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Surjectivity of Siegel Φ-operator for square free level and small weight

SURJECTIVITY OF SIEGEL Φ-OPERATOR FOR SQUARE FREE LEVEL AND SMALL WEIGHT

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ABSTRACT. — We show the surjectivity of the (global) Siegel Φ-operator for modular forms for certain congruence subgroups of $\mathrm{Sp}(2, \mathbb{Z})$ and weight $k = 4$, where the standard techniques (Poincaré series or Klingen-Eisenstein series) are no longer available. Our main tools are theta series and genus versions of basis problems.

RÉSUMÉ. — Nous démontrons la surjectivité de l'opérateur Φ de Siegel pour des formes modulaires pour certains groupes de congruence de $\mathrm{Sp}(2, \mathbb{Z})$ et de poids 4, où les techniques standards (séries de Poincaré ou séries de Klingen-Eisenstein) ne marchent pas. Nous utilisons des séries thêta et le problème de base pour plusieurs genres.

1. Introduction

The difference of dimensions between Siegel cusp forms and Siegel modular forms is well-understood when the weight is big enough. But not much is known for small weights, in particular for congruence subgroups, and we need special care in these cases. Our aim of this paper is to show surjectivity of the *global* Siegel Φ-operator for Siegel modular forms of degree two with respect to a certain discrete group for weight $k \geqslant 4$. Satake proved the same result for $k \geqslant 5$ in [17] by using the Poincaré series but the argument there does not work for $k = 4$ since the series do not converge for such a small weight. Besides, the Hecke trick seems difficult either for Poincaré series or Klingen type Eisenstein series. We prove the surjectivity by applying the affirmative answer to the precise basis problems on modular forms

of one variable for each genus and obtain a new result for $k = 4$, including the vector valued case.

Now we explain the details of our statement. For any ring (or a field) R, we denote by $\mathrm{Sp}(n, R)$ the group of $2n \times 2n$ symplectic matrices over R, i.e.

$$\mathrm{Sp}(n, R) = \left\{ g \in M_{2n}(R);\ {}^t g J_n g = J_n \right\}$$

where we put $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix} \right)$ for the $n \times n$ unit matrix 1_n. We denote by $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$ the usual Siegel modular group of degree n. For any positive integer N, we define a Hecke type subgroup of Γ_n of level N by

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n; C \equiv 0 \bmod N \right\}.$$

When $n = 1$, we omit n and simply write $\Gamma_0(N) = \Gamma_0^{(1)}(N)$. For any non-negative integer k, we denote by $A_k(\Gamma_0^{(n)}(N))$ the space of holomorphic Siegel modular forms of weight k belonging to $\Gamma_0^{(n)}(N)$. We denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n and by $\mathcal{S}(N)$ the Satake compactification of the analytic space $\Gamma_0^{(2)}(N) \backslash \mathfrak{H}_2$. We denote by $\mathrm{Bd}(N)$ the boundary of $\mathcal{S}(N)$. The boundary components of $\mathrm{Bd}(N)$ are modular curves and they intersect at various cusps of those curves. (*cf.* I. Satake [15].) The modular forms of weight k on $\mathrm{Bd}(N)$ are by definition modular forms of weight k on these components taking the same value at each intersection point. We denote their space by $A_k(\mathrm{Bd}(N))$. We denote by $S_k(\mathrm{Bd}(N))$ the subspace of cusp forms in $A_k(\mathrm{Bd}(N))$, or equivalently the space of forms which vanish at all the zero-dimensional components. One-dimensional components of $\mathrm{Bd}(N)$ are called one-dimensional cusps. They correspond bijectively with double cosets $\Gamma_0^{(2)}(N) \backslash \mathrm{Sp}(2, \mathbb{Q}) / P_{2,1}(\mathbb{Q})$, where $P_{2,1}(\mathbb{Q})$ is the maximal parabolic subgroup of $\mathrm{Sp}(2, \mathbb{Q})$ defined by

$$P_{2,1}(\mathbb{Q}) = \left\{ \begin{pmatrix} \mathbb{Q} & 0 & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 & \mathbb{Q} \end{pmatrix} \right\} \cap \mathrm{Sp}(2, \mathbb{Q}).$$

For any good function F on $\mathfrak{H}_2$, we define a function $\Phi(F)$ on $\mathfrak{H}_1$ by

$$(\Phi F)(\tau) = \lim_{\lambda \to \infty} F \begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}.$$

We choose representatives of the above double cosets as

$$\Gamma_0^{(2)}(N) \backslash \mathrm{Sp}(2, \mathbb{Q}) / P_{2,1}(\mathbb{Q}) = \bigsqcup_{i=1}^{d} \Gamma_0^{(2)}(N) g_i P_{2,1}(\mathbb{Q}).$$

For any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2, \mathbb{R})$, we write $j(g, Z) = \det(CZ + D)$. For any function F on $\mathfrak{H}_2$ and any integer k, we write

$$F|_k[g] = j(g, Z)^{-k} F(gZ)$$

where $gZ = (AZ + B)(CZ + D)^{-1}$. Then as in Satake [16], we can define a *global* Siegel Φ-operator $\widetilde{\Phi}$ of $A_k(\Gamma_0^{(2)}(N))$ to $A_k(\mathrm{Bd}(N))$ by

$$\widetilde{\Phi} F = (\Phi(F|_k[g_i]))_{1 \leqslant i \leqslant d}.$$

Here each $\Phi(F|[g_i])$ is a modular form of one variable on the modular curve corresponding to the component determined by g_i. Apparently this map depends on the choice of representatives g_i but not essentially. A Siegel modular form F such that $\widetilde{\Phi}(F) = 0$ (or equivalently $\Phi(F|_k[g]) = 0$ for any $g \in \mathrm{Sp}(2, \mathbb{Q})$) is called a cusp form. The space of cusp forms is denoted by $S_k(\Gamma_0^{(2)}(N))$.

THEOREM 1.1. — *Assume that N is squarefree and $k \geqslant 4$. Then the global Siegel operator $\widetilde{\Phi}$ is surjective, i.e.*

$$\widetilde{\Phi}(A_k(\Gamma_0^{(2)}(N))) = A_k(\mathrm{Bd}(N)).$$

Remark. — This is known for $k \geqslant 6$ by Satake. (See [17], Théorème 1.) When $k = 2$, $\widetilde{\Phi}$ is not surjective, but when N is a prime, the image is described explicitly in [10], pp. 192–194. When $k = 2$ and N is not a prime, we do not have a definite result yet. When k is odd, the operator $\widetilde{\Phi}$ is the zero map since $A_k(\mathrm{Bd}(N)) = 0$.

The above result and the explicit description of the boundary $\mathrm{Bd}(N)$ given in Section 2 gives the difference between dimensions of the space of cusp forms and the space of all modular forms as follows.

COROLLARY 1.2. — *We denote by t the number of prime divisors of the squarefree integer N. For odd k, we have $A_k(\Gamma_0^{(2)}(N)) = S_k(\Gamma_0^{(2)}(N))$. For even $k \geqslant 4$, we have*

$$\dim A_k(\Gamma_0^{(2)}(N)) - \dim S_k(\Gamma_0^{(2)}(N)) = 3^t + 2^t \dim S_k(\Gamma_0(N)).$$

One way to prove the surjectivity to $S_k(\mathrm{Bd}(N))$ is to show that there are enough theta series whose linear combinations give enough cusp forms on the boundary. The behavior at each cusp on the boundary is calculated directly by transformation formulas of the theta series. This direct method will be given in Section 6, including the vector valued case. But we have another viewpoint which seems equally interesting by which we can prove the surjectivity to the whole $A_k(\mathrm{Bd}(N))$. In order to prove the above theorem, we can use surjectivity of a map which we call *Witt projection* $\overline{W}$,

and a variant of the basis problem explained first in [10]. We explain this now. The Witt operator W is a map from holomorphic functions F on $\mathfrak{H}_2$ to those on $\mathfrak{H}_1 \times \mathfrak{H}_1$ defined by

$$(WF)(\tau, \omega) = F \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}.$$

If $F \in M_k(\Gamma_0^{(2)}(N))$, then WF is a modular form in $A_k(\Gamma_0(N))$ as a function of each τ or ω. In particular, $WF = 0$ if k is odd. Since $F|_k[\gamma] = (-1)^k F$ for $\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, we have $F \left(\begin{smallmatrix} \omega & z \\ z & \tau \end{smallmatrix} \right) = (-1)^k F \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix} \right)$. So for even k, WF is written as a linear combination of $f(\tau)g(\omega) + f(\omega)g(\tau)$ for some $f, g \in A_k(\Gamma_0(N))$. This means that $WF \in \mathrm{Sym}^2(A_k(\Gamma_0(N)))$ where for any vector space V, we denote by $\mathrm{Sym}^2(V)$ the vector space of symmetric tensors of degree two of V. We denote by $\overline{W}$ the composite map of W and the natural projection

$$\mathrm{Sym}^2(A_k(\Gamma_0(N))) \longrightarrow \mathrm{Sym}^2(A_k(\Gamma_0(N)))/\mathrm{Sym}^2(S_k(\Gamma_0(N)))$$

and call this map the *Witt projection*.

