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EQUIVARIANT DEGENERATIONS OF SPHERICAL MODULES FOR GROUPS OF TYPE A

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ABSTRACT. — V. Alexeev and M. Brion introduced, for a given a complex reductive group, a moduli scheme of affine spherical varieties with prescribed weight monoid. We provide new examples of this moduli scheme by proving that it is an affine space when the given group is of type A and the prescribed weight monoid is that of a spherical module.

RÉSUMÉ. — V. Alexeev et M. Brion ont introduit, pour un groupe complexe réductif donné, un schéma de modules de variétés sphériques affines ayant le même semi-groupe moment. Nous donnons de nouveaux exemples de ce schéma de modules en montrant qu'il est un espace affine lorsque le groupe donné est de type A et le semi-groupe moment fixé est celui d'un module sphérique.

1. Introduction and statement of results

As part of the classification of affine G-varieties X, where G is a complex connected reductive group, a natural question is to what extent the G-module structure of the ring $\mathbb{C}[X]$ of regular functions on X determines X. Put differently, to what extent does the G-module structure of $\mathbb{C}[X]$ determine its algebra structure?

In the mid 1990s, F. Knop conjectured that the answer to this question is "completely" when X is a smooth affine *spherical* variety. To be precise, *Knop's Conjecture*, which has since been proved by I. Losev [22], says that if X is a smooth affine G-variety such that the G-module $\mathbb{C}[X]$ has no multiplicities, then this G-module uniquely determines the G-variety X (up to G-equivariant isomorphism). Knop also proved [20] that the validity of his conjecture implies that of Delzant's Conjecture [12] about multiplicity-free symplectic manifolds.

Keywords: Invariant Hilbert scheme, spherical module, spherical variety, equivariant degeneration.
Math. classification: 14D22, 14C05, 14M27, 20G05.

In [1], V. Alexeev and M. Brion brought geometry to the general question. Given a maximal torus T in G and an affine T-variety Y such that all T-weights in $\mathbb{C}[Y]$ have finite multiplicity, they introduced a moduli scheme M_Y which parametrizes (equivalence classes of) pairs (X, φ), where X is an affine G-variety and $\varphi\colon X/\!\!/U \to Y$ is a T-equivariant isomorphism (here $U \subseteq G$ is a fixed maximal unipotent subgroup normalized by T and $X/\!\!/U := \operatorname{Spec}\mathbb{C}[X]^U$ is the categorical quotient). They also proved that M_Y is an affine connected scheme, of finite type over $\mathbb{C}$, and that the orbits of the natural action of $\mathrm{Aut}^T(Y)$ on M_Y are in bijection with the isomorphism classes of affine G-varieties X such that $X/\!\!/U \simeq Y$. See also [8, Section 4.3] for more information on M_Y.

The first examples of M_Y were obtained by S. Jansou [16]. He dealt with the following situation. Suppose Λ^+ is the set of dominant weights of G (with respect to the Borel subgroup $B = TU$ of G) and let $\lambda \in \Lambda^+$. Jansou proved that if $Y = \mathbb{C}$ with T acting linearly with weight $-\lambda$, then $\mathrm{M}_\lambda := \mathrm{M}_Y$ is a (reduced) point or an affine line. Moreover, he linked M_Y to the theory of *wonderful varieties* (see, *e.g.*, [5] or [27]) by showing that M_λ is an affine line if and only if λ is a spherical root for G.

P. Bravi and S. Cupit-Foutou [3] generalized Jansou's result as follows. Given a free submonoid $\mathcal{S}$ of Λ^+ such that

$$\langle \mathcal{S} \rangle_{\mathbb{Z}} \cap \Lambda^+ = \mathcal{S}, \tag{1.1}$$

put $Y := \operatorname{Spec}\mathbb{C}[\mathcal{S}]$ and $\mathrm{M}_{\mathcal{S}} := \mathrm{M}_Y$. Bravi and Cupit-Foutou proved that $\mathrm{M}_{\mathcal{S}}$ is isomorphic to an affine space. More precisely, the map $T \to \mathrm{Aut}^T(Y)$ coming from the action of T on Y induces an action of T on $\mathrm{M}_{\mathcal{S}}$, and they proved that $\mathrm{M}_{\mathcal{S}}$ is (isomorphic to) a multiplicity-free representation of T whose weight set is the set of spherical roots of a wonderful variety associated to $\mathcal{S}$. The connections between the moduli schemes M_Y and wonderful varieties have been studied further in [10, 11].

In this paper we compute examples of $\mathrm{M}_{\mathcal{S}}$ where $\mathcal{S}$ is a free submonoid of Λ^+, but does not necessarily satisfy (1.1). To be more precise, we prove that M_Y is (again) isomorphic to an affine space whenever $Y = W/\!\!/U$ with W a spherical G-module and G of type A (see Theorem 1.1 below for the precise statement). The reason we chose to work with spherical modules is that they have been classified ("up to central tori") and that many of their combinatorial invariants have been computed (see [19]). We prove Theorem 1.1 by reducing it to a case-by-case verification (Theorem 1.2). It turns out that in most of our cases, condition (1.1) is not satisfied. The fact that the classification of spherical modules is "up to central tori" means that this verification needs some care, see Section 4 and Remark 4.4. In this paper we restrict ourselves to groups of type A because the work needed is already quite lengthy. The reduction of the proof of Theorem 1.1 to the case-by-case analysis is independent of the type of G.

The main consequence of the absence of condition (1.1) is that computing the tangent space to $\mathrm{M}_{\mathcal{S}}$ at its unique T-fixed point and unique closed T-orbit X_0, which is also the first step in the work of Jansou, and Bravi and Cupit-Foutou, becomes more involved (see Section 3 below). On the other hand, we know, by definition, that our moduli schemes $\mathrm{M}_{\mathcal{S}} = \mathrm{M}_Y$ (where $Y = W/\!\!/U$) contain the closed point (W, π) where $\pi\colon W/\!\!/U \to Y$ is the identity map. By general results from [1] this point has an (open) T-orbit of which we know the dimension d_W. This implies that once we have determined that $\dim T_{X_0}\mathrm{M}_{\mathcal{S}} \leqslant d_W$, our main result follows. Jansou and especially Bravi–Cupit-Foutou have to do quite a bit more work (involving the existence of a certain wonderful variety depending on $\mathcal{S}$) to prove that $\mathrm{M}_{\mathcal{S}}$ contains a T-orbit of the same dimension as $T_{X_0}\mathrm{M}_{\mathcal{S}}$.

1.1. Notation and preliminaries

We will consider algebraic groups and schemes over $\mathbb{C}$. In addition, like in [1], all schemes will be assumed to be Noetherian. By a variety, we mean an integral separated scheme of finite type over $\mathbb{C}$. In particular, varieties are irreducible.

In this paper, unless stated otherwise, G will be a connected reductive linear algebraic group over $\mathbb{C}$ in which we have chosen a (fixed) maximal torus T and a (fixed) Borel subgroup B containing T. We will use U for the unipotent radical of B; it is a maximal unipotent subgroup of G. For an algebraic group H, we denote $X(H)$ the group of characters, that is, the set of all homomorphisms of algebraic groups $H \to \mathbb{G}_m$, where $\mathbb{G}_m$ denotes the multiplicative group $\mathbb{C}^\times$. By a G-module or a representation of G we will always mean a (possibly infinite dimensional) *rational* G-module (sometimes also called a locally finite G-module). For the definition, which applies to non-reductive groups too, see for example [1, p. 86]. Because G is reductive, every G-module E is the direct sum of irreducible (or *simple*) G-submodules. We call E *multiplicity-free* if it is the direct sum of pairwise non-isomorphic simple G-modules.

We will use Λ for the weight lattice $X(T)$ of G, which is naturally identified with $X(B)$, and Λ^+ for the submonoid of $X(T)$ of dominant weights (with respect to B). Every $\lambda \in \Lambda^+$ corresponds to a unique irreducible representation of G, which we will denote $V(\lambda)$. It is specified by the property that λ is its unique B-weight. Conversely every irreducible representation of G is of the form $V(\lambda)$ for a unique $\lambda \in \Lambda^+$. Furthermore, we will use v_λ for a highest weight vector in $V(\lambda)$. It is defined up to nonzero scalar: $V(\lambda)^U = \mathbb{C}v_\lambda$. For $\lambda \in \Lambda^+$, we will use λ^* for the highest weight of the dual $V(\lambda)^*$ of $V(\lambda)$. We then have that $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element of the Weyl group $N_G(T)/T$ of G. For a G-module M and $\lambda \in \Lambda^+$, we will use $M_{(\lambda)}$ for the isotypical component of M of type $V(\lambda)$.

We denote the center of G by $Z(G)$ and use T_{ad} for the adjoint torus $T/Z(G)$ of G. The set of simple roots of G (with respect to T and B) will be denoted Π, the set of positive roots R^+ and the root lattice Λ_R. When α is a root, $\alpha^\vee \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ will stand for its coroot. In particular, $\langle \alpha, \alpha^\vee \rangle = 2$ where $\langle \cdot, \cdot \rangle$ is the natural pairing between Λ and its dual $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ (which is naturally identified with the group of one-parameter subgroups of T).

The Lie algebra of an algebraic group G, H, T, B, U *etc.* will be denoted by the corresponding fraktur letter $\mathfrak{g}, \mathfrak{h}, \mathfrak{t}, \mathfrak{b}, \mathfrak{u}$, *etc.* At times, we will also use $\mathrm{Lie}(H)$ for the Lie algebra of H. For a reductive group G, we will use G' for its derived group (G, G). It is a semisimple group and its Lie algebra is $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. When G acts on a set X and $x \in X$, then G_x stands for the isotropy group of x. We adopt the convention that $G'_x := (G')_x$ and analogous notations for $\mathfrak{g}$-actions. For every root α of G, we choose a nonzero element X_α of the (one-dimensional) root space $\mathfrak{g}^\alpha \subseteq \mathfrak{g}$. We call X_α a *root operator*.

A reductive group G is said to be of *type* A if $\mathfrak{g}'$ is 0 or isomorphic to a direct sum

$$\mathfrak{sl}(n_1) \oplus \mathfrak{sl}(n_2) \oplus \cdots \oplus \mathfrak{sl}(n_k)$$

for some positive integer k and integers $n_i \geqslant 2$ $(1 \leqslant i \leqslant k)$.

When $G = \mathrm{SL}(n)$ and $i \in \{1, \ldots, n-1\}$, we denote ω_i the highest weight of the module $\bigwedge^i \mathbb{C}^n$. In addition, for $\mathrm{SL}(n)$ we put $\omega_n = \omega_0 = 0$. Similarly, when $G = \mathrm{GL}(n)$ and $i \in \{1, \ldots, n\}$, the highest weight of the module $\bigwedge^i \mathbb{C}^n$ will also be denoted ω_i. The set $\{\omega_1, \ldots, \omega_n\}$ forms a basis of the weight lattice Λ of $\mathrm{GL}(n)$. Moreover, we put $\omega_0 = 0$. It is well-known that the simple roots of $\mathrm{GL}(n)$ have the following expressions in terms of the ω_i:

$$\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1} \quad \text{for } i \in \{1, 2, \ldots, n-1\}, \tag{1.2}$$

and that the same formulas also hold for $\mathrm{SL}(n)$. The representations $V(\omega_i)$ are called the *fundamental representations* of $\mathrm{GL}(n)$ (resp. $\mathrm{SL}(n)$).

A finitely generated $\mathbb{C}$-algebra A is called a *G-algebra* if it comes equipped with an action of G (by automorphisms) for which A is a rational G-module. The *weight set* of A is then defined as

$$\Lambda_A^+ := \left\{ \lambda \in \Lambda^+ \colon A_{(\lambda)} \neq 0 \right\}.$$

Such an algebra A is called *multiplicity-free* if it is multiplicity-free as a G-module. When the G-algebra A is an integral domain, the multiplication on A induces a monoid structure on Λ^+_A, which we then call the *weight monoid* of A; it is a finitely generated submonoid of Λ^+ (see *e.g.* [7, Corollary 2.8]).

For an affine scheme X, we will use $\mathbb{C}[X]$ for its ring of regular functions. In particular, $X = \operatorname{Spec}\mathbb{C}[X]$. As in [1], an *affine G-scheme* is an affine scheme X of finite type equipped with an action of G. Then $\mathbb{C}[X]$ is a G-algebra for the following action:

$$(g \cdot f)(x) = f(g^{-1} \cdot x) \quad \text{for } f \in \mathbb{C}[X],\ g \in G \text{ and } x \in X.$$

We remark that even when G is abelian we use this action on $\mathbb{C}[X]$. A *G-variety* is a variety equipped with an action of G. If X is an affine G-scheme, then its *weight set* $\Lambda^+_{(G,X)}$ is defined, like in [1, p. 87], as the weight set of the G-algebra $\mathbb{C}[X]$. If X is an affine G-variety, then we call $\Lambda^+_{(G,X)}$ its *weight monoid*, and the *weight group* $\Lambda_{(G,X)}$ of X is defined as the subgroup of $X(T)$ generated by $\Lambda^+_{(G,X)}$. It is well–known that $\Lambda_{(G,X)}$ is also equal to the set of B-weights in the function field of X (see *e.g.* [7, p. 17]). When no confusion can arise about the group G in question, we will use Λ^+_X and Λ_X for $\Lambda^+_{(G,X)}$ and $\Lambda_{(G,X)}$, respectively. An affine G-scheme X is called *multiplicity-free* if $\mathbb{C}[X]$ is multiplicity-free as a G-module. An affine G-variety is multiplicity-free if and only if it has a dense B-orbit. We call a G-variety *spherical* if it is normal and has a dense orbit for B. A *spherical G-module* is a finite-dimensional G-module that is spherical as a G-variety. We remark that if W is a spherical G-module, then any two distinct simple G-submodules of W are non-isomorphic. For general information on spherical varieties we refer to [7, Section 2] and [27].

The indecomposable saturated spherical modules were classified up to geometric equivalence by Kac, Benson-Ratcliff and Leahy [17, 2, 21], see [19] for an overview or Section 4 for the definitions of these terms. We will use Knop's presentation in [19, Section 5] of this classification and refer to it as **Knop's List**. For groups of type A we recall the classification in List 5.1 on page 1797.

When H is a torus and M is a finite-dimensional H-module, then by the *H-weight set* of M, we mean the (finite) set of elements λ of $X(H)$ such that $M_{(\lambda)} \neq 0$. For the weight monoid Λ^+_M of M (seen as an H-variety) we then have that

$$\Lambda^+_M = \big\langle -\lambda | \lambda \text{ is an element of the } H\text{-weight set of } M \big\rangle_{\mathbb{N}}.$$

Given an affine T-scheme Y such that each T-eigenspace in $\mathbb{C}[Y]$ is finite-dimensional, Alexeev and Brion [1] introduced a moduli scheme M_Y

which classifies (equivalence classes of) pairs (X, φ), where X is an affine G-scheme and $\varphi\colon X/\!\!/U \to Y$ is a T-equivariant isomorphism. Here $X/\!\!/U := \operatorname{Spec}(\mathbb{C}[X]^U)$ is the categorical quotient. Moreover, they proved that M_Y is a connected, affine scheme of finite type over $\mathbb{C}$ and they equipped it with an action by T_{ad}, induced by the action of $\operatorname{Aut}^T(Y)$ on M_Y and the map $T \to \operatorname{Aut}^T(Y)$. We call γ this action of T_{ad} on M_Y (see Section 2.1 for details).

Now, suppose $\mathcal{S}$ is a finitely generated submonoid of Λ^+ and $Y = \operatorname{Spec}\mathbb{C}[\mathcal{S}]$ is the multiplicity-free T-variety with weight monoid $\mathcal{S}$. Like [1], we then put

$$\mathrm{M}_{\mathcal{S}} := \mathrm{M}_Y.$$

We will use $\mathrm{M}_{\mathcal{S}}^G$ for $\mathrm{M}_{\mathcal{S}}$ when we want to stress the group under consideration.

We need to define one more combinatorial invariant of affine G-varieties. Let X be such a variety. Put $R := \mathbb{C}[X]$ and define the *root monoid* Σ_X of X as the submonoid of $X(T)$ generated by

$$\Big\{\lambda + \mu - \nu \in \Lambda \mid \lambda, \mu, \nu \in \Lambda^+ \colon \big\langle R_{(\lambda)} R_{(\mu)} \big\rangle_{\mathbb{C}} \cap R_{(\nu)} \neq 0\Big\},$$

where $\langle R_{(\lambda)} R_{(\mu)} \rangle_{\mathbb{C}}$ denotes the $\mathbb{C}$-vector subspace of R spanned by the set $\{fg \mid f \in R_{(\lambda)}, g \in R_{(\mu)}\}$. Note that $\Sigma_X \subseteq \langle \Pi \rangle_{\mathbb{N}}$. We call d_X the rank of the (free) abelian group generated (in $X(T)$) by Σ_X, that is,

$$d_X := \operatorname{rk}\langle \Sigma_X \rangle_{\mathbb{Z}}.$$

We remark that for a given spherical module W, the invariant d_W is easy to calculate from the rank of Λ_W, see Lemma 2.7.

1.2. Main results

The main result of the present paper is the following theorem. Its formal proof will be given in Section 1.3.

Theorem 1.1. — *Assume W is a spherical G-module, where G is a connected reductive algebraic group of type* A. *Let $\mathcal{S}$ be the weight monoid of W. Then*

(a) *Σ_W is a freely generated monoid; and*
(b) *the T_{ad}-scheme $\mathrm{M}_{\mathcal{S}}$, where the action is γ, is T_{ad}-equivariantly isomorphic to the T_{ad}-module with weight monoid Σ_W. In particular, the scheme $\mathrm{M}_{\mathcal{S}}$ is isomorphic to the affine space $\mathbb{A}^{d_W}$, hence it is irreducible and smooth.*

Our strategy for the proof of Theorem 1.1 is as follows. Suppose W is a spherical module with weight monoid $\mathcal{S}$. Because $\dim \mathrm{M}_{\mathcal{S}} \geqslant d_W$, it is sufficient to prove that $\dim T_{X_0}\mathrm{M}_{\mathcal{S}} \leqslant d_W$, where X_0 is the unique T_{ad}-fixed point and the unique closed T_{ad}-orbit in $\mathrm{M}_{\mathcal{S}}$ (see Corollary 2.6). In Section 4 (see Corollary 4.17) we further reduce the proof of Theorem 1.1 to the following theorem.

