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NUMERICAL CHARACTER OF THE EFFECTIVITY OF ADJOINT LINE BUNDLES

by Frédéric CAMPANA, Vincent KOZIARZ & Mihai PĂUN

Abstract. — In this note we show that, for any log-canonical pair (X, Δ), $K_X + \Delta$ is $\mathbb{Q}$-effective if its Chern class contains an effective $\mathbb{Q}$-divisor. Then, we derive some direct corollaries.

Résumé. — Dans cette note nous montrons que le système linéaire adjoint associé à une paire log-canonique est non-vide dés que la classe de Chern de ce système contient un diviseur effectif dont les coefficients sont rationnels. Nous en déduisons quelques corollaires immédiats.

Introduction

For the notions of klt and log-canonical pairs, we refer to [13]. The main result we will prove is the following

Theorem 0.1. — *Let X be a smooth, connected (complex) projective manifold, and let Δ be an effective $\mathbb{Q}$-divisor on X, such that the pair (X, Δ) is lc. Assume that there exists a line bundle ρ on X such that $c_1(\rho) = 0$, and such that $H^0\big(X, m(K_X + \Delta) + \rho\big) \neq 0$, for some integer m, divisible enough (i.e., such that all coefficients appearing in Δ become integral).*

Then $h^0\big(X, m'(K_X+\Delta)\big) \geqslant h^0\big(X, m(K_X+\Delta)+\rho\big) > 0$, for some suitable multiple m' of m.

In the first part of this note we treat the theorem for klt pairs, and in the second part we prove the result in full generality, by combining the techniques of proof in the klt case with the special case of a reduced boundary with simple normal crossings, established by Y. Kawamata in

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[11]. Finally, in the third part, we draw some immediate consequences of Theorem 0.1.

The special case $\Delta = 0$ was settled in the article [5] by F. Campana and T. Peternell (with an appendix by M. Toma). The main steps of their approach are as follows: the case $m = 1$ is treated by using the fundamental result of C. Simpson in [17], and Serre duality. The general case was reduced to the $m = 1$ case by means of ramified cover techniques, central in additivity results, due to H. Esnault and E. Viehweg.

The main ingredient of the proof we present here is a generalization by N. Budur (*cf.* [4] and the references there) of C. Simpson's result, also based on the study of ramified covers à la Viehweg; this is discussed in Section 1.D. We also introduce a simple and very effective technique for the reduction to the $m = 1$ case, see 1.C below.

If the boundary Δ is reduced and has simple normal crossings, Theorem 0.1 has been recently established in [11]. The arguments invoked are the same as the ones in [5], with the notable exception that the mixed Hodge decomposition of P. Deligne for quasi-projective manifolds is used.

An easy consequence of Theorem 0.1 is that $\kappa(X, K_X + \Delta) = 0$, provided that the numerical dimension of $K_X + \Delta$ is equal to zero: this particular case of the abundance conjecture is well-known when (X, Δ) is klt, and is due to N. Nakayama in [14]. However, the methods of [14] do not seem to apply in the more general context of the previous theorem. Notice that our result shows that the abundance conjecture is of numerical character. Also, O. Fujino suggested that the results proved here can be used in order to derive Corollary 3.5 (at the end of this text).

When the boundary Δ has some ample part, Theorem 0.1 follows from a classical argument of V. Shokurov, which combines the Kawamata-Viehweg vanishing theorem with the fact that the Euler characteristic of a line bundle is a topological invariant, see [16], [15].

Part of the results of [11] by Y. Kawamata also rest on Simpson's result in an essential way. Notice that the general log-canonical version of Theorem 0.1 above is stated in the last Section of [11] of Kawamata's text. Despite the apparences, the proof given there however seems to differ significantly from ours, given below in Section 2.

After posting the present text, we were informed by C. Hacon that a statement analogous to Lemma 1.1 below is formulated in [6], Theorem 3.2.

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1. Proof of Theorem 0.1 in the klt case

First, we set $d := h^0\big(X, m(K_X+\Delta)+\rho\big) > 0$ and, after possibly blowing-up X, we denote by F the fixed part of the linear system $|m(K_X+\Delta)+\rho|$ so that for any non-zero $u \in H^0\big(X, m(K_X+\Delta)+\rho\big)$, $[u=0] = F + M_u$ where M_u is an effective divisor (the "movable part" of $[u=0]$). Once for all, we choose a generic u_0, meaning that M_{u_0} is reduced and that the support of M_{u_0} has no common component with the supports of F and Δ.