THEOREM 1.3. — *When $N = p$ is a prime, for $k \geqslant 4$, the Witt projection is surjective, i.e.*

$$\overline{W}(A_k(\Gamma_0^{(2)}(p))) = \mathrm{Sym}^2(A_k(\Gamma_0(p)))/\mathrm{Sym}^2(S_k(\Gamma_0(p))).$$

For a squarefree N which is not a prime, there are some constraints coming from the cusp configuration on symmetric tensors of Eisenstein series when they are in the image of $\overline{W}$, and $\overline{W}$ is not surjective in the sense above, but we can describe the image of $\overline{W}$ explicitly. This will be given as Theorem 3.1 in Section 3. The above theorem is proved by solving the variant of basis problem proposed in [10], p. 194. It seems interesting that we can prove Theorem 1.1 related with various cusps by using Theorem 3.1 related only with the behavior on the diagonals; the ultimate reason for this is that we can represent the cusps for $\Gamma_0^{(n)}(N)$ by diagonally embedded matrices of type $SL(2)$.

By the way, the Witt operator W itself is not surjective to $\mathrm{Sym}^2(A_k(\Gamma_0^{(2)}(N)))$ in general. Indeed Cris Poor informed us of the following counter-example. Poor and Yuen have shown that $\dim S_2(\Gamma_0^{(2)}(37)) = 2$ (*cf.* [14], Theorem 1.3.) But we have $\dim S_2(\Gamma_0(37)) = 2$ (*e.g.* [18], [13]), so we have $\mathrm{Sym}^2(S_2(\Gamma_0(37))) = 3$. We can show that if $W(F)$ is in $\mathrm{Sym}^2(S_k(\Gamma_0(N)))$ for $F \in A_2(\Gamma_0^{(2)}(N))$ for squarefree N, then F is a cusp form (see Section 3). So by dimensional reason, the Witt operator is not surjective for $N = 37$ and $k = 2$.

Traditionally, the Witt operator was often used to describe the structures of the ring $A(\Gamma)$ of Siegel modular forms, reducing it to the structure of $\mathrm{Ker}(W)$ and $A(\Gamma)/\mathrm{Ker}(W)$. So, it is a natural question to ask to what extent W is surjective. Furthermore, we should point out that the image of cusp forms under the Witt operator is closely related to the Gross-Prasad conjecture as considered in [4]: Indeed, if $WF = \sum c(i,j) f_i \otimes g_j$ for cuspidal Hecke-eigenforms $F \in S_k(\Gamma_0^{(2)}(N))$ and $f_i, g_j \in S_k(\Gamma_0(N))$, then the coefficients $c(i,j)$ should be expressed by square roots of the degree 16 L-function $L(F \otimes f_i \otimes g_j, s)$ at the center point $s = 1/2$. So, we believe that our results on the Witt projection answer some natural problems to be clarified and are worth to be mentioned.

Now we write a short explanation of each section. In Section 2, we quote the explicit description of irreducible components of the boundary of the Satake compactification $\mathcal{S}(N)$ in [6] and then see how one-dimensional cusps intersects with each other at zero-dimensional cusps. In Section 3, we describe natural constraints on the image of the Witt projection $\overline{W}$ and state Theorem 3.1 that $\overline{W}$ is surjective to that space. Taking this theorem for granted, we prove the surjectivity of $\widetilde{\Phi}$. The proof of Theorem 3.1 will be given in Section 4 and 5. In Section 4, we prove that the symmetric tensors of cusp forms and Eisenstein series modulo symmetric tensors of cusp forms are contained in the image of $\overline{W}$. In Section 5, we show that linear spans of Eisenstein series tensor Eisenstein series with some constraints modulo symmetric tensors of cusp forms are contained in the image of $\overline{W}$ and complete the proof. These claims are proved by using theta representability of modular forms of one variables by lattices in a fixed genus. In Section 6, we shortly show the surjectivity of Φ-operator again, including the vector valued case of weight $\det^k \otimes \mathrm{Sym}(j)$ with $k \geqslant 4$. Here we give arguments without using the Witt operator. In Section 7, we give a remark how we can generalize these to the case of Siegel modular forms with characters.

2. Structures of cusps of $\Gamma_0^{(2)}(N)$

The definition of boundary components of the Satake compactification is explained in [15]. We shortly review the general theory of cusps, and then give explicit descriptions of the cusp configuration for $\Gamma_0^{(2)}(N)$ for squarefree N.

2.1. General theory

Let Γ be an arithmetic discrete subgroup of $\mathrm{Sp}(2,\mathbb{Q})$. The one-dimensional cusps of the Satake compactification $\mathcal{S}(\Gamma\backslash\mathfrak{H}_2)$ of $\Gamma\backslash\mathfrak{H}_2$ corresponds bijectively with $\Gamma\backslash\mathrm{Sp}(2,\mathbb{Q})/P_{2,1}(\mathbb{Q})$ and zero-dimensional cusps with $\Gamma\backslash\mathrm{Sp}(2,\mathbb{Q})/P_{2,0}(\mathbb{Q})$, where $P_{n,0}(\mathbb{Q})$ is the so-called Siegel parabolic subgroup for any degree n defined by

$$P_{n,0}(\mathbb{Q}) = \left\{ g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Q}) \right\}.$$

Sometimes it is more convenient to take

$$\mathrm{GSp}(2,\mathbb{Q}) = \left\{ g \in M_4(\mathbb{Q});{}^t gJg = n(g)J, n(g) \in \mathbb{Q}^\times \right\}$$

and define $P_{2,1}^*(\mathbb{Q})$ or $P_{2,0}^*(\mathbb{Q})$ by the maximal parabolic subgroup of $\mathrm{GSp}(2,\mathbb{Q})$ containing $P_{2,1}(\mathbb{Q})$ or $P_{2,0}(\mathbb{Q})$, respectively. Since $\mathrm{GSp}(2,\mathbb{Q}) = \mathrm{Sp}(2,\mathbb{Q})P_{2,i}^*(\mathbb{Q})$ for $i = 0$, 1, in the above double cosets we can replace $\mathrm{Sp}(2,\mathbb{Q})$ by $\mathrm{GSp}(2,\mathbb{Q})$ and $P_{2,i}(\mathbb{Q})$ by $P_{2,i}^*(\mathbb{Q})$ respectively. We put

$$\mathrm{GSp}(2,\mathbb{Q}) = \bigsqcup_{i=1}^{d} \Gamma g_i P_{2,1}^*(\mathbb{Q}).$$

Here we assume that $n(g_i) > 0$. For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$, we put

$$\iota_1(g) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \iota_2(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

When $g \in SL_2(\mathbb{Q})$, then $\iota_i(g) \in \mathrm{Sp}(2,\mathbb{Q})$ for $i = 1$, 2. If g_1, $g_2 \in \mathrm{GL}_2(\mathbb{Q})$ and $\det(g_1) = \det(g_2)$, then $\iota_1(g_1)\iota_2(g_2) \in \mathrm{GSp}(2,\mathbb{Q})$. We also define a mapping ω_1 of $P_{2,1}(\mathbb{Q})$ to $SL_2(\mathbb{Q})$ by

$$\omega_1 \begin{pmatrix} a & 0 & b & * \\ * & * & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & * \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We put

$$\Gamma_i = \omega_1(g_i^{-1}\Gamma g_i \cap P_{2,1}(\mathbb{Q})).$$

For a general discrete group, this might be different from $g_i^{-1}\Gamma g_i \cap \iota_1(SL_2(\mathbb{Q}))$. The one-dimensional cusp corresponding to g_i is isomorphic to $V_i = \Gamma_i\backslash\mathfrak{H}_1$. The usual zero-dimensional cusps of V_i correspond with

$\Gamma_i \backslash SL_2(\mathbb{Q})/P_{1,0}(\mathbb{Q})$. Take a representative h_j of this set. Then the zero-dimensional cusp of V_i corresponding to h_j is identified with the zero-dimensional cusp of $\mathcal{S}(\Gamma \backslash \mathfrak{H}_2)$ corresponding to $\Gamma g_i \iota_1(h_j) P_{2,0}(\mathbb{Q})$. If this is the same as the double coset for different i, then it means that two one-dimensional components intersect at this zero-dimensional cusp.