Theorem 1.2. — *Suppose $(\overline{G}, W)$ is an entry in Knop's List of saturated indecomposable spherical modules with $\overline{G}$ of type* A *(see List 5.1 on page 1797). If G is a connected reductive group such that*

(1) $\overline{G}' \subseteq G \subseteq \overline{G}$; *and*

(2) *W is spherical as a G-module*

then

$$\dim T_{X_0}\mathrm{M}_{\mathcal{S}}^G = d_W,$$

where $\mathcal{S}$ is the weight monoid of (G, W).

In Section 5 we will prove this theorem case-by-case for the 8 families of spherical modules in Knop's List with $\overline{G}$ of type A.

For that purpose X_0 is identified in Section 2.1 with the closure of a certain orbit $G \cdot x_0$ in a certain G-module V and $T_{X_0}\mathrm{M}_{\mathcal{S}}$ with the vector space of G-invariant global sections of the normal sheaf of X_0 in V. This is a subspace of the space of G-invariant sections of the same sheaf over $G \cdot x_0$. This latter space is naturally identified with $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. In Section 5 we use the T_{ad}-action (more precisely a variation of it) to bound $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ by explicit computations for the pairs (G, W) in the statement of Theorem 1.2. In most cases we find that already $\dim(V/\mathfrak{g} \cdot x_0)^{G_{x_0}} \leqslant d_W$. To obtain the desired inequality for $\dim T_{X_0}\mathrm{M}_{\mathcal{S}}$ in the remaining cases we use the exclusion criterion of Section 3, which was suggested to us by M. Brion, to prove that enough sections over $G \cdot x_0$ do not extend to X_0.

1.3. Formal proof of Theorem 1.1

We now give the proof of Theorem 1.1. Corollary 2.6 and Corollary 4.17 reduce the proof to Theorem 1.2, which we prove by a case-by-case verification in Section 5.

1.4. Structure of the paper

In Section 2 we present known results, mostly from [1] and [3], in the form we need them. In Section 3, which may be of independent interest, we formulate a criterion about non-extension of invariant sections of the normal sheaf. In Section 4 we review the known classification of spherical modules [17, 2, 21] as presented in [19] and reduce the proof of Theorem 1.1 to a case-by-case verification. We perform this case-by-case analysis in Section 5, using results from [3] mentioned in Section 2 and, for the most involved cases, also the exclusion criterion of Section 3.

2. From the literature

In this section we gather known results, mostly from [1] and [3], together with immediate consequences relevant to our purposes. In particular we explain that to prove Theorem 1.1 it is sufficient to show that $\mathrm{M}_{\mathcal{S}}$ is smooth when $\mathcal{S}$ is the weight monoid of a spherical module W for G of type A. Indeed, [1, Corollary 2.14] then implies Theorem 1.1 (see Corollary 2.6). That result of Alexeev and Brion's also tells us that $\dim \mathrm{M}_{\mathcal{S}} \geqslant d_W$. Moreover, by [1, Theorem 2.7], we only have to prove smoothness at a specific point X_0 of $\mathrm{M}_{\mathcal{S}}$ (see Corollary 2.4), and for that it is enough to show that

$$\dim T_{X_0}\mathrm{M}_{\mathcal{S}} \leqslant d_W. \tag{2.1}$$

Here is an overview of the content of this section. In Sections 2.1 and 2.2 we recall known facts (mostly from [1]) about the moduli scheme $\mathrm{M}_{\mathcal{S}}$ when $\mathcal{S}$ is a freely generated submonoid of Λ^+ and apply them to the case where $\mathcal{S}$ is the weight monoid Λ^+_W of a spherical G-module W. More specifically, in Section 2.1 we identify $\mathrm{M}_{\mathcal{S}}$ with a certain open subscheme of an invariant Hilbert scheme $\mathrm{Hilb}^G_{\mathcal{S}}(V)$, where V is a specific finite-dimensional G-module determined by $\mathcal{S}$. Under this identification, the point X_0 of $\mathrm{M}_{\mathcal{S}}$ corresponds to a certain G-stable subvariety of V, which we also denote X_0. Moreover, X_0 is the closure of the G-orbit of a certain point $x_0 \in V$. We then have that

$$T_{X_0}\mathrm{M}_{\mathcal{S}} \simeq H^0(X_0, \mathcal{N}_{X_0})^G \hookrightarrow H^0(G \cdot x_0, \mathcal{N}_{X_0})^G \simeq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}},$$

where $\mathcal{N}_{X_0}$ is the normal sheaf of X_0 in V. In addition, following [1] we introduce an action of T_{ad} on $\mathrm{M}_{\mathcal{S}}$. In Section 2.2 we give some more details about the inclusion $H^0(X_0, \mathcal{N}_{X_0})^G \hookrightarrow (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ which will be of use in Section 3 and in the case-by-case analysis of Section 5. In Section 2.3 we collect some elementary technical lemmas on $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ and the T_{ad}-action. Finally, in Section 2.4 we recall some results from [3] about $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

2.1. Embedding of $\mathrm{M}_{\mathcal{S}}$ into an invariant Hilbert scheme and the T_{ad}-action

Here we recall, from [1], that if $\mathcal{S}$ is a freely generated submonoid of Λ^+, then $\mathrm{M}_{\mathcal{S}}$ can be identified with an open subscheme of a certain invariant Hilbert scheme $\mathrm{Hilb}^G_{\mathcal{S}}(V)$. We also review the T_{ad}-action on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ defined in [1], its relation to the natural action of $\mathrm{GL}(V)^G$ on that Hilbert scheme and how it allows us to reduce the question of the smoothness of $\mathrm{M}_{\mathcal{S}}$ to the question whether $\mathrm{M}_{\mathcal{S}}$ is smooth at a specific point X_0.

Like the results in Sections 2.2, 2.3 and 3 everything in this section applies to any $\mathrm{M}_{\mathcal{S}}$ with $\mathcal{S}$ freely generated. In particular, by the following well-known proposition it applies to $\mathrm{M}_{\mathcal{S}}$ with $\mathcal{S} = \Lambda^+_W$ when (G, W) is a spherical module. For a proof, see [19, Theorem 3.2].

Proposition 2.1. — *The weight monoid of a spherical module is freely generated; that is, it is generated by a set of linearly independent dominant weights.*

For the following, we fix a freely generated submonoid $\mathcal{S}$ of Λ^+ and let E^* be its (unique) basis. Put $E = \{\lambda^* \mid \lambda \in E^*\}$ and

$$V = \oplus_{\lambda\in E} V(\lambda).$$

Alexeev and Brion [1] introduced the invariant Hilbert scheme $\mathrm{Hilb}^G_{\mathcal{S}}(V)$, which parametrizes all multiplicity-free closed G-stable subschemes X of V with weight set $\mathcal{S}$ (they actually introduced the invariant Hilbert scheme in a more general setting; for more information on this object, see the survey [8]). They also defined an action of T_{ad} on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$, see [1, Section 2.1], which we call γ and now briefly review. It is obtained by lifting the natural action of $GL(V)^G$ on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ to T. First, define the following homomorphism:

$$h\colon T \to \mathrm{GL}(V)^G, \quad t \mapsto (-\lambda^*(t))_{\lambda\in E}. \tag{2.2}$$

Composing the natural action of $\mathrm{GL}(V)^G$ on V with h yields an action ϕ of T on V:

$$\phi(t, v) = h(t) \cdot v \quad \text{for } t \in T \text{ and } v \in V.$$

Note that ϕ is a linear action on V and that each G-isotypical component $V(\lambda^*)$ of V^* (with $\lambda \in E$) is the T-weight space for ϕ of weight λ^*. Since $\mathrm{GL}(V)^G$ acts naturally on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$, ϕ induces an action of T on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$. We call this last action γ. It has $Z(G)$ in its kernel and so descends to an action of $T_{\mathrm{ad}} = T/Z(G)$ on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ which we also call γ. Indeed, if $\rho\colon G \to \mathrm{GL}(V)$ is the (linear) action of G on V, then for every $z \in Z(G)$,

$\rho(z) = h(z)$, because $-\lambda^* = w_0\lambda$ is the lowest weight of $V(\lambda)$ and therefore differs from all other weights in $V(\lambda)$ by an element of $\langle\Pi\rangle_{\mathbb{N}}$. This implies that if I is a G-stable ideal in $\mathbb{C}[V]$, then $h(z) \cdot I = \rho(z) \cdot I = I$. More generally, if S is a scheme with trivial G-action and $\mathcal{I}$ is a G-stable ideal sheaf on $V \times S$, then $\mathcal{I}$ is also stable under the action induced by h on the structure sheaf $\mathcal{O}_{V\times S}$, since $\rho|_{Z(G)} = h|_{Z(G)}$. Because $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ represents the functor (Schemes) $\to$ (Sets) that associates to a scheme S the set of flat families $Z \subseteq V \times S$ with invariant Hilbert function the characteristic function of $\mathcal{S} \subseteq \Lambda^+$ (see [1, Section 1.2] or [8, Section 2.4]), this implies our claim.

From [1, Corollary 1.17] we know that the open subscheme $\mathrm{Hilb}^G_{E^*}$ of $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ that classifies the (irreducible) non-degenerate subvarieties $X \subseteq V$ with $\Lambda^+_X = \mathcal{S}$ is stable under $\mathrm{GL}(V)^G$ and therefore under the T_{ad}-action γ. Recall from [1, Definition 1.14] that a closed G-stable subvariety of V is called *non-degenerate* if its projections to the simple components $V(\lambda)$ of V, where $\lambda \in E$, are all nonzero. We call a closed G-stable subvariety of V *degenerate* if it is not non-degenerate.

Next suppose $Y = \operatorname{Spec}\mathbb{C}[\mathcal{S}]$, the multiplicity-free T-variety with weight monoid $\mathcal{S}$. Recall that $\mathrm{M}_{\mathcal{S}} = \mathrm{M}_Y$ classifies (equivalence classes of) pairs (X, φ) where X is an affine G-variety and $\varphi\colon X/\!\!/U \to Y$ is a T-equivariant isomorphism. The action of T on Y through $T \to \mathrm{Aut}^T(Y)$ induces an action of T on $\mathrm{M}_{\mathcal{S}}$. From [1, Lemma 2.2] we know that this action descends to an action of T_{ad} on $\mathrm{M}_{\mathcal{S}}$. By Corollary 1.17 and Lemma 2.2 in [1] the moduli scheme $\mathrm{M}_{\mathcal{S}}$ is T_{ad}-equivariantly isomorphic to $\mathrm{Hilb}^G_{E^*}$, where the T_{ad}-action on $\mathrm{Hilb}^G_{E^*}$ is γ. **From now on**, we will identify $\mathrm{M}_{\mathcal{S}}$ with $\mathrm{Hilb}^G_{E^*}$. As in [3], the T_{ad}-action it carries will play a fundamental role in what follows.

Remark 2.2.

(a) Let (G, W) be a spherical module with weight monoid $\mathcal{S}$, put $Y = W/\!\!/U$ and let $\pi\colon W/\!\!/U \to Y$ be the identity map. Then (W, π) corresponds to a closed point of $\mathrm{M}_Y = \mathrm{M}_{\mathcal{S}} = \mathrm{Hilb}^G_{E^*} \subseteq \mathrm{Hilb}^G_{\mathcal{S}}(V)$. On the other hand, note that the highest weights of W belong to E. Put $E_1 = \{\lambda \in \Lambda^+ : W_{(\lambda)} \neq 0\} \subseteq E$ and $E_2 = E \smallsetminus E_1$. Then

$$\begin{aligned} V = \oplus_{\lambda\in E} V(\lambda) &= [\oplus_{\lambda\in E_1} V(\lambda)] \oplus [\oplus_{\lambda\in E_2} V(\lambda)] \\ &\simeq W \oplus [\oplus_{\lambda\in E_2} V(\lambda)]. \end{aligned}$$

Identifying W with $\oplus_{\lambda\in E_1} V(\lambda) \subseteq V$ we see that W corresponds to a closed point of $\mathrm{Hilb}^G_{\mathcal{S}}(V)$. As soon as $E_2 \neq \emptyset$, $W \subseteq V$ is a degenerate subvariety of V, that is, it corresponds to a closed point of $\mathrm{Hilb}^G_{\mathcal{S}}(V) \smallsetminus \mathrm{Hilb}^G_{E^*}$.

(b) The subvariety of V corresponding to the closed point (W,π) of $\mathrm{M}_{\mathcal{S}} = \mathrm{Hilb}^G_{E^*} \subseteq \mathrm{Hilb}^G_{\mathcal{S}}(V)$ can be described as follows. Let $\mathrm{Mor}^G(W, V(\lambda))$ be the set of G-equivariant morphisms of algebraic varieties $W \to V(\lambda)$. We consider $\mathrm{Mor}^G(W, V(\lambda))$ with vector space structure induced from the one of $V(\lambda)$. Note that, by Schur's lemma and because W is spherical,

$$\mathrm{Mor}^G(W, V(\lambda)) \simeq (\mathbb{C}[W] \otimes_{\mathbb{C}} V(\lambda))^G \simeq (V(\lambda^*) \otimes V(\lambda))^G$$

is one-dimensional for every dominant weight λ with $\lambda^* \in \mathcal{S}$. After choosing, for every $\lambda \in E_2$, a nonzero $f_\lambda \in \mathrm{Mor}^G(W, V(\lambda))$, we can define the following G-equivariant closed embedding of W into V:

$$\varphi\colon W \to V,\ w \mapsto w + \big(\oplus_{\lambda \in E_2} f_\lambda(w)\big).$$

Its image corresponds to a closed point of $\mathrm{Hilb}^G_{E^*}$. An appropriate choice of the functions f_λ (which depends on the identification $\mathrm{M}_{\mathcal{S}} = \mathrm{Hilb}^G_{E^*}$) yields the closed point of $\mathrm{Hilb}^G_{E^*}$ corresponding to (W,π).

The next proposition, taken from [1, Theorem 2.7], means we can verify the smoothness of $\mathrm{M}_{\mathcal{S}}$ at just one of its points. It also implies that $\mathrm{M}_{\mathcal{S}}$ is connected.

Proposition 2.3. — *The affine scheme $\mathrm{M}_{\mathcal{S}}$ has a unique T_{ad}-fixed point X_0, which is also its only closed orbit.*

Using the well-known fact that every orbit closure contains a closed orbit, we have the following corollary.

Corollary 2.4. — *$\mathrm{M}_{\mathcal{S}}$ is smooth if and only if it is smooth at X_0.*

Under the identification of $\mathrm{M}_{\mathcal{S}}$ with $\mathrm{Hilb}^G_{E^*}$ the distinguished point X_0 of $\mathrm{M}_{\mathcal{S}}$ corresponds to a certain subvariety of V, which we also denote X_0 (see [1, p. 99]). It is the closure of the G-orbit in V of

$$x_0 := \sum_{\lambda \in E} v_\lambda \in \oplus_{\lambda \in E} V(\lambda) = V.$$

Indeed this orbit closure has the right weight monoid by [29, Theorem 6] and is fixed under the action of $\mathrm{GL}(V)^G$. Yet another result of Alexeev and Brion's gives us an a priori lower bound on the dimension of the moduli schemes we are considering. We first recall a result of F. Knop. Suppose X

is an affine G-variety. Let $\widetilde{\Sigma}_X$ be the saturated monoid generated by Σ_X, that is

$$\widetilde{\Sigma}_X := \mathbb{Q}_{\geqslant 0}\Sigma_X \cap \langle \Sigma_X \rangle_{\mathbb{Z}} \subseteq X(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then by [18, Theorem 1.3] the monoid $\widetilde{\Sigma}_X$ is free. In the following proposition we apply some standard facts about (not necessarily normal) toric varieties.

Proposition 2.5 (Cor 2.9, Prop 2.13 and Cor 2.14 in [1]). — *Suppose X is a spherical affine G-variety. We view X as a closed point of $\mathrm{M}_{\Lambda_X^+}$.*

1. *The weight monoid of the closure of the T_{ad}-orbit of X in $\mathrm{M}_{\Lambda_X^+}$ is Σ_X. Consequently $\dim \mathrm{M}_{\Lambda_X^+} \geqslant d_X$.*
2. *The normalization of the T_{ad}-orbit closure of X in $\mathrm{M}_{\Lambda_X^+}$ has weight monoid $\widetilde{\Sigma}_X$. Consequently, it is T_{ad}-equivariantly isomorphic to a multiplicity-free T_{ad}-module of dimension d_X.*
3. *Suppose X is a smooth variety. Then its T_{ad}-orbit is open in $\mathrm{M}_{\Lambda_X^+}$ and, consequently, $\mathrm{M}_{\Lambda_X^+}$ is smooth if and only if $\dim T_{X_0}\mathrm{M}_{\Lambda_X^+} \leqslant d_X$.*

Applying this proposition to our situation we immediately obtain the following corollary. It reduces the proof of Theorem 1.1 to Corollary 4.17 and Theorem 1.2.

Corollary 2.6. — *Let W be a spherical G-module and let $\mathcal{S}$ be its weight monoid. Then the following are equivalent*

1. *$\mathrm{M}_{\mathcal{S}}$ is smooth;*
2. *$\dim T_{X_0}\mathrm{M}_{\mathcal{S}} = d_W$;*
3. *$\dim T_{X_0}\mathrm{M}_{\mathcal{S}} \leqslant d_W$.*

Moreover, if $\mathrm{M}_{\mathcal{S}}$ is smooth then $\Sigma_W = \widetilde{\Sigma}_W$ and $\mathrm{M}_{\mathcal{S}}$ is T_{ad}-equivariantly isomorphic to the multiplicity-free T_{ad}-module with T_{ad}-weight set $-\Psi_W$, where Ψ_W is the (unique) basis of Σ_W.

