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The cofinal property of the reflexive indecomposable Banach spaces

THE COFINAL PROPERTY OF THE REFLEXIVE INDECOMPOSABLE BANACH SPACES

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ABSTRACT. — It is shown that every separable reflexive Banach space is a quotient of a reflexive hereditarily indecomposable space, which yields that every separable reflexive Banach is isomorphic to a subspace of a reflexive indecomposable space. Furthermore, every separable reflexive Banach space is a quotient of a reflexive complementably ℓ_p-saturated space with $1 < p < \infty$ and of a c_0 saturated space.

RÉSUMÉ. — On démontre que tout espace de Banach séparable réflexif est quotient d'un espace réflexif héréditairement indécomposable, ce qui implique que tout espace de Banach séparable réflexif est isomorphe à un sous-espace d'un espace réflexif indécomposable. De plus, tout espace de Banach séparable réflexif est quotient d'un espace réflexif complémentablement ℓ_p-saturé, où $1 < p < +\infty$, et d'un espace c_0-saturé.

1. Introduction

An infinite dimensional Banach space X is said to be indecomposable if it is not the topological direct sum of two infinite dimensional subspaces. In the 70^s J. Lindenstrauss [17] had asked if every infinite dimensional space is decomposable. Note that it was already known that the aforementioned problem has a positive answer for the members of a variety of classes of Banach spaces. For example, Banach spaces with an unconditional basis, nonseparable reflexive spaces [16] (or more generally nonseparable WCG spaces [2]), separable Banach spaces containing c_0 [22] are all decomposable spaces.

On the other hand since 1991 it is known that Lindenstrauss' problem has an emphatically negative answer. Indeed W.T. Gowers and B. Maurey's

discovery of Hereditarily Indecomposable (HI) spaces ([13]) has provided examples of Banach spaces with no decomposable infinite dimensional subspace. Since the seminal work of Gowers and Maurey the classes of HI and Indecomposable spaces have been extensively studied leading to some remarkable results. In particular, new techniques have been developed concerning the existence of HI spaces having as a quotient a desired Banach space. These techniques follow two distinct directions.

The first one, which appeared in [4], is closely related to the DFJP interpolation method ([10]) and makes heavy use of the geometric aspect of thin sets, which can be traced back to A. Grothendieck's work ([14]) and was explicitly defined in R. Neidinger's PhD thesis ([20]). This method yielded that every reflexive space with an unconditional basis has a subspace which is a quotient of a reflexive HI space. In particular, separable Hilbert spaces and more generally any ℓ_p for $1 < p < \infty$ are quotients of reflexive HI spaces. Using duality arguments, one may also conclude that reflexive ℓ_p spaces can be embedded into a reflexive indecomposable.

The second method is based on saturated and HI extensions of a ground norming set and led to the most general result concerning quotients of HI spaces. Namely, as is shown in [9], every separable Banach space not containing ℓ_1 is a quotient of a separable HI space. Comparing the aforementioned techniques, one should point out that the second leads to more general results but by its own nature the dual of the resulting HI space is decomposable. Thus, for a given separable reflexive space X the corresponding HI space Y which has X as a quotient is never reflexive. On the contrary, whenever the first method is applicable it leads to HI spaces with structure similar to the starting one (*i.e.*, starting with a reflexive space the obtained HI space remains reflexive).

The aim of the present work is to prove the following:

THEOREM 1.1. — *Let X be a separable reflexive space then,*

   (1) *X is a quotient of a reflexive HI space.*
   (2) *X is isomorphic to a subspace of a reflexive indecomposable space.*

Since the dual of a reflexive HI is indecomposable (2) is a direct consequence of (1). The proof of the theorem is based on a combination of the aforementioned methods and uses certain auxiliary spaces which are constructed either by interpolation or extension. More precisely, starting with a reflexive space X with a Schauder basis first we define a space $\mathfrak{X}_0$ with a Schauder tree basis $(e_t)_{t \in \mathcal{T}}$, a weakly compact symmetric subset W of $\mathfrak{X}_0$ and a map $\Phi : \mathfrak{X}_0 \to X$ such that $\Phi(W)$ is $\frac{1}{8}$ dense in the unit ball of X, which implies that X is a quotient of $\mathfrak{X}_0$. The definition of $\mathfrak{X}_0$ shares

common features with the corresponding one in [4], however it requires certain modifications as in the present case the basis of X is not necessarily unconditional. It is worth mentioning that if we were able to show that W is a thin or $(a_n)_n$-thin set (*cf.* [4]) of $\mathfrak{X}_0$ then applying a HI interpolation on the pair $(\mathfrak{X}_0, W)$ we would arrive to the reflexive HI space $\mathfrak{X}$ which has X as a quotient. This remains unclear and we proceed as follows.

In the second step using DFJP ℓ_2 interpolation on the pair $(\mathfrak{X}_0, W)$ we obtain a reflexive space $\mathfrak{X}_1$ with a Schauder tree basis $(\tilde{e}_t)_{t \in \mathcal{T}_K}$, where $\mathcal{T}_K$ is a complete subtree of $\mathcal{T}$ and a bounded closed convex set $\tilde{W}$ such that $\tilde{W} = J_1^{-1}(W)$. Here, $J_1 : \mathfrak{X}_1 \to \mathfrak{X}_0$ is the usual operator mapping the diagonal space to the original one. Note that the composition operator $\Phi \circ J_1$ maps $\tilde{W}$ onto a $\frac{1}{8}$ dense subset of B_X and thus X is a quotient of $\mathfrak{X}_1$. As in the case of $\mathfrak{X}_0$ it remains unclear whether the set $\tilde{W}$ is a thin subset of $\mathfrak{X}_1$. Since $\mathfrak{X}_1$ is a separable reflexive space there exists a countable ordinal ξ such that every weakly null sequence in $\mathfrak{X}_1$ does not admit a ℓ^1_ξ spreading model (*cf.* [5]).

The next step is the most critical. Here, using a ξ-saturated extension method ([9]) we pass to a new space denoted as $\mathfrak{X}_\xi$ and $I_\xi : \mathfrak{X}_\xi \to \mathfrak{X}_1$ a bounded linear injection such that the set $W_\xi = I_\xi^{-1}(\tilde{W})$ is a weakly compact and also thin set. Let us note that the structure of $\mathfrak{X}_\xi$ resembles the generalized Tsirelson space T_ξ (*cf.* [9]). In that sense $\mathfrak{X}_\xi$ has a much richer local ℓ_1 structure than $\mathfrak{X}_1$. Thus the thinness is established in a space with a strong presence of local ℓ_1 structure which a-priori seems contradictory or at least peculiar. The final step is the expected one. Namely, we apply a HI interpolation on the pair $(\mathfrak{X}_\xi, W_\xi)$ to obtain a diagonal reflexive space $\mathfrak{X}$ and a bounded convex set $\tilde{W}_\xi$ such that the natural operator $J_\xi : \mathfrak{X} \to \mathfrak{X}_\xi$ satisfies $J_\xi(\tilde{W}_\xi) = W_\xi$. As the set W_ξ is thin the space $\mathfrak{X}$ is HI and is the desired one. Indeed the operator $Q = \Phi \circ J_1 \circ I_\xi \circ J_\xi$ maps the set $\tilde{W}_\xi$ to a $\frac{1}{8}$ dense subset of X which yields that $Q : \mathfrak{X} \to X$ is a quotient map.

The paper is organized as follows. Section 2 concerns preliminaries. Section 3 is devoted to the definition of the space $\mathfrak{X}_0$ which as was mentioned has a Schauder tree basis $(e_t)_{\mathcal{T}}$ equipped with a partial form of unconditionality defined as "segment-complete unconditional" tree basis. (Def. 3.1). The main result of this section is that when X is reflexive although the space $\mathfrak{X}_0$ is not necessarily reflexive the set W is weakly compact. In section 4 we prove the following:

THEOREM 1.2. — *Let $(X, \| \cdot \|)$ be a reflexive space with a segment complete unconditional tree basis $(e_t)_{t \in T}$ and K be a bounded subset of X such that for $x \in$ K, supp x is a segment of T. Let also $\xi < \omega_1$ such that*

X does not admit a ℓ_ξ^1 spreading model. Then, there exists a norm $\|\cdot\|_\xi$ on $c_{00}(T)$ such that setting $\mathfrak{X}_\xi$ to be the completion of $(c_{00}(T), \|\cdot\|_\xi)$ the following hold:

(1) *For every $x \in c_{00}(T)$ $\|x\| \leqslant \|x\|_\xi$.*
(2) *For every $x \in$ K, $\|x\| = \|x\|_\xi$.*
(3) *Denoting by W_ξ the set $\overline{\mathrm{co}}(\mathrm{K} \cup -\mathrm{K})$ in $\mathfrak{X}_\xi$ we obtain that it is weakly compact and thin.*

This theorem provides a tool for constructing thin sets in spaces with a Schauder tree basis. The norm of the space $\mathfrak{X}_\xi$ is defined via a norming set G_ξ which contains as ground set a norming subset of the dual of the space X and which is $\mathcal{S}_\xi$ saturated for finite sequences of functionals with pairwise incomparable, segment complete supports. In section 5 we show that the diagonal space in the DFJP ℓ_2 interpolation applied on the pair $(\mathfrak{X}_0, W)$ has a segment complete unconditional tree basis. In section 7 combining the results of sections 3, 4 and 5 we prove the following:

THEOREM 1.3. — *Let Y be a separable reflexive space. Then for every $p \in (1, \infty)$ there exists a reflexive space X_p such that every subspace contains an isomorphic copy of ℓ_p, complemented in the whole space and X_p has Y as a quotient. Additionally, Y is a quotient of a separable c_0 saturated space.*

It is worth mentioning that the first result in this direction was done by D.H. Leung in [15] proving that every separable Hilbert space is a quotient of a c_0 saturated space and it was followed by the results in [4] mentioned earlier. More recently, following different techniques, I. Gasparis in [11] and [12] has shown that certain members of the class of separable reflexive spaces are quotients of c_0 saturated spaces. Let us also point out that Theorem 1.3 provides examples of reflexive Banach spaces with divergent structure between the spaces and their duals. For example, there exist spaces X_p as above such that their dual contains HI subspaces.

In Section 7 we present a variant of the HI interpolation method appearing in [4] which is traced to [3]. The necessity for modifying the initial HI interpolation method is the requirement that the diagonal space admits a Schauder basis. Similarly to [4] applying the new HI interpolation to a pair (X, W) with W being convex, symmetric, weakly compact and thin subset of X we obtain that the diagonal is reflexive HI and this proves Theorem 1.1 in the case where X has a Schauder basis. The general case of a separable reflexive space mentioned in Theorem 1.1 follows by the classical result of

M. Zippin that every separable reflexive space embeds into a reflexive space with a Schauder basis ([23]).

The research included in the present paper was carried out in 2006. In April 2007 Richard Haydon visited us in Athens and with his collaboration we were able to prove that there exists an indecomposable space X containing ℓ_1. After the solution of the "scalar plus compact" problem ([6]) the aforementioned result was adapted to the $\mathcal{L}_\infty$ frame as follows:

THEOREM 1.4. — *There exists a $\mathcal{L}_\infty$ space X with the scalar plus compact property containing ℓ_1.*

As it is also mentioned in [6] the ultimate problem concerning the cofinal properties of Indecomposable Banach spaces is the following:

PROBLEM. — *Does every separable Banach space not containing c_0 embed into a separable indecomposable space?*

We made an effort to make the present paper as self contained as possible. Thus, except for a few technical or well known results all the other proofs are included.

2. Preliminaries

Let us recall some standard notation and definitions for trees.

Notation 2.1.
1. Let Λ be a countable set. By $[\Lambda]^{<\omega}$ and $[\Lambda]$ we denote its finite and infinite sequences respectively. We consider $[\Lambda]^{<\omega}$ to be equipped with the partial ordering of the initial segment denoted by $\sqsubseteq$.
2. By a tree on Λ we mean a subset T of $[\Lambda]^{<\omega}$ which is backwards closed under $\sqsubseteq$.
3. Let T be a tree on Λ. A segment of T is a subset of T of the form $\{t \in \mathrm{T} : t_1 \sqsubseteq t \sqsubseteq t_2\}$ with $t_1, t_2 \in \mathrm{T}$. We will usually denote segments of this form by $[t_1, t_2]$ or more generally by s. For $t_1 \in \mathrm{T}$ we denote by $\hat{t}_1$ the set $\{t \in \mathrm{T} : t \sqsubseteq t_1\}$. For a segment $s \subset \mathrm{T}$, $\hat{s}$ has a similar meaning, namely $\hat{s} = \{t \in \mathrm{T} : \exists\, t' \in s \text{ such that } t \sqsubseteq t'\}$. For $t \in \mathrm{T}$ we set $|t|$ the cardinality of the set $\{t' \in \mathrm{T} : t' \sqsubseteq t\}$ to be the height of t. For every $n \in \mathbb{N}$ the nth-level of T is the set $\{t \in \mathrm{T} : |t| = n\}$.
4. We identify the branches of a tree T with the elements of the set $\{(a_i)_{i=1}^\infty : (a_i)_{i=1}^k \in \mathrm{T}, \forall\, k \in \mathbb{N}\}$ and we denote this set by $[\mathrm{T}]$. For every $b \in [\mathrm{T}]$ with $b = (a_i)_{i=1}^\infty$ we set $b|n = (a_1, ..., a_n)$.

5. Two nodes $t_1, t_2 \in \mathrm{T}$ are called comparable if either $t_1 \sqsubseteq t_2$ or $t_2 \sqsubseteq t_1$. More generally, if $A_1, A_2 \subseteq \mathrm{T}$ then A_1, A_2 are called comparable if there exist $t_1 \in A_1$ and $t_2 \in A_2$ which are comparable. Otherwise A_1, A_2 are called incomparable. We will write $A_1 \perp A_2$ to denote the fact that A_1, A_2 are incomparable.

6. For $t \in \mathrm{T}$ by $\mathcal{O}_t$ we denote the set $\mathcal{O}_t = \{b \in [\mathrm{T}] : \exists n \in \mathbb{N} \text{ such that } b|n = t\}$. The sets $\{\mathcal{O}_t : t \in \mathrm{T}\}$ form the usual basis of the topology of $[\mathrm{T}]$.

7. For every $t \in \mathrm{T}$ and $b \in [\mathrm{T}]$ we will write $t \in b$ if $\exists\, n \in \mathbb{N}$ such that $b|n = t$. For every segment s of T and $b \in \mathrm{T}$ we will write $s \subseteq b$ if $\forall t \in s$ it holds $t \in b$.

In the following sections all trees are considered countable, finitely splitting and with nonempty bodies. For the next definition we assume that Λ is a totally ordered set which later is going to be a subset of the real numbers with the natural order.

DEFINITION 2.2. — *For every such tree* T *we fix a bijection* $h_{\mathrm{T}} : \mathrm{T} \mapsto \mathbb{N}$ *such that the following hold:*

i. $h_{\mathrm{T}}(t_1) < h_{\mathrm{T}}(t_2)$, *whenever* $|t_1| < |t_2|$.

ii. *If* $t_1, t_2 \in \mathrm{T}$ *and* $|t_1| = |t_2|$ *i.e.,* $t_1 = (a_1, ..., a_n)$ *and* $t_2 = (b_1, ..., b_n)$ *then* $h_{\mathrm{T}}(t_1) < h_{\mathrm{T}}(t_2)$, *whenever* $a_n < b_n$.

When the tree T is understood we will refer to h_{T} simply as h. We denote by $c_{00}(\mathrm{T})$ the linear space of all functions $f : \mathrm{T} \mapsto \mathbb{R}$ such that $\mathrm{supp}(f) = \{t \in \mathrm{T} : f(t) \neq 0\}$ is a finite set. We also denote by $(e_t)_{t \in \mathrm{T}}$ the standard Hamel basis of $c_{00}(\mathrm{T})$ consisting of the characteristic functions of all singletons $\{t\} \subseteq \mathrm{T}$.

DEFINITION 2.3. — *Let* $(A_i)_{i=1}^{\infty}$ *be a sequence of finite subsets of* T. *We will say that* $(A_i)_{i=1}^{\infty}$ *is*

i. *a block sequence if* $\max\{h(t) : t \in A_i\} < \min\{h(t) : t \in A_{i+1}\}$ *and we will write* $A_1 < A_2 < ... < A_n < ...$

ii. *a level-block sequence if* $\max\{|t| : t \in A_i\} < \min\{|(t| : t \in A_{i+1}\}$ *and we will write* $A_1 \prec^l A_2 \prec^l ... \prec^l A_n \prec^l ....$

iii. *For a sequence* $(f_i)_{i=1}^{\infty}$ *of elements of* $c_{00}(\mathrm{T})$ *we will say that* $(f_i)_{i=1}^{\infty}$ *is block (level-block) if* $(\mathrm{supp}\, f_i)_{i=1}^{\infty}$ *is a block (level-block, respectively) sequence of subsets of* T.

For the sake of simplicity of notation if A is a subset of T we will write $\min A$ for $\min\{h(t) : t \in A\}$ and if A is finite $\max A$ for $\max\{h(t) : t \in A\}$. We will also write $\min^l A$ for $= \min\{|t| : t \in A\}$ and $\max^l A$ for $\max\{t : t \in A\}$.

DEFINITION 2.4. — *Let A, I be subsets of* T.

    i. *We will call A segment complete if for every t_1, t_2 in A, with $t_1 \sqsubseteq t_2$*
      $[t_1, t_2] \subseteq A$.
    ii. *I will be called an interval of* T *if $h(I) = \{h(t) : t \in$ T$\}$ is an interval*
      of $\mathbb{N}$.

We note that every interval I of T is segment complete.

DEFINITION 2.5. — *For every $f \in c_{00}($T$)$ we define ran f to be the minimal interval I of* T *such that supp $f \subseteq I$. Similarly, we set $\operatorname{ran}^l f$ to be the minimal interval I^l of* T *of the form $I^l = \{t \in$ T $: m \leqslant |t| \leqslant M\}$ such that supp $f \subseteq I^l$.*

Remark 2.6. — It is clear that a sequence $(f_i)_{i=1}^{\infty}$ in $c_{00}($T$)$ is

    i. block if $\operatorname{ran} f_i < \operatorname{ran} f_{i+1} \ \forall i \in \mathbb{N}$
    ii. level-block if $\operatorname{ran}^l f_i < \operatorname{ran}^l f_{i+1} \ \forall i \in \mathbb{N}$.

