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ALBANESE VARIETIES WITH MODULUS AND HODGE THEORY

by Kazuya KATO & Henrik RUSSELL (*)

Abstract. — Let X be a proper smooth variety over a field k of characteristic 0 and Y an effective divisor on X with multiplicity. We introduce a generalized Albanese variety $\mathrm{Alb}(X,Y)$ of X of modulus Y, as higher dimensional analogue of the generalized Jacobian with modulus of Rosenlicht-Serre. Our construction is algebraic. For $k=\mathbb{C}$ we give a Hodge theoretic description.

Résumé. — Soient X une variété propre et lisse sur un corps k de caractéristique 0 et Y un diviseur effectif avec multiplicité sur X. Nous introduisons une variété d'Albanese généralisée $\mathrm{Alb}(X,Y)$ de X, de module Y, comme analogue en dimension supérieure de la jacobienne généralisée avec module de Rosenlicht-Serre. Notre construction est algébrique. Si $k=\mathbb{C}$, nous donnons une description en termes de théorie de Hodge.

1. Introduction

1.1. Let X be a proper smooth variety over a field k of characteristic 0, and let $\mathrm{Alb}(X)$ be the Albanese variety of X. In the work [10], the second author constructed generalized Albanese varieties $\mathrm{Alb}_{\mathcal{F}}(X)$, which are commutative connected algebraic groups over k with surjective homomorphisms $\mathrm{Alb}_{\mathcal{F}}(X) \to \mathrm{Alb}(X)$ (see Section 5 for a review). If Y is an effective divisor on X, a special case of $\mathrm{Alb}_{\mathcal{F}}(X)$ becomes the generalized Albanese variety $\mathrm{Alb}(X,Y)$ of X of modulus Y (*cf.*, Section 5). This is a higher dimensional analogue of the generalized Jacobian variety with modulus of Rosenlicht-Serre. Note that the divisor Y can have multiplicity, and so the algebraic group $\mathrm{Alb}(X,Y)$ can have an additive part.

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Assume now $k = \mathbb{C}$. The purpose of this paper is to give Hodge theoretic presentations (Theorem 1.1) of $\mathrm{Alb}(X, Y)$.

The case when Y has no multiplicity was studied in the work [3] of Barbieri-Viale and Srinivas. A Hodge theoretic presentation of a generalized Albanese variety in the case without modulus but allowing singularities on X was given in the work [6] of Esnault, Srinivas and Viehweg.

1.2. First we review the curve case. Let X be a proper smooth curve over $\mathbb{C}$ and let Y be an effective divisor on X. In this case, the Albanese variety $\mathrm{Alb}(X, Y)$ of X relative to Y coincides with the generalized Jacobian variety $J(X, Y)$ of X relative to Y. In the following, we will write the complex analytic space associated to X simply by X, and the sheaf of holomorphic functions on it by $\mathcal{O}_X$. Let $I = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$ be the ideal of $\mathcal{O}_X$ which defines Y. The cohomology below is for the topology of the analytic space X (not for Zariski topology).

The generalized Jacobian variety $J(X, Y)$ is the kernel of the degree map $\mathrm{Pic}(X, Y) \to \mathbb{Z}$ where $\mathrm{Pic}(X, Y) = H^1(X, \mathrm{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times))$. Let $j : X - Y \longrightarrow X$ be the inclusion map and let $j_!\mathbb{Z}(1)$ be the 0-extension of the constant sheaf $\mathbb{Z}(1)$ of $X - Y$ to X. (For $r \in \mathbb{Z}$, $\mathbb{Z}(r)$ denotes $\mathbb{Z}(2\pi i)^r$ as usual.) Then we have an exact sequence

$$0 \longrightarrow j_!\mathbb{Z}(1) \longrightarrow I \xrightarrow{\exp} \mathrm{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times) \longrightarrow 0$$

and hence we have an isomorphism

$$\mathrm{Pic}(X, Y) \cong H^2(X, [j_!\mathbb{Z}(1) \to I]). \tag{1.1}$$

Here in the complex $[j_!\mathbb{Z}(1) \to I]$, $j_!\mathbb{Z}(1)$ is put in degree 0.

We have another presentation of $J(X, Y)$ given in (2) below. Let I_1 be the ideal of $\mathcal{O}_X$ which defines the reduced part of Y and let $J = II_1^{-1} \subset \mathcal{O}_X$. Note that the composition of the two inclusion maps of complexes

$$[I \xrightarrow{d} J\Omega_X^1] \longrightarrow [I \xrightarrow{d} \Omega_X^1] \longrightarrow [I_1 \xrightarrow{d} \Omega_X^1]$$

is a quasi-isomorphism. Hence we have an isomorphism in the derived category

$$[I \xrightarrow{d} \Omega_X^1] \cong [I_1 \xrightarrow{d} \Omega_X^1] \oplus (\Omega_X^1/J\Omega_X^1)[-1].$$

Since $j_!\mathbb{C} \longrightarrow [I_1 \xrightarrow{d} \Omega_X^1]$ is a quasi-isomorphism, we have an exact sequence

$$H^0(X, \Omega_X^1) \longrightarrow H_c^1\big(X - Y, \mathbb{C}/\mathbb{Z}(1)\big) \oplus H^0(X, \Omega_X^1/J\Omega_X^1) \longrightarrow J(X, Y) \longrightarrow 0. \tag{1.2}$$

(Here H_c is the cohomology with compact supports.)

1.3. Now let X be a proper smooth variety over $\mathbb{C}$ of dimension n and let Y be an effective divisor on X.

Again in the following theorem, cohomology groups are for the topology of the complex analytic spaces, and the notation $\mathcal{O}$ and Ω stand for analytic sheaves.

Let I be the ideal of $\mathcal{O}_X$ which defines Y, let I_1 be the ideal of $\mathcal{O}_X$ which defines the reduced part of Y, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

Theorem 1.1.

(1) *We have an exact sequence*

$$0 \longrightarrow \mathrm{Alb}(X,Y) \longrightarrow H^{2n}\big(X, \mathcal{D}_{X,Y}(n)\big) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

where for $r \in \mathbb{Z}$, $\mathcal{D}_{X,Y}(r)$ denotes the kernel of the surjective homomorphism of complexes $\mathcal{D}_X(r) \to \mathcal{D}_Y(r)$ with $D_X(r)$ the Deligne complex

$$[\mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_X]$$

and $\mathcal{D}_Y(r)$ the similar complex

$$[\mathbb{Z}(r)_Y \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_Y].$$

(2) *We have an exact sequence*

$$H^{n-1}(X, \Omega^n_X) \longrightarrow H^{2n-1}_c\big(X - Y, \mathbb{C}/\mathbb{Z}(n)\big) \oplus H^{n-1}(X, \Omega^n_X / J\Omega^n_X) \longrightarrow \mathrm{Alb}(X,Y) \longrightarrow 0.$$

Note that the case $n = 1$ of Theorem 1.1 (1) (resp. (2)) becomes the presentation of $J(X,Y)$ given by (1) (resp. (2)) in No. 1.2.

Remark 1.2. — We give some remarks on this theorem.

(a) The case $Y = 0$ of Theorem 1.1 (1) is nothing but the well known exact sequence

$$0 \longrightarrow \mathrm{Alb}(X) \longrightarrow H^{2n}\big(X, \mathcal{D}_X(n)\big) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 \tag{1.3}$$

by using the Deligne cohomology $H^{2n}\big(X, \mathcal{D}_X(n)\big)$. (Usually the Deligne cohomology $H^m(X, \mathcal{D}_X(r))$ is denoted by $H^m_D\big(X, \mathbb{Z}(r)\big)$.)

The case $Y = 0$ of Theorem 1.1 (2) is nothing but the usual presentation

$$\mathrm{Alb}(X) \cong H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^0 H_{\mathbb{C}} \tag{1.4}$$

of the Albanese variety $\mathrm{Alb}(X)$ of X, where $(H_{\mathbb{Z}}, H_{\mathbb{C}}, F^\bullet)$ is the following Hodge structure of weight -1. $H_{\mathbb{Z}} = H^{2n-1}(X, \mathbb{Z}(n))/(\text{torsion part})$, $H_{\mathbb{C}} =$

$\mathbb{C}\otimes_{\mathbb{Z}} H_{\mathbb{Z}} = H^{2n-1}(X,\Omega_X^{\bullet})$, and $F^{\bullet}$ is the Hodge filtration on $H_{\mathbb{C}}$ defined as

$$F^{-1} = H_{\mathbb{C}}, \quad F^0 = H^{n-1}(X,\Omega_X^n), \quad F^1 = 0.$$

(b) Recall that the presentations (3) and (4) of $\mathrm{Alb}(X)$ are related as follows. Consider the exact sequence of complexes $0 \to \Omega_X^{\leqslant n-1}[-1] \to \mathcal{D}_X(n) \to \mathbb{Z}(n) \to 0$, where $\Omega_X^{\leqslant n-1}$ denotes the part of degree $\leqslant n-1$ of the de Rham complex $\Omega_X^{\bullet}$, which is actually a quotient complex of $\Omega_X^{\bullet}$. By taking the cohomology associated to this exact sequence, we have an exact sequence

$$H^{2n-1}\big(X,\mathbb{Z}(n)\big) \longrightarrow H^{2n-1}(X,\Omega_X^{\leqslant n-1}) \longrightarrow H_D^{2n}\big(X,\mathbb{Z}(n)\big) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Since

$$\begin{aligned} H^{2n-1}(X,\Omega_X^{\leqslant n-1}) &\cong H^{2n-1}(X,\Omega_X^{\bullet})/H^{n-1}(X,\Omega_X^n) \\ &\cong H^{2n-1}(X,\mathbb{C})/H^{n-1}(X,\Omega_X^n), \end{aligned}$$

the exact sequence (4) is equivalent to (3).