The following formula for d_W, which is an immediate consequence of [9, Lemme 5.3], will be of use. For the convenience of the reader, we provide a proof suggested by the referee.

Lemma 2.7. — *If W is a spherical G-module, then $d_W = a - b$, where a is the rank of the (free) abelian group Λ_W and b is the number of summands in the decomposition of W into simple G-modules.*

Proof. — Let G/H be the open orbit in W. From [6, Théorème 4.3], we have that $d_W = \operatorname{rk}\Lambda_W - \dim N_G(H)/H$. Since $N_G(H)/H$ is isomorphic to the group of G-equivariant automorphisms $\mathrm{Aut}^G(G/H)$ of G/H, we obtain by [23, Lemma 3.1.2] that $N_G(H)/H \simeq \mathrm{Aut}^G(W)$. Moreover, $\mathrm{Aut}^G(W) =$

$\mathrm{GL}(W)^G$, because a G-automorphism of the multiplicity-free G-algebra $\mathbb{C}[W]$ preserves all irreducible submodules of $\mathbb{C}[W]$ and therefore sends W^* to W^*. Since $\dim \mathrm{GL}(W)^G = b$, it follows that $\dim N_G(H)/H = b$, which finishes the proof. □

Remark 2.8. — In [19] Knop computed the simple reflections of the so-called "little Weyl group" of W^*, whenever W is a saturated indecomposable spherical module. This entry in Knop's List is equivalent to giving the basis of the free monoid $\widetilde{\Sigma}_{W^*} = -w_0\widetilde{\Sigma}_W$: that basis is the set of simple roots of a certain root system of which the "little Weyl group" is the Weyl group (see [18, Section 1], [22, Section 3] or [4, Appendix A] for details). Knop's List also contains the basis of $\Lambda^+_{W^*} = -w_0\Lambda^+_W$ for the same modules W. Those were computed in [14] and [21].

Here now is a proposition which provides a concrete description of the tangent space $T_{X_0}\mathrm{M}_{\mathcal{S}}$.

PROPOSITION 2.9 ([1], Proposition 1.13). — *Let V be a finite dimensional G-module and suppose X is a multiplicity-free closed G-subvariety of V. Also writing X for the corresponding closed point in $\mathrm{Hilb}^G_{\Lambda^+_X}(V)$, we have that the Zariski tangent space $T_X\mathrm{Hilb}^G_{\Lambda^+_X}(V)$ is canonically isomorphic to $H^0(X,\mathcal{N}_X)^G$, where $\mathcal{N}_X$ is the normal sheaf of X in V.*

2.2. $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ as a first estimate of $T_{X_0}\mathrm{M}_{\mathcal{S}}$

In this section we describe a natural inclusion of $T_{X_0}\mathrm{M}_{\mathcal{S}}$ into $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, see Corollary 2.14. For calculational purposes we introduce a second T_{ad}-action on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ denoted $\widehat{\psi}$, which is a twist of the action γ defined in Section 2.1, and also the infinitesimal version of $\widehat{\psi}$ on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ denoted α. The action α is the one used throughout [3]. The main ideas of this section come from the proof of [1, Proposition 1.15]. We continue to use the notation of Section 2.1.

Because $G\cdot x_0$ is dense in X_0, we have an injective restriction map

$$H^0(X_0,\mathcal{N}_{X_0}) \hookrightarrow H^0(G\cdot x_0,\mathcal{N}_{X_0}) = H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0}),$$

where $\mathcal{N}_{G\cdot x_0}$ is defined as the restriction of $\mathcal{N}_{X_0}$ to the open subset $G\cdot x_0 \subseteq X_0$. This map is $G\times \mathrm{GL}(V)^G$-equivariant because X_0 and $G\cdot x_0$ are stable under the natural action of $G\times\mathrm{GL}(V)^G$ on V. Restricting to G-invariants we obtain a $\mathrm{GL}(V)^G$-equivariant inclusion

$$H^0(X_0,\mathcal{N}_{X_0})^G \hookrightarrow H^0(G\cdot x_0,\mathcal{N}_{X_0})^G = H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0})^G. \tag{2.3}$$

Since $G\cdot x_0$ is homogeneous, $\mathcal{N}_{G\cdot x_0}$ is the G-linearized sheaf on G/G_{x_0} associated with the G_{x_0}-module $V/\mathfrak{g}\cdot x_0$, that is, the vector bundle associated to $\mathcal{N}_{G\cdot x_0}$ is G-equivariantly isomorphic to $G\times_{G_{x_0}}(V/\mathfrak{g}\cdot x_0)$. In particular, we have a canonical isomorphism

$$H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0})^G \to (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}, \quad s\mapsto s(x_0) \tag{2.4}$$

which is the precise way of saying that G-invariant global sections of $\mathcal{N}_{G\cdot x_0}$ are determined by their value at x_0.

The T-action ϕ on V defined in Section 2.1 induces an action on $H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0})^G$ and we could use the isomorphism (2.4) to induce an action on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. Because it is better suited to our calculations, we prefer to work with a slightly different action. Recall that ϕ was obtained by composing the natural action of $\mathrm{GL}(V)^G$ with the homomorphism h of (2.2). Instead, we obtain a T-action, denoted ψ, on V by composing the action of $\mathrm{GL}(V)^G$ with the homomorphism

$$f\colon T\to \mathrm{GL}(V)^G,\ t\mapsto (\lambda(t))_{\lambda\in E}. \tag{2.5}$$

In other words, ψ is the following action:

$$\psi\colon T\times V\to V,\ \psi(t,v)=f(t)\cdot v.$$

Remark 2.10. — We note that since the elements of E are linearly independent, f is surjective.

Since ψ commutes with the action of G on V, it induces an action of T on $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ and on $H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0})^G$. By slight abuse of notation we call both of these actions $\widehat{\psi}$. Using the isomorphism of equation (2.4) we now translate this action into an action of T on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. The relationship (via f) between the action of T on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ and the action of $\mathrm{GL}(V)^G$ on $H^0(G\cdot x_0,\mathcal{N}_{G\cdot x_0})^G \simeq (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ will play a part in the proof of Proposition 3.4. Let $\rho\colon T\times V\to V$ be the action of T on V induced by restriction of the action of G.

Definition 2.11. — *We denote α the action of T on V given by*

$$\alpha(t,v):=\psi(t,\rho(t^{-1},v))\quad \text{for } t\in T \text{ and } v\in V.$$

Remark 2.12. — One immediately checks that for all $\lambda\in E$ and every $v\in V(\lambda)\subseteq V$,

$$\alpha(t,v)=\lambda(t)t^{-1}v. \tag{2.6}$$

Proposition 2.13. — *The action α induces an action of T on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, which we also call α. For $H^0(G\cdot x_0,\mathcal{N}_{X_0})^G$ equipped with the action $\widehat{\psi}$ and $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ with the action α, the isomorphism (2.4) is*

T-equivariant. Moreover, both actions α and $\widehat{\psi}$ have $Z(G)$ in their kernel, whence the isomorphism (2.4) is T_{ad}-equivariant.

Proof. — In this proof, we will write $\mathcal{N}$ for $\mathcal{N}_{X_0}$. Suppose $t \in T$ and $s \in H^0(G\cdot x_0, \mathcal{N}_{G\cdot x_0})^G$. Then

$$\begin{aligned}\widehat{\psi}(t,s)(x_0) &= (f(t)\cdot s)(x_0) = f(t)\cdot s(f(t)^{-1}\cdot x_0)\\ &= f(t)\cdot s(\psi(t^{-1},x_0)).\end{aligned}$$

Now note that $\psi(t^{-1},x_0) = \rho(t^{-1},x_0)$ by the definitions of f and x_0. In other words, we have that $\widehat{\psi}(t,s)(x_0) = f(t)\cdot s(\rho(t^{-1},x_0))$. Let v be an element of V such that $s(x_0) = [v] \in \mathcal{N}|_{x_0} = V/\mathfrak{g}\cdot x_0$. Then $s(\psi(t^{-1},x_0)) = [\rho(t^{-1},v)] \in \mathcal{N}|_{\rho(t^{-1},x_0)}$ because s is G-invariant and therefore T-invariant. It follows that

$$f(t)\cdot s(\rho(t^{-1},x_0)) = [\psi(t,\rho(t^{-1},v))] = [\alpha(t,v)] \in V/\mathfrak{g}\cdot x_0. \tag{2.7}$$

This shows that α induces a well-defined action on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. From (2.7) we can also conclude that the isomorphism (2.4) is T-equivariant. Because $f(z) = h(z)$ for all $z \in Z(G)$, where h is the homomorphism (2.2), $Z(G)$ is contained in the kernel of $\widehat{\psi}$ (see page 1773) and of α. □

From now on, the T_{ad}-*action on* V (and on $V/\mathfrak{g}\cdot x_0$, $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, *etc.*) will refer to the action given by α, and the T_{ad}-*action on* $\mathrm{Hilb}^G_{\mathcal{S}}(V)$ (and on $\mathrm{M}_{\mathcal{S}}$) will refer to the action given by $\widehat{\psi}$. Combining Proposition 2.9 and equations (2.3) and (2.4) we obtain a natural injection $T_{X_0}\mathrm{M}_{\mathcal{S}} \hookrightarrow (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$.

COROLLARY 2.14. — *The natural injection $T_{X_0}\mathrm{M}_{\mathcal{S}} \hookrightarrow (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ just defined is T_{ad}-equivariant, where we consider $T_{X_0}\mathrm{M}_{\mathcal{S}}$ as a T_{ad}-module via $\widehat{\psi}$ and $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ via α.*

Remark 2.15. — Thanks to [1, Proposition 1.15 (iii)] and Lemma 3.2 below, we know that the injection in Corollary 2.14 is an isomorphism when $X_0 \setminus G\cdot x_0$ has codimension at least 2 in X_0. This condition is often not met in our situation. Even when it is not, the injection is often an isomorphism, but we also have a number of cases where the injection is not surjective; see, for example, Remark 5.20.

2.3. Auxiliary lemmas on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ and the T_{ad}-action

We continue to use the notation of Sections 2.1 and 2.2. Let $G \rtimes T_{\mathrm{ad}}$ be the semidirect product of G and T_{ad}, where T_{ad} acts on G as follows:

$$T_{\mathrm{ad}} \times G \to G, (t,g) \mapsto t^{-1}gt. \tag{2.8}$$

As explained in [1, p. 102], the linear actions of T_{ad} and G on V can be extended together to a linear action of $G \rtimes T_{\mathrm{ad}}$ on V as follows. Suppose $(g,t) \in G \rtimes T_{\mathrm{ad}}$ and $v \in V$, then

$$(g,t)\cdot v := g\cdot \alpha(t,v) = \alpha(t,(tgt^{-1})\cdot v), \tag{2.9}$$

where α is the T_{ad}-action. Since T_{ad} fixes x_0, we have that $(G \rtimes T_{\mathrm{ad}})_{x_0} = G_{x_0} \rtimes T_{\mathrm{ad}}$ and $(G \rtimes T_{\mathrm{ad}})\cdot x_0 = G\cdot x_0$. It follows that $(G \rtimes T_{\mathrm{ad}})_{x_0}$ acts on $\mathfrak{g}\cdot x_0 = T_{x_0}(G\cdot x_0)$ and we have an exact sequence of $(G_{x_0} \rtimes T_{\mathrm{ad}})$-modules

$$0 \longrightarrow \mathfrak{g}\cdot x_0 \longrightarrow V \longrightarrow V/\mathfrak{g}\cdot x_0 \longrightarrow 0. \tag{2.10}$$

The next lemma gathers some elementary facts about G_{x_0} and $\mathfrak{g}\cdot x_0$. They will be of use in Sections 3 and 5.

Lemma 2.16. — *Let E be a finite subset of Λ^+, and define V and x_0 as before, that is, $x_0 := \sum_{\lambda\in E} v_\lambda \in V := \oplus_{\lambda\in E} V(\lambda)$. Then the following hold:*

1. *$G_{x_0} = T_{x_0}.G^\circ_{x_0}$, where $G^\circ_{x_0}$ is the connected component of G_{x_0} containing the identity;*
2. *$T_{x_0} = \cap_{\lambda\in E}\ker\lambda$;*
3. *$\mathfrak{g}_{x_0} = \mathfrak{u}\oplus\mathfrak{t}_{x_0}\oplus\bigoplus_{\alpha\in E^\perp}\mathfrak{g}_{-\alpha}$, where $E^\perp := \{\alpha \in R^+ \mid \langle\lambda,\alpha^\vee\rangle = 0$ for all $\lambda\in E\}$;*
4. *The T_{ad}-weight set of $\mathfrak{g}\cdot x_0$ is $(R^+ \smallsetminus E^\perp)\cup\{0\}$.*

Proof. — The proof of (1) just requires replacing v_λ by x_0 in the proof of [16, Lemme 1.7]. (2) is immediate. (3) follows form the well-known properties of the action of root operators on highest weight vectors. For (4) just note that $\mathfrak{g}\cdot x_0 = \mathfrak{b}^-\cdot x_0$, where $\mathfrak{b}^-$ is the Lie algebra of the Borel subgroup B^- opposite to B with respect to T. □

In addition to the facts listed in Lemma 2.16, the following will be useful too in Section 5. Recall our convention that $G'_{x_0} := (G')_{x_0}$ and $\mathfrak{g}'_{x_0} := (\mathfrak{g}')_{x_0}$. Recall also that if $\mathfrak{k}$ is a Lie-subalgebra of $\mathfrak{g}_{x_0}$, then $(V/\mathfrak{g}\cdot x_0)^{\mathfrak{k}} = \{[v] \in V/\mathfrak{g}\cdot x_0 \mid Xv \in \mathfrak{g}\cdot x_0$ for all $X\in\mathfrak{k}\}$, by definition.

Lemma 2.17. — *Using the notations of this section, the following hold:*

(a) *The inclusions $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}} \subseteq (V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} \subseteq (V/\mathfrak{g}\cdot x_0)^{\mathfrak{g}'_{x_0}}$ are inclusions of T_{ad}-modules;*

(b) *Let H be a subgroup of G and let T_H be a subtorus of $T\cap H$. Let Γ be the subgroup of $X(T_H)$ generated by the image of E under the restriction map $p\colon X(T) \twoheadrightarrow X(T_H)$. Suppose $v\in V$ is a T_{ad}-eigenvector of weight β so that $[v]$ is a nonzero element of $(V/\mathfrak{g}\cdot x_0)^{H_{x_0}}$. Then $p(\beta)$ belongs to Γ;*

(c) *If* $\mathfrak{h}$ *is a Lie-subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{g}'$*, then*

$$(V/\mathfrak{g}x_0)^{G_{x_0}} = (V/\mathfrak{g}x_0)^{\mathfrak{h}_{x_0}}_{\langle E\rangle},$$

where $(V/\mathfrak{g}x_0)^{\mathfrak{h}_{x_0}}_{\langle E\rangle}$ *is the subspace of* $(V/\mathfrak{g}x_0)^{\mathfrak{h}_{x_0}}$ *spanned by* $\Big\{[v] \in (V/\mathfrak{g}x_0)^{\mathfrak{h}_{x_0}} \mid v$ *is a* T_{ad}*-eigenvector with weight in* $\langle E\rangle_{\mathbb{Z}}\Big\}$.

Proof. — For assertion (a) we first note that the subgroups G'_{x_0} and $(G'_{x_0})^\circ$ of G are stable under the action of T_{ad} on G in (2.8), so that the $(G_{x_0} \rtimes T_{\mathrm{ad}})$-action on $V/\mathfrak{g}\cdot x_0$ restricts to $G'_{x_0} \rtimes T_{\mathrm{ad}}$ and $(G'_{x_0})^\circ \rtimes T_{\mathrm{ad}}$. The assertion now follows since $\mathrm{Lie}(G'_{x_0}) = \mathfrak{g}'_{x_0}$. We now prove (b). Let β be the T_{ad}-weight of v and for every $\lambda \in E$, let x_λ be the projection of v onto $V(\lambda) \subseteq V$. Then $v = \sum_{\lambda\in E} x_\lambda$. Since v is nonzero, at least one of the x_λ is nonzero. Choose one. Then x_λ is a T-eigenvector of weight $\lambda-\beta$. Since v is fixed by $(T_H)_{x_0}$ it follows that x_λ is and so $(\lambda-\beta)|_{(T_H)_{x_0}} = 0$. Since $(T_H)_{x_0} = \cap_{\lambda\in E}\ker p(\lambda)$ this implies that $p(\lambda-\beta)$ and therefore $p(\beta)$ lie in Γ. Assertion (c), finally, is a consequence of parts (1) and (2) of Lemma 2.16. □

Lemma 2.18. — *We use the notations of this section. Let* $v \in V$ *be a* T_{ad}*-eigenvector. If* $[v]$ *is a nonzero element of* $(V/\mathfrak{g}x_0)^{\mathfrak{g}'_{x_0}}$*, then the following two statements hold.*

(A) *For every positive root* α *one of the following situations occurs*
   (1) $X_\alpha v = 0$*;*
   (2) $X_\alpha v$ *is a* T_{ad}*-eigenvector of weight* 0*;*
   (3) $X_\alpha v$ *is a* T_{ad}*-eigenvector with weight in* $R^+ \smallsetminus E^\perp$*;*

(B) *There is at least one simple root* α *such that* $X_\alpha v \neq 0$.

Proof. — Part (A) follows from the fact that $\mathfrak{u} \subseteq \mathfrak{g}'_{x_0}$ and part (4) of Lemma 2.16. For (B) first note that the linear independence of E implies that the subspace $\mathfrak{t}\cdot x_0$ of $\mathfrak{g}\cdot x_0$ contains all the highest weight vectors of V. Therefore $[v] \neq 0$ implies that v is not a sum of highest weight vectors. □

Lemma 2.19. — *Let* $(\overline{G}, W)$ *be a spherical* $\overline{G}$*-module and let* G *be a reductive subgroup of* $\overline{G}$ *containing* $\overline{G}'$ *and such that* (G, W) *is spherical. Then* $\mathfrak{g}\cdot x_0 = \overline{\mathfrak{g}}\cdot x_0$.

