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LINEAR MAPS PRESERVING ORBITS

by Gerald W. SCHWARZ

Abstract. — Let $H \subset \mathrm{GL}(V)$ be a connected complex reductive group where V is a finite-dimensional complex vector space. Let $v \in V$ and let $G = \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Following Raïs we say that the orbit Hv is *characteristic for H* if the identity component of G is H. If H is semisimple, we say that Hv is *semi-characteristic* for H if the identity component of G is an extension of H by a torus. We classify the H-orbits which are not (semi)-characteristic in many cases.

Résumé. — Soit $H \subset \mathrm{GL}(V)$ un groupe complexe réductif connexe où V est un espace vectoriel complexe de dimension finie. Soient $v \in V$ et $G = \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. D'aprés Raïs nous disons que l'orbite Hv est *caractéristique pour H* si la composante connexe de l'identité de G est H. Si H est semi-simple, nous disons que Hv est *semi-caractéristique pour H* si la composante connexe de l'identité de G est une extension de H par un tore. Nous classifions les orbites de H qui ne sont pas (semi)-caractéristiques dans plusieurs cas.

1. Introduction

Let K be a field. Then $H := \mathrm{PGL}_n(K)$ acts on $V := M(n,K)$ via conjugation. There is a large literature on solving *linear preserver problems*, that is, on finding the subgroups of $\mathrm{GL}(V)$ which preserve a certain set F of H-orbits in V. See [13] for a survey. One method of solving such problems is to classify all possible subgroups of $\mathrm{GL}(V)$ containing H and then check to see if these subgroups preserve F. This idea goes back at least to Dynkin [3] and has been used in many papers, *e.g.*, [5, 6, 7, 2, 1, 19]. We generalize the problem (but only in characteristic zero) by letting H be a reductive complex algebraic group, letting V be an arbitrary finite dimensional representation of H and letting F be an H-orbit Hv. The question then becomes: What is the subgroup G of $\mathrm{GL}(V)$ which preserves

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Hv? The method of solution is often to look at the possible G and possible G_v such that $G = HG_v$ (which implies that $Gv = Hv$). We are able to answer the question in many circumstances. We are particularly interested in identifying those cases where G^0 is the image of H, which, in the language of Raïs [20], means identifying those H-orbits which are *characteristic*.

Our base field is $\mathbb{C}$, the field of complex numbers. Let V be a finite dimensional H-module where H is a connected reductive group. Let $0 \neq v \in V$ and set $G := \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then G is a closed algebraic subgroup of $\mathrm{GL}(V)$ (see 2.1 below), We say that *Hv is characteristic for H* (or simply that v is characteristic for H or just that v is characteristic) if G^0 is the image of H in $\mathrm{GL}(V)$. (From now on we will not distinguish H from its image in $\mathrm{GL}(V)$, so we will say that v is characteristic if $G^0 = H$, even though this is not quite correct.) The definition that Hv is *semi-characteristic* is as above, except that we require only that G^0 is an extension of H by a torus (so G has to be reductive). In general, G is not reductive (see Examples 6.12, 6.13, 7.8 and 7.30). We say that v is *almost characteristic* if H is a Levi factor of G^0 and that *v is almost semi-characteristic* if H contains the semisimple part of a Levi factor of G^0.

In § 2 we consider some elementary properties of our definitions. We see that one has a chance for $G^0 = H$ only in the case that $v \in V$ is *generic*, which is equivalent to saying that Hv spans V. In § 3 we consider what can happen to G if we add a trivial factor to V. We show that Hv is characteristic if H is a torus and $v \in V$ is generic. In § 4 we consider the case that H is simple of rank at least 2 and V is irreducible. We recall some fundamental results of A. Onishchik which apply. We are then able to classify the irreducible H-modules V and $v \in V$ such that Hv is not semi-characteristic. We determine which orbits are semi-characteristic in the adjoint representation of a semisimple group. In § 5 we consider the case that H is simple of rank at least 2 and V is reducible. We determine the possible semisimple G containing H such that $Gv = Hv$ for $v \in V$. In § 6 we consider the case that H is semisimple and V is irreducible. In § 7 we determine the structure of G when $H = \mathrm{SL}_2$. In an appendix we prove branching rules which we need to establish our results.

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2. Elementary remarks

We consider when we can remove the prefixes "almost" and "semi." We also reduce to the case that Hv spans V. First we show that G is closed in $\mathrm{GL}(V)$.

Lemma 2.1. — *Let V be a finite-dimensional H-module where H is algebraic. Let $G = \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then G is a closed subgroup of $\mathrm{GL}(V)$.*

Proof. — Let $G_1 = \{g \in \mathrm{GL}(V) \mid g\overline{Hv} = \overline{Hv}\}$ and $G_2 = \{g \in \mathrm{GL}(V) \mid g(\overline{Hv} \setminus Hv) = (\overline{Hv} \setminus Hv)\}$. Then G_1 and G_2 are closed subgroups of $\mathrm{GL}(V)$ and $G = G_1 \cap G_2$. □

We now consider complexifications of compact group actions. Let C be a compact Lie group and W a real C-module. Let $w \in W$ and assume that Cw spans W.

Proposition 2.2. — *Let C, W and w be as above. Let $L = \{g \in \mathrm{GL}(W) \mid gCw = Cw\}$. Then L is compact.*

Proof. — Fix a basis $w_1, \dots, w_n$ of W lying in Cw and let $||\cdot||$ be a norm on W. Then for $g \in L$ and $1 \leqslant i \leqslant n$, $||gw_i||$ is bounded by a constant which is independent of g. Thus L is a closed bounded subset of $\mathrm{GL}(W)$, hence compact. □

Corollary 2.3. — *Let $H = C_{\mathbb{C}}$ be the complexification of C acting on $V = W \otimes_{\mathbb{R}} \mathbb{C}$. Let $G = \{g \in \mathrm{GL}(V) \mid gHw = Hw\}$. Then G is the complexification of L, hence reductive.*

Proof. — Since Cw is real algebraic [21, Lemma 4.3], it is defined by an ideal $I \subset \mathbb{R}[W]$, and clearly the complex zeroes of I are Hw. Let I_s denote the subspace of I of elements of degree at most s, $s \in \mathbb{N}$. Then I is generated by some I_s. Let $f_1, \dots, f_m$ be a basis for I_s. Then $g \in \mathrm{GL}(W)$ lies in L if and only if $g^* f_i \in I_s$ for all i. This gives a set of real equations defining the compact Lie group L, and the complex solutions of these equations are $L_{\mathbb{C}}$. But the complex solutions of the equations are clearly G. Thus $G = L_{\mathbb{C}}$. □

Recall ([11, 12, Ch. II Theorem 11]) that if $G \subset \mathrm{GL}(V)$ acts irreducibly on V, then G is reductive.

Corollary 2.4. — *Let H be reductive, let V be an H-module and let $v \in V$. Suppose that v is almost semi-characteristic for H. Then v is semi-characteristic in the following two cases.*

(1) *V is an irreducible representation of H.*
(2) *There is a compact Lie group C and real C-module W such that $V = W \otimes_{\mathbb{R}} \mathbb{C}$, $v \in W$ and $H = C_{\mathbb{C}}$.*

The following result characterizes when Hv is a cone.

Proposition 2.5. — *Let $0 \neq v \in V$ where V is an H-module. Suppose that Hv is a cone. Then there is a 1-parameter subgroup $\sigma\colon \mathbb{C}^* \to H$ such that v is an eigenvector of σ with nonzero weight.*

Proof. — Since Hv is a cone, $v \in T_v(Hv)$ and there is an $X \in \mathfrak{h}$ such that $X(v) = v$. Applying an element of H we can assume that $X \in \mathfrak{b}$, a Borel subalgebra of $\mathfrak{h}$. Write $X = s + n$ (Jordan decomposition) where s is semisimple and n is nilpotent. Then s and n are in $\mathfrak{b}$. We can assume that $s \in \mathfrak{t} \subset \mathfrak{b}$ where $\mathfrak{t}$ is the Lie algebra of T, a maximal torus of H. Write $v = \sum_{\lambda \in \Lambda} v_\lambda$ as a sum of nonzero weight vectors where Λ is the set of weights of V relative to T such that $v_\lambda \neq 0$. Let Φ be the set of positive roots. Then for $\lambda \in \Lambda$ we have $(s+n)v_\lambda = sv_\lambda$ modulo $\sum_{\mu > \lambda} V_\mu$ where $\mu > \lambda$ means that $\mu \in \lambda + \mathbb{N}\Phi$. Thus by an easy induction we get that $sv_\lambda = v_\lambda$ for all $\lambda \in \Lambda$ so that $sv = v$. Hence $S := \{t \in T \mid tv \in \mathbb{C}^* v\}^0$ is a subtorus of T which acts nontrivially on v. It follows that there is a one-parameter subgroup $\sigma\colon \mathbb{C}^* \to S$ as desired. □

Proposition 2.6. — *Let $0 \neq v \in V$ where V is an irreducible H-module. Suppose that $\mathbb{C}^* v \not\subset Hv$. Then v is characteristic if it is semi-characteristic. In particular, this holds if v is not in the null cone of V.*

Proof. — The group G is reductive and its center is contained in the scalar matrices. Under our hypotheses on v, the center must be finite. □

Let $V = \bigoplus_{i=1}^k n_i V_i$ be the isotypic decomposition of an H-module where H is nontrivial reductive. Let $v \in V$. Then $v = (v_{ij})$ where v_{ij} belongs to the jth copy of V_i, $j = 1, \dots, n_i$, $i = 1, \dots, k$. Let S denote $\mathrm{GL}(V)^H = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$. Let $U_i \subset V_i$ be the linear subspace of V_i generated by the v_{ij}, $j = 1, \dots, n_i$. If $\dim U_i = n_i$ for all i, then we say that v is *generic*.

Proposition 2.7. — *Let H be reductive, let V be an H-module and let $v \in V$. Then the following are equivalent.*

(1) *The span of Hv is V.*
(2) *There is no nontrivial one parameter subgroup of $S = \mathrm{GL}(V)^H$ which fixes v.*
(3) *The vector v is generic.*

Proof. — If $s \in S$, then v satisfies one of the conditions if and only if sv does. Clearly, if (1) or (3) fails, we can find an s and an i such that $(sv)_{i1} = 0$. If W denotes the first copy of V_i in $n_i V_i$, we have that $\mathbb{C}^* = GL(W)^H \subset \mathrm{GL}(V)^H$ is a one-parameter subgroup fixing v, so (2) fails. Conversely, if (2) fails, the fixed point set of a "bad" one-parameter subgroup is a proper H-submodule of V containing v and (1) and (3) fail. □

COROLLARY 2.8. — *Let $v \in V$ and let G be a Levi component of $\{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then v is generic for H if and only if it is generic for G.*

If v is not generic, then G is in a rather trivial way larger than H. To avoid this case, we usually assume from now on that v is generic.

3. Trivial factors and tori

Let $v \in V$ and suppose that $V^H = (0)$. Consider $\tilde{v} = (v, 1) \in \tilde{V} := V \oplus \mathbb{C}$. Let $\tilde{G} = \{g \in \mathrm{GL}(\tilde{V}) \mid gH\tilde{v} = H\tilde{v}\}$. We conjecture that $\tilde{G} = G$, where $G \subset \mathrm{GL}(\tilde{V})$ in the canonical way. Equivalently, we conjecture that the subgroup of the affine group $\mathrm{Aff}(V)$ preserving Hv lies in $\mathrm{GL}(V)$. Note that v generic implies that $\tilde{v}$ is generic (we can add at most a one-dimensional fixed point set). The following example shows that the conjecture fails if H is not reductive.

Example 3.1. — Let $H = (\mathbb{C}, +)$ act on $\mathbb{C}^2$ by sending $(a, b) \in \mathbb{C}^2$ to $(a, ta + b)$, $t \in H$. Let $\tilde{H} = H \times \mathbb{C}$ where $(t, s) \cdot (a, b) = (a, ta + s + b)$, $(t, s) \in \tilde{H}$, $(a, b) \in \mathbb{C}^2$. Then for $a \neq 0$, the H and $\tilde{H}$ orbits of (a, b) are the same, where $H \subset \mathrm{GL}(\mathbb{C}^2)$, $\tilde{H} \subset \mathrm{Aff}(\mathbb{C}^2)$ and $\tilde{H} \not\subset \mathrm{GL}(\mathbb{C}^2)$.

For this section only G will denote the subgroup of $\mathrm{Aff}(V)$ preserving Hv (rather than the corresponding subgroup of $\mathrm{GL}(V)$). It is easy to see that we can always reduce to the case that $V^H = (0)$, which we assume holds for the rest of this section.