2.2. The cusps of $\Gamma_0^n(N)$ and the Atkin-Lehner involution

Now we assume that N is squarefree throughout the paper. We denote by t the number of prime divisors of N. Then the representatives of cusps of $\Gamma_0^n(N)$ for each dimension are given explicitly in [6], Lemma 8.1. In this subsection, first we review this for $n = 1$ and 2, and then describe how one-dimensional cusps intersect with each other. First we review the Atkin-Lehner involution of $\Gamma_0(N)$ (for $n = 1$). The main reference is T. Miyake [13], §4.6. For any positive divisor $N_1|N$, there exists an element $\gamma_{N_1} \in SL_2(\mathbb{Z})$ such that

$$\gamma_{N_1} \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \mod N_1^2, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mod (N/N_1)^2. \end{cases}$$

We put

$$\eta_{N_1} = \gamma_{N_1} \begin{pmatrix} N_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

PROPOSITION 2.1. — *The set $\{\eta_{N_1}; 0 < N_1 | N\}$ is a complete set of representatives of $\Gamma_0(N) \backslash \mathrm{GL}_2(\mathbb{Q})/P_{1,0}^*(\mathbb{Q})$, i.e. cusps of $\Gamma_0(N)$. The number of cusps of $\Gamma_0(N)$ is 2^t.*

For the sake of simplicity, we denote by $\kappa(l)$ the cusp of $\Gamma_0(N)$ represented by η_l with $l|N$.

For later use, we add one more formula. Let m, l be positive divisors of N. Let c be the greatest common divisor of l and m, i.e. $c = \gcd(l, m)$ and put $l_1 = l/c$, $m_1 = m/c$. Then we have

$$\Gamma_0(N)\eta_l \eta_m = \Gamma_0(N)\eta_{l_1 m_1} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Now we treat the case when $n = 2$. By virtue of [6], one-dimensional cusps of $\Gamma_0^{(2)}(N)$ are represented by elements $\gamma \in \mathrm{Sp}(2, \mathbb{Z})$ such that $\gamma \equiv$

$\iota_1(1_2)\iota_2(J_1^{e_p})$ mod p for every $p|N$ where $e_p = 0$ or 1. The zero-dimensional cusps are by γ such that $\gamma \equiv \iota_1(J_1^{e_{1,p}})\iota_2(J_1^{e_{2,p}})$ mod p for every $p|N$ with $0 \leqslant e_{1,p} \leqslant e_{2,p} \leqslant 1$. Since the Atkin-Lehner involutions are more convenient for our purpose than γ_m, we rewrite the above claim in the following way.

PROPOSITION 2.2. — *We assume that N is squarefree.*

(1) *A complete set of representatives of the equivalence classes of one-dimensional cusps of $\mathcal{S}(N)$ are given by $\iota_1(\eta_m)\iota_2(\eta_m)$ for positive divisors $m|N$. The number of one-dimensional cusps is equal to 2^t. All the one-dimensional cusps are isomorphic to $\Gamma_0(N)\backslash\mathfrak{H}_1$.*

(2) *A complete set of representatives of the zero-dimensional cusps of $\mathcal{S}(N)$ are given by $\iota_1(\gamma_l)\iota_2(\gamma_m)$ where $l|m|N$. The number of zero-dimensional cusps is equal to 3^t.*

We denote by $\kappa_1(m,m)$ the one-dimensional cusp corresponding to $\iota_1(\eta_m)\iota_2(\eta_m)$ and by $\kappa_0(l,m)$ the zero-dimensional cusp corresponding to $\iota_1(\gamma_l)\iota_2(\gamma_m)$.

For the configuration of cusps, we have the following proposition.

PROPOSITION 2.3.

(1) *For any l, m with $l|N$, $m|N$ and any divisor c of l which is coprime to m, we have*

$$\Gamma_0^{(2)}(N)\iota_1(\gamma_l)\iota_2(\gamma_m)P_{2,0}(\mathbb{Q}) = \Gamma_0^{(2)}(N)\iota_1(\gamma_{l/c})\iota_2(\iota_{cm})P_{2,0}(\mathbb{Q}).$$

(2) *The zero-dimensional cusp $\kappa(l)$ of the one-dimensional cusp $\kappa_1(m,m) \cong \Gamma_0(N)\backslash\mathfrak{H}_1$ in the usual sense coincides in the Satake compactification $\mathcal{S}(N)$ with the zero-dimensional cusp $\kappa_0(m_1, m_1l_1c)$ where $c = \gcd(m,l)$, $m_1 = m/c$, $l_1 = l/c$. In particular, any two cusps on the same one-dimensional cusp do not intersect.*

(3) *The zero-dimensional cusp $\kappa_0(N_1, N_2)$ with $N_1|N_2|N$ is on the one-dimensional cusp $\kappa_1(m,m)$ with $m|N$ if and only if there exists $c|(N_2/N_1)$ such that $m = N_1c$. In particular, the number of one-dimensional cusp passing through $\kappa(N_1, N_2)$ is the number of positive divisors of N_2/N_1.*

Before proving the proposition, we check the consistency of numbers. Any cusp $\kappa(m,m)$ has 2^t cusps. So if we count all the zero dimensional cusps on each one-dimensional cusp separately, then the total is $2^t \times 2^t = 4^t$. On the other hand, denote by the number of prime divisors of N_2 or N_1 by a or b, respectively. The choice of N_2 is $\binom{t}{a}$ and for fixed N_2 the choice of $N_1|N_2$ is $\binom{a}{b}$. For fixed such N_1 and N_2, the number of one-dimensional

cusps passing through $\kappa(N_1, N_2)$ is 2^{a-b}. So the total of zero-dimensional cusps counted separately for each component is $\sum_{a=0}^{t} \binom{t}{a} \sum_{b=0}^{a} \binom{a}{b} 2^{a-b} = \sum_{a=0}^{t} \binom{t}{a} 3^a = 4^t$. So the number is consistent.

Proof of the proposition. — The proof of (1) is obvious since

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \times \iota_1(1_2)\iota_2(J_1) = \iota_1(J_1)\iota_2(1_2) \times \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the double coset is determined only by modulo N. To prove the rest, we consider everything in $\mathrm{GSp}(2, \mathbb{Q})$. So the one-dimensional cusp corresponds with $\Gamma_0^{(2)}(N)\iota_1(\eta_m)\iota_2(\eta_m)P_{2,1}^*(\mathbb{Q})$ and the zero-dimensional cusp with $\Gamma_0^{(2)}(N)\iota_1(\eta_l)\iota_2(\eta_m)\iota_1\begin{pmatrix} m/l & 0 \\ 0 & 1 \end{pmatrix} P_{2,0}^*(\mathbb{Q})$. Now we have

$$\Gamma_0^{(2)}(N)\iota_1(\eta_m)\eta_2(\eta_m)\iota_1(\eta_l)\iota_2 \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} P_{2,0}(\mathbb{Q})$$

$$= \Gamma_0^{(2)}(N)\iota_1(\eta_m\eta_l)\iota_2(\eta_m)\iota_2 \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} P_{2,0}(\mathbb{Q})$$

$$= \Gamma_0^{(2)}(N)\iota_1(\eta_{m_1 l_1})\iota_1(c1_2)\iota_2(\gamma_m)\iota_2 \begin{pmatrix} ml & 0 \\ 0 & 1 \end{pmatrix} P_{2,0}(\mathbb{Q}),$$

where $c = \gcd(m, l)$ and $l_1 = l/c$, $m_1 = m/c$. The above double coset is equal to

$$\Gamma_0^{(2)}(N)\iota_1(\gamma_{m_1 l_1})\iota_2(\gamma_m)\iota_1 \begin{pmatrix} m_1 l_1 c & 0 \\ 0 & c \end{pmatrix} \iota_2 \begin{pmatrix} ml & 0 \\ 0 & 1 \end{pmatrix} P_{2,0}(\mathbb{Q}).$$

Since $m_1 l_1 c^2 = ml$, we have

$$\iota_1 \begin{pmatrix} m_1 l_1 c & 0 \\ 0 & c \end{pmatrix} \iota_2 \begin{pmatrix} ml & 0 \\ 0 & 1 \end{pmatrix} \in P_{2,0}(\mathbb{Q})$$

and the above double coset is represented by $\iota_1(\gamma_{m_1 l_1})\iota_2(\gamma_m)$. By the assertion (1), this is also represented by $\iota_1(\gamma_{m_1})\iota_2(\gamma_{l_1 m}) = \iota_1(\gamma_{m_1})\iota_2(\gamma_{l_1 m_1 c})$. Since this gives a different zero-dimensional cusp for a different choice of l, we prove (2).