Proof. — We have that $\mathfrak{g}\cdot x_0 = \mathfrak{t}\cdot x_0 + \mathfrak{g}'\cdot x_0$. By hypothesis, $\mathfrak{g}' = \overline{\mathfrak{g}}'$. Finally $\mathfrak{t}\cdot x_0 = \langle v_\lambda \colon \lambda \in E\rangle_{\mathbb{C}} = \overline{\mathfrak{t}}\cdot x_0$ because the elements of E are linearly independent (for both G and $\overline{G}$). □

2.4. Further results and notions from [3]

We continue to use the notation of Sections 2.1 and 2.2. In this section we recall results from [3] about $\mathrm{M}_{\mathcal{S}}$ and $T_{X_0}\mathrm{M}_{\mathcal{S}}$ under the condition that $\mathcal{S}$ is G-saturated (see Definition 2.20), and we mention some immediate consequences.

The following condition on submonoids of Λ^+ was considered by D. Panyushev in [25]. It also occurs in [29]. We will use the terminology of [8, Section 4.5].

Definition 2.20. — *A submonoid $\mathcal{S}$ of Λ^+ is called G-*saturated *if $\langle\mathcal{S}\rangle_{\mathbb{Z}} \cap \Lambda^+ = \mathcal{S}$.*

Remark 2.21. — As explained in [3, Section 3] the injection $T_{X_0}\mathrm{M}_{\mathcal{S}} \hookrightarrow (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ of Corollary 2.14 is an isomorphism when $\mathcal{S}$ is G-saturated. The reason is that, by Theorem 9 of [29], $X_0 \smallsetminus G\cdot x_0$ then has codimension at least 2 in X_0, which is a normal variety (cf. Lemma 3.2); see also Remark 2.15.

Remark 2.22. — Clearly, a submonoid $\mathcal{S} \subseteq \Lambda^+$ is G-saturated if and only if $-w_0(\mathcal{S})$ is. This fact will be used in Section 5, because if $\mathcal{S}$ is the weight monoid of a spherical module (G, W), then $-w_0(\mathcal{S})$ is the weight monoid of the dual module (G, W^*).

Lemma 2.23 (Lemma 2.1 in [3]). — *Let $\lambda_1, \ldots, \lambda_k$ be linearly independent dominant weights. The following are equivalent:*

(a) *$\mathcal{S} = \langle\lambda_1, \ldots, \lambda_k\rangle_{\mathbb{N}}$ is G-saturated;*

(b) *there exist k simple roots $\alpha_{t_1}, .., \alpha_{t_k}$ such that $\langle\lambda_i, \alpha_{t_j}^\vee\rangle \neq 0$ if and only if $i = j$.*

Theorem 2.24 (Theorem 2.2 and Corollary 2.4 in [3]). — *Suppose G is a semisimple group and $\mathcal{S}$ is a G-saturated and freely generated submonoid of Λ^+. Then*

(1) *the tangent space $T_{X_0}\mathrm{M}_{\mathcal{S}}^G \simeq (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ is a multiplicity-free T_{ad}-module whose T_{ad}-weights belong to Table 1 of* [3, p. 2810]*;*

(2) *the moduli scheme $\mathrm{M}_{\mathcal{S}}^G$ is isomorphic as a T_{ad}-scheme to $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$.*

Remark 2.25. — When G is of type A, the T_{ad}-weights which can occur in the space $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ of Theorem 2.24 are (see [3, Table 1, p. 2810]):

(SR1) $\alpha + \alpha'$ with $\alpha, \alpha' \in \Pi$ and $\alpha \perp \alpha'$;

(SR2) 2α with $\alpha \in \Pi$;

(SR3) $\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_{i+r}$ with $r \geqslant 2$ and $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+r}$ simple roots that correspond to consecutive vertices in a connected component of the Dynkin diagram of G;

(SR4) $\alpha_i + 2\alpha_{i+1} + \alpha_{i+2}$ with $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ simple roots that correspond to consecutive vertices in a connected component of the Dynkin diagram of G.

For several cases in Knop's List, Theorem 1.2 is a consequence of Bravi and Cupit-Foutou's result mentioned above, thanks to Corollary 2.27 below. We first establish a lemma needed in the proof of Corollary 2.27 and of Proposition 4.11.

Lemma 2.26. — *Suppose X is an affine G-variety and let H be a connected subgroup of G containing G'. Let B_H be the Borel subgroup $B \cap H$ of H and let $p\colon X(B) \twoheadrightarrow X(B_H)$ be the restriction map. Let Σ_X be the root monoid of the G-variety X and let Σ'_X be the root monoid of X considered as an H-variety (where H acts as a subgroup of G). If the restriction of p to $\Lambda_{(G,X)} \subseteq X(B)$ is injective, then $\Sigma'_X = p(\Sigma_X)$. Consequently, the invariant d_X is the same for (G, X) as for (H, X).*

Proof. — By Lemma 4.6 below, $p(\Lambda^+_{(G,X)}) = \Lambda^+_{(H,X)}$. Put $R = \mathbb{C}[X]$ and let $R = \oplus_{\lambda \in \Lambda^+_{(G,X)}} R_{(\lambda)}$ be its decomposition into isotypical components as a G-module. Then, because $p|_{\Lambda^+_{(G,X)}}$ is injective and $G' \subseteq H$, we have that for every $\lambda \in \Lambda^+_{(G,X)}$, $R_{(\lambda)} \subseteq R$ is the H-isotypical component of R of type $V(p(\lambda))$. The lemma now follows from the definitions of Σ_X and d_X. □

Corollary 2.27. — *Let G be a connected reductive group and let X be a smooth affine spherical G-variety with weight monoid $\mathcal{S}$. Suppose X is spherical for the restriction of the G-action to G'. Put $T' = T \cap G'$. Let $\mathcal{S}'$ be the image*[1] *of $\mathcal{S}$ under the restriction map $p\colon X(T) \twoheadrightarrow X(T')$.*

If $\mathcal{S}'$ is freely generated then so is $\mathcal{S}$. Suppose $\mathcal{S}'$ is freely generated and G'-saturated. Then $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_X$ and, consequently, $\dim T_{X_0} \mathrm{M}^G_{\mathcal{S}} = d_X$.

Proof. — The fact that X is spherical for G' implies that the restriction of p to $\mathcal{S}$ is injective (see Lemma 4.6 below). This proves that $\mathcal{S}$ is freely generated when $\mathcal{S}'$ is.

We now assume that $\mathcal{S}'$ is freely generated and G'-saturated. First note that

$$V \simeq \oplus_{\lambda \in E} V(p(\lambda)) \tag{2.11}$$

[1] By Lemma 4.6 below, $\mathcal{S}'$ is the weight monoid of the G'-variety X.

as a G'-module and that, because the sets $E \subseteq X(T)$ and $p(E) \subseteq X(T')$ are linearly independent,

$$\mathfrak{g} \cdot x_0 = \mathfrak{t} \cdot x_0 + \mathfrak{u}^- \cdot x_0 = \mathfrak{t}' \cdot x_0 + \mathfrak{u}^- \cdot x_0 = \mathfrak{g}' \cdot x_0. \tag{2.12}$$

where $\mathfrak{u}^-$ is the sum of the negative root spaces of $\mathfrak{g}'$.

Now consider X as a closed point of $\mathrm{M}^{G'}_{\mathcal{S}'}$. By Theorem 2.24, $\mathrm{M}^{G'}_{\mathcal{S}'}$ is smooth, and so Proposition 2.5 (with Lemma 2.26) tells us that $\dim T_{X_0}\mathrm{M}^{G'}_{\mathcal{S}'} = d_X$. Since $T_{X_0}\mathrm{M}^{G'}_{\mathcal{S}'} \simeq (V/\mathfrak{g}' \cdot x_0)^{G'_{x_0}}$ (using (2.11)) and, since from (2.12) we have that $(V/\mathfrak{g} \cdot x_0) = (V/\mathfrak{g}' \cdot x_0)$ and therefore that $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = (V/\mathfrak{g}' \cdot x_0)^{G'_{x_0}}$, it follows that $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_X$. By Corollary 2.14, $T_{X_0}\mathrm{M}^{G}_{\mathcal{S}} \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}} \subseteq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$, and Proposition 2.5 now finishes the proof. □

3. Criterion for non-extension of sections

We continue to use the notation of Sections 2.1 and 2.2. In particular, by the T_{ad}-action on V and $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ we mean the action α of Definition 2.11. The criterion we give here (Proposition 3.4) for excluding certain T_{ad}-weight spaces of $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ from $T_{X_0}\mathrm{M}_{\mathcal{S}}$ was suggested to us by M. Brion. It consists of sufficient conditions on a section $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0})^G \simeq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ for it not to extend to X_0. The basic idea is that the conditions guarantee that there is a point $z_0 \in X_0$ (which depends on s) whose G-orbit has codimension 1 in X_0 and such that s does not extend to z_0 along the line joining x_0 and z_0.

Before we prove the criterion we recall some facts. We begin with the orbit structure of X_0. It is known (see [29, Theorem 8]) that the following map describes a one-to-one correspondence between the set of subsets of E and the set of G-orbits in X_0:

$$(D \subseteq E) \mapsto G \cdot v_D \quad \text{where } v_D := \sum_{\lambda \in D} v_\lambda.$$

Recall that $\mathrm{GL}(V)^G \simeq \mathbb{G}_m^{|E|}$ and that an element $(t_\lambda)_{\lambda \in E} \in \mathrm{GL}(V)^G$ acts on $V = \oplus_{\lambda \in E} V(\lambda)$ by scalar multiplication by $t_\lambda \in \mathbb{G}_m$ on the submodule $V(\lambda)$. Given $D \subseteq E$, define the one-parameter subgroup σ_D of $\mathrm{GL}(V)^G$ as follows:

$$\sigma_D \colon \mathbb{G}_m \to \mathrm{GL}(V)^G, t \mapsto (p_\lambda(t))_{\lambda \in E}$$

where $p_\lambda(t) = t$ if $\lambda \notin D$ and $p_\lambda(t) = 1$ otherwise. Then $\lim_{t \to 0} \sigma_D(t) \cdot x_0 = v_D$. We also put $z_t := \sigma_D(t) \cdot x_0$ for $t \in \mathbb{G}_m$ and $z_0 := v_D$ so that $\lim_{t \to 0} z_t = z_0$. The orbits (of codimension 1) that will play a part in the criterion correspond to subsets $D = E \smallsetminus \{\lambda\}$ where $\lambda \in E$ is a judiciously chosen element, depending on the section to be excluded.

The following proposition tells us which subsets $D \subsetneq E$ correspond to orbits of codimension 1 in X_0.

Proposition 3.1. — *Let E, V and x_0 be as before. Suppose $\lambda_0 \in E$. Put $z_0 = \sum_{\lambda \in E, \lambda \neq \lambda_0} v_\lambda$. Then $\dim \mathfrak{t}_{z_0} = \dim \mathfrak{t}_{x_0} + 1$. Consequently, the following are equivalent:*

(a) $\dim \mathfrak{g}_{z_0} = \dim \mathfrak{g}_{x_0} + 1$;
(b) $E^\perp = (E \smallsetminus \{\lambda_0\})^\perp$ *(see Lemma 2.16 (3) for the definition of $\perp$);*
(c) $E^\perp \cap \Pi = (E \smallsetminus \{\lambda_0\})^\perp \cap \Pi$.

Proof. — The first assertion follows from (the Lie-algebra version of) Lemma 2.16 (2) and the fact that E is linearly independent. The equivalence of (a) and (b) is an immediate consequence of Lemma 2.16 (3). For (b) $\Leftrightarrow$ (c) we use a standard fact about parabolic subgroups containing B. Indeed, let $\mathbb{P}(V)$ be the projective space of lines through 0 in V and $V \smallsetminus \{0\} \to \mathbb{P}(V), v \mapsto [v]$ the canonical map. Define the parabolic subgroup P of G by $P := G_{[x_0]}$. Then $-E^\perp$ is the set of negative roots of P. As is well known (see, *e.g.* [15, Theorem 30.1]), $-E^\perp$ is the set of negative roots of G that are $\mathbb{Z}$-linear combinations of the simple roots in $E^\perp \cap \Pi$. Consequently, $E^\perp$ is completely determined by $E^\perp \cap \Pi$. Similarly, $(E \smallsetminus \{\lambda_0\})^\perp \cap \Pi$ determines $(E \smallsetminus \{\lambda_0\})^\perp$. □

Lemma 3.2. — *The G-variety X_0 is normal.*

Proof. — Because $\mathcal{S}$ is freely generated, we have that $\langle \mathcal{S} \rangle_{\mathbb{Z}} \cap \mathbb{Q}_{\geqslant 0} \mathcal{S} = \mathcal{S}$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We then apply [29, Theorem 10] or the general fact [28, Theorem 6] that X_0 is normal if and only if $X_0/\!\!/U$ is a normal T-variety (recall that $X_0/\!\!/U \simeq \operatorname{Spec} \mathbb{C}[\mathcal{S}]$). □

Lemma 3.3. — *Suppose $\lambda \in E$ is such that for $D = E \smallsetminus \{\lambda\}$, the G-orbit of $z_0 = v_D$ has codimension 1 in X_0. Then $T_{z_0} X_0 = \mathfrak{g} \cdot z_0 \oplus \mathbb{C} v_\lambda$.*

Proof. — By Lemma 3.2, X_0 is normal. Therefore its singular locus has codimension at least 2. Since the singular locus is G-stable and $G \cdot z_0$ has codimension 1, it follows that X_0 is smooth at z_0. Therefore, $\dim T_{z_0} X_0 = \dim \mathfrak{g} \cdot z_0 + 1$. Moreover $t \mapsto z_t = \sigma_D(t) \cdot x_0$ is an irreducible curve in X_0 (because the elements of E are linearly independent) and $z_t = t \cdot v_\lambda + z_0$. Thus $\frac{d}{dt}|_{t=0} z_t = v_\lambda$ and so $v_\lambda \in T_{z_0} X_0$. Further $v_\lambda \notin \mathfrak{g} \cdot z_0$ since $\mathfrak{g} \cdot z_0$ lies in the complement of $V(\lambda) \subseteq V$. □

Now let $[v]$ be a T_{ad}-eigenvector in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. We denote the corresponding section in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ by s, that is, $s(x_0) = [v]$. Recall from Proposition 2.13 that the T_{ad}-action on $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ comes from the action of T on $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ through $f \colon T \to GL(V)^G$, defined in (2.5). Since

f is surjective (see Remark 2.10), we can also consider s as an eigenvector for $GL(V)^G$. Because it will play a part in what follows, we remark that if the $GL(V)^G$-weight of s is δ, then the T_{ad}-weight of $s(x_0) = [v]$ is $f^*(\delta)$. By definition, we have that for $a \in GL(V)^G$

$$s^a(x_0) := a \cdot s(a^{-1} \cdot x_0) = \delta(a)s(x_0).$$

This implies that for every $D \subseteq E$ and $t \in \mathbb{G}_m$,

$$\begin{aligned} s(z_t) = s(\sigma_D(t) \cdot x_0) &= \delta(\sigma_D(t))^{-1}\sigma_D(t) \cdot s(x_0) \\ &= [\delta(\sigma_D(t))^{-1}\sigma_D(t)v] \in V/\mathfrak{g} \cdot z_t. \end{aligned} \tag{3.1}$$

We need one final ingredient for the proof of Proposition 3.4. Recall that any $v \in V$ defines a global section $s_v \in H^0(X_0, \mathcal{N}_{X_0})$ by

$$s_v(x) = [v] \in V/T_xX_0 \text{ for all } x \in X_0.$$

Here then is the proposition we will use in Sections 5.5, 5.6 and 5.7 to prove that certain sections in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ do not extend to X_0. As mentioned at the beginning of this section, by the T_{ad}-action on V we mean α. Recall also that Λ_R stands for the root lattice.

Proposition 3.4. — *Suppose $v \in V$ is a T_{ad}-eigenvector of weight $\beta \in \Lambda_R$ such that $[v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Let $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ be defined by $s(x_0) = [v]$. If there exists $\lambda \in E$ so that*

- (ES1) *the coefficient of λ in the unique expression of $\beta \in \langle E \rangle_{\mathbb{Z}}$ as a $\mathbb{Z}$-linear combination of the elements of E is positive;*
- (ES2) *the projection of $v \in V$ onto $V(\lambda) \subseteq V$ is zero;*
- (ES3) *if η is a simple root so that $\langle \lambda, \eta^\vee \rangle \neq 0$ then there exists $\widetilde{\lambda} \in E \smallsetminus \{\lambda\}$ so that $\langle \widetilde{\lambda}, \eta^\vee \rangle \neq 0$;*
- (ES4) *if $\beta \in R_+ \smallsetminus E^\perp$ (see Lemma 2.16 for the definition of $E^\perp$), then there exists ξ in $E \smallsetminus \{\lambda\}$ so that $\langle \xi, \beta^\vee \rangle \neq 0$ and the projection of v onto $V(\xi)$ is zero;*

then s does not extend to X_0.

Proof. — The idea of the proof is to “compare” the section s to the section $s_v \in H^0(X_0, \mathcal{N}_{X_0})$. Put $D = E \smallsetminus \{\lambda\}$. We first show that

- (i) there exists a positive integer k so that $s(\sigma_D(t) \cdot x_0) = t^{-k} s_v(\sigma_D(t) \cdot x_0)$ for all $t \in \mathbb{G}_m$;
- (ii) $s_v(z_0) \neq 0$,

where $z_0 = v_D = \lim_{t \to 0} \sigma_D(t) \cdot x_0$. We then show that (i) and (ii) imply that $\lim_{t \to 0} s(\sigma_D(t) \cdot x_0)$ does not exist, *i.e.* that $s(z_0)$ does not exist.

We first prove (i). Let $f \colon T \to \mathrm{GL}(V)^G$ be the map (2.5) on page 1778. Since it is surjective, $f^* \colon X(\mathrm{GL}(V)^G) \to X(T), \delta \mapsto \delta \circ f$ is injective.