From now on, we denote by $(W_j)_{j\in J}$ the set of irreducible components appearing either in the support of $\Delta = \sum_{j\in J} \mu^j W_j$, or of $[u_0 = 0]$. In particular, we can write $F = \sum_{j\in J} a^j W_j$ and $M_{u_0} = \sum_{j\in J} b^j W_j$.

We divide the proof of Theorem 0.1 in a few steps.

1.A. *We can assume that the $(W_j)_{j\in J}$ are non-singular and have normal crossings*

Indeed this is completely standard: we consider a log-resolution $\mu : \widehat{X} \to X$ of (X, Δ) such that the proper transforms of $(W_j)_{j\in J}$, and the exceptional divisors are non-singular, and have normal crossings. The change of variables formula reads as

$$\mu^*(K_X+\Delta) + E = K_{\widehat{X}} + \widehat{\Delta}$$

where E is effective and μ-exceptional, $\widehat{\Delta}$ is effective, and $(\widehat{X}, \widehat{\Delta})$ is klt (see *e.g.*, [13]).

Together with Hartogs principle, the above formula shows that the statement we want to prove is preserved by the modification μ, *i.e.*, it is sufficient to prove that some multiple of $K_{\widehat{X}} + \widehat{\Delta}$ has at least d linearly independent sections in order to conclude. Remark also that the assumptions concerning u_0 are not affected.

In what follows we will not change the notations, but we keep in mind that we have the transversality property 1.A.

1.B. *We can assume that Δ and the zero divisor of u_0 have no common component*

Recall that we wrote $F = \sum_{j\in J} a^j W_j$. We have

$$m\Big(\sum_{j\in J} \mu^j W_j\Big) - \sum_{j\in J} a^j W_j = m\Big(\sum_{j\in J_1} \mu_0^j W_j\Big) - \sum_{j\in J_2} a_0^j W_j$$

where $\mu_0^j := \max\Big(\mu^j - \frac{1}{m}a^j, 0\Big)$ and $a_0^j := \max\big(a^j - m\mu^j, 0\big)$, hence the sets J_1 and J_2 corresponding to non-zero coefficients are disjoint.

Now, if $u \in H^0\big(X, m(K_X+\Delta)+\rho\big)$, we have

$$m\Big(K_X + \sum_{j\in J} \mu^j W_j\Big) + \rho \sim \sum_{j\in J} a^j W_j + M_u$$

and thus

$$m\Big(K_X + \sum_{j\in J_1} \mu_0^j W_j\Big) + \rho \sim \sum_{j\in J_2} a_0^j W_j + M_u =: E_u \tag{1.1}$$

where we denote by the symbol "$\sim$" the linear equivalence of the bundles in question. In particular, if $M_{u_0} = \sum_{j\in J_3} W_j$ (recall that M_{u_0} is reduced) where J_3 is the set of non-zero coefficients, it follows from our choice of u_0 that the J_i are pairwise disjoint, and the claim is proved. Indeed, if we are able to produce d linearly independent sections of some multiple of the $\mathbb{Q}$-bundle $K_X + \sum_{j\in J_1} \mu_0^j W_j$, the conclusion follows.

For the rest of this note, we replace the divisor Δ in Theorem 0.1 with $\sum_{j\in J_1} \mu_0^j W_j$ and we denote the divisor E_{u_0} by E so that

$$m\Big(K_X + \sum_{j\in J_1} \mu_0^j W_j\Big) + \rho \sim \sum_{j\in J_2} a_0^j W_j + \sum_{j\in J_3} W_j = E. \tag{1.2}$$

Observe that the supports of E and Δ have no common component, and that their union is snc.