3. Tree representation of the ball of a Banach space

Let X be an arbitrary Banach space with a bimonotone, normalized Schauder basis $(x_i)_{i \in \mathbb{N}}$ and $(x_i^*)_{i \in \mathbb{N}}$ the biorthogonal functionals of $(x_i)_{i \in \mathbb{N}}$ in X^*. In this section we define a tree $\mathcal{T}$ and, for a given X, a norm on $c_{00}(\mathcal{T})$ that will help us "spread" along the branches of $\mathcal{T}$ a set K which is isometric (via a map Φ to be defined later) to a $\frac{1}{8}$-net in the unit ball of X. We will denote by $\mathfrak{X}_0$ the completion of $c_{00}(\mathcal{T})$ with respect to this norm. This technique gives X as a quotient of $\mathfrak{X}_0$. In addition we show that if X is reflexive then the set K is weakly compact in $\mathfrak{X}_0$. We start with the following general definition.

3.1. The space $\mathfrak{X}_0$

DEFINITION 3.1. — *Let T be a tree and a norm $\| \cdot \|$ defined on $c_{00}(T)$ such that the sequence $(e_t)_{t \in T}$ is a Schauder basis, with the order induced by the function h, for the completion of $c_{00}(T)$ denoted by X_T. Then*

    1. *The norm $\| \cdot \|$ and the basis $(e_t)_{t \in T}$ will be called $\mathcal{SC}$-unconditional if for every $A \subseteq T$ segment complete and $x = \sum_{t \in T} \lambda_t e_t \in X_T$ we have:*

$$\left\| \sum_{t \in A} \lambda_t e_t \right\| \leqslant \left\| \sum_{t \in T} \lambda_t e_t \right\|.$$

2. *Let $\psi : T \to [-1, 1]$ be a function assigning to each node of T a scalar $\psi(t) \in [-1, 1]$. Let also $C > 0$. We consider for each $t \in T$ the vector $y_t = \sum_{t' \sqsubseteq t} \psi(t')e_{t'}$ and set $\mathrm{K}_\psi^1 = \{y_t \in X_T : \|y_t\| \leqslant C\}$, $\mathrm{K}_\psi^2 = \{y_t|_I : \|y_t\| \leqslant C$ and I is an interval of $T\}$ and $\mathrm{K}_\psi = \overline{\mathrm{K}_\psi^2}^{\|\cdot\|}$.*

For the rest of this section we assume that X is a fixed Banach space with a Schauder basis $(x_i)_i$ and $(x_i^*)_i$ the biorthogonal functionals in X^*. We pass on to define a $\mathcal{SC}$-unconditional norm on $c_{00}(\mathcal{T})$ where $\mathcal{T}$ is an appropriately defined tree such that the completion of this space with respect to this norm has a quotient isomorphic to X. We start with the definition of $\mathcal{T}$.

DEFINITION 3.2. — *Let $(F_n)_{n=1}^\infty$ be the following sequence in $([-1,1] \cap \mathbb{Q})^{<\omega}$*

$$F_n = \left\{ \pm \frac{i}{8^n} : 1 \leqslant i \leqslant 8^n \right\} \forall n \in \mathbb{N}.$$

We set

$$\mathcal{T} = \{(a_1, a_2, ..., a_k) : a_i \in F_i, i \leqslant k, k \in \mathbb{N}\}.$$

It can be readily seen that $\mathcal{T}$ is a countable, finitely splitting tree such that every $t \in \mathcal{T}$ with $|t| = n$, has $2 \cdot 8^{n+1}$ immediate successors.

The norming set $\mathrm{G}_0(X)$ of $\mathfrak{X}_0$ is defined as follows:

DEFINITION 3.3. — *Let $\mathrm{G}_0^1(X)$ be the following subset of $c_{00}(\mathcal{T})$*

$$\mathrm{G}_0^1(X) = \left\{ \sum_{i=1}^n a_i \sum_{|t|=i} e_t^* : \|\sum_{i=1}^n a_i x_i^*\|_{X^*} \leqslant 1 \right\}.$$

Set

$$\mathrm{G}_0(X) = \{f|_A : f \in \mathrm{G}_0^1(X) \text{ and } A \text{ is a segment complete subset of } \mathcal{T}\}$$

where $f|_A$ denotes the restriction of f on A.

We consider the norm on $c_{00}(\mathcal{T})$ induced by the set $\mathrm{G}_0(X)$. Namely,

$$\forall x \in c_{00}(\mathcal{T}) \text{ we set } \|x\| = \sup\{f(x) : f \in \mathrm{G}_0(X)\}.$$

The space $\mathfrak{X}_0$ is the completion of $c_{00}(\mathcal{T})$ under the norm defined above. It can be readily verified that the sequence $(e_t)_{t\in\mathcal{T}}$ (enumerated through h) becomes a bimonotone, normalized Schauder basis of $\mathfrak{X}_0$. We also have the following easy observation:

Remark 3.4. — For every $A \subseteq \mathcal{T}$ segment complete the natural projection $P_A : \mathfrak{X}_0 \mapsto \mathfrak{X}_0$ defined by $P_A(\sum_{t\in\mathcal{T}} \lambda_t e_t) = \sum_{t\in A} \lambda_t e_t$ has norm one.

We will need the following Lemma that gives a description of the pointwise closure of the norming set $G_0(X)$.

LEMMA 3.5. — *Let*

$$G = \left\{ f|A : f = \sum_{i=1}^{\infty} a_i \sum_{|t|=i} e_t^*, \ \left\| \sum_{i=1}^{\infty} a_i x_i^* \right\|_{X^*} \leqslant 1 \right.$$

$$\left. \text{and } A \text{ is a segment complete subset of } \mathcal{T} \right\}$$

where all limits are taken with respect to the w^*-topology. Then $\overline{G_0(X)}^p = G$.

Proof. — It is easy to see that $G \subseteq \overline{G_0(X)}^p$. Let $f \in \overline{G_0(X)}^p$. Then there exists a sequence $(f_n)_n$ in $G_0(X)$ such that $f_n \xrightarrow{p} f$. Each f_n is of the form $f_n = \mathcal{X}_{A_n} \cdot (\sum_{i=1}^{k_n} a_i^n \sum_{|t|=i} e_t^*)$, where A_n are finite segment complete subsets of $\mathcal{T}$, $\mathcal{X}_{A_n}$ is the characteristic function of A_n and $\|\sum_{i=1}^{k_n} a_i^n x_i^*\| \leqslant 1$. Let $g_n = \sum_{i=1}^{k_n} a_i^n x_i^*$. Then there exists a $M \in [\mathbb{N}]$ such that the sequence $(g_n)_{n \in M}$ converges w^* to a $g \in B_{X^*}$. Let $i \in \mathbb{N}$. Denote by a_i the limit $\lim_{n \in M} a_i^n$ and set $A = \liminf_{n \in M} A_n$ which can be readily seen to be a segment complete subset of $\mathcal{T}$. Considering the functional $\hat{f} = \mathcal{X}_A \sum_{i=1}^{\infty} a_i \sum_{|t|=i} e_t^*$, we claim that the sequence $(f_n)_{n \in M}$ converges w^* to $\hat{f}$. Indeed, let $t \in \mathcal{T}$ with $|t| = i$. If $t \in A$, then $\lim_{n \in M} f_n(e_t) = a_i = \hat{f}(e_t)$. Assuming that $t \notin A$ then $\lim_{n \in M} f_n(e_t) = \hat{f}(e_t) = 0$. Hence, $\hat{f} = f \in G$ and the proof is complete. $\square$

We pass on to define the map $\psi : \mathcal{T} \to \mathbb{N}$ that will give us the corresponding set K_ψ

3.2. The set K

The set K and the map Φ are defined as follows:

DEFINITION 3.6. — *Let $\psi : \mathcal{T} :\to \mathbb{N}$ be the following assignment. For every $t = (a_1, a_2, ..., a_n) \in \mathcal{T}$ we set $\psi(t) = a_t = a_n$ and $y_t = \sum_{t' \sqsubseteq t} a_{t'} e_{t'}$. We set $K = K_\psi$ as in Definition 3.1 with constant $C = 1 + \frac{1}{7}$.*

DEFINITION 3.7. — *We consider a map $\Phi : \mathfrak{X}_0 \mapsto X$ defined as $\Phi(\sum_{t \in \mathcal{T}} \lambda_t e_t) = \sum_{i=1}^{\infty} (\sum_{|t|=i} \lambda_t) x_i$.*

Remark 3.8. — We can see that Φ is a bounded linear operator with $\|\Phi\| \leqslant 1$. In addition, for $b \in [\mathcal{T}]$ if we denote by X_b the subspace $X_b = \overline{< e_t : t \in b >}^{\|\cdot\|}$ then we have that Φ restricted to X_b is an isometry.

We also have the following

PROPOSITION 3.9. — *The set* $\Phi(\mathrm{K})$ *is a* $\frac{1}{8}$*-net in the unit ball* B_X *of* X. *Moreover, the map* Φ *is onto.*

Proof. — Let $x = \sum_n b_n x_n \in B_X$. Since the basis of X is bimonotone we have that $|b_n| \leqslant 1$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we can choose a_n in the set $F_n = \{\pm\frac{i}{8^n} : 1 \leqslant i \leqslant 8^n\}$ such that $|b_n - a_n| \leqslant \frac{1}{8^n}$. If we set $\sigma = (a_n)_{n=1}^\infty \in [\mathcal{T}]$ then $y_\sigma = \sum_n a_n e_{\sigma|n} \in \mathrm{K}$ and $\|\Phi(y_\sigma) - x\|_X \leqslant \frac{1}{8}$. Indeed, if we let $X_\sigma = \overline{< e_t : t \in \sigma >}$, then by Remark 3.8 $\Phi : X_\sigma \mapsto X$ is an isometry. Thus if we denote the restriction of Φ on X_σ by Φ_σ we have that $\Phi_\sigma^{-1}(\sum_n b_n x_n) = \sum_n b_n e_{\sigma|n}$ and $\|\sum_n b_n e_{\sigma|n}\|_{\mathfrak{X}_0} = \|\Phi(\sum_n b_n e_{\sigma|n})\|_X = \|\sum_n b_n x_n\|_X \leqslant 1$. Thus,

$$\Big\|\sum_n a_n e_{\sigma|n}\Big\|_{\mathfrak{X}_0} \leqslant \Big\|\sum_n b_n e_{\sigma|n}\Big\|_{\mathfrak{X}_0} + \sum_n |a_n - b_n| \leqslant 1 + \frac{1}{8}.$$

This gives us that y_σ is an element of K. Finally,

$$\Big\|\sum_n b_n e_{\sigma|n} - \sum_n a_n e_{\sigma|n}\Big\|_{\mathfrak{X}_0} = \Big\|\Phi\Big(\sum_n b_n e_{\sigma|n}\Big) - \Phi\Big(\sum_n a_n e_{\sigma|n}\Big)\Big\|_X$$

$$= \|\Phi(y_\sigma) - x\|_X \leqslant \frac{1}{8}.$$

$\square$

The following Lemma shows the behavior of incomparably supported sequences of vectors in K if we assume that X has a shrinking basis.

LEMMA 3.10. — *Suppose* $(x_i)_i$ *is a shrinking basis for* X, *then for every sequence* $(y_n)_n$ *in* K *such that* $(\operatorname{supp} y_n)_n$ *are finite and mutually incomparable subsets of* $\mathcal{T}$ *we have that* $y_n \xrightarrow{w} 0$.

Proof. — Let $(y_n)_n$ be as above. In order to prove that $(y_n)_n$ is weakly null it is enough to show that $f(y_n) \to 0, \forall f \in \overline{\mathrm{G}_0}^p$. Choose $f \in \overline{\mathrm{G}_0}^p$. By Lemma 3.5 there exist a $g = \sum_{i=1}^\infty b_i x_i^* \in B_{X^*}$ and a segment complete $A \subset \mathcal{T}$ so that $f = \mathcal{X}_A \sum_{i=1}^\infty b_i \sum_{|t|=i} e_t^*$. Let $s_n^A = \operatorname{supp} y_n \cap A$, $z_n = y_n|_A = \sum_{t \in s_n^A} a_t e_t$ and observe the following,

$$f(y_n) = \sum_{i=1}^\infty b_i \sum_{|t|=i} e_t^*(z_n) = g(\Phi(z_n)).$$

Hence, it is enough to show that $g(\Phi(z_n)) \to 0$. As $\Phi(z_n) = \sum_{t \in s_n^A} a_t x_{|t|}$ it can be seen that for $i \in \mathbb{N}$, $x_i^*(\Phi(z_n)) = a_t$, if $s_n^A \cap L_i \neq \emptyset$ and zero otherwise. Since $\mathcal{T}$ is finitely branching and $s_n^A \perp s_m^A$ for all $n \neq m \in \mathbb{N}$, we deduce that for a fixed $i \in \mathbb{N}$ the set $\{n \in \mathbb{N} : s_n^A \cap L_i \neq \emptyset\}$ is finite. Thus,

$x_i^*(\Phi(z_n)) \to 0$. Finally, as $(x_i)_i$ is shrinking we obtain that $g(\Phi(z_n)) \to 0$ and this completes the proof.   □

PROPOSITION 3.11. — *For every reflexive Banach space X with a bi-monotone normalized Schauder basis $(x_i)_i$ the set* K *is weakly compact.*

Proof. — Let $(w_n)_n$ be a sequence in K. Up to an arbitrarily small perturbation we may assume that $\operatorname{supp} w_n$ is finite for all $n \in \mathbb{N}$. We set $s_n = \operatorname{supp} w_n$ and we observe that each s_n is a finite segment of $\mathcal{T}$. So, $w_n = \sum_{t \in s_n} a_t e_t$. We may also assume (by passing to a subsequence if needed) that for each $t \in \mathcal{T}$, $e_t^*(w_n) \to w(t) \in \mathbb{R}$. We set $S = \{t \in \mathcal{T} : w(t) \neq 0\}$. We know that S is a segment of $\mathcal{T}$ (finite or infinite). Thus, we may assume that each y_n has a decomposition $y_n = u_n + y_n$ where

    i. $(\operatorname{supp} u_n)_n$ is an increasing (with respect to $\subseteq$) sequence of segments of S

    ii. $(\operatorname{supp} y_n)_n$ is a sequence of incomparable segments of $\mathcal{T}$.

We set $w_s = \sum_{t \in S} a_t e_t$ if $S \neq \emptyset$ and 0 otherwise. We claim that $w_n \overset{w}{\to} w_s$. Lemma 3.10 yields that $(y_n)_n$ is weakly null. To finish the proof we shall show that $u_n \overset{\|\cdot\|}{\to} w_s$. Indeed, for every $n \in \mathbb{N}$ we have that $\operatorname{supp} u_n \sqsubseteq S$ and since $w_n \in$ K we have that $u_n \in$ K for all $n \in \mathbb{N}$. Thus, as the basis of X is boundedly complete, $u_n \overset{p}{\to} w_s$ and $\|\Phi(u_n)\|_X = \|u_n\|_{\mathfrak{X}_0}$ we have that $w_s \in$ K and $u_n \overset{\|\cdot\|}{\to} w_s$.   □

Remark 3.12. — The connection between the set K and B_X when X is an arbitrary Banach space with a basis is not completely clear to us. For example, if one considers X to be c_0 with the summing basis then it turns out that K contains a sequence equivalent to the standard ℓ^1 basis. Indeed, we notice that the norming set in this case becomes $\overline{\mathrm{G}_0}^p = \{\pm \mathcal{X}_A : A \subseteq \mathcal{T}$ segment complete$\}$ where by $\mathcal{X}_A$ we denote the characteristic function of A. We can construct a sequence $(w_n)_n$ in K which has no weakly Cauchy subsequence. To see this choose a sequence $(t_n)_n \subset \mathcal{T}$ with the following properties,

    i. Each t_n is of the form $t_n = (a_1, ..., a_{k_n})$ and $a_{k_n} = a_{t_n} = \frac{1}{2}$;

    ii. $t_n \perp t_m$ for all $n \neq m \in \mathbb{N}$.

For each $n \in \mathbb{N}$ set $t_n' = t_n \frown \frac{-1}{2} = (a_1, ..., a_{k_n}, \frac{-1}{2})$ and $w_n = a_{t_n} e_{t_n} + a_{t_n'} e_{t_n'}$. To see that $(w_n)_n$ satisfies the desired property, choose a subsequence $(w_{m_i})_{i \in \mathbb{N}}$. Let $t(i) = t_{m_i}$ when $i = 2k$ and $t(i) = t_{m_i}'$ when $i = 2k - 1$. It is clear that $(t(i))_i$ are mutually incomparable. Therefore, the set $A = \cup_{i=1}^{\infty} \{t(i)\}$ is segment complete. So the functional $f = \mathcal{X}_A \in \overline{\mathrm{G}_0}^p$

estimates $|f(w_{m_i}) - f(w_{m_{i+1}})| = 1$ for all i. By Rosenthal's ℓ^1 theorem [21] we obtain that $(w_n)_n$ is equivalent to the ℓ^1 basis.

4. Thin subsets of Banach spaces

Let T be a tree and $\|\cdot\|_{X_T}$ be a $\mathcal{SC}$-unconditional norm defined on $c_{00}(T)$. Denote by X_T the completion of $c_{00}(T)$ with respect to $\|\cdot\|_{X_T}$. Fix also a function $\psi : T \to [-1, 1]$ and K_ψ (referred to as K for simplicity) as in Definition 3.1.

In this section we present a general method for extending the norm of X_T to a new norm defined on $c_{00}(T)$ such that the completion of this space contains K as a *thin* subset. Namely, the entire section is devoted to the proof of the following theorem:

THEOREM 4.1. — *Suppose that X_T is reflexive and K is a weakly compact subset of X_T. Then there exists a space $\mathfrak{X}_\xi$ such that the following hold:*

1. *The identity operator $I : \mathfrak{X}_\xi \to X_T$ is continuous.*
2. *K $\subset I(\mathfrak{X}_\xi)$ and the closed convex hull of $(I^{-1}(K) \cup I^{-1}(-K))$ is a weakly compact and thin subset of $\mathfrak{X}_\xi$.*

The notion of thinness was introduced in [19] and was extensively used in [4] where several methods for proving that a set satisfies this property were developed. We give the corresponding definition in subsection 4.3 where we also prove the aforementioned Theorem. Before doing so though we need some preparatory work which is done in the following subsection.

4.1. Tsirelson type spaces and norms

We start with some preliminary results concerning families of finite subsets of $\mathbb{N}$ and Tsirelson type norms. Most of these results are well known and have been extensively used in the relevant literature (see [5] and the references therein), with the exception of Lemmas 4.6, 4.9, 4.11 and Remark 4.7 which can be found in [18] and were brought to our attention by the authors. We include this subsection in order to make the text as self-contained as possible. We start by recalling the following notions concerning families of finite subsets of $\mathbb{N}$.