Moreover$\beta \in \operatorname{im}(f^*)$. Put $\delta := (f^*)^{-1}(\beta)$, the $\mathrm{GL}(V)^G$-weight of s. From equation (3.1) we have that $s(z_t) = [\delta(\sigma_D(t))^{-1}\sigma_D(t)v]$ for every $t \in \mathbb{G}_m$. Using (ES2), $\sigma_D(t)v = v$ for every $t \in \mathbb{G}_m$. Therefore

$$s(z_t) = [\delta(\sigma_D(t))^{-1}v] = \delta(\sigma_D(t))^{-1}[v] = \delta(\sigma_D(t))^{-1}s_v(z_t)$$

for all $t \in \mathbb{G}_m$. Let k be the coefficient of λ in the expression of β as a $\mathbb{Z}$-linear combination of the elements of E. Then $\delta(\sigma_D(t)) = t^k$ for every $t \in \mathbb{G}_m$. Consequently $s(z_t) = t^{-k}s_v(z_t)$ for all $t \in \mathbb{G}_m$. By (ES1) $k > 0$, and we have proved (i).

We now prove (ii). Condition (ES3) together with Proposition 3.1 tells us that $G \cdot z_0$ has codimension 1 in X_0. It follows from Lemma 3.3 that $T_{z_0}X_0 = \mathfrak{g} \cdot z_0 \oplus \mathbb{C}v_\lambda$. We now proceed by contradiction. Indeed, if $s_v(z_0) = [v]$ were zero than we would have $v \in \mathfrak{g} \cdot z_0 \oplus v_\lambda$. Since, by (ES1), v has nonzero T_{ad}-weight this would imply that $v \in \mathfrak{g} \cdot z_0$. The nonzero T_{ad}-weights in $\mathfrak{g} \cdot z_0$ are (by (ES3)) the same as those in $\mathfrak{g} \cdot x_0$, that is, they are the elements of $R^+ \smallsetminus E^\perp$ (by (4) of Lemma 2.16). So if $\beta \notin R^+ \smallsetminus E^\perp$ we are done. We only need to deal with the case where $\beta \in R^+ \smallsetminus E^\perp$. Then the T_{ad}-weight space in $\mathfrak{g} \cdot z_0$ of weight β is the line spanned by $X_{-\beta}z_0$. Now (ES4) tells us that v cannot belong to that line: $X_{-\beta}z_0$ has a nonzero projection to $V(\xi)$, whereas v does not.

We now prove the claim that (i) and (ii) establish the proposition. Denote by $X_0^{\leqslant 1}$ the union of $G \cdot x_0$ and all G-orbits of codimension 1 in X_0. Then $X_0^{\leqslant 1}$ is open because X_0 has finitely many orbits, and it is smooth because X_0 is normal. Again by the normality of X_0, s extends to X_0 if and only if it extends to $X_0^{\leqslant 1}$ (cf. [8, Lemma 3.7]). Since $X_0^{\leqslant 1}$ is smooth, the normal sheaf $\mathcal{N}_{X_0^{\leqslant 1}}$ of $X_0^{\leqslant 1}$ in V, which is the restriction of $\mathcal{N}_{X_0}$ to $X_0^{\leqslant 1}$, is locally free. The claim follows. $\square$

4. Reduction to classification of spherical modules

In this section we reduce the proof of Theorem 1.1 to a case-by-case verification, that is, we reduce it to Theorem 1.2. This reduction (formally, Corollary 4.17) does not use the fact that G is of type A: if Theorem 1.2 holds for groups of arbitrary type, then so does Theorem 1.1. We first introduce some more notation. We will use R for the radical of G; since G is reductive, R is the connected component $Z(G)^\circ$ of $Z(G)$ containing the identity. When (G, W) is a spherical module and $\mathcal{S}$ is its weight monoid, we will use M_W^G for the moduli scheme $\mathrm{M}_{\mathcal{S}}$ (in fact, it is easy to check that $\mathrm{M}_{\mathcal{S}}^G$ is, up to isomorphism (of schemes), independent of the choice of maximal torus T and Borel subgroup B and therefore determined by the pair (G, W), see [26, Lemma 4.13]). We introduce this notation because we will have to relate moduli schemes for different modules and different groups to one another. Given a spherical module (G, W) we will also use $\rho\colon G \to \mathrm{GL}(W)$ for the representation and we put

$$G^{\mathrm{st}} := G' \times \mathrm{GL}(W)^G.$$

We begin with an overview of the reduction. To make the classification of spherical modules in [17, 2, 21] possible, several issues had to be dealt with (see [19, Section 5]). Indeed, Knop's List gives the *saturated indecomposable* spherical modules up to *geometric equivalence*. We begin by recalling the definitions of these terms from [19, Section 5].

Definition 4.1.

(a) *Two finite-dimensional representations* $\rho_1\colon G_1 \to \mathrm{GL}(W_1)$ *and* $\rho_2\colon G_2 \to \mathrm{GL}(W_2)$ *are called* geometrically equivalent *if there is an isomorphism of vector spaces* $\phi\colon W_1 \to W_2$ *such that for the induced map*[(2)] $\mathrm{GL}(\phi)\colon \mathrm{GL}(W_1) \to \mathrm{GL}(W_2)$ *we have*

$$\mathrm{GL}(\phi)(\rho_1(G_1)) = \rho_2(G_2).$$

(b) *By the* product *of the representations* $(G_1, W_1), \dots, (G_n, W_n)$ *we mean the representation* $(G_1 \times \cdots \times G_n, W_1 \oplus \cdots \oplus W_n)$.
(c) *A finite-dimensional representation* (G, W) *is* decomposable *if it is geometrically equivalent to a representation of the form* $(G_1 \times G_2, W_1 \oplus W_2)$ *with* W_1 *a non-zero* G_1*-module and* W_2 *a non-zero* G_2*-module. It is called* indecomposable *if it is not decomposable.*
(d) *A finite-dimensional representation* $\rho\colon G \to \mathrm{GL}(W)$ *is called* saturated *if the dimension of the center of* $\rho(G)$ *equals the number of irreducible summands of* W*.*

Remark 4.2.

(a) If ρ is saturated and multiplicity-free, then the center of $\rho(G)$ is equal to the centralizer $\mathrm{GL}(W)^G$.
(b) Suppose (G_1, W_1) and (G_2, W_2) are geometrically equivalent representations. Then (G_1, W_1) is spherical if and only if (G_2, W_2) is, and (G_1, W_1) is saturated if and only if (G_2, W_2) is.

(2) By definition, $\mathrm{GL}(\phi)(f) = \phi \circ f \circ \phi^{-1}$ for every $f \in \mathrm{GL}(W_1)$.

Example 4.3 ([19], p. 311). — The spherical modules $(\mathrm{SL}(2), S^2\mathbb{C}^2)$ and $(\mathrm{SO}(3), \mathbb{C}^3)$ are geometrically equivalent. Every finite-dimensional representation is geometrically equivalent to its dual representation. The spherical module

$$(\mathrm{SL}(2) \times \mathbb{G}_m \times \mathrm{SL}(2)) \times (\mathbb{C}^2 \oplus \mathbb{C}^2) \longrightarrow \mathbb{C}^2 \oplus \mathbb{C}^2 \colon ((A, t, B), (x, y)) \longmapsto (tAx, tBy)$$

is indecomposable but not saturated.

For our reduction to Theorem 1.2, we deal with geometric equivalence and products of spherical modules in a straightforward matter. Indeed, we prove in Proposition 4.9 that if (G_1, W_1) and (G_2, W_2) are geometrically equivalent spherical modules, then $\mathrm{M}^{G_1}_{W_1} \simeq \mathrm{M}^{G_2}_{W_2}$ as schemes. That the tangent space to M^G_W behaves as expected under products is proved in Proposition 4.12. Dealing with the fact that the classification consists of *saturated* spherical modules requires a bit more effort. Indeed, we could not establish an *a priori* isomorphism between $\mathrm{M}^{\overline{G}}_W$ and M^G_W, where $(\overline{G}, W)$ is a (saturated) spherical module and G is a subgroup of $\overline{G}$ containing $\overline{G}'$ such that (G, W) is spherical. This is why in Theorem 1.2 we cannot restrict ourselves to the modules $(\overline{G}, W)$ of Knop's List. We circumvent this difficulty by proving in Proposition 4.15 that even when (G^{st}, W) is decomposable Theorem 1.2 implies the equality

$$\dim T_{X_0}\mathrm{M}^{G^{\mathrm{st}}}_W = \dim T_{X_0}\mathrm{M}^{G' \times \rho(R)}_W \tag{4.1}$$

for a spherical module $\rho\colon G \to \mathrm{GL}(W)$ with G of type A. In (4.1), by abuse of notation, X_0 on each side denotes the unique closed orbit of the corresponding moduli scheme. From Proposition 4.5 we have that $(G' \times \rho(R), W)$ is geometrically equivalent to (G, W). Using Theorem 1.2 and Lemma 2.26 we then deduce that $\dim T_{X_0}\mathrm{M}^{G^{\mathrm{st}}}_W = d_W$, thus proving Corollary 4.17.

Remark 4.4. — Theorem 1.1 proves, *a posteriori*, that $\mathrm{M}^{\overline{G}}_W$ and M^G_W are isomorphic, when $\overline{G}$ is of type A, $(\overline{G}, W)$ is a (saturated) spherical module and G is a subgroup of $\overline{G}$ containing $\overline{G}'$ such that (G, W) is spherical. We note that Remarks 5.18 and 5.20 show that, contrary to the tangent space $T_{X_0}\mathrm{M}^G_W$, the T_{ad}-module $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ that contains it does in general depend on the subgroup G of $\overline{G}$ as above: these remarks give instances where the inclusion $(V/\overline{\mathfrak{g}} \cdot x_0)^{\overline{G}_{x_0}} \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is strict. (Recall that $V = \oplus_{\lambda \in E} V(\lambda)$ with E the basis of the weight monoid of the dual module W^*.) Furthermore, we expect that the isomorphism $\mathrm{M}^G_W \simeq \mathrm{M}^{\overline{G}}_W$ cannot follow from "very general" considerations, as the following example, where $\mathcal{S}$ is not the weight monoid of a spherical module W, illustrates. Take $\overline{G} = \mathrm{SL}(3) \times \mathbb{G}_m$, $G = \mathrm{SL}(3)$ and $\mathcal{S} = \langle \omega_1 + \varepsilon, \omega_2 + \varepsilon \rangle$, where ε is a nonzero character of $\mathbb{G}_m$. Set $V = V(\omega_1+\varepsilon)^* \oplus V(\omega_2+\varepsilon)^*$ as in Section 2.1. Since $\mathcal{S}$ is G-saturated, $T_{X_0}\mathrm{M}_{\mathcal{S}}^{\overline{G}} \simeq (V/\overline{\mathfrak{g}} \cdot x_0)^{\overline{G}_{x_0}}$ and $T_{X_0}\mathrm{M}_{\mathcal{S}}^{G} \simeq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ by Remark 2.21. A direct calculation shows that $\dim(V/\mathfrak{g}\cdot x_0)^{G_{x_0}} = 1$, whereas $\dim(V/\overline{\mathfrak{g}} \cdot x_0)^{\overline{G}_{x_0}} = 0$.

The following proposition explains how a general spherical module (G, W) fits into the classification of spherical modules. It is (somewhat implicitly) contained in [21, Section 2] and [9, Section 5.1]. Recall that given a spherical module (G, W), we put $G^{\mathrm{st}} := G' \times \mathrm{GL}(W)^G$.

Proposition 4.5 (Leahy). — *Suppose $\rho\colon G \to \mathrm{GL}(W)$ is a spherical module. Then the following hold:*

(i) *If (G, W) is saturated and indecomposable, then (G, W) is geometrically equivalent to an entry in Knop's List;*
(ii) *(G^{st}, W) is a saturated spherical module;*
(iii) *(G^{st}, W) is geometrically equivalent to a product of indecomposable saturated spherical modules;*
(iv) *$\rho(R) \subseteq \mathrm{GL}(W)^G$ and $\rho(G) = \rho(G')\rho(R) \subseteq \mathrm{GL}(W)$;*
(v) *Suppose $(G_1, W_1), (G_2, W_2), \ldots, (G_n, W_n)$ are spherical modules and let (K, E) be their product. Suppose (K, E) and (G^{st}, W) are geometrically equivalent and denote by $\phi\colon W \to E$ a linear isomorphism establishing their geometric equivalence (see Definition 4.1). If $A = \mathrm{GL}(\phi)(\rho(R))$, then $A \subseteq \mathrm{GL}(E)^K$ and (G, W) is geometrically equivalent to $(K' \times A, E)$.*

Proof. — Assertion (i) just says that Knop's List contains all indecomposable saturated spherical modules up to geometric equivalence (see [21, Theorem 2.5] or [2, Theorem 2]). Next, let b be the number of irreducible components of (G, W). Assertion (ii) follows from the fact that $\mathrm{GL}(W)^G \simeq \mathbb{G}_m^b$ (because W is a multiplicity-free G-module). Assertion (iii) follows from the fact that if $(G_1 \times G_2, W_1 \oplus W_2)$ is saturated (resp. spherical) then (G_1, W_1) and (G_2, W_2) are saturated (resp. spherical). We come to (iv). Note that R commutes with G and so $\rho(R)$ commutes with $\rho(G)$ hence the first assertion. For the second, we use a well-known decomposition of reductive groups: $G = G'R$. Finally we prove (v). Let us call $\psi\colon K \to \mathrm{GL}(E)$ and $\rho^{\mathrm{st}}\colon G^{\mathrm{st}} \to \mathrm{GL}(W)$ the representations. Then $\mathrm{GL}(\phi)\colon \rho^{\mathrm{st}}(G^{\mathrm{st}}) \to \psi(K)$ is an isomorphism of algebraic groups. As $\mathrm{GL}(W)^G \subseteq Z(G^{\mathrm{st}})$, its image $\mathrm{GL}(\phi)(\mathrm{GL}(W)^G)$ belongs to the center of $\psi(K)$, which is a subset of $\mathrm{GL}(E)^K$. Because $\rho(R) \subseteq GL(W)^G$, it follows that $A = \mathrm{GL}(\phi)(\rho(R)) \subseteq$

$\mathrm{GL}(E)^K$. This proves the first assertion. Next, note that $\mathrm{GL}(\phi)(\rho(G)) = \mathrm{GL}(\phi)(\rho(G')) \cdot \mathrm{GL}(\phi)(\rho(R))$. Moreover,

$$\mathrm{GL}(\phi)(\rho(G')) = \mathrm{GL}(\phi)(\rho((G^{\mathrm{st}})')) = [\mathrm{GL}(\phi)(\rho^{\mathrm{st}}(G^{\mathrm{st}}))]' = \psi(K)' = \psi(K')$$

and the second assertion follows. □

The following lemma is well-known and straightforward. For a proof, see *e.g.* [26, Lemma 4.6].

Lemma 4.6. — *Let X be an affine G-variety and let H be a connected subgroup of G containing G'. Let B_H be the Borel subgroup $B \cap H$ of H and let $p\colon X(B) \twoheadrightarrow X(B_H)$ be the restriction map. If we consider X as an H-variety, then its weight monoid is $p(\Lambda^+_{(G,X)})$.*

If, moreover, X is an affine spherical G-variety, *then the following are equivalent*

(i) *X is spherical as an H-variety;*
(ii) *the restriction of p to $\Lambda^+_{(G,X)}$ is injective*
(iii) *the restriction of p to $\Lambda_{(G,X)}$ is injective.*

Remark 4.7.

(1) Theorem 5.1 of [19] is a somewhat refined version of Lemma 4.6.
(2) For every saturated indecomposable spherical module (G, W), Knop's List, following [21], gives a basis for $\langle \ker p\rangle_{\mathbb{C}} \cap \langle \Lambda_W\rangle_{\mathbb{C}} \subseteq \mathfrak{t}^*$, where p is as in Lemma 4.6. In Knop's List, $\langle \ker p\rangle_{\mathbb{C}}$ is denoted $\mathfrak{z}^*$ and $\langle \Lambda_W\rangle_{\mathbb{C}}$ is denoted $\mathfrak{a}^*$.

Using that when $f\colon G \to H$ is a surjective homomorphism, representations of H are the same as representations of G with kernel containing $\ker f$, it is straightforward to prove the following proposition (for details, see [26, Proposition 4.10]).

Proposition 4.8. — *Suppose $f\colon G \twoheadrightarrow H$ is a surjective group homomorphism between connected reductive groups. Put $T_H := f(T)$ and $B_H = f(B)$ and write f^* for the map $X(T_H) \hookrightarrow X(T)$ given by $\lambda \to \lambda \circ f$. Let $\mathcal{S} \subseteq X(T_H)$ be the weight monoid of an affine spherical H-variety (with respect to the Borel subgroup B_H). Then $\mathrm{M}^H_{\mathcal{S}} \simeq \mathrm{M}^G_{f^*(\mathcal{S})}$ as schemes.*

Straightforward arguments using Proposition 4.8 prove that geometrically equivalent spherical modules have isomorphic moduli schemes (again, for details see [26, Proposition 4.15]).

Proposition 4.9. — *Suppose* $\rho_1\colon G_1 \to \mathrm{GL}(W_1)$ *and* $\rho_2\colon G_2 \to \mathrm{GL}(W_2)$ *are geometrically equivalent spherical modules. Then we have an isomorphism of schemes* $\mathrm{M}_{W_1}^{G_1} \simeq \mathrm{M}_{W_2}^{G_2}$. *Consequently,*

$$\dim T_{X_0}\mathrm{M}_{W_1}^{G_1} = \dim T_{X_0}\mathrm{M}_{W_2}^{G_2},$$

where by abuse of notation, X_0 *on each side denotes the unique closed orbit of the corresponding moduli scheme.*

The next lemma states how the invariant d_W behaves under restriction of center, geometric equivalence and taking products. We need two of its assertions in the proof of Proposition 4.11. It will also be of use later.

Lemma 4.10.

