11) and (5.15) continue to hold on Y_{i_7} (although we may have that some 2-curves for D are blown up which do not contract to points of $S(i_0)$).

We now apply Proposition 3.14 to the union H of irreducible components E of D for Y_{i_7} which contain a point of $\mathrm{Sing}_r(Y_{i_7} \setminus W(i_7))$, with

$$A = \{q \in H \mid Y_{i_7} \text{ is not 3-prepared at } q \text{ (which are necessarily 1-points of } D)\}$$

being sure that none of the finitely many 2-points for D

$$B = \{\Lambda(Y_{i_7}, \nu_p^t) \mid p \in S(i_0)\}$$

are in the image of the general curves blown up, to construct an admissible transformation $Y_{i_8} \to Y_{i_7}$ of type 5, so that if E is an irreducible component of D for Y_{i_8} which contains a point of $\mathrm{Sing}_r(Y_{i_8} \setminus W(i_8))$, then all points of $E \setminus W(i_8)$ are 3-prepared. We also will have that the conditions of (5.8), (5.9), (5.10), (5.15) and (5.15) hold on points of E.

We now perform a sequence of admissible transformations of type 4, introducing local resolvers at all 2-points $p \in Y_{i_8} \setminus W(i_8)$ such that $\sigma_D(p) = r$ (the finite set of these points are all necessarily 3-prepared). □

Lemma 5.8. — *Suppose that $Y_{i_0} \to X$ is an admissible sequence, and C is a curve contained in D_{Y_i} such that C is not a 2-curve and $C \not\subset W(i_0)$. Let η be the generic point of C. Then there exists an admissible sequence*

$Y_{i_j} \to Y_{i_0}$, consisting of admissible transformations of types 2, 3, 4 and 5, such that if C_j is the strict transform of C in Y_{i_j}, then

1. *If $\sigma_D(\eta) > 0$, then all points of $C_j \setminus W(i_j)$ are 3-prepared 1-points.*
2. *If $\sigma_D(\eta) = 0$, then all points q of $C_j \setminus W(i_j)$ are 1-points or 2-points with $\sigma_D(q) = 0$.*

Proof. — The proof follows from the arguments of the proof of Lemma 5.7, applied only to the component E of D containing C. In the case where $\sigma_D(\eta) = 0$, the set A of the hypotheses of Proposition 3.14 used in the construction, will be the union of the set of 1-points of the strict transform of E which are not 3-prepared, and the 1-points q on the strict transform of C such that $\sigma_D(q) > 0$. □

Lemma 5.9. — *Suppose that $Y_{i_0} \to X$ is an admissible sequence, and C is a curve in $D_{Y_{i_0}}$ which contains a 1-point. Suppose that $p \in C \cap W(i_0)$ is a 2-point for D. Then there exists an admissible sequence $Y_{i_j} \to Y_{i_0}$, consisting entirely of transformations of types 2 and 3, such that if C_j is the strict transform of C in Y_{i_j}, then the following holds. Suppose that $q \in$ Preimage$(Y_{i_j}, p) \cap C_j$. Then q is a 2-point for D, and we further have that if $q \in W(i_j)$, then $q = \Lambda(Y_{i_j}, \nu_b^t)$ and $C_j = C(Y_{i_j}, \nu_b^t)$ for some $b \in S(i_j)$.*

Proof. — We have that $b \in C \cap W(Y_{i_0}, b)$ for some $b \in S(i_0)$. If $C = C(Y_{i_0}, \nu_b^t)$ for some t, then we have obtained the conclusions of the lemma, so suppose that $C \neq C(Y_{i_0}, \nu_b^t)$ for any t. Since C is not a 2-curve for D, there exists a sequence of blow ups of 3-points for $\overline{D}_b$, $Y_{i_1} \to Y_{i_0}$, such that the strict transform C_1 of C on Y_{i_1} has the property that the set $C_1 \cap \mathrm{Preimage}(Y_{i_1}, p)$ consists of 2-points for D. We further may obtain that either $C_1 \cap \mathrm{Preimage}(Y_{i_1}, p)$ is disjoint from $W(Y_{i_1}, b)$, in which case we have achieved the conclusions of the lemma, or that

$$C_1 \cap \mathrm{Preimage}(Y_{i_1}, p) \text{ has non trivial intersection with } W(Y_{i_1}, b),$$

but $\Lambda(Y_{i_1}, \nu_b^t) \notin C_1$ for any t. Assume that this last case holds, and $q \in C_1 \cap \mathrm{Preimage}(Y_{i_1}, p)$. Then there is a unique 2-curve γ of $\overline{D}_b$, which is also a 2-curve for D, such that $q \in \gamma$. There is a finite sequence of blow ups $Y_{i_2} \to Y_{i_1}$ of 2-curves for $\overline{D}_b$, which are sections over γ, such that if C_2 is the strict transform of C_1 in Y_{i_2}, and $a \in C_2 \cap \mathrm{Preimage}(Y_{i_2}, q)$, then $I_b\mathcal{O}_{Y_{i_2},a}$ is principal, so that $C_2 \cap \mathrm{Preimage}(Y_{i_2}, q)$ is disjoint from $W(i_2)$.