DEFINITION 4.2. — *Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$. $\mathcal{M}$ is called*

   i. *Compact if the set of characteristic functions $\{\mathcal{X}_A : A \in \mathcal{M}\}$ is a compact subset of $\{0,1\}^{\mathbb{N}}$.*

   ii. *Hereditary if for every $A \in \mathcal{M}$ and $B \subseteq A$ we have $B \in \mathcal{M}$.*

   iii. *Spreading if for every $A = \{t_1 < t_2 < ... < t_r\} \in \mathcal{M}$ and $B = \{t_1' < t_2' < ... < t_r'\}$ with $t_i \leqslant t_i' \ \forall i = 1, ..., r$ we have $B \in \mathcal{M}$.*

DEFINITION 4.3. — *Let $\mathcal{M} \subseteq [\mathbb{N}]^{<\omega}$.*

   i. *A finite sequence $(E_1, ..., E_n)$ of successive and finite subsets of $\mathbb{N}$ is called $\mathcal{M}$-admissible if there exists $F \in \mathcal{M}$ with $F = \{m_1 < m_2 < ... < m_n\}$ such that $m_1 \leqslant E_1 < m_2 \leqslant E_3 < ... < m_n \leqslant E_n$.*

   ii. *A finite sequence $(f_1, ..., f_n)$ of vectors in $c_{00}(\mathbb{N})$ is called $\mathcal{M}$-admissible if $(\operatorname{supp} f_i)_{i=1}^n$ is $\mathcal{M}$-admissible.*

DEFINITION 4.4. — *Let $\mathcal{F}, \mathcal{G}$ be two families of finite subsets of $\mathbb{N}$ we define:*

   i. *The block sum $\mathcal{F} \oplus \mathcal{G} = \{M \cup N : M < N, M \in \mathcal{G}, N \in \mathcal{F}\}$.*

   ii. *The convolution $\mathcal{F} \otimes \mathcal{G} = \{\cup_{i=1}^n F_i : F_1 < ... < F_n, F_i \in \mathcal{F}, i = 1, ..., n$ and $\{\min F_i\}_{i=1}^n \in \mathcal{G}\}$.*

DEFINITION 4.5. — *The Schreier hierarchy was first defined in [1]. It is a set of families $(\mathcal{S}_\xi)_{\xi < \omega_1}$ of finite subsets of $\mathbb{N}$ which can be defined recursively as follows:*

$$\mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leqslant \min F\} \cup \{\emptyset\}$$

Let $\xi < \omega_1$ and suppose that $\mathcal{S}_\xi$ have been defined for all $\zeta < \xi$ Then

   i. *If $\xi = \zeta + 1$ we set $\mathcal{S}_\xi = \mathcal{S}_\zeta \otimes \mathcal{S}_1$.*

   ii. *If ξ is a limit ordinal then we fix a strictly increasing sequence of non-limit ordinals $(\xi_n)_n$ with $\sup \xi_n = \xi$ and set $\mathcal{S}_\xi = \bigcup_{n=1}^\infty \{F \in \mathcal{S}_{\xi_n} : F \geqslant n\}$.*

It can be verified by transfinite induction that each $\mathcal{S}_\xi$ for $\xi < \omega_1$ is compact hereditary and spreading. We will need the following two results found in [18] concerning the families $(\mathcal{S}_\xi)_{\xi < \omega_1}$ which can be proved by transfinite induction.

LEMMA 4.6. — *For every ordinal $\xi < \omega_1$ and $M \in [\mathbb{N}]$ we have*

   i. *$[M]^{\leqslant 3} \otimes \mathcal{S}_\xi \subseteq \mathcal{S}_\xi \otimes [M]^{\leqslant 2}$.*

   ii. *If $\min M \geqslant 3$, then $[M]^{\leqslant 3} \otimes (\mathcal{S}_\xi \oplus [M]^{\leqslant 1}) \subseteq \mathcal{S}_\xi \otimes [M]^{\leqslant 3}$.*

Remark 4.7. — By Lemma 4.6 we have that $\forall M \in [\mathbb{N}]$ and $\xi < \omega_1$

$$[M]^{\leqslant 8} \otimes (\mathcal{S}_\xi \otimes [M]^{\leqslant 2}) \subseteq ([M]^{\leqslant 3} \otimes ([M]^{\leqslant 3} \otimes \mathcal{S}_\xi)) \otimes [M]^{\leqslant 2}$$

$$\subseteq (\mathcal{S}_\xi \otimes [M]^{\leqslant 4}) \otimes [M]^{\leqslant 2} \subseteq \mathcal{S}_\xi \otimes [M]^{\leqslant 8}.$$

The Spaces $T(\theta, \mathcal{F})$**.** Let $0 < \theta < 1$ and $\mathcal{F}$ be a compact hereditary family of finite subsets of $\mathbb{N}$.

DEFINITION 4.8. — *Let $G_{\theta, \mathcal{F}}$ be the minimal subset of $c_{00}(\mathbb{N})$ such that*

    i. $\pm e_n \in G_{\theta, \mathcal{F}}$ $\forall n \in \mathbb{N}$;
    ii. $G_{\theta, \mathcal{F}}$ *is closed under the* $(\theta, \mathcal{F})$*-operation. That is if* $(f_i)_{i=1}^d$ *is an* $\mathcal{F}$*-admissible family in $c_{00}(\mathbb{N})$ then $\theta \sum_{i=1}^d f_i \in G_{\theta, \mathcal{F}}$.*

The space $T(\theta, \mathcal{F})$ is the completion of $c_{00}(\mathbb{N})$ under the following norm

$$\forall x \in c_{00}(\mathbb{N}) \quad \text{we set } \|x\|_{(\theta, \mathcal{F})} = \sup\{f(x) : f \in G_{\theta, \mathcal{F}}\}.$$

Detailed expositions of the $T(\theta, \mathcal{F})$ type spaces can be found in [8]. In the sequel we shall denote $T_\xi = T(\frac{1}{2}, \mathcal{S}_\xi)$, $T_\xi^1 = T(\frac{1}{2}, \mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 2})$ and $T_\xi^2 = T(\frac{1}{2}, \mathcal{S}_\xi \oplus [\mathbb{N}]^{\leqslant 1})$. The following two Lemmas are results in [18] but for the sake of completion we include their proofs here.

LEMMA 4.9. — *Let $\xi < \omega_1$. Then for every finite sequence $(b_i)_{i=1}^k$ of scalars we have*

    I. $\| \sum_{i=1}^k b_i e_i \|_{T_\xi^1} \leqslant 8 \| \sum_{i=1}^k b_i e_i \|_{T_\xi}$
    II. $\| \sum_{i=1}^k b_i e_i \|_{T_\xi^2} \leqslant 3 \| \sum_{i=1}^k b_i e_i \|_{T_\xi}$

where by $(e_i)_{i \in \mathbb{N}}$ we denote the standard Hamel basis of $c_{00}(\mathbb{N})$.

Proof. — I. By Remark 4.7 we have that $[\mathbb{N}]^{\leqslant 8} \otimes (\mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 2}) \subseteq \mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 8}$. Let $G_{T_\xi^1}$ be the norming set of T_ξ^1 and let $f \in G_{T_\xi^1}$. We will define $g_1 < ... < g_l$ with $l \leqslant 8$ and $g_i \in G_{T_\xi}, i = 1, ..., l$ such that $f = \sum_{i=1}^l g_i$. We use induction on the complexity of f. Let $f = \pm e_n$ for some $n \in \mathbb{N}$, then there is nothing to prove. Let $f = \frac{1}{2} \sum_{i=1}^d$ such that

    i. $f_1 < ... < f_d$.
    ii. $(f_i)_{i=1}^d$ is an $\mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 2}$ admissible sequence.
    iii. For every f_i there exists a sequence $g_1^i < ... < g_{l_i}^i$ such that $g_j^i \in G_{T_\xi}$ for $j = 1, ..., l_i$ and $f_i = \sum_{j=1}^{l_i} g_j^i$.

Now since $\{\min f_i\}_{i=1}^d \in \mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 2}$ by Remark 4.7 that $\bigcup_{i=1}^d \{\min g_j^i : j \leqslant l_i\} \in \mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 8}$. Thus there exist $B_1, ..., B_k$ with $k \leqslant 8$ such that

    1. $B_m \in \mathcal{S}_\xi$ for all $m = 1, ..., k$ and
    2. $B_1 < ... < B_k$

such that $\bigcup_{i=1}^d \{\min g_j^i : j \leqslant l_i\} = \bigcup_{m=1}^k B_m$. By setting $g^{(m)} = \frac{1}{2} (\sum_{\min g_j^i \in B_m} g_i^j)$ we get

    a. $f = \sum_{m=1}^k g^{(m)}$ and
    b. $g^{(m)} \in G_{T_\xi}$

as desired.

II. By Lemma 4.6 we have $[\mathbb{N}]^{\leqslant 3} \otimes (\mathcal{S}_\xi \oplus [\mathbb{N}]^{\leqslant 1}) \subseteq \mathcal{S}_\xi \otimes [\mathbb{N}]^{\leqslant 3}$. Using the same arguments as in the proof of I. we conclude that for every $f \in G_{T_\xi^2}$ there exist $g_1 < g_2 < g_3$ with $g_i \in G_{T_\xi}$ for $i = 1, 2, 3$ such that $f = \sum_{i=1}^{3} g_i$. □

DEFINITION 4.10. — *Let $M \in [\mathbb{N}]$ with $M = \{m_1 < m_2 < ...\}$.*

i. *For every $m \in M$ we set m^+ to be the immediate successor of m in M, that is $m_i^+ = m_{i+1}$ for all $i \in \mathbb{N}$.*

ii. *If $A \in [M]^{\leqslant \omega}$ then we set $A^+ = \{m^+ : m \in A\}$.*

LEMMA 4.11. — *Let $\xi < \omega_1$ and $M \in [\mathbb{N}]$ with $M = \{m_1 < m_2 < ...\}$. Then for every finite sequence $(b_i)_{i=1}^{k}$ of scalars we have $\|\sum_{i=1}^{k} b_i e_{m_i}\| \leqslant \|\sum_{i=1}^{k} b_i e_{m_i^+}\| \leqslant 3\|\sum_{i=1}^{k} b_i e_{m_i}\|$ where all norms are considered in the space T_ξ.*

Proof. — Let $A \in [M]^{\leqslant \omega}$. Suppose that $A^+ = \{m^+ : m \in A\} \in \mathcal{S}_\xi$. Since $\mathcal{S}_\xi$ is a spreading family it follows that $A \setminus \min A \in \mathcal{S}_\xi$. Thus, $A \in \mathcal{S}_\xi \oplus [M]^{\leqslant 1}$. So, if we consider $f \in G_{T_\xi}$ with the property $\operatorname{supp} f \subseteq \{m_i^+ : i = 1, ..., k\}$ there is an $f' \in G_{T_\xi^2}$ such that $f(\sum_{i=1}^{k} b_i e_{m_i^+}) \leqslant f'(\sum_{i=1}^{k} b_i e_{m_i})$ and this gives

i. $\|\sum_{i=1}^{k} b_i e_{m_i^+}\|_{T_\xi} \leqslant \|\sum_{i=1}^{k} b_i e_{m_i}\|_{T_\xi^2}$.

On the other hand since $\mathcal{S}_\xi$ is spreading it is easily verified that

ii. $\|\sum_{i=1}^{k} b_i e_{m_i}\|_{T_\xi} \leqslant \|\sum_{i=1}^{k} b_i e_{m_i^+}\|_{T_\xi}$

combining i. and ii. we have

$$\left\|\sum_{i=1}^{k} b_i e_{m_i}\right\|_{T_\xi} \leqslant \left\|\sum_{i=1}^{k} b_i e_{m_i^+}\right\|_{T_\xi} \leqslant \left\|\sum_{i=1}^{k} b_i e_{m_i}\right\|_{T_\xi^2}$$

and by Lemma 4.9 we get the desired. □

4.2. The norming set G_ξ

In this subsection starting with X_T, as in the introductory paragraph of the section, assuming that X_T does not contain an isomorphic copy of ℓ^1 we define $\mathfrak{X}_\xi$ and prove that it satisfies the first two properties of Y mentioned in Theorem 4.1. Namely, we show that the identity map $I : \mathfrak{X}_\xi \to X_T$ is continuous and that the set $I^{-1}(K)$ is a weakly compact subset of $\mathfrak{X}_\xi$. We start with some well known results concerning ℓ^1-spreading models.

DEFINITION 4.12. — *A bounded sequence $(x_n)_n$ in a Banach space Y is an ℓ^1_ξ-spreading model, for $\xi < \omega_1$, if there exists a constant $C > 0$ such that*

$$\left\| \sum_{i \in F} a_i x_i \right\| \geqslant C \sum_{i \in F} |a_i|$$

for every $F \in \mathcal{S}_\xi$ and all choices of scalars $(a_i)_{i \in F}$.

The following is a well known result and for its proof we refer the interested reader to [7].

LEMMA 4.13. — *If a separable Banach space Y contains ℓ^1_ξ-spreading model, for every $\xi < \omega_1$, then Y contains an isomorphic copy of ℓ^1.*

As we have supposed, the space X_T does not contain ℓ^1 therefore it follows that there is $\xi < \omega_1$ such that X_T contains no ℓ^1_ξ-spreading model. We fix this countable ordinal $\xi < \omega_1$ and we use the following norming set for X_T:

$$G_1 = \left\{ \sum_{t \in F} b_t e_t^* : \left\| \sum_{t \in F} b_t e_t^* \right\|_{X_T^*} \leqslant 1 \text{ and } F \subseteq T \text{ finite and segment complete} \right\}.$$

We also consider a bijection $h : T \to \mathbb{N}$ as in Definition 2.2 and make use of the following piece of notation:

Notation 4.14. — For every sequence $(f_i)_{i=1}^d$ in $c_{00}(T)$ such that

    i. $(f_i)_{i=1}^d$ is block
    ii. $\left\{ \min\{h(t) : t \in \operatorname{supp} f_i\} \right\}_{i=1}^d \in \mathcal{S}_\xi$
    iii. $\{\operatorname{supp} f_i\}_{i=1}^d$ are incomparable subsets of T.

We will call $(f_i)_{i=1}^d$ a (T, ξ)-admissible sequence.

The definition of the norming set is the following

DEFINITION 4.15. — *Let G_ξ be the minimal subset of $c_{00}(T)$ such that*

    1. $G_1 \subseteq G_\xi$
    2. G_ξ *is closed under the $(\frac{1}{2}, \mathcal{S}_\xi)$-operation on (T, ξ)-admissible sequences. That is, for every (T, ξ)-admissible sequence $f_1, ..., f_d$ in G_ξ we have that $\frac{1}{2} \sum_{i=1}^d f_i$ is an element of G_ξ.*

We define a norm on $c_{00}(T)$ as follows:

    For every $x \in c_{00}(T)$ we let $\|x\|_{\mathfrak{X}_\xi} = \sup\{f(x) : f \in G_\xi\}$

and set

$$\mathfrak{X}_\xi = \overline{< e_t : t \in T >}^{\|\cdot\|_{\mathfrak{X}_\xi}}.$$

Remark 4.16. — It can be readily verified that $(e_t)_{t \in T}$ (enumerated via h) becomes a bimonotone Schauder basis for $\mathfrak{X}_\xi$. In addition as $G_1 \subset G_\xi$ it is evident that the identity operator $I : \mathfrak{X}_\xi \to X_T$ is continuous.

We also have the following,

LEMMA 4.17. — *The set G_ξ is closed under restrictions of its elements on segment complete subsets of T and thus for every segment complete $A \subseteq T$ the natural projection $P_A : \mathfrak{X}_\xi \mapsto \mathfrak{X}_\xi$ defined by $P_A(\sum_{t \in T} \lambda_t e_t) = \sum_{t \in A} \lambda_t e_t$ has norm 1.*

Proof. — Let $f \in G_\xi$ and $A \subset T$ segment complete. We will show that $f|A \in G_\xi$ by using induction on the complexity of f. Suppose that $f \in G_1$. By our assumptions we have $f|A \in G_1 \subset G_\xi$. Now let $f = \frac{1}{2} \sum_{i=1}^d f_i \in G_\xi$ and assume that $f_i|A \in G_\xi$ for all $i = 1, ..., d$. Then $f|A = \frac{1}{2} \sum_{i=1}^d f_i|A$ and the following properties of $(f_i|A)_{i=1}^d$ can be readily verified

  i. $\{f_i|A\}_{i=1}^d$ is a block sequence.
  ii. $\{\operatorname{supp} f_i|A\}_{i=1}^d$ are pairwise incomparable subsets of T.
  iii. $\{\min\{h(t) : t \in \operatorname{supp} f_i|A\}\}_{i=1}^d \in \mathcal{S}_\xi$, since $\mathcal{S}_\xi$ is hereditary and spreading.

Thus $f|A \in G_\xi$. $\qquad\qquad\square$

DEFINITION 4.18. — *Let $f \in G_\xi$. By a tree analysis of f we mean a finite family $(f_a)_{a \in \mathcal{A}}$ indexed by a finite tree $\mathcal{A}$ with a unique root $0 \in \mathcal{A}$ such that*

  1. $f_0 = f$ *and* $f_a \in G_\xi$ *for every* $a \in \mathcal{A}$
  2. *An* $a \in \mathcal{A}$ *is maximal if and only if* $f_a \in G_1$
  3. *For every* $a \in \mathcal{A}$ *not maximal we denote by* S_a *the set of immediate successors of* a *in* A *and define an ordering denoted by* $<$ *on* S_a *with* $b_1 < b_2$ *if and only if* $f_{b_1} < f_{b_2}$ *for all* $b_1, b_2 \in S_a$. *Then we have that* $(f_b)_{b \in S_a}$ *ordered by* $<$ *is a* (T, ξ)-*admissible sequence and* $f_a = \frac{1}{2} \sum_{b \in S_a} f_b$

It is straightforward that by the minimality of G_ξ that every $f \in G_\xi$ admits a tree analysis.

Remark 4.19. — We note that the definition of the norming set G_ξ uses a Tsirelson type extension technique but only on functionals with incomparable supports. Therefore, if we consider any branch $b \in [T]$ and a vector $x \in \mathfrak{X}_\xi$ such that $\operatorname{supp} x \subset b$ then we can observe that for every $f \in G_\xi$ with a tree analysis $(f_a)_{a \in A}$ there exists at most one maximal $a \in A$ such that $\operatorname{supp} f_a \cap \operatorname{supp} x \neq \emptyset$. Hence, for every such vector it follows that

$\|x\|_{X_T} = \|x\|_{\mathfrak{X}_\xi}$. This fact allows us to identify the sets K $\subset X_T$ and $I^{-1}(\mathrm{K}) \subset \mathfrak{X}_\xi$. We will use this for what follows.

