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LOCAL RIGIDITY OF ASPHERICAL THREE-MANIFOLDS

by Pierre DERBEZ (*)

Abstract. — In this paper we construct, for each aspherical oriented 3-manifold M, a 2-dimensional class in the l_1-homology of M whose norm combined with the Gromov simplicial volume of M gives a characterization of those nonzero degree maps from M to N which are homotopic to a covering map. As an application we characterize those degree one maps which are homotopic to a homeomorphism in term of isometries between the bounded cohomology groups of M and N.

Résumé. — Dans ce papier nous construisons, pour chaque variété de dimension trois close orientable et asphérique M, une classe d'homologie l_1 de dimension deux dans M dont la norme permet avec le volume simplicial de M de caractériser les applications de degré non-nul de M dans N qui sont homotopes à un revêtement. Comme conséquence, nous donnons un critère d'homéomorphisme pour les applications de degré un en terme d'isométries entre les groupes de cohomologie bornée de M et N.

1. Introduction

Throughout this paper all manifolds are orientable. Given a topological space X we denote by $(C_*(X), \partial)$ its real singular chain complex endowed with the l_1-norm defined by $\|\sigma\|_1 = \sum_i |a_i|$ if $\sigma = \sum_i a_i\sigma_i$, where σ_i are singular simplices.

Any finite covering map $f\colon M \to N$ between closed orientable 3-manifolds induces an isometry $f_\sharp\colon H_3(M;\mathbf{R}) \to H_3(N;\mathbf{R})$ with respect to the l_1 (semi) norm induced by the l_1-norms on the real singular chains of M and N.

For hyperbolic manifolds this condition is sufficient to characterize covering maps by Gromov and Thurston's works. However, since the Gromov

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simplicial volume of a 3-manifold M, which is the l_1-norm $\|[M]\|_1$ of a generator $[M]$ of $H_3(M;\mathbf{Z}) \subset H_3(M;\mathbf{R})$, does not detect the "non-hyperbolic part" of 3-manifolds one can construct, using pinching maps, many pairwise non-homeomorphic 3-manifolds with the same Gromov simplicial volume related by a degree one map.

When M is a surface bundle over the circle with a fiber of negative Euler characteristic, M. Boileau and S. Wang gave in [3, Theorem 2.1, Corollary 2.3] a characterization of nonzero degree maps $f\colon M \to N$ into an irreducible 3-manifold which are homotopic to a covering map in terms of isometry with respect to the Thurston's norm in the second homology group of the manifolds. The purpose of this paper is to extend [3, Theorem 2.1] to aspherical 3-manifolds.

According to the Geometrization Theorem of Perelman, any closed aspherical 3-manifold M admits a JSJ-splitting along a family of characteristic tori $\mathcal{T}_M$ such that each component of $M \setminus \mathcal{T}_M$ either admits a Seifert fibration or has a complete finite volume hyperbolic interior.

We say that M is *orientable** if M is orientable and if each Seifert component of $M\setminus\mathcal{T}_M$ admits a fibration over an orientable surface. This condition is satisfied for example when M contains no embedded Klein bottle or when M is obtained from a holomorphic function $f\colon (\mathbf{C}^3,0) \to (\mathbf{C},0)$ with an isolated singularity at 0 by taking the boundary of the singularity of f at 0 defined by $f^{-1}(0) \cap S(\varepsilon)$, where $S(\varepsilon)$ is a Milnor sphere centered at 0 in $\mathbf{C}^3$ with radius ε (see [18]). Notice that this orientation* condition is also satisfied when M is a surface bundle with a fiber of negative Euler characteristic ([3]).

In [3, Theorem 2.1], a key point, is that when M is a surface bundle, there there exists a class $\alpha_M \in H_2(M)\setminus\{0\}$, namely the class of the fiber, "passing non-trivially through the whole manifold". Of course, such a *fiber class*, does not exist in the homology of a general 3-manifold because if we try to define local classes in M there are often homological obstructions which do not allow to glue them together in order to define a global class. However these obstructions disappear considering the l_1-completion $H_2^{l_1}(M)$ of $H_2(M)$ and a *fiber class* α_M can be defined in $H_2^{l_1}(M)$ as follows. Let M be a closed orientable aspherical 3-manifold :

When M is a geometric 3-manifold, set $\alpha_M = 0$ excepted when M is a $\widetilde{\mathrm{SL}}_2(\mathbf{R})$-manifold. In this case, M admits a finite covering $p\colon \widetilde{M} \to M$ which is a (true) circle bundle $\xi\colon \widetilde{M} \to \widetilde{F}$ over a smooth surface. Then we set

$$\alpha_M = p_\sharp \circ \xi_\sharp^{-1}\left(\left[\widetilde{F}\right]_1\right)$$

where $\left[\widetilde{F}\right]_1$ denotes the l_1-class of the l_1-cycle $\widetilde{F}$. This makes sense since by [9, Mapping Theorem] ξ induces an isometric isomorphism $\xi_\sharp\colon H_2^{l_1}(\widetilde{M}) \to H_2^{l_1}(\widetilde{F})$.

When M is not a geometric 3-manifold, each Seifert component of $M\setminus\mathcal{T}_M$ admits either a Euclidean geometry or a $\mathbf{H}^2\times\mathbf{R}$-geometry. For each $\mathbf{H}^2\times\mathbf{R}$-component P_i, $i = 1,...,l$, of $M \setminus \mathcal{T}_M$ we choose a horizontal properly embedded incompressible surface $\mathcal{F}_i$ in P_i and we set

$$\alpha_M = \sum_{i=1}^{l} \frac{1}{k_i}\alpha_M\left(\mathcal{F}_i\right)$$

where k_i denotes the intersection number between $\mathcal{F}_i$ and the generic fiber of P_i and where $\alpha_M\left(\mathcal{F}_i\right)$ denote the l_1-class of $\mathcal{F}_i$ in M which makes sense since the relative cycle $\mathcal{F}_i$ of P_i can be "filled" in a natural way giving a l_1-cycle in M (see paragraph 2). If $M\setminus\mathcal{T}_M$ contains no $\mathbf{H}^2\times\mathbf{R}$-components we just set $\alpha_M = 0$.

Remark 1.1. — Obviously, it follows from our construction that our fiber class does not need to be unique, as well as the fiber class of a surface bundle when the rank of the homology is distinct from 1, by a result of [17]. On the other hand, it follows from our proof of Theorem 1.2 that our results hold for any choices of a fiber class.

The main result of this paper states as follows

THEOREM 1.2. — *Let $f\colon M \to N$ be a nonzero degree map from a closed orientable* aspherical* 3*-manifold into a closed orientable irreducible* 3*-manifold such that $\|f_\sharp([M])\|_1 = \|[M]\|_1$ and $\|f_\sharp(\alpha_M)\|_1 = \|\alpha_M\|_1$ for some fiber class α_M. Then f is homotopic to a* $\deg(f)$*-fold covering map.*

To make the hypothesis $\|f_\sharp(\alpha_M)\|_1 = \|\alpha_M\|_1$ more concrete one can compare it with a condition given in [6] where we indroduce an invariant denoted by $\mathrm{vol}(M)$ and defined as the sum of the absolute value of the orbifolds Euler characteristic of the Seifert pieces of M. This volume is used to state rigidity results, see [6, Theorems 1.3 and 1.6]. Using sections 2 and 3 of this paper and results in [6] one can easily check that $\|\alpha_M\|_1 = \mathrm{vol}(M)$ and if $\|f_\sharp([M])\|_1 = \|[M]\|_1$, meaning that $\|M\| = |\deg(f)|\|N\|$, then $\|\alpha_M\|_1 = \mathrm{vol}(M) \geqslant \|f_\sharp\alpha_M\|_1 \geqslant \|\alpha_N\|_1 = \mathrm{vol}(N)$.

As far as we know, there are no general results to characterize local isometries for aspherical 3-manifolds excepted when the sectional curvature is negative. In the situation we deal with, the best metric we can hope, in many cases, is a metric with non-positive curvature by [14] and our manifolds contain many totally geodesic surfaces where the curvature vanishes. From the point of view of maps $f\colon M \to N$ there are more flexibility when the sectional curvature of M vanishes and so it is more difficult to control the behavior of f. On the other hand, we hope that our results offer an application of the theory of bounded cohomology and l_1-homology.

Notice that if M and N are both orientable* then the isometry condition is also necessary (see Lemma 2.2 and Proposition 2.4). If N is not orientable* the condition is not necessary. Indeed, consider for N the trivial orientable $\mathbf{S}^1$-bundle over the genus -3 surface $\mathbf{R}P(2)\sharp\mathbf{R}P(2)\sharp\mathbf{R}P(2)$ and for M the trivial bundle $\Sigma_2 \times \mathbf{S}^1$ which is a 2-fold covering $p\colon M \to N$, where Σ_2 is the genus 2-surface. Let α_M denote the class of Σ_2 in $H_2^{l_1}(M;\mathbf{R})$. Then it follows from the arguments of section 2 that $\|\alpha_M\|_1 > 0$ and $p_\sharp(\alpha_M) = 0$.