We now apply this procedure above any other points of $C_1 \cap \mathrm{Preimage}(Y_{i_1}, p)$, to construct a further sequence of blow ups of 2-curves $Y_{i_3} \to Y_{i_2}$ such that the strict transform C_3 of C_2 on Y_{i_3} satisfies the condition that $C_3 \cap \mathrm{Preimage}(Y_{i_3}, p)$ is disjoint from $W(i_3)$. □

Lemma 5.10. — *Suppose that $Y_{i_0}\to X$ is an admissible sequence and C is a curve in $D_{Y_{i_0}}$ which contains a 1-point. Suppose that $p\in C$ is a 2-point. Then there exists an admissible sequence $Y_{i_j}\to Y_{i_0}$, consisting entirely of transformations of types 2, 3 and 4, satisfying the following properties. Let C_j be the strict transform of C in Y_{i_j}. Suppose that $q\in$ Preimage$(Y_{i_j},p)\cap C_j$. Then q is a 2-point for D, and one of the following holds:*

1. *There exists $a\in S(i_j)$ such that $q=\Lambda(Y_{i_j},\nu_a^t)$ and $C_j=C(Y_{i_j},\nu_a^t)$ for some t, or*
2. *$\sigma_D(q)=0$ and $q\not\in W(i_j)$.*

Proof. — First suppose that $p\in W(i_0)$. Then there exists a point $b\in S(i_0)$ such that $p\in W(Y_{i_0},b)$. Perform Lemma 5.9 to construct an allowable sequence $Y_{i_1}\to Y_{i_0}$ such that if C_1 is the strict transform of C on Y_{i_1}, and $q\in C_1\cap\text{Preimage}(Y_{i_0},p)$ is contained in $W(i_1)$, then there exists $a\in S(i_1)$ such that $q=\Lambda(Y_{i_1},\nu_a^t)$ and $C_1=C(Y_{i_1},\nu_a^t)$ for some a. Let

$$\lambda(i_1):=\max\{\sigma_D(q)\mid q\in(C_1\cap\text{Preimage}(Y_{i_1},p))\setminus W(i_1)\}.$$

We have that

$$\lambda(i_1)<\sigma_D(p).$$

If $p\not\in W(i_0)$, then we let $Y_{i_1}=Y_{i_0}$, $S(i_1)=S(i_0)$ and $\lambda(i_1)=\sigma_D(p)$.

The rest of the proof is the same for both cases considered above ($p\in W(i_0)$ and $p\not\in W(i_0)$).

Now perform Lemma 3.4 to construct a sequence of blow ups of 2-curves for D, $Y_{i_2}\to Y_{i_1}$, such that if C_2 is the strict transform of C_1 on Y_{i_2}, then all points of $(\text{Preimage}(Y_{i_2},p)\cap C_2)\setminus W(i_2)$ (which are necessarily 2-points for D) are 3-prepared. Let

$$R(i_2)=\{q\in(\text{Preimage}(Y_{i_2},p)\cap C_2)\setminus W(i_2)\mid q\text{ is a 2-point and }\sigma_D(q)>0\}.$$

Write $R(i_2)=\{q_1,\dots,q_m\}$. For each $q_i\in R(i_2)$, let $(U_{q_i},\overline{D}_{q_i},I_{q_i},\nu_{q_i}^1,\nu_{q_i}^2)$ be a local resolver in Y_{i_2}. Let $Y_{i_3}\to Y_{i_2}$ be the admissible sequence consisting of transformations of type 4, where $S(i_3)=S(i_2)\cup R(i_2)$. Let $C_3=C_2$, the strict transform of C on Y_{i_3}. If $q\in(\text{Preimage}(Y_{i_3},p)\cap C_3)\setminus W(i_3)$. then $\sigma_D(q)=0$. If $q\in(\text{Preimage}(Y_{i_3},p)\cap C_3)$ and $q\in R(i_2)=S(i_3)\setminus S(i_2)$, then we have

$$\sigma_D(q)\leqslant\lambda(i_1).$$

Now again perform Lemma 5.9, to construct $Y_{i_4}\to Y_{i_3}$ such that if C_4 be the strict transform of C_3 on Y_{i_4}, and $q\in(\text{Preimage}(Y_{i_4},p)\cap C_4)\cap$