(c) (1) and (2) of Theorem 1.1 are related similarly. Let S be the subcomplex of the de Rham complex $\Omega_X^{\bullet}$ of X defined by $S^p = \mathrm{Ker}(\Omega_X^p \to \Omega_Y^p)$ for $0 \leqslant p \leqslant n-1$ and $S^n = \Omega_X^n$. Then Theorem 1.1 (1) is equivalent to

$$\mathrm{Alb}(X,Y) \cong H_{\mathbb{Z}}\backslash H^{2n-1}(X,S)/H^{n-1}(X,\Omega_X^n)$$

where $H_{\mathbb{Z}} = H_c^{2n-1}(X-Y,\mathbb{Z}(n))/(\text{torsion part})$. As shown in § 6, we have a commutative diagram with an isomorphism in the lower row

$$\begin{array}{ccc} H^{n-1}(X,\Omega_X^n) & = & H^{n-1}(X,\Omega_X^n) \\ \downarrow & & \downarrow \\ H^{2n-1}(X,S) & \cong & H_c^{2n-1}(X-Y,\mathbb{C}) \oplus H^{n-1}(X,\Omega_X^n/J\Omega_X^n). \end{array}$$

Thus (1) and (2) of Theorem 1.1 are deduced from each other.

1.4. As mentioned above, Theorem 1.1 shows that $\mathrm{Alb}(X,Y)$ is expressed as $H_{\mathbb{Z}}\backslash H_V/F^0$ where:

$$\begin{aligned} H_{\mathbb{Z}} &= H_c^{2n-1}\big(X-Y,\mathbb{Z}(n)\big)\big/(\text{torsion part}), \\ H_V &= H_{\mathbb{C}} \oplus H^{n-1}(X,\Omega_X^n/J\Omega_X^n) \cong H^{2n-1}(X,S) \end{aligned}$$

$$(H_{\mathbb{C}} = \mathbb{C}\otimes H_{\mathbb{Z}} \text{ and } S \text{ is as in 1.5 (d)}),$$

$F^{\bullet}$ is the decreasing filtration on H_V given by

$$F^{-1} = H_V, \quad F^0 = H^{n-1}(X,\Omega_X^n), \quad F^1 = 0.$$

Note that H_V can be different from $H_{\mathbb{C}}$ here, and so $(H_{\mathbb{Z}}, H_V, F^\bullet)$ here need not be a Hodge structure. It is some kind of "mixed Hodge structure with additive part". This object $(H_{\mathbb{Z}}, H_V, F^\bullet)$ with a weight filtration, which we will denote by $H^{2n-1}(X, Y_-)(n)$ in Section 6, belongs to a category $\mathcal{H}$ introduced in Section 2 which contains the category of mixed Hodge structures but is larger than that. In the proof of Theorem 1.1, it is essential to consider such an object. This category $\mathcal{H}$ is related to the category of "enriched Hodge structures" of Bloch-Srinivas [4] and to the category of "formal Hodge structures" of Barbieri-Viale [1]. However, the relations between these three categories are not trivial, see 4.6 and [2, 4.2]. Our definition of $\mathcal{H}$ aims to stick close to the classical language of Hodge structures and to express duality in a simplest possible way. In the proof of Theorem 1.1, we use a Hodge theoretic description of the category of "1-motives with additive parts" over $\mathbb{C}$ in terms of $\mathcal{H}$. This description is similar to the result of Barbieri-Viale in [1].

1.5. The theory of generalized Albanese varieties in characteristic $p > 0$ is given in [11], basing on duality theory of "1-motives with unipotent parts".

In characteristic $p > 0$, *syntomic cohomology* is an analogue of *Deligne cohomology*. We expect that we can have presentations of the p-adic completion of $\mathrm{Alb}(X, Y)(k)$ (k is the base field), which is similar to Theorem 1.1, by using crystalline cohomology theory and syntomic cohomology theory.

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2. Mixed Hodge structures with additive parts

2.1. For a proper smooth variety X over $\mathbb{C}$ of dimension n and for an effective divisor Y on X, we will have in Section 6 certain structures $H^1(X, Y_+)$ and $H^{2n-1}(X, Y_-)$ which are kinds of "mixed Hodge structures with additive parts". (These structures for the case when X is a curve are explained in Example 2.1 below.) The authors imagine that there is a nice definition of the category of "mixed Hodge structures with additive parts", which contains these $H^1(X, Y_+)$ and $H^{2n-1}(X, Y_-)$ as objects, but can not define it. Instead, we define a category $\mathcal{H}$ containing these objects, which may be a very simple approximation of such a nice category.

2.2. The category $\mathcal{H}$. An object of $\mathcal{H}$ is by definition a tuple $H = (H_{\mathbb{Z}}, H_V, W_\bullet H_{\mathbb{Q}}, W_\bullet H_V, F^\bullet H_V, a, b)$, where $H_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module, H_V is a finite dimensional $\mathbb{C}$-vector space, $W_\bullet H_{\mathbb{Q}}$ is an increasing filtration on $H_{\mathbb{Q}} := \mathbb{Q} \otimes H_{\mathbb{Z}}$ (called weight filtration), $W_\bullet H_V$ is an increasing filtration on H_V (called weight filtration), $F^\bullet$ is a decreasing filtration on H_V (called Hodge filtration), a is a $\mathbb{C}$-linear map $H_{\mathbb{C}} := \mathbb{C} \otimes H_{\mathbb{Z}} \to H_V$ which sends $W_w H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Q}} W_w H_{\mathbb{Q}}$ into $W_w H_V$ for any $w \in \mathbb{Z}$, and b is a $\mathbb{C}$-linear map $H_V \to H_{\mathbb{C}}$ which sends $W_w H_V$ into $W_w H_{\mathbb{C}}$ for any $w \in \mathbb{Z}$ such that $b \circ a$ is the identity map of $H_{\mathbb{C}}$. We sometimes denote an object H of $\mathcal{H}$ simply by $(H_{\mathbb{Z}}, H_V)$.

A morphism $f : H \to H'$ in $\mathcal{H}$ is a pair of homomorphisms $(f_{\mathbb{Z}}, f_V)$, where $f_{\mathbb{Z}} : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ is compatible with the weight filtrations and $f_V : H_V \to H'_V$ is compatible with weight filtrations and Hodge filtrations, which is compatible with the maps a, b and a', b'.

The category of mixed Hodge structures is naturally embedded into $\mathcal{H}$ as a full subcategory, by putting $H_V = H_{\mathbb{C}}$.

Similarly as for mixed Hodge structures we can give $\underline{\mathrm{Hom}}(H, H')$ the structure of an object of $\mathcal{H}$ for $H, H' \in \mathrm{Ob}(\mathcal{H})$. We call $\underline{\mathrm{Hom}}(H, \mathbb{Z})$ the object *dual to* H. The full subcategory of $\mathcal{H}$ consisting of all objects H such that $H_{\mathbb{Z}}$ are torsion free is clearly self-dual.

We will say that a sequence $H' \to H \to H''$ in $\mathcal{H}$ is exact, if and only if the following sequences are all exact:

$$H'_{\mathbb{Z}} \to H_{\mathbb{Z}} \to H''_{\mathbb{Z}}, \qquad H'_V \to H_V \to H''_V,$$

$$W_w H'_{\mathbb{Q}} \to W_w H_{\mathbb{Q}} \to W_w H''_{\mathbb{Q}}, \qquad W_w H'_V \to W_w H_V \to W_w H''_V,$$

$$F^p H'_V \to F^p H_V \to F^p H''_V,$$

for all $w, p \in \mathbb{Z}$.

See No. 4.6 for the relation of this category $\mathcal{H}$ to the category of enriched Hodge structures of Bloch-Srinivas [4] and to the category of formal Hodge structures of Barbieri-Viale [1].

Example 2.1. — Let X be a proper smooth curve over $\mathbb{C}$ and let Y be an effective divisor on X. Let I be the ideal of $\mathcal{O}_X$ which defines Y, let I_1 be the ideal of $\mathcal{O}_X$ which defines the reduced part of Y, and let $J = I I_1^{-1} \subset \mathcal{O}_X$.

We define objects $H^1(X, Y_+)$ and $H^1(X, Y_-)$ of $\mathcal{H}$.