Now we prove (3). If $\kappa_0(N_1, N_2)$ is on $\kappa_1(m, m)$, then this means that for some l, we have $N_1 = m_1$ and $N_2 = m_1 l_1 c$ where $c = \gcd(m, l)$ and $l_1 = l/c$, $m_1 = m/c$. If so, then $l = l_1 c = N_2/N_1$ and $m = cm_1 = cN_1$. So m is determined by a divisor c of N_2/N_1. This proves (3). $\square$

As a corollary to the above proposition, we have

PROPOSITION 2.4. — *For $k \geqslant 4$, we have*

$$\dim A_k(\mathrm{Bd}(N)) = 3^t + 2^t \dim S_k(\Gamma_0(N)).$$

Proof. — Since $k \geqslant 4$, there exists an Eisenstein series $E_m \in A_k(\Gamma_0(N))$ such that this does not vanish at the cusp of $\Gamma_0(N)\backslash\mathfrak{H}_1$ corresponding to η_m and vanishes at all the other cusps. Now we fix a zero-dimensional cusp $\kappa_0(l, m)$ of $\mathcal{S}(N)$. Then for a one-dimensional cusp $\kappa_1(N_1, N_1)$ passing through $\kappa_0(l, m)$, there exists a modular form f_{N_1} on $\kappa_1(N_1, N_1)$ which takes the value one at $\kappa_0(l, m)$ and vanishes at all the other cusps of $\kappa_1(N_1, N_1)$. So we can define a modular form $f \in A_k(\mathrm{Bd}(N))$ such that $f = f_{N_1}$ for all such one-dimensional cusps passing through $\kappa_0(l, m)$ and $f = 0$ on all the other one-dimensional cusps. This is a modular form in $A_k(\mathrm{Bd}(N))$ which does not vanish only at $\kappa_0(l, m)$ and vanishes at all the other zero-dimensional cusps. So subtracting a suitable linear combination of such modular forms from an element of $A_k(\mathrm{Bd}(N))$ we get an element which vanishes at all the zero-dimensional cusps. The space of such functions is just the direct sum of $S_k(\Gamma_0(N))$ on each one-dimensional cusp. $\square$

3. The Witt projection $\overline{W}$ and the operator $\widetilde{\Phi}$

First we see consistency between action and restriction. For $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in$ $\mathrm{GL}_2(\mathbb{Q})$ with $\det(g_1) = \det(g_2) > 0$ and a holomorphic function $F(Z)$ on $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathfrak{H}_2$, we have

$$\left(F|_k \left[\iota_1(g_1)\iota_2(g_2)\right]\right)(Z)$$
$$= \left(\left(c_1\tau + d_1\right)\left(c_2\omega + d_2\right) - c_1 c_2 z^2\right)^{-k}$$

$$\times F \begin{pmatrix} \dfrac{(a_1\tau + b_1)(c_2\omega + d_2) - a_1 c_2 z^2}{(c_1\tau + d_1)(c_2\omega + d_2) - c_1 c_2 z^2} & \dfrac{\det(g_1)z}{(c_1\tau + d_1)(c_2\omega + d_2) - c_1 c_2 z^2} \\ \dfrac{\det(g_2)z}{(c_1\tau + d_1)(c_2\omega + d_2) - c_1 c_2 z^2} & \dfrac{(a_2\omega + b_2)(c_1\tau + d_1) - a_2 c_1 z^2}{(c_1\tau + d_1)(c_2\omega + d_2) - c_1 c_2 z^2} \end{pmatrix}.$$

So we have

$$\left(W\left(F|_k \left[\iota_1(g_1)\iota_2(g_2)\right]\right)\right)(\tau, \omega)$$
$$= (WF)\left(\dfrac{a_1\tau + b_1}{c_1\tau + d_1}, \dfrac{a_2\omega + b_2}{c_2\omega + d_2}\right)(c_1\tau + d_1)^{-k}(c_2\omega + d_2)^{-k}.$$

Now we see relations between the Witt projection and the Φ-operator. We assume that $k \geqslant 4$. For any $m|N$, we denote by E_m the Eisenstein series

in $A_k(\Gamma_0(N))$ such that

$$\lim_{\lambda \to \infty} (E_m|_k[\eta_m])(i\lambda) = 1 \quad \text{and} \quad \lim_{\lambda \to \infty} (E_m|_k[\eta_l])(i\lambda) = 0$$

for any other $l \neq m$ with $l|N$. We may write

$$(WF)(\tau, \omega) = \sum_{m|N} \big(F_m(\tau)E_m(\omega) + F_m(\omega)E_m(\tau)\big)$$
$$+ \sum_i \big(g_i(\tau)h_i(\omega) + g_i(\omega)h_i(\tau)\big)$$

where $F_m \in A_k(\Gamma_0(N))$ and g_i, $h_i \in S_k(\Gamma_0(N))$. For any $m'|N$, the limit of F to the one-dimensional cusp $\kappa_1(m', m')$ is given by

$$\Big(\Phi\big(F|_k[\iota_1(\eta_{m'})\iota_2(\eta_{m'})]\big)\Big)(\tau) = \sum_{m|N}(F_m|_k[\eta_{m'}])(\tau).$$

In particular if $WF \in \mathrm{Sym}^2(S_k(\Gamma_0(N)))$, then $F \in S_k(\Gamma_0^{(2)}(N))$ obviously. Since $\eta_{m'}$ normalizes $\Gamma_0(N)$, we have again $F_m|_k[\eta_{m'}] \in A_k(\Gamma_0(N))$ and if F_m is a cusp form, then this is also a cusp form. Now we see the constraints coming from the fact that these modular forms should take the same value at each zero-dimensional cusp $\kappa_0(l, m)$ with $l|m|N$. To describe this, we rewrite F_m as a sum of Eisenstein series and cusp forms. Then we may write

$$(\overline{W}F)(\tau, \omega) = \sum_{m|N}(f_m(\tau)E_m(\omega) + E_m(\tau)f_m(\omega))$$
$$+ \sum_{l,m|N} c(l, m)(E_l(\tau)E_m(\omega) + E_m(\tau)E_l(\omega))$$

where $f_m \in S_k(\Gamma_0(N))$ and $c(l, m) \in \mathbb{C}$. Then the image of F at the one-dimensional cusp $\kappa_1(m, m)$ is $\sum_{l|N} c(l, m)E_l|_k[\eta_m]$ up to cusp forms. The value of this at the cusp of $\kappa_1(m, m)$ corresponding to η_l is the value of $\sum_{l'|N} c(l', m)E_{l'}|_k[\eta_m\eta_l]$ at $i\infty$. If we write $m_1 = m/\gcd(l, m)$ and $l_1 = l/\gcd(l, m)$, then we have $E_{l'}|_k[\eta_m\eta_l] = E_{l'}|_k[\eta_{l_1 m_1}]$ so the value at $i\infty$ is 1 for $l' = l_1 m_1$ and 0 otherwise. So the value of the whole function is $c(l_1 m_1, m)$. The above cusp of $\kappa_1(m, m)$ corresponds to the zero-dimensional cusp $\kappa_0(m_1, lm_1)$. So if we fix zero-dimensional cusp $\kappa_0(N_1, N_2)$ with $N_1|N_2|N$ instead, then if we put $l = N_2/N_1$, $m = cN_1$ and $m_1 = N_1$ for any $c|(N_2/N_1)$, then $\gcd(l, m) = c$ because N is squarefree and so N_2/N_1 and N_1 is coprime, and $c(l_1 m_1, cN_1) = c(N_2/c, cN_1)$ must be the same for any $c|(N_2/N_1)$. Since N is squarefree and $c|(N_2/N_1)$, we have $\gcd(N_2/c, cN_1) = N_1$. This means that for divisors l, m, l', m' of N, we have

$$c(l, m) = c(l', m')$$

if $lm = l'm'$ and $\gcd(l, m) = \gcd(l', m')$.

So we define the subspace consisting of the elements which satisfy the constraints as above. More precisely we define a linear subspace $V_k(N)$ of $\mathrm{Sym}^2(A_k(\Gamma_0(N)))/\mathrm{Sym}^2(S_k(\Gamma_0(N)))$ generated by $f_m(\tau)E_m(\omega) + f_m(\omega)E_m(\tau)$ for $m|N$, $f_m \in S_k(\Gamma_0(N))$ and $\sum_{l,m|N} c(l, m)(E_l(\tau)E_m(\omega) + E_l(\omega)E_m(\tau))$ with $c(l, m) = c(l', m')$ if $\gcd(l, m) = \gcd(l', m')$ and $lm = l'm'$.