$W(i_4)$, then $q = \Lambda(Y_{i_4}, a)$ and $C_4 = C(Y_{i_4}, \nu_a^t)$ for some $a \in S(i_4)$ and t. If $(\mathrm{Preimage}(Y_{i_4}, p) \cap C_4) \cap W(i_4) \neq \emptyset$, we have that

$$\lambda(i_3) := \max\{\sigma_D(q) | q \in (\mathrm{Preimage}(Y_{i_4}, p) \cap C_4) \cap W(i_4)\} < \lambda(i_1).$$

Iterate the above, performing Lemma 3.4 followed by a sequence of adimissible transformations of type 4, and then performing Lemma 5.9, to eventually obtain $Y_{i_j} \to Y_{i_0}$ such that if C_{i_j} is the strict transform of C on Y_{i_j}, then $\sigma_D(q) = 0$ if $q \in \left(\mathrm{Preimage}(Y_{i_j}, p) \cap C_{i_j}\right) \setminus W(i_j)$, and if $q \in \left(\mathrm{Preimage}(Y_{i_j}, p) \cap C_{i_j}\right) \cap W(i_j)$, then $q = \Lambda(Y_{i_j}, b)$ and $C_{i_j} = C(Y_{i_j}, \nu_b^t)$ for some $b \in S(i_j)$ and t. $\square$

We now define an admissible transformation of type 6. Suppose that $Y_{i_0} \to X$ is an admissible sequence, and C is a good curve on Y_{i_0} (Definition 5.4).

First assume that $\sigma_D(\eta) = 0$, where η is the generic point of C. By Lemmas 3.9 - 3.11, there exists a sequence of transformations of type 1 $Y_{i_1} \to Y_{i_0}$ such that the strict transform C_1 of C in Y_{i_1} is such that $\sigma_D(q) = 0$ and the other assumptions of Lemma 3.12 hold for all $q \in C_1 \setminus W(i_1)$. Let $Y_{i_2} \to Y_{i_1}$ be the blow up of C which is an admissible blow up. We have that $\sigma_D(q) = 0$ for all $q \in \mathrm{Preimage}(Y_{i_2}, C_1 \setminus W(i_1))$ by Lemma 3.12. We define the morphism $Y_{i_2} \to Y_{i_0}$ to be the transformation of type 6 associated to C.

Now assume that $\sigma_D(\eta) > 0$, where η is the generic point of C. Let $Z \to Y_{i_0} \setminus \left(W(i_0) \cup D_{Y_{i_0}}\right)$ be the canonical sequence of blow ups above $C \setminus W(i_0)$ defined in 1) of Definition 5.2. $Z \to Y_{i_0} \setminus \left(W(i_0) \cup D_{Y_{i_0}}\right)$ has a factorization

$$Z = Z_m \to Z_{m-1} \to \cdots \to Z_1 \to Z_0 = Y_{i_0} \setminus \left(W(i_0) \cup D_{Y_{i_0}}\right)$$

where each $Z_{j+1} \to Z_j$ is the blow up of a curve A_j which is a section over $C \setminus W(i_0)$, and is permissible for D (thus A_j is either a 2-curve, or consists entirely of 1-points). We will inductively extend these morphisms (to an admissible sequence

$$X_m \to V_{m-1} \to X_{m-1} \to \cdots X_3 \to V_2 \to X_2 \to V_1 \to X_1 \to Y_{i_0},$$

so that

$$\begin{aligned}\mathrm{Preimage}(V_j, Y_{i_0} \setminus \left(W(i_0) \cup D_{Y_{i_0}}\right)) &= \mathrm{Preimage}(X_j, Y_{i_0} \setminus \left(W(i_0) \cup D_{Y_{i_0}}\right)) \\ &= Z_j\end{aligned}$$

for all j.