1.C. *Reduction to the case $m = 1$*

We write the bundle (1.2) in adjoint form, as follows:

$$m(K_X+\Delta)+\rho \sim K_X + \Delta + \frac{m-1}{m}E + \frac{1}{m}\rho. \tag{1.3}$$

In order to simplify the writing, we introduce the next notations:

$$\Big[\frac{m-1}{m}E\Big] := \sum_{j\in J_2} \Big[\frac{m-1}{m}a_0^j\Big] W_j,$$

and

$$\Big\{\frac{m-1}{m}E\Big\} := \sum_{j\in J_2} \Big\{\frac{m-1}{m}a_0^j\Big\} W_j + \sum_{j\in J_3} \frac{m-1}{m} W_j;$$

here we denote by $\{x\}$ and $[x]$ respectively the fractional part and the integer part of the real number x. We therefore have the decomposition

$$\frac{m-1}{m}E = \Big[\frac{m-1}{m}E\Big] + \Big\{\frac{m-1}{m}E\Big\}.$$

We subtract next the divisor $\big[\frac{m-1}{m}E\big]$ from both sides of (1.3), and we define

$$\Delta^+ := \Delta + \frac{m-1}{m}E - \Big[\frac{m-1}{m}E\Big] = \Delta + \Big\{\frac{m-1}{m}E\Big\}.$$

Henceforth, we have the identity

$$E - \left[\frac{m-1}{m}E\right] \sim K_X + \Delta^+ + \frac{1}{m}\rho. \tag{1.4}$$

Moreover remark that by (1.1), for any $u \in H^0\big(X, m(K_X + \Delta) + \rho\big)$ the divisor

$$E_u - \left[\frac{m-1}{m}E\right] \sim K_X + \Delta^+ + \frac{1}{m}\rho$$

is also effective. Notice indeed that $\left[\frac{m-1}{m}E_u\right] = \left[\frac{m-1}{m}E\right]$ arises from the fixed part.

In conclusion, we can define a *line bundle* L_1 associated to the effective $\mathbb{Q}$-divisor Δ^+ such that:

- The pair (X, Δ^+) is klt;
- The adjoint bundle $K_X + L_1 + \frac{1}{m}\rho$ has d linearly independent sections.

1.D. *A lemma by N. Budur*

The following result could be extracted from N. Budur's article [4]; for the convenience of the reader, we include a direct proof as well. We denote by $\mathrm{Pic}^\tau(X) \subset \mathrm{Pic}(X)$ the subgroup of line bundles whose first Chern class is torsion.

Lemma 1.1 ([4]). — *Let X be a connected complex projective manifold, and Δ^+ an effective $\mathbb{Q}$-divisor on X, with simple normal crossings support, and (X, Δ^+) klt. Assume also that $\Delta^+ \sim L_1$, for some $L_1 \in \mathrm{Pic}(X)$. For each integer $k \geqslant 0$, define $L_k := kL_1 - [k\Delta^+] \sim \{k\Delta^+\}$ (we remark that this is consistent with the previous assumption).*

Then for each k, i and q the set

$$V_i^q(f, L_k) = \{\lambda \in \mathrm{Pic}^\tau(X) \,:\, h^q(X, K_X + L_k + \lambda) \geqslant i\}$$

is a finite union of torsion translates of complex subtori of $\mathrm{Pic}^0(X)$.

Proof. — We write $\Delta^+ = \sum_{j\in J} \alpha^j D_j$, and let N be the smallest positive integer such that $N\alpha^j \in \mathbb{N}$ for all j. Then NL_1 has a section whose zero divisor is $\sum_{j\in J} N\alpha^j D_j$. We take the N-th root and normalize it in order to obtain a normal cyclic cover $\pi : \widetilde{X} \to X$ of order N. Let $\eta : \widehat{X} \to \widetilde{X}$ be a resolution and $f := \pi \circ \eta$. After fundamental results of Esnault-Viehweg [7] we know that $\widetilde{X}$ has rational singularities hence $\eta_* K_{\widehat{X}} = K_{\widetilde{X}}$ and $R^i\eta_* K_{\widehat{X}} = 0$ for all $i > 0$ (see [13], Theorem 5.10 for example). Moreover,

$$\pi_* K_{\widetilde{X}} = K_X \otimes \bigoplus_{k=0}^{N-1} L_k,$$

and $R^i\pi_* K_{\widetilde{X}} = 0$ for all $i > 0$, since π is finite.

Finally, for any line bundle λ on X and any q, we have

$$H^q(\widehat{X}, K_{\widehat{X}} + f^*\lambda) \simeq H^q(\widetilde{X}, K_{\widetilde{X}} + \pi^*\lambda) \simeq \bigoplus_{k=0}^{N-1} H^q(X, K_X + L_k + \lambda)$$

by the Leray spectral sequence.