DEFINITION 4.20. — *Let $(y_n)_{n\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_\xi$ and $f \in \mathrm{G}_\xi$.*

1. *We set $M_f = \{n \in \mathbb{N} : \mathrm{supp}\, f \cap \mathrm{ran}\, y_n \neq \emptyset\}$ and if $(f_a)_{a\in\mathcal{A}}$ is a tree analysis of f we define a correspondence $\lambda_f : M_f \mapsto \mathcal{A}$ with $\lambda_f(n)$ to be the $\sqsubseteq_\mathcal{A}$-maximal element of $\mathcal{A}$ such that $\mathrm{supp}\, f_{\lambda_f(n)} \cap \mathrm{ran}\, y_n = \mathrm{supp}\, f \cap \mathrm{ran}\, y_n$.*
2. *For all $a \in \mathcal{A}$ we define $D_a = \bigcup_{b\in S_a}\{n \in \mathbb{N} : b = \lambda_f(n)\}$, equivalently, $D_a = \{n \in \mathbb{N} : \emptyset \neq \mathrm{supp}\, f_a \cap \mathrm{ran}\, y_n = \mathrm{supp}\, f \cap \mathrm{ran}\, y_n\}$ and $E_a = D_a \setminus \bigcup_{b\in S_a} D_b$, or equivalently, $E_a = \{n \in \mathbb{N} : a = \lambda_f(n)\}$.*
3. *For $a \in \mathcal{A}$ not $\sqsubseteq_\mathcal{A}$-maximal we set $b_L(n) = \min\{b \in S_a : \mathrm{supp}\, f_b \cap \mathrm{ran}\, y_n \neq \emptyset\}$ and $b_R(n) = \max\{b \in S_a : \mathrm{supp}\, f_b \cap \mathrm{ran}\, y_n \neq \emptyset\}$ where the maximum and minimum are taken with respect to the ordering on S_a defined above.*
4. *For a block sequence $(y_n)_n$ in $\mathfrak{X}_\xi$ we set $p_n = \min \mathrm{supp}\, y_n$ and $q_n = \max \mathrm{supp}\, y_n$, for all $n \in \mathbb{N}$.*

We start with an easy but crucial observation that will be used extensively in what follows.

Remark 4.21. — Let $(y_n)_n$ be a seminormalized level block sequence such that $\mathrm{supp}\, y_n \perp \mathrm{supp}\, y_m$ for all $n \neq m \in \mathbb{N}$. Then $(y_n)_n$ is a ℓ^1_ξ spreading model.

Proof. — Let $r > 0$ be such that $\|y_n\| > r$ for all $n \in \mathbb{N}$. Choose a sequence of functionals $(f_n)_n$ in G_ξ such that for each $n \in \mathbb{N}$ the following hold:

i. $\mathrm{ran}\, f_n \subset \mathrm{ran}\, y_n$;
ii. $f_n(y_n) > \frac{r}{2}$.

Let $F \in \mathcal{S}_\xi$ and $(b_i)_{i\in F} \in c_{00}(\mathbb{N})$. It is easy to see that the functional $f = \frac{1}{2}\sum_{i\in F}\mathrm{sgn}(b_i)f_i$ belongs to G_ξ. In addition,

$$\left\|\sum_{i\in F} b_i y_i\right\| \geqslant f\left(\sum_{i\in F} b_i y_i\right) \geqslant \frac{r}{2}\sum_{i\in F}|b_i|,$$

which proves that $(y_n)_n$ is a ℓ^1_ξ spreading model.   □

The next proposition is the basic tool for proving that K is weakly compact in $\mathfrak{X}_\xi$. It is also used in the next section where we show that the closed convex hull of K is thin in $\mathfrak{X}_\xi$.

PROPOSITION 4.22. — *Let $(y_n)_n$ be a bounded level-block sequence in $\mathfrak{X}_\xi$ such that*

$$\lim_{n\to\infty} \|y_n\|_{X_T} = 0.$$

Then there exists subsequence of $(y_n)_n$ which satisfies an upper T_ξ-estimate, that is, there exist a constant $C > 0$ and $M \in [\mathbb{N}]$ such that for every choice of scalars $(\lambda_i)_{i=1}^k \in c_{00}(\mathbb{N})$ we have

$$\Big\| \sum_{i=1}^k \lambda_i y_{m_i} \Big\|_{\mathfrak{X}_\xi} \leqslant C \Big\| \sum_{i=1}^k \lambda_i e_{p_{m_i}} \Big\|_{T_\xi}$$

where $M = \{m_1 < m_2 < \dots\}$.

In order to prove Proposition 4.22 we need the following Lemma.

LEMMA 4.23. — *Let $(y_n)_n$ be a bounded level-block sequence in $\mathfrak{X}_\xi$ with $\|y_n\|_{\mathfrak{X}_\xi} \leqslant r \;\forall n \in \mathbb{N}$. Suppose also that $\sum_{n\in\mathbb{N}} \|y_n\|_{X_T} < 2r$. Then for every $f \in G_\xi$ there exists a $g \in B_{(T_\xi^1)^*}$ such that for every $k \in \mathbb{N}$ and every choice of scalars $(\lambda_i)_{i=1}^k \in c_{00}(\mathbb{N})$ we have that $|f(\sum_{i=1}^k \lambda_i y_i)| \leqslant 2rg(\sum_{i=1}^k \lambda_i e_{q_i})$.*

Proof. — Let $f \in G_\xi$, $(f_a)_{a\in\mathcal{A}}$ a tree analysis of f and $(\lambda_i)_{i=1}^d \in c_{00}(\mathbb{N})$. For each $a \in \mathcal{A}$ with $D_a \neq \emptyset$ we will recursively define g_a such that the following hold

    i. $g_a \in B_{(T_\xi^1)^*}$;
    ii. $\operatorname{supp} g_a \subseteq \{q_n : n \in D_a\}$;
    iii. $|f_a(\sum_{i\in D_a} \lambda_i y_i)| \leqslant 2rg_a(\sum_{i\in D_a} \lambda_i e_{q_i})$.

Let $a \in \mathcal{A}$ be a maximal element of $\mathcal{A}$ such that $D_a \neq \emptyset$. Let also $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} = \max_{i\in D_a} |\lambda_i|$. We set $g_a = \operatorname{sgn} \lambda_{n_0} e_{q_{n_0}}^*$. Clearly $g_a \in B_{(T_\xi^1)^*}$ and

$$\Big| f_a\Big(\sum_{i\in D_a} \lambda_i y_i \Big) \Big| \leqslant \max_{i\in D_a} |\lambda_i| \cdot \sum_{i\in D_a} |f_a(y_i)| \leqslant 2rg_a\Big(\sum_{i\in D_a} \lambda_i e_{q_i} \Big).$$

Let now $a \in \mathcal{A}$ not maximal. Suppose also that for every $b \in S_a$ the functionals $(g_b)_{b\in S_a}$ have been defined satisfying conditions i. ii and iii above. Let $\{b_1 < \dots < b_l\}$ be the enumeration of S_a as it was given in Definition 4.20. Pick $b_i \in S_a$ and suppose that $D_{b_i} \neq \emptyset$. Then,

$$\min \operatorname{supp} f_{b_i} \leqslant \operatorname{supp} g_{b_i} < \min \operatorname{supp} f_{b_{i+1}}.$$

The left inequality holds because if we pick $k \in D_{b_i}$ then $\max \operatorname{supp} y_k \geqslant \min \operatorname{supp} f_{b_i}$ and $\operatorname{supp} g_{b_i} \subseteq \{q_n : n \in D_{b_i}\}$. On the other hand assume that there exists $k \in D_{b_i}$ such that $q_k > \min \operatorname{supp} f_{b_{i+1}}$ then $\operatorname{ran} y_k \cap \operatorname{supp} f_{b_{i+1}} \neq \emptyset$. This contradicts the definition of D_{b_i} and proves the right hand inequality. Similarly, we can see that for every $n \in E_a$ such that

$b_R(n) \neq b_l$ we have $\min \operatorname{supp} f_{b_R(n)} \leqslant q_n < \min \operatorname{supp} f_{b_R(n)+1}$. For every i with $1 \leqslant i \leqslant l$ we set,

$$M_i = \operatorname{supp} g_{b_i} \cup \{q_n : n \in E_a \text{ and } b_i = b_R(n)\}.$$

We can readily observe the following,

  i. For every $b \in S_a$ it holds $|\{q_n : n \in E_a \text{ and } b = b_R(n)\}| \leqslant 1$.
  ii. For $i < l$ we get $\{q_n : n \in E_a \text{ and } b_i = b_R(n)\} < \operatorname{supp} g_{b_i}$, while for $i = l$ we have the converse.
  iii. $\bigcup_{b \in S_a} M_b = \left(\bigcup_{b \in S_a} \operatorname{supp} g_b\right) \cup \{q_n : n \in E_a\} \subseteq \{q_n : n \in D_a\}$.
  iv. $\min \operatorname{supp} f_{b_i} \leqslant M_{b_i} < \min \operatorname{supp} f_{b_{i+1}}$ for all $i = 1, ..., l-1$.

Combining these four facts we conclude that the functionals $(e_{q_n})_{n \in E_a}$ and $(g_{b_i})_{1 \leqslant i \leqslant l}$ together form a $[\mathbb{N}]^{\leqslant 2} \otimes \mathcal{S}_\xi$-admissible family. Consequently the functional $g_a = \frac{1}{2}(\sum_{n \in E_a} e_{q_n}^* + \sum_{b \in S_a} g_b)$ is an element of $B_{T_\xi^{1*}}$. Finally,

$$\left| f_a\left(\sum_{i \in D_a} \lambda_i y_i \right) \right| \leqslant \left| f_a\left(\sum_{i \in E_a} \lambda_i y_i \right) \right| + \left| f_a\left(\sum_{i \in D_a \setminus E_a} \lambda_i y_i \right) \right|$$

$$\leqslant r \cdot \sum_{i \in E_a} |\lambda_i| + \left| \frac{1}{2} \sum_{b \in S_a} f_b\left(\sum_{i \in D_b} \lambda_i y_i \right) \right|$$

$$\leqslant r \sum_{i \in E_a} |\lambda_i| + r \sum_{b \in S_a} g_b\left(\sum_{i \in D_b} \lambda_i e_{q_i} \right) = 2r g_a\left(\sum_{i \in D_a} \lambda_i e_{q_i} \right)$$

$\square$

We are now ready to prove Proposition 4.22.

Proof of Proposition 4.22. — Since $(y_n)_n$ is a bounded block sequence in $\mathfrak{X}_\xi$ such that

$$\lim_{n \to \infty} \|y_n\|_{X_T} = 0$$

we may choose $M \in [\mathbb{N}]$ and a subsequence $(y_n)_{n \in M}$ such that

  i. $\sum_{n \in M} \|y_n\|_{X_T} < 2r$
  ii. $(y_n)_{n \in M}$ is level-block.

For simplicity we denote the subsequence by $(y_n)_n$ again. Lemma 4.23 yields,

$$\left\| \sum_{i=1}^k \lambda_i y_i \right\|_{\mathfrak{X}_\xi} \leqslant 2r \left\| \sum_{i=1}^k \lambda_i e_{q_i} \right\|_{T_\xi^1}.$$

By Lemma 4.9 we have,

$$\left\| \sum_{i=1}^k \lambda_i y_i \right\|_{\mathfrak{X}_\xi} \leqslant 2r \left\| \sum_{i=1}^k \lambda_i e_{q_i} \right\|_{T_\xi^1} \leqslant 16r \left\| \sum_{i=1}^k \lambda_i e_{q_i} \right\|_{T_\xi}.$$

Finally, applying Lemma 4.11 we obtain,

$$\Big\| \sum_{i=1}^{k} \lambda_i y_i \Big\|_{\mathfrak{X}_\xi} \leqslant 16r \Big\| \sum_{i=1}^{k} \lambda_i e_{q_i} \Big\|_{T_\xi} \leqslant 48r \Big\| \sum_{i=1}^{k} \lambda_i e_{p_i} \Big\|_{T_\xi},$$

for all choices of $k \in \mathbb{N}$ and $(\lambda_i)_{i=1}^{k} \in c_{00}(\mathbb{N})$ completing the proof. $\qquad \square$

Notation 4.24. — We set $W_\xi^0 = \mathrm{co}(\mathrm{K} \cup -\mathrm{K})$ and $W_\xi = \overline{W_\xi^0}^{\|\cdot\|_{\mathfrak{X}_\xi}}$.

PROPOSITION 4.25. — *The set K is weakly compact in $\mathfrak{X}_\xi$*

Proof. — Let $(y_n)_n$ be a sequence in K. Clearly we may assume that each y_n is finitely supported. First we prove the following

CLAIM. — *If $(y_n)_n$ consists of incomparably supported vectors then it is weakly null.*

$\square$

Proof of Claim. — Let $r > 0$ be such that $\|y_n\| \geqslant r > 0$ and suppose towards a contradiction that there exist an $\epsilon > 0$ and a functional $x^* \in (\mathfrak{X}_\xi)^*$ with $\|x^*\| = 1$ such that $x^*(y_n) > \epsilon$ for all n. Now, since $\mathrm{supp}\, y_n$ are incomparable segments of T we may also assume (by passing to a subsequence) that $(y_n)_n$ is a level block sequence. Remark 4.21 yields that $(y_n)_n$ is a ℓ_ξ^1 spreading model. As X_T does not contain any ℓ_ξ^1-spreading model, there exists a sequence $(z_n)_n$ of block convex combinations of $(y_n)_n$ such that $\|z_n\|_{X_T} \to 0$. By Proposition 4.22, there exists a subsequence of $(z_n)_n$ (denoted by $(z_n)_n$ again) which satisfies an upper-T_ξ estimate. As the space T_ξ is reflexive this implies that $(z_n)_n$ is weakly null. Hence, there are further convex combinations of $(z_n)_n$ that converges norm to zero. This is clearly a contradiction since we have assumed that $x^*(y_n) > \epsilon$ for all $n \in \mathbb{N}$ and it completes the proof of the claim.

Now if $(y_n)_n$ is arbitrary we can assume, by passing to a subsequence if necessary, that $e_t^*(y_n) \xrightarrow{n} y_t$ for all $t \in T$. Observe that $S = \{t \in T : y_t \neq 0\}$ is a segment (finite or infinite) of T. Thus, we may assume that each y_n has a decomposition as $y_n = u_n + v_n$ where

    i. $(u_n)_n$ is a $\subseteq$-increasing sequence of segments of S
    ii. $(v_n)_n$ is a sequence of incomparable segments of T.

The previous claim yields that $(v_n)_n$ is weakly null. To finish the proof set $y = \sum_{t \in S} y_t e_t$ and observe that $(u_n)_n$ has a subsequence which converges to y in the norm topology of $\mathfrak{X}_\xi$. Indeed, by the weak compactness of K in X_T there exists a subsequence (which we denote by $(u_n)_n$ again) such that

$u_n \overset{(X_T, w)}{\to} y' \in \mathrm{K}$. As $e_t^*(u_n) \to e_t^*(y)$ we deduce that $y = y'$. It is easy to see that the definition of the set K implies, in fact, that $u_n \overset{\|\cdot\|_{X_T}}{\to} y$. Since, $\operatorname{supp} u_n \subset S$ for all $n \in \mathbb{N}$ we deduce by Remark 4.19 that $u_n \overset{\|\cdot\|_{\mathfrak{X}_\xi}}{\to} y$ which completes the proof. $\qquad \square$

4.3. The set K is thin in $\mathfrak{X}_\xi$

In this subsection use Lemma 4.29, Proposition 4.29 and Proposition 4.22 in order to show that the closed convex hull W_ξ of K is thin in $\mathfrak{X}_\xi$. Lemma 4.29 and Proposition 4.29 use techniques developed in [4], adapted to this setting, which are crucial for the proof. We note that for the sake of simplicity of notation hereby all norms are considered in $\mathfrak{X}_\xi$ unless stated otherwise. We start with the definition of a thin subset of a Banach space.

DEFINITION 4.26. — *Let A, Γ be two subsets of a Banach space Y*

  i. *Let $\epsilon > 0$. We say that Γ ϵ-absorbs A if there exists $\lambda > 0$ such that $A \subseteq \lambda\Gamma + \epsilon B_Y$.*
  ii. *We say that Γ almost absorbs A if Γ ϵ-absorbs A for every $\epsilon > 0$.*
  iii. *We say that A is thin in Y if A does not almost absorb the ball of any infinite dimensional closed subspace of Y.*

DEFINITION 4.27. — *Let A be a segment complete subset of T and $\epsilon > 0$.*

  1. *For each $x \in \mathfrak{X}_\xi$ we denote by $P_A x$ the natural projection of x onto A and for s segment of T denote by $x_s = P_s x_t = \sum_{t \in s} \psi(t) e_t$.*
  2. *We set $A_{\mathrm{K}} = \{t \in A : x_t \in \mathrm{K}\}$ and $A^\epsilon = \{t \in A_{\mathrm{K}} : \|P_A x_t\| \geqslant \epsilon\}$. Let also $A' = A \setminus A^\epsilon$.*
  3. *We set $\operatorname{seg}(A) = \{s \text{ segment} : t \in A_{\mathrm{K}} \text{ for all } t \in s\}$. Clearly, for all $s \in \operatorname{seg}(A)$ it follows that $s \subseteq A$.*

We observe the following:

LEMMA 4.28. — *For every A segment complete and $\epsilon > 0$ the set A^ϵ is also segment complete.*

Proof. — Let $t_1, t_2 \in A^\epsilon$ with $t_1 \sqsubseteq t_2$. Then we have that $\|x_{t_2}\| \leqslant 1$ thus for every $t \in [t_1, t_2]$ we obtain $\|x_t\| \leqslant \|x_{t_2}\| \leqslant C$ and at the same time $\|P_A x_t\| \geqslant \|P_A x_{t_1}\| > \epsilon$. These facts imply $t \in A^\epsilon$. $\qquad \square$

LEMMA 4.29. — *Let $\epsilon > 0$ and E a subset of T of the form $E = \{t \in T : m \leqslant |t| \leqslant M\}$. Then there exists a decomposition of E into two disjoint subsets E', E'' such that*

    i. $\|P_{E'}w\| < \epsilon$ for every $w \in W_\xi^0$ and
    ii. For $t \in T$ with $|t| \geqslant M$ we have $\|P_{E''}x_t\| \geqslant \epsilon$.

Proof. — Set $E'' = E^\epsilon$ as in Definition 4.27 and $E' = E \setminus E''$. Notice that $ii.$ above is clear. Observe also that for every $w \in W_\xi^0$, $P_E w$ can be written as $P_E w = \sum_{s \in L} \lambda_s x_s$ where $L \subseteq \mathrm{seg}(E)$ and $\sum_{s \in L} |\lambda_s| \leqslant 1$. It is clear that for every $s \in L$ the set $s' = s \cap E'$ is either empty or a segment of E such that $\|x_{s'}\| < \epsilon$. Therefore $\|P'_E w\| = \|\sum_{s \in L} \lambda_s x_{s'}\| < \epsilon$.   □

    The following Proposition is the key ingredient for proving that the set W_ξ is a thin subset of $\mathfrak{X}_\xi$. It is an adaptation of the techniques developed in [4] and for the sake of completeness we include its proof here.