We have a homomorphism $\mathrm{Aff}(V) \to \mathrm{GL}(V)$ which sends an element $(g, c) \in G \subset \mathrm{GL}(V) \ltimes V$ to $g \in \mathrm{GL}(V)$. Let G' denote the image of G in $\mathrm{GL}(V)$.

LEMMA 3.2. — *The homomorphism $G \to G'$ is injective.*

Proof. — The kernel K of $G \to G'$ consists of the pure translations in G, *i.e.*, the homomorphisms $x \mapsto x+c$ where $x, c \in V$. Clearly K is isomorphic to a closed subgroup of the additive group $(V,+)$ of V. Now $(V,+)$ has Lie algebra V (trivial bracket) and the exponential map is the identity. Thus $\mathfrak{k}$ is a vector subspace W of V and K/K^0 is isomorphic to a finite subgroup of $(V/W,+)$. Hence K is connected and $K = (W,+)$ where W must be H-stable. Let $\pi\colon V \to W$ be an H-equivariant projection (here we use that H is reductive). Then there are elements of G which translate v to v' where $\pi(v')$ is arbitrary. Since H preserves W and $\operatorname{Ker}\pi$, this is not possible for elements of H, unless $W = 0$. Hence K is the trivial group. □

Note that injectivity fails in the case of Example 3.1.

LEMMA 3.3. — *Let M be a reductive subgroup of the affine group $\operatorname{Aff}(V)$. Then there is an $\alpha \in \operatorname{Aff}(V)$ such that $\alpha M \alpha^{-1} \subset \mathrm{GL}(V)$.*

Proof. — We use transcendental methods. Choose a hermitian metric on V so that we have a unitary group $\mathrm{U}(V) \subset \mathrm{GL}(V)$. Let K be a maximal compact subgroup of M. Then M is the complexification $K_{\mathbb{C}}$ of K. Now any compact subgroup of $\operatorname{Aff}(V)$ is contained in a maximal compact subgroup of $\operatorname{Aff}(V)$ and all the maximal compact subgroups of $\operatorname{Aff}(V)$ are conjugate [9, Ch. XV Theorem 3.1]. But clearly $\mathrm{U}(V) \subset \operatorname{Aff}(V)$ is maximally compact. Thus K is conjugate to a subgroup of $\mathrm{U}(V)$, hence M is conjugate to a subgroup of $\mathrm{U}(V)_{\mathbb{C}} = \mathrm{GL}(V)$. □

PROPOSITION 3.4. — *In the following cases $G \subset \mathrm{GL}(V)$.*

(1) *The image $G' \subset \mathrm{GL}(V)$ is reductive.*
(2) *There is an $h' \in H$ such that $h'v = \lambda v$, $\lambda \in \mathbb{C}$, $\lambda \neq 1$.*

Proof. — If (1) holds, then G is reductive and there is an element $\alpha \in \operatorname{Aff}(V)$ such that $\alpha G \alpha^{-1} \subset \mathrm{GL}(V)$, hence $\alpha H \alpha^{-1} \subset \mathrm{GL}(V)$. But one easily sees that any affine transformation conjugating H into $\mathrm{GL}(V)$ must have translation part which is fixed by H. But V contains no nonzero H-fixed vectors. Hence $G \subset \mathrm{GL}(V)$.

Assume (2). Let $x \mapsto c + A(x)$ be an element of $\mathfrak{g} = \operatorname{Lie}(G)$ where $0 \neq c \in V$ and $A \in \mathfrak{gl}(V)$. Then the difference of $c + A(hv)$ and $c + A(hh'v)$ is a nonzero multiple of $A(hv)$ and lies in $\mathfrak{h}(hv)$ for any $h \in H$. Thus A itself lies in $\mathfrak{g}$ and $\mathfrak{g}$ contains the linear and translation parts of its elements. But $\mathfrak{g}$ cannot contain pure translations, as we saw above. Thus $\mathfrak{g} \subset \mathfrak{gl}(V)$ and $G \subset \mathrm{GL}(V)$. □

COROLLARY 3.5. — *We have that $G \subset \mathrm{GL}(V)$ in the following cases.*

(1) *V is an irreducible H-module.*
(2) *V is an SL_2-module whose irreducible components are all of even dimension, i.e., a module all of whose weights are odd.*

Remark 3.6. — Suppose that C, W, $w \in W$ are as in Proposition 2.2 where $W^C = 0$. Let L denote the subgroup of the real affine group of W stabilizing Cw. Then one can show that L is compact, and as above one sees that $L \subset \mathrm{GL}(W)$. Complexifying, we see that the subgroup of the affine group of $V = W \otimes_{\mathbb{R}} \mathbb{C}$ preserving Hw, where $H = C_{\mathbb{C}}$, is again just the complexification of L, a subgroup of $\mathrm{GL}(V)$.

Proposition 3.7. — *Let $V = \oplus_i n_i V_i$ be the isotypic decomposition of V. Suppose that for no i and j do we have that V_i occurs in $\mathrm{Hom}(V_j, V_i)$. Then $G \subset \mathrm{GL}(V)$.*

Proof. — Suppose that $G \not\subset \mathrm{GL}(V)$. Then we would have a subspace of $\mathfrak{g}$ consisting of elements $A_w + w$, $w \in W$, where $W \simeq V_i$ is an irreducible submodule of V, $A_w \in \mathfrak{gl}(V)$ and $hA_wh^{-1} = A_{hw}$ for $h \in H$. Our hypotheses imply that A_w followed by projection to n_iV_i is zero. Thus $\exp(A_w + w)(v)$ has the same projection to W as $v+w$. Hence we cannot have $Hv = Gv$. □

Example 3.8. — Let $V := \sum_{i=1}^n m_i\varphi_i$ and $H = \mathsf{A}_n$, $n \geqslant 1$, where φ_i is the ith fundamental representation of H, $i = 1, \ldots, n$. Then $G \subset \mathrm{GL}(V)$.

Theorem 3.9. — *Let H be a torus. Then*
(1) *$G \subset \mathrm{GL}(V)$.*
(2) *If $v \in V$ is generic, then $G^0 = H$.*

Proof. — We may assume that $V^H = (0)$. First consider (1) in the case that $H = \mathbb{C}^*$. Let W be the subspace of V spanned by $H \cdot v$. Then any $g \in G$ must preserve W, so we can replace V by W. Thus we can reduce to the case that $v \in V$ is generic. This implies that the weight spaces of H are one-dimensional. We have a weight basis $v_1, \ldots, v_n$ of V such that $v = (v_1, \ldots, v_n)$ where the weight of v_i is $0 \neq a_i \in \mathbb{Z}$. Suppose that the orbit of v is preserved by a transformation (g, c) where $(g, c)(v) = (\sum_j a_{ij}v_j + c_i)$. Here the a_{ij} and c_i are scalars. Then the ith component of $g(\lambda \cdot v)$ (where λ is a parameter in $H = \mathbb{C}^*$) is $\sum_j a_{ij}\lambda^{a_j}v_j + c_i$. Now the powers of λ that occur are distinct, hence the Laurent polynomial in λ that gives the ith component has some nonzero coefficient for a nonzero power of λ. If $c_i \neq 0$, then one can see that the polynomial takes on the value 0 for some $\lambda \neq 0$. But the $\mathbb{C}^*$-orbit of v is nonzero in the ith slot. Thus $c_i = 0$ for all i and g lies in $\mathrm{GL}(V)$ so we have (1). The reasoning above also shows that for each i there is a unique j such that $a_{ij} \neq 0$. Thus a power g^k of g preserves the weight spaces. Then $g^k v = hv$ for some $h \in H$, and it follows that $g^k = h$. Thus we have (2). Note that g normalizes $H = \mathbb{C}^*$, so that we actually have $g^2 \in H$.

Now suppose that H is a torus. As before, to prove (1), we can assume that v is generic. Let $(g = (a_{ij}), c) \in G$. Choose a 1-parameter subgroup λ of H such that all the characters of V, restricted to λ, are distinct. It follows, as above, that $c = 0$ and that a power of g lies in H. □

3.10. — Let G_0 denote a Levi component of G containing H. Then as we saw before, we must have that $G_0 \subset \mathrm{GL}(V)$. We can write G' as $G_0 \ltimes G'_u$ where G'_u is the unipotent radical of G'. Then we have the corresponding decompostion of $\mathfrak{g}'$ as $\mathfrak{g}_0 \ltimes \mathfrak{g}'_u$. As H-module, $\mathfrak{g}'_u$ is completely reducible. Assuming that G is not contained in $\mathrm{GL}(V)$ we can choose an irreducible H-module $W \subset \mathfrak{g}'_u$ whose inverse image in $\mathfrak{g}$ is not contained in $\mathfrak{gl}(V)$. Then we have a copy of W in V and elements $A_w \in \mathfrak{gl}(V)$, $w \in W$, such that $x \mapsto A_w(x) + w$ lies in $\mathfrak{g}$ and $\{A_w\}_{w\in W}$ maps to our copy of W in $\mathfrak{g}'_u$. For all $h \in H$ we have $hA_w h^{-1} = A_{hw}$.

THEOREM 3.11. — *Suppose that $v \in V$ is generic and in the null cone. Then $G \subset \mathrm{GL}(V)$.*

Proof. — Suppose the contrary. Let $V = \oplus_i n_i V_i$ be the isotypic decomposition of V as H-module. Let $A_w + w \in \mathfrak{g}$, $w \in W$ be as above where we may assume that $W = V_i$ (first copy) for some i. Let $\pi\colon V \to W$ be an equivariant projection and set $v_i = \pi(v)$. Since v is generic, $v_i \neq 0$. The projection of $\exp(w + A_w)(v)$ to W has the form $v_i + w + p(v, w)$ where $p(v, w)$ is a polynomial which has no linear factors in w and such that the coefficients of the various monomials in w are polynomials in v without constant term. By applying elements $h \in H$ we can make the coefficients of hw in $p(hv, hw)$ as small as we want. But there is no loss if we replace hw by w since we are able to consider all possible w. Thus we can assume that the coefficients of the monomials in w in $p(v, w)$ are very small, in which case the inverse function theorem tells us that $w \mapsto w + p(v, w)$ covers a ball around $0 \in W$ whose radius we can choose to be independent of v (for v close to zero). Then we see that $w \mapsto v_i + w + p(v, w)$ takes on the value 0. Thus Gv contains a point which projects to $0 \in W$, which is impossible. Hence $G \subset \mathrm{GL}(V)$. □

Recall that V is called *stable* if it contains a nonempty Zariski open subset of closed orbits.

COROLLARY 3.12. — *Let V be stable with a one-dimensional quotient. Then $G \subset \mathrm{GL}(V)$.*

Proof. — We have that $\mathbb{C}[V]^H = \mathbb{C}[f]$ where f is homogeneous of degree $d > 1$. Moreover, $f^{-1}(f(v)) = Hv = Gv$ if $f(v) \neq 0$. Now the case that $f(v) = 0$ follows from Theorem 3.11 and if $f(v) \neq 0$, then $Gv \supset \Gamma v$ where $\Gamma \subset G$ is a finite subgroup isomorphic to $\mathbb{Z}/d\mathbb{Z} \subset \mathbb{C}^*$ acting via scalar multiplication on V. Then $G \subset \mathrm{GL}(V)$ by Proposition 3.4(2). □

Remark 3.13. — A case by case check shows that H simple and $\dim /\!\!/ VH = 1$ implies that $G \subset \mathrm{GL}(V)$.

THEOREM 3.14. — *If $H = \mathrm{SL}_2$, then $G \subset \mathrm{GL}(V)$.*

We prove the theorem by contradiction, so assume that we have $A_w + w$ as in 3.10. Then the A_w lie in a Lie algebra of nilpotent matrices, and by Engel's theorem we can find a partial flag $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ such that V_1 is the joint kernel of the A_w, $V_2/V_1 \subset V/V_1$ is the joint kernel of the A_w, etc. Note that the V_j are H-stable.