The next theorem is a natural generalization of Theorem 1.3 in the introduction.

THEOREM 3.1. — *For squarefree N and $k \geqslant 4$, the Witt projection on $A_k(\Gamma_0^{(2)}(N))$ is surjective to $V_k(N)$.*

The proof of this theorem will be given in Section 4 and 5.

Proof of Theorem 1.1. — Assuming the above theorem for a while, we prove the surjectivity of $\widetilde{\Phi}$. Fix $m|N$ and take any $f \in S_k(\Gamma_0(N))$. By the above theorem there exists $F \in A_k(\Gamma_0(N))$ such that

$$\overline{W}(F) = \left(f|_k\left[\eta_m^{-1}\right]\right)(\tau)E_m(\omega) + E_m(\tau)\left(f|_k\left[\eta_m^{-1}\right]\right)(\omega).$$

Then we have $\Phi(F|_k[\iota_1(\eta_m)\iota_2(\eta_m)]) = f$ and $\Phi(F|_k[\iota_1(\eta_l)\iota_2(\eta_l)]) = 0$ for any $l \neq m$, $l|N$. So the image of $\widetilde{\Phi}$ contains the direct sum of cusp forms on each one-dimensional component. Now fix natural numbers N_1 and N_2 such that $N_1|N_2|N$ and take $F \in A_k(\Gamma_0^{(2)}(N))$ such that $\overline{W}F = \sum_{c|(N_2/N_1)}(E_{N_2/c}(\tau)E_{cN_1}(\omega) + E_{cN_1}(\tau)E_{N_2/c}(\omega))$. Then $\widetilde{\Phi}(F)$ does not vanish at the zero-dimensional cusp $\kappa_0(N_1, N_2)$ and vanishes at any other zero-dimensional cusps. So we prove the theorem. $\qquad\square$

By the way, for the sake of completeness, we count here the dimension of $V_k(N)$ separately. The number of pairs (l, m) up to the above relation is exactly 3^t. Also there are 2^t Eisenstein series in $A_k(\Gamma_0(N))$ so from the tensors of cusp forms and Eisenstein series, we have $2^t \dim S_k(\Gamma_0(N))$. So the total is equal to the dimension of $A_k(\mathrm{Bd}(N))$.

We add here an example of configuration of cusps below. Since the picture of cusps is complicated, we give an example as a table in the case $N = pq$ where p, q are primes. In the following table, the first row means the usual four cusps of $\Gamma_0(pq)\backslash\mathfrak{H}_1$. The identification of these on each one dimensional cusp with zero dimensional cusps of $\mathcal{S}(N)$ is written in each column. Each row below means which zero dimensional cusps are on the one dimensional cusp in the first column.

one dim. cusps\usual cusps	$\kappa(1)$	$\kappa(p)$	$\kappa(q)$	$\kappa(pq)$
$\kappa_1(1,1)$	$\kappa_0(1,1)$	$\kappa_0(1,p)$	$\kappa_0(1,q)$	$\kappa_0(1,pq)$
$\kappa_1(p,p)$	$\kappa_0(p,p)$	$\kappa_0(1,p)$	$\kappa_0(p,pq)$	$\kappa_0(1,pq)$
$\kappa_1(q,q)$	$\kappa_0(q,q)$	$\kappa_0(q,pq)$	$\kappa_0(1,q)$	$\kappa_0(1,pq)$
$\kappa_1(pq,pq)$	$\kappa_0(pq,pq)$	$\kappa_0(q,pq)$	$\kappa_0(p,pq)$	$\kappa_0(1,pq)$

4. The image of $\overline{W}$

For a given $f \in S_k(\Gamma_0(N))$ let us consider more generally the space

$$\mathcal{E}(f) := \big\{e \in E_k(\Gamma_0(N)) \mid e \otimes f + f \otimes e \in \mathrm{Im}(\overline{W})\big\},$$

where $E_k(\Gamma_0(N)) \subseteq A_k(\Gamma_0(N))$ denotes the space generated by Eisenstein series. We will study this space using theta series.

4.1. Some properties of theta series

We first recall some properties of quadratic forms and theta series in a slightly more general setting. For a natural number k, let S be a $2k \times 2k$ positive definite integral symmetric matrix. If all the diagonal components of S are even, S is called even. The minimum of all natural numbers N such that NS^{-1} is also even is called the level of S. We call two such quadratic forms S and T equivalent (over $\mathbb{Z}$) if $A^t S A = T$ for some $A \in \mathrm{GL}(2k, \mathbb{Z})$. We say that S and T are in the same genus $\mathfrak{g}$, if they are equivalent over $\mathbb{Z}_p$ for all primes p. Whenever convenient, we freely switch from the language of quadratic forms and symmetric matrices to the language of lattices in a quadratic space; we just recall that starting from a lattice L we get a matrix S by choosing a $\mathbb{Z}$-basis of L and considering the (Gram)-matrix of scalar products of these basis elements.

The degree n theta series attached to S is then defined by

$$\Theta_S^n(Z) := \sum_{X \in M_{2k,n}(\mathbb{Z})} \exp(\pi i Tr(X^t S X Z)) \quad (Z \in \mathfrak{H}_n).$$

This theta series is known to define a modular form of weight k for $\Gamma_0^{(n)}(N)$ with character defined by the quadratic character $\chi_S := \big(\frac{(-1)^k \det(S)}{\cdot}\big)$.

For a genus $\mathfrak{g}$ of positive even integral quadratic forms of 2k variables, we define the (degree one) unnormalized genus theta series by

$$e(\mathfrak{g})(\tau) := \sum \frac{1}{A(S_i)} \Theta_{S_i}(\tau) \quad (\tau \in \mathfrak{H}_1),$$

where the S_i are representatives of the equivalence classes in $\mathfrak{g}$ and $A(S_i)$ denotes the number of integral automorphisms of S_i. By a famous result of Siegel [19] , this modular form is in the space of Eisenstein series.

BASIC OBSERVATION. — *Assume that two positive definite integral quadratic forms S_1 and S_2 are in the same genus, then*

$$\Theta_{S_1} \otimes \Theta_{S_2} + \Theta_{S_2} \otimes \Theta_{S_1} \in \text{Im}(\overline{W}).$$

Proof. — We write the expression above as

$$\Theta_{S_1}(\tau) \cdot \Theta_{S_1}(w) + \Theta_{S_2}(\tau) \cdot \Theta_{S_2}(w) - (\Theta_{S_1}(\tau) - \Theta_{S_2}(\tau)) \cdot (\Theta_{S_1}(w) - \Theta_{S_2}(w)).$$

The values of theta series in the cusps are given essentially by exponential sums, in particular they depend only on the genus; this statement seems to be due to Siegel [19, p. 376]. Therefore the last summand is an element of $\text{Sym}^2(S_k(\Gamma_0(N)))$. Clearly products of type $\Theta_S(\tau) \cdot \Theta_S(w) = \Theta_S^2 \left(\begin{smallmatrix} \tau & 0 \\ 0 & z \end{smallmatrix} \right)$ are in the image of the Witt operator. $\qquad\square$

As a consequence of the observation above we obtain

PROPOSITION 4.1. — *Let $f \in S_k(\Gamma_0(N))$ be a fixed cusp form.*

a) *If f can be written as a linear combination of theta series Θ_S with all S in a fixed genus $\mathfrak{g}$ of level N, then*

$$e(\mathfrak{g}) \otimes f + f \otimes e(\mathfrak{g}) \in \text{Im}(\overline{W}).$$

b) *We denote by $\mathcal{E}^\theta(f)$ the space generated by all genus theta series $e(\mathfrak{g})$ such that f is a linear combination of theta series Θ_S with $S \in \mathfrak{g}$. Then we have*

$$\mathcal{E}^\theta(f) \subseteq \mathcal{E}(f).$$

Proof. — If $\mathfrak{g}$ is a genus satisfying the conditions of a), then the genus Eisenstein series $e(\mathfrak{g})$ as well as the cusp form f can be written as linear combination of theta series Θ_S with quadratic forms $S \in \mathfrak{g}$. We can then apply the basic observation. $\qquad\square$

Remark. — So far, everything works for arbitrary level, arbitrary weight (using a slightly more general notation, we may also include arbitrary real nebentypus and half-integral weight modular forms); the formulation was chosen in such a way that it may be also applied to cases, where $\mathcal{E}(f)$ depends on f.

4.2. Witt projection and basis problems for several genera

We recall that if all cusp forms in $S_k(\Gamma_0(N))$ can be written as linear combinations of theta series Θ_S with $S \in \mathfrak{g}$, then we say that $\mathfrak{g}$ solves the basis problem for $S_k(\Gamma_0(N))$ (more precisely: it solves the *genus version* of the basis problem).

