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STRONG q -VARIATION INEQUALITIES FOR ANALYTIC SEMIGROUPS

by Christian LE MERDY & Quanhua XU (*)

ABSTRACT. — Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a positive contraction, with $1 < p < \infty$. Assume that T is analytic, that is, there exists a constant $K \geq 0$ such that $\|T^n - T^{n-1}\| \leq K/n$ for any integer $n \geq 1$. Let $2 < q < \infty$ and let v^q be the space of all complex sequences with a finite strong q -variation. We show that for any $x \in L^p(\Omega)$, the sequence $([T^n(x)](\lambda))_{n \geq 0}$ belongs to v^q for almost every $\lambda \in \Omega$, with an estimate $\|(T^n(x))_{n \geq 0}\|_{L^p(v^q)} \leq C\|x\|_p$. If we remove the analyticity assumption, we obtain an estimate $\|(M_n(T)x)_{n \geq 0}\|_{L^p(v^q)} \leq C\|x\|_p$, where $M_n(T) = (n+1)^{-1} \sum_{k=0}^n T^k$ denotes the ergodic average of T . We also obtain similar results for strongly continuous semigroups $(T_t)_{t \geq 0}$ of positive contractions on L^p -spaces.

RÉSUMÉ. — Soit $T: L^p(\Omega) \rightarrow L^p(\Omega)$ une contraction positive, avec $1 < p < \infty$. Supposons T analytique, au sens où il existe une constante $K \geq 0$ telle que $\|T^n - T^{n-1}\| \leq K/n$ pour tout entier $n \geq 1$. Soit $2 < q < \infty$ et soit v^q l'espace des suites complexes à q -variation forte bornée. On montre que pour tout $x \in L^p(\Omega)$, la suite $([T^n(x)](\lambda))_{n \geq 0}$ appartient à v^q pour presque tout $\lambda \in \Omega$, avec la majoration $\|(T^n(x))_{n \geq 0}\|_{L^p(v^q)} \leq C\|x\|_p$. Si l'on supprime l'hypothèse d'analyticité, on obtient une majoration $\|(M_n(T)x)_{n \geq 0}\|_{L^p(v^q)} \leq C\|x\|_p$, où $M_n(T) = (n+1)^{-1} \sum_{k=0}^n T^k$ désigne la moyenne ergodique de T . On obtient également des résultats similaires pour les semi-groupes fortement continus $(T_t)_{t \geq 0}$ de contractions positives sur L^p .

1. Introduction

Variational inequalities in probability, ergodic theory and harmonic analysis have been the subject of many recent research papers. One important character of these inequalities is the fact that they can be used to measure the speed of convergence for the family of operators in consideration. To be

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more precise, consider, for instance, a measure space (Ω, μ) and an operator T on $L^1(\Omega) + L^\infty(\Omega)$. Form the ergodic averages of T :

$$M_n(T) = \frac{1}{n+1} \sum_{k=0}^n T^k, \quad n \geq 0.$$

A fundamental theorem in ergodic theory states that if T is a contraction on $L^p(\Omega)$ for every $1 \leq p \leq \infty$, then the limit $\lim_{n \rightarrow \infty} M_n(T)x$ exists a.e. for every $x \in L^p(\Omega)$. One can naturally ask what is the speed of convergence of this limit. A classical tool for measuring that speed is the following square function

$$S(x) = \left(\sum_{n \geq 0} n |M_{n+1}(T)x - M_n(T)x|^2 \right)^{\frac{1}{2}},$$

the problem being to estimate its norm $\|S(x)\|_p$. This issue goes back to Stein [30], who proved that if T as above is positive on $L^2(\Omega)$ (in the Hilbertian sense), then a square function inequality

$$\|S(x)\|_p \leq C_p \|x\|_p, \quad x \in L^p(\Omega),$$

holds for any $1 < p < \infty$. Stein's inequality is closely related to Dunford-Schwartz's maximal ergodic inequality,

$$\left\| \sup_{n \geq 0} |M_n(T)x| \right\|_p \leq C_p \|x\|_p, \quad x \in L^p(\Omega), \quad 1 < p \leq \infty.$$

This maximal inequality and its weak type $(1, 1)$ substitute for $p = 1$ are key ingredients in the proof of the previous pointwise ergodic theorem.

The strong q -variation is another (better) tool to measure the speed of $\lim_{n \rightarrow \infty} M_n(T)x$. Bourgain was the first to consider variational inequalities in ergodic theory. To state his inequality we need to recall the definition of the strong q -variation. Given a sequence $(a_n)_{n \geq 0}$ of complex numbers and a number $1 \leq q < \infty$, the strong q -variation norm is defined as

$$\|(a_n)_{n \geq 0}\|_{v^q} = \sup \left\{ \left(|a_0|^q + \sum_{k \geq 1} |a_{n_k} - a_{n_{k-1}}|^q \right)^{\frac{1}{q}} \right\},$$

where the supremum runs over all increasing sequences $(n_k)_{k \geq 0}$ of integers such that $n_0 = 0$. It is clear that the set v^q of all sequences with a finite strong q -variation is a Banach space for the norm $\|\cdot\|_{v^q}$.

Bourgain [4] proved that if T is induced by a measure preserving transformation on (Ω, μ) , then for any $2 < q < \infty$,

$$\left\| (M_n(T)x)_{n \geq 0} \right\|_{L^2(v^q)} \leq C_q \|x\|_2, \quad x \in L^2(\Omega).$$

This inequality was then extended to $L^p(\Omega)$ for any $1 < p < \infty$ by Jones, Kaufman, Rosenblatt and Wierdl [15]. The latter paper contains many other interesting results on the subject. Note that the predecessor of Bourgain's inequality is Lépingle's variational inequality for martingales [21]. The latter says that if $(\mathbb{E}_n)_{n \geq 0}$ is an increasing sequence of conditional expectations on a probability space Ω , then we have an estimate $\|(\mathbb{E}_n(x))_{n \geq 0}\|_{L^p(v^q)} \leq C \|x\|_p$ (see also [29] for further results on that theme).

Since [4], variational type inequalities have been extensively studied in ergodic theory and harmonic analysis. Many classical sequences of operators and semigroups have been proved to satisfy strong variational bounds, see in particular [9, 16, 17, 18, 25] and references therein. The main purpose of this paper is to exhibit a large class of operators T on $L^p(\Omega)$ for a fixed $1 < p < \infty$ with the following property: for any $2 < q < \infty$, there exists a constant $C > 0$ (which may depend on q and T) such that for any $x \in L^p(\Omega)$, the sequence $(T^n(x))_{n \geq 0}$ belongs to $L^p(\Omega; v^q)$, and

$$(1.1) \quad \|(T^n(x))_{n \geq 0}\|_{L^p(v^q)} \leq C \|x\|_p.$$

We show that this holds true provided that T is a positive contraction (more generally, a contractively regular operator) and T is analytic, in the sense that

$$\sup_{n \geq 1} n \|T^n - T^{n-1}\| < \infty.$$

Inequality (1.1) implies, of course, a similar variational inequality for the ergodic averages $M_n(T)$. However, in this latter case, the analytic assumption above can be removed. Namely, for a positive contraction T on $L^p(\Omega)$ with $1 < p < \infty$ we have an estimate

$$(1.2) \quad \|(M_n(T)x)_{n \geq 0}\|_{L^p(v^q)} \leq C \|x\|_p, \quad x \in L^p(\Omega),$$

for any $2 < q < \infty$. This result extends those of [4] and [15] quoted previously.

Note that inequality (1.1) for positive analytic contractions considerably improves our previous maximal ergodic inequality for such operators T proved in [19]. In this sense, this paper is a continuation of [19]. On the other hand, our proof of (1.1) heavily relies on the square function inequalities of [19].

We also establish results similar to (1.1) and (1.2) for strongly continuous semigroups. This requires the following continuous analog of v^q . Given a complex family $(a_t)_{t>0}$, define

$$\|(a_t)_{t>0}\|_{V^q} = \sup \left\{ \left(|a_{t_0}|^q + \sum_{k \geq 1} |a_{t_k} - a_{t_{k-1}}|^q \right)^{\frac{1}{q}} \right\},$$

where the supremum runs over all increasing sequences $(t_k)_{k \geq 0}$ of positive real numbers. Then we let V^q be the resulting Banach space of all $(a_t)_{t>0}$ such that $\|(a_t)_{t>0}\|_{V^q} < \infty$.

Consider a bounded analytic semigroup $(T_t)_{t \geq 0}$ on $L^p(\Omega)$, with $1 < p < \infty$. We show that if T_t is a positive contraction (more generally, a contractively regular operator) for any $t \geq 0$, then for any $2 < q < \infty$ and any $x \in L^p(\Omega)$, the family $([T_t(x)](\lambda))_{t>0}$ belongs to V^q for almost every $\lambda \in \Omega$, and we have an estimate

$$(1.3) \quad \left\| \lambda \mapsto \left\| ([T_t(x)](\lambda))_{t>0} \right\|_{V^q} \right\|_p \leq C \|x\|_p, \quad x \in L^p(\Omega).$$

We mention that Jones and Reinhold [16] proved (1.3) for positive, unital symmetric diffusion semigroups and (1.1) for a certain class of convolution operators (only in the case $p = 2$). Our results turn out to extend these contributions in various directions.