First, we define $H = H^1(X, Y_+)$. Let

$$H_{\mathbb{Z}} = H^1(X - Y, \mathbb{Z}), \quad H_V = H^1(X, [\mathcal{O}_X \xrightarrow{d} I^{-1}\Omega^1_X]).$$

The map $a : H_{\mathbb{C}} \to H_V$ is

$$H^1(X - Y, \mathbb{C}) \cong H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega^1_X]) \longrightarrow H^1(X, [\mathcal{O}_X \to I^{-1}\Omega^1_X]).$$

The map $b : H_V \to H_{\mathbb{C}}$ is the composition

$$H^1(X, [\mathcal{O}_X \to I^{-1}\Omega^1_X]) \longrightarrow H^1(X, [J^{-1} \to I^{-1}\Omega^1_X]) \xleftarrow{\simeq} H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega_X]) \cong H^1(X - Y, \mathbb{C}).$$

The weight filtrations and the Hodge filtration are given by

$$W_2 H_{\mathbb{Q}} = H_{\mathbb{Q}}, \quad W_1 H_{\mathbb{Q}} = H^1(X, \mathbb{Q}), \quad W_0 H_{\mathbb{Q}} = 0,$$
$$W_2 H_V = H_V, \quad W_1 H_V = H^1(X, \mathbb{C}), \quad W_0 H_V = 0,$$

where $H^1(X, \mathbb{C})$ is embedded in H_V via a, and

$$F^0 H_V = H_V, \quad F^1 H_V = H^1(X, \mathbb{C}), \quad F^2 H_{\mathbb{C}} = 0.$$

Next, we define $H = H^1(X, Y_-)$. Let

$$H_{\mathbb{Z}} = H^1_c(X - Y, \mathbb{Z}), \quad H_V = H^1(X, [I \xrightarrow{d} \Omega^1_X]).$$

The map $a : H_{\mathbb{C}} \to H_V$ is the composition

$$H^1_c(X - Y, \mathbb{C}) \cong H^1(X, [I_1 \to \Omega^1_X]) \xleftarrow{\simeq} H^1(X, [I \to J\Omega^1_X]) \longrightarrow H^1(X, [I \to \Omega^1_X]).$$

The map $b : H_V \to H_{\mathbb{C}}$ is

$$H^1(X, [I \to \Omega^1_X]) \longrightarrow H^1(X, [I_1 \to \Omega^1_X]) \cong H^1_c(X - Y, \mathbb{C}).$$

The weight filtrations and the Hodge filtration are given by

$$W_1 H_{\mathbb{Q}} = H_{\mathbb{Q}}, \quad W_0 H_Q = \operatorname{Ker}\big(H_{\mathbb{Q}} \to H^1(X, \mathbb{Q})\big), \quad W_{-1} H_{\mathbb{Q}} = 0,$$
$$W_1 H_V = H_V, \quad W_0 H_V = \operatorname{Ker}\big(H_V \to H^1(X, \mathbb{C})\big), \quad W_{-1} H_V = 0,$$

where $H^1(X, \mathbb{C})$ is regarded as quotient of H_V via b, and

$$F^0 H_V = H_V, \quad F^1 H_V = \operatorname{Ker}\big(H_V \to H^1(X, \mathcal{O}_X)\big), \quad F^2 H_{\mathbb{C}} = 0.$$

Then we have exact sequences in $\mathcal{H}$

$$0 \longrightarrow H^1(X) \longrightarrow H^1(X, Y_+) \longrightarrow H^0(Y)(-1) \longrightarrow \mathbb{Z}(-1) \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow H^0(Y) \longrightarrow H^1(X, Y_-) \longrightarrow H^1(X) \longrightarrow 0.$$

Here for $r \in \mathbb{Z}$, $\mathbb{Z}(r)$ is the usual Hodge structure $\mathbb{Z}(r)$ regarded as an object of $\mathcal{H}$. $H^1(X)$ is also the usual Hodge structure of weight 1 associated to the first cohomology of X, regarded as an object of $\mathcal{H}$. Finally the object $H^0(Y)$ of $\mathcal{H}$ is defined as below, and $H^0(Y)(-1)$ is the -1 Tate twist.

The definition of $H = H^0(Y)$ is as follows. $H_{\mathbb{Z}} = H^0(Y, \mathbb{Z}) = \oplus_{y \in Y} \mathbb{Z}$. $H_V = H^0(Y, \mathcal{O}_Y)$. a is the canonical map $H^0(Y, \mathbb{C}) \to H^0(Y, \mathcal{O}_Y)$. b is the canonical map $H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathbb{C})$ given by $\mathcal{O}_Y \to \mathbb{C}$ which is

$\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_y = \mathbb{C}$ at each $y \in Y$ (m_y denotes the maximal ideal of $\mathcal{O}_{Y,y}$). The weight filtration and the Hodge filtration are given by

$$W_0H = H, \qquad W_{-1}H = 0,$$
$$F^0H_V = H_V, \qquad F^1H_V = 0.$$

Note that $H_{\mathbb{C}} \to H_V$ can be like $\mathbb{C} \to \mathbb{C}[T]/(T^n)$, and need not be an isomorphism.

The evident self-duality $\underline{\mathrm{Hom}}(\ ,\mathbb{Z})$ for torsion free objects in $\mathcal{H}$ induces

$$H^1(X, Y_-) \cong \underline{\mathrm{Hom}}\big(H^1(X, Y_+), \mathbb{Z}\big)(-1).$$

3. 1-motives with additive parts

In [9], Laumon formulated the notion of a "1-motive with additive part" over a field of characteristic 0. We give a short review assuming that the base field is algebraically closed for simplicity.

Fix an algebraically closed field k of characteristic 0.

3.1. Let $\mathcal{A}b/k$ be the category of sheaves of abelian groups on the fppf-site of the category of affine schemes over k. Let $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$ be the abelian category of complexes in $\mathcal{A}b/k$ concentrated in degrees -1 and 0.

A 1-*motive with additive part* over k is an object of $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$ of the form $[\mathcal{F} \to G]$, where G is a commutative connected algebraic group over k and $\mathcal{F} \cong \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some t and s. (*cf.*, [9, Def. (5.1.1)].) Here $\mathbb{Z}$ is regarded as a constant sheaf and $\widehat{\mathbb{G}}_a$ denotes the formal completion of the additive group $\mathbb{G}_a$ at 0. Recall that for any commutative ring R, $\widehat{\mathbb{G}}_a(R)$ is the subgroup of the additive group R consisting of all nilpotent elements. We have $\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}}$, where $\mathcal{F}_{\text{ét}}$ is the étale part of $\mathcal{F}$ which corresponds to $\mathbb{Z}^t$ in the above isomorphism and $\mathcal{F}_{\text{inf}}$ is the infinitesimal part of $\mathcal{F}$ which corresponds to $(\widehat{\mathbb{G}}_a)^s$.

We denote the category of 1-motives with additive parts over k by $\mathcal{M}_1$.

3.2. The category $\mathcal{M}_1$ admits a notion of duality (called "Cartier duality"). Let $[\mathcal{F} \to G]$ be a 1-motive with additive part over k. Then we have the "Cartier dual" $[\mathcal{F}' \to G']$ of $[\mathcal{F} \to G]$ which is an object of $\mathcal{M}_1$ obtained as follows. Let $0 \to L \to G \to A \to 0$ be the canonical decomposition of G as an extension of an abelian variety A by a commutative connected affine algebraic group L. Note that $L \cong (\mathbb{G}_m)^t \oplus (\mathbb{G}_a)^s$ for some t and s. We have

$$\mathcal{F}' = \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m), \quad G' = \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m)$$

and the homomorphism $\mathcal{F}' \to G'$ is the connecting homomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m)$$

associated to the short exact sequence $0 \longrightarrow L \longrightarrow [\mathcal{F} \to G] \longrightarrow [\mathcal{F} \to A] \longrightarrow 0$ in $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$. Since

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}, \quad \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{G}_a, \mathbb{G}_m) \cong \widehat{\mathbb{G}}_a,$$

we have $\mathcal{F}' \simeq \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some t and s. We have an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathcal{F}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{A}b/k}(A, \mathbb{G}_m) \longrightarrow 0,$$

$\underline{\mathrm{Ext}}^1_{\mathcal{A}b/k}(A, \mathbb{G}_m)$ is the dual abelian variety of A, and since

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{Z}, \mathbb{G}_m) \cong \mathbb{G}_m, \quad \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \cong \mathbb{G}_a,$$

$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathcal{F}, \mathbb{G}_m) \cong (\mathbb{G}_m)^t \oplus (\mathbb{G}_a)^s$ for some t and s. Hence G' is a commutative connected algebraic group over k. Thus $[\mathcal{F}' \to G']$ is a 1-motive with additive part. The Cartier dual of $[\mathcal{F}' \to G']$ is canonically isomorphic to $[\mathcal{F} \to G]$.

See [9, Section 5] for details or [10, Section 1] for another review.

3.3. Let $\mathcal{M}_{1,\{-1,-2\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that $\mathcal{F} = 0$.

Let $\mathcal{M}_{1,\{0,-1\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that G is an abelian variety.

Then the self-duality of $\mathcal{M}_1$ in No. 3.2 induces an anti-equivalence between the categories $\mathcal{M}_{1,\{-1,-2\}}$ and $\mathcal{M}_{1,\{0,-1\}}$.

4. Equivalences of categories

In [1], Barbieri-Viale constructed a Hodge theoretic category and proved that in the case when the base field is $\mathbb{C}$, the category $\mathcal{M}_1$ is equivalent to his Hodge theoretic category. Here we reformulate his equivalence in the style which is convenient for us, by using the category $\mathcal{H}$ from Section 2.

4.1. The category $\mathcal{H}_1$. An object of $\mathcal{H}_1$ is an object H of $\mathcal{H}$ endowed with a splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ satisfying the following conditions (i)–(iv).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-3}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight -1. That is, $\mathrm{gr}^W_{-1} H_{\mathbb{C}} = \mathrm{gr}^W_{-1} H_V$ and $\mathrm{gr}^W_{-1} H_{\mathbb{Z}}$ with the Hodge filtration on $\mathrm{gr}^W_{-1} H_{\mathbb{C}}$ is a polarizable Hodge structure of weight -1.

(iii) $F^0 \mathrm{gr}^W_0 H_V = \mathrm{gr}^W_0 H_V$.

(iv) $F^0 W_{-2} H_V = 0$.

Morphisms of $\mathcal{H}_1$ are the evident ones.

The category $\mathcal{H}_1$ is self-dual by the functor $\underline{\mathrm{Hom}}(\ ,\mathbb{Z})(1)$.

4.2. For a subset Δ of $\{0,-1,-2\}$, let $\mathcal{H}_{1,\Delta}$ be the full subcategory of $\mathcal{H}_1$ consisting of all objects H such that $\mathrm{gr}^W_w H = 0$ unless $w \in \Delta$.