(a) *Suppose* $\overline{G}$ *is a connected reductive group and let* $(\overline{G}, W)$ *be a spherical module. Let* G *be a connected (reductive) subgroup of* $\overline{G}$ *containing* $\overline{G}'$. *Assume that the restriction* (G, W) *of* $(\overline{G}, W)$ *is also spherical. Then both modules have the same invariant* d_W.

(b) *Suppose* (G_1, W_1) *and* (G_2, W_2) *are geometrically equivalent spherical modules. Then* $d_{W_1} = d_{W_2}$.

(c) *Let* $(G_1, W_1), (G_2, W_2), \ldots, (G_n, W_n)$ *be spherical modules and let* (G, W) *be their product. Then* $d_W = d_{W_1} + \cdots + d_{W_n}$.

Proof. — For (a) just combine Lemma 4.6 with Lemma 2.26 (or with Lemma 2.7). Next, to prove (b), let $\rho_1\colon G_1 \to \mathrm{GL}(W_1)$ and $\rho_2\colon G_2 \to \mathrm{GL}(W_2)$ be the representations. Suppose $\phi\colon W_1 \to W_2$ is a linear isomorphism establishing the geometric equivalence. Then $\mathrm{GL}(\phi)\colon \rho_1(G_1) \to \rho_2(G_2)$ is an isomorphism of algebraic groups. Let U be a maximal unipotent subgroup of G_1. Then $U_1 := \rho_1(U)$ is a maximal unipotent subgroup of $\rho_1(G_1)$ and $U_2 := \mathrm{GL}(\phi)(\rho_1(U))$ is a maximal unipotent subgroup of $\rho_2(G_2)$. Moreover, ϕ induces an isomorphism of vector spaces $W_1^{U_1} \simeq W_2^{U_2}$ and an isomorphism of algebras $\mathbb{C}[W_1]^{U_1} \simeq \mathbb{C}[W_2]^{U_2}$. Since, for $i \in \{1,2\}$, $\dim W_i^{U_i}$ is the number of irreducible components of W_i and $\dim \mathrm{Spec}(\mathbb{C}[W_i]^{U_i})$ is the rank of the weight group of W_i, Lemma 2.7 proves assertion (b). We turn to (c). This assertion follows by combining Lemma 2.7 with the fact that $\Lambda^+_{(G,W)} = \Lambda^+_{(G_1,W_1)} \oplus \cdots \oplus \Lambda^+_{(G_n,W_n)}$. □

Proposition 4.11. — *Let* (G, W) *be an indecomposable saturated spherical module. Suppose that* $G = G^{\mathrm{st}}$ *(hence* $Z(G)^\circ = \mathrm{GL}(W)^G$*) and that* $H \subseteq Z(G)^\circ$ *is a subtorus such that* W *is spherical for* $G' \times H$. *Assume that the conclusion of Theorem 1.2 holds for every pair* $(\overline{G}, W)$ *in Knop's List with* $\overline{G}$ *of a type that occurs in the decomposition of* G' *into almost simple components. Then* $\dim T_{X_0}\mathrm{M}_W^{G'\times H} = \dim T_{X_0}\mathrm{M}_W^G = d_W$, *where by abuse of notation each X_0 stands for the unique closed orbit of the corresponding moduli scheme.*

Proof. — By Proposition 4.5 (i), (G,W) is geometrically equivalent to an entry in Knop's List, say $(\overline{K},E)$. Suppose $\phi\colon W\to E$ is a map establishing the geometric equivalence (see Definition 4.1) between (G,W) and $(\overline{K},E)$. We first claim that there exists a connected reductive subgroup $K\subseteq\overline{K}$ containing $\overline{K}'$ for which E is still spherical and so that ϕ also establishes the geometric equivalence of $(G'\times H,W)$ and (K,E). Indeed, let $\rho\colon G\to\mathrm{GL}(W)$ and $\psi\colon\overline{K}\to\mathrm{GL}(E)$ be the representations and put $\rho_1=\rho|_{G'\times H}$. Then $F:=\mathrm{GL}(\phi)(\operatorname{im}\rho_1)$ is a connected subgroup of $\psi(\overline{K})$ containing $\psi(\overline{K})'=\psi(\overline{K}')$. The reason is that $\mathrm{GL}(\phi)(\operatorname{im}\rho_1)$ contains $\mathrm{GL}(\phi)((\operatorname{im}\rho)')=(\mathrm{GL}(\phi)(\operatorname{im}\rho))'=(\psi(\overline{K}))'$, since $\operatorname{im}\rho_1$ contains $(\operatorname{im}\rho)'$. Now set $\widetilde{K}:=\psi^{-1}(F)$ and let K be the identity component of $\widetilde{K}$. Then $\widetilde{K}$ is a subgroup of $\overline{K}$ containing $\overline{K}'$ and therefore so is K. Moreover, K is reductive. Clearly $\psi(\widetilde{K})=F=\mathrm{GL}(\phi)(\operatorname{im}\rho_1)$ (since $F\subseteq\operatorname{im}\psi$). Since $\psi(\widetilde{K})=\psi(K)$ because $\psi(\widetilde{K})$ is connected (see *e.g.* [15, Proposition B of §7.4]), ϕ establishes the geometric equivalence of ρ_1 and $\psi|_K$. It also follows (by Remark 4.2 (b)) that E is a spherical module for K. This proves the claim.

By Lemma 4.10 (a), (G,W) and $(G'\times H,W)$ have the same invariant d_W, and $(\overline{K},E)$ and (K,E) have the same invariant d_E. By assumption, the conclusion of Theorem 1.2 holds for $(\overline{K},E)$ and so $\dim T_{X_0}\mathrm{M}_E^{\overline{K}}=\dim T_{X_0}\mathrm{M}_E^{K}=d_E$. Thanks to Lemma 4.10 (b), $d_E=d_W$. Finally, by Proposition 4.9, $\dim T_{X_0}\mathrm{M}_E^{\overline{K}}=\dim T_{X_0}\mathrm{M}_W^{G}$ and $\dim T_{X_0}\mathrm{M}_E^{K}=\dim T_{X_0}\mathrm{M}_W^{G'\times H}$, and we have proved the proposition. □

The next proposition reminds us that the normal sheaf behaves as expected with respect to products.

Proposition 4.12. — *Let n be a positive integer. Suppose that for every positive integer $i\leqslant n$ we have a finite-dimensional G-module V_i and a G-stable closed subscheme X_i of V_i. For every i, we put $R_i:=\mathbb{C}[V_i]$, $I_i:=I(X_i)\subseteq R_i$ (the ideal of X_i in V_i) and $N_i:=\mathrm{Hom}_{R_i}(I_i,R_i/I_i)$. We also put $V:=\oplus_i V_i$, $R:=\mathbb{C}[V]$, $X:=X_1\times\cdots\times X_n$, $I:=I(X)\subseteq\mathbb{C}[V]$ and $N:=\mathrm{Hom}_R(I,R/I)$. We then have a canonical isomorphism of R-G-modules:*

$$N\simeq\oplus_i(N_i\otimes_{R_i}R)$$

Proof. — It is clear that, for $1\leqslant j\leqslant n$, we can consider I_j as a subset of I. For $1\leqslant i\leqslant n$ we define the G-stable R-submodule $\widetilde{N}_i\subseteq N$ by

$$\widetilde{N}_i=\big\{\phi\in N \text{ such that } \phi(a)=0 \text{ when } a \text{ is in } I_j \text{ and } j\neq i\big\}.$$

Using [24, Lemma 9] it follows that $N = \oplus_{i=1}^{n} \widetilde{N}_i$, and that $\widetilde{N}_i$ is canonically isomorphic to $N_i \otimes_{R_i} R$ as an R-module with the isomorphism being G-equivariant. $\square$

We note that, with the notation of Proposition 4.12, there is a canonical isomorphism $N_i \otimes_{R_i} R \simeq N_i \otimes_{\mathbb{C}} \widehat{R}_i$ for every $i \in \{1, \ldots, n\}$, where $\widehat{R}_i := \otimes_{j \neq i} R_j / I_j$ (tensor product over $\mathbb{C}$). We will use this formulation of the proposition in what follows.

Corollary 4.13. — *Let n be a positive integer and suppose that for every positive integer $i \leqslant n$, G_i is a connected reductive group, V_i is a finite-dimensional G_i-module and X_i is a* multiplicity-free *G_i-stable closed subscheme of V_i. Put $\overline{G} := G_1 \times \cdots \times G_n$. Define N and N_i as in Proposition 4.12. Then we have a canonical isomorphism of $\mathbb{C}$-vector spaces*

$$N^{\overline{G}} \simeq \oplus_i N_i^{G_i}. \tag{4.2}$$

Proof. — In this proof all the tensor products are over $\mathbb{C}$. We introduce the following notation for every $i \in \{1, \ldots, n\}$: $\widehat{G}_i := \times_{j \neq i} G_j$. Using Proposition 4.12 we have that

$$N^{\overline{G}} \simeq \oplus_i \big(N_i \otimes \widehat{R}_i\big)^{\overline{G}} = \oplus_i \big(N_i^{G_i} \otimes \widehat{R}_i^{\widehat{G}_i}\big) \simeq \oplus_i N_i^{G_i}, \tag{4.3}$$

where the last isomorphism uses that $\widehat{R}_i^{\widehat{G}_i} = \mathbb{C}$ by the multiplicity-freeness of $\widehat{R}_i$. $\square$

Remark 4.14. — An immediate consequence of this corollary is that if (G_1, W_1) and (G_2, W_2) are spherical modules and (G, W) is their product, then $\dim T_{X_0} \mathrm{M}_W = \dim T_{X_0} \mathrm{M}_{W_1} + \dim T_{X_0} \mathrm{M}_{W_2}$, where by abuse of notation each X_0 denotes the unique closed orbit of the corresponding moduli scheme. This is how we will use the corollary (in the proof of Corollary 4.17).

Proposition 4.15. — *Suppose that for every $i \in \{1, \ldots, n\}$ we have an indecomposable saturated spherical module (G_i, W_i). For every i, assume that $G_i = G_i^{\mathrm{st}}$ and that the conclusion of Theorem 1.2 holds for every pair $(\overline{G}, W)$ in Knop's List with $\overline{G}$ of a type that occurs in the decomposition of G_i' into almost simple components. For every i we put $Z_i := Z(G_i)^\circ = \mathrm{GL}(W_i)^{G_i}$, $E_i := \Lambda_{W_i}^+$, $V_i := \oplus_{\lambda \in E_i} V(\lambda)$, $X_i = \overline{G_i x_i}$, where $x_i = \sum_{\lambda \in E_i} v_\lambda$. Put $\overline{G} := G_1 \times \cdots \times G_n$. We also define N_i and N as in Proposition 4.12. Finally suppose that A is a subtorus of $Z_1 \times \cdots \times Z_n$ such that $W_1 \oplus \cdots \oplus W_n$ is spherical for $G := G_1' \times \cdots \times G_n' \times A$. Then*

$$N^G = N^{\overline{G}}. \tag{4.4}$$

Proof. — We continue to use the notation $\widehat{G}_i$ introduced in the proof of Corollary 4.13. In this proof all the tensor products are over $\mathbb{C}$. To prove (4.4) it is sufficient (by Proposition 4.12) to prove that $(N_i \otimes \widehat{R}_i)^G = (N_i \otimes \widehat{R}_i)^{\overline{G}}$ for every i. We clearly have that $(N_i \otimes \widehat{R}_i)^G = (N_i^{G'_i} \otimes \widehat{R}_i^{\widehat{G'_i}})^A$. Recall from equation (4.3) that $(N_i \otimes \widehat{R}_i)^{\overline{G}} = N_i^{G_i} \otimes F_0$, where $F_0 := \widehat{R}_i^{\widehat{G}_i} \simeq \mathbb{C}$. We will prove that

$$F := (N_i^{G'_i} \otimes \widehat{R}_i^{\widehat{G'_i}})^A = N_i^{G_i} \otimes F_0.$$

The inclusion $N_i^{G_i} \otimes F_0 \subseteq F$ is clear. For the other inclusion, assume, by contradiction, that F is not a subspace of $N_i^{G_i} \otimes F_0$. Then there exist a character $\lambda \in X(A)$, a nonzero vector v in $N_i^{G'_i}$ of weight $-\lambda$ and a nonzero vector w of weight λ in $\widehat{R}_i^{\widehat{G'_i}}$ such that $v \otimes w \notin N_i^{G_i} \otimes F_0$. It follows that $\lambda \neq 0$, for otherwise

$$v \otimes w \in N_i^{G'_i \times A} \otimes \widehat{R}_i^{\widehat{G'_i} \times A} = N_i^{G'_i \times A} \otimes F_0 = N_i^{G'_i \times p(A)} \otimes F_0$$

where $p\colon \times_j Z_j \to Z_i$ is the projection, while Proposition 4.11 tells us that $N_i^{G'_i \times p(A)} = N_i^{G_i}$ (because W_i is spherical for $G'_i \times p(A)$).

Now, by Lemma 4.16 below, we have that X_i is spherical for $G'_i \times \ker \lambda$, hence for $G'_i \times p(\ker \lambda)$, since A acts on X_i through the factor Z_i. Again by Proposition 4.11, we have that $N_i^{G'_i \times p(\ker \lambda)} = N_i^{G_i}$. We obtain a contradiction: $v \in N_i^{G'_i \times p(\ker \lambda)}$ since v has A-weight λ, but $v \notin N_i^{G_i}$ since λ is nonzero and therefore $p(A) \subseteq G_i$ does not fix v. □

LEMMA 4.16. — *Let G_1 and G_2 be connected reductive groups and let A_1 and A_2 be tori. Suppose that for every $i \in \{1,2\}$ we have a normal affine $G_i \times A_i$-variety X_i. Let $A \subseteq A_1 \times A_2$ be a subtorus such that $X_1 \times X_2$ is spherical for the action restricted to $G_1 \times A \times G_2 \subseteq G_1 \times A_1 \times A_2 \times G_2$. If $\lambda \in X(A)$ is such that the eigenspace $\mathbb{C}[X_2]^{G_2}$ contains a nonzero A-eigenvector of weight λ, then X_1 is spherical for $G_1 \times \ker \lambda$.*

Proof. — Pick Borel subgroups and maximal tori $T_1 \subseteq B_1 \subseteq G_1$ and $T_2 \subseteq B_2 \subseteq G_2$. In this proof we identify $X(A)$ with its image under the canonical embeddings into $X(A \times T_i)$ for $i \in \{1,2\}$ and into $X(A \times T_1 \times T_2)$.

Clearly, X_1 is spherical for $G_1 \times A$. If X_1 is not spherical for the subgroup $G_1 \times \ker \lambda$, then there are highest weight vectors $f_\alpha, f_\beta \in \mathbb{C}[X_1]^{(B_1 \times A)}$ of weight α and β respectively such that $\alpha \neq \beta$ and $\alpha = \beta$ on $\ker \lambda \subseteq T_1 \times A$. This implies that $\alpha - \beta = d\lambda$ for some integer d. Reversing the roles of α and β if necessary, we assume d nonnegative.

It is given that there is a g_λ in $\mathbb{C}[X_2]^{(A\times B_2)}$ of weight λ. We then have that the two $(B_1\times A\times B_2)$-eigenvectors $f_\alpha\otimes 1$ and $f_\beta\otimes g_\lambda^d$ in $\mathbb{C}[X_1]\otimes\mathbb{C}[X_2]$ have the same weight. This contradicts the sphericality of $X_1\times X_2$ for the action of $G_1\times A\times G_2$. □

Corollary 4.17. — *Let (G,W) be a spherical module and let $\mathcal{S}$ be its weight monoid. Assume that the conclusion of Theorem 1.2 holds for every pair $(\overline{G},W)$ in Knop's List with $\overline{G}$ of a type that occurs in the decomposition of G' into almost simple components. Then*

$$\dim T_{X_0}\mathrm{M}_{\mathcal{S}} = d_W. \tag{4.5}$$

Proof. — In this proof, by abuse of notation, X_0 will stand for the unique closed orbit of the relevant moduli scheme. By Proposition 4.5 there exist indecomposable saturated spherical modules (G_i,W_i) in Knop's List, with $i\in\{1,2,\dots,n\}$, such that (G^{st},W) is geometrically equivalent to the product (K,E) of the (G_i,W_i), and such that (G,W) is geometrically equivalent to $(K'\times A,E)$ where A is a subtorus of $\mathrm{GL}(E)^K$. By assumption, the conclusion of Theorem 1.2 holds for each (G_i,W_i) and so

$$\dim T_{X_0}\mathrm{M}_{W_i}^{G_i} = d_{W_i} \text{ for every } i\in\{1,2,\dots,n\}.$$

As a consequence, Corollary 4.13 and Lemma 4.10 (c) yield that

$$\dim T_{X_0}\mathrm{M}_E^K = d_E. \tag{4.6}$$

On the other hand, using that $\mathrm{GL}(E)^K = \times_i \mathrm{GL}(W_i)^{G_i}$, Proposition 4.15 tells us that

$$\dim T_{X_0}\mathrm{M}_E^{K'\times A} = \dim T_{X_0}\mathrm{M}_E^K, \tag{4.7}$$

whereas by Proposition 4.9, $\dim T_{X_0}\mathrm{M}_W^G = \dim T_{X_0}\mathrm{M}_E^{K'\times A}$. With equations (4.6) and (4.7) and Lemma 4.10 (a,b) this implies equation (4.5), as desired. □

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 through case-by-case verification. Formally the proof runs as follows. We have to check the theorem for the 8 families in List 5.1 below. For families (1), (2) and (3), the arguments are given in Sections 5.1, 5.2 and 5.3, respectively. For family (4), the theorem follows from Proposition 5.11 on page 1801; for family (5) it follows from Proposition 5.12 on page 1802; for family (6) from Proposition 5.19 on page 1806; for family (7) from Proposition 5.21 on page 1807; and for family (8) from Proposition 5.22 on page 1807. Thus, all cases are covered.