As in the discrete case, we obtain similar results for the averages of the semigroup if we remove the analyticity assumption.

Inequalities for ergodic averages, such as (1.2), will be established in Section 3. Then our main results leading to (1.1) and (1.3) will be established in Section 4, using the above mentioned results from Section 3, as well as square function estimates from [19].

Section 5 is devoted to various complements. On the one hand, we establish individual (= pointwise) ergodic theorems in our context. For example, if $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is a positive analytic contraction, then $\lim_{n \rightarrow \infty} T^n(x)$ exists a.e. for every $x \in L^p(\Omega)$. On the other hand, it is well-known that (1.2) cannot be extended to the case $q = 2$. Then we show analogs of (1.2) and (1.1) when v^q is replaced by the oscillation space o^2 . We give examples and applications in Section 6.

We end this introduction with a few notation. If X is a Banach space, we let $B(X)$ denote the algebra of all bounded operators on X and we let I_X denote the identity operator on X (or simply I if there is no ambiguity on X). For any $T \in B(X)$, we let $\sigma(T)$ denote the spectrum of T . Also we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of \mathbb{C} .

For a measurable function $x: \Omega \rightarrow \mathbb{C}$ acting on a measure space (Ω, μ) , $\|x\|_p$ denotes the L^p -norm of x .

We refer to [26] and [14] for background on strongly continuous and analytic semigroups on Banach space.

2. Preliminaries on q -variation

The aim of this section is to provide some elementary background on the spaces v^q and V^q defined above. We fix some $1 \leq q < \infty$.

Let $(a_n)_{n \geq 0}$ be an element of v^q . Then the sequence $(a_n)_{n \geq 0}$ is bounded, with

$$\|(a_n)_{n \geq 0}\|_\infty \leq 2\|(a_n)_{n \geq 0}\|_{v^q},$$

and

$$\lim_{m \rightarrow \infty} \|(0, \dots, 0, a_{m+1} - a_m, \dots, a_n - a_m, \dots)\|_{v^q} = 0.$$

Thus the space of eventually constant sequences is dense in v^q . Note that any element of v^q is a converging sequence.

For any integer $m \geq 1$, let v_m^q be the space of $(m+1)$ -tuples (a_0, a_1, \dots, a_m) of complex numbers, equipped with the norm

$$\|(a_0, a_1, \dots, a_m)\|_{v_m^q} = \|(a_0, a_1, \dots, a_m, a_m, \dots)\|_{v^q}.$$

It follows from above that an infinite sequence $(a_n)_{n \geq 0}$ belongs to v^q if and only if there is a constant $C \geq 0$ such that $\|(a_0, a_1, \dots, a_m)\|_{v_m^q} \leq C$ for any $m \geq 1$ and in this case,

$$(2.1) \quad \|(a_n)_{n \geq 0}\|_{v^q} = \lim_{m \rightarrow \infty} \|(a_0, a_1, \dots, a_m)\|_{v_m^q}.$$

Let (Ω, μ) be a measure space, let $1 < p < \infty$ and let $L^p(\Omega; v^q)$ denote the corresponding Bochner space. Any element of that space can be naturally regarded as a sequence of $L^p(\Omega)$. The following is a direct consequence of the above approximation properties.

LEMMA 2.1. — *Let $(x_n)_{n \geq 0}$ be a sequence of $L^p(\Omega)$, the following assertions are equivalent.*

- (i) *The sequence $(x_n)_{n \geq 0}$ belongs to $L^p(\Omega; v^q)$.*
- (ii) *The sequence $(x_n(\lambda))_{n \geq 0}$ belongs to v^q for almost every $\lambda \in \Omega$ and the function $\lambda \mapsto \|(x_n(\lambda))_{n \geq 0}\|_{v^q}$ belongs to $L^p(\Omega)$.*
- (iii) *There is a constant $C \geq 0$ such that*

$$\|(x_0, x_1, \dots, x_m)\|_{L^p(v_m^q)} \leq C$$

for any $m \geq 1$.

In this case,

$$\begin{aligned} \|\lambda \mapsto \|(x_n(\lambda))_{n \geq 0}\|_{V^q}\|_p &= \|(x_n)_{n \geq 0}\|_{L^p(V^q)} \\ &= \lim_{m \rightarrow \infty} \|(x_0, x_1, \dots, x_m)\|_{L^p(V_m^q)}. \end{aligned}$$

We now consider the continuous case. We note for further use that any element $(a_t)_{t > 0}$ of V^q admits limits

$$\lim_{t \rightarrow 0^+} a_t \quad \text{and} \quad \lim_{t \rightarrow \infty} a_t.$$

The following is a continuous analog of Lemma 2.1. Note however that families satisfying the next statement do not necessarily belong to the Bochner space $L^p(\Omega; V^q)$.

LEMMA 2.2. — *Let $(x_t)_{t > 0}$ be a family of $L^p(\Omega)$ and assume that:*

- (1) *For a.e. $\lambda \in \Omega$, the function $t \mapsto x_t(\lambda)$ is continuous on $(0, \infty)$.*
- (2) *There exists a constant $C \geq 0$ such that whenever $t_0 < t_1 < \dots < t_m$ is a finite increasing sequence of positive real numbers, we have*

$$\|(x_{t_0}, x_{t_1}, \dots, x_{t_m})\|_{L^p(\Omega; V_m^q)} \leq C.$$

Then $(x_t(\lambda))_{t > 0}$ belongs to V^q for a.e. $\lambda \in \Omega$, the function $\lambda \mapsto \|(x_t(\lambda))_{t > 0}\|_{V^q}$ belongs to $L^p(\Omega)$ and

$$\|\lambda \mapsto \|(x_t(\lambda))_{t > 0}\|_{V^q}\|_p \leq C.$$

Proof. — For any integer $N \geq 1$, define

$$\varphi_N(\lambda) = \sup_{k \geq 1} \|(x_{n2^{-N}}(\lambda))_{n \geq k}\|_{V^q}, \quad \lambda \in \Omega.$$

It follows from (2.1) that φ_N is measurable. Moreover the sequence $(\varphi_N)_{N \geq 1}$ is nondecreasing, and we may therefore define

$$\varphi(\lambda) = \lim_{N \rightarrow \infty} \varphi_N(\lambda), \quad \lambda \in \Omega.$$

By construction, φ is measurable and by the monotone convergence theorem, its L^p -norm is equal to $\lim_N \|\varphi_N\|_p$. According to the assumption (2) and the approximation property (2.1), the L^p -norm of φ_N is $\leq C$ for any $N \geq 1$. Hence

$$\int_{\Omega} \varphi(\lambda)^p d\mu(\lambda) \leq C^p.$$

This implies that $\varphi(\lambda) < \infty$ for a.e. $\lambda \in \Omega$.

If $\lambda \in \Omega$ is such that $t \mapsto x_t(\lambda)$ is continuous on $(0, \infty)$, then

$$\varphi(\lambda) = \left\| (x_t(\lambda))_{t>0} \right\|_{V^q}.$$

According to the assumption (1), this holds true almost everywhere. Hence the quantity $\left\| (x_t(\lambda))_{t>0} \right\|_{V^q}$ is finite for a.e. $\lambda \in \Omega$. The lemma clearly follows from these properties. \square

3. Variation of ergodic averages

Throughout we let (Ω, μ) be a measure space and we let $1 < p < \infty$. We first recall the notion of regular operators on $L^p(\Omega)$ and some of their basic properties which will be used in this paper. We refer e.g. to [22, Chap. 1] and to [27, 28] for more details and complements.

An operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is called regular if there is a constant $C \geq 0$ such that

$$\left\| \sup_{k \geq 1} |T(x_k)| \right\|_p \leq C \left\| \sup_{k \geq 1} |x_k| \right\|_p$$

for any finite sequence $(x_k)_{k \geq 1}$ in $L^p(\Omega)$. Then we let $\|T\|_r$ denote the smallest C for which this holds. The set of all regular operators on $L^p(\Omega)$ is a vector space on which $\|\cdot\|_r$ is a norm. We say that T is contractively regular if $\|T\|_r \leq 1$.

Let E be a Banach space. If an operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is regular, then the operator $T \otimes I_E: L^p(\Omega) \otimes E \rightarrow L^p(\Omega) \otimes E$ extends to a bounded operator on the Bochner space $L^p(\Omega; E)$, and

$$(3.1) \quad \left\| T \otimes I_E: L^p(\Omega; E) \longrightarrow L^p(\Omega; E) \right\| \leq \|T\|_r.$$

Indeed by definition this holds true when $E = \ell_n^\infty$ for any $n \geq 1$, and the general case follows from the fact that for any $\varepsilon > 0$, any finite dimensional Banach space is $(1 + \varepsilon)$ -isomorphic to a subspace of ℓ_n^∞ for some large enough $n \geq 1$.