PROPOSITION 4.30. — *Let $(w_n)_n$ be a level-block sequence in W_ξ^0, $\epsilon > 0$ and $(E_n)_n$ a level-block sequence of subsets of T where each one is of the form $E_n = \{t \in T : m_n \leqslant |t| \leqslant M_n\}$ such that $\mathrm{ran}\, w_n \subseteq E_n$. Then there exist a $L \in [\mathbb{N}]$ and a sequence $(F_n)_{n \in L}$ with the following properties:*

    i. $F_n \subseteq E_n$ for $n \in L$.
    ii. $(F_n)_{n \in L}$ *are pairwise incomparable and segment complete subsets of T.*
    iii. $\|P_{E_n}w_n - P_{F_n}w_n\| < \epsilon$.

Proof. — We apply Lemma 4.29 to find a decomposition of E_n into two disjoint subsets E'_n, E''_n such that

    i. $\|P_{E'_n}w_n\|_{\mathfrak{X}_\xi} < \frac{\epsilon}{2}$.
    ii. If s is a segment with $s = [t_1, t_2]$ and $|t_1| \leqslant m_n, |t_2| \geqslant M_n$ as well as $s \cap E''_n \neq \emptyset$ we have $\|P_{E''_n}x_s\| \geqslant \frac{\epsilon}{2}$.

Now for every $n \in \mathbb{N}$ $E_n w_n = \sum_{s \in L_n} \lambda_s x_s$, where $L_n \subseteq \mathrm{seg}(E_n)$ and $\sum_{s \in L_n} |\lambda_s| \leqslant 1$. This representation defines a positive measure on $\mathrm{seg}(E_n)$ with $\mu_n(A) = \sum_{s \in A \cap L_n} |\lambda_s|$ for $A \subseteq \mathrm{seg}(E_n)$. Now let us consider a probability measure ν on the compact metrizable space of the branches $[T]$ of the tree T such that for every segment s and every $\mathcal{O}_s$ basic clopen neighborhood of $[T]$ that contains s of the form $\mathcal{O}_s = \{b \in [T] : s \sqsubseteq b\}$ we have that $\nu(\mathcal{O}_s) > 0$. With the help of ν we define a measure μ on $[T]$ as follows. For every clopen $B \subseteq [T]$ we set

$$\mu(B) = \lim_{n \to \mathcal{U}} \sum_{s \in L_n} |\lambda_s| \frac{\nu(\mathcal{O}_s \cap B)}{\nu(\mathcal{O}_s)}$$

where the limit is taken with respect to a non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Using a diagonal argument we may assume that this is an ordinary limit. Now for every $n < k$ in $\mathbb{N}$ we define

    i. $B_n = \{b \in [T] : b \cap E''_n \neq \emptyset\}$.

  ii. $A_n^k = \{s \subseteq \operatorname{seg}(E_k) : \hat{s} \cap E_n'' \neq \emptyset\}$.

By our definition of μ we have that

$$\mu(B_n) = \lim_k \mu_k(A_n^k).$$

$\square$

  CLAIM. — *For every $M \in [\mathbb{N}]$ and $\delta > 0$ the set $I_\delta = \{n \in M : \mu(B_n) < \delta\}$ is an infinite subset of M.*

  Proof of claim. — Suppose not. Then there exists an $M \in [\mathbb{N}]$ such that for every $n \in M$ it holds $\mu(B_n) \geqslant \delta$ and therefore there exists a branch $b \in [T]$ such that $b \in B_n, \forall n \in M$. This implies $b \cap E_n'' \neq \emptyset$ for all $n \in M$. Thus the sequence $(x_{b|n})_{n \in \mathbb{N}}$ converges norm to $x_b = \sum_{n \in \mathbb{N}} \psi(b|n) e_{b|n}$ and $\|P_{E_n''} x_b\| \geqslant \frac{\epsilon}{2}$ for all $n \in M$ which is a contradiction and proves the claim.

  Now we define the following sets. First we set $I_0 = \mathbb{N}$ and inductively for $k \geqslant 0$

$$\tilde{I}_{k+1} = \{n \in I_k : \mu(B_n) < \frac{\epsilon}{C \cdot 2^{k+2}}\}, n_{k+1} = \min \tilde{I}_{k+1}$$

$$I_{k+1} = \{l \in \tilde{I}_{k+1} : \mu_{n_{k+1}}(A_{n_{k+1}}^l) < \frac{\epsilon}{C \cdot 2^{k+2}}\}$$

$$F_{n_{k+1}} = \{t \in E_{n_{k+1}}'' : \hat{t} \cap (\cup_{i=1}^k E_{n_i}'') = \emptyset\} = \{t \in E_{n_{k+1}}'' : t \perp E_i'' \text{ for } i \leqslant n_k\}.$$

Where by s_t we denote the unique initial segment that contains t. By the previous claim the sets $\tilde{I}_{k+1}, I_{k+1}$ are infinite and since $n_{k+1} \in I_k$ we have that $\mu_{n_{k+1}}(A_{n_k}^{n_{k+1}}) < \frac{\epsilon}{2^{k+1}}$. The set $I_\infty = \{n_1, n_2, ...\}$ is infinite and we observe that $\{F_n\}_{n \in I_\infty}$ are incomparable by definition and are also segment complete. Recall also that for every $x \in K$ it holds $\|x\| \leqslant C$. Now let $k \in \mathbb{N}$ then it remains to show that $\|P_{(E_{n_k} \setminus F_{n_k})} w_{n_k}\| < \epsilon$. For $k > 1$ we set $r = n_{k+1}$. We consider the set $A_r = A_{n_1}^r \cup ... \cup A_{n_k}^r$. Then $\mu_r(A_r) \leqslant \sum_{i=2}^{k+2} \frac{\epsilon}{C \cdot 2^{k+2}} < \frac{\epsilon}{4C}$. Let $s \notin A_r$ then $\forall t \in s$ and $i = 1, ..., k$ we have $s_t \cap E_{n_i}'' = \emptyset$ and thus $s \subseteq F_r$ and if $s \in A_r$ then $s \cap F_r = \emptyset$. So

$$\|P_{(E_r'' \setminus F_r)} w_r\| \leqslant \sum_{s \in L_r \cap A_r} |\lambda_s| \cdot \|P_{(E_r'' \setminus F_r)} x_s\| \leqslant \mu_r(A_r) < \frac{\epsilon}{2}.$$

Thus

$$\|P_{(E_r \setminus F_r)} w_r\| \leqslant \|P_{E_r'} w_r\| + \|P_{(E_r'' \setminus F_r)} w_r\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

  PROPOSITION 4.31. — *Let $(z_n)_n$ be a normalized level-block sequence in $\mathfrak{X}_\xi$ such that the unit ball B_Z of the subspace $Z = \overline{< z_n : n \in \mathbb{N} >}^{\|\cdot\|}$ is almost absorbed by W_ξ. Then every normalized block sequence in Z has a*

subsequence which is a ℓ_ξ^1 spreading model. Moreover, the identity operator $I : Z \mapsto X_T$ is strictly singular.

Proof. — Let $(y_n)_n$ be a normalized level-block sequence in Z. By our hypothesis there exists a $\lambda > 0$ such that $B_Z \subseteq \lambda W_\xi + \frac{1}{8} B_{\mathfrak{X}_\xi}$. Thus for every $k \in \mathbb{N}$ there exists $w_k \in W_\xi^0$ such that $\|y_k - \lambda w_k\| < \frac{1}{8}$. By Proposition 4.30 there exists a $M \in [\mathbb{N}]$ and a sequence $(E_k)_{k \in M}$ of subsets of T such that the following hold:

  i. $\|w_k - P_{E_k} w_k\| < \frac{1}{8\lambda}$.
  ii. $(E_k)_k$ are pairwise incomparable and segment complete subsets of T, for all $k \in M$.

For the sake of simplicity of notation we assume that $M = \mathbb{N}$. Now if we set $w_k' = P_{E_k} w_k$ and $w_k'' = \lambda w_k'$, we have that $\|y_k - w_k''\| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. Thus $\|w_k''\|_{\mathfrak{X}_\xi} \geqslant \frac{3}{4}$. Remark 4.21 yields that $(w_k'')_k$ is a ℓ_ξ^1 spreading model. It is easy to see that this property is transferred to $(y_n)_n$ as well. In addition, as X_T does not contain ℓ_ξ^1 spreading models, it is immediate that $I : Z \mapsto X_T$ is strictly singular. $\qquad\square$

An immediate consequence of the preceding Proposition is the following.

COROLLARY 4.32. — *Let $(z_n)_n$ be a normalized level-block sequence in $\mathfrak{X}_\xi$ such that the unit ball B_Z of the subspace $Z = \overline{< z_n : n \in \mathbb{N} >}^{\|\cdot\|}$ is almost absorbed by W_ξ. Then every normalized block sequence in Z has a further block subsequence which satisfies an upper (T, ξ) estimate.*

Proof. — Proposition 4.31 yields that for every normalized block sequence $(y_n)_n$ in Z there exists a further normalized block subsequence $(x_n)_n$ of $(y_n)_n$ such that $\|x_n\|_{X_T} \to 0$. A direct application of Proposition 4.22 yields the result. $\qquad\square$

We are now ready to prove the main result of this section.

THEOREM 4.33. — *The set W_ξ is thin in $\mathfrak{X}_\xi$.*

Proof. — Suppose not. Then there exists a normalized block sequence $(y_n)_n$ in $\mathfrak{X}_\xi$ such that B_Y is almost absorbed by W_ξ, where by Y we denote $Y = \overline{< y_n : n \in \mathbb{N} >}^{\|\cdot\|}$. By Corollary 4.32 we can find a normalized block sequence $(z_n)_n$ in Y such that

$$\left\| \sum_{i=1}^{k} b_i z_n \right\| \leqslant 48 \left\| \sum_{i=1}^{k} b_i e_{p_n} \right\|_{T_\xi}$$

for every choice of scalars $(b_i)_{i=1}^k$ and $k \in \mathbb{N}$. Since by our hypothesis the unit ball of $Z = \overline{< z_n : n \in \mathbb{N} >}^{\|\cdot\|}$ is almost absorbed by W_ξ we can apply

the same arguments as in Proposition 4.31 to obtain a sequence $(z^*_{m_i})_i$ satisfying the following

    i. supp $z^*_{m_i} \subseteq \operatorname{ran} z_{m_i}$ for all $i \in \mathbb{N}$.

    ii. $(\operatorname{supp} z^*_{m_i})_n$ are pairwise incomparable, segment complete subsets of T.

    iii. $z^*_{m_i}(z_{m_i}) > \frac{1}{4}$ for all $i \in \mathbb{N}$.

    iv. $z^*_{m_i} \in \mathrm{G}_\xi$ for all $i \in \mathbb{N}$.

For the sake of simplicity we enumerate $(z_{m_i})_i$ as $(z_n)_n$. Define an operator $P : \mathfrak{X}_\xi \mapsto \overline{< z_n : n \in \mathbb{N} >}^{\|\cdot\|}$ by $P(x) = \sum_{n=1}^{\infty} z^*_n(x) z_n$. We will show first that P is bounded.

To see this, let $x \in B_{\mathfrak{X}_\xi}$. It is enough to prove that $\| \sum_n z^*_n(x) e_{p_n} \|_{T_\xi} \leqslant 1$. Indeed, Let f be a functional in the norming set of T_ξ and $(f_a)_{a \in A}$ a tree analysis of f. We can assume without loss of generality that supp $f \subseteq \{p_n : n \in \mathbb{N}\}$. Let $a \in \mathcal{A}$ be a $\sqsubseteq_{\mathcal{A}}$-maximal node. Then $f_a = \pm e_{p_k}$ for a $k \in \mathbb{N}$. Thus $|f_a(\sum_n z^*_n(x) e_{p_n})| = |z^*_k(x)| \leqslant 1$. We move on to recursively define for each $a \in \mathcal{A}$ a functional $g_a \in \mathrm{G}_\xi$ such that $f_a(\sum_n z^*_n(x) e_{p_n}) = g_a(x)$. If a is maximal and $f_a = \pm e^*_{p_k}$ we set $g_a = z^*_k$ or $g_a = -z^*_k$ respectively. We observe that if $\{\pm e^*_{p_{i_1}} < ... < \pm e^*_{p_{i_l}}\}$ is an $\mathcal{S}_\xi$-admissible sequence of functional in $B^*_{T_\xi}$ then $\{\pm z^*_{p_{i_1}} < ... < \pm z^*_{p_{i_l}}\}$ is an (T, ξ)-admissible family of functionals as well. Now suppose that $a \in \mathcal{A}$ is not maximal such that for every $\beta \in S_a$ we have defined a functional $g_\beta \in \mathrm{G}_\xi$ such that $(g_\beta)_{\beta \in S_a}$ are successive and their supports are pairwise incomparable subsets of T and each g_β satisfies $f_\beta(\sum_n z^*_n(x) e_{p_n}) = g_\beta(x)$ for all $\beta \in S_a$. Then the functional $g_a = \frac{1}{2} \sum_{\beta \in S_a} g_\beta$ is an element of G_ξ and

$$f_a \Big(\sum_n z^*_n(x) e_{p_n} \Big) = \frac{1}{2} \sum_{\beta \in S_a} f_\beta \Big(\sum_n z^*_n(x) e_{p_n} \Big) = \frac{1}{2} \sum g_\beta(x) = g_a(x).$$

Thus, by following the structure of the tree $\mathcal{A}$ we arrive at a functional $g \in \mathrm{G}_\xi$ with $g(x) = f(\sum_n z^*_n(x) e_{p_n})$ and this gives us

$$\Big\| \sum_n z^*_n(x) e_{p_n} \Big\|_{T_\xi} \leqslant \|x\| \leqslant 1.$$

Therefore, $\|P\| \leqslant C$.

Suppose now, that $\forall \epsilon > 0$, $\exists \lambda > 0$ such that

$$B_Y \subseteq \lambda W_\xi + \epsilon B_{\mathfrak{X}_\xi}.$$

Then it is clear that $B_Z \subseteq \lambda W_\xi + \epsilon B_{\mathfrak{X}_\xi}$ where $Z = \overline{< z_n : n \in \mathbb{N} >}^{\|\cdot\|}$. Since $|z^*_n(z_n)| > \frac{1}{4}$ we obtain that

$$\frac{1}{4} B_Z \subseteq P(B_Z)$$

and by setting $\epsilon = \frac{1}{8\|P\|}$ we have

$$P(B_Z) \subseteq \lambda P(W_\xi) + \frac{1}{8}B_Z.$$

Thus $B_Z \subseteq 8\lambda \overline{P(W_\xi)}^{\|.\|}$. Since the operator P is defined by the sequence $(z_n^*)_n$ and $(\operatorname{supp} z_n^*)_n$ are pairwise incomparable we have that $P(W_\xi) \subseteq \|P\|\overline{co}[(\pm z_n) : n \in \mathbb{N}]$ and finally

$$B_Z \subseteq 8\lambda\|P\|\overline{co}[(\pm z_n) : n \in \mathbb{N}].$$

Now since, as is well known, the basis of T_ξ is weakly null we can select a convex combination $x = \sum_{i=1}^n k_i e_{p_i}$ such that $\|x\|_{T_\xi} < \frac{C}{16\lambda\|P\|}$ and we obtain $\|\sum_{i=1}^n k_i z_i\|_{\mathfrak{X}_\xi} < \frac{1}{16\lambda\|P\|}$. We observe that if $z = \sum_{i=1}^n k_i z_i$ then $(16\lambda\|P\|)z \in B_Z \subseteq 8\lambda\|P\|\overline{co}[(\pm z_n) : n \in \mathbb{N}]$. We conclude that W_ξ does *not* almost absorb B_Y. This is a contradiction which yields the proof of the Theorem. $\qquad\square$

5. Classical Interpolation Spaces

We fix T, X_{T} with a $\mathcal{SC}$-unconditional basis $(e_t)_{t\in\mathrm{T}}$ and K as in the previous section. We also set $W = \overline{co}^{\|\cdot\|_{X_{\mathrm{T}}}}(\mathrm{K} \cup -\mathrm{K})$. In this section we use the classical Davies-Figiel-Johnson-Pelczynski iterpolation method [10] for the pair (X_{T}, W) to produce a new space $\mathfrak{X}_1$ in which the structure of the set K is preserved and study the properties of this new space. We begin by recalling the (DFJP)-interpolation method:

DEFINITION 5.1. — *Let* T, X_{T} *and* W *be as above. We set* $W_n = 2^n W + \frac{1}{2^n}B_{X_{\mathrm{T}}}$ *and define a sequence of equivalent norms* $(\|\cdot\|_n)_n$ *on* X_{T}, *each induced by the Minkowski gauge of the respective* W_n. *We consider* $(\sum_{n=1}^\infty \oplus(X_{\mathrm{T}}, \|\cdot\|_n))_2$ *to be the* ℓ_2-*Schauder sum of the spaces* $(X_{\mathrm{T}}, \|\cdot\|_n)_n$. *Finally, we set* $\mathfrak{X}_1$ *to be the diagonal space of this* ℓ_2-*Schauder sum. That is, the (closed) subspace consisting of all elements of the form* $\tilde{x} = (x, x, ...x, ...)$, *for* $x \in X_{\mathrm{T}}$. *We also denote by* J_1 *the* $1 - 1$ *bounded linear operator* $J_1 : \mathfrak{X}_1 \to X_{\mathrm{T}}$ *defined as* $J_1(\tilde{x}) = x$ *and by* $\tilde{\mathrm{K}} = \{\tilde{x} : x \in \mathrm{K}\}$.

Remark 5.2. — In [10] it was proved that if the set W is weakly compact then the space $\mathfrak{X}_1$ is reflexive. In addition, by its construction the space $\mathfrak{X}_1$ consists of all elements $\tilde{x} = (x, x, ..., x, ...)$ such that $\sum_{n=1}^\infty \|x\|_n^2 < \infty$ and $\|\tilde{x}\|_{\mathfrak{X}_1} = (\sum_{n=1}^\infty \|x\|_n^2)^{\frac{1}{2}}$. Therefore, we can observe that for $w \in W$ we have $\|w\|_n \leqslant \frac{1}{2^n}$ and thus $\|\tilde{w}\|_{\mathfrak{X}_1} \leqslant 1$. It follows that $\tilde{\mathrm{K}}$ is a closed subset of $\mathfrak{X}_1$ and $J_1(\tilde{\mathrm{K}}) = \mathrm{K}$.

We pass now to show that the space $\mathfrak{X}_1$ has a $\mathcal{SC}$-unconditional basis. We start with the following Lemma.

LEMMA 5.3. — *Let* $T_K = \{t \in T : y_t = \sum_{t' \sqsubseteq t} \psi(t')e_{t'} \in K\}$. *Then* T_K *is a backwards closed subtree of* T *and for every* $t \in T_K$ *we have that* $\tilde{e}_t = (e_t, e_t, ...) \in \mathfrak{X}_1$.