Each subsection of this section corresponds to one of the eight families given in the following list. We provide the full argument for only two representative cases (family (1) in Section 5.1 and family (5) in Section 5.5) and refer the reader to [26] for details about the similar verifications required for the remaining six families.

List 5.1. — *The* 8 *families of saturated indecomposable spherical modules* $(\overline{G}, W)$ *with* $\overline{G}$ *of type* A *in Knop's List are*

(1) $(\mathrm{GL}(m) \times \mathrm{GL}(n), \mathbb{C}^m \otimes \mathbb{C}^n)$ *with* $1 \leqslant m \leqslant n$;
(2) $(\mathrm{GL}(n), \mathrm{Sym}^2 \mathbb{C}^n)$ *with* $1 \leqslant n$;
(3) $(\mathrm{GL}(n), \bigwedge^2 \mathbb{C}^n)$ *with* $2 \leqslant n$;
(4) $(\mathrm{GL}(n) \times \mathbb{G}_m, \bigwedge^2 \mathbb{C}^n \oplus \mathbb{C}^n)$ *with* $4 \leqslant n$;
(5) $(\mathrm{GL}(n) \times \mathbb{G}_m, \bigwedge^2 \mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ *with* $4 \leqslant n$;
(6) $(\mathrm{GL}(m) \times \mathrm{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus \mathbb{C}^n)$ *with* $1 \leqslant m, 2 \leqslant n$;
(7) $(\mathrm{GL}(m) \times \mathrm{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus (\mathbb{C}^n)^*)$ *with* $1 \leqslant m, 2 \leqslant n$;
(8) $(\mathrm{GL}(m) \times \mathrm{SL}(2) \times \mathrm{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n))$ *with* $2 \leqslant m \leqslant n$.

Remark 5.2. — The indices m and n in family (6) and family (7) run through a larger set than that given in Knop's List. Knop communicated the revised range of indices for these families to the second author. We remark that these cases do appear in the lists of [21] and [2].

Remark 5.3.

(i) Recall from Lemma 2.7 that for a given spherical module W it is easy to compute d_W from the rank of Λ_W.
(ii) Recall that by Corollary 2.6 it is enough to prove that $\dim T_{X_0} \mathrm{M}_{\mathcal{S}}^{G} \leqslant d_W$ for every (G, W) as in Theorem 1.2 to establish the theorem.

In each subsection, $(\overline{G}, W)$ will denote a member of the family from List 5.1 under consideration. Here is some more notation we will use for the rest of this section. Given a spherical module $(\overline{G}, W)$ from Knop's List,

– E denotes the basis of the weight monoid $\Lambda^+_{(\overline{G}, W^*)}$ of W^* (the elements of E are called the "basic weights" in Knop's List);
– $V = \oplus_{\lambda \in E} V(\lambda)$;
– $x_0 = \sum_{\lambda \in E} v_\lambda$.

Except if stated otherwise, G will denote a connected subgroup of $\overline{G}$ containing $\overline{G}'$ such that (G, W) is spherical. Recall that such a group G is necessarily reductive. To lighten notation, we will use G' for the derived subgroup $\overline{G}'$ of $\overline{G}$. This should not cause confusion since $(\overline{G}, \overline{G}) = (G, G) = G'$. We will use $\overline{T}$ for a fixed maximal torus in $\overline{G}$ and put $T = \overline{T} \cap G$ and $T' = \overline{T} \cap G'$.

Then $T \subseteq G$ and $T' \subseteq G'$ are maximal tori. We will use $p\colon X(T) \twoheadrightarrow X(T')$, $q\colon X(\overline{T}) \twoheadrightarrow X(T)$ and $r\colon X(\overline{T}) \twoheadrightarrow X(T')$ for the restriction maps. Similarly, $\overline{B}$ is a fixed Borel subgroup of $\overline{G}$ containing $\overline{T}$ and we put $B = \overline{B} \cap G$ and $B' = \overline{B} \cap G'$. Then B and B' are Borel subgroups of G and G', respectively. Note that the restriction of p to Λ_R is injective and we can, and will, identify the root lattices of $\overline{G}, G$ and G'. Moreover, our choice of Borel subgroups allows us to identify the sets of positive roots (which we denote R^+) and the sets of simple roots (which we denote Π) of $\overline{G}, G$ and G'. Note also that since $Z(G') = Z(G) \cap T'$, we have that $T' \hookrightarrow T$ induces an isomorphism $T'/Z(G') \simeq T/Z(G)$. We therefore can (and will) identify the adjoint torus of $\overline{G}, G$ and of G' and we denote it T_{ad}. We will use $\omega, \omega', \omega''$ for weights of the first, second and third non-abelian factor of G, while ε will refer to the character $\mathbb{G}_m \to \mathbb{G}_m, z \mapsto z$ of $\mathbb{G}_m$.

Recall our convention that by the T_{ad}-action on V (and on $V/\mathfrak{g} \cdot x_0$, $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, *etc.*) we mean the action given by α (see Definition 2.11). The T_{ad}-action on $\mathrm{M}_{\mathcal{S}}$ refers to the action given by $\widehat{\psi}$, see page 1778.

Remark 5.4. — A consequence of using the action α is that the T_{ad}-weight set we obtain below for each $T_{X_0}\mathrm{M}_{\mathcal{S}}^G$ is the basis of the free monoid $\widetilde{\Sigma}_{W^*} = -w_0\widetilde{\Sigma}_W$ (instead of $-\widetilde{\Sigma}_W$ as in Theorem 1.1 where the action γ was used).

Remark 5.5. — We have the following isomorphism of G-modules (where G acts on V as a subgroup of $\overline{G}$): $V \simeq \oplus_{\lambda \in E} V(q(\lambda))$. Using Lemma 2.19 it follows that the T_{ad}-module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ only depends on $(\overline{G}, W)$ (that is, it does not depend on the particular subgroup G).

We will also use $\overline{\mathcal{S}}$ for the weight monoid of $(\overline{G}, W)$, $\mathcal{S}$ for the weight monoid of (G, W), $\overline{\Delta}$ for the weight group of $(\overline{G}, W^*)$, and Δ for the weight group of (G, W^*). Note that $\overline{\Delta} = \langle E \rangle_{\mathbb{Z}} \subseteq X(\overline{T})$, $\Delta = q(\overline{\Delta})$, $\mathcal{S} = q(\overline{\mathcal{S}})$, that the weight group of (G', W^*) (which is not necessarily spherical) is $r(\overline{\Delta}) = p(\Delta)$ and that the weight monoid of (G', W) is $r(\overline{\mathcal{S}}) = p(\mathcal{S})$.

Remark 5.6. — In proving Theorem 1.2 for families (5), (6) and (7) we exclude certain T_{ad}-weight spaces in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ from belonging to the subspace $T_{X_0}\mathrm{M}_{\mathcal{S}}^G$. Comparing with the simple reflections of the little Weyl group of W^* computed in Knop's List suggested which T_{ad}-weights we had to exclude. Logically however, that information from Knop's List plays no part in our proof. In fact, because $\dim T_{X_0}\mathrm{M}_{\mathcal{S}}^G$ is minimal (by Theorem 1.2), the computations of the T_{ad}-weights in $T_{X_0}\mathrm{M}_{\mathcal{S}}^G$ we perform in this section confirm Knop's computations of the little Weyl group of the spherical modules under consideration. For the relationship between the

T_{ad}-weights in $T_{X_0}\mathrm{M}^G_{\mathcal{S}}$ and the little Weyl group of W^*, see Remarks 2.8 and 5.4.

5.1. The modules $(\mathrm{GL}(m) \times \mathrm{GL}(n), \mathbb{C}^m \otimes \mathbb{C}^n)$ with $1 \leqslant m \leqslant n$

Here

$$E = \left\{\omega_1 + \omega_1', \omega_2 + \omega_2', \ldots, \omega_m + \omega_m'\right\} \quad \text{and} \quad d_W = m - 1.$$

When $m < n$ the module W is spherical for $G' = \mathrm{SL}(m) \times \mathrm{SL}(n)$ and its weight monoid $p(\mathcal{S})$ is G'-saturated. Corollary 2.27 therefore takes care of these cases. The only case that remains is when $m = n$. Then W is not spherical for G' because the determinant yields a non-constant invariant function (after identifying W with the space of m-by-m matrices). Since $\omega_m + \omega_m' \in E$, $\mathcal{S}$ is not G-saturated for any intermediate group G for which W is spherical. We prove that in that case too $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ has dimension d_W.

Proposition 5.7. — *Suppose $m = n$. Then the T_{ad}-module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and its weight set is*

$$\left\{\alpha_1 + \alpha_1', \alpha_2 + \alpha_2', \ldots, \alpha_{m-1} + \alpha_{m-1}'\right\}.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. *Consequently,* $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

Proof. — First note that $p(\Delta) = \langle \omega_1 + \omega_1', \ldots, \omega_{m-1} + \omega_{m-1}'\rangle_{\mathbb{Z}} \subseteq X(T')$. Suppose v is a T_{ad}-eigenvector in V of weight γ so that $[v]$ is a nonzero element of $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$. Then

$$\gamma \in p(\Delta) \cap \Lambda_R \tag{5.1}$$

by Lemma 2.17 (b). Clearly, $p(\Delta) \cap \Lambda_R$ is the diagonal of Λ_R, that is, the group

$$\left\langle \alpha_1 + \alpha_1', \alpha_2 + \alpha_2', \ldots, \alpha_{m-1} + \alpha_{m-1}'\right\rangle_{\mathbb{Z}} \subseteq \Lambda_R.$$

Moreover, Lemma 2.18 (B) implies that there exists a simple root δ of G' so that

$$\gamma - \delta \text{ (which is the weight of } X_\delta v) \text{ belongs to } R^+ \cup \{0\}. \tag{5.2}$$

Equations (5.1) and (5.2) imply that $\gamma = \alpha_i + \alpha_i'$ for some i with $1 \leqslant i \leqslant m - 1$.

We next claim that the T_{ad}-eigenspace of weight $\alpha_i + \alpha_i'$ in V is one dimensional for every i with $1 \leqslant i \leqslant m-1$. Indeed, the only G'-submodule of V which contains an eigenvector of that weight is $V(\omega_i + \omega_i')$ and the eigenspace is the line spanned by $X_{-\alpha_i}X_{-\alpha_i'}x_0 = X_{-\alpha_i'}X_{-\alpha_i}x_0$. This finishes the proof. □

Example 5.8. — We illustrate Proposition 5.7 for $m = n = 3$ and $G = \overline{G} = \mathrm{GL}(3)\times\mathrm{GL}(3)$. Consider two copies of $\mathbb{C}^3$, one with basis e_1, e_2, e_3, the other with basis f_1, f_2, f_3, and with the first (resp. second) copy of GL(3) acting on the first (resp. second) copy of $\mathbb{C}^3$ by the defining representation. Then we can take

$$V = \mathbb{C}^3 \otimes \mathbb{C}^3 \oplus \wedge^2\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^3 \oplus \wedge^3\mathbb{C}^3 \otimes \wedge^3\mathbb{C}^3;$$
$$x_0 = e_1 \otimes f_1 + e_1 \wedge e_2 \otimes f_1 \wedge f_2 + e_1 \wedge e_2 \wedge e_3 \otimes f_1 \wedge f_2 \wedge f_3.$$

Consequently,

$$\begin{aligned}\mathfrak{g}\cdot x_0 = \big\langle & e_1 \otimes f_1, e_1 \wedge e_2 \otimes f_1 \wedge f_2, e_1 \wedge e_2 \wedge e_3 \otimes f_1 \wedge f_2 \wedge f_3,\\ & e_2 \otimes f_1, e_3 \otimes f_1 - e_2 \wedge e_3 \otimes f_1 \wedge f_2, e_1 \wedge e_3 \otimes f_1 \wedge f_2,\\ & e_1 \otimes f_2, e_1 \otimes f_3 - e_1 \wedge e_2 \otimes f_2 \wedge f_3, e_1 \wedge e_2 \otimes f_1 \wedge f_3\big\rangle_{\mathbb{C}},\end{aligned}$$

$$G'_{x_0} = \left\{ \left(\begin{pmatrix} a & c_1 & c_2 \\ 0 & b & c_3 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}, \begin{pmatrix} a^{-1} & c_4 & c_5 \\ 0 & b^{-1} & c_6 \\ 0 & 0 & ab \end{pmatrix} \right) \;\middle|\; a, b \in \mathbb{C}^\times, c_i \in \mathbb{C} \right\}$$

and $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = \big\langle [e_2 \otimes f_2], [e_1 \wedge e_3 \otimes f_1 \wedge f_3]\big\rangle_{\mathbb{C}}$.

5.2. The modules $(\mathrm{GL}(n), \mathrm{Sym}^2\,\mathbb{C}^n)$ with $1 \leqslant n$

Here

$$E = \big\{2\omega_1, 2\omega_2, \ldots, 2\omega_n\big\} \quad \text{and} \quad d_W = n - 1.$$

Because $2\omega_n \in E$, there is no group G with $G' \subseteq G \subsetneq \overline{G}$ for which (G, W) is spherical. Hence we assume that $G = \overline{G} = \mathrm{GL}(n)$. For the same reason, $\mathcal{S} = \overline{\mathcal{S}}$ is not G-saturated. For the proof of the following proposition, see [26, Proposition 5.9].

Proposition 5.9. — *The T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ is multiplicity-free and has T_{ad}-weight set*

$$\big\{2\alpha_1, 2\alpha_2, \ldots, 2\alpha_{n-1}\big\}.$$

In particular, its dimension is d_W. Consequently, $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

5.3. The modules $(\mathrm{GL}(n), \bigwedge^2\mathbb{C}^n)$ with $2 \leqslant n$

Here

$$E = \left\{\omega_{2i} \colon 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad \text{and} \quad d_W = \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

When n is odd this module is spherical for $G' = \mathrm{SL}(n)$, because $\langle\omega_n\rangle_{\mathbb{Z}} \cap \overline{\Delta} = 0$, and $p(\mathcal{S})$ is G'-saturated. Corollary 2.27 therefore takes care of these cases.

On the other hand, when n is even, $\omega_n \in E$, and so there is no group G with $G' \subseteq G \subsetneq \overline{G}$ for which (G, W) is spherical. Moreover, for the same reason, $\mathcal{S} = \overline{\mathcal{S}}$ is not G-saturated. As it needs no extra work compared to $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, we show that $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ has dimension d_W. For the proof of the following proposition, see [26, Proposition 5.11].

Proposition 5.10. — *Suppose $n \geqslant 2$ is even. Then the T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is multiplicity-free and has T_{ad}-weight set*

$$\{\alpha_i + 2\alpha_{i+1} + \alpha_{i+2} \colon 1 \leqslant i \leqslant n-3 \text{ and } i \text{ is odd}\}.$$

In particular, $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = \frac{n}{2} - 1 = d_W$. *Consequently,* $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

5.4. The modules $(\mathrm{GL}(n)\times\mathbb{G}_m, \bigwedge^2\mathbb{C}^n \oplus \mathbb{C}^n)$ with $4 \leqslant n$

We now have

$$E = \left\{\omega_{2i-1}+\varepsilon \colon 1 \leqslant i \leqslant \left\lceil\frac{n}{2}\right\rceil\right\} \cup \left\{\omega_{2i} \colon 1 \leqslant i \leqslant \left\lfloor\frac{n}{2}\right\rfloor\right\} \quad \text{and} \quad d_W = n-2.$$

The modules W are not spherical for G' because $\Delta\cap\langle\omega_n,\varepsilon\rangle_{\mathbb{Z}} \neq 0$. Moreover, for the same reason, $\mathcal{S}$ is not G-saturated for any intermediate group G for which W is spherical. For the proof of the following proposition, see [26, Proposition 5.13].

Proposition 5.11. — *The T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is multiplicity-free with T_{ad}-weight set*

$$\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-2\}.$$

In particular, $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = d_W$. *Consequently,* $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

5.5. The modules $(\mathrm{GL}(n)\times\mathbb{G}_m, \bigwedge^2\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ with $4 \leqslant n$

For these modules we have

$$E = \{\lambda_i \colon 1 \leqslant i \leqslant n-2, i \text{ odd}\} \cup \{\lambda_j \colon 1 \leqslant j \leqslant n, j \text{ even}\} \cup \{\mu\} \quad \text{and } d_W = n-2,$$

where $\lambda_i := \omega_i + \varepsilon$ for $1 \leqslant i \leqslant n-2$ with i odd, $\lambda_j := \omega_j$ for $1 \leqslant j \leqslant n$ with j even, and $\mu := \omega_{n-1} - \omega_n + \varepsilon$.

These modules are not spherical for G' because $\Delta \cap \langle \omega_n, \varepsilon\rangle_{\mathbb{Z}} \neq 0$. Moreover, for the same reason, $\mathcal{S}$ is not G-saturated for any intermediate group G for which W is spherical.

Proposition 5.12. — *Suppose $n \geqslant 4$. The T_{ad}-module $T_{X_0}\mathrm{M}^G_{\mathcal{S}}$ is multiplicity-free and has T_{ad}-weight set*

$$\big\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-2\big\} \quad \textit{when } n \textit{ is even}; \tag{5.3}$$

$$\big\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-3\big\} \cup \big\{\alpha_{n-1}\big\} \quad \textit{when } n \textit{ is odd}. \tag{5.4}$$

In particular, $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

Proof. — When n is even, we are done by Proposition 5.13, because $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = d_W$. On the other hand, when n is odd, let J be the set (5.4) and put $\beta = \alpha_{n-2} + \alpha_{n-1}$. We prove in Proposition 5.14 that $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is multiplicity-free, and that its T_{ad}-weight set is $J \cup \{\beta\}$. In particular, $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = d_W + 1$. When β is not a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, it follows that $\dim(V/\mathfrak{g}\cdot x_0)^{G_{x_0}} \leqslant d_W$ and we are done. We show in Proposition 5.17 that even when β is a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, the corresponding section in $H^0(G\cdot x_0, \mathcal{N}_{X_0})^G$ does not extend to X_0. Consequently $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} \leqslant d_W$ and the proposition follows. □

Proposition 5.13. — *Suppose $n \geqslant 4$ is even. Then $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is a multiplicity-free T_{ad}-module with T_{ad}-weight set*

$$\big\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-2\big\}.$$

In particular, $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = d_W$.