Any positive operator T (in the lattice sense) is regular and $\|T\|_r = \|T\|$ in this case. Thus all statements given for contractively regular operators apply to positive contractions. It is well-known that conversely, T is regular with $\|T\|_r \leq C$ if and only if there is a positive operator $S: L^p(\Omega) \rightarrow L^p(\Omega)$ with $\|S\| \leq C$, such that $|T(x)| \leq S(|x|)$ for any $x \in L^p(\Omega)$.

Finally, following [27], we say that an operator $T: L^1(\Omega) + L^\infty(\Omega) \rightarrow L^1(\Omega) + L^\infty(\Omega)$ is an absolute contraction if it induces two contractions

$$(3.2) \quad T: L^1(\Omega) \longrightarrow L^1(\Omega) \quad \text{and} \quad T: L^\infty(\Omega) \longrightarrow L^\infty(\Omega).$$

Then the resulting operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is contractively regular.

The main result of this section is the following theorem, which might be known to experts. Its proof relies on the transference principle, already used in [4].

THEOREM 3.1. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator, with $1 < p < \infty$, and let $2 < q < \infty$. Then we have*

$$\|(M_n(T)x)_{n \geq 0}\|_{L^p(v^q)} \leq C_{p,q} \|x\|_p, \quad x \in L^p(\Omega),$$

for some constant $C_{p,q}$ only depending on p and q .

Let

$$s_p: \ell_{\mathbb{Z}}^p \longrightarrow \ell_{\mathbb{Z}}^p, \quad s_p((c_n)_n) = (c_{n-1})_n,$$

denote the shift operator on $\ell_{\mathbb{Z}}^p$. According to [15, Thm. B], s_p satisfies Theorem 3.1 (the crucial case $p = 2$ going back to [4]). Thus for any $1 < p < \infty$ and $2 < q < \infty$ we have a constant

$$(3.3) \quad C_{p,q} = \|c \mapsto (M_n(s_p)c)_{n \geq 0}\|_{\ell_{\mathbb{Z}}^p \rightarrow \ell_{\mathbb{Z}}^p(v^q)}.$$

The following lemma is a variant of the well-known Coifman-Weiss transference Theorem [7, Thm. 2.4] and is closely related to [31] and [2].

LEMMA 3.2. — *Let $U: L^p(\Omega) \rightarrow L^p(\Omega)$ be an invertible operator such that U^j is regular for any $j \in \mathbb{Z}$ and suppose that $C = \sup\{\|U^j\|_r : j \in \mathbb{Z}\} < \infty$. Let (e_1, \dots, e_N) be a basis of a finite dimensional Banach space E , and let $a(1), \dots, a(N)$ be N elements of $\ell_{\mathbb{Z}}^1$. Consider the two operators*

$$K: \ell_{\mathbb{Z}}^p \longrightarrow \ell_{\mathbb{Z}}^p(E), \quad K(c) = \sum_{k=1}^N \left(\sum_{j \in \mathbb{Z}} a(k)_j s_p^j(c) \right) \otimes e_k,$$

and

$$R: L^p(\Omega) \longrightarrow L^p(\Omega; E), \quad R(x) = \sum_{k=1}^N \left(\sum_{j \in \mathbb{Z}} a(k)_j U^j(x) \right) \otimes e_k.$$

Then

$$\|R\| \leq C^2 \|K\|.$$

Note that in the above statement, we use the group structure of \mathbb{Z} . Indeed for any $k = 1, \dots, N$, the sum $\sum_{j \in \mathbb{Z}} a(k)_j s_p^j(c)$ is equal to the convolution product $a(k) * c$.

Proof of Lemma 3.2. The operator $I_{L^p(\Omega)} \otimes K$ extends to a bounded operator $L^p(\Omega; \ell_{\mathbb{Z}}^p) \rightarrow L^p(\Omega; \ell_{\mathbb{Z}}^p(E))$, whose norm is equal to $\|K\|$. (Nothing special about K is required for this tensor extension property.) By Fubini's Theorem,

$$L^p(\Omega; \ell_{\mathbb{Z}}^p) \simeq \ell_{\mathbb{Z}}^p(L^p(\Omega)) \quad \text{and} \quad L^p(\Omega; \ell_{\mathbb{Z}}^p(E)) \simeq \ell_{\mathbb{Z}}^p(L^p(\Omega; E))$$

isometrically. Further, under these identifications, the extension of $I_{L^p(\Omega)} \otimes K$ corresponds to the operator

$$\begin{aligned} \tilde{K} : \ell_{\mathbb{Z}}^p(L^p(\Omega)) &\longrightarrow \ell_{\mathbb{Z}}^p(L^p(\Omega; E)), \\ \tilde{K}(z) &= \sum_{k=1}^N \left(\sum_j a(k)_j (s_p^j \otimes I_{L^p(\Omega)})(z) \right) \otimes e_k \end{aligned}$$

Thus we have $\|\tilde{K}\| = \|K\|$.

By approximation, we may suppose that $a(1), \dots, a(N)$ are finitely supported. Let $m \geq 1$ be chosen such that $a(k)_j = 0$ for any k and any $|j| > m$. Let $x \in L^p(\Omega)$ and let

$$y_k = \sum_{j=-m}^m a(k)_j U^j(x)$$

for any $k = 1, \dots, N$. Our aim is to estimate the norm of $\sum_k y_k \otimes e_k$ in $L^p(\Omega; E)$. For any $i \in \mathbb{Z}$, we have

$$\sum_{k=1}^N y_k \otimes e_k = (U^i \otimes I_E) \left(\sum_{k=1}^N U^{-i} y_k \otimes e_k \right).$$

Hence applying (3.1) to U^i , we derive that

$$\left\| \sum_{k=1}^N y_k \otimes e_k \right\|_{L^p(\Omega; E)} \leq C \left\| \sum_{k=1}^N U^{-i} y_k \otimes e_k \right\|_{L^p(\Omega; E)}$$

Let $n \geq 1$ be an arbitrary integer and let χ be the characteristic function of the interval $[-(n+m), (n+m)]$. We deduce from the above estimate that

$$\begin{aligned} (2n+1) \left\| \sum_{k=1}^N y_k \otimes e_k \right\|_{L^p(\Omega; E)}^p &\leq C^p \sum_{i=-n}^n \left\| \sum_{k=1}^N \sum_{j=-m}^m a(k)_j U^{j-i}(x) \otimes e_k \right\|_{L^p(\Omega; E)}^p \\ &\leq C^p \sum_i \left\| \sum_{k=1}^N \sum_j a(k)_j \chi(j-i) U^{j-i}(x) \otimes e_k \right\|_{L^p(\Omega; E)}^p. \end{aligned}$$

Since

$$\begin{aligned} \sum_i \left\| \sum_{k=1}^N \sum_j a(k)_j \chi(j - \hat{i}) U^{j-i}(x) \otimes e_k \right\|_{L^p(\Omega; E)}^p \\ = \left\| \tilde{K} \left[(\chi(-i)U^{-i}(x))_i \right] \right\|_{\ell^p_2(L^p(\Omega; E))}^p \end{aligned}$$

and $\|\tilde{K}\| = \|K\|$, this yields

$$\begin{aligned} (2n + 1) \left\| \sum_{k=1}^N y_k \otimes e_k \right\|_{L^p(\Omega; E)}^p &\leq C^p \|K\|^p \sum_{i=-(n+m)}^{n+m} \|U^{-i}(x)\|_p^p \\ &\leq (2(n + m) + 1) C^{2p} \|K\|^p \|x\|_p^p. \end{aligned}$$

Letting $n \rightarrow \infty$, we get the result. □

Proof of Theorem 3.1. — Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator. There exists another measure space $(\widehat{\Omega}, \widehat{\mu})$, two positive contractions $J: L^p(\Omega) \rightarrow L^p(\widehat{\Omega})$ and $Q: L^p(\widehat{\Omega}) \rightarrow L^p(\Omega)$ and an isometric invertible operator $U: L^p(\widehat{\Omega}) \rightarrow L^p(\widehat{\Omega})$ such that

$$T^k = QU^k J, \quad k \geq 0.$$

In the case when T is positive, this is Akcoglu’s famous dilation Theorem (see [1]). The extension to regular operators stated here is from [27] or [6]. Moreover U can be chosen so that U and U^{-1} are both contractively regular. Thus

$$\forall j \in \mathbb{Z}, \quad \|U^j\|_r = 1.$$

Note that when $1 < p \neq 2 < \infty$, any isometry on L^p is contractively regular, so the latter information is relevant only when $p = 2$.