By the Hahn-Banach Theorem, for each fiber class α_M with $\|\alpha_M\|_1 > 0$, there exists a class β_M in the second bounded cohomology group of M, denoted by $H_b^2(M;\mathbf{R})$ and endowed with the semi-norm $\|.\|_\infty$, such that $\langle\beta_M, \alpha_M\rangle = 1$ and $\|\beta_M\|_\infty = \frac{1}{\|\alpha_M\|_1}$. When $\|\alpha_M\|_1 = 0$, just set $\beta_M = 0$. Thus we deduce the following

Corollary 1.3. — *Let $f\colon M \to N$ be a nonzero degree map from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold such that $\|f_\sharp([M])\|_1 = \|[M]\|_1$. If there exists a class $\beta \in H_b^2(N;\mathbf{R})$ such that $f^\sharp(\beta) = \beta_M$ and $\|\beta_M\|_\infty = \|\beta\|_\infty$ then f is homotopic to a covering map.*

We give the following corollary answering positively to a question of Professor M. Boileau.

Corollary 1.4. — *A degree one map $f\colon M \to N$ from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a homeomorphism iff*

(i) $f_\sharp\colon H_3(M;\mathbf{R}) \to H_3(N;\mathbf{R})$ is an isometry with respect to the l_1-norms and

(ii) f induces an isometric isomorphism $f^\sharp\colon H_b^2(N;\mathbf{R})) \to H_b^2(M;\mathbf{R})$, resp. an isometry $f_\sharp\colon H_2^{l_1}(M;\mathbf{R}) \to H_2^{l_1}(N;\mathbf{R})$.

Theorem 1.5. — *A nonzero degree map $f\colon M \to N$ from a closed orientable aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a covering map iff it induces a homomorphism $f_*\colon \pi_1 M \to \pi_1 N$ with amenable kernel.*

We end this section by mentioning a related result for self maps which is a direct consequence of [24] and [13, Theorem 0.7] using a standard covering space argument suggested by Professor W. Lück:

Theorem 1.6. — *Any nonzero degree map $f\colon M \to M$ from a closed orientable aspherical 3-manifold to itself is homotopic to a $\deg(f)$-fold covering.*

Organization of the paper. This paper is organized as follows. In Section 2 we collect some technical results which will be used in the proof of the theorem. More precisely we compute the l_1-norm of certain classes in $H_2^{l_1}(M)$ which come from classical integral homology classes and we study some isometric properties of finite coverings with respect to the l_1-norms. Section 3 is devoted to the proof of Theorems 1.2 and 1.5.

2. Norm of surfaces in aspherical 3-manifolds

To fix the notations we recall the basic definitions of l_1-homology and bounded cohomology according to the main papers of [16] and [9]. For a topological space X, denote by $C_*^{l_1}(X)$ the l_1-completion of the real singular chains $C_*(X)$. Then

$$C_n^{l_1}(X) = \left\{ c = \sum_{i=1}^{\infty} a_i\sigma_i \text{ s.t. } \|c\|_1 = \sum_{i=1}^{\infty} |a_i| < \infty \right\}$$

where $a_i \in \mathbf{R}$ and $\sigma_i\colon \Delta_n \to X$ is a singular n-simplex. We will denote by $S_n(X)$ the set of singular n-simplices. The topological dual of $C_*^{l_1}(X)$ is given by the set

$$C_b^n(X) = \left\{ w \in C^n(X) \text{ s.t. } \|w\|_\infty = \sup_{\sigma\in S_n(X)} |\langle w, \sigma\rangle| < \infty \right\}$$

Note that the ∂ and δ operators are bounded so that $(C_*^{l_1}(X), \partial)$ and $(C_b^*(X), \delta)$ are chain, resp. cochain, complexes. We denote by $H_*^{l_1}(X)$, resp. by $H_b^*(X)$, the homology, resp. cohomology, of this chain, resp. cochain, complex. The vector spaces $H_*^{l_1}(X)$ and $H_b^*(X)$ are endowed with the quotient semi-norm that we still denote by $\|.\|_1$ and $\|.\|_\infty$ respectively. In the same way it is a standard fact that one can define the l_1-homology and bounded cohomology of a pair of topological spaces (X, A). Denote by $i\colon A \to X$ the natural inclusion and by $j\colon C_*^{l_1}(X) \to C_*^{l_1}(X, A)$ the projection. Then we get the classic long exact sequence

$$\ldots \to H_n^{l_1}(A) \stackrel{i_\sharp}{\to} H_n^{l_1}(X) \stackrel{j_\sharp}{\to} H_n^{l_1}(X, A) \stackrel{\partial_\sharp}{\to} H_{n-1}^{l_1}(A) \to \ldots$$

If moreover each component of A has an amenable fundamantal group then by [16, Corollary 2.5] we know that $H_n^{l_1}(A) = \{0\}$ for any $n \geqslant 1$ and thus $j_\sharp$ admits an inverse $j_\sharp^{-1}\colon H_n^{l_1}(X, A) \to H_n^{l_1}(X)$ for $n \geqslant 2$ defined by $j_\sharp^{-1}([z]) = [z + u]$ where z is a relative cycle in (X, A) and u is any l_1-chain in A such that $\partial u = -\partial z$. It follows from the definition that any continuous map of pairs $f\colon (X, A) \to (Y, B)$ induces a bounded homomorphism $f_\sharp\colon H_n(X, A) \to H_n(Y, B)$ such that $\|f_\sharp\| \leqslant 1$. On the other hand, when M is compact orientable n-manifold with (possibly empty) boundary we will denote by $[M]$ its fundamental class in $H_n(M, \partial M)$, by $[M]_1$ the image of $[M]$ under the homomorphism $H_n(M, \partial M) \to H_n^{l_1}(M, \partial M)$ induced by the completion and by $\|M\|$ its Gromov simplicial volume. For technical reasons we need the following

Lemma 2.1. — *Let $p\colon \widetilde{X} \to X$ be a regular covering map with finite Galois group Γ. For any Γ-invariant class $\alpha \in H_n^{l_1}(\widetilde{X})$ then $\|p_\sharp(\alpha)\|_1 = \|\alpha\|_1$.*

Proof. — We use the averaging retraction $A\colon C_b^n(\widetilde{X}) \to C_b^n(X)$ defined in [9] by

$$\langle A(\gamma), \sigma\rangle = \frac{\sum_{g\in\Gamma} \langle g^\sharp\gamma, \widetilde{\sigma}\rangle}{\mathrm{Card}(\Gamma)}$$

where $\widetilde{\sigma}\colon \Delta^n \to \widetilde{X}$ denotes a lifting of $\sigma\colon \Delta^n \to X$. This definition does not depend one the choice of the lifting $\widetilde{\sigma}$ since the covering is regular. By construction, A commutes with the differentials so that it induces a homomorphism $\widehat{A}\colon H_b^n(\widetilde{X}) \to H_b^n(X)$ such that $\|\widehat{A}\| \leqslant 1$. Let $\alpha \in H_n^{l_1}(\widetilde{X})$ such that $g_\sharp(\alpha) = \alpha$ for any $g \in \Gamma$. If $\|\alpha\|_1 \neq 0$ then by the Hahn-Banach Theorem, there exists $\beta \in H_b^n(\widetilde{X})$ such that $\langle \beta, \alpha\rangle = 1$ and $\|\beta\|_\infty = \frac{1}{\|\alpha\|_1}$. Since α is Γ-invariant then by the definition of the averaging we have $\left\langle \widehat{A}(\beta), p_\sharp(\alpha)\right\rangle = 1$ and thus using the Hölder inequality and the fact that $\|\widehat{A}\| \leqslant 1$ we deduce $\|p_\sharp(\alpha)\|_1 \geqslant \|\alpha\|_1$. This proves the lemma. □

2.1. $\widetilde{\mathrm{SL}_2}(\mathbf{R})$-manifolds

Let M be an orientable* 3-manifold admitting a $\widetilde{\mathrm{SL}_2}(\mathbf{R})$-geometry. If moreover M is a (true) circle bundle, with projection ξ and base F then by [9, Mapping Theorem] ξ induces an isometric isomorphism $\xi_\sharp\colon H_2^{l_1}(M) \to H_2^{l_1}(F)$. Denote by $\alpha_M(F)$ the class $\xi_\sharp^{-1}([F]_1)$.

Lemma 2.2. — *Let M be an orientable* $\widetilde{\mathrm{SL}}_2(\mathbf{R})$-manifold.*
(i) If M is a (true) circle bundle with base F then

$$\|\alpha_M(F)\|_1 = \|F\|$$

(ii) Otherwise, for any finite covering $p\colon\ \widetilde{M} \to M$ such that $\widetilde{M}$ is a (true) circle bundle over a surface F and projection $\xi\colon\ \widetilde{M} \to F$ then $\|p_\sharp \alpha_{\widetilde{M}}(F)\| = \|F\|$.

(iii) Moreover when $\widetilde{M}$ is a circle bundle, the vector space generated by $p_\sharp \alpha_{\widetilde{M}}(F)$ does not depend on the choice of the finite covering $p\colon\ \widetilde{M} \to M$.