LEMMA 3.15. — *We have $W \subset V_{k-1}$.*

Proof. — Since $A_w(v) \in V_{k-1}$, we have that $(A_w + w)(v + V_{k-1}) = w + V_{k-1}$. Thus $\exp(A_w + w)(v + V_{k-1}) = v + w + V_{k-1}$. Let π be the projection of V to V/V_{k-1} with kernel V_{k-1}. If $W \not\subset V_{k-1}$, then $\pi(Gv)$ contains a nontrivial H-stable subspace of V/V_{k-1}. This is not possible for the H-orbit, hence $W \subset V_{k-1}$. □

LEMMA 3.16. — *Suppose that for some $j \geqslant 1$ we have $W \subset V_{k-j}$ and $A_w(v) \in V_{k-j}$ for all $w \in W$. Then the stabilizer of $v + V_{k-j}$ in H is infinite.*

Proof. — Suppose that $v + V_{k-j}$ has finite stabilizer. Since $A_w(v) + w$ projects to zero in V/V_{k-j}, we must have that $A_w(v) + w = 0$, else the G-orbit of v has dimension greater than $\dim H$. Now for $h \in H$, $A_w(hv) + w = h(A_{h^{-1}w} + h^{-1}w)(v) = 0$, so that the average of $A_w(hv) + w$ over a maximal compact subgroup K of H is zero. Since $V^H = 0$ and A_w is linear, the average of $w + A_w(hv)$ over K is w. Thus $W = 0$, a contradiction. □

LEMMA 3.17. — *Suppose that $v + V_{k-j}$ is in the null cone of V/V_{k-j} for some $j \geqslant 1$. Then $W \subset V_{k-j-1}$.*

Proof. — Assume that $W \not\subset V_{k-j-1}$. Our argument in 3.11 shows that the G-orbit of v projected to the image of W in V/V_{k-j-1} contains zero, which is not possible for the H-orbit. Hence $W \subset V_{k-j-1}$. □

LEMMA 3.18. — *Suppose that for some $j \geqslant 1$ we have that $A_w(v) \in V_{k-j}$ for all $w \in W$ and that $W \subset V_{k-j}$. Then $v + V_{k-j}$ is in the null cone of V/V_{k-j}.*

Proof. — Suppose not. Consider $V' := \mathbb{C}\cdot v + V_{k-j} \subset V$. Then for $z \in \mathbb{C}$, $z(A_w + w)$ exponentiates to an element $g(z)$ of $\mathrm{Aff}(V')$ which fixes $v' := v + V_{k-j} \in V/V_{k-j}$. By Lemma 3.16 we know that v' has an infinite stabilizer S in H. Since v' is not in the null cone, S has identity component $T \simeq \mathbb{C}^*$. For any $s \in S$ there is an $h_{z,s} \in H$ such that $h_{z,s}v = g(z)sv$. Then $h_{z,s} \in S$ since $g(z)s$ fixes v' so that the $g(z)$ preserve the S-orbit of v. The group generated by T and the $g(z)$ is connected, so it preserves the T-orbit of v. By Theorem 3.9 we see that $w = 0$, a contradiction. □

Proof of Theorem 3.14. — We have that $A_w(v) \in V_{k-1}$ and $W \subset V_{k-1}$. Suppose that we have $A_w(v) \in V_{k-j}$ and $W \subset V_{k-j}$ for some $j \geqslant 1$. By Lemmas 3.16 and 3.18 and genericity of v we may assume that $v + V_{k-j}$ is a sum of highest weight vectors. By Lemma 3.17, $W \subset V_{k-j-1}$, so that if $A_w(v) \in V_{k-j-1}$ we can continue. We eventually arrive at a case where $A_w(v) \notin V_{k-j-1}$ (we cannot have a pure translation in $\mathfrak{g}$). Since $v + V_{k-j}$ is a sum of highest weight vectors there are unique elements $B_w \in \mathfrak{u}$ such that $A_w(v) + w + B_w(v) \in V_{k-j-1}$. Here $\mathfrak{u}$ is the Lie algebra of the standard unipotent subgroup of H. Since $A_w(v) + w \notin V_{k-j-1}$, $B_w(v) \notin V_{k-j-1}$ and $v + V_{k-j-1}$ is not a sum of highest weight vectors. Since $v + V_{k-j}$ is a nonzero sum of highest weight vectors, the H-isotropy group of $v + V_{k-j-1}$, which is a subgroup of the H-isotropy group of $v + V_{k-j}$, is finite. Arguing as in Lemma 3.16 we obtain that $w' := A_w(v) + w + B_w(v) = 0$ and that $W = 0$, a contradiction. Hence $G \subset \mathrm{GL}(V)$. □

From now on we will assume that $V^H = 0$, even though we have only established our conjecture for SL_2 or the case that V is irreducible.

4. The case H is simple and V is irreducible

Our goal in this section is to find the possible $G \subset \mathrm{GL}(V)$ preserving an orbit Hv where V is an irreducible H-module and H is simple of rank at least two. We will see that perforce G is simple. We begin by recalling some important results of Onishchik.

Let $H \subset G$ where G and H are linear algebraic groups. Let V be an H-module. If $v \in V$ and $Gv = Hv$, then $G = HK$ where $K = G_v$. Conversely, $G = HK$ implies that $Gv = Hv$ for $v \in V^K$. There is a rather restricted class of possibilities for H and K when G is simple and H is semisimple, as follows from the work of Onishchik [16, 17].

If K is a connected complex linear algebraic group, let $\mathfrak{k}$ denote its Lie algebra and let $L(K)$ denote a Levi subgroup of K. The next two theorems follow from [16] and [17] (see also [18] and [4]).

THEOREM 4.1. — *Let H and K be connected algebraic subgroups of the connected reductive group G. Then the following are equivalent.*

(1) $G = HK$.
(2) $G = \sigma(H)\tau(K)$ *where σ and τ are any automorphisms of G.*
(3) $G = L(H)L(K)$.
(4) $G_0 = H_0K_0$ *where H_0 and K_0 are maximal compact subgroups of H and K contained in a maximal compact subgroup G_0 of G.*
(5) $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ *(if H and K are reductive).*

COROLLARY 4.2. — *Suppose that $G = HK$ where all the groups are connected algebraic. Choose Levi factors $L(G) \supset L(H), L(K)$. Then $L(G) = L(H)L(K)$.*

Now assume that $\mathfrak{h}$ and $\mathfrak{k}$ are reductive subalgebras of the reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_s$ be the sum of the simple components of $\mathfrak{g}$ of rank at least 2 (the *strongly semisimple* part of $\mathfrak{g}$). Let G_s be the corresponding subgroup of G. Let $r(\mathfrak{g})$ be the sum of the center and simple components of rank 1 of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_s \oplus r(\mathfrak{g})$.

THEOREM 4.3. — *Let $\mathfrak{h}$ and $\mathfrak{k}$ be reductive subalgebras of the reductive Lie algebra $\mathfrak{g}$. Then the following are equivalent.*

(1) $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$.
(2) $\mathfrak{g}_s = \mathfrak{h}_s + \mathfrak{k}_s$ *and $r(\mathfrak{g})$ is the sum of the projections of $r(\mathfrak{h})$ and $r(\mathfrak{k})$ to $r(\mathfrak{g})$.*

COROLLARY 4.4. — *Suppose that $v \in V$ is not semi-characteristic and that H contains a strongly semisimple subgroup. Then so does G_v.*

From the above and [16] we have the following

THEOREM 4.5. — *Let G be connected, simple and simply connected of rank at least 2. Let H and K be connected semisimple subgroups of G such that $G = HK$. Then, up to switching the roles of H and K and replacing each of them by their image under an automorphism of G, all possibilities are listed in Table 1.*

Table 1.

	G	H	$\varphi_1(G)\|H$	K	$\varphi_1(G)\|K$	$H \cap K$
1	A_{2n-1}	C_n	φ_1	A_{2n-2}	$\varphi_1+\theta_1$	C_{n-1}
2	D_{n+1}	B_n	$\varphi_1+\theta_1$	A_n	$\varphi_1+\varphi_n$	A_{n-1}
3.1	D_{2n}	B_{2n-1}	$\varphi_1+\theta_1$	C_n	$2\varphi_1$	C_{n-1}
3.2	D_{2n}	B_{2n-1}	$\varphi_1+\theta_1$	$\mathsf{C}_n \times \mathsf{A}_1$	$\varphi_1 \otimes \varphi_1$	$\mathsf{C}_{n-1} \times \mathsf{A}_1$
4.1	B_3	G_2	φ_2	B_2	$\varphi_1+\theta_2$	A_1
4.2	B_3	G_2	φ_2	D_3	$\varphi_1+\theta_1$	A_2
5.1	D_4	B_3	φ_3	B_2	$\varphi_1+\theta_3$	A_1
5.2	D_4	B_3	φ_3	$\mathsf{B}_2 \times \mathsf{A}_1$	$\varphi_1+\varphi_1^2$	$\mathsf{A}_1 \times \mathsf{A}_1$
5.3	D_4	B_3	φ_3	D_3	$\varphi_1+\theta_2$	A_2
5.4	D_4	B_3	φ_3	B_3	$\varphi_1+\theta_1$	G_2
6	D_8	B_7	$\varphi_1+\theta_1$	B_4	φ_4	B_3

In our tables, we always have $n > 1$ and $k \geqslant 1$. We use θ_k to denote a trivial representation of dimension k. Corresponding to an ordering of the simple roots of G we have fundamental representations $\varphi_i = \varphi_i(G)$, $i = 1, \dots, \operatorname{rank} G$. We use the ordering of the roots of the simple groups of Dynkin [3]. Note that entries (5.1), (5.2) and (5.3) of Table 1 are special cases of (3.1), (3.2) and (2), but we have included them for completeness.

Corollary 4.6. — *Let (G, H, K) be a triple in Table 1.*

(1) *If $L \subset G$ is a reductive subgroup commuting with H or K, then L has rank at most 1.*
(2) *We have $G = H_s K_s$ where K_s and H_s are simple.*

Now that we know the possibilities for G, H and K, our task is to find the irreducible representations of G which remain irreducible when restricted to H. This can be read off from [3, Table 5]. However, given that one knows the possibilities for (G, H, K), it is relatively easy to see which irreducible representations of G are possible. Note that we can sometimes gain an irreducible representation by adding a group of rank 1 to H (Table 2 (3.5)).

Theorem 4.7. — *Let $G = G_s$ be simple and let H and $K = K_s$ be proper semisimple subgroups of G such that $G = HK$. Assume that V is an irreducible representation of G which is also irreducible when restricted to H. Then, up to automorphisms of G, all possibilities are listed in Table 2.*

Table 2.

	G	V	H	$V\|H$	K	$V\|K$	V^K
1	A_{2n-1}	φ_1^k	C_n	φ_1^k	A_{2n-2}	$\varphi_1^k + \varphi_1^{k-1} + \cdots + \theta_1$	θ_1
2.1	D_{2n+1}	φ_{2n}^k	B_{2n}	φ_{2n}^k	A_{2n}	$\mathcal{S}^k(\varphi_1 + \varphi_3 + \cdots + \varphi_{2n-1} + \theta_1)$	θ_1
2.2	D_{2n+1}	φ_{2n+1}^k	B_{2n}	φ_{2n}^k	A_{2n}	$\mathcal{S}^k(\varphi_2 + \varphi_4 + \cdots + \varphi_{2n} + \theta_1)$	θ_1
3.1	D_{2n}	φ_{2n-1}^k	B_{2n-1}	φ_{2n-1}^k	A_{2n-1}	$\mathcal{S}^k(\varphi_2 + \varphi_4 + \cdots + \varphi_{2n-2} + \theta_2)$	$S^k(\mathbb{C}^2)$
3.2	D_{2n}	φ_{2n-1}^k	B_{2n-1}	φ_{2n-1}^k	C_n	$*$	$S^k(\mathbb{C}^{n+1})$
3.3	D_{2n}	φ_{2n}^k	B_{2n-1}	φ_{2n-1}^k	A_{2n-1}	$\mathcal{S}^k(\varphi_1 + \varphi_3 + \cdots + \varphi_{2n-1})$	(0)
3.4	D_{2n}	φ_{2n}^k	B_{2n-1}	φ_{2n-1}^k	C_n	$*$	(0)
3.5	D_{2n}	φ_1	$\mathsf{C}_n \times \mathsf{A}_1$	$\varphi_1 \otimes \varphi_1$	B_{2n-1}	$\varphi_1 + \theta_1$	θ_1
4.1	B_3	φ_1^k	G_2	φ_2^k	B_2	$\mathcal{S}^k(\varphi_1 + \theta_2)$	$S^k(\mathbb{C}^2)$
4.2	B_3	φ_1^k	G_2	φ_2^k	D_3	$\mathcal{S}^k(\varphi_1 + \theta_1)$	θ_1
5.1	D_4	φ_1^k	B_3	φ_3^k	B_2	$\mathcal{S}^k(\varphi_1 + \theta_3)$	$S^k(\mathbb{C}^3)$
5.2	D_4	φ_1^k	B_3	φ_3^k	D_3	$\mathcal{S}^k(\varphi_1 + \theta_2)$	$S^k(\mathbb{C}^2)$
5.3	D_4	φ_1^k	B_3	φ_3^k	B_3	$\mathcal{S}^k(\varphi_1 + \theta_1)$	θ_1
5.4	D_4	φ_1	$\mathsf{C}_2 \times \mathsf{A}_1$	$\varphi_1 \otimes \varphi_1$	B_3	$\varphi_1 + \theta_1$	θ_1
6.1	D_8	φ_1	B_4	φ_4	B_7	$\varphi_1 + \theta_1$	θ_1
6.2	D_8	φ_7	B_4	$\varphi_1\varphi_4$	B_7	φ_7	(0)
6.3	D_8	φ_7^k	B_7	φ_7^k	B_4	$*$	$S(f_4)_k$
6.4	D_8	φ_8^k	B_7	φ_7^k	B_4	$*$	$S(f_2, f_3)_k$

In Table 2, if V is a K-module, then $\mathcal{S}^k(V)$ denotes the K-subspace of $S^k(V)$ generated by $S^k(V^U)$ where U is a maximal unipotent subgroup of K. In other words, in $\mathcal{S}^k(V)$ we take only the Cartan components of products. In column V^K the notation $S(f_2, f_3)_k$ means the span of the monomials in f_2 and f_3 of degree k where f_i has degree i, $i = 2$, 3. Here the $f_i \in \mathbb{C}[\varphi_8(D_8)]^{B_4}$. A similar interpretation applies to $S(f_4)_k$ where $f_4 \in \mathbb{C}[\varphi_7(D_8)]^{B_4}$ has degree 4. The justification of the entries $V|K$ and V^K can be found in the Appendix.

