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## DE RHAM THEORY FOR TAME STACKS AND SCHEMES WITH LINEARLY REDUCTIVE SINGULARITIES

by Matthew SATTRIANO

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ABSTRACT. — We prove that the Hodge-de Rham spectral sequence for smooth proper tame Artin stacks in characteristic  $p$  (as defined by Abramovich, Olsson, and Vistoli) which lift mod  $p^2$  degenerates. We push the result to the coarse spaces of such stacks, thereby obtaining a degeneracy result for schemes which are étale locally the quotient of a smooth scheme by a finite linearly reductive group scheme.

RÉSUMÉ. — Nous démontrons que la suite spectrale de Hodge-De Rham d'un champ d'Artin propre modéré en caractéristique  $p$  (d'après Abramovich, Olsson et Vistoli) qui se relève mod  $p^2$  dégénère. Nous étendons ce résultat à des schémas quotients d'un schéma lisse par un schéma en groupes linéaires réductifs.

Given a scheme  $X$  smooth and proper over a field  $k$ , the cohomology of the algebraic de Rham complex  $\Omega_{X/k}^\bullet$  is an important invariant of  $X$ , which, when  $k = \mathbb{C}$ , recovers the singular cohomology of  $X(\mathbb{C})$ . When the Hodge-de Rham spectral sequence

$$E_1^{st} = H^t(\Omega_{X/k}^s) \Rightarrow H^n(\Omega_{X/k}^\bullet)$$

degenerates, the invariants  $\dim_k H^n(\Omega_{X/k}^\bullet)$  break up into sums of the finer invariants  $\dim_k H^t(\Omega_{X/k}^s)$ . The degeneracy of this spectral sequence for smooth proper schemes in characteristic 0 was first proved via analytic methods. It was not until much later that Faltings [6] gave a purely algebraic proof by means of  $p$ -adic Hodge Theory. Soon afterwards, Deligne and Illusie [5] gave a substantially simpler algebraic proof by showing that the degeneracy of the Hodge-de Rham spectral sequence in characteristic 0 is implied by its degeneracy for smooth proper schemes in characteristic  $p$  that lift mod  $p^2$ . Their method therefore extends de Rham Theory to

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the class of smooth proper schemes in positive characteristic which lift. A version of de Rham Theory also exists for certain singular schemes. Steenbrink showed [18, Thm 1.12] that if  $k$  is a field of characteristic 0,  $M$  a proper  $k$ -scheme with quotient singularities, and  $j : M^0 \hookrightarrow M$  its smooth locus, then the hypercohomology spectral sequence

$$E_1^{st} = H^t(j_*\Omega_{M^0/k}^s) \Rightarrow H^n(j_*\Omega_{M^0/k}^\bullet)$$

degenerates and  $H^n(j_*\Omega_{M^0/k}^\bullet)$  agrees with  $H^n(M(\mathbb{C}), \mathbb{C})$  when  $k = \mathbb{C}$ . As we explain in this paper, a version of this theorem is true in positive characteristic as well: if  $k$  has characteristic  $p$  and  $M$  is proper with quotient singularities by groups whose orders are prime to  $p$ , then the above spectral sequence degenerates for  $s + t < p$  provided a certain liftability criterion is satisfied (see Theorem 1.15 for precise hypotheses).

As a warm-up for the rest of the paper, we begin by showing how Steenbrink's result can be reproved using the theory of stacks. The idea is as follows. Every scheme  $M$  as above is the coarse space of a smooth Deligne-Mumford stack  $\mathfrak{X}$  whose stacky structure is supported at the singular locus of  $M$ . We show that the de Rham cohomology  $H^n(\Omega_{\mathfrak{X}/k}^\bullet)$  of the stack agrees with  $H^n(j_*\Omega_{M^0/k}^\bullet)$ . After checking that the method of Deligne-Illusie extends to Deligne-Mumford stacks, we recover Steenbrink's result as a consequence of the degeneracy of the Hodge-de Rham spectral sequence for  $\mathfrak{X}$ .

The above extends de Rham Theory to the class of schemes with quotient singularities by groups whose orders are prime to the characteristic, but in positive characteristic this class of schemes contains certain "gaps" and it is natural to ask if de Rham Theory can be extended further. For example, in all characteristics except for 2, the affine quadric cone  $\text{Spec } k[x, y, z]/(xy - z^2)$  can be realized as the quotient of  $\mathbb{A}^2$  by  $\mathbb{Z}/2\mathbb{Z}$  under the action  $x \mapsto -x$ ,  $y \mapsto -y$ . In characteristic 2, however, this action is trivial. If we allow quotients not just by finite groups, but rather finite group schemes, then we can realize the cone as  $\mathbb{A}^2/\mu_2$  where  $\zeta \in \mu_2(T)$  acts as  $x \mapsto \zeta x$ ,  $y \mapsto \zeta y$ . This is an example of what we call a scheme with *linearly reductive singularities*; that is, a scheme which is étale locally the quotient of a smooth scheme by a finite flat linearly reductive group scheme.

One of the main results of this paper is that de Rham Theory can be extended to the class of schemes with isolated linearly reductive singularities. As with Steenbrink's result, we prove this by passing through stacks. Just as schemes with quotient singularities are coarse spaces of smooth Deligne-Mumford stacks whose stacky structure is supported at the singular locus, we show in Theorem 4.2 that schemes with linearly reductive singularities are coarse spaces of smooth tame (Artin) stacks (as introduced in [1])

whose stacky structure is supported at the singular locus. So, to extend de Rham Theory to schemes with isolated linearly reductive singularities, we first show the degeneracy of a type of Hodge-de Rham spectral sequence for tame stacks. We should emphasize that, unlike in the case of Deligne-Mumford stacks, there are technical barriers to extending the method of Deligne-Illusie to Artin stacks or even tame stacks, first and foremost being that relative Frobenius does not behave well under smooth base change. It should also be noted that it is *a priori* not clear what the definition of the de Rham complex of a tame stack  $\mathfrak{X}$  should be. One can use the cotangent complex  $L_{\mathfrak{X}}$  of the stack (see [12, §15] and [16, §8]) to define the derived de Rham complex  $\bigwedge^{\bullet} L_{\mathfrak{X}}$ ; alternatively, one can use a more naive sheaf of differentials  $\varpi_{\mathfrak{X}}^1$  on the lisse-étale site of  $\mathfrak{X}$  whose restriction to each  $U_{et}$  is  $\Omega_U^1$ , for every  $U$  smooth over  $\mathfrak{X}$ . The latter has the advantage that it is simpler, but it is *not coherent*; the cotangent complex, on the other hand, has coherent cohomology sheaves. We take the naive de Rham complex as our definition, but it is by comparing this complex with the derived de Rham complex that we prove our main result for tame stacks:

**THEOREM 3.7.** — *Let  $\mathfrak{X}$  be a smooth proper tame stack over a perfect field  $k$  of characteristic  $p$ . If  $\mathfrak{X}$  lifts mod  $p^2$ , then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\varpi_{\mathfrak{X}/k}^s) \Rightarrow H^n(\varpi_{\mathfrak{X}/k}^{\bullet})$$

*degenerates for  $s + t < p$  (see the Notation section below).*

>From Theorem 3.7 and Theorem 4.2, we are able to deduce

**THEOREM 4.8.** — *Let  $M$  be a proper  $k$ -scheme with isolated linearly reductive singularities, where  $k$  is a perfect field of characteristic  $p$ . Let  $j : M^0 \hookrightarrow M$  be the smooth locus of  $M$  and let  $\mathfrak{X}$  be as in Theorem 4.2. If  $\mathfrak{X}$  lifts mod  $p^2$ , then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_*\Omega_{M^0/k}^s) \Rightarrow H^n(j_*\Omega_{M^0/k}^{\bullet})$$

*degenerates for  $s + t < p$ .*

We should mention that unlike in the case of quotient singularities, the cohomology groups  $H^n(\varpi_{\mathfrak{X}/k}^{\bullet})$  and  $H^n(j_*\Omega_{M^0/k}^{\bullet})$  no longer agree, so some care is needed in showing how Theorem 4.8 follows from the degeneracy of the Hodge-de Rham spectral sequence of the stack.

It is desirable, of course, to remove the stack from the statement Theorem 4.8. We show in Theorem 4.9 that if the dimension of  $M$  is at least 4, the liftability of  $M$  implies the liftability of  $\mathfrak{X}$ . In this case, we therefore have

a purely scheme-theoretic statement of Theorem 4.8. We end the paper by proving a type of Kodaira vanishing theorem within this setting.

This paper is organized as follows. In Section 1, we begin by reviewing some background material and giving an outline of [5, Thm 2.1] as some of the technical details will be used later. We then consider de Rham Theory for Deligne-Mumford stacks and show how stacks can be used to recast Steenbrink’s result. The purpose of Section 2 is to find a way around the problem that the method of Deligne and Illusie does not carry over directly to the lisse-étale site of Artin stacks. Since relative Frobenius *does* behave well under étale base change, our solution is to prove a Deligne-Illusie result on the étale site of  $X_\bullet$ , where  $X \rightarrow \mathfrak{X}$  is a smooth cover of a smooth tame stack by a scheme, and  $X_\bullet$  is the simplicial scheme obtained by taking fiber products over  $\mathfrak{X}$ . The key technical point here is showing that étale locally on the coarse space of  $\mathfrak{X}$ , the relative Frobenius for  $\mathfrak{X}$  lifts mod  $p^2$ . In Section 3, we prove that the naive de Rham complex and the derived de Rham complex above compute the same cohomology, and show how this result implies the degeneracy of the Hodge-de Rham spectral sequence for smooth proper tame stacks which lift mod  $p^2$ . In Section 4, we prove Theorem 4.8.

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*Notation.* — Unless otherwise stated, all Artin stacks are assumed to have finite diagonal. If  $\mathfrak{X}$  is an Artin stack over a scheme  $S$ , we let  $\mathfrak{X}'$  denote the pullback of  $\mathfrak{X}$  by the absolute Frobenius  $F_S$ . We usually drop the subscript on the relative Frobenius  $F_{\mathfrak{X}/S}$ , denoting it by  $F$ . Given a morphism  $g : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  of  $S$ -stacks, we denote by  $g' : \mathfrak{X}'_1 \rightarrow \mathfrak{X}'_2$  the induced morphism.

Given a morphism  $g : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  of Artin stacks and complex of sheaves  $\mathcal{F}^\bullet$  on  $\mathfrak{X}_1$ , we do not use the shorthand  $g_*\mathcal{F}^\bullet$  when we mean  $Rg_*\mathcal{F}^\bullet$ . For us,  $g_*\mathcal{F}^\bullet$  always denotes the complex obtained by applying the functor  $g_*$  to the complex  $\mathcal{F}^\bullet$ .

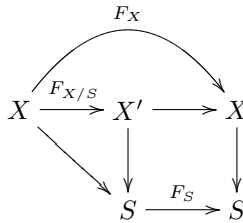
Lastly, we say a first quadrant spectral sequence  $E_{r_0}$  “degenerates for  $s + t < N$ ” if for all  $r \geq r_0$  and all  $s$  and  $t$  satisfying  $s + t < N$ , all of the differentials to and from the  $E_r^{st}$  are zero.

# 1. Steenbrink’s Result via Stacks

## 1.1. Review of Deligne-Illusie

We briefly review the proof of [5, Thm 2.1] and explain how it generalizes to Deligne-Mumford stacks. Having an outline of this proof will be useful for us in Section 2.

Let  $S = \text{Spec } k$  be a perfect field of characteristic  $p$ . For any  $S$ -scheme  $X$ , let  $F_X : X \rightarrow X$  be the absolute Frobenius, which acts as the identity on topological spaces and sends a local section  $s \in \mathcal{O}_X(U)$  to  $s^p$ . We have the following commutative diagram, where  $F_{X/S}$  is the relative Frobenius and the square is cartesian



We drop the subscript on the relative Frobenius  $F_{X/S}$ , denoting it by  $F$ . If  $X$  is locally of finite type over  $S$ , so that it is locally  $\text{Spec } k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ , where  $f_j = \sum a_{j,I} x^I$ , then  $X'$  is locally  $\text{Spec } k[x_1, \dots, x_n]/(f_1^{(p)}, \dots, f_m^{(p)})$ , where  $f_j^{(p)} := \sum a_{j,I}^p x^I$ . The relative Frobenius morphism is then given by sending  $x_i$  to  $x_i^p$  and  $a \in k$  to  $a$ .

Our primary object of study is the de Rham complex  $\Omega_{X/S}^\bullet$ . The maps in this complex are not  $\mathcal{O}_X$ -linear. To correct this “problem” we instead consider  $F_*\Omega_{X/S}^\bullet$ , whose maps are  $\mathcal{O}_{X'}$ -linear. It is now reasonable to ask how the cohomology of this new complex compares with the cohomology of the de Rham complex on  $X'$ . An answer is given by:

**THEOREM 1.1 (Cartier isomorphism).** — *If  $X$  is smooth over  $S$ , then there is a unique isomorphism of  $\mathcal{O}_{X'}$ -graded algebras*

$$C^{-1} : \bigoplus \Omega_{X'/S}^i \longrightarrow \bigoplus \mathcal{H}^i(F_*\Omega_{X/S}^\bullet)$$

such that  $C^{-1}d(x \otimes 1)$  is the class of  $x^{p-1}dx$  for all local sections  $x$  of  $\mathcal{O}_{X'}$ .

Note that once  $C^{-1}$  is shown to exist, uniqueness is automatic. For a proof of this theorem, see [10, Thm 7.2].

We are now ready to discuss [5, Thm 2.1].

THEOREM 1.2. — *Let  $W_2(k)$  be the ring of truncated Witt vectors and let  $\tilde{S} = \text{Spec } W_2(k)$ . If  $X$  is smooth over  $S$ , then to every smooth lift  $\tilde{X}$  of  $X$  to  $\tilde{S}$ , there is an associated isomorphism*

$$\varphi : \bigoplus_{i < p} \Omega_{X'/S}^i[-i] \longrightarrow \tau_{<p} F_* \Omega_{X/S}^\bullet$$

in the derived category of  $\mathcal{O}_{X'}$ -modules such that  $\mathcal{H}^i(\varphi) = C^{-1}$  for all  $i < p$ .

We give a sketch of the argument. To define  $\varphi$ , we need only define  $\varphi^i : \Omega_{X'/S}^i[-i] \rightarrow \tau_{<p} F_* \Omega_{X/S}^\bullet$  such that  $\mathcal{H}^i(\varphi) = C^{-1}$  for each  $i < p$ . We take  $\varphi^0$  to be the composite

$$\mathcal{O}_{X'} \xrightarrow{C^{-1}} \mathcal{H}^0 F_* \Omega_{X/S}^\bullet \longrightarrow F_* \Omega_{X/S}^\bullet.$$

Suppose for the moment that  $\varphi^1$  has already been defined. For  $i > 1$ , we can then define  $\varphi^i$  to be the composite

$$\Omega_{X/S}^i[-i] \xrightarrow{a[-i]} (\Omega_{X/S}^1)^{\otimes i}[-i] \xrightarrow{(\varphi^1)^{\otimes i}} (F_* \Omega_{X/S}^\bullet)^{\otimes i} \xrightarrow{b} F_* \Omega_{X/S}^\bullet,$$

where

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = \frac{1}{i!} \sum_{\sigma \in S_i} (\text{sign } \sigma) \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}$$

and  $b(\omega_1 \otimes \cdots \otimes \omega_i) = \omega_1 \wedge \cdots \wedge \omega_i$ .