We define X_1 to be the blow up of C (which is an admissible blow up). If A_j is a 2-curve for D, then $V_j \to X_j$ will just be the identity map (with $S(V_j) = S(X_j)$).

if A_j is not a 2-curve, then let γ_0 be the Zariski closure of A_j in X_j. $\gamma_0 \setminus Z_j$ is a set of 2-points and 3-points for D. First define a sequence $T_1 \to X_j$ of blow ups of 3-points for D, so that the Zariski closure γ_1 of A_j in T_1 is such that $\gamma_1 \setminus A_j$ consists only of 2-points. Now successively apply Lemma 5.10 to the points of $\gamma_1 \setminus A_j$ to construct an admissible sequence $T_2 \to T_1$ consisting of transformations of types 2, 3 and 4, so that if γ_2 is the Zariski closure of A_j in T_2, and $q \in \gamma_2 \setminus A_j$, then either $\sigma_D(q) = 0$ and $q \notin W(T_2)$, or there exists $a \in S(T_2)$ such that $q = \Lambda(T_2, \nu_a^t)$ and $C_j = C(T_2, \nu_a^t)$.

A point in $\gamma_1 \setminus A_j$ cannot be contained in a 2-curve which is a section over C, since $\gamma_1 \setminus A_j$ contains no 3-points, and the points of $C \cap W(i_0)$ are all 2-points for D. Thus $T_2 \to T_1$ has the property that Preimage$(T_2, Y_{i_0} \setminus \big(W(i_0) \cup D_{Y_{i_0}})\big) = Z_j$.

Let η_j be the generic point of A_j. Then $\sigma_D(\eta_j) > 0$ (by Theorem 4.2). Thus all points of $q \in \gamma_2$ satisfy $\sigma_D(q) \geqslant \sigma_D(\eta_j) > 0$. We then define V_j to be T_2.

We now define $X_{j+1} \to V_j$ to be the blow up of γ_2, which is an admissible blow up.

Lemma 5.11. — *Suppose that (5.2) is an admissible sequence consisting of admissible transformations of types 1 - 7. Then for any transformation $Y_{i_{j+1}} \to Y_{i_j}$ in (5.2), the conditions (5.3) - (5.7) hold.*

The proof of Lemma 5.11 follows from our construction of an admissible transformation of type 6, and Theorem 4.2, Lemma 3.12 and Lemma 5.5.

Proposition 5.12. — *Suppose that $Y_{i_0} \to X$ is an admissible sequence. Let $r = \sigma(Y_i) > 0$. Then there exists an admissible sequence $Y_{i_j} \to Y_{i_0}$ such that $\sigma(Y_{i_j}) \leqslant r$ and $\sigma_D(p) < r$ for all $p \in Y_{i_j} \setminus W(i_j)$.*

Proof. — First perform Lemma 5.7, to obtain an admissible sequence $Y_{i_1} \to Y_{i_0}$ such that $\Gamma(Y_{i_1}) = \mathrm{Sing}_r(Y_{i_1} \setminus W(i_1))$ consists of 3-prepared 1-points. By Lemma 5.1, $\Gamma(Y_{i_1})$ is a disjoint union of nonsingular curves.

Suppose that C is the closure in Y_{i_1} of a curve in $\Gamma(Y_{i_1})$. By Lemma 5.10, there exists an admissible sequence $Y_{i_2} \to Y_{i_2}$ consisting of transformations of types 2, 3 and 4 such that the strict transform C_2 of C in Y_{i_2} is a good curve. We may thus perform an admissible transformation of type 6, $Y_{i_3} \to Y_{i_2}$ to get that all points q of Preimage$(Y_{i_3}, C_2 \setminus W(i_2))$ are 2-prepared for D with $\sigma_D(q) \leqslant r-1$ (by Theorem 4.2). Further, $\sigma_D(q) \leqslant r-1$ for $q \in$ Preimage$(Y_{i_3}, W(i_1)) \setminus W(i_3)$. We now apply Lemma 5.10 followed by an admissible transformation of type 6 for the other curves of $\Gamma(Y_{i_1})$, to obtain the conclusions of the Proposition. $\square$

PROPOSITION 5.13. — *Suppose that $Y_{i_0} \to X$ is an admissible sequence, $r = \sigma(Y_{i_0}) > 0$ and $\sigma_D(p) < r$ if $p \in Y_{i_0} \setminus W(i_0)$. Then there exists an admissible sequence $Y_{i_j} \to Y_{i_0}$ such that $\sigma(Y_{i_j}) < r$.*