Remark 4.8. — In Table 2, we have chosen not to remove all redundancies due to automorphisms of groups of type D_n. In column $V|K$ we have omitted the decompositions in (3.2) and (3.4) which are obtained by restricting V to C_n. To determine this one needs the branching rule for restrictions of SL_{2n}-representations to C_n. As determined by Weyl [27], one proceeds as follows. Let $\omega \in \bigwedge^2((\mathbb{C}^{2n})^*)$ be nonzero and C_n-invariant. Then from the exterior powers of ω we obtain invariants in the duals of $\varphi_2^{b_2} \ldots \varphi_{2n-2}^{b_{2n-2}}$ for nonnegative b_j. Then the restriction of an irreducible representation φ of SL_{2n} to C_n is obtained by taking all possible complete contractions of the duals of our invariants in the $\varphi_2^{b_2} \ldots \varphi_{2n-2}^{b_{2n-2}}$ with φ. For example, $\varphi_1\varphi_5(\mathrm{SL}_8)$ contracted with ω gives rise to φ_4 and $\varphi_1\varphi_3$ while contraction with $\omega \wedge \omega \in \wedge^4(\mathbb{C}^8)^*$ gives rise to φ_1^2 and φ_2. Note that the only representations of SL_{2n} which can give rise to the trivial representation of C_n are those of the form $\varphi_2^{a_2} \ldots \varphi_{2n-2}^{a_{2n-2}}$. Hence in Table 2 (3.4) the trivial C_n-representation does not occur in the column $V|K$ for any $k \geqslant 1$

and in (3.2) the occurrences of the trivial representation are the symmetric algebra in θ_2 and the subspaces $\varphi_2^{\mathsf{C}_n}, \dots, \varphi_{2n-2}^{\mathsf{C}_n}$.

Remark 4.9. — We do not know what to put in the column $V|K$ in cases (6.3) and (6.4) of Table 2. However, in the Appendix we are able to compute V^K.

Suppose that $H \subset G$ where G and H are semisimple, and connected and V is a G-module which is irreducible as an H-module. Then the inclusion of H in G has a very special form, as shown by Dynkin [3, Theorem 2.2].

THEOREM 4.10. — *Let $G_1, \dots, G_k$ be the simple components of G. Then $H = H_1 \cdots H_k$ where the H_i are nontrivial semisimple subgroups of the G_i, $i = 1, \dots, k$.*

We are interested in the case that H is simple. Then Theorem 4.10 tells us that G has to be simple and from Table 2 we get the following theorem.

THEOREM 4.11. — *Let H be simple of rank at least two and let $\varphi\colon H \to \mathrm{GL}(V)$ be an irreducible representation. Then every nonzero H-orbit Hv is semi-characteristic, except for the following cases (where $n \geqslant 2$ and $k \geqslant 1$).*

(1) *$H = \mathsf{C}_n$, $\varphi = \varphi_1^k$ and v is a highest weight vector. Equivalently, v is fixed by A_{2n-2} where A_{2n-2}, C_n, A_{2n-1} and V are as in Tables 1(1) and 2(1).*

(2) *$H = \mathsf{B}_{2n}$, $\varphi = \varphi_{2n}^k$ and v is a highest weight vector. Equivalently, v is fixed by A_{2n} where A_{2n}, B_{2n}, D_{2n+1} and V are as in Tables 1(2) and 2(2.1) or 2(2.2).*

(3) *$H = \mathsf{B}_{2n-1}$, $\varphi = \varphi_{2n-1}^k$ and v is fixed by C_n where C_n, B_{2n-1}, D_{2n} and V are as in Tables 1(3.1) and 2(3.2).*

(4) *$H = \mathsf{G}_2$, $\varphi = \varphi_2^k$ and v is fixed by B_2 where B_2, G_2, B_3 and V are as in Tables 1(4.1) and 2(4.1).*

(5) *$H = \mathsf{B}_4$, $\varphi = \varphi_4$ and Hv is closed. Equivalently, v is fixed by B_7 where B_4, B_7, D_8 and V are as in Tables 1(6) and 2(6.1).*

(6) *$H = \mathsf{B}_7$, $\varphi = \varphi_7^k$ and v is fixed by B_4 where B_4, B_7, D_8 and V are as in Tables 1(6) and 2(6.3) or 2(6.4).*

For special direct sums of representations we have the following result.

PROPOSITION 4.12. — *Let V_i be an irreducible H_i-module where the H_i are semisimple, $i = 1, \dots, k$. Let $V = V_1 \oplus \cdots \oplus V_k$ be the corresponding $H = H_1 \times \cdots \times H_k$-module. Suppose that $0 \neq v_i \in V_i$ such that v_i is (semi)-characteristic for H_i for each i. Then v is (semi)-characteristic for H where $v = (v_1, \dots, v_k) \in V$.*

Proof. — Let G be as usual. First suppose that V is an irreducible G^0-module. Then, up to a cover and scalar matrices, $G^0 = G_1 \times \cdots \times G_r$ where the G_j are simple and $V \simeq U_1 \otimes \cdots \otimes U_r$ where the U_j are irreducible G_j-modules. By Theorem 4.10 each simple factor of each H_i must project nontrivially to a single G_j. But given the structure of V as H-module, this implies that $k = 1$, where the theorem is trivial.

We may now assume that there is a maximal flag $W_1 \subset \cdots \subset W_r \subset W_{r+1} = V$ of G^0-stable subspaces where $r \geqslant 1$. We may assume that, as H-module, $W_r = V_1 \oplus \cdots \oplus V_p$ so that $V/W_r \simeq V_{p+1} \oplus \cdots \oplus V_k$. The image of G in $\mathrm{GL}(V/W_r)$ is reductive. Let G' denote its semisimple part. Then for $g \in G'$, we have $g(v_{p+1}, \ldots, v_k) \in (H_{p+1} \times \cdots \times H_k)(v_{p+1}, \ldots, v_k)$. By induction on k, $G' = H_{p+1} \times \cdots \times H_k$. But by maximality of the flag, we must have that $p = k - 1$, *i.e.*, $V/W_r \simeq V_k$. If V_k is not G-stable, then $\mathfrak{g}$ contains a nonzero linear map of V_k to W_r. Since $\mathfrak{g}$ is stable under the action of H, we may assume that it contains $\mathrm{Hom}(V_k, V_1)$. Thus the G-orbit of v contains a point $(0, v_2, \ldots, v_k)$. Such a point is not in Hv, so we have a contradiction. Thus V_k is G-stable and we have a G-module direct sum decomposition $V = W_r \oplus V_k$. It follows by induction on k that Hv is (semi)-characteristic. □

If one considers the adjoint representation $\mathfrak{h}$ of a simple H, the only case that appears in Theorem 4.11 is $H = \mathsf{C}_n$, $n \geqslant 2$, where $\mathfrak{h} = \varphi_1^2$. Thus we have

COROLLARY 4.13. — *Let $H = H_1 \times \cdots \times H_k$ where the H_i are simple, and let $\varphi\colon H \to \mathrm{GL}(\mathfrak{h})$ be the adjoint representation. Let $v = (v_1, \ldots, v_k) \in \oplus_i \mathfrak{h}_i$ where no v_i is zero. Then v is semi-characteristic if and only if for every simple component H_i of type C_n, $n \geqslant 2$, $v_i \in \mathfrak{c}_n$ is not on the highest weight orbit.*

5. The case H is simple

We now consider the case where H is simple of rank at least two and our H-module V may be reducible. We consider the possible semisimple G which can act on V such that $Gv = Hv$ where $v \in V$ is generic for the action of H.

Here are some examples to keep in mind.

Example 5.1. — Let H be simple and let V be an H-module. Let $G = H \times H$ and let v be the identity in $V \otimes V^*$. Then $Gv = Hv$ where the diagonal copy of H in G plays the role of K.

Example 5.2. — Let (G, H, K) or (G, K, H) be an entry of Table 1. Let $V = \bigoplus_{i=1}^{n} m_i V_i$ be the isotypic decomposition of a G-module such that $\dim V_i^K \geqslant m_i \geqslant 1$ for all i. Let $v \in V^K$ be generic for the action of G. Then $Gv = Hv$.

Example 5.3. — Here $H \simeq \mathsf{A}_{2n-1}$. Let $(G_1, H_1, K_1) = (\mathsf{D}_{2n}, \mathsf{A}_{2n-1}, \mathsf{B}_{2n-1})$ and $(G_2, H_2, K_2) = (\mathsf{A}_{2n-1}, \mathsf{A}_{2n-2}, \mathsf{C}_n)$. Let $G = G_1 \times G_2$, let H denote the diagonal copy of A_{2n-1} and let $K = K_1 \times K_2$. Then $G = HK$. Let V be a representation of G which contains a generic vector $v \in V^K$. If V is irreducible, then $V = V_1 \otimes V_2$ where V_i is an irreducible representation of G_i, $i = 1, 2$, and $v \in V_1^{K_1} \otimes V_2^{K_2}$. The only possibilities allowing nontrivial fixed points are $V_1 = \varphi_1^k$, $k \geqslant 0$, and $V_2 = \varphi_2^{a_2} \varphi_4^{a_4} \cdots \varphi_{2n-2}^{a_{2n-2}}$ where the a_{2i} are in $\mathbb{Z}^+$. In both cases, $\dim V_i^{K_i} = 1$. Thus for v to be generic, V must be a sum of representations (each of multiplicity one) of the form $V_1 \otimes V_2$ where $\dim V_i^{K_i} = 1$ for all i. For V to be almost faithful the sum must contain a nontrivial V_1 and a nontrivial V_2.

Example 5.4. — Here $H \simeq \mathsf{A}_3$. Let $(G_1, H_1, K_1) = (\mathsf{B}_3, \mathsf{D}_3, \mathsf{G}_2)$ and $(G_2, H_2, K_2) = (\mathsf{A}_3, \mathsf{A}_2, \mathsf{C}_2)$. Let $G = G_1 \times G_2$, let H be the diagonal copy of $\mathsf{A}_3 = \mathsf{D}_3$ and let $K = K_1 \times K_2$. Then $G = HK$. If V_i is an irreducible representation of G_i, $i = 1, 2$, with $V_1^{K_1} \neq (0) \neq V_2^{K_2}$, then $V_1 = \varphi_3^k$, $k \geqslant 0$ and $V_2 = \varphi_2^\ell$, $\ell \geqslant 0$. Again, $\dim V_i^{K_i} = 1$, $i = 1, 2$, and the conditions for v generic and V almost faithful are as in the case above.

Example 5.5. — Here $H \simeq \mathsf{B}_3$. Let $(G_1, H_1, K_1) = (\mathsf{D}_4, \mathsf{B}_3, \mathsf{B}_3)$ and $(G_2, H_2, K_2) = (\mathsf{B}_3, \mathsf{G}_2, \mathsf{B}_2 \text{ or } \mathsf{D}_3)$. Let $G = G_1 \times G_2$, let $H \subset H_1 \times G_2$ be the diagonal copy of B_3 and let $K = K_1 \times K_2$. Then $G = HK$. The possibilities for the V_i having nontrivial K_i-fixed points are $V_1 = \varphi_1^k$, $k \geqslant 0$ and $V_2 = \varphi_1^a \varphi_2^b$ if $K_2 = \mathsf{B}_2$ and $V_2 = \varphi_1^a$ if $K_2 = \mathsf{D}_3$ where a and b are nonnegative. While $V_1^{K_1}$ has dimension 1, this is not true for $V_2^{K_2}$, in general, if $K_2 = \mathsf{B}_2$. Let $v \in V$ be generic and K-fixed. Then irreducible G-modules which can occur in V are sums of tensor products of modules $V_1 \otimes V_2$ where $\dim V_i^{K_i} \geqslant 1$, $i = 1, 2$.