We fix an integer $m \geq 1$ and we consider v_m^q as defined in Section 2. For any $n \geq 0$, we clearly have

$$M_n(T) = QM_n(U)J.$$

Since $\|Q\|_r \leq 1$, it follows from (3.1) that

$$(3.4) \quad \left\| (M_n(T)x)_{0 \leq n \leq m} \right\|_{L^p(\Omega; v_m^q)} \leq \left\| (M_n(U)J(x))_{0 \leq n \leq m} \right\|_{L^p(\widehat{\Omega}; v_m^q)}, \quad x \in L^p(\Omega).$$

For any $n = 0, 1, \dots, m$, let $a(n) \in \ell^1_{\mathbb{Z}}$ be defined by letting $a(n)_j = (n + 1)^{-1}$ if $0 \leq j \leq n$ and $a(n)_j = 0$ otherwise. Then

$$\sum_{j \in \mathbb{Z}} a(n)_j U^j = M_n(U) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} a(n)_j s_p^j = M_n(s_p).$$

Applying Lemma 3.2 with $E = v_m^q$ and recalling (3.3), we therefore deduce that

$$\| (M_n(U)z)_{0 \leq n \leq m} \|_{L^p(\widehat{\Omega}; v_m^q)} \leq C_{p,q} \|z\|_p$$

for any $z \in L^p(\widehat{\Omega})$. Combining with the inequality (3.4) we obtain that

$$\| (M_n(T)x)_{0 \leq n \leq m} \|_{L^p(\Omega; v_m^q)} \leq C_{p,q} \|x\|_p$$

for any $x \in L^p(\Omega)$. Then the result follows from Lemma 2.1. □

Remark 3.3. — The above Lemma 3.2 extends without any difficulty to amenable groups, as follows. Let G be a locally compact amenable group, with left Haar measure dt , let $\pi: G \rightarrow B(L^p(\Omega))$ be a strongly continuous representation valued in the space of regular operators on $L^p(\Omega)$, and assume that

$$C = \sup\{\|\pi(t)\|_r : t \in G\} < \infty.$$

Next let h_1, \dots, h_N be N elements of $L^1(G)$ and let $K: L^p(G) \rightarrow L^p(G; E)$ be defined by letting $K(f) = \sum_k (h_k * f) \otimes e_k$ for any $f \in L^p(G)$. Then for any $x \in L^p(\Omega)$, we have

$$\left\| \sum_k \left(\int_G h_k(t) \pi(t)x dt \right) \otimes e_k \right\|_{L^p(\Omega; E)} \leq C^2 \|K\| \|x\|_p.$$

We conclude this section with a continuous version of Theorem 3.1. Given a strongly continuous semigroup $T = (T_t)_{t \geq 0}$ on $L^p(\Omega)$, we let

$$M_t(T) = \frac{1}{t} \int_0^t T_s ds, \quad t > 0,$$

defined in the strong sense.

COROLLARY 3.4. — *Let $T = (T_t)_{t \geq 0}$ be a strongly continuous semigroup on $L^p(\Omega)$ and assume that $T_t: L^p(\Omega) \rightarrow L^p(\Omega)$ is contractively regular for any $t \geq 0$. Let $2 < q < \infty$ and let $x \in L^p(\Omega)$. Then for a.e. $\lambda \in \Omega$, the family $([M_t(T)x](\lambda))_{t > 0}$ belongs to V^q and*

$$\left\| \lambda \mapsto \left\| ([M_t(T)x](\lambda))_{t > 0} \right\|_{V^q} \right\|_p \leq C_{p,q} \|x\|_p.$$

Proof. — Consider $x \in L^p(\Omega)$. According to [11, Section VIII.7], the function $t \mapsto [M_t(T)x](\lambda)$ is continuous for a.e. $\lambda \in \Omega$. Let $t_0 < t_1 < \dots < t_m$ be positive real numbers and let $\varepsilon > 0$. It follows from the strong continuity of $T = (T_t)_{t \geq 0}$ that there exist $\alpha > 0$ and integers n_0, n_1, \dots, n_m such that

$$\forall k = 0, \dots, m, \quad \left\| M_{t_k}(T)x - M_{n_k}(T_\alpha)x \right\|_p < \varepsilon.$$

Hence applying Theorem 3.1 and a limit argument, we deduce that

$$(3.5) \quad \|(M_{t_0}(T)x, M_{t_1}(T)x, \dots, M_{t_m}(T)x)\|_{L^p(\Omega; v_m^q)} \leq C_{pq} \|x\|_p.$$

The result therefore follows from Lemma 2.2. □

An alternative proof of (3.5) consists in using Fendler’s dilation Theorem for semigroups (see [12]), and then arguing as in the proof Theorem 3.1. This only requires knowing that the result of Corollary 3.4 holds true for the translation group on $L^p(\mathbb{R})$, which follows from [4, 15, 16], and using Remark 3.3 for $G = \mathbb{R}$ to transfer that result to strongly continuous groups of contractively regular isometries.

4. The analytic case

Let X be an arbitrary Banach space, and let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup on X . We call it a bounded analytic semigroup if there exists a positive angle $\omega \in (0, \frac{\pi}{2})$ and a bounded analytic family $z \in \Sigma_\omega \mapsto T_z \in B(X)$ extending $(T_t)_{t > 0}$, where

$$\Sigma_\omega = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \omega\}$$

is the open sector of angle 2ω around $(0, \infty)$. We refer to [14, 26] for various characterizations and properties of bounded analytic semigroups. We simply recall that if we let A denote the infinitesimal generator of $(T_t)_{t \geq 0}$, then the latter is a bounded analytic semigroup if and only if T_t maps X into the domain of A for any $t > 0$ and there exist two constants $C_0, C_1 > 0$ such that

$$(4.1) \quad \forall t > 0, \quad \|T_t\| \leq C_0 \quad \text{and} \quad \|tAT_t\| \leq C_1.$$

The definition of analyticity for discrete semigroups parallels (4.1). Let $T \in B(X)$. We say that T is power bounded if

$$\sup_{n \geq 0} \|T^n\| < \infty$$

and that it is analytic if moreover,

$$\sup_{n \geq 1} n \|T^n - T^{n-1}\| < \infty.$$

This notion goes back to [8] and has been studied in various contexts so far. We gather here a few spectral properties of these operators and refer to [3, 20, 23, 24] for proofs and complements. The most important result is the following: an operator T is power bounded and analytic if and only if

$$(4.2) \quad \sigma(T) \subset \overline{\mathbb{D}} \quad \text{and} \quad \{(z - 1)(zI - T)^{-1} : |z| > 1\} \text{ is bounded.}$$

This property is called the ‘Ritt condition’. The key argument for this characterization is due to O. Nevanlinna [24].

For any angle $\gamma \in (0, \frac{\pi}{2})$, let B_γ be the interior of the convex hull of 1 and the closed disc $\overline{D}(0, \sin \gamma)$.

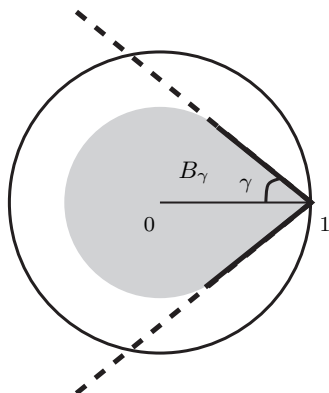


Figure 4.1.

Then (4.2) implies that

$$(4.3) \quad \exists \gamma \in (0, \frac{\pi}{2}) \mid \sigma(T) \subset \overline{B_\gamma}.$$

Furthermore, (4.3) is equivalent to

$$(4.4) \quad \exists K > 0 \mid \forall z \in \sigma(T), \quad |1 - z| \leq K(1 - |z|).$$

The aim of this section is to show that under an analyticity assumption, the ergodic averages can be replaced by the semigroup itself in either Theorem 3.1 (discrete case) or Corollary 3.4 (continuous case).

As in Section 3, we consider a measure space (Ω, μ) and a number $1 < p < \infty$. We will consider an operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$ and we let

$$\Delta_n^m = T^n(T - I)^m$$

for any integers $n, m \geq 0$. Note that $(\Delta_n^m)_{n \geq 0}$ is the m -difference sequence of $(T^n)_{n \geq 0}$.

We will need the following Littlewood-Paley type inequalities which were established in [19].

PROPOSITION 4.1. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator, with $1 < p < \infty$, and assume that T is analytic. Then for any integer $m \geq 0$, there is a constant $C_m > 0$ such that*

$$\left\| \left(\sum_{n=0}^{\infty} (n+1)^{2m+1} |\Delta_n^{m+1}(x)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_m \|x\|_p, \quad x \in L^p(\Omega).$$

We will also use the following elementary estimates, whose proofs are left to the reader.

LEMMA 4.2. — *For any integer $m \geq 0$, there exists a constant K_m such that for any $n \geq 1$,*

$$\left(\sum_{j=n}^{2n} (j+1)^{1-2m} \right)^{\frac{1}{2}} \leq K_m n^{-m+1}.$$

LEMMA 4.3. — *For any sequences $(\delta_n)_{n \geq 0} \in v^1$ and $(z_n)_{n \geq 0} \in L^p(\Omega; v^q)$, we have $(\delta_n z_n)_{n \geq 0} \in L^p(\Omega; v^q)$ and*

$$\|(\delta_n z_n)_{n \geq 0}\|_{L^p(v^q)} \leq 3 \|(\delta_n)_{n \geq 0}\|_{v^1} \| (z_n)_{n \geq 0} \|_{L^p(v^q)}.$$

In the next statements and their proofs, \lesssim will stand for an inequality up to a constant which may depend on T , q and m , but not on x .