From now on we stick to the case N squarefree with t prime factors, k even and $k \geqslant 4$. Then in this situation, by our solution of the basis problem [3, Theorem] we may choose in Proposition 4.1 (independently of $f \in S_k(\Gamma_0(N))$) as $\mathfrak{g}$ any genus of exact level N satisfying for all $p|N$ the extra condition

$$L \otimes \mathbb{Z}_p \quad \text{and} \quad p \cdot L^\sharp \otimes \mathbb{Z}_p \quad \text{are not maximal;}$$

here L denotes any lattice from $\mathfrak{g}$ and $L^\sharp$ is the lattice dual to L.

We must assure that we obtain enough Eisenstein series in this way: We put $\mathcal{L}_k(N)$ the set of all genera of rank $2k$, which satisfy the conditions as above.

PROPOSITION 4.2. — Let N be squarefree, then for even $k \geqslant 4$

$$\mathbb{C}\{e(\mathfrak{g}) \mid \mathfrak{g} \in \mathcal{L}_k(N)\} = E_k(\Gamma_0(N)).$$

Proof. — We have to assure the existence of sufficiently many such genera: If $4|k$ there exists a genus $\mathfrak{g}_{0,2k}$ of even unimodular lattices of rank $2k$; this genus is locally everywhere at finite primes a direct sum of hyperbolic planes $\mathbb{H}$. We consider for all divisors $M|N$ the positive definite genus $\mathfrak{g}(N,M)$ which looks locally like

$$N \cdot \mathbb{H} \perp M \cdot \mathbb{H} \perp \mathbb{H} \perp \cdots \perp \mathbb{H}.$$

Note that such a genus exists indeed, because all its local invariants are the same as those of $\mathfrak{g}_{0,2k}$. In this way we define 2^t many genera with the requested property; their Eisenstein series are linearly independent; this can be either read off from the considerations in [11] or [5]; a more direct proof follows from considering the constant terms of these 2^t many Eisenstein series in the 2^t many cusps. The square matrix describing these values is then (for $N = p_1 \cdot p_2 \cdots p_t$) a Kronecker product

$$\begin{pmatrix} 1 & 1 \\ p_1^{-1} & p_1^{-2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ p_2^{-1} & p_2^{-2} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ p_t^{-1} & p_t^{-2} \end{pmatrix},$$

in particular it is of full rank 2^t. Therefore, using these genus Eisenstein series we get the full space of Eisenstein series.

If $4 < k \equiv 2 \bmod 4$ we start from a genus of lattices $L \perp M$ where $M \in \mathfrak{g}_{0,2k-4}$ and L is any quaternary lattice of level N and determinant N^2. We may now apply essentially the same procedure as for the case $4|k$. □

Remark. — We should emphasize that in our construction of many genera it is important to have 3 hyperbolic planes: The first one we need to assure (by scaling using N) that the true level is N and the lattice is not maximal at any prime $p|N$. The second hyperbolic plane we need to

produce (by scaling using $R|N$) many genera; the third hyperbolic plane assures that also $N \cdot L^\sharp$ is not maximal at any prime $p|N$.

COROLLARY 4.3. — *For N squarefree, k even with $k \geqslant 4$, all tensors of type*

$$(e \otimes f) + (f \otimes e), \quad f \in S_k(\Gamma_0(N)), \ e \in E_k(\Gamma_0(N))$$

are in the image of $\overline{W}$.

5. The Eisenstein part

5.1. General aspects

The problem can be studied in a quite general context. The relevant parabolic subgroup here is the Siegel parabolic $P_{n,0}$; For any subgroup G of $\mathrm{Sp}(n, \mathbb{Z})$ we put $G_\infty := G \cap P_{n,0}(\mathbb{Z})$. Let Γ be an arbitrary congruence subgroup of $\mathrm{Sp}(n, \mathbb{Z})$. We consider only even weights here. The first observation is that for a modular form $F \in A_k(\Gamma)$ the value at a zero dimensional cusp, *i.e.*

$$(F|_k R)(i\infty)$$

for $R \in \mathrm{Sp}(n, \mathbb{Z})$ depends only on the double coset $\Gamma \cdot R \cdot P_{n,0}(\mathbb{Z})$.

On the other hand, we can define Eisenstein series

$$E_k^n(\Gamma, R)(Z) := \sum_{\gamma \in (R^{-1}\Gamma R)_\infty \backslash R^{-1} \cdot \Gamma} j(\gamma, Z)^{-k}, \quad \left(Z \in \mathfrak{H}_n, R \in \mathrm{Sp}(n, \mathbb{Z})\right).$$

They are known to converge absolutely (and uniformly in Siegel domains for Γ) if the weight is large, *i.e.* $k > n+1$. The matrices R and R' define the same series if R and R' define the same cusp. Moreover, for two matrices $R, S \in \mathrm{Sp}(n, \mathbb{Z})$:

$$(E_k^n(\Gamma, R)|_k[S])(i\infty) = \begin{cases} 1 & \text{if } \Gamma R P_{n,0}(\mathbb{Z}) = \Gamma S P_{n,0}(\mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

For weights $k > n+1$ we get a direct sum decomposition

$$A_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus A_k(\Gamma)^0,$$

where $\mathcal{E}_k(\Gamma)$ is the space generated by these Eisenstein series and $A_k(\Gamma)^0$ the subspace of modular forms vanishing at all zero dimensional cusps.

Now we start from a congruence subgroup Γ of $\mathrm{Sp}(2n, \mathbb{Z})$ (size $4n$) and we consider the generalized Witt operator

$$W \colon A_k(\Gamma) \longrightarrow A_k(\Gamma') \otimes A_k(\Gamma'')$$

defined by

$$(WF)(\tau, \omega) := F \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}, \quad (\tau, \omega \in \mathfrak{H}_n)$$

where the congruence subgroups Γ' and Γ'' are defined by

$$\Gamma' \times 1_{2n} = \Gamma \cap \left(\mathrm{Sp}(n) \times 1_{2n} \right)$$

$$1_{2n} \times \Gamma'' = \left(1_{2n} \times \mathrm{Sp}(n) \right) \cap \Gamma.$$

Here we identify $\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ with a subgroup of $\mathrm{Sp}(2n)$ in the usual way via

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \right) \longmapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix}.$$

We will apply slash-operators $|_k$ to images under Witt-operators; we use then an upper index to indicate the variable (τ or ω) which is relevant.

For $F \in A_k(\Gamma)$ the value

$$\left((WF)|_k^\tau[\gamma] |_k^\omega[\delta] \right)(i\infty, i\infty)$$

does in general not really depend on the double cosets $\Gamma' \gamma P_{n,0}(\mathbb{Z})$ and $\Gamma'' \delta P_{n,0}(\mathbb{Z})$ but only on the double cosets

$$\Gamma \cdot (\gamma \times \delta) \cdot P_{2n,0}(\mathbb{Z});$$

this relation may sometimes be weaker.

If we have convergent Eisenstein series (*i.e.* for $k > n+1$) we can rephrase this as follows. Let (γ_i) and (δ_j) be a system of representatives for the zero dimensional cusps for Γ' and Γ''. Then we get a decomposition

$$WF(\tau, \omega) = \sum_{i,j} c(i,j) E_k^n(\Gamma', \gamma_i) \otimes E_k^n(\Gamma'', \delta_j) + \mathcal{R}_0$$

for some $\mathcal{R}_0 \in \mathcal{R} = A_k(\Gamma') \otimes A_k(\Gamma'')^0 + A_k(\Gamma')^0 \otimes A_k(\Gamma'')$. Then the coefficients $c(i,j)$ are not completely free, they must always satisfy the relations

$$\Gamma \cdot (\gamma_i \times \delta_j) \cdot P_{2n,0}(\mathbb{Z}) = \Gamma \cdot (\gamma_{i'} \times \delta_{j'}) \cdot P_{2n,0}(\mathbb{Z}) \implies A(i,j) = A(i',j'). \quad (*)$$

Thus we have shown that the Eisenstein part of the image of the Witt operator lies modulo $\mathcal{R}$ in a subspace $\mathcal{E}_k^{2n,n}(\Gamma)$ of $\mathcal{E}_k(\Gamma') \otimes \mathcal{E}_k(\Gamma'')$ defined by the relation $(*)$ above. For large weights we can say more:

PROPOSITION 5.1. — *For $k > 2n+1$ the Eisenstein part of the image of the Witt operator is modulo $\mathcal{R}$ equal to $\mathcal{E}_k^{2n,n}(\Gamma)$.*

Proof. — For γ_i, δ_j as above we consider the degree $2n$ Eisenstein series attached to $R := \gamma_i \times \delta_j$. Then we see that for any $\gamma_{i'}, \delta_{j'}$

$$\left(WE_k^{2n}(\tau, \omega)|_k^\tau[\gamma_{i'}]|_k^\omega[\delta_{j'}] \right)(i\infty, i\infty)$$

is equal to 1 if $\Gamma(\gamma, \delta)P_{2n,0}(\mathbb{Z}) = \Gamma(\gamma', \delta')P_{2n,0}(\mathbb{Z})$, i.e. $i = i', j = j'$ and zero otherwise. $\qquad\square$

Remarks.