Example 5.6. — Let (G_1, H_1, K_1) be an entry of Table 1. Let $G = G_1 \times G_1$, let H be the diagonal copy of G_1 and let $K = H_1 \times K_1 \subset G$. Then $G = HK$. It is usually easy to determine the almost faithful $V_1 \otimes V_2$ with K-fixed points. For example, in the case of $(\mathsf{A}_{2n-1}, \mathsf{C}_n, \mathsf{A}_{2n-2})$, V_1 has to be of the form $\varphi_2^{a_2} \cdots \varphi_{2n-2}^{a_{2n-2}}$ where the a_{2i} are nonnegative and V_2 has to be of the form φ_1^k or φ_{2n-1}^k for $k \geqslant 0$. On the other hand, if we have $(\mathsf{D}_8, \mathsf{B}_7, \mathsf{B}_4)$, then V_1 is of the form φ_1^k, $k \geqslant 0$, and we have been unable to pin down exactly which V_2 have B_4 fixed points.

Looking at Table 1 one easily sees the following.

Proposition 5.7. — *Suppose that* (G, H, K) *and* (G', H', K') *appear in Table 1 where* $H \cap K \simeq H' \subset L \subset G'$ *or* $H \cap K \simeq K' \subset L \subset G'$ *and* L *is isomorphic to* H *or* K*. Then* G' *is isomorphic to* L*.*

5.8. — Let H be simple of rank at least 2 and let V be an almost faithful H-module. Let $v \in V$ be generic. Let $G = G_1 \times \cdots \times G_r$ where the G_i are simple and simply connected and G acts almost faithfully on V such that $Gv = Hv$ where $H \subset G$. Let K denote a Levi factor of G_v. Then $G = HK$. Let $\mathrm{pr}_i \colon G \to G_i$ denote projection on the ith factor, $i = 1, \ldots, r$. We may assume that $\mathrm{pr}_i(H) \neq \{e\}$ for $1 \leqslant i \leqslant s$ and $\mathrm{pr}_i(H) = \{e\}$ for $s < i \leqslant r$ where $s \geqslant 1$. Let $G' = G_1 \times \cdots \times G_s$ and $G'' = G_{s+1} \times \cdots \times G_r$. For $r \geqslant j > s$ there is a unique simple component K_j of K such that $\mathrm{pr}_j(K_j) = G_j$ and clearly $K'' := K_{s+1} \times \cdots \times K_r$ covers G''. The kernel of $K'' \to G'$ commutes with H and fixes v. Since Hv spans v (Proposition 2.7), the kernel must be finite. Hence K'' covers its image in G'. Let K' denote the product of the simple components of K not in K''. Then $K' \subset G'$ and the projection of K'' to G' centralizes K'. We must have that $HK' = G'$.

We may write $V = \bigoplus V_i \otimes W_i$ where the V_i are pairwise nonisomorphic irreducible representations of G' and the W_i are representations of G''. Then the projection v_i of v to each $V_i \otimes W_i$ is a tensor of rank $\dim W_i$ since v is generic. Let U_i denote the smallest subspace of V_i such that $v_i \in U_i \otimes W_i$. Then $\dim U_i = \dim W_i$ and $v_i \in U_i \otimes W_i$ corresponds to a K''-equivariant isomorphism of W_i^* onto $U_i \subset V_i$. In the sense of the following definition, v_i corresponds to a *subordination* $\alpha_i \colon (W_i^*, G'') \to (V_i, G')$.

Definition 5.9. — *Let* Z_i *be an* L_i*-module* $i = 1, 2$*, where the* L_i *are reductive. We say that* Z_1 *is subordinate to* Z_2 *if there is a linear injection* $\alpha \colon Z_1 \to Z_2$ *and a reductive subgroup* $L \subset L_1 \times L_2$ *such that* α *is* L*-equivariant (for the* L*-module structures on* Z_1 *and* Z_2*). Moreover, we require that* $\mathrm{pr}_1 \colon L \to L_1$ *be a cover. We say that* $\alpha \colon Z_1 \to Z_2$ *is a subordination of* Z_1 *to* Z_2*. We sometimes use the notation* $\alpha \colon (Z_1, L_1) \to (Z_2, L_2)$ *to specify the groups involved.*

We now consider the possibilities for K'.

Lemma 5.10. — *Let* H*, etc. be as in* (5.8)*. Then for* $1 \leqslant i \leqslant s$ *we have* $\mathrm{pr}_i(K') \neq G_i$*.*

Proof. — Suppose that $\mathrm{pr}_i(K') = G_i$. Then there is a unique simple component K_i of K' such that $\mathrm{pr}_i(K_i) = G_i$. If $\mathrm{pr}_j(K_i) = \{e\}$ for $j \neq i$, then G_i acts trivially on Gv, which is not possible. Hence $\mathrm{pr}_j(K_i) \neq \{e\}$

for some $j \neq i$, $1 \leqslant j \leqslant s$. Suppose that $\mathrm{pr}_j(K_i) = G_j$. Then no simple component of K' other than K_i can project nontrivially to G_i and G_j. Consider the projections H' of H and K_i' of K_i to $G_i \times G_j$. Then $H'K_i' = G_i \times G_j$, and by reason of dimension we must have that $\mathrm{pr}_i(H') = G_i$ and $\mathrm{pr}_j(H') = G_j$. On the level of Lie algebras this says that we have a simple Lie algebra $\mathfrak{g}$ and two subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g} \oplus \mathfrak{g}$ which project isomorphically to each $\mathfrak{g}$ factor such that $\mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{g} \oplus \mathfrak{g}$. But it follows from [10, Theorem 9] that $\mathfrak{h}_1 \cap \mathfrak{h}_2 \neq (0)$, a contradiction.

We may thus assume that $\dim \mathrm{pr}_j(K_i) < \dim G_j$, hence $\dim H < \dim G_j$ as well. By Corollary 4.6 we have that $\mathrm{pr}_j(K_i)\,\mathrm{pr}_j(H) = G_j$ and that $\mathrm{pr}_j(K')$ differs from $\mathrm{pr}_j(K_i)$ by at most a factor of rank 1. Hence, with H' and K_i' as above, we again have $H'K_i' = G_i \times G_j$ which is not possible by reason of dimension. Hence we have that $\mathrm{pr}_i(K') \neq G_i$ for $1 \leqslant i \leqslant s$. □

Corollary 5.11. — *Suppose that $1 \leqslant i \leqslant s$ and that $K_i \subset K'$ is a simple factor of rank at least two such that $\mathrm{pr}_i(K_i) \neq \{e\}$. Then $\mathrm{pr}_j(K_i) = \{e\}$ for $i \neq j$, $1 \leqslant j \leqslant s$.*

Proof. — Suppose that $\mathrm{pr}_j(K_i) \neq \{e\}$. If $\dim H < \dim G_i$ and $\dim H < \dim G_j$, then Corollary 4.6 shows that $\mathrm{pr}_i(K_i)$ and $\mathrm{pr}_j(K_i)$ differ from $\mathrm{pr}_i(K')$ and $\mathrm{pr}_j(K')$ by at most groups of rank 1, so that with H' and K_i' as above, we have $H'K_i' = G_i \times G_j$, which is not possible by reason of dimension. Thus $\mathrm{pr}_i(H) = G_i$ (or $\mathrm{pr}_j(H) = G_j$) which forces $\mathrm{pr}_j(K_i) = G_j$ (or $\mathrm{pr}_i(K_i) = G_i$), contradicting the lemma above. □

Theorem 5.12. — *Let H, s, r, $V = \bigoplus_i V_i \otimes W_i$, etc. be as above. Then one of the following occurs.*

(1) *$s = 1$ and $H = G_1$. Then $G'' = G_2 \times \cdots \times G_r$ and v corresponds to a subordination $(\bigoplus_i W_i^*, G'') \to (\bigoplus_i V_i, H)$ where W_i^* is sent to $V_i^{K'}$ for each i.*

(2) *$s = 1$, $H \neq G_1$ and $G_1 = HK_1$ where $K_1 \subset G_1$ is a simple component of K. If $r > 1$, then $r = 2$, $G_2 = \mathrm{SL}_2$ (up to a cover) and $(G_1, H, K_1 \times \mathrm{SL}_2)$ is case 3.2 of Table 1. We have a subordination $(\bigoplus W_i^*, \mathrm{SL}_2) \to (\bigoplus V_i, H)$ where the image of W_i^* is a subset of $V_i^{K_1}$ for each i.*

(3) *$s = 2$, $\mathrm{pr}_1(H) = G_1$, $\mathrm{pr}_2(H) = G_2$ and there are subgroups K_1', K_2' of H such that (H, K_1', K_2') occurs in Table 1. We have $K = K_1 \times K_2$ where $K_i = \mathrm{pr}_i(K_i')$, $i = 1$, 2 and $G = HK$ (Example 5.6). If $r > 2$, then $r = 3$, $(H, K_1', K_2' \times \mathrm{SL}_2)$ or $(H, K_2', K_1' \times \mathrm{SL}_2)$ is entry 3.2 of Table 1, $G'' = \mathrm{SL}_2$ and v corresponds to a subordination*

$(\bigoplus_i W_i^*, \mathrm{SL}_2) \to (\bigoplus_i V_i, H)$ *where* W_i^* *has image in* $V_i^{K_1 \times K_2}$ *for all* i.

(4) $r = s = 2$ *where* $\mathrm{pr}_1(H) \neq G_1$ *and* $\mathrm{pr}_2(H) = G_2$. *Then we are in the case of Example 5.3, 5.4 or 5.5.*

Proof. — The cases where $s = 1$ are quite easy and we leave them to the reader. Now suppose that $s > 1$. Suppose that $\mathrm{pr}_1(H) \neq G_1$ and that $\mathrm{pr}_2(H) \neq G_2$. Then there are strongly semisimple factors K_i of K' such that $\mathrm{pr}_i(K_i)\,\mathrm{pr}_i(H) = G_i$, $i = 1, 2$. By Lemma 5.10 and Corollary 5.11 the $\mathrm{pr}_i(K_i)$ are proper subgroups of the G_i where $\mathrm{pr}_2(K_1) = \mathrm{pr}_1(K_2) = \{e\}$. Applying Proposition 5.7 we obtain that each of the G_i is isomorphic to H, a contradiction. Thus we may assume that $\mathrm{pr}_1(H) = G_1$.

Let L' (resp. K_1) be the product of the strongly simple components of K' which map trivially (resp. nontrivially) to G_1. Then $L' \subset G_2 \times \cdots \times G_s$. Since $G' = HK'$ and $\mathrm{pr}_1(H) = G_1$, we must have that $G_2 \times \cdots \times G_s = H'L'$ where H' is the inverse image of $\mathrm{pr}_1(K_1)$ in H projected to $G_2 \times \cdots \times G_s$. Since H' is a proper subgroup of H, $\mathrm{pr}_i(H') \neq G_i$, $i = 2, \ldots, s$. Since $\mathrm{pr}_2(H')\,\mathrm{pr}_2(L') = G_2$, we must have that H' is simple, by Table 1. By our argument above, we must have that $s = 2$.

Now suppose that $\mathrm{pr}_1(H) = G_1$ and that $\mathrm{pr}_2(H) = G_2$. Then by Lemma 5.10 and Corollary 5.11 we have that $H \times H \simeq G_1 \times G_2 = H(K_1 \times K_2)$ where the $K_i \subset G_i$ are images of simple subgroups K_1' and K_2' of K. Then $H = K_1' K_2'$ so that (H, K_1', K_2') occurs in Table 1, as claimed. Suppose that $r > 2$. Then for $j > 2$, $K_j \subset G_1 \times G_2 \times G_j$ projects onto G_j and commutes with K_1 and K_2. But the centralizer of $K_1 \times K_2$ in $G_1 \times G_2$ is trivial unless (H, K_1', K_2') or (H, K_2', K_1') is entry 3.1 of Table 1, in which case the centralizer is SL_2. Thus $r = 3$ and there is a subordination as claimed.