THEOREM 4.4. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator, with $1 < p < \infty$, and assume that T is analytic. Then for any $2 < q < \infty$, we have an estimate*

$$(4.5) \quad \|(T^n(x))_{n \geq 0}\|_{L^p(v^q)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

More generally, for any integer $m \geq 0$, we have an estimate

$$(4.6) \quad \|(n^m \Delta_n^m(x))_{n \geq 1}\|_{L^p(v^q)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

Proof. — It will be convenient to set

$$\Delta_n^{-1} = nM_{n-1}(T) = \sum_{j=0}^{n-1} T^j, \quad n \geq 1.$$

Then for any $n < N$ and any $m \geq -1$, we have

$$(4.7) \quad \Delta_N^m - \Delta_n^m = \sum_{j=n}^{N-1} \Delta_j^{m+1}.$$

With the above notation, (4.6) holds true for $m = -1$, by Theorem 3.1.

We will proceed by induction. We fix an integer $m \geq 0$ and assume that (4.6) holds true for $(m - 1)$. Thus using Lemma 4.3, we both have

$$(4.8) \quad \left\| (n^{m-1} \Delta_{n+1}^{m-1}(x))_{n \geq 1} \right\|_{L^p(v^q)} \lesssim \|x\|_p$$

and

$$(4.9) \quad \left\| (n^{m-1} \Delta_{2n+1}^{m-1}(x))_{n \geq 1} \right\|_{L^p(v^q)} \lesssim \|x\|_p.$$

Next for any $n \geq 1$, we write

$$\begin{aligned} \sum_{j=n}^{2n} (j+1) \Delta_j^{m+1} &= \sum_{j=n}^{2n} (j+1) (\Delta_{j+1}^m - \Delta_j^m) \\ &= \sum_{j=n+1}^{2n+1} j \Delta_j^m - \sum_{j=n}^{2n} (j+1) \Delta_j^m \\ &= - \sum_{j=n+1}^{2n} \Delta_j^m + (2n+1) \Delta_{2n+1}^m - (n+1) \Delta_n^m \\ &= n \Delta_{2n+1}^m + (n+1) (\Delta_{2n+1}^m - \Delta_n^m) + \Delta_{n+1}^{m-1} - \Delta_{2n+1}^{m-1}, \end{aligned}$$

using (4.7) in due places. Hence

$$(4.10) \quad \begin{aligned} n^m \Delta_{2n+1}^m &= n^{m-1} \sum_{j=n}^{2n} (j+1) \Delta_j^{m+1} - n^{m-1} (n+1) (\Delta_{2n+1}^m - \Delta_n^m) \\ &\quad + n^{m-1} \Delta_{2n+1}^{m-1} - n^{m-1} \Delta_{n+1}^{m-1}. \end{aligned}$$

This identity suggests the introduction of the following two sequences of operators. For any $n \geq 1$, we set

$$A_n = n^{m-1} \sum_{j=n}^{2n} (j+1) \Delta_j^{m+1} \quad \text{and} \quad B_n = n^m (\Delta_{2n+1}^m - \Delta_n^m).$$

Also for any $x \in L^p(\Omega)$, we set

$$\Phi_m(x) = \left(\sum_{j=1}^{\infty} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}}.$$

According to Proposition 4.1, this function is an element of $L^p(\Omega)$.

Let $(n_k)_{k \geq 0}$ be an increasing sequence of integers, with $n_0 = 1$. For any $k \geq 1$, we set

$$\begin{aligned}
 a_k &= \begin{cases} n_k^{m-1} \sum_{j=2n_{k-1}+1}^{2n_k} (j+1)\Delta_j^{m+1} & \text{if } 2n_{k-1} \geq n_k \\ n_k^{m-1} \sum_{j=n_k}^{2n_k} (j+1)\Delta_j^{m+1} & \text{if } 2n_{k-1} < n_k, \end{cases} \\
 b_k &= \begin{cases} -n_{k-1}^{m-1} \sum_{j=n_{k-1}}^{n_k-1} (j+1)\Delta_j^{m+1} & \text{if } 2n_{k-1} \geq n_k \\ -n_{k-1}^{m-1} \sum_{j=n_{k-1}}^{2n_{k-1}-1} (j+1)\Delta_j^{m+1} & \text{if } 2n_{k-1} < n_k, \end{cases} \\
 c_k &= \begin{cases} (n_k^{m-1} - n_{k-1}^{m-1}) \sum_{j=n_k}^{2n_{k-1}} (j+1)\Delta_j^{m+1} & \text{if } 2n_{k-1} \geq n_k \\ 0 & \text{if } 2n_{k-1} < n_k. \end{cases}
 \end{aligned}$$

This yields a decomposition

$$(4.11) \quad A_{n_k} - A_{n_{k-1}} = a_k + b_k + c_k.$$

Let $x \in L^p(\Omega)$. If $2n_{k-1} \geq n_k$, we have, using Cauchy-Schwarz,

$$\begin{aligned}
 |a_k(x)| &\leq n_k^{m-1} \sum_{j=2n_{k-1}+1}^{2n_k} (j+1) |\Delta_j^{m+1}(x)| \\
 &\leq n_k^{m-1} \left(\sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}} \\
 &\leq n_k^{m-1} \left(\sum_{j=n_k}^{2n_k} (j+1)^{1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Similarly if $2n_{k-1} < n_k$, we have

$$\begin{aligned}
 |a_k(x)| &\leq n_k^{m-1} \left(\sum_{j=n_k}^{2n_k} (j+1)^{1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=n_k}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}} \\
 &\leq n_k^{m-1} \left(\sum_{j=n_k}^{2n_k} (j+1)^{1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Hence in both cases, we have

$$|a_k(x)|^2 \leq K_m^2 \sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2,$$

by Lemma 4.2. Summing up, we deduce that

$$(4.12) \quad \sum_{k=1}^{\infty} |a_k(x)|^2 \leq K_m^2 \Phi_m(x)^2.$$

Likewise we have

$$(4.13) \quad \sum_{k=1}^{\infty} |b_k(x)|^2 \leq K_m^2 \Phi_m(x)^2.$$

We now turn to $c_k(x)$. Assume that $2n_{k-1} \geq n_k$. Then using again Cauchy-Schwarz and Lemma 4.2, we have

$$|c_k(x)| \leq |n_k^{m-1} - n_{k-1}^{m-1}| \left(\sum_{j=n_k}^{2n_{k-1}} (j+1)^{1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=n_k}^{2n_{k-1}} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}},$$

hence

$$|c_k(x)|^2 \leq K_m^2 \left(\frac{n_k^{m-1} - n_{k-1}^{m-1}}{n_k^{m-1}} \right)^2 \sum_{j=n_k}^{2n_{k-1}} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2.$$

For any integer $j \geq 1$, define

$$J_j = \{k \geq 1 : n_k \leq j \leq 2n_{k-1}\},$$

and set

$$\Lambda_j = \sum_{k \in J_j} \left(\frac{n_k^{m-1} - n_{k-1}^{m-1}}{n_k^{m-1}} \right)^2.$$

Then it follows from the above calculation that

$$\sum_{k=1}^{\infty} |c_k(x)|^2 \leq K_m^2 \sum_{j=1}^{\infty} \Lambda_j (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2.$$

Let us now estimate the Λ_j 's. Observe that if J_j is a non empty set, then it is a finite interval of integers. Thus it reads as

$$J_j = \{k_j - N + 1, k_j - N + 2, \dots, k_j - 1, k_j\},$$

where k_j is the biggest element of J_j and N is its cardinal. Then

$$\sum_{k \in J_j} n_k^{m-1} - n_{k-1}^{m-1} = \sum_{r=0}^{N-1} n_{k_j-r}^{m-1} - n_{k_j-r-1}^{m-1} = n_{k_j}^{m-1} - n_{k_j-N}^{m-1} \leq n_{k_j}^{m-1}.$$

Suppose that $m \geq 2$. Since $k_j \in J_j$, we have $n_{k_j} \leq j$, hence

$$\sum_{k \in J_j} n_k^{m-1} - n_{k-1}^{m-1} \leq j^{m-1}.$$

Moreover we have $j \leq 2n_{k-1} \leq 2n_k$ for any $k \in J_j$, hence

$$\sum_{k \in J_j} \frac{n_k^{m-1} - n_{k-1}^{m-1}}{n_k^{m-1}} \leq \left(\frac{2}{j}\right)^{m-1} \sum_{k \in J_j} n_k^{m-1} - n_{k-1}^{m-1} \leq 2^{m-1}.$$

We immediatly deduce that

$$\Lambda_j \leq 4^{m-1}.$$

In the case when $m = 0$, we have similarly

$$\sum_{k \in J_j} \frac{n_{k-1}^{-1} - n_k^{-1}}{n_k^{-1}} \leq j \sum_{k \in J_j} n_{k-1}^{-1} - n_k^{-1} \leq j n_{k_j-N}^{-1} \leq 2.$$

Hence $\Lambda_j \leq 4$ in this case. Lastly, it is plain that if $m = 1$, we have $\Lambda_j = 0$.