1) The considerations above also work in essentially the same way for Witt operators for more general restrictions from $\mathfrak{H}_{n+r}$ to $\mathfrak{H}_n \times \mathfrak{H}_r$.

2) There are cases, where $W(E_k^{2n}(\Gamma, R))$ contains no Eisenstein part: To give a simple example we consider for $n = 2$, $\Gamma = \Gamma(p)$ the principal congruence subgroup of level p and the Eisenstein series associated to $R = \left(\begin{smallmatrix} 1_2 & 0_2 \\ X & 1_2 \end{smallmatrix} \right)$, where X is integral symmetric and not diagonal mod p. Then the double coset $\Gamma(p) \cdot R \cdot P_{2,0}$ cannot be represented by an element of the form $\gamma \times \delta$ with $\gamma, \delta \in SL(2, \mathbb{Z})$.

3) The case $\Gamma = \Gamma_0^{(2n)}(N)$ is much simpler, because here all zero dimensional cusps can be represented by elements of the form $(\gamma \times \delta) \in \mathrm{Sp}(n, \mathbb{Z}) \times \mathrm{Sp}(n, \mathbb{Z})$, in fact one can choose representatives in $SL(2, \mathbb{Z})^{2n} \subseteq \mathrm{Sp}(2n, \mathbb{Z})$, see [12, 5]. If N is squarefree, then the situation simplifies still further (see [6]).

5.2. The special case $n = 1$

Our original aim was to determine the Eisenstein part of our Witt projection (with level N squarefree). Here we can parametrize the degree one Eisenstein series by divisors M of N: we associate to M first any matrix $R := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL(2, \mathbb{Z})$ such that c is divisible by M and coprime to N/M. We then write $e_{k,M}$ instead of $E_k(\Gamma_0(N), R)$.

Essentially as a special case of the Proposition above we obtain

COROLLARY 5.2. — *For squarefree N and even $k \geqslant 4$ the Eisenstein part of $\mathrm{Im}(\overline{W})$ is the space*

$$\left\{ \sum_{M_1, M_2 | N} A(M_1, M_2) e_{k, M_1} \otimes e_{k, M_2} \,\middle|\, A(M_1, M_2) \in \mathbb{C} \text{ with } (**) \right\}$$

where $(**)$ *means that*

$$A(M_1, M_2) = A(M_1', M_2')$$

if $\gcd(M_1, M_2) = \gcd(M_1', M_2')$ *and* $M_1 \cdot M_2 = M_1' \cdot M_2'$.

The dimension of this space is 3^t.

Proof. — The condition above is a reformulation of

$$\operatorname{rank}_{\mathbb{F}_p} \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \operatorname{rank}_{\mathbb{F}_p} \begin{pmatrix} M_1' & 0 \\ 0 & M_2' \end{pmatrix}, \quad (\forall p | N).$$

Clearly the condition above is of multiplicative structure; to compute the dimension, we only have to consider the case of N a prime, where the dimension is indeed equal to 3. □

We also note that due to our results about tensors of type $e \otimes f + f \otimes e$, where f is cuspidal and e is an Eisenstein series, the corollary does not need to be considered modulo $\mathcal{R}$, it is enough to factor out symmetric tensors of cusp forms.

Proof of Theorem 3.1. — This is now obvious by Corollary 4.3 and Corollary 5.2. □

Remarks.

1) Compared with the general results on "pullbacks of Eisenstein series" a la Garrett [7, 8] and others, the statements above are quite weak, because we consider only the Eisenstein part; on the other hand we avoid the double coset decompositions used in the investigation of pullbacks; they can be quite complicated for Eisenstein series for congruence subgroups, see *e.g.* [2]

2) In [10] a variant of the basis problem was proposed:
   Is $\operatorname{Sym}^2(A_k(\Gamma_0(N)))/\operatorname{Sym}^2(S_k(\Gamma_0(N)))$ *spanned by theta series* $\Theta_S(\tau) \times \Theta_S(w)$ *with S running over even integral positive definite matrices of size $2k \times 2k$ and level N?*
   Surprisingly, it is the Eisenstein series part, which is responsible for a negative answer in general: $\operatorname{Sym}^2(E_k(\Gamma_0(N)))$ has dimension $2^{t-1}(2^t + 1)$ whereas $\dim \mathcal{E}_k^{2,1}(\Gamma_0) = 3^t$. This fits well to the fact that our "basic observation" cannot be applied to tensors of type $e(\mathfrak{g}) \otimes e(\mathfrak{g}') + e(\mathfrak{g}') \otimes e(\mathfrak{g})$ for two different genera $\mathfrak{g}$ and $\mathfrak{g}'$.

3) Our consideration shows that up to cusp forms the space $A_k(\Gamma_0^{(2)}(N))$ is generated by theta series if $k \geqslant 4$: This is clear for forms which map under $\widetilde{\Phi}$ to cusp forms (by Corollary 4.3), while theta series generate the full space of Eisenstein series by referring to [5] or by applying a genus 2 version of the method used in the proof of Proposition 4.2.

6. The Φ-operator again (vector-valued case)

Here we define vector valued Siegel modular forms only for very special weights $\det^k \otimes \operatorname{Sym}^\nu$ where Sym^ν is the symmetric tensor representations

of $\mathrm{GL}(n,\mathbb{C})$ of degree ν. This is enough for our setting since only these representations occur for $n = 1$ and 2. We realize these representations in the usual way on the space $V_{n,\nu} := \mathbb{C}[u_1,\ldots,u_n]_\nu$ of homogeneous polynomials of degree ν. We denote by $A_{k,\nu}(\Gamma_0^{(n)}(N))$ the space of Siegel modular forms for $\Gamma_0^{(n)}(N)$ with values in $\mathbb{C}[u_1,\ldots,u_n]_\nu$ and with automorphy factor $\det^k \otimes \mathrm{Sym}^\nu$. More precisely, $F \in A_{k,\nu}(\Gamma_0^{(n)}(N))$ is a $\mathbb{C}[u_1,\ldots,u_n]_\nu$ valued holomorphic function $F(Z,u)$ ($Z \in \mathfrak{H}_n$, $u = (u_1,\ldots,u_n)$) such that $F(\gamma Z, u) = j(\gamma, Z)^k F(Z, u\gamma)$ for any $\gamma \in \Gamma_0^{(n)}(N)$ (+holomorphy at cusps if $n = 1$). Note that for $n = 1$ this is the space of modular forms of weight $k + \nu$ if we identify $f(\tau) \in A_{k+\nu}(\Gamma_0(N))$ with $f(\tau)u_1^\nu$.

For a divisor R of N we define the Φ-operator Φ_R by

$$
\Phi_R \colon \begin{cases} A_{k,\nu}(\Gamma_0^{(2)}(N)) & \longrightarrow & A_{k+\nu}(\Gamma_0(N)) \\[2mm] F & \longmapsto & \displaystyle\lim_{\lambda \to \infty} (F\,|_{k,\nu}\,\iota_1(\mathrm{id})\iota_2(\eta_R)) \begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}. \end{cases}
$$

Here we note that in the limit for $n = 2$ above all the coefficients of $u_1^a u_2^b$ vanish unless $b = 0$ and hence the image is in $A_{k+\nu}(\Gamma_0(N))u_1^\nu \cong A_{k+\nu}(\Gamma_0(N))$. This can be proved in the same way as in Arakawa [1]. We first describe this Φ-operator on theta series: We denote by $\mathcal{P}_{m,n,\nu}$ the space of pluriharmonic polynomial maps P from $\mathbb{C}^{m,n}$ to $V_{n,\nu}$ satisfying

$$
P(XA) = \mathrm{Sym}^\nu(A^t)P(X), \quad \left(X \in \mathbb{C}^{(m,n)}, A \in \mathrm{GL}(n,\mathbb{C})\right)
$$

For an even positive definite symmetric matrix S of size $m = 2k$ and a polynomial $P \in \mathcal{P}_{m,n,\nu}$ we define a degree n theta series by

$$
\Theta_{S,P}^n(Z) := \sum_{X \in M_{2k,n}(\mathbb{Z})} P\big(S^{1/2}X\big) \exp\big(\pi i Tr(X^t SXZ)\big),
$$

where $S^{1/2}$ denotes a positive definite symmetric matrix whose square is S. This theta series is known to define a Siegel modular form of level N for the automorphy factor $\det^k \otimes \mathrm{Sym}^\nu$, where N is the level of S.