Finally, suppose that $s = 2$ and that $\mathrm{pr}_2(H) = G_2$ and $\mathrm{pr}_1(H) \neq G_1$. Using Lemma 5.10 and Corollary 5.11 and the fact that $HK' = G_1 G_2$, we see that there are simple subgroups $K_i \subset G_i$, $i = 1, 2$, such that $H(K_1 K_2) = G_1 G_2$. We have entries $(G_1, \mathrm{pr}_1(H), K_1)$ and $(\mathrm{pr}_2(H), \mathrm{pr}_2(L), K_2)$ in Table 1 where L is the preimage in H of $\mathrm{pr}_1(H) \cap K_1$. Then H must be B_3 or of type A_{2n-1}. If $H = \mathsf{B}_3$, then one easily sees that we are in Example 5.5 and that we cannot have $r > 2$. The remaining possibilities are that $H = \mathsf{A}_{2n-1}$ and $G_1 = \mathsf{D}_{2n}$ or B_3 giving Examples 5.3 and 5.4 where $r = 2$ is forced. □

The theorem above gives one the possibilities for the semisimple part of the Levi factor of $\{g \in \mathrm{GL}(V) \mid Gv = Hv\}$. Preferable would be a theorem which starts with a representation V of H and a generic $v \in V$ and tells you when v is almost semi-characteristic. In general, it is rather cumbersome to give such a theorem (for SL_2 see section 7). We content ourselves with working out the following example.

Example 5.13. — Let $H = \mathsf{D}_{2n+1}$, $n \geqslant 2$. Let $V = \bigoplus_{i=1}^{k} n_i V_i$ be the isotypic decomposition of the H-module V. Let $v = (v_1, \ldots, v_k) \in V$ be generic. We find conditions which guarantee that v is almost semi-characteristic.

Each v_i is $(v_{i1}, \ldots, v_{i,n_i})$ where v_{ij} lies in the jth copy of V_i, and the v_{ij} span a subspace $U_i \subset V_i$ of dimension n_i. In order to avoid case (1) of Theorem 5.12 we have to assume that the intersection of the stabilizers of the subspaces U_i in H contains no nontrivial semisimple group. Cases (2) and (4) do not apply, so we are left with case (3), where we have $G = H \times H$, $K_1 = \mathsf{B}_{2n}$ and $K_2 = \mathsf{A}_{2n}$. But then there is a copy of A_{2n-1} in D_{2n+1} which fixes our point. We have already ruled this out.

6. Semisimple groups

We turn our attention to the case that $H \subset G$ where G and H are connected semisimple, V is an irreducible H-module, G acts almost faithfully on V and $Gv = Hv$ for some nonzero $v \in V$. Let $G_1, \ldots, G_k$ be the simple components of G. Then Theorem 4.10 tells us that $H = H_1 \cdots H_k$ where the H_i are semisimple and lie in G_i, $i = 1, \ldots, k$. Note that no H_i is trivial, else G_i acts trivially on V. Thus if G_i has rank 1, then $G_i = H_i$. We have $V = V_1 \otimes \cdots \otimes V_k$ where V_i is an almost faithful irreducible representation of both G_i and H_i, $i = 1, \ldots, k$.

6.1. — Suppose that $G = HK$ where K is semisimple and G, H and V are as above. (Think of $K \subset G_v$.) Let pr_i denote the projection of G to G_i, $i = 1, \ldots, k$. Let K' be a simple component of K and set $I' := \{i \mid H_i \,\mathrm{pr}_i(K') = G_i \text{ and } H_i \neq G_i\}$. We may assume that K contains no simple component of rank 1.

Proposition 6.2. — *Let $G = HK$ as above. Let K', K'' be distinct simple components of K and let I' and I'' be as above. Then*

(1) $I' \cap I'' = \emptyset$.

(2) $|I'| \leqslant 2$. *If* $|I'| = 2$, *then* $\mathrm{pr}_i(K') = G_i$ *for some* $i \in I'$.

Proof. — For any $i \in I' \cap I''$, the (nontrivial) images of K' and K'' in G_i commute. This is clearly not possible if $\mathrm{pr}_i(K')$ is G_i. If not, then we are in one of the entries of Table 1, and commutativity is not possible if $\mathrm{pr}_i(K')$ is one of the groups occurring there. Hence (1) holds. Suppose that i, $j \in I'$, $i \neq j$ and $\mathrm{pr}_i(K') \neq G_i$ and $\mathrm{pr}_j(K') \neq G_j$. Then we must have that $H_j L = G_j$ where $L = \mathrm{pr}_j(\mathrm{pr}_i^{-1}(H_i) \cap K')$ is a proper subgroup of $\mathrm{pr}_j(K')$ as in the last column of Table 1. But then, by inspection, we cannot have $H_j L = G_j$. If i, j and k are distinct elements of I', then we can assume that $\mathrm{pr}_i(K') = G_i$, and we derive a contradiction as before by considering the non-surjective projections of $\mathrm{pr}_i^{-1}(H_i) \cap K'$ to G_j and G_k. Thus we have (2). □

THEOREM 6.3. — *Suppose that $k = 2$ and that $H_1 \neq G_1$, $H_2 \neq G_2$ and $Gv = Hv$ for a nonzero $v \in V$. Then we are in one of the following cases.*

(1) *There are tuples (G_i, V_i, H_i, K_i) in Table 2, $i = 1, 2$, and $v \in V_1^{K_1} \otimes V_2^{K_2}$.*
(2) *The tuple (G_1, V_1, H_1, K_1) is entry (1) of Table 2, (G_2, V_2, H_2, K_2) is entry (3.3) (with the same k) and v generates the one-dimensional space of A_{2n-1} fixed vectors in $V_1 \otimes (V_2|\mathsf{A}_{2n-1})$.*
(3) *The tuple (G_1, V_1, H_1, K_1) is entry (6.2) of Table 2, (G_2, V_2, H_2, K_2) is entry (6.3) (with $k = 1$) and v generates the one-dimensional space of D_8 fixed points in $V_1 \otimes V_2$.*

Proof. — Let K_1 denote a maximal strongly semisimple subgroup of G_v. Suppose that $K_1 \subset G_1$. Then Table 1 implies that $H_1 K_1 = G_1$. Let K_2 be a strongly semisimple subgroup of G_v such that $\mathrm{pr}_2(K_2) H_2 = G_2$. Then we must have $\mathrm{pr}_1(K_2) = \{e\}$ (again by Table 1), hence $K_2 \subset G_2$ and we are in case (1). Thus we may suppose that any maximal strongly semisimple subgroup L of G_v lies diagonally in $G_1 \times G_2$. Since we are not in case (1), pr_2 restricted to L is almost faithful (and so is pr_1). By Table 1, L must be simple. It follows from Proposition 6.2 that we have two cases:
Case 1: $\mathrm{pr}_1(L) = G_1$ and $\mathrm{pr}_2(L) = K_2$ where (G_2, H_2, K_2) is in Table 1. Moreover, (G_2, H_2, K_2') is in Table 1, where $K_2' = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(H_1) \cap L)$. Thus we are in the case $(G_2, H_2, K_2) = (\mathsf{D}_{2n}, \mathsf{B}_{2n-1}, \mathsf{A}_{2n-1})$ and $(G_2, H_2, K_2') = (\mathsf{D}_{2n}, \mathsf{B}_{2n-1}, \mathsf{C}_n)$, $n \geqslant 2$, where $H_1 \simeq \mathsf{C}_n$. Then from Table 2 we see that $V_1 \simeq \varphi_1^k(\mathsf{A}_{2n-1})$ (or its dual). From Table 2(3.3) we get possibility (2) of our theorem. From (3.1) we get nothing since A_{2n-1} has no fixed vectors in $V_1 \otimes V_2$. The possibilities (4.2) and (5.2) fail for the same reason. Hence we only get (2).
Case 2: Here we have that $\mathrm{pr}_1(L) = G_1$ and $\mathrm{pr}_2(L) = G_2$. Then $L = H_1' H_2'$ where $H_i' = \mathrm{pr}_i^{-1}(H_i) \cap L$, $i = 1, 2$. Moreover, there are irreducible representations V_i of L whose restrictions to H_i' are irreducible, $i = 1, 2$. Table 2 tells us that we may have possibilities from entry (5.3) (and isomorphic entries), but then there are no D_4-fixed points in $V_1 \otimes V_2$. Finally, from (6.2) and (6.3) we get possibility (3) above. □

6.4. — Let $H \subset \mathrm{GL}(V)$ where H is semisimple connected and V is an irreducible H-module. Suppose that $v \in V$ is a nonzero orbit such that the connected semisimple part G of $\{g \in \mathrm{GL}(V) \mid gHv = Hv\}$ is strictly larger than H. Let K denote the strongly semisimple part of G_v. We are then in the situation of 6.1. For each simple component K_j of K, let $I_j \subset \{1, \dots, k\}$ be as in 6.1. Set $I' = \cup_j I_j$, $V' = \otimes_{i \in I'} V_i$ and $V'' = \otimes_{i \notin I'} V_i$. Define G' and G'' analogously. Then $G'' = H'' = \Pi_{i \notin I'} H_i$. Let K' be the product of the K_j such that $K_j \subset G'$ and let K'' be the product of the other simple factors of K so we have $K = K'K''$. Via the projections to G' and H'' we have K''-module structures on V' and V''. Let $W' \subset V'$ be the minimal subspace such that $v \in W' \otimes V''$. Then $W' \subset (V')^{K'}$ and v is K''-fixed.

Remark 6.5. — It follows from Theorem 6.3 that each simple component of K' arises from an entry of Table 2 as in Theorem 6.3(1), is a group A_{2n-1} as in Theorem 6.3(2) or is the group D_8 in Theorem 6.3(3).

We restate the discussion in (6.4) as follows.

Theorem 6.6. — *Let $v \in V$. If Hv is not semi-characteristic, then there are K', K'', etc. as in* (6.4) *and a minimal K''-stable subspace $W' \subset (V')^{K'}$ such that $v \in W' \otimes V''$. If $K'' \neq \{e\}$, then there is a subordination $\alpha\colon ((W')^*, K'') \to (V'', H'')$.*

Now we would like to find some simple sufficient criteria for all generic v to be semi-characteristic. For this, we only need to avoid the case that $Gv = Hv$ where G_i differs from H_i for only one i. Then after renumbering we have that $G_1 \neq H_1$ and $G_i = H_i$ for $i > 1$. We have that $V'' = V_2 \otimes \cdots \otimes V_k$ and $H'' = H_2 \times \cdots \times H_k$. Note that H_1 may be any semisimple subgroup of H.

Proposition 6.7. — *Let $G_1 \supset H_1$ be as above. Then one of the following occurs.*

1. *There is a subordination $\alpha\colon (V_1^*, G_1) \to (V'', H'')$ where K'' projects onto G_1.*
2. *The tuple (V_1, G_1, H_1, K_1) occurs in Table 2 where K_1 is the projection of K'' to G_1, and we have a subordination $(W_1^*, K_1) \to (V'', H'')$ where W_1 is minimal such that $v \in W_1 \otimes V''$.*
3. *The group K'' projects trivially to G_1 and the tuple (V_1, G_1, H_1, K_1) occurs in Table 2 for some K_1 where $V_1^{K_1} \neq (0)$.*

Proof. — If $\mathrm{pr}_1(K'') = \{e\}$, then we are in case (3). Suppose that $\mathrm{pr}_1(K'')$ is nontrivial. Then it follows from Table 1 that $K' = \{e\}$. If the projection of K'' to G_1 is G_1, then v corresponds to a subordination of V_1^* to V'' and we are in case (1). The only other possibility is that the projection of K'' is K_1 where (V_1, G_1, H_1, K_1) occurs in Table 2 and we are in case (2). □

Example 6.8. — Suppose that $k = 2$, $\dim V_2 \geqslant \dim V_1$ and that $H_2 = \mathrm{SL}(V_2)$. Let $v \in V_1 \otimes V_2$ have maximal rank. Then case (1) applies. If (V_1, G_1, H_1, K_1) occurs in Table 2, let W_1 be any nontrivial K_1-subspace of V_1 and let $v \in W_1 \otimes V_2$ have maximal rank. Then case (2) applies.

From Proposition 6.7 we get the following criterion for all nonzero orbits Hv to be semi-characteristic.

Corollary 6.9. — *Let V be an irreducible H-module where H is semisimple. Write $H = H_1 \times \cdots \times H_k$ where the H_i are simple for $i > 1$, and let $V = V_1 \otimes \cdots \otimes V_k$ be the corresponding decomposition of V. Let V'' denote $V_2 \otimes \cdots \otimes V_k$ and set $H'' = H_2 \cdots H_k$. Suppose that none of the following occurs for any decomposition $H = H_1 \times \cdots \times H_k$.*

1. *There is a subordination $(V_1^*, H_1) \to (V'', H'')$ where $H_1 \neq \mathrm{SL}(V_1)$.*
2. *There is a tuple (V_1, G_1, H_1, K_1) in Table 2 and a subordination $(W^*, K_1) \to (V'', H'')$ where $W \subset V_1$ is K_1-stable.*
3. *There is a tuple (V_1, G_1, H_1, K_1) in Table 2 where $V_1^{K_1} \neq (0)$.*

Then every nonzero $v \in V$ is semi-characteristic.