Thus, we are reduced to defining  $\varphi^1$ . Suppose first that Frobenius lifts; that is, there exists  $\tilde{F}$  filling in the diagram

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow F & & \downarrow \tilde{F} \\ X' & \longrightarrow & \tilde{X}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & \tilde{S} \end{array}$$

where  $\tilde{X}' = \tilde{X} \times_{\tilde{S}, \sigma} \tilde{S}$  and  $\sigma$  is the Witt vector Frobenius automorphism.

Let  $\mathbf{p} : \mathcal{O}_X \xrightarrow{\cong} p\mathcal{O}_{\tilde{X}}$  be the morphism sending  $x_0$  to  $px$  for any local section  $x$  of  $\mathcal{O}_{\tilde{X}}$  reducing mod  $p$  to  $x_0$ . Note that if  $x \otimes 1$  is a local section of  $\mathcal{O}_{\tilde{X}} \otimes_{W_2(k), \sigma} W_2(k) = \mathcal{O}_{\tilde{X}'}$ , then  $\tilde{F}^*(x \otimes 1) = x^p + \mathbf{p}(u(x))$  for a unique local section  $u(x)$  of  $\mathcal{O}_X$ . We define a morphism  $f : \Omega_{X'/S}^1 \rightarrow F_* \Omega_{X/S}^1$  by

$$f(dx_0 \otimes 1) = x_0^{p-1} dx_0 + du(x).$$

Deligne and Illusie show that  $\varphi^1$  can be taken to be  $f$ . Given two different choices  $\tilde{F}_1$  and  $\tilde{F}_2$  of  $F$ , we obtain a homotopy  $h_{12}$  relating  $f_1$  and  $f_2$ , defined by  $h_{12}(dx_0 \otimes 1) = u_2(x) - u_1(x)$ .

Note that  $F$  lifts locally since the obstruction to lifting it lies in

$$\text{Ext}^1(F^*\Omega_{X'/S}^1, \mathcal{O}_X) = H^1(X, F^*T_{X'/S}).$$

So, to define  $\varphi^1$  in general, we need only patch together the local choices. This is done as follows. Let  $\mathcal{U} = \{U_i\}$  be a cover on which Frobenius lifts and let  $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$  denote the sheafified version of the Čech complex of a sheaf  $\mathcal{F}$ . We define  $\varphi^1$  to be the morphism in the derived category

$$\Omega_{X'/S}^1[-1] \xrightarrow{\Phi} \text{Tot}(F_*\check{C}^\bullet(\mathcal{U}, \Omega_{X/S}^\bullet)) \xleftarrow{\cong} F_*\Omega_{X/S}^\bullet,$$

where

$$\Phi = (\Phi_1, \Phi_2) : \Omega_{X'/S}^1 \rightarrow F_*\check{C}^1(\mathcal{U}, \mathcal{O}_X) \oplus F_*\check{C}^0(\mathcal{U}, \Omega_{X/S}^\bullet)$$

is given by  $(\Phi_1(\omega))_{ij} = h_{ij}(\omega|_{U'_{ij}})$  and  $(\Phi_2(\omega))_i = f_i(\omega|_{U'_i})$ . Deligne and Illusie further show that this is independent of the choice of covering. This completes the proof.

*Remark 1.3.* — In the local case where Frobenius lifts,  $\varphi$  is a morphism of complexes. It is only when patching together the local choices that we need to pass to the derived category.

*Remark 1.4.* — Using the fact that for any étale morphism  $g : Y \rightarrow Z$  of  $S$ -schemes, the pullback of  $F_{Z/S} : Z \rightarrow Z'$  by  $g$  is  $F_{Y/S}$ , one can check that the proof of Theorem 1.2 works when  $X$  is a Deligne-Mumford stack. Alternatively, this will follow from the proof of Theorem 2.5 below.

Given any abelian category  $\mathcal{A}$  with enough injectives, a left exact functor  $G : \mathcal{A} \rightarrow \mathcal{B}$  to another abelian category, and a bounded below complex of objects  $A^\bullet$  of  $\mathcal{A}$ , we obtain a hypercohomology spectral sequence

$$E_1^{st} = R^tG(A^s) \Rightarrow R^nG(A^\bullet).$$

If  $\mathfrak{X}$  is a Deligne-Mumford stack over a scheme  $Y$ , the hypercohomology spectral sequence  $E_1^{st} = H^t(\Omega_{\mathfrak{X}/Y}^s) \Rightarrow H^n(\Omega_{\mathfrak{X}/Y}^\bullet)$  obtained in this way is called the Hodge-de Rham spectral sequence.

As Deligne and Illusie show, Theorem 1.2 implies the degeneracy of the Hodge-de Rham spectral sequence for smooth proper schemes. We reproduce their proof, which requires no modification to handle the case of Deligne-Mumford stacks, after first isolating the following useful fact from homological algebra.



LEMMA 1.5. — *Let  $K$  be a field and  $r_0$  a positive integer. Let  $E_{r_0}^{st} \Rightarrow E^{s+t}$  be a first quadrant spectral sequence whose terms are finite-dimensional  $K$ -vector spaces and whose morphisms are  $K$ -linear. If  $n$  is a non-negative integer and*

$$\sum_{s+t=n} \dim_K E_{r_0}^{st} = \dim_K E^n,$$

*then for all  $r \geq r_0$  the differentials to and from the  $E_r^{s,n-s}$  are zero. Hence, if the above equality holds for all  $n < N$ , then the spectral sequence degenerates for  $s + t < N$ .*

*Proof.* — Note that for all  $r \geq r_0$

$$\sum_{s+t=n} \dim_K E_{r+1}^{st} \leq \sum_{s+t=n} \dim_K E_r^{st}$$

with equality if and only if all of the differentials to and from the  $E_r^{s,n-s}$  are zero. Hence

$$\sum_{s+t=n} \dim_K E_\infty^{st} \leq \sum_{s+t=n} \dim_K E_{r_0}^{st}$$

with equality if and only if the differentials to and from the  $E_r^{s,n-s}$  are zero for all  $r \geq r_0$ . Since the  $E_\infty$  terms are  $K$ -vector spaces, the extension problem is trivial, and so

$$\dim_K E^n = \sum_{s+t=n} \dim_K E_\infty^{st} \leq \sum_{s+t=n} \dim_K E_{r_0}^{st} = \dim_K E^n,$$

which completes the proof. □

COROLLARY 1.6 ([5, Cor 2.5]). — *If  $\mathfrak{X}$  is a Deligne-Mumford stack over  $S$ , which is smooth, proper, and lifts mod  $p^2$ , then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\Omega_{\mathfrak{X}/S}^s) \Rightarrow H^n(\Omega_{\mathfrak{X}/S}^\bullet)$$

*degenerates for  $s + t < p$ .*

*Proof.* — By Theorem 1.2 and Remark 1.4, we have an isomorphism

$$\bigoplus_{s < p} \Omega_{\mathfrak{X}'/S}^s[-s] \longrightarrow \tau_{<p} F_* \Omega_{\mathfrak{X}/S}^\bullet$$

in the derived category of  $\mathcal{O}_{\mathfrak{X}'}$ -modules. It follows that for all  $n < p$ ,

$$\bigoplus_{s+t=n} H^t(\Omega_{\mathfrak{X}'/S}^s) = H^n(\Omega_{\mathfrak{X}/S}^\bullet).$$

Using the fact that  $H^t(\Omega_{\mathfrak{X}'/S}^s) = H^t(\Omega_{\mathfrak{X}/S}^s) \otimes_{k, F_k} k$ , we see

$$\sum_{s+t=n} \dim_k H^t(\Omega_{\mathfrak{X}/S}^s) = \sum_{s+t=n} \dim_k H^t(\Omega_{\mathfrak{X}'/S}^s) = \dim_k H^n(\Omega_{\mathfrak{X}/S}^\bullet),$$

which, by Lemma 1.5, proves the degeneracy of the spectral sequence.  $\square$

Deligne and Illusie further show that the degeneracy of the Hodge-de Rham spectral sequence in positive characteristic implies the degeneracy in characteristic 0. While its degeneration in characteristic 0 had previously been known by analytic means, this provided a purely algebraic proof.

**COROLLARY 1.7.** — *Let  $\mathfrak{X}$  be a Deligne-Mumford stack which is smooth and proper over a field  $K$  of characteristic 0. Then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\Omega_{\mathfrak{X}/K}^s) \Rightarrow H^n(\Omega_{\mathfrak{X}/K}^\bullet)$$

*degenerates.*

The proof given in [5, Cor 2.7] for schemes requires only a minor modification. It uses that if  $X$  is a smooth proper scheme over a field  $K$  of characteristic 0, then there is an integral domain  $A$  of finite type over  $\mathbb{Z}$ , a morphism  $A \rightarrow K$ , and a smooth proper scheme  $Y$  over  $\text{Spec } A$  which pulls back over  $\text{Spec } K$  to  $X$ . Since this statement remains true when we allow  $X$  and  $Y$  to be Deligne-Mumford stacks ([13, p.2]), the proof given in [5, Cor 2.7] implies Corollary 1.7 above.

### 1.2. de Rham Theory for Schemes with Quotient Singularities

Let  $k$  be a field of characteristic 0 and let  $S = \text{Spec } k$ .

**DEFINITION 1.8.** — *An  $S$ -scheme  $M$  (necessarily normal) is said to have quotient singularities if there is an étale cover  $\{U_i/G_i \rightarrow M\}$ , where the  $U_i$  are smooth over  $S$  and the  $G_i$  are finite groups.*

Our goal in this subsection is to use stacks to reprove [18, Thm 1.12] which states:

**THEOREM 1.9.** — *Let  $M$  be a proper  $S$ -scheme with quotient singularities, and let  $j : M^0 \rightarrow M$  be its smooth locus. Then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_*\Omega_{M^0/S}^s) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$$

*of the complex  $j_*\Omega_{M^0/S}^\bullet$  degenerates. Furthermore, if  $k = \mathbb{C}$ , then  $H^n(j_*\Omega_{M^0/S}^\bullet)$  agrees with the Betti cohomology  $H^n(M(\mathbb{C}), \mathbb{C})$  of  $M$ .*

The following proposition gives the relationship between Deligne-Mumford stacks and schemes with quotient singularities.

PROPOSITION 1.10. — *Let  $M$  be an  $S$ -scheme and let  $j : M^0 \rightarrow M$  be its smooth locus. Then  $M$  has quotient singularities if and only if it is the coarse space of a smooth Deligne-Mumford stack  $\mathfrak{X}$  such that  $f^0$  in the diagram*

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ f^0 \downarrow & & \downarrow f \\ M^0 & \xrightarrow{j} & M \end{array}$$

is an isomorphism, where  $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$ .

For a proof, see [7, Rmk 4.9] or [20, Prop 2.8]. Vistoli’s proposition is slightly more general than the proposition above.

We give the proof of Theorem 1.9 after first proving a lemma which compares  $j_*\Omega_{M^0/S}^\bullet$  to the de Rham complex of a Deligne-Mumford stack.

LEMMA 1.11. — *If  $M$  is an  $S$ -scheme with quotient singularities and  $\mathfrak{X}$  is as in Proposition 1.10, then*

$$j_*\Omega_{M^0/S}^\bullet = f_*\Omega_{\mathfrak{X}/S}^\bullet.$$

*Proof.* — To prove this equality, we need only show  $j_*^0\Omega_{\mathfrak{X}^0/S}^\bullet = \Omega_{\mathfrak{X}/S}^\bullet$ . That is, given an étale morphism  $U \rightarrow \mathfrak{X}$ , we want to show  $i_*\Omega_{U^0/S}^\bullet = \Omega_{U/S}^\bullet$ , where  $U^0 := M^0 \times_M U$  and  $i$  is the projection to  $U$ . Since  $\Omega_{U^0/S}^k$  is locally free, hence reflexive, the following lemma completes the proof.  $\square$

LEMMA 1.12. — *Let  $X$  be a normal scheme and  $i : U \hookrightarrow X$  an open subscheme whose complement has codimension at least 2. If  $\mathcal{F}$  is a reflexive sheaf on  $X$ , then the adjunction map  $\mathcal{F} \rightarrow i_*i^*\mathcal{F}$  is an isomorphism.*

*Proof.* — Since  $\mathcal{F}$  is reflexive,  $\mathcal{F} = \mathcal{H}om(\mathcal{G}, \mathcal{O}_X)$ , where  $\mathcal{G} = \mathcal{F}^\vee$ . Therefore,

$$i_*i^*\mathcal{F} = i_*\mathcal{H}om(i^*\mathcal{G}, \mathcal{O}_U) = \mathcal{H}om(\mathcal{G}, i_*\mathcal{O}_U)$$

and since  $X$  is normal,  $i_*\mathcal{O}_U = \mathcal{O}_X$ .  $\square$

*Proof of Theorem 1.9.* — Let  $\mathfrak{X}$  be as in Proposition 1.10. From Lemma 1.11, we see that  $j_*\Omega_{M^0/S}^\bullet = f_*\Omega_{\mathfrak{X}/S}^\bullet$  and  $j_*\Omega_{M^0/S}^s = f_*\Omega_{\mathfrak{X}/S}^s$  for all  $s$ . Since the  $\Omega_{\mathfrak{X}/S}^s$  are coherent, it follows from [2, Lemma 2.3.4] that

$$j_*\Omega_{M^0/S}^\bullet = Rf_*\Omega_{\mathfrak{X}/S}^\bullet \quad \text{and} \quad j_*\Omega_{M^0/S}^s = Rf_*\Omega_{\mathfrak{X}/S}^s.$$

We see then that

$$H^n(j_*\Omega_{M^0/S}^\bullet) = H^n(\Omega_{\mathfrak{X}/S}^\bullet) \quad \text{and} \quad H^t(j_*\Omega_{M^0/S}^s) = H^t(\Omega_{\mathfrak{X}/S}^s).$$

The proof of Proposition 1.10 given in [20, Prop 2.8] shows that  $f$  is proper, and so the Hodge-de Rham spectral sequence for  $\mathfrak{X}$  degenerates by Corollary 1.7. It follows that

$$\begin{aligned} \sum_{s+t=n} \dim_k H^t(j_*\Omega_{M^0/S}^s) &= \sum_{s+t=n} \dim_k H^t(\Omega_{\mathfrak{X}/S}^s) = \dim_k H^n(\Omega_{\mathfrak{X}/S}^\bullet) \\ &= \dim_k H^n(j_*\Omega_{M^0/S}^\bullet), \end{aligned}$$

which, by Lemma 1.5, proves the degeneracy of the hypercohomology spectral sequence for  $j_*\Omega_{M^0/S}^\bullet$ .