In the case $n = 2$ we can write $P \in \mathcal{P}_{2k,2,\nu}$ as sum of monomials $u_1^j u_2^{\nu-j}$; this gives

$$
P = \sum_{j=0}^{\nu} Q_j
$$

with $Q_j \in \mathcal{P}_{2k,1,j} \otimes \mathcal{P}_{2k,1,\nu-j}$. This corresponds to a decomposition

$$
\Theta_{S,P}^2 \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \sum_{j=0}^{\nu-1} \sum_{\vec{x}_1,\vec{x}_2 \in \mathbb{Z}^{2k}} Q_j\big(\vec{x}_1,\vec{x}_2\big) \exp\big(\pi i(\vec{x}_1^t S\vec{x}_1\tau_1 + \vec{x}_2^t S\vec{x}_2\tau_2)\big)
$$
$$
+ \Theta_{S,P^0}(\tau_1)\Theta_{S,1}(\tau_2).
$$

Here $P^0 := P(u_1, 0)$ (so $Q_\nu = P^0 \otimes 1$) and we observe that all the contributions for $j < \nu$ are cuspidal for τ_2. We get

$$\Phi_R(\Theta^2_{S,P}) = (\Theta_{S,1}|_k \eta_R)(i\infty) \cdot \Theta_{S,P^0}.$$

The global $\widetilde{\Phi}$-operator can be defined as before by

$$\widetilde{\Phi}: \begin{cases} A_{k,\nu}(\Gamma_0^{(2)}(N)) & \longrightarrow & A_{k+\nu}(\Gamma_0(N))^{\omega(N)} \\ f & \longmapsto & (\Phi_R(f)_{R|N}) \end{cases}$$

where $\omega(N)$ is the number of divisors of N, $i.e.$ the number of one-dimensional cusps of $\Gamma_0^{(2)}(N)$.

THEOREM 6.1. — *The operator $\widetilde{\Phi}$ is surjective onto cusp forms $S_{k+\nu}(\Gamma_0(N))^{\omega(N)}$ for $k \geqslant 4$ and all $\nu \geqslant 0$.*

This is a generalization of [10], Theorem 5.1 where the case $N = 1$ was treated. We remark that for $\nu > 0$ only cusp forms appear in the image $\widetilde{\Phi}(A_{k,\nu}(\Gamma_0^{(2)}(N)))$ as in [1]; furthermore note that in the case $\nu = 0$ we reprove the surjectivity to $S_k(\mathrm{Bd}(N))$ obtained in previous sections by using the Witt projection.

To prove the theorem we start from an arbitrary cusp form $f \in S_{k+\nu}(\Gamma_0(N))$. We use the same notation as in Section 4.2. First we assume $4|k$; then for any divisor M of N our solution of the basis problem [3] allows us to write f as a linear combination of theta series using only quadratic forms from the genus $\mathfrak{g}(N, M)$, $i.e.$

$$f = \sum_i \Theta_{S_i, P_i}$$

with $S_i \in \mathfrak{g}(N, M)$ and $P_i \in \mathcal{P}_{2k,1,\nu}$. For each P_i, if we put $\widetilde{P}_i(X) = \widetilde{P}_i(\vec{x}_1, \vec{x}_2) = P_i(\vec{x}_1 u_1 + \vec{x}_2 u_2)$ ($X = (\vec{x}_1, \vec{x}_2)$, $\vec{x}_i \in \mathbb{C}^{2k}$), then $\widetilde{P}_i \in \mathcal{P}_{2k,2,\nu}$ and we have $\widetilde{P}_i(\vec{x}_1, 0) = P_i(\vec{x}_1)u_1^\nu$. The degree two (and $V_{2,\nu}$-valued) Siegel modular form

$$F_M := \sum \Theta^2_{S_i, \widetilde{P}_i}$$

then satisfies for any divisor $R|N$

$$\Phi_R(F_M) = \sum_i (\Theta_{S_i}|_k \eta_R)(i\infty)\, \Theta_{S_i, P_i} = c(R, M) \cdot f.$$

Here we use that $c(R, M) := (\Theta_{S_i}|_k \eta_R)(i\infty)$ depends only on R and M. As in Section 4, the square matrix $C = (c(R, M))$ (of size $2^{\omega(N)}$) is a Kronecker product of matrices $\begin{pmatrix} 1 & 1 \\ p^{-1} & p^{-2} \end{pmatrix}$ and in particular of maximal rank. This implies the surjectivity, because we get any tuple of type $(0, \ldots, f, 0, \ldots, 0)$

as an image of $\widetilde{\Phi}$. If $k \equiv 2 \bmod 4$ we use the same kind of modified argument as in Section 4.

Remark. — The surjectivity of the (suitably defined) Witt projection for $\nu > 0$ is proved in the same way as before. We omit the details.

7. The real primitive nebentypus case

We can ask the same questions about Witt projections for modular forms of nebentypus. Of course our techniques of theta series apply only to quadratic characters. We rely here on the methods used in [2]; note that in [3] only trivial characters are considered. Therefore we must restrict ourselves to primitive nebentypus, but there is no doubt that the methods and results of [3, 2] can be generalized to nonprimitive quadratic characters (as announced in [2]).

Let $N \equiv 1 \bmod 4$ be squarefree and denote by χ_N the associated (even) primitive quadratic character mod N. Whenever convenient, we identify χ_N with a character of $\Gamma_0(N)$ by

$$\chi_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \chi_N(d) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Let $\mathcal{L}_\chi(N)$ be the set of genus $\mathfrak{g}$ consisting of lattices L such that for all $p|N$,

$$p^3 \mid \det(L_p) \nmid p^{2k-1}.$$

Then we have

PROPOSITION 7.1. — *For N as above and even $k \geqslant 4$,*

$$\mathbb{C}\{e(\mathfrak{g}) \mid \mathfrak{g} \in \mathcal{L}_\chi(N)\} = E_k\big(\Gamma_0(N), \chi_N\big),$$

where $E_k\big(\Gamma_0(N), \chi_N\big)$ is the subspace of Eisenstein series inside the space $A_k\big(\Gamma_0(N), \chi_N\big)$.

Proof. — There exists a genus $\mathfrak{g}_0$ of discriminant N, which locally at all p is of the form

$$M_p \perp \mathbb{H} \perp \cdots \perp \mathbb{H}$$

where M_p is a suitable binary lattice with $|\det(M_p)| = p$; for a proof see [20], where the existence of the adjoint lattice is stated explicitly. For all divisors R of N we consider lattices, which look locally like

$$M_p \perp N \cdot \mathbb{H} \perp R \cdot \mathbb{H} \perp \cdots \perp \mathbb{H}.$$

Since locally the invariants are the same as M_p, there exist such global lattices. As before, by using the Eisenstein series of these genera, we get 2^t linearly independent genus Eisenstein series $e(\mathfrak{g})$. $\qquad\square$

We assume from now on that the basis problem for $S_k(\Gamma_0(N), \chi_N)$ has a positive answer for each given genus $\mathfrak{g} \in \mathcal{L}_\chi(N)$. Unfortunately this was proved in [2] only for prime levels, but should be true more generally.

COROLLARY 7.2. — *For $N \equiv 1 \bmod 4$, N squarefree, all tensors of type $(e \otimes f) + (f \otimes e)$, $f \in S_k(\Gamma_0(N), \chi_N)$, $e = $ Eisenstein series in $A_k(\Gamma_0(N), \chi_N)$ are in the image of $\overline{W}$ under the assumption above.*

Again we mention that this is proved without assumption only for the prime level case. (as far as it relies on [2]).

As for the Eisenstein part, we just mention that instead of the Eisenstein series $E_k^n(\Gamma_0^{(n)}(N), R)$ we should use

$$E_k^n\big(\Gamma_0^{(n)}(N), R, \chi_N\big)(Z) = \sum_{\gamma \in (R^{-1}\Gamma_0^{(n)}(N)R)_\infty \backslash R^{-1}\Gamma_0^{(n)}(N)} \chi_N(R\gamma) j(\gamma, Z)^{-k}.$$

We can then get the same results as before about the Eisenstein part of the image of $\overline{W}$. We omit details here.

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