If we apply the result of Simpson [17], we know by Serre duality (although $\widehat{X}$ might not be connected) that for any i

$$V_i^q(f) = \{\lambda \in \mathrm{Pic}^\tau(X) \,:\, h^q(\widehat{X}, K_{\widehat{X}} + f^*\lambda) \geqslant i\}$$

is a finite union of torsion translates of complex subtori of $\mathrm{Pic}^0(X)$ (if $\widehat{X}_1, \dots, \widehat{X}_r$ are the connected components of $\widehat{X}$, just write

$$V_i^q(f) = \bigcup_{i_1+\cdots+i_r=i} \left[\bigcap_{\ell=1}^{r} V_{i_\ell}^q\Big(f_{|\widehat{X}_\ell}\Big)\right]).$$

Now, an observation of Budur, Arapura, Simpson (*cf.* [4], see also [1], [2]) shows that each of the sets

$$V_i^q(f, L_k) = \{\lambda \in \mathrm{Pic}^\tau(X) \,:\, h^q(X, K_X + L_k + \lambda) \geqslant i\}$$

has the same structure. Their argument goes as follows: let V be an irreducible component of $V_i^q(f, L_k) \subset \mathrm{Pic}^\tau(X)$. For all $0 \leqslant \ell < N$, let $\iota_\ell = \max\{p \,:\, V \subset V_p^q(f, L_\ell)\}$ and $I = \sum_{\ell=0}^{N-1} \iota_\ell$. Then V is an irreducible analytic subset of $V_I^q(f)$ and it is an irreducible component of $\cap_{\ell=0}^{N-1} V_{\iota_\ell}^q(f, L_\ell)$ because $V_{\iota_k}^q(f, L_k) \subset V_i^q(f, L_k)$. Moreover,

$$V_I^q(f) = \bigcup_{\iota'_0+\cdots+\iota'_{N-1}=I} \left[\bigcap_{\ell=0}^{N-1} V_{\iota'_\ell}^q(f, L_\ell)\right]$$

and by construction, V is not included in $\cap_{\ell=0}^{N-1} V_{\iota'_\ell}^q(f, L_\ell)$ if $\iota' \neq \iota$. Therefore, V is an irreducible component of $V_I^q(f)$ for which Simpson's theorem applies, and the lemma is proved. □

1.E. *End of the proof*

We follow next the original argument in [5]: by the second bullet at the end of Section 1.C the bundle $K_X + \Delta^+ + \frac{1}{m}\rho$ has $d = h^0(X, m(K_X + \Delta) + \rho)$ linearly independent sections, which means that

$$\frac{1}{m}\rho \in V_d^0(f, L_1)$$

(in the notations of Lemma 1.1). By the results discussed in Section 1.D we infer the existence of a torsion line bundle ρ_{tor} and of an element $T \in \mathbb{T}$, where $\mathbb{T}$ is a subtorus of $\mathrm{Pic}^0(X)$ such that

$$\rho = m(\rho_{\mathrm{tor}} + T).$$

On the other hand, we also have $(1-m)T + \rho_{\mathrm{tor}} \in V^0_d(f, L_1)$, which by definition implies that the bundle

$$K_X + \Delta^+ + (1-m)T + \rho_{\mathrm{tor}} = K_X + \Delta^+ - \frac{m-1}{m}\rho + m\rho_{\mathrm{tor}}$$

admits d linearly independent sections. Now, the relation (1.3) shows that

$$K_X + \Delta^+ - \frac{m-1}{m}\rho \sim m(K_X + \Delta) - \left\lceil \frac{m-1}{m}E \right\rceil,$$

which implies that the line bundle

$$m(K_X + \Delta) - \left\lceil \frac{m-1}{m}E \right\rceil + m\rho_{\mathrm{tor}}$$

has also d linearly independent sections, and Theorem 0.1 immediately follows.

Remark 1.2. — We observe that unlike in [5], we do not use a ramified cover of X in order to reduce ourselves to the case $m = 1$. However, a ramified cover is used in 1.D in order to "remove the boundary" Δ.