This shows that in all cases, we have an estimate

$$\sum_{k=1}^{\infty} |c_k(x)|^2 \leq K'_m{}^2 \Phi_m(x)^2.$$

Now recall (4.11). Combining the above estimate with (4.12) and (4.13), we obtain that

$$\begin{aligned} \sum_{k=1}^{\infty} |A_{n_k}(x) - A_{n_{k-1}}(x)|^2 &\leq 3 \left(\sum_{k=1}^{\infty} |a_k(x)|^2 + \sum_{k=1}^{\infty} |b_k(x)|^2 + \sum_{k=1}^{\infty} |c_k(x)|^2 \right) \\ &\leq (6K_m^2 + K'_m{}^2) \Phi_m(x)^2. \end{aligned}$$

Since the upper bound does not depend on the sequence $(n_k)_{k \geq 0}$, this estimate and Proposition 4.1 imply that the sequence $(A_n(x))_{n \geq 0}$ belongs to $L^p(\Omega; v^2)$, and that we have an estimate

$$(4.14) \quad \left\| (A_n(x))_{n \geq 1} \right\|_{L^p(v^2)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

We will now apply a similar treatment to the sequence $(B_n)_n$. According to (4.7), we can write

$$B_n = n^m \sum_{j=n}^{2n} \Delta_j^{m+1}.$$

For any $k \geq 1$, we set

$$\begin{aligned} \alpha_k &= n_k^m \sum_{j=2n_{k-1}+1}^{2n_k} \Delta_j^{m+1}, \\ \beta_k &= -n_{k-1}^m \sum_{j=n_{k-1}}^{n_k-1} \Delta_j^{m+1}, \\ \gamma_k &= \left(n_k^m - n_{k-1}^m \right) \sum_{j=n_k}^{2n_{k-1}} \Delta_j^{m+1} \end{aligned}$$

if $2n_{k-1} \geq n_k$, and

$$\begin{aligned} \alpha_k &= n_k^m \sum_{j=n_k}^{2n_k} \Delta_j^{m+1}, \\ \beta_k &= -n_{k-1}^m \sum_{j=n_{k-1}}^{2n_{k-1}} \Delta_j^{m+1}, \\ \gamma_k &= 0 \end{aligned}$$

if $2n_{k-1} < n_k$.

Arguing as above, we obtain that for any $x \in L^p(\Omega)$, we have

$$|\alpha_k(x)| \leq n_k^m \left(\sum_{j=n_k}^{2n_k} (j+1)^{-1-2m} \right)^{\frac{1}{2}} \left(\sum_{j=2n_{k-1}+1}^{2n_k} (j+1)^{2m+1} |\Delta_j^{m+1}(x)|^2 \right)^{\frac{1}{2}},$$

and then

$$\sum_{k=1}^{\infty} |\alpha_k(x)|^2 \leq K_{m+1}^2 \Phi_m(x)^2.$$

Likewise we have

$$\sum_{k=1}^{\infty} |\beta_k(x)|^2 \leq K_{m+1}^2 \Phi_m(x)^2,$$

as well as an estimate

$$\sum_{k=1}^{\infty} |\gamma_k(x)|^2 \leq K_{m+1}'^2 \Phi_m(x)^2.$$

These three inequalities imply that the sequence $(B_n(x))_{n \geq 0}$ belongs to $L^p(\Omega; v^2)$, and that we have an estimate

$$(4.15) \quad \left\| (B_n(x))_{n \geq 1} \right\|_{L^p(v^2)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

We can now conclude our proof. Recall that $q > 2$, so that $v^2 \subset v^q$. Then using (4.8), (4.9), (4.14), (4.15) and Lemma 4.3, it follows from the

decomposition formula (4.10) that for any $x \in L^p(\Omega)$, $(n^m \Delta_{2n+1}^m(x))_{n \geq 1}$ belongs to $L^p(\Omega; v^q)$ and that we have an estimate

$$\| (n^m \Delta_{2n+1}^m(x))_{n \geq 1} \|_{L^p(v^q)} \lesssim \|x\|_p.$$

Since $n^m \Delta_n^m = n^m \Delta_{2n+1}^m - B_n$, a second application of (4.15) yields (4.6). □

The following is an analog of Theorem 4.4 for continuous semigroups.

COROLLARY 4.5. — *Let $(T_t)_{t \geq 0}$ be a bounded analytic semigroup on $L^p(\Omega)$ and assume that $T_t : L^p(\Omega) \rightarrow L^p(\Omega)$ is contractively regular for any $t \geq 0$. Let $2 < q < \infty$ and let $x \in L^p(\Omega)$. Then for a.e. $\lambda \in \Omega$, the family $([T_t(x)](\lambda))_{t > 0}$ belongs to V^q and we have an estimate*

$$(4.16) \quad \left\| \lambda \mapsto \left\| ([T_t(x)](\lambda))_{t > 0} \right\|_{V^q} \right\|_p \lesssim \|x\|_p.$$

More generally, for any integer $m \geq 0$, the family $\left(t^m \frac{\partial^m}{\partial t^m} (T_t(x))(\lambda) \right)_{t > 0}$ belongs to V^q for a.e. $\lambda \in \Omega$ and we have an estimate

$$(4.17) \quad \left\| \left\| \lambda \mapsto \left(t^m \frac{\partial^m}{\partial t^m} (T_t(x))(\lambda) \right)_{t > 0} \right\|_{V^q} \right\|_p \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

Proof. — Let $m \geq 0$ be an integer. It follows from [30, Lemma, p. 72] that for any $x \in L^p(\Omega)$, the function

$$t \mapsto t^m \frac{\partial^m}{\partial t^m} (T_t(x))(\lambda)$$

is continuous for a.e. $\lambda \in \Omega$.

Next it follows from the proof of [19, Cor. 4.2] that there exists a constant $C_m > 0$ such that

$$\left\| \left(\sum_{n=0}^{\infty} (n+1)^{2m-1} |T_t^n (T_t - I)^m(x)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_m \|x\|_p$$

for any $t > 0$ and any $x \in L^p(\Omega)$. That is, the operators T_t satisfy Proposition 4.1 uniformly. Since they also satisfy Theorem 3.1 uniformly, it follows from the proof of Theorem 4.4 that they satisfy the estimate (4.6) uniformly. Hence using an approximation argument as in the proof of [19, Cor. 4.2], we deduce that for any $x \in L^p(\Omega)$ and for any $0 < t_0 < t_1 < \dots < t_m$, we have

$$\| (T_{t_0}(x), T_{t_1}(x), \dots, T_{t_m}(x)) \|_{L^p(\Omega; v_m^q)} \leq C.$$

The result therefore follows from Lemma 2.2. □

5. Additional properties

We give here further properties of contractively regular operators and contractively regular semigroups, in connection with variational inequalities. Let (Ω, μ) be a measure space and let $1 < p < \infty$.

5.1. Individual ergodic theorems

Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contraction. According to the Mean Ergodic Theorem, we have a direct sum decomposition

$$L^p(\Omega) = N(I - T) \oplus \overline{R(I - T)},$$

where $N(\cdot)$ and $R(\cdot)$ denote the kernel and the range, respectively. Moreover if we let $P_T: L^p(\Omega) \rightarrow L^p(\Omega)$ denote the corresponding projection onto $N(I - T)$, then

$$(5.1) \quad M_n(T)x \xrightarrow{L^p} P_T(x)$$

for any $x \in L^p(\Omega)$. It is well-known that if T is an absolute contraction (see (3.2) for the definition), then $M_n(T)x \rightarrow P_T(x)$ almost everywhere (see e.g. [11, Section VIII.6]). We extend this classical result, as follows.

COROLLARY 5.1. — *Assume that $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is contractively regular. Then for any $x \in L^p(\Omega)$,*

$$[M_n(T)x](\lambda) \longrightarrow [P_T(x)](\lambda) \quad \text{for a.e. } \lambda \in \Omega.$$

Proof. — Let $x \in L^p(\Omega)$ and let $2 < q < \infty$. According to Theorem 3.1, the sequence $([M_n(T)x](\lambda))_{n \geq 0}$ belongs to v^q for almost every $\lambda \in \Omega$. Hence $([M_n(T)x](\lambda))_{n \geq 0}$ converges for almost every $\lambda \in \Omega$. Combining with (5.1), we obtain the result. \square

If a contraction $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is analytic, then $\|T^n - T^{n+1}\| \rightarrow 0$, hence $T^n(x) \rightarrow 0$ for any $x \in R(I - T)$. Consequently,

$$T^n(x) \xrightarrow{L^p} P_T(x)$$

for any $x \in L^p(\Omega)$. Using Theorem 4.4 and arguing as above, we obtain the following.