We now show that if  $k = \mathbb{C}$ , then  $H^n(j_*\Omega_{M^0/S}^\bullet) = H^n(M(\mathbb{C}), \mathbb{C})$ . We have shown  $H^n(j_*\Omega_{M^0/S}^\bullet) = H^n(\Omega_{\mathfrak{X}}^\bullet)$ , and GAGA for Deligne-Mumford stacks ([19, Thm 5.10]) shows

$$H^n(\Omega_{\mathfrak{X}}^\bullet) = H^n(\Omega_{\mathfrak{X}^{an}}^\bullet),$$

where  $\mathfrak{X}^{an}$  is defined in [19, Def 5.6]. Note that  $\mathbb{C} \rightarrow \Omega_{\mathfrak{X}^{an}}^\bullet$  is a quasi-isomorphism since this can be checked étale locally. It follows that

$$H^n(\Omega_{\mathfrak{X}^{an}}^\bullet) = H^n(\mathfrak{X}^{an}, \mathbb{C}).$$

Lastly, the singular cohomology of  $\mathfrak{X}^{an}$  and that of its coarse space,  $M(\mathbb{C})$ , are the same. This is shown in [3, Prop 36] for topological Deligne-Mumford stacks with  $\mathbb{Q}$ -coefficients, but the proof works equally well in our situation once it is combined with [19, Prop 5.7], which states  $[U^{an}/G] = [U/G]^{an}$ . □

We end this section with some remarks about the situation in positive characteristic. Suppose  $k$  is a perfect field of characteristic  $p$  and let  $S = \text{Spec } k$ .

**DEFINITION 1.13.** — *We say an  $S$ -scheme  $M$  (necessarily normal) has good quotient singularities if it has an étale cover  $\{U_i/G_i \rightarrow M\}$ , where the  $U_i$  are smooth over  $S$  and the  $G_i$  are finite groups of order prime to  $p$ .*

Both the proof in [7] and in [20] (along with Vistoli’s Remark 2.9) cited above work in positive characteristic. So, we have the following generalization of Proposition 1.10.

**PROPOSITION 1.14.** — *Let  $M$  be an  $S$ -scheme, and let  $j : M^0 \rightarrow M$  be its smooth locus. Then  $M$  has good quotient singularities if and only*

if it is the coarse space of a smooth tame Deligne-Mumford stack  $\mathfrak{X}$  ([2, Def. 2.3.1]) such that  $f^0$  in the diagram

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ f^0 \downarrow & & \downarrow f \\ M^0 & \xrightarrow{j} & M \end{array}$$

is an isomorphism, where  $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$ .

If  $\mathfrak{X}$  is a smooth proper tame Deligne-Mumford stack, then the Hodge-Rham spectral sequence for  $\mathfrak{X}$  degenerates by Corollary 1.6, and  $f_*\mathcal{F} = Rf_*\mathcal{F}$  for any quasi-coherent sheaf on  $\mathfrak{X}$  by [2, Lemma 2.3.4]. The proof of Theorem 1.9 therefore gives the following result as well.

**THEOREM 1.15.** — *Let  $M$  be a proper  $S$ -scheme with good quotient singularities, and let  $j : M^0 \rightarrow M$  be its smooth locus. If  $\mathfrak{X}$ , as in Proposition 1.14, lifts mod  $p^2$ , then*

$$E_1^{st} = H^t(j_*\Omega_{M^0/S}^s) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$$

degenerates for  $s + t < p$ .

As will follow from Theorem 4.9 below, if  $M$  has dimension at least 4, lifts mod  $p^2$ , and has isolated singularities, then  $\mathfrak{X}$  automatically lifts mod  $p^2$ .

## 2. Deligne-Illusie for Simplicial Schemes

Let  $k$  be a perfect field of characteristic  $p$  and let  $S = \text{Spec } k$ . In this section, we prove a Deligne-Illusie result at the simplicial level. To do so, we must first make sense of the Cartier isomorphism for simplicial schemes.

**LEMMA 2.1.** — *Let  $X$  and  $Y$  be smooth schemes over  $S$  and let  $\rho : X \rightarrow Y$  be a morphism of  $S$ -schemes. If  $C^{-1}$  denotes the Cartier isomorphism, then the following diagram commutes*

$$\begin{array}{ccc} \rho'^*\Omega_{X'/S}^i & \xrightarrow{\rho'^*C^{-1}} & \rho'^*\mathcal{H}^i(F_*\Omega_{X/S}^\bullet) \\ \downarrow & & \downarrow \\ \Omega_{Y'/S}^i & \xrightarrow{C^{-1}} & \mathcal{H}^i(F_*\Omega_{Y/S}^\bullet) \end{array}$$

*Proof.* — Using the canonical morphism  $\rho'^* \mathcal{H}^i(F_* \Omega_{X/S}^\bullet) \rightarrow \mathcal{H}^i(\rho'^* F_* \Omega_{X/S}^\bullet)$  and the multiplicativity property of the Cartier isomorphism, we need only check that the diagram commutes for  $i = 0, 1$ . For  $i = 0$ , the Cartier isomorphism is simply the kernel map, so the  $i = 0$  case follows from the commutativity of

$$\begin{CD} \rho'^* F_* \mathcal{O}_X @>\rho'^* d>> \rho'^* F_* \Omega_{Y/S}^1 \\ @VVV @VVV \\ F_* \mathcal{O}_Y @>d>> F_* \Omega_{X/S}^1 \end{CD}$$

To handle the  $i = 1$  case, let  $f$  be a local section of  $\mathcal{O}_X$  and note that

$$\begin{CD} df @>>> f^{p-1} df \\ @VVV @VVV \\ d(\rho(f)) @>>> \rho(f)^{p-1} d(\rho(f)) \end{CD}$$

□

**COROLLARY 2.2.** — *Let  $\mathfrak{X}$  be a smooth Artin stack over  $S$  and let  $X_0 \rightarrow \mathfrak{X}$  be a smooth cover by a scheme. If  $X_\bullet$  is the simplicial scheme obtained by taking fiber products of  $X_0$  over  $\mathfrak{X}$ , and  $X'_\bullet$  is its pullback by  $F_S$ , then there exists a unique isomorphism*

$$C^{-1} : \Omega_{X'_\bullet/S}^i \rightarrow \mathcal{H}^i(F_* \Omega_{X_\bullet/S}^\bullet)$$

such that  $C^{-1}(1) = 1$ ,  $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$ , and  $C^{-1}(df)$  is the class of  $f^{p-1} df$ .

*Proof.* — If such a  $C^{-1}$  exists, then its restriction to the  $n^{th}$  level of the simplicial scheme is the Cartier isomorphism for  $X_n$ . Therefore, we need only show existence, which follows from Lemma 2.1. □

We have now proved the Cartier isomorphism for simplicial schemes. The other main ingredient in extending Deligne-Illusie to simplicial schemes  $X_\bullet$ , is showing that relative Frobenius for  $X_\bullet$  lifts locally. We note that there are, in fact, simplicial schemes for which relative Frobenius does not lift.

**Example 2.3.** — Let  $X_\bullet$  be obtained by taking fiber products of  $S$  over  $B\mathbb{G}_a$ . Lifting Frobenius for  $X_\bullet$  is then equivalent to lifting Frobenius  $F$  of  $\mathbb{G}_a$  to a morphism  $\tilde{F}$  of group schemes

$$\text{Spec } W_2(k)[x] = \mathbb{G}_{a,S} \rightarrow \mathbb{G}_{a,S} = \text{Spec } W_2(k)[x].$$

Since  $\tilde{F}$  reduces to  $F$ , we must have  $\tilde{F}(x) = x^p + pf(x)$  for some  $f(x) \in W_2(k)[x]$ . The condition that  $\tilde{F}$  be a group scheme homomorphism implies

$$(x + y)^p + pf(x + y) = x^p + y^p + p(f(x) + f(y)),$$

and an easy check shows that this is not possible.

Although the above example shows that relative Frobenius need not lift locally for an arbitrary simplicial scheme, we show that relative Frobenius does lift locally for those simplicial schemes which come from smooth tame stacks. This is the key technical point of this section.

PROPOSITION 2.4. — *Let  $\mathfrak{X}$  be a smooth tame stack over  $S$  with coarse space  $M$ . Then étale locally on  $M$ , both  $\mathfrak{X}$  and the relative Frobenius  $F_{\mathfrak{X}/S}$  lift mod  $p^2$ .*

*Proof.* — Since the statement of the proposition is étale local, by [17, Prop 5.2] we can assume that  $M$  is affine and  $\mathfrak{X} = [U/G]$ , where  $G = \mathbb{G}_{m,S}^r \rtimes H$  and  $H$  is a finite étale constant group scheme. Note that  $U$  is affine and that the smoothness of  $G$  and  $\mathfrak{X}$  imply that  $U$  is smooth over  $S$ .

As a first step in showing that  $\mathfrak{X}$  and  $F_{\mathfrak{X}/S}$  lift mod  $p^2$ , we begin by showing that  $BG$  and its relative Frobenius lift. Since the underlying scheme of  $G$  is  $\mathbb{G}_{m,S}^r \times_S H$  and its group structure is determined by the action

$$H \rightarrow \text{Aut}(\mathbb{G}_m^r) = \text{Aut}(\mathbb{Z}^r),$$

we can use this same action to define a group scheme  $\tilde{G} = \mathbb{G}_{m,\tilde{S}}^r \rtimes H$  which lifts  $G$ . It follows that  $B\tilde{G}$  is a lift of  $BG$ . Lifting the relative Frobenius of  $BG$  is the same as lifting the relative Frobenius  $F_{G/S} : \mathbb{G}_{m,S}^r \times H \rightarrow \mathbb{G}_{m,S}^r \times H$  to a group scheme homomorphism. Note that  $F_{G/S}$  is given by the identity on  $H$  and component-wise multiplication by  $p$  on  $\mathbb{G}_{m,S}^r$ . It therefore has a natural lift mod  $p^2$  to the group scheme homomorphism given by the identity on  $H$  and component-wise multiplication by  $p$  on  $\mathbb{G}_{m,\tilde{S}}^r$ .

We now prove that  $\mathfrak{X}$  and  $F_{\mathfrak{X}/S}$  lift. There is a natural map  $\pi : \mathfrak{X} \rightarrow BG$  which makes

$$\begin{array}{ccc} U & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\pi} & BG \end{array}$$

a cartesian diagram. To lift  $\mathfrak{X} \bmod p^2$ , it suffices to show that there a stack  $\tilde{\mathfrak{X}}$  and a cartesian diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \tilde{\mathfrak{X}} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ BG & \longrightarrow & B\tilde{G} \end{array}$$

The obstruction to the existence of such a diagram lies in  $\text{Ext}^2(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}})$ ; here  $L_{\mathfrak{X}/BG}$  denotes the cotangent complex. Since  $\pi$  is representable and smooth,  $L_{\mathfrak{X}/BG}$  is a locally free sheaf. It follows that

$$R\mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}),$$

which is a quasi-coherent sheaf. Since  $\pi$  is affine and  $G$  is linearly reductive, for any quasi-coherent sheaf  $\mathcal{F}$  on  $\mathfrak{X}$ , we have

$$R\Gamma(\mathfrak{X}, \mathcal{F}) = R\Gamma(BG, R\pi_*\mathcal{F}) = \Gamma(BG, \pi_*\mathcal{F}).$$

In particular,

$$R\mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = \Gamma(\mathfrak{X}, \mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}))$$

and so  $\text{Ext}^2(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = 0$ .

To show that  $F_{\mathfrak{X}/S}$  lifts mod  $p^2$ , it suffices to show that it lifts over our choice  $\tilde{F}_{BG/S}$ . That is, it suffices to show that there exists a dotted arrow making the diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \tilde{\mathfrak{X}} \\ F_{\mathfrak{X}/S} \downarrow & & \downarrow \text{dotted} \\ \mathfrak{X}' & \longrightarrow & \tilde{\mathfrak{X}}' \\ \pi' \downarrow & & \downarrow \tilde{\pi}' \\ BG' & \longrightarrow & B\tilde{G}' \end{array} \quad \begin{array}{l} \curvearrowright \\ \tilde{F}_{BG/S} \circ \tilde{\pi} \end{array}$$

commute. The obstruction to finding such a dotted arrow lies in  $\text{Ext}^1(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}})$ . As before, we have

$$R\mathcal{H}om(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}}) = \mathcal{H}om(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}}),$$

which is again a quasi-coherent sheaf. An argument similar to the one above then shows  $\text{Ext}^1(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}}) = 0$ , thereby completing the proof.  $\square$

We now prove Deligne-Illusie for simplicial schemes coming from smooth tame stacks.



**THEOREM 2.5.** — *Let  $\mathfrak{X}$  be a smooth tame stack over  $S$ . Let  $X_0 \rightarrow \mathfrak{X}$  be a smooth cover by a scheme and let  $X_\bullet$  be the simplicial scheme obtained by taking fiber products of  $X_0$  over  $\mathfrak{X}$ . Then, to every lift  $\tilde{X}_0 \rightarrow \tilde{\mathfrak{X}}$  of  $X_0 \rightarrow \mathfrak{X}$ , there is a canonically associated isomorphism*

$$\varphi : \bigoplus_{i < p} \Omega_{X'_i/S}^i[-i] \rightarrow \tau_{< p} F_* \Omega_{X_\bullet/S}^\bullet$$

in the derived category of  $\mathcal{O}_{X'_i}$ -modules such that  $\mathcal{H}^i(\varphi) = C^{-1}$  for  $i < p$ .

*Proof.* — To prove this theorem we simply check that all of the morphisms in the proof of Deligne-Illusie extend to morphisms on the simplicial level (see Section 1.1 for an outline of Deligne-Illusie and relevant notation).

Let  $\rho : X_n \rightarrow X_m$  be a face or a degeneracy map of  $X_\bullet$ . To ease notation, we denote  $X_n$  by  $Y$  and  $X_m$  by  $X$ . In addition, we use  $F$  to denote all relative Frobenii.