Proof. — We first check point (i). The inequality $\|\alpha_M(F)\|_1 \leqslant \|F\|$ follows from the definition. To check the converse inequality we use exactly the same construction as in [2]. Fix a complete hyperbolic metric on F. Since F is orientable we can define a bounded 2-cocyle Ω_F in F in the following way: for each 2-simplex $\sigma\colon\ \Delta^2 \to F$, where Δ^2 denotes the standard 2-simplex, we set $\langle \Omega_F, \sigma\rangle = \mathcal{A}(\mathrm{st}(\sigma))$, where $\mathrm{st}(\sigma)$ denotes the geodesic simplex obtained from σ after straightening and $\mathcal{A}$ denotes the algebraic area with respect to the fixed hyperbolic metric. In particular we get, if z denotes a 2-cycle representing the fundamental class of F

$$\left\langle \xi^\sharp([\Omega_F]), \xi_\sharp^{-1}([F]_1)\right\rangle = \langle [\Omega_F], [F]_{l_1}\rangle = \langle \Omega_F, z\rangle = \mathrm{Area}(F)$$

where $[\Omega_F] \in H^2_b(F)$ and $[F]_{l_1} \in H_2^{l_1}(F)$. Since by the construction $\|\xi^\sharp([\Omega_F])\|_\infty = \|[\Omega_F]\|_\infty \leqslant \pi$ then by the Hölder inequality we get $\|\alpha_M(F)\|_1 \geqslant \|F\|$. This proves point (i). We now prove point (ii). We consider two casis depending on whether the covering is regular or not.

Case 1. Assume that p is regular. Denote by Γ the Galois group of the covering. Note that since M is a Seifert bundle with orientable base 2-orbifold then any $g \in \Gamma$ induces an orientation preserving homeomorphism $\overline{g}\colon\ F \to F$ such that $\xi \circ g = \overline{g} \circ \xi$ and thus $\alpha_{\widetilde{M}}(F)$ is Γ-invariant and point (ii) of the lemma follows from Lemma 2.1 and point (i). This completes the proof of point (ii) in Case 1.

Case 2. If p is not regular then consider a finite covering $q\colon\ \widehat{M} \to \widetilde{M}$ such that $p \circ q$ is regular. Since $q_\sharp\left(\left\langle \alpha_{\widehat{M}}\left(\widehat{F}\right)\right\rangle\right) = \left\langle \alpha_{\widetilde{M}}(F)\right\rangle$, where $\langle v\rangle$ denotes the vector space generated by the vector v and where $\widehat{F}$ is the base of the bundle $\widehat{M}$, then point (ii) in Case 2 follows from Case 1.

To check point (iii) it suffices to consider a common covering $\widetilde{M}$ to $\widetilde{M}_1$ and $\widetilde{M}_2$ (which corresponds for example to $(p_1)_*(\pi_1\widetilde{M}_1) \cap (p_2)_*(\pi_1\widetilde{M}_2)$). This completes the proof of the lemma. □

2.2. Aspherical 3-manifolds

Let M be a closed orientable* aspherical 3-manifold. We fix an orientation on M. In the following we will assume that $\mathbf{H}^2$ and $\mathbf{R}$ are oriented with the usual convention. Let P denote a component of $M \setminus \mathcal{T}_M$ whose interior admits a $\mathbf{H}^2 \times \mathbf{R}$-geometry. Since M is orientable* then P admits a Seifert fibration over an orientable basis and we denote by h_P the fiber of P. We orient the fiber h_P in such a way that the universal covering $p\colon \mathbf{H}^2 \times \mathbf{R} \to P$ induces an orientation preserving map $\mathbf{R} \to h_P$. Let $\mathcal{F}$ be an oriented surface and let $f\colon (\mathcal{F}, \partial\mathcal{F}) \to (P, \partial P)$ be a proper map. For any $x \in \mathbf{R}$ we denote by $\alpha_M(x\mathcal{F}, f)$ the class defined by $k_\sharp j_\sharp^{-1} f_\sharp(x[\mathcal{F}]_1)$ following the composition of homomorphisms:

$$H_2^{l_1}(\mathcal{F}, \partial\mathcal{F}) \overset{f_\sharp}{\to} H_2^{l_1}(P, \partial P) \overset{j_\sharp^{-1}}{\to} H_2^{l_1}(P) \overset{k_\sharp}{\to} H_2^{l_1}(M)$$

where $k\colon P \to M$ denotes the inclusion.

Lemma 2.3. — *We have* $\|\alpha_M(x\mathcal{F}, f)\|_1 \leqslant |x|\|\mathcal{F}\|$ *for any* $x \in \mathbf{R}$.

Proof. — The proof follows from [9, Equivalence Theorem] combined with [16, Theorem 2.3]. □

Consider now a proper map $f\colon (\mathcal{F}, \partial\mathcal{F}) \to (P, \partial P)$ transverse to the fibers of P. We choose always the orientation of each component of F of $\mathcal{F}$ so that so that f is *orientation preserving* which means that the orientation of $f(\mathcal{F})$ followed by the orientation of h_P matches the orientation induced by M. The main purpose of this section is to check the following

Proposition 2.4. — *Let M be a closed aspherical orientable* 3-manifold and denote by $P_1, ..., P_l$ a collection of pairwise distinct Seifert components of $M \setminus \mathcal{T}_M$ whose interior admits a $\mathbf{H}^2 \times \mathbf{R}$-geometry. For each $i = 1, ..., l$ assume that we are given an orientation preserving proper embedding $f_i\colon (\mathcal{F}_i, \partial\mathcal{F}_i) \to (P_i, \partial P_i)$. Then*

(i) Isometry: for any $i = 1, ..., l$ we have the equality

$$\|\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\mathcal{F}_i\|$$

(ii) Additivity under JSJ-splitting:

$$\|\alpha_M(x_1\mathcal{F}_1, f_1) + ... + \alpha_M(x_l\mathcal{F}_l, f_l)\|_1 = \|\alpha_M(x_1\mathcal{F}_1, f_1)\|_1 + ... + \|\alpha_M(x_l\mathcal{F}_l, f_l)\|_1$$

where $x_1, ..., x_l$ are positive real numbers.

(iii) Let $f\colon M \to N$ be a covering map with N orientable. If $\alpha = \alpha_M(x_1\mathcal{F}_1, f_1) + ... + \alpha_M(x_l\mathcal{F}_l, f_l)$ then $\|f_\sharp(\alpha)\|_1 = \|\alpha\|_1$.*

To prove this proposition we need two intermediate results. Hypothesis are the same as in Proposition 2.4.

LEMMA 2.5. — *Suppose that $\{P_i\}_{i\in I}$ is a family of circle bundles components of $M\setminus\mathcal{T}_M$ admitting a $\mathbf{H}^2\times\mathbf{R}$-geometry. For any $i\in I$ there exists a bounded 2-cocycle Ω_{P_i} in M satisfying the following properties:*

(i) $k_i^*(\Omega_{P_i})$ is a relative 2-cocycle in $(P_i,\partial P_i)$ where $k_i\colon P_i\hookrightarrow M$ denotes the natural inclusion and $k_i^*(\Omega_{P_j})=0$ if $i\neq j$,

(ii) $|\langle[\Omega_{P_i}],\alpha_M(\mathcal{F}_i,f_i)\rangle|=\mathrm{Area}(\mathcal{F}_i)$ where $\mathrm{Area}(\mathcal{F}_i)$ denotes the area of $\mathrm{int}(\mathcal{F}_i)$ endowed with a complete hyperbolic metric.

(iii) $\left\|\sum_{i\in I}[\Omega_{P_i}]\right\|_\infty=\pi$, where $[\Omega_{P_i}]$ denotes the class of Ω_{P_i} in $H_b^2(M;\mathbf{R})$.

Remark 2.6. — The above result is stated for Seifert pieces which are circle bundles only for convenience. This lemma remains true if we consider a family of Seifert pieces admitting a geometry $\mathbf{H}^2\times\mathbf{R}$ with an orientable base 2-orbifold. Notice that the bounded class Ω_{P_i} cannot be defined for Seifert pieces with non-orientable basis.

To prove this lemma we need the reduction of singular chains with respect to the JSJ-splitting of aspherical 3-manifolds. This chain map is stated for example in [8]. Since this construction is crucial for our purpose we recall it and fix notation.

Let M be a closed aspherical orientable 3-manifold. Denote by $P_1,\dots,P_l$ the components of $M\setminus\mathcal{T}_M$. As in [8], we consider a chain map $\rho\colon C_n(M)\to C_n(M)$ defined as follows:

0-*simplices.* If $n=0$ then ρ is the identity.

1-*simplices.* If $n=1$ let $\tau\colon[v_0,v_1]\to M$ be a 1-simplex. Since $\mathcal{T}_M$ is incompressible, the map τ is homotopic, rel. $\{v_0,v_1\}$, to a *reduced* 1-simplex i.e. a map $\tau_1\colon[v_0,v_1]\to M$ such that either $\tau_1([v_0,v_1])\subset\mathcal{T}_M$ or $\tau_1|(v_0,v_1)$ is transverse to $\mathcal{T}_M$ and for each component J of $\tau_1^{-1}(P_i)$ then $\tau_1|J$ is not homotopic rel. ∂J into ∂P_i. Then we set $\rho(\tau)=\tau_1$ and we extend ρ by linearity.

2-*simplices.* If $n=2$ let $\sigma\colon\Delta^2=[v_0,v_1,v_2]\to M$ be a 2-simplex. Then σ is homotopic rel. $V(\Delta^2)=\{v_0,v_1,v_2\}$ to a reduced 2-simplex σ_1 such that either $\sigma_1(\Delta^2)\subset\mathcal{T}_M$ or $\sigma_1|\mathrm{int}(\Delta^2)$ is transverse to $\mathcal{T}_M$, the 1-simplex $\sigma_1|e$ is reduced for each edge e of Δ^2 and $\sigma_1^{-1}(\mathcal{T}_M)$ contains no loop components. Thus if J is a component of $\sigma_1^{-1}(\mathcal{T}_M)$ such that $J\cap\mathrm{int}(\Delta^2)\neq\emptyset$ then J is a proper arc in Δ^2 connecting two distinct edges (see figure 2.1). Then we set $\rho(\sigma)=\sigma_1$ and we extend ρ by linearity.