Admittedly, the corollary is a little unwieldy, but in any concrete case it is quite easy to apply. We see what we can say in the case of isotropy representations of symmetric spaces.

Example 6.10. — Let $H = \mathsf{A}_5 \times \mathsf{A}_1$ acting on $V = \varphi_3 \otimes \varphi_1$. This corresponds to the symmetric space of type EII (see [8, Ch. X, Table V]). Let $0 \neq v \in V$ and let G be as usual with semisimple part G_s. If G_s contains H, then G_s cannot be simple (by Table 2), and if it is of the form $G_1 \times G_2$ where $G_1 \supset \mathsf{A}_5$ and $G_1 = \mathrm{SL}_2$, then it follows from Corollary 6.9 or Proposition 6.7 that $G_1 = \mathsf{A}_5$. Hence v is semi-characteristic.

Example 6.11. — Let $p \geqslant q \in \mathbb{N}$ where $p > 1$. Let $H = \mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q}$ act in the natural way on $V = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q}$. This corresponds to the symmetric space of type CII. Let $0 \neq v \in V$. Then one easily sees that the only possibility for a semisimple G_s containing H stabilizing Hv occurs in the case that v has rank 1, in which case $G_s = \mathrm{SL}_{2p} \times \mathrm{SL}_{2q}$. For $q > 1$ this corresponds to Theorem 6.3(1). If rank $v > 1$, then v is semi-characteristic.

Example 6.12. — Let $p \geqslant q \geqslant 1$. Let H be the intersection of the block diagonal copy of $\mathrm{GL}_p \times \mathrm{GL}_q$ in GL_{p+q} with SL_{p+q}. Then H acts naturally on $V \oplus V^*$ where $V = \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$. This is an isotropy representation corresponding to the symmetric space of type AIII. First suppose that $q \geqslant 2$. Let $v = (x, x^*)$ where $x \in V$ and $x^* \in V^*$ are nonzero. Let $G = \{g \in \mathrm{GL}(V \oplus V^*) \mid gHv = Hv\}^0$. Suppose that v is not semi-characteristic. Then $\mathfrak{g}$ is an H-stable Lie subalgebra of $\mathrm{Hom}(V \oplus V^*, V \oplus V^*)$ which properly contains $\mathfrak{h}$. If $\mathfrak{g}$ projected to $\mathrm{Hom}(V, V)$ or $\mathrm{Hom}(V^*, V^*)$ is more than a central extension of $\mathfrak{h}$, then $\mathfrak{g}$ has to contain $\mathfrak{sl}_{p+q}$. But there is no corresponding entry in Table 2. Thus we can suppose that $\mathfrak{g}$ projects nontrivially to one of the irreducible components of

$$\mathrm{Hom}(V, V^*) \simeq V^* \otimes V^* \simeq (S^2(\mathbb{C}^p) + \wedge^2(\mathbb{C}^p)) \otimes (S^2((\mathbb{C}^q)^*) + \wedge^2((\mathbb{C}^q)^*)).$$

Let us consider the case that $\mathfrak{g}$ contains $\mathfrak{g}' := \wedge^2(\mathbb{C}^p) \otimes \wedge^2((\mathbb{C}^q)^*)$. Now x has normal form $\sum_{i=1}^k e_i^* \otimes f_i$ where $e_1, \dots, e_p$ is a basis of $\mathbb{C}^p$, $f_1, \dots, f_q$ is a basis of $\mathbb{C}^q$ and $e_1^*, \dots, e_p^*$, $f_1^*, \dots, f_q^*$ denote the elements of the dual bases. Then $e_1 \wedge e_2 \otimes f_1^* \wedge f_2^*$ lies in $\mathfrak{g}'$ and applied to x gives us $y^* = e_1 \otimes f_1^* + e_2 \otimes f_2^*$ if $k \geqslant 2$. The contraction of x and y^* (an H-invariant of $V \oplus V^*$) is not zero. Thus $(x, x^* + y^*)$ cannot be in the H-orbit of v, a contradiction. If $k = 1$, then acting by the reductive part of H_x we can bring x^* to the normal form

$$ce_1 \otimes f_1^* + e_1 \otimes f^* + e \otimes f_1^* + \sum_{i=2}^{\ell} e_i \otimes f_i^*$$

where $c \in \mathbb{C}$, $f^* \in \mathrm{span}\{f_2^*, \dots, f_q^*\}$ and $e \in \mathrm{span}\{e_2, \dots, e_p\}$. If $c \neq 0$, then acting by unipotent elements of H_x we can arrange that e and f^* are zero. Then $\ell + 1$ is an invariant of x^* (its rank) under the action of H_x. But we can change ℓ by adding elements of $\mathfrak{g}'$ applied to x, again giving a contradiction. If $c = 0$ and $q > 2$, then one similarly sees that we can change the rank of x^*. If $c = 0$, $q = 2$ and e or $f^* \neq 0$, however, the G'-orbit of v is contained in the H_x-orbit of v and v is not semi-characteristic. The other three possible components of $\mathfrak{g}$ give nothing new. Thus v is possibly not semi-characteristic only when $q = 2$, v is in the null cone and one of x and x^* has rank 1.

If $p = 1$ and $q = 1$ we have a torus action in which case $G = H$. If $q = 1$ and $p \geqslant 2$ we have the action of GL_p on $\mathbb{C}^p \oplus (\mathbb{C}^p)^*$. If Hv is not closed, then v is semi-characteristic. If Hv is closed, then $G \simeq \mathrm{SO}_{2p}$ and v is not semi-characteristic.

Example 6.13. — In general, one has a good chance to have points v which are not semi-characteristic in case your representation is reducible. One can calculate that this actually occurs for the following isotropy representations of symmetric spaces.

(1) $(V, H) = (\mathbb{C}^p \otimes \mathbb{C}^2, \mathrm{SO}_p \times \mathrm{SO}_2)$, $p \geqslant 3$. This is of type BDI and of type CI for $p = 3$.
(2) $(V, H) = (\wedge^2(\mathbb{C}^n) \oplus \wedge^2((\mathbb{C}^n)^*), \mathrm{GL}_n)$, $3 \leqslant n \leqslant 5$. This is of type DIII.
(3) $(V, H) = (\varphi_4 \otimes \nu_1 + \varphi_5 \otimes \nu_{-1}, \mathsf{D}_5 \times \mathbb{C}^*)$. This is of type EIII. Here ν_j denotes the one-dimensional representation of $\mathbb{C}^*$ of weight j.
(4) $(V, H) = (\varphi_1 \otimes \nu_1 + \varphi_5 \otimes \nu_{-1}, \mathsf{E}_6 \times \mathbb{C}^*)$. This is of type EVII.

Our discussion above establishes

Proposition 6.14. — *Let (V, H) be the isotropy representation of an irreducible symmetric space. Then, with the exception of the adjoint representation of C_n, $n \geqslant 2$ and the exceptions noted in Examples 6.11, 6.12 and 6.13, every orbit Hv, v generic, is semi-characteristic.*

The proposition applies to some of the questions of Raïs in [20].

7. Representations of $\mathbf{SL}_2$

We consider the case of H-modules V where $H := \mathrm{SL}_2$ and $V^H = 0$. We have a generic $v \in V$ and $G := \{g \in \mathrm{GL}(V) \mid gHv = Hv\}^0$ is not equal to H. We denote by R_n the H-module of binary forms of degree n. Then $R_n \simeq S^n(\mathbb{C}^2)$ has basis $x^n, x^{n-1}y, \ldots, y^n$ where x, y are the usual basis of $\mathbb{C}^2$ and x^n is a highest weight vector. Let $N_G(H)$ denote the connected normalizer of H in G.

To determine G, we show that it suffices to determine $N_G(H)$ and $\mathfrak{g}_u$. We determine $N_G(H)$ in Theorem 7.4 below. We show that $\mathfrak{g}_u$ is abelian and a multiplicity free H-module (Proposition 7.15). We give necessary and sufficient conditions for $\mathfrak{g}_u$ to contain a copy of R_p, $p > 0$ (Theorem 7.27). We then find some simple conditions that guarantee that $\mathfrak{g}_u$ is zero or the trivial H-module for every generic $v \in V$ (Corollary 7.29).

Lemma 7.1. — *Let $\tilde{G}$ be a Levi component of G containing H. Then $\tilde{G} \subset N_G(H)$.*

Proof. — It follows from Theorem 4.3 that $\tilde{G}_v$ contains the simple components of $\tilde{G}$ of rank at least 2. These components are normalized by H so they fix the whole orbit Hv which spans V. Thus all the components of $\tilde{G}$ have rank at most 1 and $\tilde{G} \subset N_G(H)$. □

Corollary 7.2. — *We have $\mathfrak{g} \simeq \tilde{\mathfrak{g}} \ltimes \mathfrak{g}_u$. Hence $G \neq N_G(H)$ if and only if $\mathfrak{g}_u$, as H-module, contains R_p for some $p > 0$. To determine G it suffices to determine $N_G(H)$ and $\mathfrak{g}_u$.*

We now consider the possibilities for $N_G(H)$.

Examples 7.3.

(1) Let $\bar{H}$ be another copy of SL_2 and let $V_k = R_k \otimes \bar{R}_k$, $k \geqslant 1$, where $\bar{R}_k$ is the $\bar{H}$-module of binary forms of degree k. Let $v_k \in V_k$ be a nonzero fixed vector of $\{(h,h)\} \subset H \times \bar{H}$. Then $Hv_k = (H \times \bar{H})v_k$ and v_k is generic. We can also take $V = V_{k_1} \oplus \cdots \oplus V_{k_\ell}$ and $v = (v_{k_1}, \dots, v_{k_\ell})$ where $k_1 < \cdots < k_\ell$. Then v is generic and $N_G(H) \supset H\bar{H}$.

(2) Let $V = \bigoplus_{k\in F} m_k R_k$ where $1 \leqslant m_k \leqslant k+1$ for all k and F is a nonempty finite subset of $\mathbb{N}$. Let B and $\bar{B}$ be the standard Borel subgroups of H and $\bar{H}$, respectively. Let v_k be the highest weight vector of the copy of $R_{2k+2-2m_k}$ in $V_k = R_k \otimes \bar{R}_k$ for the diagonal H-action. Then v_k lies in R_k tensored with the span of the weight vectors of $\bar{R}_k$ of weight at least $k-2m_k+2$. Now v_k is an eigenvector for the diagonal copy of B, with weight $2k+2-2m_k$. For $\bar{b} \in \bar{B}$, let $\chi(\bar{b})$ denote its upper left hand entry. Let $\bar{b}$ act on $\bar{R}_k$ as the tensor product of the usual action and the scalar action $\bar{b} \mapsto \chi(\bar{b})^{2m_k-2k-2}$. Then v_k is fixed by the diagonal in $B \times \bar{B}$. Assume that $m_k \geqslant 2$ for some k so that $\bar{B}$ acts effectively. Set $v = \bigoplus_{k\in F} v_k$. Then v is generic and $N_G(H) \supset H\bar{B}$.

(3) Let $V = \bigoplus_{k\in F} m_k R_k$ as above. Let $v \in V$ be a generic vector whose projection $v_{k,j}$ to the jth copy of R_k is a weight vector. If the weight is not zero, then there is an obvious $\mathbb{C}^*$-action on this copy of R_k such that $v_{k,j}$ is fixed by the product of the standard torus in H and our external copy of $\mathbb{C}^*$. If v is not a sum of zero weight vectors we have $N_G(H) \supset H\mathbb{C}^*$.