COROLLARY 5.2. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator and assume that T is analytic. Then for any $x \in L^p(\Omega)$,*

$$[T^n(x)](\lambda) \longrightarrow [P_T(x)](\lambda) \quad \text{for a.e. } \lambda \in \Omega.$$

We now consider the continuous case. The situation is essentially similar, except that we can also consider the behaviour when the parameter t tends to 0^+ . Let $T = (T_t)_{t \geq 0}$ be a strongly continuous semigroup of contractions. By definition, for any $x \in L^p(\Omega)$, $T_t(x) \rightarrow x$ in the L^p -norm when $t \rightarrow 0^+$. This implies that $M_t(T)x \rightarrow x$ when $t \rightarrow 0^+$. Let A denote the infinitesimal generator of $T = (T_t)_{t \geq 0}$. As in the discrete case, we have a direct sum decomposition

$$L^p(\Omega) = N(A) \oplus \overline{R(A)}.$$

Moreover if we let $P_A: L^p(\Omega) \rightarrow L^p(\Omega)$ denote the corresponding projection onto $N(A)$, then

$$M_t(T)x \xrightarrow{L^p} P_A(x) \quad \text{when } t \rightarrow \infty$$

for any $x \in L^p(\Omega)$. If further $(T_t)_{t \geq 0}$ is a bounded analytic semigroup, then

$$T_t(x) \xrightarrow{L^p} P_A(x)$$

for any $x \in L^p(\Omega)$.

Now applying Corollaries 3.4 and 4.5, we deduce the following individual ergodic theorems.

COROLLARY 5.3. — *Let $T = (T_t)_{t \geq 0}$ be a strongly continuous semigroup of contractively regular operators on $L^p(\Omega)$, and let $x \in L^p(\Omega)$. Then for almost every $\lambda \in \Omega$,*

$$[M_t(T)x](\lambda) \rightarrow [P_A(x)](\lambda) \quad \text{when } t \rightarrow \infty$$

and

$$[M_t(T)x](\lambda) \rightarrow x(\lambda) \quad \text{when } t \rightarrow 0^+.$$

COROLLARY 5.4. — *Let $T = (T_t)_{t \geq 0}$ be a bounded analytic semigroup and assume that T_t is contractively regular for any $t \geq 0$. Let $x \in L^p(\Omega)$. Then for almost every $\lambda \in \Omega$,*

$$[T_t(x)](\lambda) \rightarrow [P_A(x)](\lambda) \quad \text{when } t \rightarrow \infty$$

and

$$[T_t(x)](\lambda) \rightarrow x(\lambda) \quad \text{when } t \rightarrow 0^+.$$

5.2. The case $q = 2$

In this section, we fix a increasing sequence $(n_k)_{k \geq 0}$ of integers, with $n_0 = 0$. Given any sequence $(a_n)_{n \geq 0}$ of complex numbers, we define the so-called oscillation norm

$$\|(a_n)_{n \geq 0}\|_{o^2} = (|a_0|^2 + \sum_{k \geq 0} \max_{n_k \leq n, m \leq n_{k+1}} |a_n - a_m|^2)^{\frac{1}{2}},$$

and we let o^2 denote the Banach space of all sequences with a finite oscillation norm, equipped with $\| \cdot \|_{o^2}$. This space (whose definition depends on the sequence $(n_k)_{k \geq 0}$) was used in [4, 15, 16] as a substitute to v^q in the case $q = 2$ (see also [13]). Indeed, neither Theorem 3.1 nor Theorem 4.4 holds true for $q = 2$, see [15] and [18, Section 8].

Recall the shift operator $s_p: \ell_{\mathbb{Z}}^p \rightarrow \ell_{\mathbb{Z}}^p$ for any $1 < p < \infty$ (see Section 3). According to [15, Thm. A], there is a constant $C_{p,2}$ such that

$$\| (M_n(s_p)c)_{n \geq 0} \|_{L^p(o^2)} \leq C_{p,2} \|c\|_p$$

for any $c \in \ell_{\mathbb{Z}}^p$. Hence arguing as in the proof of Theorem 3.1, we obtain the following o^2 -version of the latter statement.

THEOREM 5.5. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator, with $1 < p < \infty$. Then we have*

$$\| (M_n(T)x)_{n \geq 0} \|_{L^p(o^2)} \leq C_{p,2} \|x\|_p, \quad x \in L^p(\Omega).$$

We also have an o^2 -version of Theorem 4.4, as follows.

THEOREM 5.6. — *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a contractively regular operator, with $1 < p < \infty$, and assume that T is analytic. Then we have an estimate*

$$(5.2) \quad \| (T^n(x))_{n \geq 0} \|_{L^p(o^2)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

More generally, for any integer $m \geq 0$, we have an estimate

$$(5.3) \quad \| (n^m \Delta_n^m(x))_{n \geq 1} \|_{L^p(o^2)} \lesssim \|x\|_p, \quad x \in L^p(\Omega).$$

Proof. — The proof is a variant of the one written for Theorem 4.4, let us explain this briefly. We use the notation from Section 4. For any $m \geq -1$, consider the following three properties:

- (i)_m $\| (n^m \Delta_n^m(x))_{n \geq 1} \|_{L^p(o^2)} \lesssim \|x\|_p$;
- (ii)_m $\| (n^m \Delta_{n+1}^m(x))_{n \geq 1} \|_{L^p(o^2)} \lesssim \|x\|_p$;
- (iii)_m $\| (n^m \Delta_{2n+1}^m(x))_{n \geq 1} \|_{L^p(o^2)} \lesssim \|x\|_p$.

Property (i)_m is the result we wish to prove. Our strategy is to show, by induction, that these three estimates hold true.

By Lemma 4.3, the sequence $(\frac{z_n}{\sqrt{n+1}})_{n \geq 0}$ belongs to $L^p(v^2)$ for any $(z_n)_{n \geq 0}$ in $L^p(v^2)$. Since $\ell^2 \subset v^2 \subset o^2$, this implies that $(\frac{z_n}{\sqrt{n+1}})_{n \geq 0}$ belongs to $L^p(o^2)$ for any $(z_n)_{n \geq 0}$ in $L^p(\ell^2)$. Applying Proposition 4.1, we deduce from this observation that (i)_m and (ii)_m are equivalent for any $m \geq 0$.

Next it follows from (4.14), (4.15) and (4.10) that (ii)_{m-1} and (iii)_{m-1} imply (iii)_m and (i)_m. Indeed $v^2 \subset o^2$ and $n^m \Delta_n^m = n^m \Delta_{2n+1}^m - B_n$. Hence it suffices to show (ii)₋₁ and (iii)₋₁. This is obtained by applying Theorem 5.5 twice, the first time for the o^2 -space associated with the sequence

$(n_k)_{k \geq 1}$, the second time for the o^2 -space associated with the sequence $(2n_k)_{k \geq 1}$. □

There are also o^2 -versions of Corollary 3.4 and Corollary 4.5, whose statements are left to the reader.

5.3. Jump functions

It is well known that variational inequalities for a sequence of operators have consequences in terms of jump functions. For any $\tau > 0$ and any sequence $a = (a_n)_{n \geq 0}$ of complex numbers, let $N(a, \tau)$ denote the number of τ -jumps of a , defined as the supremum of all integers $N \geq 0$ for which there exist integers

$$0 \leq n_1 < m_1 \leq n_2 < m_2 \leq \dots \leq n_N < m_N,$$

such that $|a_{m_k} - a_{n_k}| > \tau$ for each $k = 1, \dots, N$. It is clear that for any $1 \leq q < \infty$,

$$\tau^q N(a, \tau) \leq \|a\|_{v^q}^q.$$

Combining with Theorem 3.1 and Theorem 4.4, we immediately obtain (as in [16, Thm. 3.15]) the following jump estimates.

COROLLARY 5.7. — *Consider $1 < p < \infty$ and $2 < q < \infty$. Let $T: L^p(\Omega, \mu) \rightarrow L^p(\Omega, \mu)$ be a contractively regular operator.*

(1) *We have an estimate*

$$\left\| \lambda \mapsto N\left(\left([M_n(T)x](\lambda)\right)_{n \geq 0}, \tau\right) \right\|_p^{\frac{1}{q}} \lesssim \frac{\|x\|_p}{\tau},$$

and for any $K > 0$, we also have

$$\mu\left\{ \lambda \in \Omega \mid N\left(\left([M_n(T)x](\lambda)\right)_{n \geq 0}, \tau\right) > K \right\} \lesssim \frac{\|x\|_p^p}{\tau^p K^{\frac{p}{q}}},$$

(2) *Assume moreover that T is analytic. Then we have similar estimates*

$$\left\| \lambda \mapsto N\left(\left([T^n(x)](\lambda)\right)_{n \geq 0}, \tau\right) \right\|_p^{\frac{1}{q}} \lesssim \frac{\|x\|_p}{\tau},$$

and

$$\mu\left\{ \lambda \in \Omega \mid N\left(\left([T^n(x)](\lambda)\right)_{n \geq 0}, \tau\right) > K \right\} \lesssim \frac{\|x\|_p^p}{\tau^p K^{\frac{p}{q}}},$$

Furthermore for any integer $m \geq 0$, similar results hold with $n^m \Delta_n^m$ instead of T^n .