The categories $\mathcal{H}_{1,\{-1,-2\}}$ and $\mathcal{H}_{1,\{0,-1\}}$ are important for us. These categories are in fact defined as full subcategories of $\mathcal{H}$ without reference to the splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$, for the weight filtrations on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ of objects of these categories are pure.

Thus $\mathcal{H}_{1,\{-1,-2\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects H satisfying the following conditions (i)–(iii).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_{-1}H = H$, $W_{-3}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight -1.

(iii) $F^0 W_{-2} H_V = 0$.

For example, the Tate twist $H^1(X, Y_-)(1)$ of the object $H^1(X, Y_-)$ of $\mathcal{H}$ in Example 2.1 belongs to $\mathcal{H}_{1,\{-1,-2\}}$.

Similarly, $\mathcal{H}_{1,\{0,-1\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects H satisfying the following conditions (i)–(iii).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-2}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight -1.

(iii) $F^0 \mathrm{gr}^W_0 H_V = \mathrm{gr}^W_0 H_V$.

For example, the Tate twist $H^1(X, Y_+)(1)$ of the object $H^1(X, Y_+)$ of $\mathcal{H}$ in Example 2.1 belongs to $\mathcal{H}_{1,\{0,-1\}}$.

The self-duality $\underline{\mathrm{Hom}}(\ ,\mathbb{Z})(1)$ of $\mathcal{H}_1$ induces an anti-equivalence between the categories $\mathcal{H}_{1,\{-1,-2\}}$ and $\mathcal{H}_{1,\{0,-1\}}$.

Theorem 4.1. — *(This is an analogue of the equivalence of categories proved by Barbieri-Viale in* [1]*.) We have an equivalence of categories $\mathcal{H}_1 \simeq \mathcal{M}_1$ which is compatible with dualities, and which induces the equivalences*

$$\mathcal{H}_{1,\{-1,0\}} \simeq \mathcal{M}_{1,\{-1,0\}}, \quad \mathcal{H}_{1,\{-2,-1\}} \simeq \mathcal{M}_{1,\{-2,-1\}}.$$

The equivalence $\mathcal{H}_1 \simeq \mathcal{M}_1$ is described in No.s 4.3 and 4.4 below.

4.3. First we define the functor $\mathcal{H}_1 \to \mathcal{M}_1$.

Let H be an object of $\mathcal{H}_1$. The corresponding object $[\mathcal{F} \to G]$ of $\mathcal{M}_1$ is as follows.

$$\begin{aligned} G &= W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_V/F^0W_{-1}H_V, \\ \mathcal{F}_{\text{ét}} &= \operatorname{gr}_0^W(H_{\mathbb{Z}}), \\ \mathcal{F}_{\text{inf}} &= \text{the formal completion of } \operatorname{Ker}\big(\operatorname{gr}_0^W(H_V) \to \operatorname{gr}_0^W(H_{\mathbb{C}})\big). \end{aligned}$$

Here $\mathcal{F}_{\text{ét}}$ is the étale part of $\mathcal{F}$ and $\mathcal{F}_{\text{inf}}$ is the infinitesimal part of $\mathcal{F}$. The homomorphism $\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}} \to G$ is given as follows.

The part $\mathcal{F}_{\text{ét}} \to G$: Let $x \in \mathcal{F}_{\text{ét}} = \operatorname{gr}_0^W H_{\mathbb{Z}}$. Since the sequence $0 \to W_{-1}H_{\mathbb{Z}} \to H_{\mathbb{Z}} \to \operatorname{gr}_0^W H_{\mathbb{Z}} \to 0$ is exact, we can lift x to an element y of $H_{\mathbb{Z}}$ and this lifting is unique modulo $W_{-1}H_{\mathbb{Z}}$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\operatorname{gr}_0^W H_V \to 0$ is exact, we can lift x to an element z of F^0H_V and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\mathcal{F}_{\text{ét}} = \operatorname{gr}_0^W H_{\mathbb{Z}} \longrightarrow W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_V/F^0W_{-1}H_V = G\,; \quad x \longmapsto y - z.$$

The part $\mathcal{F}_{\text{inf}} \to G$: Identify $\operatorname{Hom}(\mathcal{F}_{\text{inf}}, G)$ with $\operatorname{Hom}_{\mathbb{C}}\big(\operatorname{Lie}(\mathcal{F}_{\text{inf}}), \operatorname{Lie}(G)\big)$. We give the corresponding homomorphism $\operatorname{Lie}(\mathcal{F}_{\text{inf}}) = \operatorname{Ker}\big(\operatorname{gr}_0^W(H_V) \to \operatorname{gr}_0^W(H_{\mathbb{C}})\big) \to \operatorname{Lie}(G) = W_{-1}H_V/F^0W_{-1}H_V$. Let $x \in \operatorname{Ker}\big(\operatorname{gr}_0^W(H_V) \to \operatorname{gr}_0^W(H_{\mathbb{C}})\big)$. The given splitting of the weight filtration on $\operatorname{Ker}(H_V \to H_{\mathbb{C}})$ sends x to an element y of $\operatorname{Ker}(H_V \to H_{\mathbb{C}})$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\operatorname{gr}_0^W H_V \to 0$ is exact, we can lift x to an element z of F^0H_V and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\operatorname{Ker}(\operatorname{gr}_0^W H_V \to \operatorname{gr}_0^W H_{\mathbb{C}}) \longrightarrow W_{-1}H_V/F^0W_{-1}H_V = \operatorname{Lie}(G)\,; \quad x \longmapsto y - z.$$

4.4. We give the functor $\mathcal{M}_1 \to \mathcal{H}_1$.

Let $[\mathcal{F} \to G]$ be an object of $\mathcal{M}_1$. The corresponding object H of $\mathcal{H}_1$ is as follows. Let $0 \to L \to G \to A \to 0$ be the exact sequence of commutative algebraic groups where A is an abelian variety and L is affine. Let $\mathcal{F}_{\text{ét}}$ be the étale part of $\mathcal{F}$ and let $\mathcal{F}_{\text{inf}}$ be the infinitesimal part of $\mathcal{F}$.

First, $H_{\mathbb{Z}}$ is the fiber product of $\mathcal{F}_{\text{ét}} \to G \leftarrow \operatorname{Lie}(G)$, where $\operatorname{Lie}(G) \to G$ is the exponential map, so we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H_1(G,\mathbb{Z}) & \to & H_{\mathbb{Z}} & \to & \mathcal{F}_{\text{ét}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_1(G,\mathbb{Z}) & \to & \operatorname{Lie}(G) & \to & G & \to & 0. \end{array}$$

The weight filtration on $H_{\mathbb{Z}}$ is given as follows.

$$\begin{aligned} W_0 H_{\mathbb{Z}} &= H_{\mathbb{Z}}, \\ W_{-1} H_{\mathbb{Z}} &= H_1(G,\mathbb{Z}), \\ W_{-2} H_{\mathbb{Z}} &= H_1(L,\mathbb{Z}) = \operatorname{Ker}\big(H_1(G,\mathbb{Z}) \to H_1(A,\mathbb{Z})\big), \\ W_{-3} H_{\mathbb{Z}} &= 0. \end{aligned}$$

Next,

$$H_V = H_{\mathbb{C}} \oplus \operatorname{Lie}(L_a) \oplus \operatorname{Lie}(\mathcal{F}_{\mathrm{inf}})$$

where L_a is the additive part of L. The weight filtration on H_V is as follows.

$$\begin{aligned} W_0 H_V &= H_V, \\ W_{-1} H_V &= H_1(G,\mathbb{C}) \oplus \operatorname{Lie}(L_a), \\ W_{-2} H_V &= H_1(L,\mathbb{C}) \oplus \operatorname{Lie}(L_a), \\ W_{-3} H_V &= 0. \end{aligned}$$

The splitting of the weight filtration on $\operatorname{Ker}(H_V \to H_{\mathbb{C}}) = \operatorname{Lie}(L_a) \oplus \operatorname{Lie}(\mathcal{F}_{\mathrm{inf}})$ is by definition this direct decomposition.

The Hodge filtration on H_V is given as follows.

$$\begin{aligned} F^{-1} H_V &= H_V, \\ F^1 H_V &= 0, \\ F^0 H_V &= \operatorname{Ker}\big(H_V \to \operatorname{Lie}(G)\big) \end{aligned}$$

where $H_V \to \operatorname{Lie}(G)$ is defined as follows. The part $H_{\mathbb{C}} \to \operatorname{Lie}(G)$ of it is the $\mathbb{C}$-linear map induced by the canonical map $H_{\mathbb{Z}} \to \operatorname{Lie}(G)$. The part $\operatorname{Lie}(L_a) \to \operatorname{Lie}(G)$ of it is the inclusion map. The part $\operatorname{Lie}(\mathcal{F}_{\mathrm{inf}}) \to \operatorname{Lie}(G)$ of it is the homomorphism induced by $\mathcal{F}_{\mathrm{inf}} \to G$. We have hence $H_V/F^0 H_V \cong \operatorname{Lie}(G)$.

It is easy to see that this functor $\mathcal{M}_1 \to \mathcal{H}_1$ is quasi-inverse to the functor $\mathcal{H}_1 \to \mathcal{M}_1$ in No. 4.3.