Proof. — Consider the G-submodule V' of V defined as

$$V' := V(\lambda_1) \oplus V(\lambda_2) \oplus \cdots \oplus V(\lambda_{n-2}) \oplus V(\mu).$$

Note that as a G'-module, V' is the direct sum of the fundamental representations. Furthermore, $V = V' \oplus V(\lambda_n)$ and $V(\lambda_n)$ is one-dimensional. If we put $x_0' := x_0 - v_{\lambda_n}$, then we have $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} \simeq (V'/\mathfrak{g}'\cdot x_0')^{G'_{x_0'}}$, and by [3, Corollary 3.9 and Theorem 3.10] we know that $(V'/\mathfrak{g}'\cdot x_0')^{G'_{x_0'}}$ is a multiplicity-free T_{ad}-module whose T_{ad}-weight set is $\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-2\}$. □

When n is odd, determining $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ requires a little more care, because $V(\lambda_{n-1}) \simeq V(\mu)$ as G'-modules.

Proposition 5.14. — *Suppose $n \geqslant 5$ is odd. Then $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is a multiplicity-free T_{ad}-module. Its T_{ad}-weight set is*

$$\big\{\alpha_i + \alpha_{i+1} \colon 1 \leqslant i \leqslant n-2\big\} \cup \big\{\alpha_{n-1}\big\}. \tag{5.5}$$

The eigenspace of the weight $\beta = \alpha_{n-2} + \alpha_{n-1}$ is spanned by the vector

$$\big[X_{-\beta} v_{\lambda_{n-2}}\big] = -\big[X_{-\beta}(v_{\lambda_{n-1}} + v_\mu)\big].$$

Proof. — Let V' be the following G'-submodule of V:

$$V' := V(\lambda_1) \oplus V(\lambda_2) \oplus \cdots \oplus V(\lambda_{n-2}) \oplus V_{n-1}$$

where $V_{n-1} := \langle G' \cdot (v_{\lambda_{n-1}} + v_\mu) \rangle_{\mathbb{C}}$. Then

$$V = V' \oplus Z_{n-1}, \tag{5.6}$$

where $Z_{n-1} := \langle G' \cdot (v_{\lambda_{n-1}} - v_\mu) \rangle_{\mathbb{C}}$, and

$$\mathfrak{g} \cdot x_0 = \mathfrak{g}' \cdot x_0 \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu). \tag{5.7}$$

Moreover, we have an inclusion of $G'_{x_0} \rtimes T_{\mathrm{ad}}$-modules $\mathfrak{g} \cdot x_0 \subseteq V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu) \subseteq V$ and so an exact sequence

$$0 \longrightarrow \frac{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}{\mathfrak{g} \cdot x_0} \longrightarrow V/\mathfrak{g} \cdot x_0 \longrightarrow \frac{V}{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)} \longrightarrow 0.$$

Taking G'_{x_0}-invariants, we obtain an exact sequence of T_{ad}-modules

$$0 \longrightarrow \Big(\frac{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}{\mathfrak{g} \cdot x_0}\Big)^{G'_{x_0}} \longrightarrow (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} \longrightarrow \Big(\frac{V}{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}\Big)^{G'_{x_0}} \tag{5.8}$$

From (5.7) we have that

$$\frac{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}{\mathfrak{g} \cdot x_0} \simeq \frac{V'}{\mathfrak{g}' \cdot x_0}$$

as $G'_{x_0} \rtimes T_{\mathrm{ad}}$-modules. Clearly, as a G'-module, V' is the direct sum of the fundamental representations, and $\mathfrak{g}' \cdot x_0$ is the tangent space to the orbit of the sum of the highest weight vectors in V'. Therefore [3, Cor 3.9 and Thm 3.10] tells us that $\Big(\frac{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}{\mathfrak{g} \cdot x_0}\Big)^{G'_{x_0}}$ is a multiplicity-free T_{ad}-module with weight set $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{n-2} + \alpha_{n-1}\}$. On the other hand, (5.6) tells us that

$$\frac{V}{V' \oplus \mathbb{C}(v_{\lambda_{n-1}} - v_\mu)} \simeq \frac{Z_{n-1}}{\mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}.$$

Furthermore, we claim that

$$\Big(\frac{Z_{n-1}}{\mathbb{C}(v_{\lambda_{n-1}} - v_\mu)}\Big)^{G'_{x_0}} = \mathbb{C}\big[X_{-\alpha_{n-1}}(v_{\lambda_{n-1}} - v_\mu)\big]. \tag{5.9}$$

Indeed, if $[v]$ is a nonzero T_{ad}-eigenvector in $\left(\frac{Z_{n-1}}{\mathbb{C}(v_{\lambda_{n-1}}-v_\mu)}\right)^{G'_{x_0}}$ then there exists a simple root α so that $X_\alpha v \neq 0$ (because v is not a highest weight vector) and $X_\alpha v \in \mathbb{C}(v_{\lambda_{n-1}} - v_\mu) = Z^U_{n-1}$. Hence $X_\alpha v$ has trivial T_{ad}-weight and therefore v has weight α. Since $Z_{n-1} \simeq V(\omega_{n-1})$, this implies that $\alpha = \alpha_{n-1}$ and the claim (5.9).

From the sequence (5.8) and the description of its first and third term above, we know that the T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is multiplicity-free, and that its T_{ad}-weight set is a subset of (5.5) and contains all its weights except possibly α_{n-1}. But α_{n-1} belongs to the T_{ad}-weight set because $[X_{-\alpha_{n-1}}v_{\lambda_{n-1}}] = -[X_{-\alpha_{n-1}}v_\mu] \in (V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ by a straightforward verification (or because $s_{n-1\,n}$ is a "simple reflection" in Knop's List). The assertion about the eigenspace of weight β merely needs a straightforward verification. □

The next lemma determines for which groups G the weight $\beta = \alpha_{n-2} + \alpha_{n-1}$ is a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$.

Lemma 5.15. — *Suppose $n \geqslant 5$ is odd and let β be defined as in Proposition 5.14. Then the following are equivalent (recall that, by assumption, (G,W) is spherical)*

1. *β is a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$;*
2. *$\beta \in \Delta$;*
3. *$\mathfrak{t} = \ker[(a+1)\omega_n - (a-1)\varepsilon] \subseteq \mathrm{Lie}(\overline{T})$ for some integer a.*

If $\mathfrak{t} = \ker[(a+1)\omega_n - (a-1)\varepsilon]$ for some integer a, then we have the following equality in Δ:

$$\beta = \lambda_{n-2} + (a+1)\lambda_{n-1} - a\mu - \lambda_{n-3}. \tag{5.10}$$

Remark 5.16. — We use $\mathfrak{t} = \mathrm{Lie}(T)$ in Lemma 5.15 instead of T because $\ker[(a+1)\omega_n - (a-1)\varepsilon] \subseteq \overline{T}$ is not necessarily connected (for example, it is disconnected when $a = 1$).

Proof. — Since β is a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ by Proposition 5.13, the fact that (1) and (2) are equivalent follows from Lemma 2.17 (c). We now prove that (2) and (3) are equivalent. Recall that $r\colon X(\overline{T}) \twoheadrightarrow X(T')$ and $q\colon X(\overline{T}) \twoheadrightarrow X(T)$ are the restriction maps. Recall further that $\Delta = q(\overline{\Delta})$ and note that $\ker q \subseteq \ker r = \langle \omega_n, \varepsilon\rangle_{\mathbb{Z}}$. Now $\beta = -\omega_{n-3} + \omega_{n-2} + \omega_{n-1} - \omega_n \in X(\overline{T})$. So $q(\beta) \in \Delta$ if and only if $q(\beta + \lambda_{n-3} - \lambda_{n-2} - \lambda_{n-1}) = q(-\omega_n - \varepsilon) \in \Delta$. In other words, $q(\beta) \in \Delta$ if and only if there exists $\gamma \in \overline{\Delta}$ so that $q(-\omega_n - \varepsilon) = q(\gamma)$, that is, so that $\gamma + \omega_n + \varepsilon \in \ker q$. Since $\omega_n + \varepsilon \in \ker r$ this is equivalent to the existence of $\gamma \in \overline{\Delta} \cap \ker r$ so that $q(\gamma + \omega_n + \varepsilon) = 0$.

Next we claim that $\overline{\Delta}\cap\ker r = \langle\omega_n-\varepsilon\rangle$. The inclusion "$\supseteq$" is immediate: $\omega_n-\varepsilon=\lambda_{n-1}-\mu$. The other inclusion follows from a direct calculation, or from Knop's List which tells us that[(3)] $\langle\overline{\Delta}\rangle_{\mathbb{C}}\cap\langle\ker r\rangle_{\mathbb{C}}=\langle\omega_n-\varepsilon\rangle_{\mathbb{C}}$ as subspaces of $\mathrm{Lie}(\overline{T})^*$.

Consequently, $q(\beta)\in\Delta$ if and only if there exists an integer a so that

$$a(\omega_n-\varepsilon)+\omega_n+\varepsilon=(a+1)\omega_n-(a-1)\varepsilon$$

belongs to $\ker q$. Equivalently, $T\subseteq\ker[(a+1)\omega_n-(a-1)\varepsilon]$, or (since T is connected)

$$\mathfrak{t}\subseteq\ker\big[(a+1)\omega_n-(a-1)\varepsilon\big]. \tag{5.11}$$

On the other hand, [19, Theorem 5.1] tells us that W is spherical as a G-module if and only if

$$\mathfrak{t}\not\subseteq\ker(\omega_n-\varepsilon). \tag{5.12}$$

Because $\mathfrak{t}'=\langle\omega_n,\varepsilon\rangle_{\mathbb{C}}^{\perp}$ is of codimension 2 in $\mathrm{Lie}(\overline{T})$, and for every integer a, the two vectors $(a+1)\omega_n-(a-1)\varepsilon$ and $\omega_n-\varepsilon$ in $\mathrm{Lie}(\overline{T})^*$ are linearly independent, $\mathfrak{t}$ satisfies (5.12) and (5.11) for some integer a if and only if $\mathfrak{t}=\ker[(a+1)\omega_n-(a-1)\varepsilon]$. The equivalence of (2) and (3) follows. The straightforward verification of (5.10) is left to the reader. □

Proposition 5.17. — *Suppose $n\geqslant 5$ is odd and let β be defined as in Proposition 5.14. Let a be an integer and suppose that the maximal torus T of G satisfies $\mathfrak{t}=\ker[(a+1)\omega_n-(a-1)\varepsilon]$. Then the section $s\in H^0(G\cdot x_0,\mathcal{N}_{X_0})^G$ defined*[(4)] *by*

$$s(x_0)=\big[X_{-\beta}v_{\lambda_{n-2}}\big]=-\big[X_{-\beta}(v_{\lambda_{n-1}}+v_\mu)\big]\in(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$$

does not extend to X_0.

Proof. — We consider two cases: $a<0$ and $a\geqslant 0$.

(i) If $a<0$, we apply Proposition 3.4 with $\lambda=\mu$ and $v=X_{-\beta}v_{\lambda_{n-2}}$. We check the four conditions: (ES1) follows from equation (5.10); (ES2) is clear from the description of v given above; (ES3) follows from the equalities $\mu=\omega_{n-1}-\omega_n+\varepsilon$ and $\langle\lambda_{n-1},\alpha_{n-1}^\vee\rangle=1$; for (ES4) take $\delta=\lambda_{n-1}$.

(ii) If $a\geqslant 0$, we apply Proposition 3.4 with $\lambda=\lambda_{n-1}$ and the same v. We check the four conditions: (ES1) follows from equation (5.10); (ES2) is clear from the description of v given above; (ES3) follows from the equalities $\lambda_{n-1}=\omega_{n-1}$ and $\langle\mu,\alpha_{n-1}^\vee\rangle=1$; for (ES4) take $\delta=\mu$. □

(3) In the notation of Knop's List, $\mathfrak{a}^*\cap\mathfrak{z}^*$ is used for $\langle\overline{\Delta}\rangle_{\mathbb{C}}\cap\langle\ker r\rangle_{\mathbb{C}}$.

(4) The fact that this formula defines a section of $H^0(G\cdot x_0,\mathcal{N}_{X_0})^G\simeq(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ uses Lemma 5.15.

Remark 5.18. — We now obtain a description of the T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$; for details see [26, Remark 5.22]. For n even, this is done in Proposition 5.13 since $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}} = (V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$. For n odd, the T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is described in Proposition 5.14. Call its T_{ad}-weight set F. Now $(V/\mathfrak{g}\cdot x_0)^{\overline{G}_{x_0}}$ is the T_{ad}-submodule of $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ with T_{ad}-weight set $F \smallsetminus \{\beta\}$. Since $(V/\mathfrak{g}\cdot x_0)^{\overline{G}_{x_0}} \subseteq (V/\mathfrak{g}\cdot x_0)^{G_{x_0}} \subseteq (V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$, the T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ is completely determined by our characterization in Lemma 5.15 of those intermediate groups G for which β is a T_{ad}-weight of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$.

5.6. The modules $(\mathrm{GL}(m)\times\mathrm{GL}(n), (\mathbb{C}^m\otimes\mathbb{C}^n)\oplus\mathbb{C}^n)$ with $1\leqslant m, 2\leqslant n$

We begin with some notation. Put $K = \min(m+1, n)$ and $L = \min(m, n)$. We also put $\lambda_i = \omega_{i-1}+\omega'_i$ for $i\in\{1,\dots,K\}$ (with $\omega_0 = 0$), and $\lambda'_i = \omega_i+\omega'_i$ for $i\in\{1,\dots,L\}$. For the modules under consideration,

$$E = \left\{\lambda_i\colon 1\leqslant i\leqslant K\right\}\cup\left\{\lambda'_i\colon 1\leqslant i\leqslant L\right\}$$
$$d_W = K+L-2 = \min(2m+1, 2n)-2.$$

These modules are not spherical for G' because $\Delta\cap\langle\omega_m,\omega'_n\rangle_{\mathbb{Z}}\neq 0$. Moreover, for the same reason, $\mathcal{S}$ is not G-saturated for any intermediate group G for which W is spherical. For the proof of the following proposition, see [26, Proposition 5.23].

Proposition 5.19. — *The T_{ad}-module $T_{X_0}\mathrm{M}^G_{\mathcal{S}}$ is multiplicity-free and has T_{ad}-weight set*

$$\left\{\alpha_i\colon 1\leqslant i\leqslant L-1\right\}\cup\left\{\alpha'_j\colon 1\leqslant j\leqslant K-1\right\}. \tag{5.13}$$

In particular, $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

Remark 5.20. — We remark that except for a few small values of m and n, the inclusion $T_{X_0}\mathrm{M}^G_{\mathcal{S}} \subseteq (V/\mathfrak{g}\cdot x_0)^{\overline{G}_{x_0}}$ turns out to be strict. Moreover, for $n = m-1$ and for $m = n-2$ there exist groups $G\subseteq\overline{G}$, containing $\overline{G}'$, for which W is spherical and for which the inclusion $(V/\mathfrak{g}\cdot x_0)^{\overline{G}_{x_0}}\subseteq(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ is strict. For details see [26, Remark 5.24].

5.7. The modules $(\mathrm{GL}(m)\times\mathrm{GL}(n), (\mathbb{C}^m\otimes\mathbb{C}^n)\oplus(\mathbb{C}^n)^*)$ with $1\leqslant m, 2\leqslant n$

We begin with some notation. Put $K = \min(m, n-1)$ and $L = \min(m, n)$. We also put $\lambda_i = \omega_i+\omega'_{i-1}$ for $i\in\{1,\dots,K\}$ (with $\omega'_0 = 0$), $\mu = \omega'_{n-1}-\omega'_n$, and $\lambda'_i = \omega_i + \omega'_i$ for $i \in \{1, \ldots, L\}$. For the modules under consideration,

$$E = \big\{\lambda_i \colon 1 \leqslant i \leqslant K\big\} \cup \big\{\lambda'_i \colon 1 \leqslant i \leqslant L\big\} \cup \{\mu\};$$
$$d_W = K + L - 1 = \min(2m+1, 2n) - 2.$$

These modules are not spherical for G' because $\Delta \cap \langle \omega_m, \omega'_n \rangle_{\mathbb{Z}} \neq 0$. Moreover, for the same reason, $\mathcal{S}$ is not G-saturated for any intermediate group G for which W is spherical. For the proof of the following proposition, see [26, Proposition 5.46].

Proposition 5.21. — *The T_{ad}-module $T_{X_0}\mathrm{M}^G_{\mathcal{S}}$ is multiplicity-free. Its T_{ad}-weight set is*

$$\big\{\alpha_i \colon 1 \leqslant i \leqslant L-1\big\} \cup \big\{\alpha'_j \colon 1 \leqslant j \leqslant K-1\big\} \cup \big\{\alpha'_K + \alpha'_{K+1} + \cdots + \alpha'_{n-1}\big\}. \tag{5.14}$$

In particular, $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

5.8. The modules $(\mathrm{GL}(m) \times \mathrm{SL}(2) \times \mathrm{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n))$ with $2 \leqslant m \leqslant n$

Here

$$E = \big\{\omega_1 + \omega', \omega' + \omega''_1, \omega_1 + \omega''_1, \omega_2, \omega''_2\big\} \quad \text{and} \quad d_W = 3.$$

In this case $\mathcal{S}$ is not G-saturated for any group G for which W is spherical as one easily checks using Lemma 2.23. The module W is spherical for G' if and only if $m > 2$. For the proof of the following proposition, see [26, Proposition 5.57].

Proposition 5.22. — *The T_{ad}-module $(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}}$ is multiplicity-free and its T_{ad}-weight set is $\{\alpha_1, \alpha', \alpha''_1\}$. In particular,* $\dim(V/\mathfrak{g}\cdot x_0)^{G'_{x_0}} = d_W$. *Consequently,* $\dim T_{X_0}\mathrm{M}^G_{\mathcal{S}} = d_W$.

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