2. Proof of Theorem 0.1 in the log-canonical case

In this section, our goal is to prove Theorem 0.1 in full generality. We shall now denote by D the boundary denoted Δ in § 1, and assume the pair (X, D) to be log-canonical.

The first remark is that the reductions performed in Sections 1.A-1.B still apply in the current lc setting; hence we can assume that:

- We have a decomposition $D = B + \Delta$, where the support of D is snc, and $B = [D]$, so that (X, Δ) is klt.
- The union of the support of D together with the support of the zero-locus E of a chosen generic section $u_0 \in H^0\big(X, m(K_X + D) + \rho\big)$ is also a divisor which is snc.
- The divisors E and D have no common component.

We will show next that the proof of Theorem 0.1 is obtained as a consequence of two of its special cases: the case where $D = B$, treated by Y. Kawamata in [11], and (simple modifications of) the klt case treated in the previous section.

The arguments provided for the case where $D = B$ in [11] are parallel to the ones in [5], but in the mixed Hodge theoretic context of Deligne. We remark that although this result is stated in [11] only under the assumption that the numerical dimension of $K_X + D$ is equal to zero, the given arguments imply the general version.

We shall reduce next to this case, by performing the constructions made in Section 1.C above (we simply replace K_X with $K_X + B$); we have

$$m(K_X + B + \Delta) + \rho \sim K_X + B + \Delta + \frac{m-1}{m}E + \frac{1}{m}\rho \tag{2.1}$$

and

$$E - \left\lceil \frac{m-1}{m}E \right\rceil \sim K_X + B + \Delta^+ + \frac{1}{m}\rho \tag{2.2}$$

with the same notations as above.

Let $f = \pi \circ \eta : \widehat{X} \to X$ be the map associated to $N\Delta^+$ (*cf.* 1.D); recall that

$$f_* K_{\widehat{X}} = \bigoplus_{k=0}^{N-1} (K_X + L_k),$$

and thus

$$f_*\Big(K_{\widehat{X}} + f^*B + \frac{1}{m} f^*\rho\Big) = \bigoplus_{k=0}^{N-1} \Big(K_X + B + L_k + \frac{1}{m}\rho\Big). \tag{2.3}$$

We will show in the lemma below that we have

$$f_*\Big(K_{\widehat{X}} + f^*B + \frac{1}{m} f^*\rho\Big) = f_*\Big(K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + \frac{1}{m} f^*\rho\Big). \tag{2.4}$$

Granted the equality (2.4), the proof of Theorem 0.1 ends as follows.

By hypothesis, the bundle $K_X + B + L_1 + \frac{1}{m}\rho$ appearing in the right-hand side of the equality (2.3) has d linearly independent sections. We may assume that $(f^*B)_{\mathrm{red}}$ has snc support, from which we obtain by [11] that *the set of λ's in $\mathrm{Pic}^\tau(X)$ for which $K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + f^*\lambda$ has d linearly independent sections is a torsion translate of a subtorus in* $\mathrm{Pic}^\tau(X)$. From the proof of Lemma 1.1, we derive the same conclusion for each of the sets associated to the bundles $K_X + B + L_k$. The rest of the proof is the same as in the klt case.

We prove now the equality (2.4).

Lemma 2.1. — *With the above notations, we have*

$$f_*\Big(K_{\widehat{X}} + f^*B + \frac{1}{m} f^*\rho\Big) = f_*\Big(K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + \frac{1}{m} f^*\rho\Big).$$

Proof. — It suffices to prove the lemma under the assumption that $\frac{1}{m}\rho$ is trivial.

We shall show that $\eta_*(K_{\widehat{X}} + (f^*B)_{\mathrm{red}}) = K_{\widetilde{X}} + \pi^*B$.

Observe first that the singularities of $\widetilde{X}$ being quotient (and in particular rational), we have $\mathcal{O}(K_{\widetilde{X}}) = \omega_{\widetilde{X}} = \eta_*(K_{\widehat{X}})$, and that any section of this sheaf defined on the complement of a set of codimension 2 or more extends since $\widetilde{X}$ is normal.