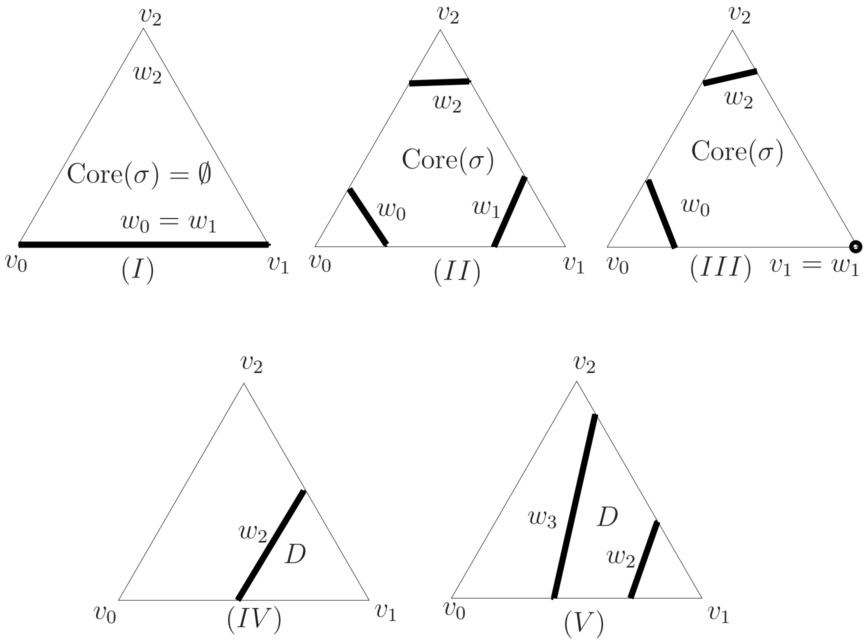


Figure 2.1

Remark 2.7. — Suppose that $\sigma\colon \Delta^2 \to M$ is a reduced 2-simplex. If $\sigma(e)$ is not contained in $\mathcal{T}_M$ for any edge e of Δ^2 then there exists a unique component, denoted by $\mathrm{Core}(\sigma)$, of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ which meets the three edges of Δ^2 (see [8]).

3-simplices. If $n = 3$ let $\sigma\colon \Delta^3 = [v_0, v_1, v_2, v_3] \to M$ be a 3-simplex. Then σ is homotopic rel. $V(\Delta^3) = \{v_0, v_1, v_2, v_3\}$ to a reduced 3-simplex σ_1 such that either $\sigma_1(\Delta^3) \subset \mathcal{T}_M$ or $\sigma_1|\mathrm{int}(\Delta^3)$ is transverse to $\mathcal{T}_M$, the 2-simplex $\sigma_1|\Delta^2_i$ is reduced for each face Δ^2_i of Δ^3 and if D is a component of $\sigma_1^{-1}(\mathcal{T}_M)$ such that $D \cap \mathrm{int}(\Delta^3) \neq \emptyset$ then D is either a normal triangle or a normal rectangle (see figure 2.2). Then we set $\rho(\sigma) = \sigma_1$ and we extend ρ by linearity. Notice that the reduction is stable under the ∂-operator.

Proof of Lemma 2.5. — We use here the technique developed in [1].

Step 1: Crushing M into P_i. Denote by $p_i\colon \widetilde{M}_i \to M$ the covering map corresponding to the subgroup $(k_i)_*(\pi_1 P_i)$ of $\pi_1 M$, fix a lifting $\widetilde{k}_i\colon P_i \to \widetilde{M}_i$ of $k_i\colon P_i \to M$ and denote by $\widetilde{P}_i$ the image of $\widetilde{k}_i$. There exists a retraction $r_i\colon \widetilde{M}_i \to P_i$ crushing each component of $\widetilde{M}_i \setminus \widetilde{P}_i$ to the corresponding component of $\partial\widetilde{P}_i$ such that $r_i|\widetilde{P}_i = \widetilde{k}_i^{-1}$. Denote by F_i the base surface

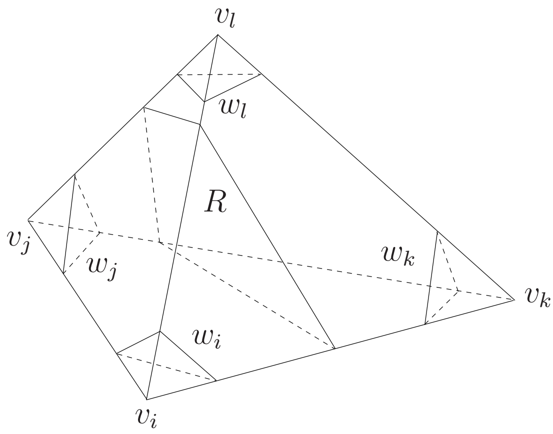
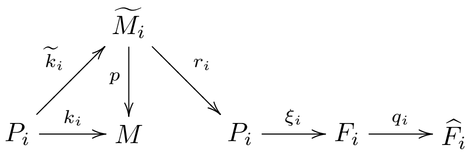


Figure 2.2. Normal triangles and rectangles

of the circle bundle P_i and by $\xi_i\colon P_i \to F_i$ the projection. Fix a complete hyperbolic metric on $\operatorname{int}(F_i)$, crush each component of ∂F_i to a point, denote by $\widehat{F}_i$ the new surface and by $q_i\colon F_i \to \widehat{F}_i$ the natural crushing map. This construction is equivalent to adding a limit parabolic point to each component C of ∂F_i. This parabolic point corresponds to the fixed point of the parabolic isometry generating $\pi_1 C$.



Step 2: Straightening simplices on surfaces with boundary. Let $\sigma\colon \Delta^2 = [v_0, v_1, v_2] \to \widehat{F}_i$ be a (singular) 2-simplex. Consider an edge $\tau = \sigma|[v_i, v_j]\colon [v_i, v_j] \to \widehat{F}_i$ and denote by $\widetilde{\tau}\colon [v_i, v_j] \to \overline{\mathbf{H}}^2$ a lifting of τ in the hyperbolic space union its boundary. Then $\widetilde{\tau}$ is homotopic by a homotopy fixing the end points to the unique geodesic arc (which may be constant) connecting the end points of $\widetilde{\tau}$. Denote by $\operatorname{st}(\widetilde{\tau})$ the new straight 1-simplex and by $\operatorname{st}(\tau)$ the projection of $\operatorname{st}(\widetilde{\tau})$ into $\widehat{F}_i$. We straighten each edge of σ and finally we homotop σ to a straight 2-simplex $\operatorname{st}(\sigma)$. As in the proof of Lemma 2.2 we define a bounded 2-cocyle $\widehat{\omega}_i$ on $\widehat{F}_i$ by setting $\langle \widehat{\omega}_i, \sigma \rangle = \mathcal{A}(\operatorname{st}(\sigma))$, the algebraic area of $\operatorname{st}(\sigma)$. Thus $q_i^\sharp(\widehat{\omega}_i)$ defines a relative 2-cocyle on $(F_i, \partial F_i)$ such that $\langle q_i^\sharp(\widehat{\omega}_i), z_i \rangle = \operatorname{Area}(F_i)$, where z_i is a relative 2-cycle representing the fundamental class of F_i.

Step 3: Lifting the singular chains. Let $\mu = \sum_l a_l\mu_l$ be a n-chain for $n = 2,3$ where $a_l \in \mathbf{R}$ and $\mu_l\colon\ \Delta^n \to M$ is a singular n-simplex. We choose a decomposition of each component of $\Delta^n \setminus \rho(\mu_l)^{-1}(\mathcal{T}_M)$ into n-simplices $\nabla_l^j, j = 1,...,n_l$ (recall that ρ denotes the reduction operator). Next we replace μ by the n-chain $\sigma = \sum_{l,j} a_l\rho(\mu_l)|\nabla_l^j$. Denote by $\widetilde{\sigma}$ the preimage of σ in $\widetilde{M_i}$. Then $\widetilde{\sigma}$ is a locally finite n-chain in $\widetilde{M_i}$. Since $\widetilde{P_i}$ is compact then we define a finite n-chain $\widetilde{\sigma}_i$ in $\widetilde{M_i}$ by taking only the simplices of $\widetilde{\sigma}$ which meet $\widetilde{P_i}$.

Step 4: Definition of a bounded cocyle satisfying the conclusion of the lemma. Keeping the same notation as in Step 3 we define a 2-cochain Ω_{P_i} in M by setting

$$\langle \Omega_{P_i}, \mu\rangle = \left\langle g_i^\sharp\widehat{w_i}, \widetilde{\sigma}_i\right\rangle$$

where μ is a singular 2-simplex and where $g_i = q_i \circ \xi_i \circ r_i$. By construction $\|\Omega_{P_i}\|_\infty \leqslant \pi$. Indeed let $\sigma\colon\ \Delta^2 \to M$ be a singular 2-simplex. By construction of Ω_{P_i} we may assume that σ is reduced. First note that it follows from the construction that for each triangle Δ of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ (given in the decomposition of Step 3) whose an edge is a component of $\sigma^{-1}(\mathcal{T}_M)$ then $\langle \Omega_{P_i}, \sigma|\Delta\rangle = 0$ (the simplices of $\widetilde{\sigma|\Delta}$ are sent into a point or a geodesic arc after straightening in $\widehat{F_i}$). On the other hand there exist at most one triangle Δ_σ of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ whose no edge is a component of $\sigma^{-1}(\mathcal{T}_M)$. This triangle necessarily lives in $\mathrm{Core}(\sigma)$. Since there exists at most one component of $\widetilde{\sigma|\Delta_\sigma}$ which meets $\widetilde{P_i}$ then the inequality $\|\Omega_{P_i}\|_\infty \leqslant \pi$ follows.