To show that  $\varphi^0$  extends to a morphism  $\mathcal{O}_{X'_i} \rightarrow F_* \Omega_{X_\bullet/S}^\bullet$ , we show

$$\begin{array}{ccccc} \rho'^* \mathcal{O}_{X'} & \xrightarrow{\rho'^* C^{-1}} & \rho'^* \mathcal{H}^0 F_* \Omega_{X/S}^\bullet & \longrightarrow & \rho'^* F_* \Omega_{X/S}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{Y'} & \xrightarrow{C^{-1}} & \mathcal{H}^0 F_* \Omega_{Y/S}^\bullet & \longrightarrow & F_* \Omega_{Y/S}^\bullet \end{array}$$

commutes. The left square commutes by Lemma 2.1. The right square commutes since for any morphism  $A^\bullet \rightarrow B^\bullet$  of complexes concentrated in non-negative degrees, the following diagram commutes

$$\begin{array}{ccc} \ker d_A^0 & \longrightarrow & A^0 \\ \downarrow & & \downarrow \\ \ker d_B^0 & \longrightarrow & B^0 \end{array}$$

To show that  $\varphi^i$  extends to a morphism on the simplicial level for  $i > 0$ , we must check

$$\begin{array}{ccccccc} \rho'^* \Omega_{X'_i/S}^i[-i] & \xrightarrow{\rho'^* a^{[-i]}} & (\Omega_{X'_i/S}^1)^{\otimes i}[-i] & \xrightarrow{\rho'^* (\varphi^1)^{\otimes i}} & (F_* \Omega_{X/S}^\bullet)^{\otimes i} & \xrightarrow{\rho'^* b} & F_* \Omega_{X/S}^\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega_{Y'_i/S}^i[-i] & \xrightarrow{a^{[-i]}} & (\Omega_{Y'_i/S}^1)^{\otimes i}[-i] & \xrightarrow{(\varphi^1)^{\otimes i}} & (F_* \Omega_{Y/S}^\bullet)^{\otimes i} & \xrightarrow{b} & F_* \Omega_{Y/S}^\bullet \end{array}$$

commutes. It is clear that the outermost squares commute, and so we are reduced to checking the commutativity of

$$\begin{CD} \rho'^* \Omega_{X'/S}^1[-1] @>{\rho'^* \varphi^1}>> \rho'^* F_* \Omega_{X/S}^\bullet \\ @VVV @VVV \\ \Omega_{Y'/S}^1[-1] @>{\varphi^1}>> F_* \Omega_{Y/S}^\bullet \end{CD}$$

Suppose now that Frobenius (for the simplicial scheme) lifts. So, we have a commutative square

$$\begin{CD} \tilde{Y} @>{\tilde{F}}>> \tilde{Y}' \\ @V{\tilde{\rho}}VV @VV{\tilde{\rho}'}V \\ \tilde{X} @>{\tilde{F}}>> \tilde{X}' \end{CD}$$

of  $\tilde{S}$ -schemes which pulls back to

$$\begin{CD} Y @>{F}>> Y' \\ @V{\rho}VV @VV{\rho'}V \\ X @>{F}>> X' \end{CD}$$

over  $S$ . In this case  $\varphi^1 = f$ , and to check that it defines a morphism  $\Omega_{X'_\bullet/S}^1[-1] \rightarrow F_* \Omega_{X_\bullet/S}^1$ , we need to check that

$$\begin{CD} \rho'^* \Omega_{X'/S}^1[-1] @>{\rho'^* f}>> \rho'^* F_* \Omega_{X/S}^1 \\ @VVV @VVV \\ \Omega_{Y'/S}^1[-1] @>{f}>> F_* \Omega_{Y/S}^1 \end{CD}$$

commutes. Under these morphisms,

$$\begin{CD} dx_0 \otimes 1 @>>> x_0^{p-1} dx_0 + du(x) \\ @VVV @VVV \\ d\rho(x_0) \otimes 1 @>>> \rho(x_0)^{p-1} d\rho(x_0) + d\rho(u(x)) \end{CD}$$

We see  $d\rho(x_0) \otimes 1$  is sent to  $\rho(x_0)^{p-1} d\rho(x_0) + d\rho(u(x))$  since

$$\begin{aligned} \tilde{F}^*(\tilde{\rho}(x) \otimes 1) &= \tilde{F} \tilde{\rho}'(x \otimes 1) = \tilde{\rho} \tilde{F}^*(x \otimes 1) = \tilde{\rho}(x^p + \mathbf{p}(u(x))) \\ &= \tilde{\rho}(x)^p + \mathbf{p}(u(\tilde{\rho}(x))). \end{aligned}$$

Given two different choices  $\tilde{F}_1$  and  $\tilde{F}_2$  of  $F$ , we obtain a homotopy  $h_{12}$  relating  $f_1$  and  $f_2$ . It is clear that  $h_{12}$  extends to a morphism on the simplicial level since  $h_{12}(dx_0 \otimes 1) = u_2(x) - u_1(x)$  and  $\tilde{\rho}(\mathbf{p}(u_i(x))) = \mathbf{p}(u_i(\tilde{\rho}(x)))$ .

We now need to handle the general case when Frobenius does not lift. We begin by proving that Frobenius lifts étale locally on  $X_\bullet$ . To do so, we can, by Proposition 2.4, assume that there is a lift  $\tilde{F}$  of  $F_{\mathfrak{X}/S}$ . Let  $\mathcal{U}_0 = \{U_i\}$  be a Zariski cover of  $X_0$  where Frobenius lifts and let  $\tilde{F}_i$  be a lift of  $F_{U_i/S}$ . Then  $\mathcal{U}_n := \{U_{i_1} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} U_{i_n}\}$  is a Zariski cover of  $X_n$  and  $\tilde{F}_{i_1} \times_{\tilde{F}} \cdots \times_{\tilde{F}} \tilde{F}_{i_n}$  is a lift of Frobenius on  $U_{i_1} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} U_{i_n}$ . Moreover, these lifts of Frobenius are compatible so we see that Frobenius for the simplicial scheme does lift étale locally.

To finish the proof of the theorem, we need only prove the commutativity of

$$\begin{array}{ccccc}
 \rho'^* F_* \Omega_{X/S}^\bullet & \xrightarrow{\cong} & \text{Tot}(\rho'^* F_* \check{C}^\bullet(\mathcal{U}_m, \Omega_{X/S}^\bullet)) & \xleftarrow{\rho'^* \Phi} & \rho'^* \Omega_{X'/S}^1[-1] \\
 \downarrow & & \downarrow & & \downarrow \\
 F_* \Omega_{Y/S}^\bullet & \xrightarrow{\cong} & \text{Tot}(F_* \check{C}^\bullet(\mathcal{U}_n, \Omega_{Y/S}^\bullet)) & \xleftarrow{\Phi} & \Omega_{Y'/S}^1[-1]
 \end{array}$$

The right square commutes because the  $\Phi$  are defined in terms of the  $f$ 's and  $h$ 's. The middle vertical map is induced by the morphism of the respective double complexes given by

$$(\omega_{1,s} \wedge \cdots \wedge \omega_{a,s})_{s \in S_m} \mapsto (\rho_s^{st}(\omega_{1,s} \wedge \cdots \wedge \omega_{a,s}))_{s \in S_m, t \in S_n}$$

where  $\rho_s^{st} : U_s \times_{\mathfrak{X}} U_t \rightarrow U_s$  and  $S_k$  is the symmetric group. So, under the morphisms in the left square,

$$\begin{array}{ccc}
 \omega_1 \wedge \cdots \omega_a & \longmapsto & (\omega_1|_{U_s} \wedge \cdots \wedge \omega_a|_{U_s})_s \\
 \downarrow & & \\
 \rho(\omega_1) \wedge \cdots \rho(\omega_a) & \longmapsto & (\rho(\omega_1)|_{U_s \times U_t} \wedge \cdots \wedge \rho(\omega_a)|_{U_s \times U_t})_{s,t}
 \end{array}$$

Under the middle vertical map,  $(\omega_1|_{U_s} \wedge \cdots \wedge \omega_a|_{U_s})_s$  is sent to  $(\rho_s^{st}(\omega_1|_{U_s}) \wedge \cdots \wedge \rho_s^{st}(\omega_a|_{U_s}))_{s,t}$ . But

$$\begin{array}{ccc}
 U_s \times_{\mathfrak{X}} U_t & \longrightarrow & Y \\
 \rho_s^{st} \downarrow & & \downarrow \rho \\
 U_s & \longrightarrow & X
 \end{array}$$

commutes, so this completes the proof. □

### 3. de Rham Theory for Tame Stacks

Let  $S$  be a scheme and  $\mathfrak{X} \rightarrow \mathcal{Y}$  a morphism of Artin stacks over  $S$ . We denote by  $\varpi_{\mathfrak{X}/\mathcal{Y}}^1$  the sheaf of  $\mathcal{O}_{\mathfrak{X}}$ -modules on the lisse-étale site of  $\mathfrak{X}$  such that  $\varpi_{\mathfrak{X}/\mathcal{Y}}^1|_{U_{et}} = \Omega_{U/\mathcal{Y}}^1$  for all  $U$  smooth over  $\mathfrak{X}$ . We define  $\varpi_{\mathfrak{X}/\mathcal{Y}}^s$  to be  $\bigwedge^s \varpi_{\mathfrak{X}/\mathcal{Y}}^1$ . Given a morphism  $f : V \rightarrow U$  of smooth  $\mathfrak{X}$ -schemes, note that the transition function

$$f^* \Omega_{U/\mathcal{Y}}^1 \longrightarrow \Omega_{V/\mathcal{Y}}^1$$

need not be an isomorphism, and so the  $\varpi_{\mathfrak{X}/\mathcal{Y}}^s$  are never coherent. Note also that  $\varpi_{\mathfrak{X}/\mathfrak{X}}^1$  is not the zero sheaf.

As mentioned in the introduction, the sheaf  $\varpi_{\mathfrak{X}/S}^1$  gives us a naive de Rham complex  $\varpi_{\mathfrak{X}/S}^\bullet$ . In this section we prove that when  $S$  is spectrum of a perfect field of characteristic  $p$ , the hypercohomology spectral sequence

$$E_1^{st} = H^t(\varpi_{\mathfrak{X}/S}^s) \Rightarrow H^n(\varpi_{\mathfrak{X}/S}^\bullet)$$

degenerates for smooth proper tame stacks  $\mathfrak{X}$  that lift mod  $p^2$ . The reason the proof of Corollary 1.6 and the Deligne-Illusie result proved in the last section do not immediately imply the degeneracy of this spectral sequence is that, as mentioned above, the  $\varpi_{\mathfrak{X}/S}^s$  are not coherent, and so we do not yet know that the  $H^t(\varpi_{\mathfrak{X}/S}^s)$  and  $H^n(\varpi_{\mathfrak{X}/S}^\bullet)$  are finite-dimensional  $k$ -vector spaces. The main goal of this section, which implies the degeneracy of the above spectral sequence, is to prove that they are by comparing them with the cohomology of the cotangent complex.

We begin by proving three general lemmas and a corollary which require no assumptions on the base scheme  $S$ . The first two lemmas are concerned with relative cohomological descent. For background material on cohomological descent, we refer the reader to [16, §2] and [4, §6].

In what follows, given a smooth hypercover  $a : X_\bullet \rightarrow \mathfrak{X}$  of an Artin stack by a simplicial algebraic space,  $\mathfrak{X}_{lis-et}|_{X_s}$  denotes the topos of sheaves over the representable sheaf defined by  $X_s$  and  $\mathfrak{X}_{lis-et}|_{X_\bullet}$  denotes the associated simplicial topos.

LEMMA 3.1. — *Let  $\mathfrak{X}$  be an Artin stack over  $S$  and let  $a : X_\bullet \rightarrow \mathfrak{X}$  be a smooth hypercover by a simplicial algebraic space. If  $f : \mathfrak{X} \rightarrow M$  is a morphism to a scheme, then for any  $\mathcal{F}_\bullet \in Ab(\mathfrak{X}_{lis-et}|_{X_\bullet})$ , there is a spectral sequence*

$$E_1^{st} = R^t(fa_s)_*(\mathcal{F}_s|_{X_{s,et}}) \Rightarrow \epsilon_* R^n(f_* a_*) \mathcal{F}_\bullet,$$

where  $\epsilon : M_{lis-et} \rightarrow M_{et}$  is the canonical morphism of topoi. If  $\mathcal{F}_\bullet = a^* \mathcal{F}$  for some  $\mathcal{F} \in Ab(\mathfrak{X}_{lis-et})$ , then  $\epsilon_* R^n(f_* a_*) \mathcal{F}_\bullet = \epsilon_* R^n f_* \mathcal{F}$ .

*Proof.* — Let  $\eta_s : \mathfrak{X}_{lis-et}|X_s \rightarrow X_{s,et}$  be the canonical morphism of topoi and note that

$$\begin{array}{ccccccc}
 Ab(X_{s,et}) & & & & & & \\
 \uparrow \eta_{s*} & \searrow (fa_s)_* & & & & & \\
 Ab(\mathfrak{X}_{lis-et}|X_s) & \xrightarrow{a_{s*}} & Ab(\mathfrak{X}_{lis-et}) & \xrightarrow{f_*} & Ab(M_{lis-et}) & \xrightarrow{\epsilon_*} & Ab(M_{et})
 \end{array}$$

commutes. By general principles (see proof of [4, Thm 6.11]), there is a spectral sequence

$$E_1^{st} = R^t(\epsilon_* f_* a_{s*})(\mathcal{F}_s) \Rightarrow R^n(\epsilon_* f_* a_*)\mathcal{F}_\bullet.$$

As  $\epsilon_*$  is exact,  $R^n(\epsilon_* f_* a_*)\mathcal{F}_\bullet = \epsilon_* R^n(f_* a_*)\mathcal{F}_\bullet$ . Since  $\eta_{s*}$  is exact and takes injectives to injectives, the commutativity of the above diagram implies that  $E_1^{st} \simeq R^t(fa_s)_*(\mathcal{F}_s|X_{s,et})$ , which shows the existence of our desired spectral sequence. Lastly, since

$$a^* : Ab(\mathfrak{X}_{lis-et}) \rightarrow Ab(\mathfrak{X}_{lis-et}|X_\bullet)$$

is fully faithful, it follows ([4, Lemma 6.8]) that  $Ra_* a^* = id$ . As a result,  $\epsilon_* R(f_* a_*)a^* \mathcal{F} = \epsilon_* Rf_* \mathcal{F}$ . □

LEMMA 3.2. — *With notation and hypotheses as in Lemma 3.1, we have*

$$R^n(fa)_*(\eta_* \mathcal{F}_\bullet) = \epsilon_* R^n(f_* a_*)\mathcal{F}_\bullet,$$

where  $\eta : \mathfrak{X}_{lis-et}|X_\bullet \rightarrow X_{\bullet,et}$  is the canonical morphism of topoi.

*Proof.* — We see that the diagram

$$\begin{array}{ccccccc}
 Ab(X_{\bullet,et}) & & & & & & \\
 \uparrow \eta_* & \searrow (fa)_* & & & & & \\
 Ab(\mathfrak{X}_{lis-et}|X_\bullet) & \xrightarrow{a_*} & Ab(\mathfrak{X}_{lis-et}) & \xrightarrow{f_*} & Ab(M_{lis-et}) & \xrightarrow{\epsilon_*} & Ab(M_{et})
 \end{array}$$

commutes. It follows that

$$R(fa)_*(\eta_* \mathcal{F}_\bullet) = R(fa)_*(R\eta_* \mathcal{F}_\bullet) = \epsilon_* R(f_* a_*)\mathcal{F}_\bullet,$$

as  $\epsilon_*$  and  $\eta_*$  are exact and take injectives to injectives. □

Using Lemma 3.1, we prove a base change result for sheaves on an Artin stack which are not necessarily quasi-coherent, but are level-by-level quasi-coherent on a smooth hypercover of the stack.

COROLLARY 3.3. — Let  $f : \mathfrak{X} \rightarrow M$  be a morphism from an Artin stack to a scheme and let  $a : X_\bullet \rightarrow \mathfrak{X}$  be a smooth hypercover by a simplicial algebraic space. Let  $h : T \rightarrow M$  be an étale morphism and consider the diagram

$$\begin{array}{ccc}
 Y_\bullet & \xrightarrow{j} & X_\bullet \\
 \downarrow b & & \downarrow a \\
 \mathcal{Y} & \xrightarrow{i} & \mathfrak{X} \\
 \downarrow g & & \downarrow f \\
 T & \xrightarrow{h} & M
 \end{array}$$

where all squares are cartesian. If  $\mathcal{F}$  is an  $\mathcal{O}_{\mathfrak{X}}$ -module such that each  $\mathcal{F}|_{X_{s,et}}$  is quasi-coherent, then the canonical map

$$h^* \epsilon_* R^n f_* \mathcal{F} \rightarrow \alpha_* R^n g_* i^* \mathcal{F}$$

is an isomorphism, where  $\epsilon$  and  $\alpha$  denote the canonical morphisms of topoi  $M_{lis-et} \rightarrow M_{et}$  and  $T_{lis-et} \rightarrow T_{et}$ , respectively.