4.5. The induced functor $\mathcal{H}_{1,\{-1,-2\}} \xrightarrow{\simeq} \mathcal{M}_{1,\{-1,-2\}}$ is especially simple. It is given by

$$H \longmapsto [0 \to H_{\mathbb{Z}} \backslash H_V / F^0 H_V].$$

4.6. For those who are familiar with formal Hodge structures from [1] we explain the relation between $\mathcal{H}_1$ and the category $\mathrm{FHS}_1^{\mathrm{fr}}$ of torsion free formal Hodge structures of level $\leqslant 1$, see [1, Def. 1.1.2]. (This No. is not used in the rest of the paper.)

The categories $\mathcal{H}_1$ and $\mathrm{FHS}_1^{\mathrm{fr}}$ are equivalent. The functor $\mathcal{H}_1 \to \mathrm{FHS}_1^{\mathrm{fr}}$ is given by $(H_{\mathbb{Z}}, H_V) \mapsto (\mathcal{F}, V)$, where $(\mathcal{F}, V)$ is the following object of $\mathrm{FHS}_1^{\mathrm{fr}}$.

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}}, \\ \mathcal{F}_{\text{ét}} &= H_{\mathbb{Z}}, \\ \mathcal{F}_{\text{inf}} &= \text{formal completion of } \operatorname{Ker}\left(\operatorname{gr}_0^W(H_V) \to \operatorname{gr}_0^W(H_{\mathbb{C}})\right), \end{aligned}$$

$$\begin{aligned} V &= W_{-1}H_V/W_{-1}F^0H_V \\ \supseteq V^1 &= W_{-2}H_V \\ \supseteq V^0 &= \operatorname{Ker}(W_{-2}H_V \to W_{-2}H_{\mathbb{C}}), \end{aligned}$$

$$v : \mathcal{F} \to V \text{ is def. by} \begin{cases} v|_{\mathcal{F}_{\text{ét}}} = a|_{H_{\mathbb{Z}}} \mod F^0H_V \text{ (we have } V = H_V/F^0H_V), \\ v|_{\mathcal{F}_{\text{inf}}} \text{ is the map } \mathcal{F}_{\text{inf}} \subset \operatorname{Lie}(\mathcal{F}_{\text{inf}}) \to \operatorname{Lie}(G) \text{ as in No. 4.3,} \end{cases}$$

$$H_{\mathbb{C}}/F^0H_{\mathbb{C}} \xrightarrow{\simeq} V/V^0 \text{ is the map induced by a.}$$

The functor $\mathrm{FHS}_1^{\mathrm{fr}} \to \mathcal{H}_1$ is given by $(\mathcal{F}, V) \mapsto (H_{\mathbb{Z}}, H_V)$, where $(H_{\mathbb{Z}}, H_V)$ is the following object of $\mathcal{H}_1$.

$$\begin{aligned} H_{\mathbb{Z}} &= \mathcal{F}_{\text{ét}}, \\ H_V &= H_{\mathbb{C}} \oplus \operatorname{Lie}(\mathcal{F}_{\text{inf}}) \oplus V^0, \end{aligned}$$

$$\begin{aligned} & & W_0H_V &= H_V, \\ F^{-1}H_V &= H_V, & W_{-1}H_V &= W_{-1}H_{\mathbb{C}} \oplus V^0, \\ F^0H_V &= \operatorname{Ker}(H_V \to V), & W_{-2}H_V &= W_{-2}H_{\mathbb{C}} \oplus V^0, \\ F^1H_V &= 0, & W_{-3}H_V &= 0, \end{aligned}$$

where $H_V \to V$ is the map given by $(v|_{\mathcal{F}_{\text{ét}}} \otimes \mathbb{C}, \operatorname{Lie}(v|_{\mathcal{F}_{\text{inf}}}), V^0 \hookrightarrow V)$.

These functors are quasi-inverse to each other and yield an equivalence of categories $\mathcal{H}_1 \simeq \mathrm{FHS}_1^{\mathrm{fr}}$. The relation between FHS_1 and the category EHS_1 of enriched Hodge structures of level $\leqslant 1$ from [4] is given in [2, 4.2] by explicit functors. Composition yields an explicit functor $\mathrm{EHS}_1^{\mathrm{fr}} \to \mathcal{H}_1$ (left to the reader). The category $\mathrm{EHS}_1^{\mathrm{fr}}$ of torsion free enriched Hodge structures of level $\leqslant 1$ is equivalent to a subcategory of $\mathrm{FHS}_1^{\mathrm{fr}}$ resp. $\mathcal{H}_1$, see [2, Prop. 4.2.3].

5. Generalized Albanese varieties

Let k be an algebraically closed field of characteristic 0 and let X be a proper smooth algebraic variety over k of dimension n. We review generalized Albanese varieties $\mathrm{Alb}_{\mathcal{F}}(X)$ defined in [10][(1)]. For an effective divisor Y on X, the generalized Albanese variety $\mathrm{Alb}(X,Y)$ of modulus Y is a special case of $\mathrm{Alb}_{\mathcal{F}}(X)$.

The Albanese variety $\mathrm{Alb}(X)$ is defined by a universal mapping property for morphisms from X to abelian varieties. Similarly, the generalized Albanese variety $\mathrm{Alb}(X,Y)$ of modulus Y is characterized by a universal property for morphisms from $X-Y$ into commutative algebraic groups with "modulus" $\leqslant Y$. See Proposition 5.1.

5.1. Let $\underline{\mathrm{Div}}_X$ be the sheaf of abelian groups on $\mathcal{A}b/k$ defined as follows. For any commutative ring R over k, $\underline{\mathrm{Div}}_X(R)$ is the group of all Cartier divisors on $X\otimes_k R$ generated locally on $\mathrm{Spec}(R)$ by effective Cartier divisors which are flat over R. Let $\underline{\mathrm{Pic}}_X$ be the Picard functor, and let $\underline{\mathrm{Pic}}^0_X \subset \underline{\mathrm{Pic}}_X$ be the Picard variety of X. We have the class map $\underline{\mathrm{Div}}_X \to \underline{\mathrm{Pic}}_X$. Let $\underline{\mathrm{Div}}^0_X \subset \underline{\mathrm{Div}}_X$ be the inverse image of $\underline{\mathrm{Pic}}^0_X$.

5.2. Let Λ be the set of all subgroup sheaves $\mathcal{F}$ of $\underline{\mathrm{Div}}^0_X$ such that $\mathcal{F} \cong \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some t and s. For $\mathcal{F}\in\Lambda$, we have an object $[\mathcal{F}\to\underline{\mathrm{Pic}}^0_X]$ of $\mathcal{M}_{1,\{0,-1\}}$. The generalized Albanese variety $\mathrm{Alb}_{\mathcal{F}}(X)$ is defined in [10] to be the Cartier dual of $[\mathcal{F}\to\underline{\mathrm{Pic}}^0_X]$. It is an object of $\mathcal{M}_{1,\{-1,-2\}}$ and hence is a commutative connected algebraic group over k.

If $\mathcal{F},\mathcal{F}'\in\Lambda$ and $\mathcal{F}\subset\mathcal{F}'$, we have a canonical surjective homomorphism $\mathrm{Alb}_{\mathcal{F}'}(X)\to\mathrm{Alb}_{\mathcal{F}}(X)$. In the case $\mathcal{F}=0$, $\mathrm{Alb}_{\mathcal{F}}(X)=\mathrm{Alb}(X)$.

5.3. Let Y be an effective divisor of X. Then the generalized Albanese variety with modulus Y is defined as $\mathrm{Alb}_{\mathcal{F}}(X)$ where $\mathcal{F}=\mathcal{F}_{X,Y}\in\Lambda$ is defined as follows. The étale part $\mathcal{F}_{\text{ét}}$ of $\mathcal{F}$ is the subgroup of $\underline{\mathrm{Div}}^0_X(k)$ consisting of all divisors whose support is contained in the support of Y. The infinitesimal part $\mathcal{F}_{\inf}$ of $\mathcal{F}$ is as follows. Let I be the ideal of $\mathcal{O}_X$ (though the notation $\mathcal{O}_X$ is often used in this paper for the sheaf of analytic functions, $\mathcal{O}_X$ here stands for the usual algebraic object on the Zariski site) defining Y, let I_1 be the ideal of $\mathcal{O}_X$ which defines the reduced part of Y, and let $J=II_1^{-1}\subset\mathcal{O}_X$. Then $\mathcal{F}_{\inf}$ is the formal completion $\widehat{\mathbb{G}}_a\otimes_k H^0(X,J^{-1}/\mathcal{O}_X)$ of the finite dimensional k-vector space $H^0(X,J^{-1}/\mathcal{O}_X)$,

(1) In [10], X was assumed to be projective. This assumption was used only for singular X, which is not our concern here. The construction of the $\mathrm{Alb}_{\mathcal{F}}(X)$ is valid in the same way for proper X.

which is embedded in $\underline{\mathrm{Div}}^0_X$ by the exponential map

$$\exp\ :\ \widehat{\mathbb{G}}_a \otimes_k H^0(X, J^{-1}/\mathcal{O}_X) \longrightarrow \underline{\mathrm{Div}}^0_X.$$

If Y' is an effective divisor on X such that $Y' \geqslant Y$, then $\mathcal{F}_{X,Y'} \supset \mathcal{F}_{X,Y}$ and hence we have a canonical surjective homomorphism $\mathrm{Alb}(X,Y') \to \mathrm{Alb}(X,Y)$. In the case $Y=0$, $\mathrm{Alb}(X,Y) = \mathrm{Alb}(X)$.

In the case when X is a curve, $\mathrm{Alb}(X,Y)$ coincides with the generalized Jacobian variety $J(X,Y)$ of X with modulus Y as is explained in [10, Exm. 2.34].