We check the cocyle condition $\langle \delta\Omega_{P_i}, \sigma\rangle = 0$ for each 3-simplex $\sigma\colon\ \Delta^3 \to M$. Since $\langle \delta\Omega_{P_i}, \sigma\rangle = \langle \Omega_{P_i}, \partial\sigma\rangle$ then we may assume that σ is reduced. Consider the 3-chain $\sum_j \sigma|\nabla_j$, where ∇_j is the decomposition (given in Step 3) of $\Delta^3 \setminus \sigma^{-1}(\mathcal{T}_M)$ into 3-simplices. The 2-faces of ∇_j are made of *interior triangles* which consist of the triangles whose interiors are in the interior of Δ^3 and of triangles which define the 2-simplices of a decomposition of $\partial\Delta^3 \setminus (\sigma^{-1}(\mathcal{T}_M) \cap \partial\Delta^3)$. Since each interior triangle is the face of two adjacent tetrahedra then one can replace σ by $\sum_j \sigma|\nabla_j$. Denote still $\sum_j \sigma|\nabla_j$ by σ. The 2-chain of $\widetilde{M_i}$ defined by

$$\partial\widetilde{\sigma}_i - \left(\widetilde{\partial\sigma}\right)_i \tag{$*$}$$

consists of an alternate sum of 2-simplices of $\partial\widetilde{\sigma}$ which does not meet $\widetilde{P_i}$. Since the retraction r_i crush each component of $\widetilde{M_i} \setminus \widetilde{P_i}$ to $\partial\widetilde{P_i}$ then by construction

$$\left\langle g_i^\sharp\widehat{w_i}, \partial\widetilde{\sigma}_i - \left(\widetilde{\partial\sigma}\right)_i\right\rangle = 0 \tag{$**$}$$

On the other hand by the definition

$$\langle \delta\Omega_{P_i}, \sigma\rangle = \left\langle g_i^\sharp \widehat{w_i}, \left(\widetilde{\partial\sigma}\right)_i\right\rangle$$

Thus using (*) and (**) we get, since $g_i^\sharp \widehat{w_i}$ is a cocyle by construction,

$$\langle \delta\Omega_{P_i}, \sigma\rangle = \left\langle g_i^\sharp \widehat{w_i}, \partial\widetilde{\sigma}_i\right\rangle = 0$$

On the other hand it is easily checked from the construction that $k_i^*\Omega_{P_i}$ is a relative cocycle of $(P_i, \partial P_i)$ and $k_i^*(\Omega_{P_j}) = 0$ for any $i \neq j$.

We check point (ii). First note that $\alpha_M(\mathcal{F}_i, f_i) = [(k_i)_\sharp((f_i)_\sharp(\mu_i) + u)]$ where μ_i is a relative 2-cycle representing the fundamental class of $\mathcal{F}_i$ and u is a l_1-chain in ∂P_i such that $\partial u = -\partial f_\sharp(\mu_i)$. Thus the construction yields

$$\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i)\rangle = \left\langle g_i^\sharp \widehat{w}_i, (\widetilde{k}_i)_\sharp (f_i)_\sharp(\mu_i)\right\rangle = \left\langle q_i^\sharp \widehat{w}_i, (\xi_i \circ f_i)_\sharp(\mu_i)\right\rangle$$

But since $\xi_i \circ f_i$ is a finite covering, with positive degree denoted by d_i then $(\xi_i \circ f_i)_\sharp([\mathcal{F}_i]) = d_i[F_i]$ and thus we get (see Step 2)

$$\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i)\rangle = d_i \left\langle q_i^\sharp \widehat{\omega}_i, z_i\right\rangle = d_i \mathrm{Area}(F_i) = \mathrm{Area}(\mathcal{F}_i)$$

To complete the proof of the lemma it remains to compute the norm of the classes defined by Ω_{P_i}. Denote by Ω the sum $\sum_i \Omega_i$. We first check that $\|\sum_i \Omega_{P_i}\|_\infty \leqslant \pi$. Let $\sigma\colon \Delta^2 \to M$ be a singular 2-simplex. If there exists an edge e of Δ^2 such that $\rho\sigma(e) \subset \mathcal{T}_M$ then $\left\langle \sum_{i\in I} \Omega_{P_i}, \sigma\right\rangle = 0$. If for any edge e of Δ^2 we have $\rho\sigma(e) \not\subset \mathcal{T}_M$ then there exists a unique component $\mathrm{Core}(\sigma)$ of $(\rho\sigma)^{-1}(M \setminus \mathcal{T}_M)$ which meets the three edges of Δ^2. Denote by P_ν the component of $M \setminus \mathcal{T}_M$ such that $\rho\sigma(\mathrm{Core}(\sigma)) \subset \mathrm{int}(P_\nu)$. If $\nu \in I$ then we have $\left|\left\langle \sum_{i\in I} \Omega_{P_i}, \sigma\right\rangle\right| = |\langle \Omega_{P_\nu}, \sigma\rangle| \leqslant \pi$ and if $\nu \notin I$ then $\left|\left\langle \sum_{i\in I} \Omega_{P_i}, \sigma\right\rangle\right| = 0$. This proves that $\| \sum_{i\in I} \Omega_{P_i}\|_\infty \leqslant \pi$. Using lemma 2.3 and points (i) and (ii) of the Lemma, we get the following equalities

$$|\langle [\Omega], \alpha_M(\mathcal{F}_i, f_i)\rangle| = \mathrm{Area}(\mathcal{F}_i) \leqslant \|[\Omega]\|_\infty \, \|\alpha_M(\mathcal{F}_i, f_i)\|_1 \leqslant \|[\Omega]\|_\infty \, \|\mathcal{F}_i\|$$

this completes the proof of Lemma 2.5 since $\mathrm{Area}(\mathcal{F}_i) = \pi\|\mathcal{F}_i\|$. □

Lemma 2.8. — *Let M be a closed aspherical orientable* 3-manifold and let $p\colon \widetilde{M} \to M$ denote a finite regular covering whose each Seifert piece is a circle bundle with $\mathbf{H}^2 \times \mathbf{R}$-geometry. Assume that we are given orientation preserving proper embeddings $f_i\colon (\widetilde{\mathcal{F}}_i, \partial\widetilde{\mathcal{F}}_i) \to (\widetilde{P}_i, \partial\widetilde{P}_i)$ where $\{\widetilde{P}_i\}_{i\in I}$ is a collection of Seifert pieces of $\widetilde{M}$. Then we have the equality*

$$\left\| p_\sharp\left(\sum_i \alpha_{\widetilde{M}}\left(x_i\widetilde{\mathcal{F}}_i, f_i\right)\right)\right\|_1 = \left\|\sum_i \alpha_{\widetilde{M}}\left(x_i\widetilde{\mathcal{F}}_i, f_i\right)\right\|_1$$

where the x_i are positive real numbers.

Proof. — Denote by Γ the automorphism group of $p\colon \widetilde{M} \to M$. Let $\widetilde{\alpha}$ be the element $\sum \alpha_{\widetilde{M}}\left(x_i\widetilde{\mathcal{F}}_i, f_i\right)$ and denote by $\mathrm{Av}(\widetilde{\alpha})$ the class obtained by *averaging* $\widetilde{\alpha}$ defined by $\mathrm{Av}(\widetilde{\alpha}) = \sum_{g\in\Gamma} g_\sharp(\widetilde{\alpha})$. For a Seifert piece $\widetilde{P}$ of $\widetilde{M}$ denote by $\Omega_{\widetilde{P}}$ the bounded 2-cocycle of $\widetilde{M}$ constructed in Lemma 2.5 and denote by Ω the sum $\sum_{\widetilde{P}} \Omega_{\widetilde{P}}$. Notice that each $g \in \Gamma$ acts one $\widetilde{M}$ as an orientation preserving homeomorphism which preserves the JSJ-splitting. In particular for each Seifert piece $\widetilde{P}$ of $\widetilde{M}$ then there exists a unique Seifert piece $\widetilde{P}'$ such that $g(\widetilde{P}) = \widetilde{P}'$ and $g|\widetilde{P}\colon (\widetilde{P}, \partial\widetilde{P}) \to (\widetilde{P}', \partial\widetilde{P}')$ is a homeomorphism. Moreover since each Seifert piece of M has an orientable basis then $g|\widetilde{P}\colon (\widetilde{P}, \partial\widetilde{P}) \to (\widetilde{P}', \partial\widetilde{P}')$ induces an orientation preserving homeomorphism between the bases of $\widetilde{P}$ and $\widetilde{P}'$. Then we get

$$\langle [\Omega], \mathrm{Av}(\widetilde{\alpha})\rangle = \mathrm{Card}(\Gamma) \sum_{i\in I} x_i \mathrm{Area}(\widetilde{\mathcal{F}}_i) \leqslant \pi \|\mathrm{Av}(\widetilde{\alpha})\|_1$$

which proves that

$$\|\mathrm{Av}(\widetilde{\alpha})\|_1 \geqslant \mathrm{Card}(\Gamma) \sum_{i\in I} x_i \left\|\widetilde{\mathcal{F}}_i\right\|$$