Proof. — Let

$$T(i_0) = \{p \in S(i_0) \mid \sigma_D(p) = r\}.$$

Suppose there exists $p \in T(i_0)$ and t such that $I_p\mathcal{O}_{Y_{i_0},\eta}$ is not principal, where η is the generic point of $C(Y_{i_0}, \nu_p^t)$. First apply Lemma 5.8 to $C(Y_{i_0}, \nu_p^t)$ to construct an admissible sequence $Y_{i_1} \to Y_{i_0}$ so that all points q of $C(Y_{i_2}, \nu_p^t)\setminus W(i_2)$ are 3-prepared 1-points if $\sigma_D(\eta) > 0$ and are 1-points or 2-points which satisfy $\sigma_D(q) = 0$ if $\sigma_D(\eta) = 0$. Then successively apply Lemma 5.10 to all 2-points q of $C(Y_{i_2}, \nu_p^t)$ which have $\sigma_D(q) > 0$, to construct an admissible sequence $Y_{i_2} \to Y_{i_1}$ such that $C(Y_{i_2}, \nu_p^t)$ (which is the strict transform of $C(Y_{i_0}, \nu_p^t)$) is a good curve. Let $Y_{i_3} \to Y_{i_2}$ be a transformation of type 6 applied to $C(Y_{i_2}, \nu_p^t)$. We continue to have $\sigma(Y_{i_3}) < r$ and if $p \in S(i_3) \setminus S(i_0)$, then $\sigma_D(p) < r$ (by Lemma 5.11). Thus

$$T(i_2) = \{p \in S(i_2) \mid \sigma_D(p) = r\} = T(i_0).$$

We may thus repeat the above construction for some $q \in T(i_2)$ and t such that $I_q\mathcal{O}_{Y_{i_3},\zeta}$ is not principal, where ζ is the generic point of $C(Y_{i_3}, \nu_q^t)$. After iterating this procedure a finite number of times, we will construct an admissible sequence $Y_{i_4} \to Y_{i_0}$ such that $\sigma(Yi_4) \leqslant r$, $\sigma(Y_{i_4} \setminus W(i_4)) < r$,

$$T(i_4) = \{p \in S(i_4) \mid \sigma_D(p) = r\} = T(i_0),$$

and for all $p \in T(i_4)$, and t, $I_p\mathcal{O}_{Y_{i_4},\eta}$ is principal, where η is the generic point of $C(Y_{i_4}, \nu_p^t)$.

Now perform a sequence of blow ups of 2-curves for D $Y_{i_5} \to Y_{i_4}$, so that $W(Y_{i_5}, p) \subset \text{Preimage}(Y_{i_5}, p)$ for all $p \in T(i_5) = T(i_0)$. Finally, we may construct an admissible sequence $Y_{i_6} \to Y_{i_5}$ consisting of transformations of type 3, so that $W(i_6) = \emptyset$ for all $p \in T(i_6) = T(i_0)$. We may then apply a transformation of type 7, $Y_{i_7} \to Y_{i_6}$, defined by $Y_{i_7} = Y_{i_6}$ and $S(i_7) = S(i_6) \setminus T(i_0)$ to obtain that $\sigma(Y_{i_7}) \leqslant r - 1$. $\square$

Now we prove Theorem 5.3, by starting with $Y_0 = X$ and $S(0) = \emptyset$. After applying successively Propositions 5.12 and then 5.13 enough times, we construct an admissible sequence $Y_n \to X$ such that $\sigma(Y_n) = 0$, so that $S(Y_n) = \emptyset$, and $\sigma_D(p) = 0$ for $p \in Y_n$.

6. Proof of Toroidalization

Theorem 6.1. — *Suppose that $\mathfrak{k}$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $\mathfrak{k}$ to a nonsingular surface S over $\mathfrak{k}$ and D_S is a reduced SNC divisor on S such that $D_X = f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth. Further suppose that f is 1-prepared. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of D_X, such that the induced morphism $f_1 : X_1 \to S$ is prepared with respect to D_S.*

Proof. — The proof is immediate from Lemma 2.2, Proposition 2.7 and Theorem 5.3. □

Theorem 6.1 is a slight restatement of Theorem 17.3 of [11]. Theorem 17.3 [11] easily follows from Lemma 2.2 and Theorem 6.1.

Theorem 6.2. — *Suppose that $\mathfrak{k}$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $\mathfrak{k}$ to a nonsingular surface S over $\mathfrak{k}$ and D_S is a reduced SNC divisor on S such that $D_X = f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of D_X, and a sequence of blow ups of points $\pi_1 : S_1 \to S$ which are in the preimage of D_S, such that the induced rational map $f_1 : X_1 \to S_1$ is a morphism which is toroidal with respect to $D_{S_1} = \pi_1^{-1}(D_S)$.*

Proof. — The proof follows immediately from Theorem 6.1, and Theorems 18.19, 19.9 and 19.10 of [11]. □

Theorem 6.2 is a slight restatement of Theorem 19.11 of [10]. Theorem 19.11 [11] easily follows from Theorem 6.2.

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