Proof. — It is easy to check that T_K is indeed a backwards closed subtree of T. Let now s be an initial segment of T (finite or infinite) such that the vector $y_s = \sum_{t \in s} \psi(t)e_t$ is an element of K. It follows by the definition of K that $\psi(t)e_t \in K$, $\forall t \in s$. Let $t_0 \in T_K$, then $\sum_{t \sqsubseteq t_0} \psi(t)e_t \in K$ and we obtain $\psi(t)e_t \in K$ $\forall t \sqsubseteq t_0$. So $a_{t_0}e_{t_0} \in K$ and thus $\tilde{e}_{t_0} \in \mathfrak{X}_1$. ☐

Remark 5.4. — We note that as T_K is a backwards closed subtree of T then for every $A \subset T_K$ segment complete we have that A is also segment complete when considered as a subset of T.

We fix a bijection $g : T_K \mapsto \mathbb{N}$ as in Definition 2.2 and we pass to show that the sequence $(\tilde{e}_t)_{t \in T_K}$ enumerated through g defines a bimonotone Schauder basis for $\mathfrak{X}_1$.

LEMMA 5.5. — *Let* $A \subseteq T$ *segment complete. If we denote by* $P_A : X_T \mapsto \overline{< e_t : t \in A >}^{\|.\|_{X_T}}$ *the natural projection induced by* A *we have that* $\|P_A(x)\|_n \leqslant \|x\|_n$ *for all* $x \in X_T$ *and* $n \in \mathbb{N}$.

Proof. — As X_T has a $\mathcal{SC}$-unconditional basis it follows that $P(B_{\mathfrak{X}_0}) \subseteq B_{\mathfrak{X}_0}$. At the same time for every $w \in W$ we have $\|P_A(w)\|_{X_T} \leqslant \|w\|_{X_T}$ and $P_A(w) \in W$. Thus, $P_A(W) \subseteq W$. Let now $n \in \mathbb{N}$ and $x \in X_T$. Let also $\lambda > 0$ such that $x \in \lambda(2^n W + \frac{1}{2^n} B_{X_T})$. All the above yield $P_A(x) \in \lambda(2^n P_a(W) + \frac{1}{2^n} P_A(B_{X_T}) \subseteq \lambda(2^n W + \frac{1}{2^n} B_{X_T})$. Thus $\|P_A(x)\|_n \leqslant \|x\|_n$ as desired. ☐

PROPOSITION 5.6. — *The sequence* $(\tilde{e}_t)_{t \in T_K}$ *is a* $\mathcal{SC}$-*unconditional Schauder basis for* $\mathfrak{X}_1$.

Proof. — By Remark 5.4 and the previous Lemma it readily follows that $(\tilde{e}_t)_{t \in T_K}$ is a $\mathcal{SC}$-unconditional Schauder basis for the subspace $E = \overline{< (\tilde{e}_t)_{t \in T_K} >}$. We shall show that E actually coincides with $\mathfrak{X}_1$. We need the following Claim:

CLAIM. — *For every element* $\tilde{x} = (\sum_{t \in T_K} \lambda_t e_t, \sum_{t \in T_K} \lambda_t e_t, ...) \in \mathfrak{X}_1$, *we have that* $\tilde{x} \in E$.

☐

Proof of claim. — Let $\tilde{x} = (\sum_{t \in T_K} \lambda_t e_t, \sum_{t \in T_K} \lambda_t e_t, ...)$ then $\|\tilde{x}\|_{\mathfrak{X}_1} = (\sum_n \|\sum_{t \in T_K} \lambda_t e_t\|_n^2)^{\frac{1}{2}}$. Let $\epsilon > 0$. There exists a $n_0 \in \mathbb{N}$ such that $(\sum_{n > n_0} \|\sum_{t \in T_K} \lambda_t e_t\|_n^2)^{\frac{1}{2}} \leqslant \frac{\epsilon}{2}$. By Lemma 5.5 and the fact that the spaces $(\mathfrak{X}_1, \|\cdot\|_n)_n$ are mutually isomorphic we can choose a finite interval I of T_K such that

$$\Big(\sum_{i=1}^{n_0} \Big\| \sum_{t \in T_K} \lambda_t e_t - \sum_{t \in I} \lambda_t e_t \Big\|_n^2 \Big)^{\frac{1}{2}} \leqslant \frac{\epsilon}{2}.$$

Thus, if we set $\tilde{x}' = \sum_{t \in I} \lambda_t \tilde{e}_t \in E$ we obtain $\|\tilde{x} - \tilde{x}'\|_{\mathfrak{X}_1} \leqslant \epsilon$.

This completes the proof of the claim.

Let now, $\tilde{x} \in \mathfrak{X}_1 \setminus E$. We know that $\tilde{x}$ is of the form $\tilde{x} = (\sum_{t \in \mathcal{T}} \lambda_t e_t, \sum_{t \in \mathcal{T}} \lambda_t e_t, ...)$. Since $x \notin E$ we have that there exists $t_0 \in \mathcal{T} \setminus T_K$ such that $\lambda_{t_0} \neq 0$. Let $w \in W$ and $n \in \mathbb{N}$. We have that w is of the form $w = \sum_{t \in T_K} \beta_t e_t$. Hence,

$$\Big\| \sum_{t \in \mathcal{T}} \lambda_t e_t - 2^n w \Big\|_{\mathfrak{X}_0} = \Big\| \sum_{t \in \mathcal{T}} \lambda_t e_t - 2^n \sum_{t \in T_K} \beta_t e_t \Big\|_{\mathfrak{X}_0} \geqslant |\lambda_{t_0}| > 0.$$

Thus, $\|\sum_{t \in \mathcal{T}} \lambda_t e_t\|_n \nrightarrow 0$ and consequently $\tilde{x} \notin \mathfrak{X}_1$. This is a contradiction completing the proof. $\qquad\square$

The following result is similar originates in [19] and for the sake of completeness we outline the main arguments of its proof.

PROPOSITION 5.7. — *If W is a thin subset of X_T then the operator J_1 is strictly singular and every infinite dimensional closed subspace Y of $\mathfrak{X}_1$ contains an isomorphic copy of ℓ^2 which is complemented in $\mathfrak{X}_1$.*

Proof. — In order to show that the operator J_1 is strictly singular we shall in fact prove something stronger, namely that $J_1(B_{\mathfrak{X}_1})$ is a thin subset of X_T. This is a direct consequence of the following:

The set $J_1(B_{\mathfrak{X}_1})$ is almost absorbed by W. To see this let $\epsilon > 0$. Fix $n_0 \in \mathbb{N}$ so that $\frac{1}{2^{n_0}} < \epsilon$ and pick an arbitrary $\tilde{x} = (x, x, ...) \in B_{\mathfrak{X}_1}$. Then, $\sum_{n \in \mathbb{N}} \|x\|_n^2 \leqslant 1$ which implies that $x \in 2^n W + \frac{1}{2^n} B_{X_T}$ for all $n \in \mathbb{N}$. Simply set $\lambda = 2^{n_0}$ and observe that $J_1(\tilde{x}) = x \in 2^{n_0} W + \frac{1}{2^{n_0}} B_{X_T} \subset \lambda W + \epsilon B_{X_T}$. Now as $J_1(B_{\mathfrak{X}_1})$ is almost absorbed by a thin subset it is straightforward that this set is also thin in X_T. Pick an arbitrary Y closed subspace of $\mathfrak{X}_1$. Since the operator J_1 is strictly singular one can apply a standard sliding hump argument to produce normalized sequences $(\tilde{y}_n)_n$ in Y and $(\tilde{z}_n)_n$ horizontally block in $\big(\sum_{n \in \mathbb{N}} \oplus (X_T, \|\cdot\|_n)\big)_2$ such that $\sum_{n=1}^{\infty} \|\tilde{z}_n - \tilde{y}_n\| < \frac{1}{2}$. As $(\tilde{z}_n)_n$ is isometric to the standard ℓ_2 basis the space $Z = \overline{\langle \tilde{z}_n : n \in \mathbb{N} \rangle}$ is 1-complemented in $\mathfrak{X}_1$, we conclude that the space generated by $(\tilde{y}_n)_n$ is isomorphic to ℓ_2 and complemented in $\mathfrak{X}_1$. $\qquad\square$

Remark 5.8. —   We note that under the obvious modifications the re-
sults presented in this section remain valid for DFJP ℓ_p interpolation.

6. Reflexive spaces as quotients of ℓ^p saturated spaces

At this point we are able to use the techniques developed in all the
previous sections in order to show that every separable reflexive Banach
space X is a quotient of a separable reflexive and ℓ^p-saturated space, for
every $p \geqslant 1$ and of a separable c_0-saturated space. This is done by using all
of the results obtained above in conjunction with the following well known
result of Zippin ([23]).

THEOREM 6.1. — *Let X be a separable reflexive Banach space. Then
there exists a reflexive Banach space Z_X with a Schauder basis $(z_i)_i$ so that
X is isomorphic to a subspace of Z.*

We pass now to show the main result of this section. We present the
arguments only in the case of $p = 2$ as for any $p \geqslant 1$ and c_0 the proof
follows exactly the same lines. Namely, we have

THEOREM 6.2. — *Let X be a separable reflexive Banach space. Then for
every $p > 1$ there exists a separable reflexive complementably ℓ^p-saturated
Banach space X_p so that X is isomorphic to a quotient space of X_p. Also
there exists a separable c_0-saturated space X_0 so that X is a quotient of
X_0.*

Proof. — Granting Zippin's theorem above we may assume that X has
a normalized and bimonotone Schauder basis $(x_i)_i$. Starting with X we
consider the space $\mathfrak{X}_0$ associated to X as it was presented in section 3. We
also consider the set K and the map $\Phi : \mathfrak{X}_0 \to X$ (see Definitions 3.6 and
3.7). By Proposition 3.9 we know that $\Phi(K)$ is a $\frac{1}{8}$-net in the unit ball of X
and hence Φ is onto. By Proposition 3.11 the set $W^0 = \overline{\mathrm{conv}}^{\|\cdot\|}(K \cup -K)$ is
a weakly compact subset of $\mathfrak{X}_0$. Therefore, the space $\mathfrak{X}_1$ as it was defined in
the previous section is a reflexive Banach space with a basis $(\tilde{e}_t)_{t \in \mathcal{T}_K}$ (see
Proposition 5.6). In addition, by Remark 5.2 we have $J_1(\tilde{K}) = K$. Hence,
the operator $\Phi \circ J_1 : \mathfrak{X}_1 \to X$ is onto. By using the extension technique of
section 4 on the space $\mathfrak{X}_1$ we arrive at a space $\mathfrak{X}_\xi$ with the properties that
$I : \mathfrak{X}_\xi \to \mathfrak{X}_1$ is continuous, $\tilde{K} \subset \mathfrak{X}_\xi$ and $W^\xi = \overline{\mathrm{conv}}^{\|\cdot\|}(\tilde{K} \cup -\tilde{K})$ is weakly
compact and thin. Finally, by applying the DFJP - interpolation to the the
space $\mathfrak{X}_\xi$ with the set W^ξ we arrive at the space X_2. The map $J_2 : X_2 \to \mathfrak{X}_\xi$
is continuous and preserves the set $\tilde{K}$. Therefore, there is a map $\Pi : X_2 \to X$

onto. To complete the proof, we point out that by Propositions 5.6 and 5.7 the space X_2 is separable reflexive and complementably ℓ^2-saturated. $\qquad\square$

Remark 6.3. — The above Theorem yields examples of pairs (X, X^*) of reflexive spaces with divergent structure. Namely, there exists spaces X_p as above such that the dual X_p^* contains HI subspaces.

7. Skew HI interpolation

In this section we present a method for applying HI interpolation to a pair (X, W) in order to achieve the diagonal space to have a Schauder basis. We start with a tree T, a reflexive space X_T which has a $\mathcal{SC}$-unconditional basis $(e_t)_{t\in T}$ and a weakly compact convex symmetric subset W. We denote by $(X_n)_n$ the sequence of (mutually isomorphic) spaces $(X_T, \|\cdot\|_n)$ where the n-th norm is defined via the Minkowski gauge of the set $2^n W + \frac{1}{2^n} B_{X_T}$ and prove the following:

THEOREM 7.1. — *Let $(X_n)_n$ be the above sequence. Then there exists a norm $\|\cdot\|_G$ defined on $c_{00}(T \times \mathbb{N})$ such that if we denote by $\mathfrak{X}_G$ the completion of $c_{00}(T \times \mathbb{N})$ under this norm the following hold:*

1. *The sequence $(X_n)_n$ is a Schauder decomposition of $\mathfrak{X}_G$.*
2. *Setting $Z_t = \overline{< (e_{(t,k)} : k \in \mathbb{N} >}^{\|\cdot\|_G}$, the sequence $(Z_t)_{t\in T}$ also defines a Schauder decomposition of $\mathfrak{X}_G$.*
3. *If for all segment complete $A \subset T$ for the natural projection P_A we have that $P_A(W) \subset W$ then the diagonal subspace $\mathfrak{X}$ of $\mathfrak{X}_G$ consisting of all elements of the form $\bar{x} = (x, x, ...)$ has a Schauder basis. Moreover, $\mathfrak{X}$ is reflexive.*
4. *If the set W is thin in X_T then the space $\mathfrak{X}$ is HI.*

Let (X_T, W) be as above and assume also that for every segment complete subset A of X_T, $P_A(W) \subset W$. Then by Lemma 5.5 we have that for every $n \in \mathbb{N}$, the sequence $(e_t)_{t\in T}$ is a $\mathcal{SC}$-unconditional Schauder basis for X_n. Thus setting:

$$G_n = \left\{ \sum_{t\in A} \lambda_t e_t^* : \lambda_t \in \mathbb{Q} \text{ and } \left\| \sum_{t\in T} \lambda_t e_t^* \right\|_{X_n^*} \leqslant 1 \right.$$

$$\left. \text{and } A \subseteq T \text{ segment complete} \right\},$$

we can readily verify that G_n is a norming set for the space (X_n) for all $\mathbb{N}$.

Notation 7.2. — We define the following

    i. We set $\pi : T \times \mathbb{N} \mapsto T$ by $\pi((t,k)) = t$ and $j : T \times \mathbb{N} \mapsto \mathbb{N}$ by $j((t,k)) = k$.

    ii. For $x \in c_{oo}(T \times \mathbb{N})$ we let $\operatorname{ran} x$ denote the minimal rectangle $I \times J$ that contains the support of x. Where by a rectangle $I \times J$ we mean the product of an interval I of T and an interval of $\mathbb{N}$.

    iii. Let $A, B \subseteq T \times \mathbb{N}$ we write $A \prec_\pi B$ if $\pi(A) < \pi(B)$ and $A \prec_j B$ if $j(A) < j(B)$ and $A \prec_{(\pi,j)} B$ if $\pi(A) < \pi(B)$ and $j(A) < j(B)$. With $A \prec_\pi^l B$ we denote the property $\pi(A) <^l \pi(B)$.

    iv. For $x, y \in c_{00}(T \times \mathbb{N})$ we write $x \prec_\pi y$ whenever $\operatorname{supp} x \prec_\pi \operatorname{supp} y$. The notations $x \prec_j y$ and $x \prec_{(\pi,j)} y$ have analogous meanings.

DEFINITION 7.3. — *A sequence $(x_n)_n$ in $c_{00}(T \times \mathbb{N})$ is said to be j-block (π-block or level-π-block) if $x_n \prec_j x_{n+1}$ ($x_n \prec_\pi x_{n+1}$ or $x_n \prec_\pi^l x_{n+1}$ respectively). The sequence $(x_n)_n$ is called diagonally block if $x_n \prec_{(\pi,j)} x_{n+1}$*

We fix two sequences of natural numbers $(m_l)_{l \in \mathbb{N}}$ and $(n_l)_{l \in \mathbb{N}}$ which are both recursively defined as follows. We set $m_1 = 2, m_{l+1} = m_l^5$ and $n_1 = 4, n_{l+1} = (5n_l)^{s_l}$ where $s_l = \log_2 m_{l+1}$.

DEFINITION 7.4. — *We consider a subset G of $c_{00}(T \times \mathbb{N})$ that is the minimal set such that the following hold.*

    i. *$\bigcup_n G_n \subseteq G$ and G is closed in the restriction on rectangles of the form $I \times J$ where I, J are intervals of T and $\mathbb{N}$ respectively.(i.e., for $f \in G$ and I, J intervals of, we have that $(I \times J) \cdot f = \chi_{I \times J} \cdot f \in G$).*

    ii. *For every $l \in \mathbb{N}$, G is closed in the $\left(\mathcal{A}_{n_{2l}}, \frac{1}{m_{2l}}\right)$-operations on j-block sequences. That is, if $f_1 \prec_j f_2 \prec_j \dots \prec_j f_{n_{2l}}$ and $f_i \in G$ for all $i = 1, \dots, n_{2l}$, then $\frac{1}{m_{2l}} \sum_{i=1}^{n_{2l}} f_i \in G$.*

    iii. *For every $l \in \mathbb{N}$, G is closed in the $\left(\mathcal{A}_{n_{2l-1}}, \frac{1}{m_{2l-1}}\right)$-operation on (n_{2l-1})-special sequences.*

    iv. *G is rationally convex.*

It remains to define the (n_{2l-1})-special sequences, defined through a coding σ. For every $l \in \mathbb{N}$ if $f \in G$ is the result of the $\left(\mathcal{A}_{n_l}, \frac{1}{m_l}\right)$-operation, then we let the weight $w(f)$ of f to be m_l. Notice that $w(f)$ is not uniquely defined.

The coding function σ. First we consider the subset of $c_{00}(T \times \mathbb{N})$ defined by

$$\mathcal{S} = \left\{ (\phi_1, \phi_2, \dots, \phi_d) \quad : \quad \phi_1 \prec_j \phi_2 \prec_j \dots \prec_j \phi_d \text{ and } \phi_i(t,k) \in \mathbb{Q} \text{ for every} \right.$$
$$\left. (t,k) \in T \times \mathbb{N} \text{ and every } i \in \{1, \dots, d\} \right\}.$$

We fix a pair Ω_1, Ω_2 of disjoint infinite subsets of $\mathbb{N}$. As $\mathcal{S}$ is countable, we are able to define an injection $\sigma : \mathcal{S} \to \{2l : l \in \Omega_2\}$ such that

$$m_{\sigma(\phi_1,...,\phi_d)} > \max \left\{ \frac{1}{|\phi_i(e_{(t,k)})|} : (t,k) \in \operatorname{supp} \phi_i \text{ and } i = 1, ..., d \right\}$$

$$\cdot \max\{k : (t,k) \in \operatorname{supp} \phi_d\}.$$

A finite sequence $(f_i)_{i=1}^{n_{2l-1}}$ is said to be a (n_{2l-1})-special sequence, provided that

(a) $(f_1, ..., f_{n_{2l-1}}) \in \mathcal{S}$ and $f_i \in G$ for every $i = 1, ..., n_{2l-1}$,

(b) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{2l-1}$ and $w(f_{i+1}) = m_{\sigma(f_1,...,f_i)}$ for every $1 \leqslant i < n_{2l-1}$.