Since $\mathrm{Av}(\widetilde{\alpha})$ is Γ-invariant then by Lemma 2.1 $\|p_\sharp(\mathrm{Av}(\widetilde{\alpha}))\|_1 = \|\mathrm{Av}(\widetilde{\alpha})\|_1$. Moreover using the definitions and Lemma 2.3

$$\|p_\sharp(\mathrm{Av}(\widetilde{\alpha}))\|_1 \leqslant \sum_{g\in\Gamma} \|p_\sharp g_\sharp(\widetilde{\alpha})\|_1 \leqslant \sum_{g\in\Gamma} \|g_\sharp(\widetilde{\alpha})\|_1 \leqslant \mathrm{Card}(\Gamma) \sum_{i\in I} x_i \left\|\widetilde{\mathcal{F}}_i\right\|$$

We deduce that $\sum_{g\in\Gamma} \|p_\sharp g_\sharp(\widetilde{\alpha})\|_1 = \sum_{g\in\Gamma} \|g_\sharp(\widetilde{\alpha})\|_1$. On the other hand, since we know that $\|p_\sharp g_\sharp(\widetilde{\alpha})\|_1 \leqslant \|g_\sharp(\widetilde{\alpha})\|_1$ for any $g \in \Gamma$ then we get in particular $\|p_\sharp(\widetilde{\alpha})\|_1 = \|\widetilde{\alpha}\|_1$. □

Proof of Proposition 2.4. — To complete the proof of Proposition 2.4 it remains to check the following points

(i) $\|\alpha_M(\mathcal{F}_i, f_i)\|_1 \geqslant \|\mathcal{F}_i\|$ for $i = 1, ..., l$,

(ii) $\|\sum \alpha_M(x_i\mathcal{F}_i, f_i)\| \geqslant \sum \|\alpha_M(x_i\mathcal{F}_i, f_i)\|$, and

(iii) the covering property.

We first check points (i) and (ii). To this purpose we consider two casis.

Case 1. Assume that each P_i, $i = 1, ..., l$ is homeomorphic to a circle bundle. By Lemma 2.5 we know that there exists a bounded 2-cocyle Ω_{P_i} such that $\|[\Omega_{P_i}]\|_\infty = \pi$ and $|\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i)\rangle| = \mathrm{Area}(\mathcal{F}_i)$. Then point (i) follows from Hölder inequality.

We check point (ii). Again, applying Lemma 2.5 we know that for each $i \in \{1, ..., l\}$ there exists a bounded 2-cocycle Ω_i in M such that

$$\langle [\Omega_i], \alpha_M(x_j\mathcal{F}_j, f_j)\rangle = \delta_{ij} x_j \mathrm{Area}(\mathcal{F}_j)$$

for any i, j in $\{1, ..., l\}$. Thus we get

$$\left\langle \sum_i [\Omega_i], \sum_j \alpha_M(x_j \mathcal{F}_j, f_j) \right\rangle = \sum_i x_i \mathrm{Area}(\mathcal{F}_i) \leqslant \pi \left\| \sum_j \alpha_M(x_j \mathcal{F}_j, f_j) \right\|_1$$

Hence

$$\left\| \sum_i \alpha_M(x_i \mathcal{F}_i, f_i) \right\|_1 \geqslant \sum_i x_i \|\mathcal{F}_i\| \geqslant \sum_i \|\alpha_M(x_i \mathcal{F}_i, f_i)\|_1$$

This proves point (ii) in Case 1.

Case 2. We consider now the general case. Let $p\colon \widetilde{M} \to M$ be a finite regular covering of M whose each Seifert piece (in particular each component $\widetilde{P}_i$ over P_i for $i = 1, ..., l$) is a circle bundle (such a covering exists by [15, Proposition 4.4]). For each $i = 1, ..., l$ consider a covering $\widetilde{f}_i\colon (\widetilde{\mathcal{F}}_i, \partial \widetilde{\mathcal{F}}_i) \to (\widetilde{P}_i, \partial \widetilde{P}_i)$ of $f_i\colon (\mathcal{F}_i, \partial \mathcal{F}_i) \to (P_i, \partial P_i)$ (obtained by considering the group $(f_i)_*^{-1}(p_*(\pi_1 \widetilde{P}_i))$. By construction $\widetilde{f}_i$ is an orientation preserving embedding. Denote by $d_i > 0$ the degree of the covering $p_i\colon \widetilde{\mathcal{F}}_i \to \mathcal{F}_i$. By Case 1 we know that $\|\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)\|_1 = \|\widetilde{\mathcal{F}}_i\|$ for $i = 1, ..., l$. On the other hand by Lemma 2.8 we know that $\|p_\sharp(\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i))\|_1 = \|\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)\|_1$. Since any continuous map induces a chain map then

$$p_\sharp(\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)) = d_i \alpha_M(\mathcal{F}_i, f_i)$$

which implies that

$$\|d_i \alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\widetilde{\mathcal{F}}_i\|$$

and thus $\|\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\mathcal{F}_i\|$ for $i = 1, ..., l$. To check point (ii) we know from Case 1 that

$$\left\| \sum \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right) \right\| = \sum \left\| \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right) \right\|$$

Then using Lemma 2.8 in the right and left hand side, we get

$$\left\| \sum \alpha_M (x_i \mathcal{F}_i, f_i) \right\| = \sum \|\alpha_M (x_i \mathcal{F}_i, f_i)\|.$$

We check point (iii). Let $f\colon M \to N$ denote a finite covering map and let $\alpha = \sum \alpha_M (x_i \mathcal{F}_i, f_i)$. Using the construction of Case 2 with the same notations then $\alpha = p_\sharp(\widetilde{\alpha})$ where $\widetilde{\alpha} = \sum \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right)$. Possibly passing to some finite covering there are no loss of generality assuming $f \circ p$ is regular. Hence we get using Lemma 2.8

$$\|f_\sharp(\alpha)\| = \|f_\sharp p_\sharp(\widetilde{\alpha})\| = \|\widetilde{\alpha}\| \geqslant \|\alpha\|$$

This completes the proof of Proposition 2.4. □

3. Characterizations of covering maps

Given a closed irreducible orientable 3-manifold M we denote by $\mathcal{H}(M)$ (resp. $\mathcal{S}(M)$) the disjoint union of the hyperbolic (resp. Seifert) components of $M \setminus \mathcal{T}_M$ (see [11], [12] and [21]). In order to prove Theorem 1.2 we first check the following technical result.

Proposition 3.1. — *Let M be a closed aspherical orientable 3-manifold. Any π_1-surjective nonzero degree map $f\colon M \to N$ into a closed irreducible orientable 3-manifold satisfying the following conditions*

(i) Each Seifert component of $M \setminus \mathcal{T}_M$, resp. of $N \setminus \mathcal{T}_N$, is homeomorphic to a product, resp. to a $\mathbf{S}^1$-bundle over an orientable surface, each Seifert component of $M \setminus \mathcal{T}_M$ has at least two boundary components (if $\mathcal{T}_M \neq \emptyset$) and each component of $\mathcal{T}_M$ is shared by two distinct components of $M \setminus \mathcal{T}_M$,

(ii) $\|f_\sharp[M]\|_1 = \|[M]\|_1$, where $[M] \in H_3(M;\mathbf{R})$ is the fundamental class

(iii) $\|f_\sharp\alpha_M(\mathcal{F},g)\|_1 = \|\alpha_M(\mathcal{F},g)\|_1$ for each orientation preserving proper embedding $g\colon \mathcal{F} \to P$ when P runs over the Seifert pieces of M

is homotopic to a homeomorphism.

3.1. Proof of Proposition 3.1

Throughout this section we always assume that the map $f\colon M \to N$ and the manifolds M, N satisfy the hypothesis of Proposition 3.1. Notice that we may assume in addition that M is not a virtual torus bundle by [23]. Thus since each Seifert piece P of M is homeomorphic to a product we may assume that P is a $\mathbf{H}^2 \times \mathbf{R}$-manifold. Hence this implies, using hypothesis (ii) and (iii), that either $\|N\| \neq 0$ or $H_2^{l_1}(N;\mathbf{R})/\ker\|.\|_1 \neq \{0\}$. Hence either N is non-geometric or admits a geometry $\mathbf{H}^3, \mathbf{H}^2 \times \mathbf{R}$ or $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. The proof of Proposition 3.1 will come from the following sequence of claims.

Claim 3.2. — *The map $f|T\colon T \to N$ is π_1-injective for any characteristic torus T in M. Moreover, $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece P of M.*

Proof. — Let T be a characteristic torus of M. From the Rigidity Theorem of Soma [20] and from hypothesis (ii) it is sufficient to consider the case when T is shared by two distinct Seifert components P and P' of M. Denote by h and h' the $\mathbf{S}^1$-fiber of P and of P' respectively. If $f|T\colon T \to N$ is not π_1-injective then we may assume, since h and h' generate a rank 2 subgroup of $\pi_1 T$ (by minimality of the JSJ-decomposition), that P (for example) contains a simple closed curve c distinct from the fiber h such that $[c] \in \ker(f|T)_*$.

Moreover since ∂P is not connected then there exists an orientation preserving proper embedding $j\colon\ (F, \partial F) \to (P, \partial P)$ where F is a connected surface such that c is a boundary component of $j(F)$.