Notice that, since (X, B) is log-canonical and π is finite, $(\widetilde{X}, \pi^*B)$ is also log-canonical (see [13], Proposition 5.20), so that there exists an effective η-exceptional integral divisor F without component contained in $(f^*B)_{\mathrm{red}}$ such that

$$K_{\widehat{X}} + f^*B + F \supset K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + F \supset \eta^*(K_{\widetilde{X}} + \pi^*B)/\mathrm{Torsion}$$

(as sheaves). Then, we have

$$\begin{aligned}
&\eta_*(K_{\widehat{X}} + (f^*B)_{\mathrm{red}}) = \eta_*(K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + F) \subset \eta_*(K_{\widehat{X}} + f^*B + F)\\
&= \eta_*(K_{\widehat{X}} + f^*B) = K_{\widetilde{X}} + \pi^*B = \eta_*(\eta^*(K_{\widetilde{X}} + \pi^*B)/\mathrm{Torsion})\\
&\subset \eta_*(K_{\widehat{X}} + (f^*B)_{\mathrm{red}} + F) = \eta_*(K_{\widehat{X}} + (f^*B)_{\mathrm{red}}).
\end{aligned}$$

The last equality holds because no component of F is contained in $(f^*B)_{\mathrm{red}}$.

□

Remark 2.2. — As we have already mentioned, Lemma 1.1 thus holds true (with the same proof) when Δ^+ is, more generally, assumed to be only log-canonical.

3. Some consequences of Theorem 0.1

We start with an elementary remark.

Remark 3.1. — As a by-product of our proof, we obtain a very precise control of the zero set of the sections produced by Theorem 0.1. We refer *e.g.*, to [15], Sections 1.G and 1.H for the relevance of this matter in the context of extension of twisted pluricanonical sections.

The set-up is as follows. We assume that there exist a set $J_0 \subset J$ together with a set of rational numbers $0 \leqslant d^j \leqslant \mu^j$ where $j \in J_0$, such that the effective divisor

$$D := m \sum_{j \in J_0} d^j W_j \leqslant m\Delta$$

is contained in the fixed part F of the linear system $|m(K_X + \Delta) + \rho|$. Then we claim that the divisor $\frac{m'}{m} D$ is contained in the zero set of each of the sections in $H^0\big(X, m'(K_X + \Delta)\big)$ that we produce. Indeed, the step 1.B of our proof consists in removing the common divisor between the boundary Δ and the fixed part F (*cf.* the definition of μ_0^j and a_0^j). This divisor is bigger than D so the claim trivially follows.

As a consequence of Theorem 0.1, we obtain first the following statement (which is analogous to results obtained in [5]).

Corollary 3.2. — *Assume (X, Δ) is lc with rational coefficients. For every $\rho \in \mathrm{Pic}^{\tau}(X)$, we have $\kappa(X, K_X + \Delta) \geqslant \kappa(X, K_X + \Delta + \rho)$.*

Proof. — We have shown that the left member of the inequality is nonnegative if so is the right member. Now assume that the right hand side is nonnegative, and consider the Moishezon-Iitaka fibrations $f : X \to B$ and $g : X \to C$ of the $\mathbb{Q}$-bundles $K_X + \Delta + \rho$ and $K_X + \Delta$ respectively. These fibrations can be assumed to be regular (by using blow-ups of X if necessary).

Let Z be the general fibre of g; we denote by Δ_Z the restriction of Δ to Z. Then the pair (Z, Δ_Z) is still lc, and $0 = \kappa(Z, K_Z + \Delta_Z) \geqslant \kappa(Z, K_Z + \Delta_Z + \rho_Z) \geqslant 0$, as we see by using Theorem 0.1, because the conditions $\kappa \geqslant 0$ and $\kappa > 0$ are characterised by an inequality on the number of linearly independent sections.

But the preceding inequality shows that $f(Z)$ is zero-dimensional.

We conclude the existence of $h : C \to B$ such that $f = h \circ g$, and thus of the claimed inequality. □

We present next an $\mathbb{R}$-version of Theorem 0.1; again, the motivation is provided by the proof of the non-vanishing theorem in [3], [15].