Remark 7.5. — As we mentioned above, the weight $w(f)$ of a functional f, when it is defined, is not in general uniquely determined. However, if $f_1, ..., f_{n_{2l-1}}$ is a (n_{2l-1})-special sequence, then for all $i \geqslant 2$ by $w(f_i)$ we shall put $m_{\sigma(f_1,...,f_{i-1})}$.

Having defined the set G, we define

i. $\|x\|_G = \sup\{f(x) : f \in G\}$, for all $x \in c_{00}(T \times \mathbb{N})$.

ii. $\mathfrak{X}_G = \overline{< c_{00}(T \times \mathbb{N}), \| \cdot \|_G >}$.

Remark 7.6. — The following are easily established.

(1) For every $n \in \mathbb{N}$, the space $\overline{< (x_{t,n})_{t \in T} >}^{\| \cdot \|_G}$ is isometric to X_n.

(2) For every I, J intervals (finite or infinite) of T and $\mathbb{N}$ respectively, the projection

$$P_{I \times J} : \mathfrak{X}_G \to \mathfrak{X}_{I \times J} = \overline{< (x_{t,k})_{t \in I, k \in J} >}^{\| \cdot \|_G}$$

has norm one. Consequently we have,

(a) The sequence $(X_n)_n$ defines a Schauder decomposition of $\mathfrak{X}_G$.

(b) Setting $Z_t = \overline{< (x_{t,k})_{k \in \mathbb{N}} >}^{\| \cdot \|_G}$, the sequence $(Z_t)_t$ also defines a Schauder decomposition of $\mathfrak{X}_G$.

(3) Every j-block sequence and every π-block sequence is a bi-monotone Schauder basic sequence. Hence every diagonally block sequence is also a bi-monotone basic sequence.

Next we shall present the basic ingredients for the proof that certain block sequences in $\mathfrak{X}_G$ generate HI spaces.

DEFINITION 7.7. — Let $x \in c_{00}(T \times \mathbb{N})$ and $C > 1$. We say that x is a $C - \ell_k^1$ average if there exists a j-block sequence $x_1 \prec_j x_2 \prec_j ... \prec_j x_k$ such that $x = \frac{x_1 + ... + x_k}{k}$, $\|x_i\|_G \leqslant C$ for $i = 1, ..., k$ and $\|x\|_G = 1$.

DEFINITION 7.8 (RIS). — *A j-block sequence $(x_q)_q$ in $\mathfrak{X}_G$ is said to be a (C, ϵ) rapidly increasing sequence, if $\|x_q\|_G \leqslant C$, and there exists a strictly increasing sequence $(l_q)_q$ of natural numbers such that*

   i. $\frac{1}{m_{l_{q+1}}} |\operatorname{supp} x_q| < \epsilon$.

   ii. *For every $q = 1, 2, \dots$ and every $f \in G$ with $w(f) = m_i$, $i < l_q$ we have that $|f(x_q)| \leqslant \frac{C}{m_i}$.*

Notation. — We denote by D_{Au} the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties

   i. $\pm e_n \in Au$, for all $n \in \mathbb{N}$.

   ii. For every block sequence $f_1 < f_2 < \dots < f_{5n_l}$ in Au we have that $\frac{1}{m_l} \sum_{i=1}^{5n_l} f_i \in Au$.

   iii. Au is closed under restrictions of its elements on intervals.

We also denote by Au the completion of $c_{00}(\mathbb{N})$ under the norm induced by the norming set D_{Au}.

We state here a Lemma concerning the behavior of certain averages of the basis of Au. For the proof we refer to Lemma II.9 in [8].

LEMMA 7.9. — *Let $l_0 \in \mathbb{N}$ and $h \in D_{Au}$. Then for every $k_1 < \dots < k_{n_{l_0}}$ we have that*

   i. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} e_{k_j})| \leqslant \frac{2}{m_i \cdot m_{l_0}}$, *if $w(h) = m_i < m_{l_0}$.*

   ii. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} e_{k_j})| \leqslant \frac{1}{m_i}$, *if $w(h) = m_i \geqslant m_{l_0}$.*

If we additionally assume that the functional h admits a tree analysis $(h_a)_{a \in A}$ such that $w(h_a) \neq m_{l_0}$ for all $a \in A$, then we have that

   i. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} e_{k_j})| \leqslant \frac{2}{m_i \cdot m_{l_0}^2}$, *if $w(h) = m_i < m_{l_0}$.*

   ii. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} e_{k_j})| \leqslant \frac{1}{m_i}$, *if $w(h) = m_i \geqslant m_{l_0}$.*

PROPOSITION 7.10 (The basic inequality). — *Let $(x_q)_q$ be a j-block (C, ϵ) RIS and let also $(\lambda_q)_q$ be a sequence of scalars. Then for every $f \in G$ we can find g_1 such that either $g_1 = h_1$ or $g_1 = e_k^* + h_1$ with $k \notin \operatorname{supp} h_1$ where $h_1 \in D_{Au}$, $w(f) = w(h_1)$,$g_2 \in c_{00}(\mathrm{T})$ with $\|g_2\|_\infty \leqslant \epsilon$ and g_1, g_2 having nonnegative coordinates such that,*

$$\left| f\left(\sum \lambda_q x_q \right) \right| \leqslant C(g_1 + g_2)\left(\sum |\lambda_q| e_q \right).$$

If we additionally assume that there exists a $l_0 \in \mathbb{N}$ such that for every $\phi \in G$ with $w(\phi) = m_{l_0}$ and every interval E of the natural numbers,

$$\left| \phi\left(\sum_{q \in E} \lambda_q x_q \right) \right| \leqslant C\left(\max_{k \in E} |\lambda_q| + \epsilon \sum |\lambda_q| \right),$$

then we can choose h_1 to have a tree analysis $(h_a)_{a \in A}$ such that $w(h_a) \neq m_{l_0}$, for all $a \in A$.

We refer the reader to Lemma II.14 of [8] for a proof of the above proposition. A direct consequence of the basic inequality and Lemma 7.9 is the following.

LEMMA 7.11. — *Let* $(x_q)_{q=1}^{n_{l_0}}$ *be a* j-*block* (C, ϵ) *RIS with* $\epsilon \leqslant \frac{2}{m_{l_0}^2}$. *Then*

1. *For every* $f \in G$ *with* $w(f) = m_i$ *we have,*
    i. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} x_j)| \leqslant \frac{3C}{m_i \cdot m_{l_0}}$, *if* $w(h) = m_i < m_{l_0}$.
    ii. $|h(\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}})| \leqslant \frac{C}{n_{l_0}} + \frac{C}{m_i} + C\epsilon$, *if* $w(h) = m_i \geqslant m_{l_0}$.
    In particular, $\|\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} x_j\| \leqslant \frac{2C}{m_{l_0}}$.
2. *If* $(b_q)_{q=1}^{n_{l_0}}$ *are scalars with* $|b_q| \leqslant 1$ *for all* q *such that for every* $\phi \in G$ *with* $w(\phi) = m_{l_0}$ *and every interval* E *of the natural numbers we have that,*

$$\left| \phi\Big(\sum_{q \in E} \lambda_q x_q \Big) \right| \leqslant C\Big(\max_{k \in E} |\lambda_k| + \epsilon \sum |\lambda_q| \Big).$$

Then, $\|\frac{1}{n_{l_0}} \sum_{j=1}^{n_{l_0}} b_j x_j\| \leqslant \frac{4C}{m_{l_0}^2}$.

DEFINITION 7.12. — *Let* $x \in c_{00}(T \times \mathbb{N})$ *and* $C > 1$. *We say that* x *is a* $C - \ell_k^1$ *average if there exists a* j-*block sequence* $x_1 \prec_j x_2 \prec_j ... \prec_j x_k$ *such that* $x = \frac{x_1 + ... + x_k}{k}$, $\|x_i\|_G \leqslant C$ *for* $i = 1, ..., k$ *and* $\|x\|_G = 1$.

LEMMA 7.13. — *Let* x *be a* $C - \ell_{n_q}^1$ *average. Then for every* $f \in G$ *with* $w(f) = m_k < m_q$ *we have that* $|f(x)| \leqslant \frac{1}{m_k} C(1 + \frac{2n_{l-1}}{n_l}) \leqslant \frac{3C}{2} \frac{1}{m_k}$.

Proof. — Let $x = \frac{1}{n_q} \sum_{i=1}^{n_q} x_i$ be a $C - \ell_{n_l}^1$ average. Let also $f = \frac{1}{m_k} \sum_{i=1}^{n_k} f_i$ with $(f_i)_{i=1}^{n_k}$ a j-block sequence of functionals and $n_k < n_q$. If we set $E_i = j(\mathrm{ran}\, f_i)$ and for $l = 1, ..., n_k$ let I_l (J_l resp.) be the set of all i such that $j(\mathrm{supp}\, x_i)$ is contained (resp. intersects) E_l. Clearly $\sum_{l=1}^{n_k} |I_l| \leqslant n_q$, while for each l we have $\|E_l x\| \leqslant \frac{1}{n_q} \sum_{i \in J_l} \|E_l x_i\| \leqslant \frac{1}{n_q} C(|I_l| + 2)$. Therefore $\sum_{l=1}^{n_k} \|E_l x\| \leqslant C \frac{1}{n_q} (\sum_{l=1}^{n_k} |I_l| + 2n_k) \leqslant C(1 + \frac{2n_k}{n_q})$ and the conclusion follows.   □

LEMMA 7.14. — *Let* $(x_q)_q$ *be a* j-*block sequence in* $\mathfrak{X}_G$ *such that each* x_q *is a* $C - \ell_{k_q}^1$ *average, where* $C > 1$ *and* k_q *increasing to infinity and* $\epsilon > 0$. *Then there exists a subsequence of* $(x_q)_q$ *which is a* $(\frac{3C}{2}, \epsilon)$ *RIS.*

Proof. — For each q we set $l_q = \max\{l : n_l \leqslant n_q\}$. There exists a subsequence of $(x_q)_q$ (we denote this subsequence by $(x_q)_q$ again) such that $(l_q)_q$ is a strictly increasing sequence and $m_{l_{q+1}} > \frac{1}{\epsilon} |\mathrm{supp}\, x_q|$ for all q. From

Lemma 7.13 we also get that for each $f \in G$ with $w(f) = m_k, k < l_q$ we have that $|f(x_q)| \leqslant \frac{3C}{2} \frac{1}{m_k}$. Therefore this subsequence is a $(\frac{3C}{2}, \epsilon)$ RIS.   □

LEMMA 7.15. — Let $(x_q)_q$ be a j-block sequence with each x_q a $C - \ell^1_{k_q}$ average, where $C > 1$ and k_q increasing to infinity. Then for every $l \in \mathbb{N}$ there exists $q_1 < q_2 < ... < q_{n_{2l}}$ such that,

$$\left\| \frac{x_{q_1} + x_{q_2} + ... + x_{q_{n_{2l}}}}{n_{2l}} \right\| \leqslant \frac{3C}{m_{2l}}.$$

This is a direct consequence of the basic inequality (Proposition 7.10). The following holds.

LEMMA 7.16. — For every j-block sequence $(y_n)_n$ and every $k \in \mathbb{N}$, there exists a $2 - \ell^1_k$ average in $< (y_n)_n >$.

For the proof we refer to [8], Lemma II.22. Combining Lemma 7.16 and 7.14 we arrive at the following.

LEMMA 7.17. — For every j-block sequence $(y_n)_n$ in $\mathfrak{X}_G$ and for every $\epsilon > 0$ there exists a $(3, \epsilon)$ RIS in $< (y_n)_n >$.

DEFINITION 7.18 (exact pair). — A pair (x, ϕ) with $x \in c_{00}(T \times \mathbb{N})$ and $\phi \in G$ is said to be a (C, l) exact pair if the following conditions are satisfied.

(1) $1 \leqslant \|x\|_G \leqslant C$ and for every $f \in G$ with $w(f) = m_q$ and $q \neq l$ we have that $|f(x)| \leqslant \frac{3C}{m_q}$ if $q < l$ while $|f(x)| \leqslant \frac{C}{m_l^2}$ if $q > l$.
(2) ϕ is the result of the $(\mathcal{A}_{n_l}, \frac{1}{m_l})$-operation and so $w(\phi) = m_l$.
(3) $\phi(x) = 1$ and $\operatorname{ran} x = \operatorname{ran} \phi$ (we recall that for $c_{00}(T \times \mathbb{N})$, the range of x is the minimal rectangle generated by intervals that contains $\operatorname{supp} x$).

The following proposition is a direct consequence of Lemmas 7.16 and 7.15.

PROPOSITION 7.19. — If $(x_q)_q$ is a j-block sequence, then for every $l \in \mathbb{N}$ there exists an $(6, 2l)$ exact pair (x, ϕ) with $x \in< (x_q)_q >$ and $\phi \in G$.

Proof. — From Lemma 7.17 we have that there exists $(y_q)_{q=1}^{n_{2l}}$ a $(3, \epsilon)$ RIS in $< (x_q)_q >$ with $\epsilon \leqslant \frac{1}{m_{2l}^3}$. Choose for each $q = 1, ..., n_{2l}$ a $y_q^* \in G$ with $y_q^*(y_q) = 1$ and $\operatorname{ran} y_q^* \subseteq \operatorname{ran} y_q$. Then the functionals $(y_q^*)_q$ form a j-block sequence and the functional $y^* = \frac{1}{m_{2l}} \sum_q y_q^*$ is an element of G and if we set $y = \frac{m_{2l}}{n_{2l}} \sum_{q=1}^{n_{2l}} y_q$ by Proposition 7.10 we get that (y, y^*) is the desired pair.   □

PROPOSITION 7.20 (The tree like property of special sequences). — *Let* $(\phi_i)_{i=1}^{n_{2l-1}}$, $(\psi_i)_{i=1}^{n_{2l-1}}$ *be two distinct special sequences in* G. *Then*

   i. *For* $1 \leqslant i < j \leqslant n_{2l-1}$ *we have that* $w(\phi_i) \neq w(\psi_j)$.
   ii. *There exists* k *such that* $\phi_i = \psi_i$ *for all* $i < k$ *and* $w(\phi_i) \neq w(\psi_i)$ *for* $i > k$.

The proof can be readily deduced from the definition of special sequences. For what follows we restrict ourselves to a specific form of j-block sequences. Namely,

DEFINITION 7.21. — *We say that a* j-*block sequence* $(x_n)_n$ *is special* j-*block if either* $(x_n)_n$ *is diagonally block or there exists some* $t \in T$ *such that* $\operatorname{supp} x_n \subseteq \{t\} \times \mathbb{N}$ *for every* $n \in \mathbb{N}$.

DEFINITION 7.22 (dependent sequences). — *A double sequence* $(x_k, \phi_k)_{k=1}^{n_{2l-1}}$ *where* $(x_k)_{k=1}^{n_{2l-1}}$ *is a special* j-*block sequence and* $\phi_k \in G$ *for every* $k = 1, ..., n_{2l-1}$, *is said to be a* $(C, 2l-1)$ *dependent sequence if there exists a sequence* $(2l_k)_{k=1}^{n_{2l-1}}$ *of even integers such that the following conditions are fulfilled.*

   (i) $(\phi_k)_{k=1}^{n_{2l-1}}$ *is a* (n_{2l-1})-*special sequence with* $w(\phi_k) = m_{2l_k}$ *for all* $k = 1, ..., n_{2l-1}$.
   (ii) *Each* (x_k, ϕ_k) *is a* $(C, 2l_k)$ *exact pair.*

Remark 7.23. — It is clear that the existence of dependent sequences in certain subspaces of $\mathfrak{X}_G$ is the main tool for proving the HI property of these subspaces. In the sequel we shall present the precise statement. Here we want to comment the use of the special j-block sequences in the definition of dependent sequences. A key ingredient for showing the second inequality in the following Proposition is the tree-like property satisfied by the (n_{2l-1})-special sequences (Proposition 7.20). Nevertheless, when we deal with norms on $c_{00}(\mathbb{N})$ then the tree-like property is also satisfied by all restrictions of the special sequences on intervals of $\mathbb{N}$ (see [8], Proposition 3.3). However this is not valid when we deal with $c_{00}(T \times \mathbb{N})$ and we consider restrictions on rectangles generated by intervals of T and $\mathbb{N}$. Notice that this problem disappears if we consider special j-block sequences and this is the reason why we introduced this concept.

PROPOSITION 7.24. — *Let* $(x_k, x_k^*)_{k=1}^{n_{2l-1}}$ *be a* $(C, 2l-1)$ *dependent sequence. Then*

(7.1)
$$\left\| \frac{1}{n_{2l-1}} \sum_{k=1}^{n_{2l-1}} x_k \right\| \geqslant \frac{1}{m_{2l-1}}$$

$$(7.2) \qquad \left\| \frac{1}{n_{2l-1}} \sum_{k=1}^{n_{2l-1}} (-1)^k x_k \right\| \leqslant \frac{8C}{m_{2l-1}^2}.$$

Proof. — (1) It can be readily seen that the special functional $f = \frac{1}{m_{2l-1}} \sum_{k=1}^{n_{2l-1}} x_k^*$ belongs to G thus $f(x_k) \geqslant \frac{1}{m_{2l-1}}$.

(2) First of all it is easy to check that the sequence $(x_k)_{k=1}^{n_{2l-1}}$ is a $(2C, \frac{1}{n_{2l-1}^2})$ RIS. The inequality follows from Proposition 7.10 after showing that for every $f \in G$ with $w(f) = m_{2l-1}$ and every interval E we have that,

$$\left| f\left(\sum_{k \in E} (-1)^{k+1} x_k \right) \right| \leqslant 2C\left(1 + \frac{2}{m_{2l-1}^2} |E| \right).$$

To see this choose $f \in G$ with $w(f) = m_{2l-1}$ and observe that such an f must have the following form: $f = \frac{1}{m_{2l-1}}(Fx_{t-1}^* + x_t^* + ... + f_{r+1} + ... + f_d)$, for some special sequence $(x_1^*, x_2^*, ..., x_r^*, f_{r+1}, ..., f_{n_{2l-1}})$ of length n_{2l-1} with $x_{r+1}^* \neq f_{r+1}$, $w(x_{r+1}^*) = w(f_{r+1})$ and F an interval of the form $[m, \max \operatorname{supp} x_{t-1}^*]$. This representation is a direct consequence of the the tree-like property discussed thoroughly above. We estimate the quantity $f(x_k)$ for each k as follows.