Indeed, denote by $T_1 = T$ the component of ∂P which contains c and by $T_2, ..., T_r$ the other components of ∂P with $r \geqslant 2$. For each $i = 1, ..., r$ fix a basis $\langle s_i, h\rangle$, where s_i is a section of T_i with respect to the $\mathbf{S}^1$-fibration of P such that $s_1 + ... + s_r$ is nul-homologous in P and where h denotes the fiber of P. Denote by (α, β) the coprime integers with $\alpha \neq 0$ such that $c = \alpha[s_1] + \beta[h]$. Then

$$[c] + \alpha[s_2] + ... + \alpha[s_r] - \beta[h] = 0 \text{ in } H_1(P; \mathbf{Z})$$

Thus there exists a nontrivial class η in $H_2(P, \partial P; \mathbf{Z})$ such that

$$\partial\eta = ((\alpha, \beta), (\alpha, 0), ..., (\alpha, 0), (\alpha, -\beta))$$

in $H_1(\partial P) = H_1(T_1) \oplus H_1(T_2) \oplus ... \oplus H_1(T_{r-1}) \oplus H_1(T_r)$. Since P is aspherical, it follows from [22] that each class in $H_2(P, \partial P; \mathbf{Z})$ can be represented by a properly embedded incompressible surface. This can be argued as follows. By the Poincaré Duality, $H_2(P, \partial P; \mathbf{Z}) \simeq H^1(P; \mathbf{Z})$, there exists a homomorphism $\rho\colon\ \pi_1 P \to \mathbf{Z} = \pi_1 S^1$ corresponding to η. Since the spaces are aspherical the homomorphism is induced by a map $f_\eta\colon\ P \to \mathbf{S}^1$. Taking the pre-image of a regular value $\theta \in \mathbf{S}^1$ and using the construction given in [10, Chapter 6] we may arrange f_η by a homotopy so that the components of $f_\eta^{-1}(\theta)$ are properly embedded incompressibe surfaces. Denote by F such a surface. Then F is a horizontal surface and c is parallel to some components of ∂F.

Denote by $T \times [-1, 1]$ a regular neighborhood of T such that $T = T \times \{0\}$ and parametrize $T = \mathbf{S}^1 \times \mathbf{S}^1$ such that $c = \mathbf{S}^1 \times \{*\}$. As in [19], consider the relation $\sim$ on M defined by $z \sim z'$ iff $z = z'$ or $z = (x, y, t) \in T \times I$, $z' = (x', y', t') \in T \times I$ and $y = y'$, $t = t'$. Denote by $X = M/\sim$ the quotient space and by $\pi\colon\ M \to X$ the quotient map. Then the map f factors through X. Denote by $g\colon\ X \to N$ the map such that $f = g \circ \pi$. Denote by $\widehat{P}$ the image of P under π. Topologically $\widehat{P}$ is obtained from P after Dehn filling along T, identifying the meridian of a solid torus V to c so that the Seifert fibration of P extends to a Seifert fibration of $\widehat{P}$ and the image $\widehat{F}$ of F is a surface in $\widehat{P}$ obtained from F after gluing a 2-disk along each component of ∂F parallel to c. Consider the following commutative diagram

$$\begin{array}{ccccccc} F & \xrightarrow{j} & P & \xrightarrow{k} & M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \pi|F} & & \downarrow{\scriptstyle \pi|P} & & \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle g} & \\ \widehat{F} & \xrightarrow{\widehat{j}} & \widehat{P} & \xrightarrow{\widehat{k}} & X & & \end{array}$$

where $\widehat{j}$ is induced by j and where $k\colon\ P \to M$ is the inclusion and $\widehat{k}\colon\ \widehat{P} \to X$ denotes the induced inclusion. Note that it follows from our construction, using standard homological arguments, that

$$\pi_\sharp(\alpha_M(F,j)) = \alpha_X(\widehat{F},\widehat{j}) \in H_2^{l_1}(X;\mathbf{R}) \tag{$*$}$$

where $\alpha_X(\widehat{F},\widehat{j})$ is defined by $\widehat{k}_\sharp \alpha_{\widehat{P}}(\widehat{F},\widehat{j})$. We deduce, using hypothesis (iii) of Proposition 3.1, the following equalities:

$$\|\alpha_M(F,j)\|_1 \geqslant \|\pi_\sharp \alpha_M(F,j)\|_1 \geqslant \|f_\sharp \alpha_M(F,j)\|_1 = \|\alpha_M(F,j)\|_1$$

Thus using Lemma 2.3, equality (*) and Proposition 2.4(i) we get :

$$\|\widehat{F}\| \geqslant \|\alpha_X(\widehat{F},\widehat{j})\|_1 = \|F\|$$

A contradiction. This proves the π_1-injectivity of the map $f|T$. It remains to check that $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece P. Assume that $f_*(\pi_1 P)$ is an abelian subgroup of $\pi_1 N$. Then the map $f|P\colon\ P \to N$ factors through a space X with abelian fundamental group. Since $H_2^{l_1}(X)$ is trivial then we get a contradiction with hypothesis (iii) of Proposition 3.1 using point (i) of Proposition 2.4. □

Claim 3.3. — *There is a map g homotopic to f such that for each Seifert piece Σ of N then each component of $g^{-1}(\Sigma)$ is a Seifert piece of M.*

Proof. — By hypothesis (ii) one can apply [20, Rigidity Theorem]. Thus one may assume that f induces a $\deg(f)$-covering map from $\mathcal{H}(M)$ to $\mathcal{H}(N)$. Next, by Claim 3.2 one can apply [11, Mapping Theorem] which implies that one can arrange f by a homotopy so that for each canonical torus U of N then $f^{-1}(U)$ is a disjoint union of canonical tori of M. Hence for each Seifert piece Σ of N the space $f^{-1}(\Sigma)$ is a canonical graph submanifold of M (i.e. a submanifold which is the union of some Seifert pieces of M). If a component G of $f^{-1}(\Sigma)$ is not a Seifert manifold then there exists a canonical torus T in the interior of G which is shared by two distinct Seifert pieces Σ_1 and Σ_2 of G. Since by Claim 3.2 the group $f_*(\pi_1\Sigma_i)$ is non-abelian, for $i=1,2$, then using [11, Addendum to Theorem VI.I.6] we know that $f|\Sigma_i\colon\ \Sigma_i \to \Sigma$ is homotopic to a fiber preserving map.

Since $f|T$ is π_1-injective we get a contradiction by the minimality of the JSJ-decomposiiton. This proves the claim. □

Since f is π_1-surjective then to complete the proof of Proposition 3.1 it remains to check the following

Claim 3.4. — *There is a map g homotopic to f, rel. to $\mathcal{H}(M)$, such that for each Seifert piece Σ of N and for each component G of $g^{-1}(\Sigma)$ then $g|G\colon G \to \Sigma$ is a covering map.*

Proof. — First of all we know that for each component G of $f^{-1}(\Sigma)$ then $f|G\colon G \to \Sigma$ is fiber preserving and non-degenerate in the sense of [11]. On the other hand, notice that Σ is necessarily homeomorphic to a product. Indeed if $\partial\Sigma \neq \emptyset$ this is obvious and if $\partial\Sigma = \emptyset$ this comes from the following argument: first note that in this case $\Sigma = N$ and $G = M$, thus if Σ is not homeomorphic to a product then the bundle has a non-zero Euler number and using the Seifert volume in [5, Theorem 3 and Lemma 3] and in [4, Theorem 4] we get a contradiction (since G has a zero Euler number and $\deg(f) \neq 0$). Thus after choosing appropriate sections we identify G with $K \times \mathbf{S}^1$, resp. Σ with $F \times \mathbf{S}^1$, where K, resp. F, is a hyperbolic surface.

Let $\mathcal{F}$ denote a component of $(f|G)^{-1}(F)$. Arrange f so that $\mathcal{F}$ is incompressible in G. Since f is non-degenerate and fiber preserving then the inclusion $i\colon \mathcal{F} \to G$ is necessarily an orientation preserving proper embedding and $f|\mathcal{F}\colon \mathcal{F} \to F$ descends to a map $\pi\colon K \to F$. Therefore we get

$$f_\sharp(\alpha_M(\mathcal{F}, i)) = \deg(f|\mathcal{F}\colon \mathcal{F} \to F)\alpha_N(F, j)$$

where $j\colon F \to \Sigma$ is the inclusion. This implies that

$$\|\mathcal{F}\| = |\deg(f|\mathcal{F}\colon \mathcal{F} \to F)| \times \|F\|$$

Thus we get the equality

$$\|K\| = \deg(\pi) \times \|F\|$$

Hence π is homotopic to a covering map which implies that $f|G$ is also homotopic to a covering map. This proves the claim and completes the proof of Proposition 3.1. □

3.2. Proof of Theorem 1.2

We first check that the condition is necessary.

When $\|\alpha_M\|_1 = 0$ there is nothing to prove. So let's assume $\|\alpha_M\|_1 > 0$

Then either M is a $\widetilde{\mathrm{SL}}_2(\mathbf{R})$-manifold and Lemma 2.2 applies or M is not a $\widetilde{\mathrm{SL}}_2(\mathbf{R})$-manifold and Proposition 2.4 applies.