Let $(W_j)_{j \in J_g \cup J_d}$ be a set of non-singular hypersurfaces of X having normal crossings, where J_g and J_d are finite and disjoint sets. Denoting by "$\equiv$" the numerical equivalence of $\mathbb{R}$-divisors, we assume that we have

$$K_X + \sum_{j \in J_g} \nu^j W_j \equiv \sum_{j \in J_d} \tau^j W_j \tag{3.1}$$

where for each $j \in J_g$ we have $\nu^j \in]0, 1]$, and for $j \in J_d$ we have $\tau^j > 0$. The numbers (ν, τ) are not necessarily rationals. We state next the following direct consequence of Theorem 0.1.

Corollary 3.3. — *Let $\eta > 0$ be a real number. Then for each $j \in J_g$ and $l \in J_d$ there exists a finite set of rational numbers $\left(\frac{p^j_{k,\eta}}{q_\eta}, \frac{r^l_{k,\eta}}{q_\eta}\right)_{k=1,\dots,N_\eta}$ such that:*

(a) *The vector $(\nu, \tau) := (\nu^j, \tau^l)$ is a convex combination of $(\nu_{k\eta}, \tau_{k\eta}) := \left(\frac{p^j_{k,\eta}}{q_\eta}, \frac{r^l_{k,\eta}}{q_\eta}\right)$;*

(b) *We have $|p^j_{k,\eta} - q_\eta \nu^j| \leqslant \eta$ and $|r^l_{k,\eta} - q_\eta \tau^l| \leqslant \eta$ for each j, l, k, η;*

(c) *The bundle*

$$K_X + \sum_{j\in J_g} \frac{p^j_{k,\eta}}{q_\eta} W_j$$

is $\mathbb{Q}$-effective.

Hence, the bundle $K_X + \sum_{j\in J_g} \nu^j W_j$ is $\mathbb{R}$-linearly equivalent with an effective $\mathbb{R}$-divisor.

Proof. — We consider the set $\mathcal{A} \subset \mathbb{R}^{|J_g|} \times \mathbb{R}^{|J_d|}$ given by the couples (x, y) such that

$$K_X + \sum_{j\in J_g} x^j W_j \equiv \sum_{j\in J_d} y^j W_j. \tag{3.2}$$

We remark that $\mathcal{A}$ is non-empty, since by relation (3.1) it contains the point (ν, τ). Also, it is an affine space defined over $\mathbb{Q}$.

The upshot is that given $\eta > 0$, we can write (ν, τ) as a convex combination of points $(\nu_{k,\eta}, \tau_{k,\eta}) \in \mathcal{A}$ having rational coordinates, in such a way that the Dirichlet conditions in (b) are satisfied (see *e.g.*, [15]).

If $\eta \ll 1$, then as a consequence of (b) we infer that the coordinates of $\nu_{k,\eta}$ belong to $]0, 1]$, and that the coordinates of $\tau_{k,\eta}$ are positive. Hence the point (c) follows from Theorem 0.1. □

Remark 3.4. — If the boundary divisor Δ contains an ample part, then we can take $m' := m$ in Theorem 0.1 (see [15], Section 1.G). It would be particularly useful to have an analogous statement in our current setting.

The following interesting corollary was brought to our attention by O. Fujino; it is a slightly more general version of a result due to S. Fukuda (see [9], in which D is supposed to be semi-ample), appeared in discussions between O. Fujino and S. Fukuda.

Corollary 3.5. — *Let (X, Δ) be a projective klt pair and let D be a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that $K_X + \Delta$ is numerically equivalent to D. Then $K_X + \Delta$ is semi-ample.*

Proof. — The *nefness* of a line bundle is clearly a numerical property, hence $K_X + \Delta$ is nef. Our next claim is that we have the following sequence of relations

$$\nu(K_X + \Delta) \geqslant \kappa(K_X + \Delta) \geqslant \kappa(D) = \nu(D) = \nu(K_X + \Delta).$$

Indeed, the first inequality is valid for any nef bundle; the second one is the content of Corollary 3.2. The third relation above is a consequence of the fact that D is nef and abundant, and the last one is due to the fact that the numerical dimension ν of a nef line bundle only depends on its first Chern class. Thus, the corollary follows as a consequence of a result due to Kawamata in [12] (see also the version by O. Fujino in [8]). □

Remark 3.6. — In the preceding corollary, if (X, Δ) is only lc, we cannot conclude (as Y. Gongyo pointed to us), since Y. Kawamata's theorem is no longer available. In dimension 4, the conclusion nevertheless still holds, as shown by Y. Gongyo in [10].

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