5.4. As in [10], for $\mathcal{F} \in \Lambda$ we have a rational map

$$\alpha_{\mathcal{F}} : X \longrightarrow \mathrm{Alb}_{\mathcal{F}}(X)$$

which is canonically defined up to translation by a k-rational point of $\mathrm{Alb}_{\mathcal{F}}(X)$. If $\mathcal{F}' \in \Lambda$ and $\mathcal{F} \subset \mathcal{F}'$, then $\alpha_{\mathcal{F}}$ and $\alpha_{\mathcal{F}'}$ are compatible via the canonical surjection $\mathrm{Alb}_{\mathcal{F}'}(X) \to \mathrm{Alb}_{\mathcal{F}}(X)$.

For an effective divisor Y on X, we denote the rational map $\alpha_{\mathcal{F}_{X,Y}}$ simply by $\alpha_{X,Y}$. In Proposition 5.1 (2) below, we give a universal property of $\alpha_{X,Y} : X \to \mathrm{Alb}(X,Y)$ concerning rational maps from X to commutative algebraic groups. This property follows from a general universal property of $\alpha_{\mathcal{F}} : X \to \mathrm{Alb}_{\mathcal{F}}(X)$ obtained in [10], as is shown in No. 5.6 below.

5.5. Let G be a commutative connected algebraic group over k and let $\varphi : X \to G$ be a rational map. We define an effective divisor $\mathrm{mod}(\varphi)$ on X which we call the modulus of φ.

We treat X as a scheme. This divisor $\mathrm{mod}(\varphi)$ is written in the form $\sum_v \mathrm{mod}_v(\varphi)\, v$, where v ranges over all points of X of codimension one and $\mathrm{mod}_v(\varphi)$ is a non-negative integer defined as follows.

Let $0 \to L \to G \to A \to 0$ be the canonical decomposition of G and take an isomorphism

(1) $\quad L_a \cong (\mathbb{G}_a)^s$

where L_a is the additive part of L.

Let K be the function field of X, and regard φ as an element of $G(K)$. Since the local ring $\mathcal{O}_{X,v}$ of X at v is a discrete valuation ring and since A is proper, we have $A(O_{X,v}) = A(K)$. By the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & L(O_{X,v}) & \to & G(\mathcal{O}_{X,v}) & \to & A(O_{X,v}) & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & L(K) & \to & G(K) & \to & A(K) & \to & 0, \end{array}$$

we have $G(K) = L(K)G(O_{X,v})$. Write $\varphi \in G(K)$ as

(2) $\varphi = lg$ with $l \in L(K)$ and $g \in G(O_{X,v})$.

Let $(l_j)_{1 \leqslant j \leqslant s}$ be the image of l in $(\mathbb{G}_a)^s(K)$.

If φ belongs to $G(O_{X,v})$, we define $\operatorname{mod}_v(\varphi) = 0$. Assume that φ does not belong to $G(O_{X,v})$. Then we define

$$\operatorname{mod}_v(\varphi) = 1 + \max(\{-\operatorname{ord}_v(l_j) \mid 1 \leqslant j \leqslant s\} \cup \{0\}).$$

This integer $\operatorname{mod}_v(\varphi)$ is independent of the choice of the isomorphism (1) and of the choice of the presentation (2) of φ.

For example, if $G = \mathbb{G}_m$, $\operatorname{mod}_v(\varphi)$ is 0 if the element φ of $G(K) = K^\times$ belongs to $\mathcal{O}_{X,v}^\times$, and is 1 otherwise. If $G = \mathbb{G}_a$, $\operatorname{mod}_v(\varphi)$ is 0 if the element φ of $G(K) = K$ belongs to $\mathcal{O}_{X,v}$, and is $m+1$ if φ has a pole of order $m \geqslant 1$ at v.

Proposition 5.1. — *Let G be a commutative connected algebraic group over k and let $\varphi : X \to G$ be a rational map.*

(1) *For a dense open set U of X, φ induces a morphism $U \to G$ (not only a rational map) if and only if the support of* $\operatorname{mod}(\varphi)$ *does not meet U.*

(2) *Let Y be an effective divisor on X. Then the following two conditions (i) and (ii) are equivalent.*
   (i) *There is a homomorphism $h : \operatorname{Alb}(X,Y) \to G$ such that φ coincides with $h \circ \alpha_{X,Y}$ modulo a translation by $G(k)$.*
   (ii) $\operatorname{mod}(\varphi) \leqslant Y$.

Furthermore, if these equivalent conditions are satisfied, such homomorphism h is unique.

It is easy to prove (1). The proof of (2) is given in No. 5.7 below after we review results on $\operatorname{Alb}_{\mathcal{F}}(X)$ from [10].

5.6. We review a general universal property of $\operatorname{Alb}_{\mathcal{F}}(X)$ proved in [10] concerning rational maps from X into commutative algebraic groups.

Let $\varphi : X \to G$ be a rational map into a commutative connected algebraic group G, and let L be the canonical connected affine subgroup such that the quotient G/L is an abelian variety. One observes that φ induces a natural transformation $\tau_\varphi : L^\vee \to \underline{\operatorname{Div}}_X^0$ (see [10, Section 2.2]), where $L^\vee = \underline{\operatorname{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m)$ is the Cartier dual of L. It is shown in [10, Section 2.3] that if $\mathcal{F} \in \Lambda$, there is a rational map $\alpha_{\mathcal{F}} : X \to \operatorname{Alb}_{\mathcal{F}}(X)$ for which the corresponding homomorphism $\tau_{\alpha_{\mathcal{F}}} : \mathcal{F} \to \underline{\operatorname{Div}}_X^0$ coincides with the inclusion map, and such rational map $\alpha_{\mathcal{F}}$ is unique up to translation by a k-rational point of $\operatorname{Alb}_{\mathcal{F}}(X)$. For a rational map $\varphi : X \to G$ into a commutative connected algebraic group G and for $\mathcal{F} \in \Lambda$, there is a homomorphism

$h : \mathrm{Alb}_{\mathcal{F}}(X) \to G$ such that f coincides with $h \circ \alpha_{\mathcal{F}}$ up to translation by an element of $G(k)$ if and only if the image of the homomorphism $\tau_\varphi : L^\vee \to \underline{\mathrm{Div}}_X^0$ is contained in $\mathcal{F}$. Furthermore, if such h exists, it is unique.

Moreover, any rational map $\varphi : X \to G$ into a commutative connected algebraic group G coincides with $h \circ \alpha_{\mathcal{F}}$ up to translation by an element of $G(k)$ for some $\mathcal{F} \in \Lambda$ and for some homomorphism $h : \mathrm{Alb}_{\mathcal{F}}(X) \to G$. This is because there is always some $\mathcal{F} \in \Lambda$ which contains the image of $L^\vee \to \underline{\mathrm{Div}}_X^0$.

5.7. We prove Proposition 5.1. By No. 5.6 we find that condition (i) of Proposition 5.1 (2) is equivalent to

(i') The image of τ_φ is contained in $\mathcal{F}_{X,Y}$.

Write

$$Y = \sum_v e_v v$$

where v ranges over all points of X of codimension one and $e_v \in \mathbb{N}$. Condition (ii) of Proposition 5.1 (2) is expressed as

(ii') $\mathrm{mod}_v(\varphi) \leqslant e_v$ for all points v of codimension one in X.

Fix an isomorphism $L \cong (\mathbb{G}_m)^t \times (\mathbb{G}_a)^s$. For each point v of X of codimension one, take a presentation $\varphi = lg$ as in (2) in No. 5.5, let $(l'_{v,j})_{1\leqslant j\leqslant t}$ be the image of l in $(\mathbb{G}_m)^t(K) = (K^\times)^t$, and as in No. 5.5, let $(l_{v,j})_{1\leqslant j\leqslant s}$ be the image of l in $(\mathbb{G}_a)^s(K) = K^s$. Note that

(a) $\varphi \in G(\mathcal{O}_{X,v})$ if and only if $l'_{v,j} \in \mathcal{O}_{X,v}^\times$ for $1 \leqslant j \leqslant t$ and $l_{v,j} \in \mathcal{O}_{X,v}$ for $1 \leqslant j \leqslant s$.

By construction of the transformation τ_φ in [10, Section 2.2], we have the following (b) and (c).

(b) The étale part of τ_φ

$$\tau_{\varphi,\text{ét}} : \mathbb{Z}^t \longrightarrow \underline{\mathrm{Div}}_X^0(k)$$

sends the j-th base of $\mathbb{Z}^t$ $(1 \leqslant j \leqslant t)$ to the divisor $\sum_v \mathrm{ord}_v(l'_{v,j})\, v$.

(c) The infinitesimal part of τ_φ

$$\tau_{\varphi,\mathrm{inf}} : (\widehat{\mathbb{G}}_a)^s \longrightarrow \underline{\mathrm{Div}}_X^0$$

has the form

$$(a_j)_{1\leqslant j\leqslant s} \longmapsto \exp\left(\sum_{j=1}^s a_j f_j\right)$$

for some $f_j \in \Gamma(X, K/\mathcal{O}_X) = \mathrm{Lie}(\underline{\mathrm{Div}}_X^0)$ $(1 \leqslant j \leqslant s)$ such that for any point v of X of codimension one, the stalk of f_j at v coincides with $l_{v,j}$ mod $\mathcal{O}_{X,v}$.

Condition (i') is equivalent to the condition that the following (i'$_{\text{ét}}$) and (i'$_{\text{inf}}$) are satisfied.

(i'$_{\text{ét}}$) The image of $\tau_{\varphi,\text{ét}}$ is contained in the étale part of $\mathcal{F}_{X,Y}$.

(i'$_{\text{inf}}$) The image of $\tau_{\varphi,\text{inf}}$ is contained in the infinitesimal part of $\mathcal{F}_{X,Y}$.