We verify now that the condition is sufficient. First of all note that according to [23] we may assume that M is not a vitual torus bundle. In the sequel it will be convenient to consider the following commutative diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{f_2} & N_2 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M_1 & \xrightarrow{f_1} & N_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r} \\ M & \xrightarrow{f} & N \end{array}$$

obtained as follows. The map $s\colon\ M_1 \to M$ is a finite covering such that each Seifert piece of M_1 is a circle bundle over an orientable surface with at least two boundary components if $\mathcal{T}_{M_1} \neq \emptyset$, and each canonical torus of M_1 is shared by two distinct components of $M_1 \setminus \mathcal{T}_{M_1}$ (for the existence of such a covering see [7, Lemmas 3.2 and 3.5]), the map $r\colon\ N_1 \to N$ is a finite covering corresponding to the subgroup $f_* s_*(\pi_1 M_1)$ in $\pi_1 N$, which is of finite index since $\deg(f) \neq 0$, the map $f_1\colon\ M_1 \to N_1$ is a lifting of $f \circ s$ which exists by our construction, the map $p\colon\ N_2 \to N_1$ is a finite covering such that each Seifert piece of N_2 is a $\mathbf{S}^1$-bundle over an orientable surface and $f_2\colon\ M_2 \to N_2$ is the finite covering of f_1 corresponding to p, and $q\colon\ M_2 \to M_1$ is the covering corresponding to the subgroup $(f_1)_*^{-1}(p_* \pi_1 N_2)$. Notice that it follows from the construction that f_1 and f_2 are π_1-surjective.

Claim 3.5. — *The map f_2 is homotopic to a homeomorphism.*

Proof. — Assume that M is a $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-manifold. Since f has nonzero degree then f is homotopic to a non degenerate fiber preserving map and N is also a $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-manifold. Thus f_2 is a π_1-surjective nonzero degree map between circle bundle with nonzero Euler numbers. Denote by F_2, resp. G_2, the base of M_2, resp. N_2. It follows from the hypothesis of the theorem combined with Lemma 2.2 that $\|(f_2)_\sharp(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$. Thus f_2 induces a map $g\colon\ F_2 \to G_2$ such that $\xi \circ f_2 = g \circ \pi$ where $\pi\colon\ M_2 \to F_2$ and $\xi\colon\ N_2 \to G_2$ denote the bundle projections. Since by definition $\alpha_{M_2} = \pi_\sharp^{-1}([F_2]_{l_1})$ then condition $\|(f_2)_\sharp(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$ implies

$$\|[F_2]_{l_1}\|_1 = \|g_\sharp([F_2]_{l_1})\|_1 = \deg(g)\|[G_2]_{l_1}\|_1$$

and thus $\|F_2\| = \deg(g)\|G_2\|$. This proves that g and hence f_2 is homotopic to a homeomorphism (recall that f_2 is π_1-surjective).

Assume now that M is not a $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-manifold. Then using point (ii) of Proposition 2.4 (additivity property) and the isometry hypothesis we have $\|f_\sharp \alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$ for any $i = 1, ..., l$.

Indeed, by hypothesis we know that $\|f_\sharp \alpha_M\|_1 = \|\alpha_M\|_1$ then by point (ii) of Proposition 2.4 (additivity property) and using the definition of α_M we have

$$\|f_\sharp \alpha_M\|_1 = \|\alpha_M\|_1 = \sum_i \left\| \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1$$

Since, by paragraph 2, any continuous map induces a contraction with respect to the l_1-norm we get

$$\begin{aligned} \|f_\sharp \alpha_M\|_1 = \|\alpha_M\|_1 &\geqslant \sum_i \left\| f_\sharp \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 \\ &\geqslant \left\| \sum_i f_\sharp \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 = \|f_\sharp \alpha_M\|_1 \end{aligned}$$

Hence we get

$$\sum_i \left(\left\| \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 - \left\| f_\sharp \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 \right) = 0$$

Again, since $f_\sharp$ is a contraction, then each term of the sum is non-negative and thus $\|f_\sharp \alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$ for any $i = 1, ..., l$.

Note that if $g_i \colon \mathcal{G}_i \to P_i$ is any orientation preserving proper map of a surface $\mathcal{G}_i$ then

$$\|f_\sharp \alpha_M(\mathcal{G}_i, g_i)\|_1 = \|\alpha_M(\mathcal{G}_i, g_i)\|_1$$

This comes from the following observation: by [25, Lemma 6] there are rational numbers r_i, s_i and a vertical surface W_i in P_i (i.e. an incompressible properly embedded surface in P_i which is fibered by the $\mathbf{S}^1$-fibers of P_i) such that

$$(g_i)_\sharp[\mathcal{G}_i] = r_i (f_i)_\sharp[\mathcal{F}_i] + s_i[W_i] \in H_2(P_i, \partial P_i)$$

and since W_i has zero simplicial volume the equality follows. In order to apply Proposition 3.1 to the map f_2 it remains to check hypothesis (iii). Let $g \colon F_2 \to P_2$ be an orientation preserving embedding of a surface into a Seifert piece P_2 of M_2. Denote by P the Seifert piece of M such that P_2 is over P. Then by the above equality, applied to $s \circ q \circ g \colon F_2 \to P$ we have

$$\|f_\sharp \alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_M(F_2, s \circ q \circ g)\|_1$$

On the other hand, using point (iii) of Proposition 2.4 we know that

$$\|\alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_{M_2}(F_2, f_2)\|_1$$

By the commutativity of the diagram we have

$$f_\sharp(\alpha_M(F_2, s\circ q\circ g)) = r_\sharp p_\sharp (f_2)_\sharp(\alpha_{M_2}(F_2, f_2))$$

Therefore, this yields

$$\begin{aligned}\|\alpha_{M_2}(F_2,f_2)\|_1 = \|r_\sharp p_\sharp (f_2)_\sharp(\alpha_{M_2}(F_2,f_2))\|_1 &\leqslant \|(f_2)_\sharp(\alpha_{M_2}(F_2,f_2)\|_1\\ &\leqslant \|\alpha_{M_2}(F_2,f_2)\|_1\end{aligned}$$

Accordingly we deduce that f_2 satisfies hypothesis of Proposition 3.1 which implies that f_2 is homotopic to a homeomorphism. □

Since M is an aspherical 3-manifolds then it has a torsion free fundamental group ([10]). Since p, q, r, s are finite covering maps then they induce injective homomorphisms at the π_1-level and since f_2 induces an isomorphism f must be π_1-injective. Consider the finite covering $\widetilde{N}\to N$ corresponding to $f_*(\pi_1 M)$. Then f lifts to a map $\widetilde{f}\colon M\to\widetilde{N}$ inducing an isomorphism at the π_1-level. We deduce from this point using [13, Theorem 0.7] that $\widetilde{f}$ is a homeomorphism. This implies that f is a covering map and completes the proof of Theorem 1.2.

3.3. Proof of Theorem 1.5

By the Mapping Theorem of [9] the map f induces an isometry $f_\sharp\colon H_3^{l_1}(M)\to H_3^{l_1}(N)$. On the other hand, using the same construction as in the proof of Lemma 2.5 in dimension three (instead of dimension 2) one deduces that the natural map $H_3(M)\to H_3^{l_1}(M)$ is an isometry. Indeed, if $\|M\| = 0$ there is nothing to prove and if $\|M\| > 0$ this means that M contains some hyperbolic pieces $\mathcal{H}_1, ..., \mathcal{H}_l$ in its geometric decomposition. Thus by the straightening technique used in the proof of Lemma 2.5 one can in the same way construct an element $\Omega\in H_b^3(M)$ such that $\langle\Omega, [M]\rangle = \mathrm{vol}(\mathcal{H}_1) + ... + \mathrm{vol}(\mathcal{H}_l)$ with $\|\Omega\|_\infty\leqslant V_3$, where V_3 denotes the supremum of the volume of geodesic 3-simplices in the hyperbolic 3-space. Hence the l_1-norm of $[M]$ is $\|M\|$ in $H_3^{l_1}(M)$, proving that $H_3(M)\to H_3^{l_1}(M)$ is an isometry. This implies that $f_\sharp\colon H_3(M;\mathbf{R})\to H_3(N;\mathbf{R})$ is an isometry.

Using the same covering argument as above one can assume, without loss of generality, that f is π_1-surjective. If M is orientable* then Corollary 1.5 follows from Theorem 1.2 by the Mapping Theorem of [9]. If M is not orientable* then there exists a 2-fold finite covering $p\colon M_2\to M$ such that M_2 is orientable*. Note that the composition $g = f\circ p_2$ is not π_1-surjective. Indeed if g is π_1-surjective then $f\circ p_2$ is homotopic to a homeomorphism because since f_* has an amenable kernel then so is $\ker(g_*)$ and thus g

induces an isometric isomorphism $g_\sharp\colon H_2^{l_1}(M_2) \to H_2^{l_1}(N)$. Since moreover $\|M_2\| = 2\|N\|$ one can apply Theorem 1.2. A contradiction. Since g is not π_1-surjective then there exists a 2-fold covering $f'\colon M_2 \to N_2$ of the map f. Again, since f_* has an amenable kernel then so is $\ker(f'_*)$. Moreover f' is π_1-surjective by construction and thus it induces an isometric isomorphism $f'_\sharp\colon H_2^{l_1}(M_2) \to H_2^{l_1}(N_2)$ and $\|M_2\| = \deg(f')\|N_2\|$. Hence by Theorem 1.2 the f' is homotopic to a homeomorphism. Hence f is homotopic to a homeomorphism. This completes the proof of the corollary.

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