*Proof.* — By Lemma 3.1, we have a spectral sequence

$$E_1^{st} = R^t (f a_s)_* (\mathcal{F}|_{X_{s,et}}) \Rightarrow \epsilon_* R^n f_* \mathcal{F}.$$

Applying  $h^*$ , we obtain another spectral sequence

$$\backslash E_1^{st} = h^* R^t (f a_s)_* (\mathcal{F}|_{X_{s,et}}) \Rightarrow h^* \epsilon_* R^n f_* \mathcal{F}.$$

Flat base change shows

$$h^* R^t (f a_s)_* (\mathcal{F}|_{X_{s,et}}) = R^t (g b_s)_* j_s^* (\mathcal{F}|_{X_{s,et}}) = R^t (g b_s)_* (i^* \mathcal{F})|_{Y_{s,et}}.$$

Another application of Lemma 3.1 then shows that  $\backslash E$  in fact abuts to  $\alpha_* R^n g_* i^* \mathcal{F}$ . □

Before stating the next lemma, we introduce the following definitions. Let  $Z$  be an  $S$ -scheme equipped with an action  $\rho : G \times_S Z \rightarrow Z$  of a smooth linearly reductive group scheme  $G$  over  $S$  and let  $p : G \times Z \rightarrow Z$  be the projection. We denote by  $(G - \text{lin } \mathcal{O}_{Z_{et}} - \text{mod})$  the category of  $G$ -linearized  $\mathcal{O}_{Z_{et}}$ -modules. That is, the category of quasi-coherent  $\mathcal{O}_{Z_{et}}$ -modules  $\mathcal{F}$  together with an isomorphism  $\phi : p^* \mathcal{F} \rightarrow \rho^* \mathcal{F}$  satisfying a cocycle condition. From such a  $\phi$  we can define a “coaction map”

$$\sigma : \mathcal{F} \rightarrow p_* p^* \mathcal{F} \xrightarrow{p_*(\phi)} p_* \rho^* \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_{Z,\rho}} \mathcal{O}_{Z \times_S G}$$

which satisfies an associativity relation as in [14, p.31]; here  $\mathcal{F} \rightarrow p_* p^* \mathcal{F}$  is the canonical map. Letting  $f : Z \rightarrow Z/G$  be the natural map, we define

the  $G$ -invariants  $\mathcal{F}^G$  of  $\mathcal{F}$  to be the equalizer of

$$f_*\sigma : f_*\mathcal{F} \longrightarrow f_*p_*\rho^*\mathcal{F} = f_*p_*p^*\mathcal{F}$$

and  $f_*$  of the canonical map  $s \mapsto s \otimes 1$ .

If  $Y$  is also an  $S$ -scheme equipped with a  $G$ -action and  $h : Z \rightarrow Y$  is a  $G$ -equivariant map over  $S$ , then for every  $G$ -linearized  $\mathcal{O}_{Z_{et}}$ -module  $\mathcal{F}$ , there is a natural  $G$ -linearization on  $h_*\mathcal{F}$ . So, we have a commutative diagram of categories

$$\begin{array}{ccc} (G\text{-lin } \mathcal{O}_{Z_{et}}\text{-mod}) & \longrightarrow & (\mathcal{O}_{Z_{et}}\text{-mod}) \\ h_* \downarrow & & \downarrow h_* \\ (G\text{-lin } \mathcal{O}_{Y_{et}}\text{-mod}) & \longrightarrow & (\mathcal{O}_{Y_{et}}\text{-mod}) \end{array}$$

where the horizontal arrows are the obvious forgetful functors. If  $g : Z/G \rightarrow Y/G$  denotes the map induced by  $h$ , then it is not hard to see that  $(h_*\mathcal{F})^G = g_*\mathcal{F}^G$ . In particular,  $\mathcal{F}^G = (f_*\mathcal{F})^G$  where  $G$  acts trivially on  $Z/G$ . Note that for any sheaf  $\mathcal{G}$  of  $\mathcal{O}_{Z/G}$ -modules,  $f^*\mathcal{G}$  comes equipped with a canonical  $G$ -linearization. If the  $G$ -action on  $Z$  is free, so that  $f$  is a  $G$ -torsor, then  $(f^*\mathcal{G})^G = (f_*f^*\mathcal{G})^G = \mathcal{G}$ .

By descent theory,  $(G\text{-lin } \mathcal{O}_{Z_{et}}\text{-mod})$  is equivalent to the category of quasi-coherent sheaves on  $[Z/G]$ . Under this equivalence, taking  $G$ -invariants in the above sense corresponds to pushing forward to the coarse space  $Z/G$ .

When the action of  $G$  on  $Z$  is trivial, we denote  $(G\text{-lin } \mathcal{O}_{Z_{et}}\text{-mod})$  by  $(G\text{-}\mathcal{O}_{Z_{et}}\text{-mod})$ . We can similarly define the categories  $(G\text{-lin } \mathcal{O}_{Z_{\bullet,et}}\text{-mod})$  and  $(G\text{-}\mathcal{O}_{Z_{\bullet,et}}\text{-mod})$  for simplicial schemes  $Z_{\bullet}$ .

LEMMA 3.4. — *Let  $U$  be a smooth  $S$ -scheme with an action of a smooth affine linearly reductive group scheme  $G$  over  $S$ . Let  $\mathfrak{X} = [U/G]$  and  $a : X_{\bullet} \rightarrow \mathfrak{X}$  be the hypercover obtained by taking fiber products of  $U$  over  $\mathfrak{X}$ . Consider the diagram*

$$\begin{array}{ccc} Y_{\bullet} & \xrightarrow{\pi} & X_{\bullet} \\ b \downarrow & & \downarrow a \\ U & \xrightarrow{a_0} & \mathfrak{X} \\ & \searrow g & \downarrow f \\ & & M \end{array}$$

where the square is cartesian and  $M$  is a scheme. Then

$$R^n(fa)_*\mathcal{F}_{\bullet} = (R^n(gb)_*\pi^*\mathcal{F}_{\bullet})^G$$

for all  $\mathcal{O}_{X_{\bullet,et}}$ -modules  $\mathcal{F}_{\bullet}$  such that the  $\mathcal{F}_s$  are quasi-coherent.

*Proof.* — Note that the following diagram

$$\begin{array}{ccc}
 (G\text{-lin } \mathcal{O}_{Y_{\bullet,et}}\text{-mod}) & \xrightarrow{(gb)_*} & (G\text{-}\mathcal{O}_{M_{et}}\text{-mod}) \\
 \pi_* \downarrow & & \downarrow (-)^G \\
 (G\text{-}\mathcal{O}_{X_{\bullet,et}}\text{-mod}) & \xrightarrow{(-)^G} (\mathcal{O}_{X_{\bullet,et}}\text{-mod}) \xrightarrow{(fa)_*} & (\mathcal{O}_{M_{et}}\text{-mod})
 \end{array}$$

of categories commutes. As a result,

$$R(fa)_*R(-)^G(R\pi_*\pi^*\mathcal{F}_{\bullet}) = R(-)^G(R(gb)_*\pi^*\mathcal{F}_{\bullet}) = (R(gb)_*\pi^*\mathcal{F}_{\bullet})^G,$$

where the second equality holds because  $R(gb)_*\pi^*\mathcal{F}_{\bullet}$  has quasi-coherent cohomology. It suffices then to prove

$$\mathcal{F}_{\bullet} = R(-)^G(R\pi_*\pi^*\mathcal{F}_{\bullet}).$$

We begin by showing  $R\pi_*\pi^*\mathcal{F}_{\bullet} = \pi_*\pi^*\mathcal{F}_{\bullet}$ . Let

$$0 \rightarrow \pi^*\mathcal{F}_{\bullet} \rightarrow \mathcal{I}_{\bullet}^0 \rightarrow \mathcal{I}_{\bullet}^1 \rightarrow \dots$$

be an injective resolution of  $\mathcal{O}_{X_{\bullet,et}}$ -modules. To show  $R^n\pi_*\pi^*\mathcal{F}_{\bullet} = 0$  for  $n > 0$ , we need only do so after restricting to each level  $X_s$ . Since the restriction functor  $\text{res}_s : Ab(X_{\bullet,et}) \rightarrow Ab(X_{s,et})$  is exact, we see

$$\text{res}_s R^n\pi_*\pi^*\mathcal{F}_{\bullet} = \text{res}_s \mathcal{H}^n(\pi_*\mathcal{I}_{\bullet}^{\bullet}) = \mathcal{H}^n(\pi_*\mathcal{I}_s^{\bullet}) = R^n\pi_*\pi^*\mathcal{F}_s = 0,$$

where the last equality holds because  $\pi$  is affine and  $\mathcal{F}_s$  is quasi-coherent.

A similar argument shows  $R(-)^G(\pi_*\pi^*\mathcal{F}_{\bullet}) = (\pi_*\pi^*\mathcal{F}_{\bullet})^G$  as every  $\pi_*\pi^*\mathcal{F}_s$  is quasi-coherent. The lemma then follows from the fact that  $\pi$  is a  $G$ -torsor, and so  $(\pi_*\pi^*\mathcal{F}_{\bullet})^G = \mathcal{F}_{\bullet}$ . □

For the rest of the section, we let  $S = \text{Spec } k$ , where  $k$  is a perfect field of characteristic  $p$ . We remind the reader that if  $\mathfrak{X}$  is a smooth Artin stack and  $X_{\bullet} \rightarrow \mathfrak{X}$  is a hypercover, then the cotangent complex  $L_{\mathfrak{X}/S}$  of the stack ([16, §8]) is the bounded complex of  $\mathcal{O}_{\mathfrak{X}}$ -modules with quasi-coherent cohomology such that

$$L_{\mathfrak{X}/S}|_{X_{\bullet,et}} = \Omega_{X_{\bullet}/S}^1 \rightarrow \Omega_{X_{\bullet}/\mathfrak{X}}^1$$

with  $\Omega_{X_{\bullet}/S}^1$  in degree 0; that is,

$$L_{\mathfrak{X}/S} = \varpi_{\mathfrak{X}/S}^1 \rightarrow \varpi_{\mathfrak{X}/\mathfrak{X}}^1.$$

In Theorem 3.5 below, we compare  $\varpi_{\mathfrak{X}/S}^s$  with  $\bigwedge^s L_{\mathfrak{X}/S}$ , the  $s^{\text{th}}$  derived exterior power of  $L_{\mathfrak{X}/S}$ . Given an abelian category  $\mathcal{A}$ , the derived exterior powers  $L\bigwedge^s$ , as well as the derived symmetric powers  $LS^s$ , of a complex  $E \in D^-(\mathcal{A})$  are defined in [9, I.4.2.2.6]. Since  $L_{\mathfrak{X}/S}$  is not concentrated in



negative degrees, we cannot directly define  $\bigwedge^s L_{\mathfrak{X}/S}$ ; however, it is shown in [9, I.4.3.2.1] that for  $E \in D^-(\mathcal{A})$ ,

$$LS^s(E[1]) = \left( L \bigwedge^s E \right)[s]$$

so we may define  $\bigwedge^s L_{\mathfrak{X}/S}$  as  $LS^s(L_{\mathfrak{X}/S}[1])[-s]$ . It follows, then, from [9, I.4.3.1.7] that

$$\bigwedge^s L_{\mathfrak{X}/S} = \varpi_{\mathfrak{X}/S}^s \rightarrow \varpi_{\mathfrak{X}/S}^{s-1} \otimes \varpi_{\mathfrak{X}/\mathfrak{X}}^1 \rightarrow \cdots \rightarrow \varpi_{\mathfrak{X}/S}^1 \otimes S^{s-1} \varpi_{\mathfrak{X}/\mathfrak{X}}^1 \rightarrow S^s \varpi_{\mathfrak{X}/\mathfrak{X}}^1$$

with  $\varpi_{\mathfrak{X}/S}^s$  in degree 0. Note that we have a canonical map from  $\bigwedge^s L_{\mathfrak{X}/S}$  to  $\varpi_{\mathfrak{X}/S}^s$ .

We remark that  $\bigwedge^s L_{\mathfrak{X}/S} \in D_{coh}^b(\mathfrak{X})$  for all  $s$ . This can be seen as follows. We have an exact triangle

$$a_0^* L_{\mathfrak{X}/S} \longrightarrow L_{X_0/S} \longrightarrow \Omega_{X/\mathfrak{X}}^1$$

By [9, II.2.3.7],  $L_{X_0/S}$  has coherent cohomology. Since  $\Omega_{X/\mathfrak{X}}^1$  is coherent and coherence can be checked smooth locally, we see  $L_{X/S}$  and hence all  $\bigwedge^s L_{\mathfrak{X}/S}$  are in  $D_{coh}^b(\mathfrak{X})$ .

We are now ready to prove the comparison theorem.

**THEOREM 3.5.** — *If  $\mathfrak{X}$  is smooth and tame over  $S$  and  $f : \mathfrak{X} \rightarrow M$  is its coarse space, then the canonical map*

$$\epsilon_* R^t f_* \left( \bigwedge^s L_{\mathfrak{X}/S} \right) \longrightarrow \epsilon_* R^t f_* \varpi_{\mathfrak{X}/S}^s$$

is an isomorphism.

*Proof.* — By [17, Prop 5.2], there exists an étale cover  $h : T \rightarrow M$  and a cartesian diagram

$$\begin{array}{ccc} [U/G] & \longrightarrow & \mathfrak{X} \\ g \downarrow & & \downarrow f \\ T & \xrightarrow{h} & M \end{array}$$

where  $G$  is linearly reductive, affine, and smooth over  $S$ . Since  $\mathfrak{X}$  and  $G$  are smooth, we see that  $U$  is as well. Let  $\mathcal{Y} = [U/G]$  and let  $\varphi : \epsilon_* R^t f_* (\bigwedge^s L_{\mathfrak{X}/S}) \rightarrow \epsilon_* R^t f_* \varpi_{\mathfrak{X}/S}^s$  be the canonical map. By Corollary 3.3, we see that  $h^* \varphi$  is the canonical map

$$\epsilon_* R^t g_* \left( \bigwedge^s L_{\mathcal{Y}/S} \right) \longrightarrow \epsilon_* R^t g_* \varpi_{\mathcal{Y}/S}^s$$

To show that  $\varphi$  is an isomorphism, we can therefore assume  $\mathfrak{X} = [U/G]$  and  $M = T$ .

To prove the theorem, it suffices to show  $\epsilon_* R^t f_*(\varpi_{\mathfrak{X}/S}^{s-k} \otimes S^k \varpi_{\mathfrak{X}/\mathfrak{X}}^1) = 0$  for all  $k > 0$  and all  $t$ . With notation as in Lemma 3.4, we see

$$\begin{aligned} \epsilon_* R^t f_*(\varpi_{\mathfrak{X}/S}^{s-k} \otimes S^k \varpi_{\mathfrak{X}/\mathfrak{X}}^1) &= \epsilon_* R^t (f_* a_*) a^*(\varpi_{\mathfrak{X}/S}^{s-k} \otimes S^k \varpi_{\mathfrak{X}/\mathfrak{X}}^1) \\ &= R^t (fa)_*(\Omega_{X_\bullet/S}^{s-k} \otimes S^k \Omega_{X_\bullet/\mathfrak{X}}^1), \end{aligned}$$

where the first equality is by Lemma 3.1 and the second is by Lemma 3.2. It now follows from Lemma 3.4 that

$$R^t (fa)_*(\Omega_{X_\bullet/S}^{s-k} \otimes S^k \Omega_{X_\bullet/\mathfrak{X}}^1) = (R^t (gb)_*(\pi^* \Omega_{X_\bullet/S}^{s-k} \otimes S^k \Omega_{Y_\bullet/U}^1))^G.$$

Fix  $t$  and  $k > 0$ . It suffices then to prove by (strong) induction on  $s$  that for every flat  $\mathcal{O}_{X_\bullet}$ -module  $\mathcal{G}$  which is restriction to  $Y_{\bullet,et}$  of some  $\mathcal{O}$ -module  $\mathcal{F}$  on the lisse-étale site of  $U$ ,

$$R^n (gb)_*(\pi^* \Omega_{X_\bullet/S}^s \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = 0.$$

We begin with the case  $s = 0$ , which is handled separately. An application of Lemmas 3.1 and 3.2 shows

$$R^n (gb)_*(\mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = \epsilon_* R^n g_*(\mathcal{F} \otimes S^k \varpi_{U/U}^1).$$

If we let  $\alpha : U_{lis-et} \rightarrow U_{et}$  be the canonical morphism of topoi, we see then that

$$\epsilon_* R^n g_*(\mathcal{F} \otimes S^k \varpi_{U/U}^1) = R^n g_*(\alpha_* \mathcal{F} \otimes S^k \Omega_{U/U}^1) = 0,$$

where the last equality holds since  $k > 0$ .