By the above (b), (i'$_{\text{ét}}$) is equivalent to the condition that the following (i'$_{\text{ét},v}$) is satisfied for any point v of X of codimension one.

(i'$_{\text{ét},v}$) If $e_v = 0$, then $l'_{v,j} \in \mathcal{O}_{X,v}^{\times}$ for $1 \leqslant j \leqslant t$.

On the other hand, by the above (c), (i'$_{\text{inf}}$) is equivalent to

$$f_j \in \Gamma(X, J^{-1}/\mathcal{O}_X) \text{ for } 1 \leqslant j \leqslant s,$$

and hence equivalent to the condition that the following (i'$_{\text{inf},v}$) is satisfied for any point v of X of codimension one.

(i'$_{\text{inf},v}$) If $e_v = 0$, then $l_{v,j} \in \mathcal{O}_{X,v}$ for $1 \leqslant j \leqslant s$.
If $e_v \geqslant 1$, then $\operatorname{ord}_v(l_{v,j}) \geqslant 1 - e_v$ for $1 \leqslant j \leqslant s$.

By (a) above, for each v, (i'$_{\text{ét},v}$) and (i'$_{\text{inf},v}$) are satisfied if and only if $\operatorname{mod}_v(\varphi) \leqslant e_v$. □

Corollary 5.2. — *For any $\mathcal{F} \in \Lambda$, there exists an effective divisor Y such that $\mathcal{F} \subset \mathcal{F}_{X,Y}$.*

Proof. — Let $Y = \operatorname{mod}(\alpha_{\mathcal{F}})$ be the modulus of the rational map $\alpha_{\mathcal{F}}$: $X \to \operatorname{Alb}_{\mathcal{F}}(X)$ associated with $\mathcal{F} \in \Lambda$. Then $\mathcal{F} = \operatorname{Image}(\tau_{\alpha_{\mathcal{F}}}) \subset \mathcal{F}_{X,Y}$. □

6. Proof of Theorem 1.1

We prove Theorem 1.1. Let X be a proper smooth algebraic variety over $\mathbb{C}$ of dimension n, and let Y be an effective divisor on X. Let I be the ideal of $\mathcal{O}_X$ which defines Y, let I_1 be the ideal of $\mathcal{O}_X$ which defines the reduced part of Y, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

6.1. Let $H^1(X, Y_+)(1)$ be the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X,Y} \to \operatorname{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$ in the equivalence of categories of Theorem 4.1. Let $H^{2n-1}(X, Y_-)(n)$ be the object of $\mathcal{H}_{1,\{-1,-2\}}$ corresponding to the object $\operatorname{Alb}(X, Y)$ of $\mathcal{M}_{1,\{-1,-2\}}$.

Since the equivalence of categories in Theorem 4.1 is compatible with dualities, we have

$$H^{2n-1}(X, Y_-)(n) \cong \underline{\operatorname{Hom}}\big(H^1(X, Y_+)(1), \mathbb{Z}\big)(1). \tag{6.1}$$

We prove Theorem 1.1 in the following way. First in No. 6.3, we give an explicit description of $H^1(X, Y_+)(1)$. From this, by (6.1), we can obtain an explicit description of $H^{2n-1}(X,Y_-)(n)$ as in No. 6.4. Since $\mathrm{Alb}(X,Y)$ corresponds to $H^{2n-1}(X,Y_-)(n)$ in the equivalence of categories $\mathcal{H}_{1,\{-1,-2\}} \simeq \mathcal{M}_{1,\{-1,-2\}}$, we can obtain from No. 6.4 the explicit descriptions of $\mathrm{Alb}(X,Y)$ as stated in Theorem 1.1.

We define objects $H^1(X,Y_+)$ and $H^{2n-1}(X,Y_-)$ of $\mathcal{H}$ as follows: $H^1(X,Y_+)$ is the Tate twist $\big(H^1(X,Y_+)(1)\big)(-1)$ of $H^1(X,Y_+)(1)$, and $H^{2n-1}(X,Y_-)$ is the Tate twist $\big(H^{2n-1}(X,Y_-)(n)\big)(-n)$ of $H^{2n-1}(X,Y_+)(n)$. These are natural generalizations of the objects of $\mathcal{H}$ for the curve case considered in Example 2.1.

6.2. We define canonical $\mathbb{C}$-linear maps

$$H^1(X-Y,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}_X), \tag{6.2}$$

$$H^{n-1}(X,\Omega_X^n) \longrightarrow H_c^{2n-1}(X-Y,\mathbb{C}) \tag{6.3}$$

First assume that Y is with normal crossings. Then by [5], we have canonical isomorphisms

$$H^m(X-Y,\mathbb{C}) \cong H^m\big(X,\Omega_X^\bullet(\log(Y))\big),$$
$$H_c^m(X-Y,\mathbb{C}) \cong H^m\big(X,\Omega_X^\bullet(-\log(Y))\big)$$

for $m\in\mathbb{Z}$, where $\Omega_X^p(\log(Y))$ is the sheaf of differential p-forms with log poles along Y, and $\Omega_X^p(-\log(Y)) = I_1\Omega_X^p(\log(Y))$. Since $\mathcal{O}_X = \Omega_X^0(\log(Y))$ and $\Omega_X^n = \Omega_X^n(-\log(Y))$, we have canonical maps of complexes $\Omega_X^\bullet(\log(Y)) \to \mathcal{O}_X$ and $\Omega_X^n[-n] \to \Omega_X^\bullet(-\log(Y))$. These maps induce the maps (6.2) and (6.3) in the case Y is with normal crossings, respectively.

In general, take a birational morphism $X' \to X$ of proper smooth algebraic varieties over $\mathbb{C}$ such that the inverse image Y' of Y on X' is with normal crossings. Then we have maps

$$H^{n-1}(X,\Omega_X^n) \longrightarrow H^{n-1}(X',\Omega_{X'}^n) \longrightarrow H_c^{2n-1}(X'-Y',\mathbb{C}) = H_c^{2n-1}(X-Y,\mathbb{C})$$

where the second arrow is the map (6.3) for X', and the composition $H^{n-1}(X,\Omega_X^n) \to H_c^{2n-1}(X-Y,\mathbb{C})$ is independent of the choice of $X' \to X$. The $\mathbb{C}$-linear dual of (6.3) with respect to the Poincaré duality and Serre duality gives the map (6.2). The map (6.2) is also obtained as the composition

$$H^1(X-Y,\mathbb{C}) = H^1(X'-Y',\mathbb{C}) \longrightarrow H^1(X',\mathcal{O}_{X'}) \xleftarrow{\simeq} H^1(X,\mathcal{O}_X).$$

6.3. Let $H = H^1(X,Y_+)(1)$, the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X,Y} \to \mathrm{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$. We describe H. By [3, Thm. 4.7] which treats the case when Y has no multiplicity, we can identify $H_\mathbb{Z}$ with $H^1\big(X-Y,\mathbb{Z}(1)\big)$ and identify the map $H_\mathbb{C} \to \mathrm{Lie}(\mathrm{Pic}^0(X)) = H^1(X,\mathcal{O}_X)$

with the map (6.2) in No. 6.2. We have $H_V = H_{\mathbb{C}} \oplus H^0(X, J^{-1}/\mathcal{O}_X)$, the maps $a : H_{\mathbb{C}} \to H_V$ and $b : H_V \to H_{\mathbb{C}}$ are the evident ones, the weight filtration is given by $W_0 H = H$, $W_{-2}H = 0$,

$$W_{-1}H_{\mathbb{Q}} = H^1(X, \mathbb{Q}(1)),$$
$$W_{-1}H_V = H^1(X, \mathbb{C}),$$

and the Hodge filtration is given by $F^{-1}H_V = H_V$, $F^1 H_V = 0$, and

$$F^0 H_V = \operatorname{Ker}\big(H^1(X-Y, \mathbb{C}) \oplus H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X)\big)$$

where the map $H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is the connecting map of the exact sequence $0 \to \mathcal{O}_X \to J^{-1} \to J^{-1}/\mathcal{O}_X \to 0$.

6.4. Let $H = H^{2n-1}(X, Y_-)(n)$, the object of $\mathcal{H}_{1,\{-1,-2\}}$ corresponding to the object $\operatorname{Alb}(X,Y)$ of $\mathcal{M}_{1,\{-1,-2\}}$. By (6.1) in No. 6.1, we obtain the following description of H from the description of $H^1(X, Y_+)(1)$ in No. 6.3.

$$H_{\mathbb{Z}} = H_c^{2n-1}(X-Y, \mathbb{Z})/(\text{torsion}),$$
$$H_V = H_{\mathbb{C}} \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n),$$

the maps $a : H_{\mathbb{C}} \to H_V$ and $b : H_V \to H_{\mathbb{C}}$ are the evident ones, the weight filtration is given by $W_{-1}H = H$, $W_{-3}H = 0$,

$$W_{-2}H_{\mathbb{Q}} = \operatorname{Ker}\big(H_{\mathbb{Q}} \to H^{2n-1}(X, \mathbb{Q}(n))\big),$$
$$W_{-2}H_V = \operatorname{Ker}\big(H_V \to H^{2n-1}(X, \mathbb{C})\big),$$

and the Hodge filtration is given by $F^{-1}H_V = H_V$, $F^1 H_V = 0$, and

$$F^0 H_V = \operatorname{Image}\big(H^{n-1}(X, \Omega_X^n) \longrightarrow H_c^{2n-1}(X-Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n)\big)$$

where the map $H^{n-1}(X, \Omega_X^n) \to H_c^{2n-1}(X-Y, \mathbb{C})$ is (6.3) in No. 6.2 and the map $H^{n-1}(X, \Omega_X^n) \to H^{n-1}(X, \Omega_X^n/J\Omega_X^n)$ is the evident one.