Similar results for the continuous case can be deduced as well from Corollary 3.4 and Corollary 4.5.

6. Examples and applications

We will now exhibit various classes of operators or semigroups to which our results from Section 4 and Section 5 apply. We focus on statements involving q -variation, although statements involving the oscillation norm from the subsection 5.2 are also possible.

We start with the continuous case. Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup on $L^2(\Omega)$ and assume that each T_t is an absolute contraction, that is,

$$(6.1) \quad \|T_t(x)\|_1 \leq \|x\|_1 \quad \text{and} \quad \|T_t(x)\|_\infty \leq \|x\|_\infty$$

for any $x \in L^1(\Omega) + L^\infty(\Omega)$ and any $t > 0$ (see Section 3). Thus for any $1 < p < \infty$, $(T_t)_{t \geq 0}$ extends to a strongly continuous semigroup of contractively regular operators. A well-known application of Stein’s interpolation principle developed in [30, III.2] says that $(T_t)_{t \geq 0}$ is a bounded analytic semigroup on $L^p(\Omega)$ for every $1 < p < \infty$ if (and only if) it is a bounded analytic semigroup on $L^p(\Omega)$ for one $1 < p < \infty$. Applying Corollary 4.5 and Corollary 5.4, we derive the following.

COROLLARY 6.1. — *Let $(T_t)_{t \geq 0}$ be a bounded analytic semigroup on $L^2(\Omega)$ satisfying (6.1). Then it satisfies estimates (4.16) and (4.17) for every $1 < p < \infty$ and every $2 < q < \infty$. Moreover for any $x \in L^p(\Omega)$, $T_t(x)$ converges almost everywhere when $t \rightarrow 0^+$ and when $t \rightarrow \infty$.*

Note that if each $T_t: L^2(\Omega) \rightarrow L^2(\Omega)$ is selfadjoint, then $(T_t)_{t \geq 0}$ is a bounded analytic semigroup on $L^2(\Omega)$ (see [30, III.2] again). Hence the above corollary applies to symmetric diffusion semigroups and extends [16, Thm. 3.3].

We now consider the so-called subordinated semigroups. Let $1 < p < \infty$ and let $(T_t)_{t \geq 0}$ be a strongly continuous bounded semigroup on $L^p(\Omega)$. Let A denote its infinitesimal generator and let $0 < \alpha < 1$. Then $-(-A)^\alpha$ generates a bounded analytic semigroup $(T_{\alpha,t})_{t \geq 0}$ on $L^p(\Omega)$, and for any $t > 0$, there exists a continuous function $f_{\alpha,t}: (0, \infty) \rightarrow \mathbb{R}$ such that

$$(6.2) \quad \forall s > 0, \quad f_{\alpha,t}(s) \geq 0; \quad \int_0^\infty f_{\alpha,t}(s) ds = 1;$$

and

$$(6.3) \quad T_{\alpha,t}(x) = \int_0^\infty f_{\alpha,t}(s) T_s(x) ds, \quad x \in L^p(\Omega).$$

See e.g. [32, IX.11] for details and complements.

COROLLARY 6.2. — Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup on $L^p(\Omega)$, with $1 < p < \infty$, and assume that $T_t: L^p(\Omega) \rightarrow L^p(\Omega)$ is contractively regular for any $t \geq 0$. Then for any $0 < \alpha < \infty$ and any $2 < q < \infty$, we have an estimate

$$\left\| \lambda \mapsto \left\| ([T_{\alpha,t}(x)](\lambda))_{t>0} \right\|_{V^q} \right\|_p \lesssim \|x\|_p$$

for the subordinated semigroup $T_{\alpha,t} = e^{-t(-A)^\alpha}$. Moreover for any $x \in L^p(\Omega)$, $T_{\alpha,t}(x)$ converges almost everywhere when $t \rightarrow 0^+$ and when $t \rightarrow \infty$.

Proof. — Let $0 < \alpha < \infty$. It follows from (6.2) and (6.3) that for any $t > 0$,

$$\|T_{\alpha,t}\|_r \leq \int_0^\infty f_{\alpha,t}(s) \|T_s\|_r ds \leq 1.$$

Hence the result follows from Corollary 4.5 and Corollary 5.4. □

We now turn to the discrete case and consider normal operators on L^2 .

LEMMA 6.3. — Let H be a Hilbert space and let $T \in B(H)$ be a normal operator. Then T is an analytic power bounded operator if and only if it satisfies (4.4).

Proof. — The ‘only if’ part holds for any operator, as discussed at the beginning of Section 4. Conversely, assume that a normal operator $T: H \rightarrow H$ satisfies (4.4). Applying the Spectral Theorem, we deduce that T is a contraction and that for some constant $K > 0$, we have

$$\begin{aligned} n\|T^n - T^{n-1}\| &= \sup_{z \in \sigma(T)} n|z^n - z^{n-1}| = \sup_{z \in \sigma(T)} n|z|^{n-1}|1 - z| \\ &\leq K \sup_{r \in [0,1]} n(r^{n-1} - r^n) = K \left(\frac{n-1}{n}\right)^{n-1} \end{aligned}$$

for any $n \geq 1$. Hence the sequence $(n(T^n - T^{n-1}))_{n \geq 1}$ is bounded. □

The following is a straightforward consequence of the above lemma, Theorem 4.4 and Corollary 5.2.

COROLLARY 6.4. — Let $T: L^2(\Omega) \rightarrow L^2(\Omega)$ be a contractively regular normal operator satisfying (4.4). Then it satisfies an estimate

$$\left\| (T^n(x))_{n \geq 0} \right\|_{L^2(V^q)} \lesssim \|x\|_2, \quad x \in L^2(\Omega)$$

for any $2 < q < \infty$. Moreover for any $x \in L^2(\Omega)$, $T^n(x)$ converges almost everywhere when $n \rightarrow \infty$.

The following is a discrete analog of Corollary 6.1.

COROLLARY 6.5. — Let $T: L^1(\Omega) + L^\infty(\Omega) \rightarrow L^1(\Omega) + L^\infty(\Omega)$ be an absolute contraction and assume that $T: L^2(\Omega) \rightarrow L^2(\Omega)$ is analytic. Then

T satisfies estimates (4.5) and (4.6) for every $1 < p < \infty$ and every $2 < q < \infty$. Moreover for any $x \in L^p(\Omega)$, $T^n(x)$ converges almost everywhere when $n \rightarrow \infty$.

Proof. — If $T: L^2(\Omega) \rightarrow L^2(\Omega)$ is analytic, then $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is analytic as well for any $1 < p < \infty$, by [3, Thm. 1.1]. Hence the result follows from Theorem 4.4 and Corollary 5.2. \square

Recall that $T: L^2(\Omega) \rightarrow L^2(\Omega)$ is analytic if it is a positive selfadjoint operator, more generally if it is normal and satisfies (4.4). Corollary 6.5 therefore applies to these cases. As a consequence, we extend the main result of [16], as follows. The next statement solves a problem raised in the latter paper.

COROLLARY 6.6. — *Let G be a locally compact abelian group and let $L^p(G)$ denote the corresponding L^p -spaces with respect to a Haar measure. Let ν be a probability measure on G and let $T: L^1(G) + L^\infty(G) \rightarrow L^1(G) + L^\infty(G)$ be the associated convolution operator,*

$$T(x) = \nu * x.$$

Assume that there exists a constant $K > 0$ such that $|1 - \widehat{\nu}(s)| \leq K(1 - |\widehat{\nu}(s)|)$ for any $s \in \widehat{G}$. Then we have an estimate

$$\|(T^n(x))_{n \geq 0}\|_{L^p(\nu^q)} \lesssim \|x\|_p, \quad x \in L^p(\Omega),$$

for any $1 < p < \infty$ and any $2 < q < \infty$.

Proof. — Regard T as an L^2 -operator. By Fourier analysis, its spectrum is equal to the essential range of $\widehat{\nu}$. It therefore follows from the assumption and Lemma 6.3 that the operator $T: L^2(G) \rightarrow L^2(G)$ is analytic. Hence T satisfies the assumptions of Corollary 6.5, which yields the result. \square

Remark 6.7. — For an operator $T \in B(L^2(\Omega))$, let $W(T) = \{\langle T(x), x \rangle : \|x\|_2 = 1\}$ denote the numerical range. We recall that $W(T)$ is a compact convex set and that $\sigma(T) \subset W(T)$. Assume that there exists $\gamma \in (0, \frac{\pi}{2})$ such that $W(T) \subset \overline{B_\gamma}$ (which is a stronger condition than (4.3) or (4.4)). Then according to [10], there exists a constant $C > 0$ such that

$$\|f(T)\| \leq C \sup\{|f(z)| : z \in B_\gamma\}$$

for any polynomial f . Arguing as in the proof of Lemma 6.3, we deduce that T is an analytic power bounded operator.

Consequently if $T: L^2(\Omega) \rightarrow L^2(\Omega)$ is contractively regular and $W(T) \subset \overline{B_\gamma}$ for some $\gamma \in (0, \frac{\pi}{2})$, then it satisfies (4.5) for $p = 2$.

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