Assume now that  $s > 0$ . Since  $\pi$  is smooth, we have a short exact sequence

$$0 \rightarrow \pi^* \Omega_{X_\bullet/S}^1 \rightarrow \Omega_{Y_\bullet/S}^1 \rightarrow \Omega_{Y_\bullet/X_\bullet}^1 \rightarrow 0.$$

As a result, we have a filtration  $\Omega_{Y_\bullet/S}^s \supset \mathcal{K}^1 \supset \dots \supset \mathcal{K}^s \supset 0$  with  $\mathcal{K}^s = \pi^* \Omega_{X_\bullet/S}^s$  and short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{K}^1 \rightarrow \Omega_{Y_\bullet/S}^s \rightarrow \Omega_{Y_\bullet/X_\bullet}^s \rightarrow 0 \\ 0 &\rightarrow \mathcal{K}^2 \rightarrow \mathcal{K}^1 \rightarrow \pi^* \Omega_{X_\bullet/S}^1 \otimes \Omega_{Y_\bullet/X_\bullet}^{s-1} \rightarrow 0 \\ &\vdots \\ 0 &\rightarrow \pi^* \Omega_{X_\bullet/S}^s \rightarrow \mathcal{K}^{s-1} \rightarrow \pi^* \Omega_{X_\bullet/S}^{s-1} \otimes \Omega_{Y_\bullet/X_\bullet}^1 \rightarrow 0. \end{aligned}$$

Since  $\mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1$  is flat, tensoring each of the above short exact sequences by it results in a new list of short exact sequences. Since

$$\Omega_{Y_\bullet/S}^1 \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1 = (\varpi_{U/S}^1 \otimes \mathcal{F} \otimes S^k \varpi_{U/U}^1)|_{Y_{\bullet,et}}$$

and

$$\Omega_{Y_\bullet/X_\bullet}^1 \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1 = (L_{U/\mathfrak{X}} \otimes \mathcal{F} \otimes S^k \varpi_{U/U}^1)|_{Y_{\bullet,et}},$$

the  $s = 0$  case shows

$$R^n(gb)_*(\Omega_{Y_\bullet/S}^1 \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = R^n(gb)_*(\Omega_{Y_\bullet/X_\bullet}^1 \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = 0.$$

As a result,  $R^n(gb)_*(\mathcal{K}^1 \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = 0$ . Using the inductive hypothesis, we conclude

$$R^n(gb)_*(\mathcal{K}^i \otimes \mathcal{G} \otimes S^k \Omega_{Y_\bullet/U}^1) = 0$$

for all  $i$ , in particular for  $i = s$ . □

**COROLLARY 3.6.** — *If  $\mathfrak{X}$  is a smooth proper tame stack over  $S$ , then  $H^t(\varpi_{\mathfrak{X}/S}^s)$  and  $H^n(\varpi_{\mathfrak{X}/S}^\bullet)$  are finite-dimensional  $k$ -vector spaces for all  $s$ ,  $t$ , and  $n$ .*

*Proof.* — Let  $f : \mathfrak{X} \rightarrow M$  be the coarse space of  $\mathfrak{X}$ . For each  $s$ , there is a Leray spectral sequence

$$E_2^{ij} = H^i(\epsilon_* R^j f_* \varpi_{\mathfrak{X}/S}^s) \Rightarrow H^t(\varpi_{\mathfrak{X}/S}^s).$$

By Theorem 3.5, the canonical map

$$\epsilon_* R^j f_* \left( \bigwedge^s L_{\mathfrak{X}/S} \right) \longrightarrow \epsilon_* R^j f_* \varpi_{\mathfrak{X}/S}^s$$

is an isomorphism. As we remarked above,  $\bigwedge^s L_{\mathfrak{X}/S} \in D_{coh}^b(\mathfrak{X})$ . Since  $f$  is proper by Keel-Mori [11], and  $M$  is proper by [15, Prop 2.10], we see the  $E_2^{ij}$  are finite-dimensional  $k$ -vector spaces. It follows that  $H^t(\varpi_{\mathfrak{X}/S}^s)$  is a finite-dimensional  $k$ -vector space for every  $s$  and  $t$ .

Since the morphisms in the complex  $\varpi_{\mathfrak{X}/S}^\bullet$  are  $k$ -linear, the hypercohomology spectral sequence

$$E_1^{st} = H^t(\varpi_{\mathfrak{X}/S}^s) \Rightarrow H^n(\varpi_{\mathfrak{X}/S}^\bullet)$$

consists of finite-dimensional  $k$ -vector spaces with  $k$ -linear maps. As a result,  $H^n(\varpi_{\mathfrak{X}/S}^\bullet)$  is a finite-dimensional  $k$ -vector space as well. □

**THEOREM 3.7.** — *Let  $\mathfrak{X}$  be a smooth proper tame stack over  $S$  that lifts mod  $p^2$ . Then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\varpi_{\mathfrak{X}/S}^s) \Rightarrow H^n(\varpi_{\mathfrak{X}/S}^\bullet)$$

*degenerates for  $s + t < p$ .*

*Proof.* — Choose a smooth cover  $X \rightarrow \mathfrak{X}$  by a smooth scheme such that the cover lifts mod  $p^2$ . Theorem 2.5 now shows

$$\bigoplus_{s < p} \Omega_{X'/S}^s[-s] \simeq \tau_{< p} F_* \Omega_{X'/S}^\bullet,$$

where  $X_\bullet$  is obtained from  $X$  by taking fiber products over  $\mathfrak{X}$ . Since  $H^t(\varpi_{\mathfrak{X}/S}^s) = H^t(\Omega_{X_\bullet/S}^s)$  and  $H^n(\varpi_{\mathfrak{X}/S}^\bullet) = H^n(\Omega_{X_\bullet/S}^\bullet)$ , we see that for  $n < p$ ,

$$\begin{aligned} \dim_k H^n(\varpi_{\mathfrak{X}/S}^\bullet) &= \sum_{s+t=n} \dim_k H^t(\Omega_{X'_\bullet/S}^s) = \sum_{s+t=n} \dim_k H^t(\Omega_{X_\bullet/S}^s) \\ &= \sum_{s+t=n} \dim_k H^t(\varpi_{\mathfrak{X}/S}^s), \end{aligned}$$

which proves the degeneracy of the spectral sequence by Lemma 1.5.  $\square$

### 4. de Rham Theory for Schemes with Isolated Linearly Reductive Singularities

Let  $k$  be a perfect field of characteristic  $p$  and let  $S = \text{Spec } k$ .

DEFINITION 4.1. — We say a scheme  $M$  over  $S$  has linearly reductive singularities if there is an étale cover  $\{U_i/G_i \rightarrow M\}$ , where the  $U_i$  are smooth over  $S$  and the  $G_i$  are linearly reductive group schemes which are finite over  $S$ .

Note that if  $M$  has linearly reductive singularities, then it is automatically normal and in fact Cohen-Macaulay by [8, p.115].

Our goal in this section is to prove that if  $M$  is proper over  $S$ , and  $j : M^0 \rightarrow M$  is its smooth locus, then under suitable liftability conditions, the hypercohomology spectral sequence  $E_1^{st} = H^t(j_*\Omega_{M^0/S}^s) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$  degenerates.

#### 4.1. Relationship with Tame Stacks, and the Cartier Isomorphism

We begin by recalling the relationship between tame stacks and schemes with linearly reductive singularities:

THEOREM 4.2 ([17, Thm 1.10]). — Let  $M$  be an  $S$ -scheme with linearly reductive singularities. Then it is the coarse space of a smooth tame stack  $\mathfrak{X}$  such that  $f^0$  in the diagram

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ f^0 \downarrow & & \downarrow f \\ M^0 & \xrightarrow{j} & M \end{array}$$

is an isomorphism, where  $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$ .

Let  $M$  and  $\mathfrak{X}$  be as in Theorem 4.2. The proof of Lemma 1.11 goes through word for word (after replacing “an étale morphism  $U \rightarrow \mathfrak{X}$ ” by “a smooth morphism  $U \rightarrow \mathfrak{X}$ ”) to show

$$j_*\Omega_{M^0/S}^\bullet = \epsilon_*f_*\varpi_{\mathfrak{X}/S}^\bullet,$$

where  $\epsilon : M_{\text{lis-ét}} \rightarrow M_{\text{ét}}$  is the canonical morphism of topoi.

*Remark 4.3.* — Since  $\epsilon_*f_*\varpi_{\mathfrak{X}/S}^s = \epsilon_*f_*\mathcal{H}^0(\bigwedge^s L_{\mathfrak{X}/S})$ , the above equality shows that  $j_*\Omega_{M^0/S}^s$  is coherent, which is not a priori obvious.

To simplify notation, throughout the rest of this subsection we suppress  $\epsilon$ .

PROPOSITION 4.4 (Cartier isomorphism). — *Let  $\mathfrak{X}$  be a smooth tame stack over  $S$  which lifts mod  $p^2$ , and let  $f : \mathfrak{X} \rightarrow M$  be its coarse space. Then there is a canonical isomorphism*

$$\mathcal{H}^t(f'_*F_*\varpi_{\mathfrak{X}/S}^\bullet) \xrightarrow{\cong} f'_*\varpi_{\mathfrak{X}'/S}^t.$$

If we further assume that  $\mathfrak{X}$  and  $M$  are as in Theorem 4.2, then

$$\mathcal{H}^t(F_*j_*\Omega_{M^0/S}^\bullet) \xrightarrow{\cong} j'_*\Omega_{M^0/S}^t.$$

*Proof.* — For any left exact functor  $G : \mathcal{A} \rightarrow \mathcal{B}$  of abelian categories and any complex  $A^\bullet$  of objects of  $\mathcal{A}$ , there is a canonical morphism  $\mathcal{H}^t(GA^\bullet) \rightarrow G\mathcal{H}^t(A^\bullet)$ : the map  $GA^\bullet \rightarrow RGA^\bullet$  induces a morphism from  $\mathcal{H}^t(GA^\bullet)$  to the  $E_2^{0t}$ -term of the spectral sequence  $E_2^{st} = R^sG\mathcal{H}^t(A^\bullet) \Rightarrow R^nG(A^\bullet)$ .

For us this yields the (global) map  $\phi : \mathcal{H}^t(f'_*F_*\varpi_{\mathfrak{X}/S}^\bullet) \rightarrow f'_*\mathcal{H}^t(F_*\varpi_{\mathfrak{X}/S}^\bullet) = f'_*\varpi_{\mathfrak{X}'/S}^t$ . To prove this is an isomorphism, we need only do so locally. So, by [17, Prop 5.2] and Proposition 2.4, we are reduced to the case  $\mathfrak{X} = [U/G]$ , where  $U$  is smooth and affine,  $G = \mathbb{G}_{m,S}^r \rtimes H$  for some finite étale constant group scheme  $H$ , and both  $\mathfrak{X}$  and the relative Frobenius lift mod  $p^2$ . Let  $X_\bullet$  be the simplicial scheme obtained by taking fiber products of  $U$  over  $\mathfrak{X}$ , and let  $a : X_\bullet \rightarrow \mathfrak{X}$  be the augmentation map. Since  $U \rightarrow \mathfrak{X}$  lifts mod  $p^2$ , Theorem 2.5 yields a quasi-isomorphism

$$\varphi : \bigoplus_{t < p} \Omega_{X'_\bullet/S}^t[-t] \xrightarrow{\cong} \tau_{< p}F_*\Omega_{X_\bullet/S}^\bullet.$$

In this local setting,  $\varphi$  is a *morphism of complexes* by Remark 1.3. We can therefore apply  $(f'a)_*$ . Subsequently taking cohomology, we have a morphism  $f'_*\varpi_{\mathfrak{X}'/S}^t \xrightarrow{f'_*\varphi^t} \mathcal{H}^t(f'_*F_*\varpi_{\mathfrak{X}/S}^\bullet)$ . We show that

$$\psi : f'_*\mathcal{H}^t(F_*\varpi_{\mathfrak{X}/S}^\bullet) \xrightarrow{(f'_*C^{-1})^{-1}} f'_*\varpi_{\mathfrak{X}'/S}^t \xrightarrow{f'_*\varphi^t} \mathcal{H}^t(f'_*F_*\varpi_{\mathfrak{X}/S}^\bullet)$$

and  $\phi$  are inverses. Note that in this local setting  $f'_*$  is simply “take  $G$ -invariants”, and that  $\phi : \mathcal{H}^t((F_*\Omega_U^\bullet)^G) \rightarrow \mathcal{H}^t(F_*\Omega_U^\bullet)^G$  is  $[\alpha] \mapsto (\alpha)$ , where we use square, resp. round brackets to denote classes in  $\mathcal{H}^t((F_*\Omega_U^\bullet)^G)$ , resp.  $\mathcal{H}^t(F_*\Omega_U^\bullet)^G$ .