6.5. We prove Theorem 1.1 (2). Let $H = H^{2n-1}(X, Y_-)(n)$. Then

$$\operatorname{Alb}(X,Y) = H_{\mathbb{Z}} \backslash H_V / F^0 H_V$$

by No. 4.5. Hence the description of $H^{2n-1}(X, Y_-)(n)$ in No. 6.4 proves Theorem 1.1 (2).

6.6. As a preparation for the proof of Theorem 1.1 (1), we review a kind of Serre-duality obtained in the appendix by Deligne of the book [8].

Let S be a proper scheme over a field k, let C be a closed subscheme of S, let $U = S - C$, and let I_C be the ideal of $\mathcal{O}_S$ which defines C. Assume U is smooth over k and purely of dimension n. Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Then for any $p \in \mathbb{Z}$, we have a canonical isomorphism

$$H^p\big(U, R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \Omega^n_U)\big) \cong \varinjlim_m \operatorname{Hom}_k\big(H^{n-p}(X, I^m_C\mathcal{F}), k\big).$$

In the case when C is empty and $\mathcal{F}$ is locally free, this is the usual Serre duality

$$H^p\big(X, \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega^n_X)\big) \cong \operatorname{Hom}_k\big(H^{n-p}(X, \mathcal{F}), k\big).$$

6.7. We start the proof of Theorem 1.1 (1).

Let C_Y be the subcomplex of $\Omega^\bullet_X$ defined as

$$C^p_Y = \ker(\Omega^p_X \to \Omega^p_Y) \text{ for } 0 \leqslant p \leqslant n-1, \quad C^n_Y = J\Omega^n_X.$$

Proposition 6.1. — *For $p = 2n, 2n-1$, the maps $H^p_c(X - Y, \mathbb{C}) \to H^p(X, C_Y)$ induced by the homomorphism $j_!\mathbb{C} \to C_Y$ are isomorphisms.*

6.8. We prove Proposition 6.1 in the case $Y = Y_1$. We have an exact sequence of complexes

$$0 \longrightarrow C_{Y_1} \longrightarrow \Omega^\bullet_X \longrightarrow \Omega^{\leqslant n-1}_{Y_1} \longrightarrow 0.$$

Since the support of $\Omega^{\leqslant n-1}_{Y_1}$ is of dimension $\leqslant n-1$ and since $\Omega^{\leqslant n-1}_{Y_1}$ has only terms of degree $\leqslant n-1$, we have $H^p(X, \Omega^{\leqslant n-1}_{Y_1}) = 0$ for $p \geqslant 2n-1$. Hence

$$H^{2n}(X, C_{Y_1}) \cong H^{2n}(X, \Omega^\bullet_X) \cong H^{2n}(X, \mathbb{C}) \cong H^{2n}_c(X - Y, \mathbb{C}).$$

The above exact sequence of complexes induces the lower row of the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H^{2n-2}(X,\mathbb{C}) & \to & H^{2n-2}(Y_1,\mathbb{C}) & \to & H^{2n-1}_c(X \setminus Y_1,\mathbb{C}) & \to & H^{2n-1}(X,\mathbb{C}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H^{2n-2}(X,\Omega^\bullet_X) & \to & H^{2n-2}\big(Y_1,\Omega^{\leqslant n-1}_{Y_1}\big) & \to & H^{2n-1}(X,C_{Y_1}) & \to & H^{2n-1}(X,\Omega^\bullet_X) & \to & 0. \end{array}$$

The vertical arrows except possibly the map $H^{2n-1}_c(X - Y_1, \mathbb{C}) \to H^{2n-1}(X, C_{Y_1})$ are isomorphisms. Hence the last map is also an isomorphism.

Lemma 6.2. — *Let Y' and Y'' be effective divisors on X whose supports coincide with Y_1 and assume $Y' \geqslant Y''$. Then the canonical map $H^{2n-1}(X, C_{Y'}) \to H^{2n-1}(X, C_{Y''})$ is surjective and the canonical map $H^{2n}(X, C_{Y'}) \to H^{2n}(X, C_{Y''})$ is an isomorphism.*

Proof. — Let $N = C_{Y''}/C_{Y'}$. We have

$$N^p = \operatorname{Ker}(\Omega^p_{Y'} \to \Omega^p_{Y''}) \quad \text{for } 0 \leqslant p \leqslant n-1, \quad N^n = J''\Omega^n_X/J'\Omega^n_X.$$

Here, $J' = I'I_1^{-1}$, $J'' = I''I_1^{-1}$ with I' (resp. I'') the ideal of $\mathcal{O}_X$ which defines Y' (resp. Y''). Since the support of N is of dimension $\leqslant n-1$ and N has only terms of degree $\leqslant n$, we have $H^{2n}(X,N)=0$. Hence it is sufficient to prove $H^{2n-1}(X,N)=0$.

Let Σ be the set of all singular points of Y_1. Then Σ is of dimension $\leqslant n-2$. Let $\Omega_X^\bullet(\log(Y_1))$ be the de Rham complex on $X-\Sigma$ with log poles along $Y_1-\Sigma$. Then, as is easily seen, the restriction of C_Y to $X-\Sigma$ coincides with $I\Omega_X^\bullet(\log(Y_1))$. Let I_Σ be the ideal of $\mathcal{O}_X$ defining Σ (here Σ is endowed with the reduced structure). For $k\geqslant 0$, let N_k be the subcomplex of N defined by $N_k^p = I_\Sigma^{\max(k-p,0)}N^p$. In particular, $N_0=N$. Then if $k\geqslant j\geqslant 0$, since the support of N_j/N_k is of dimension $\leqslant n-2$ and N_j/N_k has only terms of degree $\leqslant n$, we have $H^{2n-1}(X,N_j/N_k)=0$. Hence $H^{2n-1}(X,N_k)\to H^{2n-1}(X,N_j)$ is surjective. Applying No. 6.6 for $S=X$ and $C=\Sigma$ yields that $\varprojlim_k H^{2n-1}(X,N_k)$ is the dual vector space of $H^0\big(X-\Sigma,[(J')^{-1}/(J'')^{-1}\xrightarrow{d}(J')^{-1}\Omega_X(\log Y_1)/(J'')^{-1}\Omega_X(\log Y_1)]\big)$. Since $d:(J')^{-1}/(J'')^{-1}\to(J')^{-1}\Omega_X(\log Y_1)/(J'')^{-1}\Omega_X(\log Y_1)$ is injective, the last cohomology group is 0. Hence $H^{2n-1}(X,N_k)=0$ for all $k\geqslant 0$. In particular, $H^{2n-1}(X,N)=0$. □

6.9. We prove Proposition 6.1 in general. By Lemma 6.2, the map $\varprojlim_{Y'} H^{2n-1}(X,C_{Y'})\to H^{2n-1}(X,C_Y)$ is surjective, where Y' ranges over all effective divisors on X whose supports coincide with Y_1. By No. 6.6, which we apply by taking $S=X$ and $C=Y$, we have that $\varprojlim_{Y'} H^{2n-1}(X,C_{Y'})$ is the dual vector space of $H^1\big((X-Y)_{\mathrm{zar}},\Omega^\bullet_{X-Y,\mathrm{alg}}\big)$ where "zar" means Zariski topology and "alg" means the algebraic version. But $H^1\big((X-Y)_{\mathrm{zar}},\Omega^\bullet_{X-Y,\mathrm{alg}}\big)\simeq H^1(X-Y,\mathbb{C})$ by Grothendieck's Theorem [7, Thm. 1']. This proves $\varprojlim_{Y'} H^{2n-1}(X,C_{Y'})\cong H_c^{2n-1}(X-Y,\mathbb{C})$. Hence the map $H_c^{2n-1}(X-Y,\mathbb{C})\to H^{2n-1}(X,C_Y)$ is surjective. Since the composition $H_c^{2n-1}(X-Y,\mathbb{C})\to H^{2n-1}(X,C_Y)\to H^{2n-1}(X,C_{Y_1})\cong H_c^{2n-1}(X-Y,\mathbb{C})$ is the identity map, the map $H_c^{2n-1}(X-Y,\mathbb{C})\to H^{2n-1}(X,C_Y)$ is an isomorphism.

6.10. We prove (1) of Theorem 1.1. Let $S_Y=\mathrm{Ker}(\Omega_X^\bullet\to\Omega_Y^{\leqslant n-1})$. Then $C_Y\subset S_Y\subset C_{Y_1}$. We have an exact sequence of complexes

$$0\longrightarrow C_Y\longrightarrow S_Y\longrightarrow \Omega_X^n/J\Omega_X^n[-n]\longrightarrow 0.$$

Hence we have an exact sequence

$$H^{2n-1}(X, C_Y) \to H^{2n-1}(X, S_Y) \to H^{n-1}(X, \Omega_X^n/J\Omega_X^n) \to H^{2n}(X, C_Y) \\ \to H^{2n}(X, S_Y).$$

Note that for $p = 2n, 2n-1$, the compositions

$$H^p(X, C_Y) \longrightarrow H^p(X, S_Y) \longrightarrow H^p(X, C_{Y_1})$$

are isomorphisms by Proposition 6.1. Hence by Proposition 6.1, we have an isomorphism

$$H^{2n-1}(X, S_Y) \cong H_c^{2n-1}(X - Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n)$$

which is compatible with the maps from $H^{n-1}(X, \Omega_X^n)$. Hence (1) of Theorem 1.1 follows from (2) of Theorem 1.1.

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