1. If $k < t-1$ then $f(x_k) = 0$.
2. If $k = t-1$ we get $|f(x_{t-1})| = \frac{1}{m_{2l-1}} |Fx_{t-1}^*(x_{t-1})| \leqslant \frac{1}{m_{2l-1}} \|x_{t-1}\| \leqslant \frac{C}{m_{2l-1}}$.
3. If $k > r+1$ Proposition 7.20 yields that $w(f_i) \neq m_{2l_k}$, for all $i > r$. Using the fact that (x_k, x_k^*) is an exact pair and taking into account that $n_{2l-1}^2 < m_{2l_1} \leqslant m_{2l_k}$ we proceed in the following manner:

$$\begin{aligned}
|f(x_k)| &= \frac{1}{m_{2l-1}} |(f_r + ... + f_d)(x_k)| \\
&\leqslant \frac{1}{m_{2l-1}} \left(\sum_{w(f_i)<m_{2l_k}} |f_i(x_k)| + \sum_{w(f_i)>m_{2l_k}} |f_i(x_k)| \right. \\
&\qquad \left. + \sum_{2r+2 \leqslant 2i \leqslant d} |f_{2i}(x_{2k-1})| \right) \\
&\leqslant \frac{1}{m_{2l-1}} \left(\sum_{2l-1<j<2l_k} \frac{3C}{m_j} + n_{2l-1} \frac{C}{m_{2l_k}^2} \right) \leqslant \frac{C}{m_{2l-1}^2}.
\end{aligned}$$

4. For $k = r+1$ the same argument as in the previous case yields that $|f(x_{r+1})| \leqslant \frac{C}{m_{2l-1}} + \frac{1}{m_{2l-1}^2} < \frac{C+1}{m_{2l-1}}$.

Let E be an interval. From the above estimates we obtain,

$$\left| f\Big(\sum_{k \in E} (-1)^{k+1} x_k \Big) \right| \leqslant \left| f(x_{t-1}) \right| + \left| \sum_{k \in E \cap [t,r]} \frac{1}{m_{2l-1}} (-1)^{k+1} \right|$$

$$+ \left| f(x_{r+1}) \right| + \left| \sum_{k \in E \cap [r+2, n_{2l-1}]} f(x_k) \right|$$

$$\leqslant \frac{C}{m_{2l-1}} + \frac{1}{m_{2l-1}} + \frac{C+1}{m_{2l-1}} + \frac{C}{m_{2l-1}^2} |E|$$

$$< 2C \Big(1 + \frac{2}{m_{2l-1}^2} |E| \Big),$$

completing the proof. $\qquad\square$

The following is an easy consequence of the previous results.

PROPOSITION 7.25. — Let $(x_n)_n$, $(y_n)_n$ be two diagonally block sequences. Then for every $n \in \mathbb{N}$ there exists a $(6, 2l-1)$ dependent sequence $(z_k, \phi_k)_{k=1}^{n_{2l-1}}$ such that $z_{2k-1} \in < (x_n)_n >$ and $z_{2k} \in < (y_n)_n >$. Similar results hold if $(x_n)_n$ and $(y_n)_n$ are j-block sequences in the space Z_t for some $t \in T$.

We need the following.

PROPOSITION 7.26. — Let Y be a subspace of $\mathfrak{X}_G$. Then one of the following hold.
  (a) There exists $n \in \mathbb{N}$ such that $j_n : Y \to X_n$ is not strictly singular.
  (b) There exists $t \in T$ such that $\pi_t : Y \to Z_t$ is not strictly singular.
  (c) For every $r > 0$, there exists a normalized sequence $(y_n)_n$ in Y and a diagonally block sequence $(w_n)_n$ such that $\sum_{n \in \mathbb{N}} \| y_n - w_n \| < r$.

Proof. — Assume that neither (a) nor (b) hold. Then for every $n \in \mathbb{N}$, there exists a subspace Y' of Y such that the map $j_{\{1,\ldots,n\}} : Y' \to \sum_{i=1}^n \oplus X_n$ is also strictly singular. The same also holds for the projections $\pi_{\{t_1,\ldots,t_m\}}$. Hence for every $\varepsilon > 0$ and every $(t, m) \in T \times \mathbb{N}$ there exists a subspace Y' of Y such that $\| j_{\{1,\ldots,n\}}|_{Y'} \| < \varepsilon$ and $\| \pi_{\{t_1,\ldots,t_m\}}|_{Y'} \| < \varepsilon$. Using this and a standard sliding hump argument, we can verify that the third alternative is satisfied. $\qquad\square$

Propositions 7.25 and 7.26 yield the next result.

COROLLARY 7.27. — The following are satisfied.
  (a) For every $t \in T$ the space Z_t is HI.
  (b) For each diagonally block sequence $(y_n)_n$ the space $Y = \overline{< (y_n)_n >}$ is HI.

(c) *If Y is a subspace of $\mathfrak{X}_G$ such that $j_n : Y \to X_n$ and $\pi_t : Y \to Z_t$
are strictly singular for $(t, k) \in T \times \mathbb{N}$, then Y is HI.*

Proof. — Parts (a) and (b) are direct consequences of Proposition 7.25.
To see (c), let Y be a subspace of $\mathfrak{X}_G$ such that $j_n : Y \to X_n$ and $\pi_t :
Y \to Z_t$ are strictly singular for every $(t, k) \in T \times \mathbb{N}$. Let Y_1 and Y_2 be
subspaces of Y and $\varepsilon > 0$. By Proposition 7.26, there exist normalized
block sequences $(y_n^1)_n$, $(y_n^2)_n$ and a diagonally block sequence $(w_n)_n$ such
that the following are satisfied.

(1) For every $n \in \mathbb{N}$, $y_n^1 \in Y_1$ and $y_n^2 \in Y_2$.
(2) $\sum_{n \in \mathbb{N}} \|w_{2n-1} - y_n^1\| < \varepsilon$ and $\sum_{n \in \mathbb{N}} \|w_{2n} - y_n^2\| < \varepsilon$.

The space $W = \overline{< (w_n)_n >}$ is HI by part (b). As ε can be chosen arbitrarily
small, this shows that $d(S_{Y_1}, S_{Y_2}) = 0$. As Y_1 and Y_2 are arbitrary subspaces
of Y we get that Y is HI. □

PROPOSITION 7.28. — *We have that $\mathfrak{X}_G^* = \overline{< \bigcup_{n \in \mathbb{N}} X_n^* >}^{\|\cdot\|}$.*

Proof. — Assume not. Then there exist $x^{**} \in \mathfrak{X}_G^{**}$ and $x^* \in B_{\mathfrak{X}_G^*}$ such
that $\|x^{**}\| = 1$, $x^{**}(x^*) > 1/2$ and $\bigcup_n X_n^* \subseteq \ker x^{**}$. Choose a net $(x_i)_{i \in I}$
in $B_{\mathfrak{X}_G}$ with $x_i \xrightarrow{w^*} x^{**}$. Clearly we may assume that

$$(7.3) \qquad\qquad x^*(x_i) > \frac{1}{2} \text{ for every } i \in I.$$

Observe that $j_{\{1,\ldots,n\}}(x_i) \xrightarrow{w} 0$. Hence applying Mazur's Theorem and a
sliding hump argument, we may select two sequences $(y_n)_n$ and $(z_n)_n$ such
that the following are satisfied.

(i) For every $n \in \mathbb{N}$, $y_n \in \text{conv}\{x_i : i \in I\}$.
(ii) $(z_n)_n$ is a j-block sequence.
(iii) $\sum_n \|y_n - z_n\| < \frac{1}{8}$.

Notice that for every $n_1 < n_2 < \ldots < n_k$ we have

$$(7.4) \qquad\qquad \left\| \frac{z_{n_1} + z_{n_2} + \ldots + z_{n_k}}{k} \right\| \geqslant \frac{1}{4}.$$

Indeed, by (i) and (7.3) above we have that $x^*(y_n) > 1/2$ for every $n \in \mathbb{N}$.
Hence by (iii) we get that $x^*(z_n) > 1/4$ for every $n \in \mathbb{N}$, which clearly
implies (7.4). Hence we may select a j-block sequence $(w_k)_k$ with $w_k =
\frac{1}{k} \sum_{n \in F_k} z_k$ where $F_1 < F_2 < \ldots < F_k < \ldots$ and each F_k is a finite inter-
val of $\mathbb{N}$. As the sequence $(w_k)_k$ is a j-block sequence of $4 - \ell_k^1$ averages,
Lemma 7.15 yields that for every $l \in \mathbb{N}$ there exists $k_1 < k_2 < \ldots < k_{n_{2l}}$
with

$$(7.5) \qquad\qquad \left\| \frac{1}{n_{2l}} \sum_{i=1}^{n_{2l}} w_{k_i} \right\| \leqslant \frac{12}{m_{2l}}.$$

Let $v_l = \frac{1}{n_{2l}} \sum_{i=1}^{n_{2l}} w_{k_i}$. Then v_l is a convex combination of z_k's. Let v'_l be the corresponding convex combination of y_n's. Then by (i) and (7.3) we have $\|v'_l\| > 1/2$. By (iii), we get that $\|v_l - v'_l\| < 1/8$. On the other hand, as $m_l \to \infty$ as $l \to \infty$, by (7.5) we see that $\|v_l\| \to 0$ and this leads to a contradiction. The proof is completed. $\qquad\square$

DEFINITION 7.29. — *The HI interpolation space $\mathfrak{X}$ is the (closed) subspace of $\mathfrak{X}_G$ which contains all elements of $\mathfrak{X}_G$ of the form $(x, x, ...)$.*

Remark 7.30. — This definition is an adaptation of the corresponding definition in [4], which in turn follows the scheme of the classical Davis-Fiegel-Johnson-Pelczynski interpolation method [10].

PROPOSITION 7.31. — *For every $t \in T$ we set $\bar{e}_t = (e_t, e_t, ...) \in \mathfrak{X}$. Then $(\bar{e}_t)_t$ becomes a bi-monotone Schauder basis of $\mathfrak{X}$.*

Proof. — First we notice that for every $n \in \mathbb{N}$ we have that $\|a_t e_t\|_n \leqslant \frac{1}{2^n}$, for all $t \in T$ Hence $\bar{e}_t \in \mathfrak{X}$ for every $t \in T$. Now let $\bar{x} = (x, x, ...) \in \mathfrak{X}$ with $x = \sum_t b_t e_t$. We consider the projection $\pi_t : \mathfrak{X} \to Z_t$. We shall show that $\pi_t(\bar{x}) = b_t \bar{e}_t$. Indeed observe that $\{e_{t,k} : k \in \mathbb{N}\}$ is a Schauder basis for Z_t (not normalized) and $e_{t,k}^*\big(\pi_t(\bar{x})\big) = e_{t,k}^*(\bar{x}) = b_t$ for every $k \in \mathbb{N}$. Hence $\pi_t(\bar{x}) = \sum_{k \in \mathbb{N}} b_t e_{t,k} = b_t \bar{e}_t$. This easily yields that for every finite interval I of T we have $\pi_I(\mathfrak{X}) = < \{\bar{e}_t : t \in I\} >$ and so $\pi_I(\bar{x}) = \sum_{t \in I} b_t \bar{e}_t$. The above argument and the fact that $\|\pi_I\| = 1$ yield that $(\bar{e}_t)_t$ is a bi-monotone Schauder basis for the space $Y = \overline{< (\bar{e})_t >}$. It remains to show that Y coincides with $\mathfrak{X}$. Indeed, let $(I_n)_n$ be the intervals of $\mathbb{N}$ such that $I_n = g^{-1}\big(\{1, ...n\}\big)$ where g is defined after Remark 5.4. Let also $\bar{x} = (x, x, ...)$ with $x = \sum_t b_t e_t$. We claim that the partial sums $\sum_{t \in I_n} b_t \bar{e}_t$ weakly converge to $\bar{x}$, which immediately implies the desired result. First we observe that $\sum_{t \in I_n} b_t \bar{e}_t = \pi_{I_n}(\bar{x})$ and so $\big\|\sum_{t \in I_n} b_t \bar{e}_t\big\| \leqslant \|\bar{x}\|$. Furthermore, for every $x^* \in \bigcup_{n \in \mathbb{N}} B_{X_n}$ we have that $x^*\big(\sum_{t \in I_n} b_t \bar{e}_t\big) \to x^*(\bar{x})$. Proposition 7.28 yields that $< \bigcup_{n \in \mathbb{N}} B_{X_n^*} >$ is norm dense in $\mathfrak{X}_G^*$ and this proves the claim and the entire proof is completed. $\qquad\square$

Notation 7.32. — In the sequel we shall denote by $J_{X_T} : \mathfrak{X} \to X_T$ the 1-1, bounded linear map defined by $J_{X_T}(\bar{x}) = x$, where $\bar{x} = (x, x, ...)$.

PROPOSITION 7.33. — *Let $\mathfrak{X}$ be the HI interpolation space. Then the following hold:*

    (a) *If Y is a closed subspace of $\mathfrak{X}$ such that $J_{X_T} : Y \to X_T$ is strictly singular, then Y is a HI space.*

    (b) *If Y, Z are closed subspaces of $\mathfrak{X}$ such that $J_{X_T}|_Y$ and $J_{X_T}|_Z$ are strictly singular, then $d(S_Y, S_Z) = 0$.*

Proof. — (a) We observe, following the notation of the previous section, that for every $t \in T$ the map $\pi_t : \mathfrak{X} \to Z_t$ has dimension 1,since as shown in the proof of Proposition 7.31 $\pi_t(\mathfrak{X}) =< \bar{e}_t >$ and so π_t is strictly singular. Notice also that for every $\bar{x} \in \mathfrak{X}$ and every $n \in \mathbb{N}$ we have that $j_n(\bar{x}) = J_{X_T}(\bar{x})$. As every X_n is isomorphic to X_T, we get that $j_n|_Y$ is also strictly singular. Corollary 7.27(c) yields the result.

(b) We notice that, as in part (a), for every $t \in T$ the maps $\pi_t|_Y$ and $\pi_t|_Z$ are strictly singular. Moreover, by our assumptions, for every $n \in \mathbb{N}$ the maps $j_n|_Y$ and $j_n|_Z$ are also strictly singular. Let $\varepsilon > 0$ arbitrary. Arguing as in Corollary 7.27(c) we are able to construct two normalized sequences $(y_n)_n$ and $(z_n)_n$ and a diagonally block sequence $(w_n)_n$ such that the following are satisfied.

 (i) For every $n \in \mathbb{N}$, $y_n \in Y$ and $z_n \in Z$.
 (ii) $\sum_n \|w_{2n-1} - y_n\| < \varepsilon$ and $\sum_n \|w_{2n} - z_n\| < \varepsilon$.

The space $W = \overline{< (w_n)_n >}$ is HI by Corollary 7.27(c). Hence, if we set $W_1 = \overline{< (w_{2n-1})_n >}$ and $W_2 = \overline{< (w_{2n})_n >}$ we see that $d(S_{W_1}, S_{W_2}) = 0$. As ε can be chosen arbitrarily small, by (ii) above, we conclude that $d(S_Y, S_Z) = 0$, as desired.   □

PROPOSITION 7.34. — *The space $\mathfrak{X}$ is reflexive.*

Proof. — We recall the following well-known facts. First if $T : X \to Y$ is a Tauberian operator, then $W \subseteq X$ is relatively weakly compact if and only if $T(W)$ is (see [19]). Moreover, by a classical result of A. Grothendieck [14], we have that a set $K \subseteq X$ is relatively weakly compact if for every $\varepsilon > 0$ there exists a weakly compact set $K_\varepsilon \subseteq X$ such that $K \subseteq K_\varepsilon + \varepsilon B_X$. As we have assumed the set W is weakly compact in X_T. It is easy to see that W almost absorbs $J_{X_T}(B_\mathfrak{X})$, i.e., for every $\varepsilon > 0$ there exists $\lambda > 0$ such that $J_{X_T}(B_\mathfrak{X}) \subseteq \lambda W + \varepsilon B_{X_T}$. Hence, by Grothendieck's criterion, $J_{X_T}(B_\mathfrak{X})$ is a relatively weakly compact subset of X_T. It is a well known fact that J_{X_T} is a Tauberian operator (c.f [4]). Hence $B_\mathfrak{X}$ is also a relatively weakly compact subset of $\mathfrak{X}$ and the proof is completed.   □

The last step in this section is to prove that if W is a thin subset of X_T then the space $\mathfrak{X}$ is HI. Let us recall the notion of thin operators.

DEFINITION 7.35. — *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. T is called a thin operator if $T(B_X)$ is a thin subset of Y.*

Remark 7.36. — It can be readily seen that if T is a thin operator, then it is also strictly singular.(*cf.* [4]).

Proposition 7.33 immediately yields that if W is a thin subset of X_T then the space $\mathfrak{X}$ is HI. Combining this with Propositions 7.34, 7.31 and Remark 7.6 we obtain the proof of Theorem 7.1 stated in the beginning of this section.

8. The final results

THEOREM 8.1. — *Every separable reflexive Banach space X is a quotient of a reflexive HI Banach space $\mathfrak{X}(X)$ with a bimonotone Schauder basis.*

Proof. — Let X be a separable reflexive Banach space. By Zippin's Theorem (Theorem 6.1) we obtain that X^* can be isomorphically embedded into a reflexive Banach space with a bimonotone Schauder basis $(x_n)_n$. We denote this space Z_X. Starting now with Z_X^*, which also has a bimonotone Schauder basis, we pass to the space $\mathfrak{X}_0$. The map Φ defined in Definition 3.7 yields that $\Phi(\mathrm{K})$ is $\frac{1}{8}$-dense in the ball of Z_X^*. By passing to the space $\mathfrak{X}_1$ we have by Remark 5.2 that $J_1(\tilde{\mathrm{K}}) = \mathrm{K}$ and thus the operator $\Phi_1 = \Phi \circ J_1$ maps $\tilde{\mathrm{K}}$ onto a $\frac{1}{8}$-dense set in the ball of Z_X^*. We construct the space $\mathfrak{X}_\xi$ starting with $\mathfrak{X}_1$ and $\tilde{\mathrm{K}}$. By Theorem 4.1 the identity operator $I_\xi : \mathfrak{X}_\xi \mapsto \mathfrak{X}_1$ is continuous and maps $\tilde{\mathrm{K}}$ onto itself. Finally, the natural injection (denoted as J_{X_T} in the general case) $J_\xi : \mathfrak{X} \to \mathfrak{X}_\xi$ preserves $\tilde{\mathrm{K}}$ as does J_1. Thus, by taking the composition $Q = \Phi_1 \circ I \circ J_\xi$ we can see that it is an onto map from $\mathfrak{X}$ to Z_X^*. Thus Z_X^* is a quotient of $\mathfrak{X}$. As the set W_ξ is thin in $\mathfrak{X}_\xi$ (by Theorem 4.1) Theorem 7.1 yields that $\mathfrak{X}$ is the desired reflexive HI space for Z_X^*. As X^* is isomorphically embedded in Z_X we have that X is a quotient of Z_X^* and consequently of $\mathfrak{X}$ and this completes the proof. □

Applying a standard duality argument we obtain the following cofinal property of Indecomposable reflexive Banach spaces within the class of separable reflexive Banach spaces.

THEOREM 8.2. — *Every separable reflexive Banach space can be embedded into an Indecomposable reflexive Banach space with a Schauder basis.*

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