In general, one does not expect the map  $(\alpha) \mapsto [\alpha]$  to be well-defined, but we show here that this is precisely what  $\psi$  is. Let  $(\omega) \in \mathcal{H}^t(F_*\Omega_U^\bullet)^G$ . Via the Cartier isomorphism  $\Omega_{U'}^t \xrightarrow{(C^{-1})^G} \mathcal{H}^t(F_*\Omega_U^\bullet)^G$ , we know that  $(\omega)$  is of the form

$$\left( \sum f_{i_1, \dots, i_t} x_{i_1}^{p-1} \dots x_{i_t}^{p-1} dx_{i_1} \wedge \dots \wedge dx_{i_t} \right),$$

where

$$\sum f_{i_1, \dots, i_t} (dx_{i_1} \otimes 1) \wedge \dots \wedge (dx_{i_t} \otimes 1) \in (\Omega_{U'}^t)^G.$$

The Deligne-Illusie map  $\varphi^q$  sends this  $G$ -invariant form to

$$\eta = \sum f_{i_1, \dots, i_t} (x_{i_1}^{p-1} dx_{i_1} + du(x_{i_1})) \wedge \dots \wedge (x_{i_t}^{p-1} dx_{i_t} + du(x_{i_t})),$$

where  $u(x)$  is the reduction mod  $p$  of any  $y$  satisfying  $\tilde{F}^*(d\tilde{x} \otimes 1) = \tilde{x}^p d\tilde{x} + py$ . So,  $\psi$  sends  $(\omega)$  to  $(\eta)$ . But since  $(du(x)) = 0$ , we see that  $\psi$  is the map sending  $(\alpha)$  to  $[\alpha]$ . □

### 4.2. Degeneracy of Various Spectral Sequences and a Vanishing Theorem

Let  $\mathfrak{X}$  and  $M$  be as in Theorem 4.2. Our immediate goal is to show the degeneracy of the hypercohomology spectral sequence for  $j_*\Omega_{M^0/S}^\bullet$  when  $\mathfrak{X}$  is proper and lifts mod  $p^2$ . If  $\varpi_{\mathfrak{X}/S}^1$  were coherent, then since  $\mathfrak{X}$  is tame, we would have  $j_*\Omega_{M^0/S}^\bullet = \epsilon_* f_* \varpi_{\mathfrak{X}/S}^\bullet = \epsilon_* Rf_* \varpi_{\mathfrak{X}/S}^\bullet$ . The proof of Theorem 1.9 would then apply directly to show the degeneracy of  $E_1^{st} = H^t(j_*\Omega_{M^0/S}^s) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$ . Since  $\varpi_{\mathfrak{X}/S}^1$  is not coherent, we must take a different approach. As we explain below, the Cartier isomorphism for  $j_*\Omega_{M^0/S}^\bullet$  proved in the last subsection implies that the degeneracy of the above hypercohomology spectral sequence is equivalent to the degeneracy of the conjugate spectral sequence  $E_2^{st} = H^s(\mathcal{H}^t(j_*\Omega_{M^0/S}^\bullet)) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$ . We show that this latter spectral sequence degenerates by comparing it to the spectral sequence  $E_2^{st} = H^s(R^t f_* \varpi_{\mathfrak{X}/S}^\bullet) \Rightarrow H^n(\varpi_{\mathfrak{X}/S}^\bullet)$  over which we have more control due to the Deligne-Illusie result of Section 2.

As in the last subsection, we suppress  $\epsilon : M_{et} \rightarrow M_{lis-et}$ . The following is the key technical lemma we use to prove the degeneracy of the hypercohomology spectral sequence for  $j_*\Omega_{M^0/S}^\bullet$ .

LEMMA 4.5. — *Let  $E$  and  $\backslash E$  be two first quadrant  $E_2$  spectral sequences. Suppose that for  $s \neq 0$ , every differential  $\backslash E_r^{st} \rightarrow \backslash E_r^{s+r,t-(r-1)}$  is zero. Suppose further that we are given a morphism  $E \rightarrow \backslash E$  of spectral sequences such that the induced morphism  $E_r^{st} \rightarrow \backslash E_r^{s+r,t-(r-1)}$  is zero for all  $r, s$ , and  $t$ , and such that  $E_2^{st} \rightarrow \backslash E_2^{st}$  is an injection for all  $s$  and  $t$ . Then  $E$  degenerates.*

*Proof.* — We claim that the morphism  $E_r^{st} \rightarrow \backslash E_r^{st}$  is an injection for  $s \geq r$ . Note that this is enough to prove the lemma since for all  $s$ , the square

$$\begin{array}{ccc} E_r^{st} & \longrightarrow & \backslash E_r^{st} \\ d_r^{st} \downarrow & & \downarrow \\ E_r^{s+r,t-(r-1)} & \longrightarrow & \backslash E_r^{s+r,t-(r-1)} \end{array}$$

commutes, the composite is zero, and  $E_r^{s+r,t-(r-1)} \rightarrow \backslash E_r^{s+r,t-(r-1)}$  is an injection; this shows that all differentials  $d_r^{st}$  are zero.

We now prove the claim by induction. It is true for  $r = 2$ , so we may assume  $r > 2$ . Let  $s \geq r$  and consider the commutative diagram

$$\begin{array}{ccccc} E_{r-1}^{s-(r-1),t+(r-2)} & \longrightarrow & E_{r-1}^{st} & \longrightarrow & E_{r-1}^{s+r-1,t-(r-2)} \\ \downarrow & & \downarrow & & \downarrow \\ \backslash E_{r-1}^{s-(r-1),t+(r-2)} & \longrightarrow & \backslash E_{r-1}^{st} & \longrightarrow & \backslash E_{r-1}^{s+r-1,t-(r-2)}. \end{array}$$

By the inductive hypothesis, all vertical arrows are injective and all arrows on the bottom row are zero. It follows that all arrows on the top row are zero, and so  $E_r^{st} \rightarrow \backslash E_r^{st}$  is injective. □

THEOREM 4.6. — *Let  $\mathfrak{X}$  and  $M$  be as in Theorem 4.2. If  $M$  has isolated singularities, and  $\mathfrak{X}$  is proper and lifts mod  $p^2$ , then the conjugate spectral sequence*

$$E_2^{st} = H^s(\mathcal{H}^t(j_*\Omega_{M^0/S}^\bullet)) \Rightarrow H^n(j_*\Omega_{M^0/S}^\bullet)$$

*degenerates for  $s + t < p$ .*

*Proof.* — Choose a smooth cover  $X \rightarrow \mathfrak{X}$  by a smooth scheme such that the cover lifts mod  $p^2$ , and let  $X_\bullet$  be the simplicial scheme obtained by taking fiber products of  $X$  with itself over  $\mathfrak{X}$ . Let  $a : X_\bullet \rightarrow \mathfrak{X}$  be the augmentation map. By Theorem 2.5, we have an isomorphism  $\bigoplus_{i < p} \Omega_{X_\bullet/S}^i[-i] \xrightarrow{\sim} \tau_{< p} F_* \Omega_{X_\bullet/S}^\bullet$  in the derived category, and therefore, also an isomorphism

$$\bigoplus_{i < p} R(f'_* a_*) \Omega_{X_\bullet/S}^i[-i] \xrightarrow{\sim} \tau_{< p} R(f'_* a_*) F_* \Omega_{X_\bullet/S}^\bullet.$$

The first of these isomorphisms implies that the Leray spectral sequence

$${}^{\vee}E_2^{st} = R^s f'_* \mathcal{H}^t(F_* \varpi_{\mathfrak{X}/S}^\bullet) \Rightarrow R^n f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$$

degenerates and that the extension problem is trivial. The second of the two isomorphisms shows that the spectral sequence

$${}^{\vee}E_2^{st} = H^s(R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \Rightarrow H^n(\varpi_{\mathfrak{X}/S}^\bullet)$$

decomposes as the direct sum  $\bigoplus_i {}^{\vee}E$  of Leray spectral sequences, where

$${}^{\vee}E_2^{st} = H^s(R^{t-i} f'_* \varpi_{\mathfrak{X}'/S}^i) \Rightarrow H^{n-i}(\varpi_{\mathfrak{X}'/S}^i).$$

Note that the morphism  $f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \rightarrow Rf'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$  induces a morphism of spectral sequences  $E \rightarrow {}^{\vee}E$ , where

$$E_2^{st} = H^s(\mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet)) \Rightarrow H^n(f_* \varpi_{\mathfrak{X}/S}^\bullet).$$

By the degeneracy of  ${}^{\vee}E$ , the morphism  $\varphi : \mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \rightarrow R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$  factors as

$$\mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \rightarrow f'_* F_* \varpi_{\mathfrak{X}/S}^t \hookrightarrow R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet.$$

But this first morphism is precisely how the Cartier isomorphism of Proposition 4.4 was defined. From this and the fact that the extension problem for  ${}^{\vee}E$  is trivial, we have a split short exact sequence

$$0 \rightarrow \mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \xrightarrow{\varphi} R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \rightarrow \bigoplus_{\substack{i+j=t \\ j>0}} R^j f'_* \varpi_{\mathfrak{X}'/S}^i \rightarrow 0.$$

It follows that  $E_2^{st}$  is mapped isomorphically to the direct summand  ${}^{\vee}E_2^{st}$  of  ${}^{\vee}E_2^{st}$ . This implies that for all  $r, s$ , and  $t$ , the induced morphism  $E_r^{st} \rightarrow {}^{\vee}E_r^{s+r, t-(r-1)}$  is zero.

Note that

$$j'^* R^t f'_* \varpi_{\mathfrak{X}'/S}^i = j'^* R^t f'_* \bigwedge^i L_{\mathfrak{X}'/S} = (f^0)'_* \mathcal{H}^t(\bigwedge^i L_{(\mathfrak{X}^0)'/S}) = 0.$$

It follows that  $R^t f'_* \varpi_{\mathfrak{X}'/S}^i$  is supported at the singular locus of  $M'$ , and since  $M$  is assumed to have isolated singularities,  $H^s(R^t f'_* \varpi_{\mathfrak{X}/S}^\bullet) = 0$  for  $s$  and  $t$  positive. We see then that  ${}^{\vee}E_2^{st}$  is zero if  $t > i$  and  $s > 0$ , or if  $t < i$ . Therefore, the differential  ${}^{\vee}E_r^{st} \rightarrow {}^{\vee}E_r^{s+r, t-(r-1)}$  is zero if  $s \neq 0$ . From Lemma 4.5, it follows that  $E$  degenerates.  $\square$

*Remark 4.7.* — Let  $\mathcal{E}$  be a locally free sheaf on  $M'$ . Tensoring the isomorphism

$$\bigoplus_{i < p} R(f'_* a_*) \Omega_{X'/S}^i[-i] \xrightarrow{\cong} \tau_{< p} R(f'_* a_*) F_* \Omega_{X/S}^\bullet$$



with  $\mathcal{E}$ , we see that the Leray spectral sequence

$$\backslash E_2^{st} = H^s(R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{E}) \Rightarrow H^n(Rf'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{E})$$

decomposes as the direct sum of spectral sequences. The proof of Theorem 4.6 then shows that the spectral sequence

$$E_2^{st} = H^s(\mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \otimes \mathcal{E}) \Rightarrow H^n(Rf'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{E}).$$

degenerates for  $s + t < p$ .

**THEOREM 4.8.** — *Let  $\mathfrak{X}$  and  $M$  be as in Theorem 4.2. If  $M$  has isolated singularities, and  $\mathfrak{X}$  is proper and lifts mod  $p^2$ , then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_* \Omega_{M^0/S}^s) \Rightarrow H^n(j_* \Omega_{M^0/S}^\bullet)$$

degenerates for  $s + t < p$ .

*Proof.* — By the Cartier isomorphism

$$H^s(\mathcal{H}^t(j_* \Omega_{M^0/S}^\bullet)) = H^s(\mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet)) = H^s(f'_* \varpi_{\mathfrak{X}/S}^t).$$

But  $H^s(f'_* \varpi_{\mathfrak{X}/S}^t) = H^s(f_* \varpi_{\mathfrak{X}/S}^t) \otimes_{k, F_k} k$ ; in particular,

$$\dim_k H^s(\mathcal{H}^t(j_* \Omega_{M^0/S}^\bullet)) = \dim_k H^s(f_* \varpi_{\mathfrak{X}/S}^t).$$

By Corollary 3.7, the above cohomology groups are finite-dimensional  $k$ -vector spaces. The degeneracy of the conjugate spectral sequence shows

$$H^n(j_* \Omega_{M^0/S}^\bullet) \simeq \bigoplus_{s+t=n} H^s(\mathcal{H}^t(j_* \Omega_{M^0/S}^\bullet)),$$

and so

$$\dim_k H^n(j_* \Omega_{M^0/S}^\bullet) = \sum_{s+t=n} \dim_k H^s(f_* \varpi_{\mathfrak{X}/S}^t),$$

which implies the degeneracy of the hypercohomology spectral sequence by Lemma 1.5. □

Although our proof of Theorem 1.9 goes through stacks, the statement of the theorem is purely scheme-theoretic. We would similarly like to remove the stack from the statement of Theorem 4.8. We can do so when  $M$  has large enough dimension.

**THEOREM 4.9.** — *Let  $M$  be a proper  $S$ -scheme with isolated linearly reductive singularities. If  $\dim M \geq 4$  and  $M$  lifts mod  $p^2$ , then the stack  $\mathfrak{X}$  of Theorem 4.2 lifts mod  $p^2$ , and so*

$$E_1^{st} = H^t(j_* \Omega_{M^0/S}^s) \Rightarrow H^n(j_* \Omega_{M^0/S}^\bullet)$$

degenerates for  $s + t < p$ .

*Proof.* — Let  $m = \dim M$ . The exact triangle

$$Lf^*L_{M/S} \rightarrow L_{\mathfrak{X}/S} \rightarrow L_{\mathfrak{X}/M}$$

gives rise to the long exact sequence

$$\dots \rightarrow \text{Ext}^2(L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}}) \rightarrow \text{Ext}^2(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}}) \rightarrow \text{Ext}^2(Lf^*L_{M/S}, \mathcal{O}_{\mathfrak{X}}) \rightarrow \dots$$

Note that

$$R\text{Hom}(Lf^*L_{M/S}, \mathcal{O}_{\mathfrak{X}}) = R\text{Hom}(L_{M/S}, Rf_*\mathcal{O}_{\mathfrak{X}}) = R\text{Hom}(L_{M/S}, \mathcal{O}_M)$$

since  $Rf_*\mathcal{O}_{\mathfrak{X}} = f_*\mathcal{O}_{\mathfrak{X}}$  by tameness and  $f_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_M$  by Keel-Mori [11]. Since the obstruction to lifting  $\mathfrak{X}$  lies in  $\text{Ext}^2(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}})$ , we need only show  $\text{Ext}^2(L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}}) = 0$ . We in fact prove  $\mathcal{E}xt^s(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}}) = 0$  for  $s \leq m - 2$ .

Since  $(j^0)^*L_{\mathfrak{X}/M} = L_{\mathfrak{X}^0/M^0} = 0$ , we see

$$0 = Rj_*^0 R\mathcal{H}om((j^0)^*L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}^0}) = R\mathcal{H}om(L_{\mathfrak{X}/M}, Rj_*^0\mathcal{O}_{\mathfrak{X}^0}).$$

A local cohomology argument given below will show  $R^t j_*^0 \mathcal{O}_{\mathfrak{X}^0} \neq 0$  if and only if  $t = 0, m - 1$ . Assuming this for the moment, let us complete the proof. We have a spectral sequence

$$E_2^{st} = R^s \mathcal{H}om(L_{\mathfrak{X}/M}, R^t j_*^0 \mathcal{O}_{\mathfrak{X}^0}) \Rightarrow R^n \mathcal{H}om(L_{\mathfrak{X}/M}, Rj_*^0 \mathcal{O}_{\mathfrak{X}^0}) = 0.$$

The only page with non-zero differentials, then, is the  $m^{\text{th}}$ . Since  $L_{\mathfrak{X}/M}$  is concentrated in degrees at most 1,  $R^s \mathcal{H}om(L_{\mathfrak{X}/M}, R^t j_*^0 \mathcal{O}_{\mathfrak{X}^0}) = 0$  for  $s < -1$ . It follows that

$$R^s \mathcal{H}om(L_{\mathfrak{X}/M}, j_*^0 \mathcal{O}_{\mathfrak{X}^0}) = 0$$

for  $s \leq m - 2$ , which proves the theorem since  $j_*^0 \mathcal{O}_{\mathfrak{X}^0} = \mathcal{O}_{\mathfrak{X}}$ .

We now turn to the local cohomology argument. To prove  $R^t j_*^0 \mathcal{O}_{\mathfrak{X}^0} \neq 0$  if and only if  $t = 0, m - 1$ , we can make an étale base change. We can therefore assume  $\mathfrak{X} = [U/G]$ , where  $U$  is smooth and affine, and  $G$  is finite linearly reductive. Since  $M$  has isolated singularities, we can further assume  $U^0 = U \setminus \{x\}$ , where  $U^0$  is the pullback

$$\begin{array}{ccc} U^0 & \xrightarrow{i} & U \\ h \downarrow & & \downarrow g \\ \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \end{array}$$

The following lemma, then, completes the proof. □

LEMMA 4.10. — *Let  $U$  be a normal affine scheme of dimension  $m$  and let  $x \in U$  be Cohen-Macaulay. If  $U^0 = U \setminus \{x\}$  and  $i : U^0 \hookrightarrow U$  is the inclusion, then  $R^t i_* \mathcal{O}_{U^0} \neq 0$  if and only if  $t = 0, m - 1$ .*

*Proof.* — Note that  $R^t i_* \mathcal{O}_{U^0}$  is the skyscraper sheaf  $H^t(\mathcal{O}_{U^0})$  at  $x$ . By normality,  $H^0(\mathcal{O}_{U^0}) = H^0(\mathcal{O}_U)$ . Since  $U$  is affine, the long exact sequence

$$\dots \rightarrow H_x^n(\mathcal{O}_U) \rightarrow H^n(\mathcal{O}_U) \rightarrow H_x^n(\mathcal{O}_{U^0}) \rightarrow H_x^n(\mathcal{O}_U) \rightarrow \dots$$

shows  $H^t(\mathcal{O}_{U^0}) = H_x^{t+1}(\mathcal{O}_U)$  for  $t > 0$ . Since  $x$  is Cohen-Macaulay,  $H_x^{t+1}(\mathcal{O}_U) \neq 0$  if and only if  $t + 1 = m$ . □

We now prove an analogue of [5, Lemma 2.9] which Deligne and Illusie use to deduce Kodaira Vanishing.

LEMMA 4.11. — *Let  $\mathfrak{X}$  and  $M$  be as in Theorem 4.2. Suppose  $M$  has isolated singularities, and  $\mathfrak{X}$  is proper and lifts mod  $p^2$ . Let  $d$  be the dimension of  $M$  and let  $N$  be an integer such that  $N \leq \inf(d, p)$ . If  $\mathcal{M}$  is an invertible sheaf on  $M$  such that*

$$H^t(j_* \Omega_{M^0/S}^s \otimes \mathcal{M}^p) = 0$$

for all  $s + t < N$ , then

$$H^t(j_* \Omega_{M^0/S}^s \otimes \mathcal{M}) = 0$$

for all  $s + t < N$ .

*Proof.* — Let  $\mathcal{M}'$  be the pullback of  $\mathcal{M}$  to  $M'$ . Since  $F^* \mathcal{M}' = \mathcal{M}^p$ , the projection formula shows

$$H^t(j_* \Omega_{M^0/S}^s \otimes \mathcal{M}^p) = H^t(f'_* F_* \varpi_{\mathfrak{X}/S}^s \otimes \mathcal{M}').$$

>From the hypercohomology spectral sequence

$$E_2^{st} = H^t(f'_* F_* \varpi_{\mathfrak{X}/S}^s \otimes \mathcal{M}') \Rightarrow H^n(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{M}'),$$

we see that  $H^n(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{M}') = 0$  for all  $n < N$ . Remark 4.7 shows that the Leray spectral sequence

$$E_2^{st} = H^s(f'_* \varpi_{\mathfrak{X}'/S}^t \otimes \mathcal{M}') \Rightarrow H^n(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \otimes \mathcal{M}')$$

degenerates, and so  $H^s(f'_* \varpi_{\mathfrak{X}'/S}^t \otimes \mathcal{M}') = 0$  for all  $s + t < N$ . Since

$$\dim_k H^s(f'_* \varpi_{\mathfrak{X}'/S}^t \otimes \mathcal{M}') = \dim_k H^s(j_* \Omega_{M^0/S}^t \otimes \mathcal{M}),$$

the lemma follows. □

Unfortunately, we cannot quite deduce from Lemma 4.11 a general Kodaira Vanishing result. Following Deligne and Illusie, we would like to show that if  $M$  is a projective scheme of dimension  $d$  with isolated linearly reductive singularities and  $\mathcal{L}$  is an ample line bundle on  $M$ , then  $H^t(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-p^m}) = 0$  for  $m$  sufficiently large. Lemma 4.11 would then imply that  $m$  can be taken to be 1. The issue is that the vanishing of these cohomology groups for  $m$  large enough is not clear. Under certain hypothesis, however, we obtain a vanishing theorem.

PROPOSITION 4.12. — *Let  $M$  be a projective scheme of dimension  $d$  with isolated linearly reductive singularities. Let  $\mathcal{L}$  be an ample line bundle on  $M$ . If the stack  $\mathfrak{X}$  of Theorem 4.2 lifts mod  $p^2$  and the  $j_*\Omega_{M^0/S}^s$  are Cohen-Macaulay for all  $s$ , then*

$$H^t(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-1}) = 0$$

for all  $s + t < \inf(d, p)$ .

*Proof.* — By Lemma 4.11, we need only prove that  $H^t(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-p^m}) = 0$  for  $m$  sufficiently large. Grothendieck Duality shows

$$H^t(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-p^m})^\vee = \text{Ext}^{d-t}(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-p^m}, \omega_M).$$

Since the  $\mathcal{E}xt^{d-t}(j_*\Omega_{M^0/S}^s, \omega_M)$  are coherent, the local-global Ext spectral sequence shows that for  $m$  sufficiently large,

$$H^t(j_*\Omega_{M^0/S}^s \otimes \mathcal{L}^{-p^m})^\vee = \Gamma(\mathcal{E}xt^{d-t}(j_*\Omega_{M^0/S}^s, \omega_M) \otimes \mathcal{L}^{p^m}).$$

For all  $x \in M$ ,

$$\mathcal{E}xt^{d-t}(j_*\Omega_{M^0/S}^s, \omega_M)_x = \text{Ext}_{\mathcal{O}_x}^{d-t}((j_*\Omega_{M^0/S}^s)_x, \omega_{\mathcal{O}_x}).$$

Since  $M$  and the  $j_*\Omega_{M^0/S}^s$  are Cohen-Macaulay, local duality shows that for  $t < d$ ,

$$\mathcal{E}xt^{d-t}(j_*\Omega_{M^0/S}^s, \omega_M) = 0,$$

thereby completing the proof. □

We conclude by showing that the hypercohomology spectral sequence

$$E_1^{st} = R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s \Rightarrow R^n f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$$

degenerates at  $E_2$  and that the only potentially non-zero differentials on the first page are those on the zero-th row.

LEMMA 4.13. — *Let  $\mathcal{A}$  and  $\mathcal{B}$  be abelian categories and let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor. Suppose that  $\mathcal{A}$  has enough injectives. If  $A^\bullet$  is a complex of objects in  $\mathcal{A}$  and  $C^\bullet$  denotes the cone in the derived category*

$D(\mathcal{A})$  of the canonical morphism  $FA^\bullet \rightarrow RFA^\bullet$ , then there is a spectral sequence

$$\backslash E_1^{st} = \begin{cases} R^t FA^s & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \mathcal{H}^n(C^\bullet).$$

If in the hypercohomology spectral sequence  $E_1^{pq} = R^q FA^p \Rightarrow R^n FA^\bullet$ , the differentials  $E_r^{s-r, r-1} \rightarrow E_r^{s, 0}$  are zero for all  $r \geq 2$ , then for every  $n$ ,

$$0 \rightarrow \mathcal{H}^n(FA^\bullet) \rightarrow R^n FA^\bullet \rightarrow \mathcal{H}^n(C^\bullet) \rightarrow 0$$

is a short exact sequence.

*Proof.* — The existence of the spectral sequence  $\backslash E$  is shown as follows. Let  $A^s \rightarrow I^{s, \bullet}$  be an injective resolution of  $A^s$ . The cone  $C^\bullet$  is then quasi-isomorphic to the total complex of

$$\begin{array}{ccc} \vdots & \vdots & \\ FI^{01} & FI^{11} & \dots \\ FI^{00} & FI^{10} & \dots \\ FA^0 & FA^1 & \dots \end{array}$$

where  $FA^0$  has bidegree  $(-1, 0)$ . The spectral sequence associated to this double complex in which we begin by taking cohomology vertically is our desired  $\backslash E$ .

Note that there is a morphism of spectral sequences  $E \rightarrow \backslash E$ . If the differentials  $E_r^{s-r, r-1} \rightarrow E_r^{s, 0}$  are zero for all  $r \geq 2$ , then the morphism of spectral sequences induces an isomorphism  $E_\infty^{st} \xrightarrow{\sim} \backslash E_\infty^{st}$  for  $t \neq 0$ . It follows that  $\mathcal{H}^n(C^\bullet)$  is equal to  $R^n FA^\bullet$  modulo the bottom part of its filtration, namely  $E_\infty^{n0} = \mathcal{H}^n(FA^\bullet)$ .  $\square$

PROPOSITION 4.14. — *Let  $\mathfrak{X}$  and  $M$  be as in Theorem 4.2. If  $M$  has isolated singularities, and  $\mathfrak{X}$  is proper and lifts mod  $p^2$ , then the hypercohomology spectral sequence*

$$E_1^{st} = R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s \Rightarrow R^n f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$$

degenerates at  $E_2$ , and for  $t \neq 0$ , the differentials  $E_1^{st} \rightarrow E_1^{s+1, t}$  are zero.

*Proof.* — Let  $C^\bullet$  be the cone of the canonical morphism  $f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \rightarrow Rf'_* F_* \varpi_{\mathfrak{X}/S}^\bullet$ . Note that for  $t > 0$ , we have

$$j'^* R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s = F_* j'^* R^t f_* \bigwedge^s L_{\mathfrak{X}/S} = (f^0)'_* \mathcal{H}^t \left( \bigwedge^s L_{(\mathfrak{X}^0)' / S} \right) = 0,$$

and so  $R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s$  is supported at the singular locus of  $M'$ ; in particular, the  $R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s$  are torsion. On the other hand,  $R^0 f'_* F_* \varpi_{\mathfrak{X}/S}^s = F_* j_* \Omega_{M^0/S}^s$ , which is reflexive, and hence torsion-free. As a result, for  $r \geq 2$

every differential  $E_r^{s-r, r-1} \rightarrow E_r^{s,0}$  is zero, and  $E_r^{st}$  is supported at the singular locus of  $M'$  for all  $t \neq 0$  and all  $s$  and  $r$ . So, to prove the proposition, we need only show that the spectral sequence

$$\backslash E_1^{st} = \begin{cases} R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \mathcal{H}^n(C^\bullet)$$

of Lemma 4.13 degenerates.

Since  $M$  is assumed to have isolated singularities, for any short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

with  $\mathcal{F}$  supported at the singular locus,

$$0 \rightarrow \Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{Q}) \rightarrow 0$$

is short exact as well. Furthermore,  $\Gamma(\mathcal{F}) = \bigoplus_{x \in M} \mathcal{F}_x$ , so  $\mathcal{F}$  is zero if and only if  $\Gamma(\mathcal{F})$  is zero. It follows that we have a spectral sequence

$$\backslash E_1^{st} = \begin{cases} \Gamma(R^t f'_* F_* \varpi_{\mathfrak{X}/S}^s) & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \Gamma(\mathcal{H}^n(C^\bullet))$$

whose degeneracy is equivalent to that of  $\backslash E$ . By Lemma 4.13, there is a short exact sequence

$$0 \rightarrow \mathcal{H}^n(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \rightarrow R^n f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \rightarrow \mathcal{H}^n(C^\bullet) \rightarrow 0.$$

Comparing this with the short exact sequence

$$0 \rightarrow \mathcal{H}^t(f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet) \xrightarrow{\varphi} R^t f'_* F_* \varpi_{\mathfrak{X}/S}^\bullet \rightarrow \bigoplus_{\substack{i+j=t \\ j>0}} R^j f'_* \varpi_{\mathfrak{X}'/S}^i \rightarrow 0$$

proved in Theorem 4.6, we see

$$\mathcal{H}^n(C^\bullet) = \bigoplus_{\substack{s+t=n \\ t>0}} R^t f'_* \varpi_{\mathfrak{X}'/S}^s.$$

It follows that

$$\sum_{s+t=n} \dim_k \backslash E_1^{st} = \dim_k \Gamma(\mathcal{H}^n(C^\bullet))$$

which shows the degeneracy of  $\backslash E$  by Lemma 1.5. □

## BIBLIOGRAPHY

- [1] D. ABRAMOVICH, M. OLSSON & A. VISTOLI, “Tame stacks in positive characteristic”, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 4, p. 1057-1091.
- [2] D. ABRAMOVICH & A. VISTOLI, “Compactifying the space of stable maps”, *J. Amer. Math. Soc.* **15** (2002), no. 1, p. 27-75 (electronic).
- [3] K. BEHREND, “Cohomology of stacks”, in *Intersection theory and moduli*, ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, p. 249-294 (electronic).
- [4] B. CONRAD, “Cohomological descent”, 2009, <http://math.stanford.edu/~conrad/papers/cohdesscent.pdf>.
- [5] P. DELIGNE & L. ILLUSIE, “Relèvements modulo  $p^2$  et décomposition du complexe de de Rham”, *Invent. Math.* **89** (1987), no. 2, p. 247-270.
- [6] G. FALTINGS, “ $p$ -adic Hodge theory”, *J. Amer. Math. Soc.* **1** (1988), no. 1, p. 255-299.
- [7] B. FANTECHI, E. MANN & F. NIRONI, “Smooth toric DM stacks”, 2009, arXiv:0708.1254v2.
- [8] M. HOCHSTER & J. L. ROBERTS, “Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay”, *Advances in Math.* **13** (1974), p. 115-175.
- [9] L. ILLUSIE, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971, xv+355 pages.
- [10] N. M. KATZ, “Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin”, *Inst. Hautes Études Sci. Publ. Math.* (1970), no. 39, p. 175-232.
- [11] S. KEEL & S. MORI, “Quotients by groupoids”, *Ann. of Math. (2)* **145** (1997), no. 1, p. 193-213.
- [12] G. LAUMON & L. MORET-BAILLY, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000, xii+208 pages.
- [13] K. MATSUKI & M. OLSSON, “Kawamata-Viehweg vanishing as Kodaira vanishing for stacks”, *Math. Res. Lett.* **12** (2005), no. 2-3, p. 207-217.
- [14] D. MUMFORD, J. FOGARTY & F. KIRWAN, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994, xiv+292 pages.
- [15] M. C. OLSSON, “Hom-stacks and restriction of scalars”, *Duke Math. J.* **134** (2006), no. 1, p. 139-164.
- [16] ———, “Sheaves on Artin stacks”, *J. Reine Angew. Math.* **603** (2007), p. 55-112.
- [17] M. SATRIANO, “A generalization of the Chevalley-Shephard-Todd theorem to the case of linearly reductive group schemes”, 2009, arXiv:0911.2058v1.

- [18] J. H. M. STEENBRINK, “Mixed Hodge structure on the vanishing cohomology”, in *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, p. 525-563.
- [19] B. TOEN, “K-théorie et cohomologie des champs algébriques: Théorèmes de Riemann-Roch, D-modules et théorèmes GAGA”, 1999, arXiv:math/9908097v2.
- [20] A. VISTOLI, “Intersection theory on algebraic stacks and on their moduli spaces”, *Invent. Math.* **97** (1989), no. 3, p. 613-670.

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