Theorem 7.4. — *Let $V = \bigoplus_{k\in F} m_k R_k$ be a representation of $H = \mathrm{SL}_2$ where $V^H = 0$. Let $v \in V$ be generic. If $N_G(H) \neq H$, then, up to the action of $\prod_{k\in F} \mathrm{GL}_{m_k}$, we are in one of the cases of Example 7.3. If $G \supset H\bar{H}$ as in Example 7.3(1), then $G = H\bar{H}$.*

Proof. — We have $N_G(H) = HG'$ where G' is the identity component of the centralizer of H in G. The group G'_v fixes Hv, so it is trivial. Hence the Lie algebra of $G_v = \{hg' \mid hg'v = v\}$ projects onto $\mathfrak{g}'$ and into $\mathfrak{h}$, so that G' is locally isomorphic to a quotient of a connected subgroup of H. Hence G' is locally isomorphic to a connected subgroup of H.
Case 1: $G' = \mathrm{SL}_2$ or SO_3. Going to a cover, we can assume that $G' = \bar{H}$ so that G_v is isomorphic to the diagonal copy of H. Then V is a sum of representations $V_k = R_k \otimes S_k$ where S_k is a representation of $\bar{H}$ of dimension

at most $k+1$ and the projection of v to V_k is a fixed point of the diagonal action of H. Thus $S_k \simeq \bar{R}_k$ and v is as in Example 7.3(1). Suppose that $\mathfrak{g}_u \neq 0$. Then, as $(H \times \bar{H})$-module, $\mathfrak{g}_u$ cannot contain R_0 or $\bar{R}_0$ since the connected centralizer of H is $\bar{H}$ and vice versa. Thus $\mathfrak{g}_u$ contains a term $R_a \otimes \bar{R}_b$ where $ab \neq 0$. Hence, as H-module, $\mathfrak{g}_u$ is not multiplicity free. But this contradicts Proposition 7.15 below. Hence $\mathfrak{g}_u = 0$ and $G = H\bar{H}$.
Case 2: $G' = \mathbb{C}^*$. Then $G_v \subset H \times \mathbb{C}^*$ is a diagonal torus and the fixed subspace of G_v on each isotypic component $m_k R_k$ of V is a sum of m_k distinct weight spaces of H. Thus we are in Example 7.3(3).
Case 3: $G' \supset \bar{U}$ and $G' \not\supset \bar{H}$ where $\bar{U} \subset \bar{H}$ is the standard maximal unipotent subgroup of our second copy $\bar{H}$ of SL_2. We have $V = \bigoplus_{k \in F} R_k \otimes S_k$ where S_k is a representation of $\bar{U}$. The isotropy group of v in $H \times \bar{U}$ can be taken to be the diagonal copy of U in $U \times \bar{U}$. Then v corresponds to a subordination $S_k^* \to R_k$, hence the image of S_k^* is a U-stable subspace of R_k and S_k is a $\bar{B}$-stable subspace of $\bar{R}_k$. In fact, it is the span of x^k, $x^{k-1}y, \dots, x^{k-m_k+1}y^{m_k-1}$, and acting by elements of the various $\mathrm{GL}(m_k)$ we can arrange that v is as in Example 7.3(2). Hence $N_G(H) \simeq H\bar{B}$. □

Corollary 7.5. — *Let V be as above and $v \in V$ generic. Suppose that Example 7.3(1) does not apply so that $N_G(H) \neq H\bar{H}$. Then v is almost semi-characteristic.*

We now turn to the determination of $\mathfrak{g}_u$ when it is not zero or a trivial H-module.

Proposition 7.6. — *Let $v \in V$ be generic. Suppose that there is a copy of R_p in $\mathfrak{gl}(V)$, $p > 0$, which is a Lie subalgebra and acts nilpotently on V. Further suppose that $R_p(v) \subset \mathfrak{h}(v)$. Then $R_p \subset \mathfrak{g}_u$.*

Proof. — Consider the action $\sigma\colon R_p \otimes V \to V$. Then for $h \in H$,

$$\sigma(R_p \otimes h(v)) = h\sigma(R_p \otimes v) \subset h\mathfrak{h}(v) = \mathfrak{h}(hv).$$

Hence Hv is open in $G_p v$ where G_p is the connected group with Lie algebra $\mathfrak{h} \ltimes R_p$. Thus $G_p \subset G$ and $R_p \subset \mathfrak{g}$. The projection of R_p to $\mathrm{Lie}(N_G(H))$ is trivial (by our classification of $N_G(H)$ and the fact that R_p is nilpotent). Hence $R_p \subset \mathfrak{g}_u$. □

Remark 7.7. — The proposition remains true if $p = 0$ as long as $G \neq H\bar{H}$ as in Example 7.3(1).

Example 7.8. — Let p, $l > 0$. Let $v_{l+p} = x^{l+p}$ and let $v_l = a_0 x^l + a_1 x^{l-1} y$, $a_1 \neq 0$. Set $V = R_{l+p} + R_l$. Consider a nonzero equivariant map $\sigma\colon R_p \otimes R_{l+p} \to R_l$. Then $\sigma(x^i y^{p-i} \otimes v_{l+p})$ vanishes for $i > 0$ and $\sigma(y^p \otimes$

v_{l+p}) is a nonzero multiple of x^l. Thus we may arrange that $\sigma(y^p\otimes x^{l+p}) = a_1x^l$. If $A\in\mathfrak{sl}_2$ is $x\partial/\partial y$, then $\sigma(y^p\otimes v_{l+p}) = A(v_l)$. We may consider σ as an equivariant mapping of R_p to $\mathrm{Hom}(R_{l+p}, R_l)$. Then R_p applied to $v := v_{l+p}+v_l$ is the same as $\mathfrak{u}(v)$ where $\mathfrak{u} = \mathbb{C}\cdot A$. By Proposition 7.6, $R_p\subset\mathfrak{g}_u$. We can also have a copy of R_q in $\mathfrak{g}_u$, $q\neq p$, by adding R_{l+q} to V and adding v_{l+q} to v, where $v_{l+q} = x^{l+q}$.

We now try to pin down the structure of V and v. The situation can be quite complicated. First we need a lemma.

Lemma 7.9. — *Let $\varphi\colon R_p\otimes R_n\to R_{p+n-2i}$ be equivariant and nonzero where $0\leqslant i\leqslant\min\{p,n\}$. Then $\varphi(x^{p-j}y^j\otimes x^n)\neq 0$ for $i\leqslant j\leqslant p$.*

Proof. — If $\varphi(x^{p-j}y^j\otimes x^n) = 0$, then the $\mathfrak{sl}_2$-submodule W of $R_p\otimes R_n$ generated by $x^{p-j}y^j\otimes x^n$ lies in the kernel of φ. Applying $x\partial/\partial y\in\mathfrak{sl}_2$ repeatedly we may reduce to the case that $\varphi(x^{p-i}y^i\otimes x^n) = 0$. Suppose by induction that $x^{p-k}y^k\otimes x^{n-l}y^l$ lies in W for $k+l=i$ and $l\leqslant s$. Then applying $y\partial/\partial x$ followed by $x\partial/\partial y$ to $x^{p-k}y^k\otimes x^{n-s}y^s$ we obtain elements in W as well as $k(n-s)x^{p-k+1}y^{k-1}\otimes x^{n-s-1}y^{s+1}$. Thus W contains all the weight vectors of $R_p\otimes R_n$ of weight $p+n-2i$. This implies that $R_{p+n-2i}\subset W$, a contradiction. Thus $\varphi(x^{p-j}y^j\otimes x^n)\neq 0$. ☐

Remark 7.10. — Reversing the roles of x and y we have $\varphi(x^jy^{p-j}\otimes y^n)\neq 0$ for $i\leqslant j\leqslant p$.

Corollary 7.11. — *Let φ, etc. be as above where $p+n-2i\neq 0$. Let $w = x^n\in R_n$. Then $\dim\varphi(R_p\otimes w)\geqslant 2$ unless $i=p<n$ so that $\varphi(y^p\otimes w)$ is a highest weight vector of R_{n-p}.*

Set $W_0 = V$ and for $j>0$ set $W_j = \mathfrak{g}_u(W_{j-1})$. Then W_j is a proper H-stable subspace of W_{j-1} for $j>0$. Let k be the greatest integer j such that $W_j\neq 0$. Since $\mathfrak{g}_u$ acts nontrivially on V, we must have $k>0$. Let V_j be an H-complement to W_{j+1} in W_j for $0\leqslant j\leqslant k$. Then $V = \oplus_j V_j$. Write $v = v_0+v_1+w_2$ where $v_i\in V_i$, $i=1$, 2, and $w_2\in W_2$. As before, let A denote $x\partial/\partial y\in\mathfrak{h}$.

Lemma 7.12. — *Perhaps replacing v by hv for some $h\in H$ we have the following.*

1. *The vector v_0 is a sum of highest weight vectors.*
2. *The dimension of $\mathfrak{g}_u(v)$ is one with basis $A(v)$.*
3. *Suppose that $R_p\subset\mathfrak{g}_u$ where $p>0$. Then for $p\geqslant i>0$, $x^iy^{p-i}\in R_p$ annihilates v.*

Proof. — Since $\mathfrak{g}_u$ acts nontrivially on V and v is generic, there has to be a $C \in \mathfrak{g}_u$ such that $C(v) \in W_1$ and $C(v) \notin W_2$. Then there must be a $D \in \mathfrak{h}$ such that $D(v_0 + v_1) = C(v)$ modulo W_2. Since D preserves V_0 and V_1, we must have that $D(v_1) = C(v_0)$ modulo W_2 and that D annihilates v_0. Up to the action of H, we may thus assume that v_0 is a sum of highest weight vectors or a sum of zero weight vectors. We assume the latter and derive a contradiction. Since $\mathfrak{g}_u$ is H-stable, we may assume that C is a weight vector for the action of $\mathbb{C}^* \subset H$. Note that D generates the Lie algebra of $\mathbb{C}^* \subset H$. If C has weight zero, then so does $C(v) + W_2 = C(v_0) + W_2$ and we cannot have that $C(v) = D(v)$ modulo W_2. Thus C has weight j for some $j \neq 0$ so that $C(v_0) + W_2 = D(v_1) + W_2$ also has weight j. Hence $v_1 = v_1' + v_1''$ where $v_1' + W_2 = C(v_0) + W_2$ and v_1' has weight j while v_1'' has weight 0. Now let Z be the two-dimensional vector space generated by v_0 and v_1', all modulo W_2. The groups generated by $\exp(tD)$ and $\exp(tC)$, $t \in \mathbb{C}$, act on Z and the orbits of (v_0, v_1') are the same. But $\exp(tC)(v_0, v_1')$ contains the point $(v_0, 0)$ while $\exp(tD)(v_0, v_1')$ clearly does not. Hence we have (1), *i.e.*, v_0 is a sum of highest weight vectors. Moreover, $\mathfrak{g}_u(v) + W_2$ is one-dimensional and generated by $A(v_1) + W_2$.

Let $C \in \mathfrak{g}_u$ as above. Then $C(v) = D(v)$ for some $D \in \mathfrak{h}_{v_0}$, where D is a multiple of A. Hence we have (2). Finally, suppose that $R_p \subset \mathfrak{g}_u$ where $p > 0$. By Corollary 7.11, for $i > 0$, $x^i y^{p-i}$ annihlates v, modulo W_2, while y^p sends v to a multiple of $A(v)$, modulo W_2. If $x^i y^{p-i}$ acts nontrivially on v it follows that $\dim \mathfrak{g}_u(v) > 1$. Hence we have (3). □

Let $\sigma\colon R_p \otimes V \to V$ be the action of some $R_p \subset \mathfrak{g}_u$ where $p \geqslant 0$. Let $\mu\colon V \to V$ be the action of y^p via σ. We may assume that $\mu(v) = A(v)$.

Corollary 7.13.

(1) *For all* $j \geqslant 1$, $\mu^j(v) = A^j(v)$.
(2) *If* $p > 0$, *then for all* $1 \leqslant i \leqslant p$, $j \geqslant 0$, $\sigma(x^i y^{p-i} \otimes A^j(v)) = 0$.

Proof. — Suppose that $p > 0$. We prove (1) and (2) simultaneously by induction on j. Assume that $\mu^j(v) = A^j(v)$ for $1 \leqslant j \leqslant m$ and that $\sigma(x^i y^{p-i} \otimes A^j v) = 0$ for $0 \leqslant j < m$, $i > 0$. We certainly have the case that $m = 1$. Apply A to the equation $\sigma(y^p \otimes A^{m-1}(v)) = A^m(v)$. Since σ is equivariant, one obtains that

$$\sigma(pxy^{p-1} \otimes A^{m-1}(v)) + \sigma(y^p \otimes A^m(v)) = A^{m+1}(v).$$

Since the first term above is zero, we have that $\mu(A^m(v)) = A^{m+1}v$ so that, by induction, we have $\mu^{m+1}(v) = A^{m+1}(v)$. Now apply A to the equation

$\sigma(x^i y^{p-i} \otimes A^{m-1}(v)) = 0$. One obtains that

$$\sigma((p-i)x^{i+1}y^{p-i-1} \otimes A^{m-1}(v)) + \sigma(x^i y^{p-i} \otimes A^m(v)) = 0$$

so that $\sigma(x^i y^{p-i} \otimes A^m(v)) = 0$. This completes the induction. In case $p = 0$, A commutes with the generator of R_0, so that (1) is immediate. □

Remark 7.14. — Suppose that $p > 0$ and that we have (1) above. Then applying A to the equations of (1) and using induction we obtain (2).

Proposition 7.15. — *The Lie algebra $\mathfrak{g}_u$ is abelian and as H-module is multiplicity free.*

Proof. — Suppose that we have copies of R_p and R_q in $\mathfrak{g}_u$ where we allow $p = q$ (in which case we have two copies of R_p). If $[R_p.R_q] \neq 0$, then we have a copy of some R_s in $\mathfrak{g}_u$ which maps V to W_2. Thus $R_s(v) \neq 0$ while $R_s(v) \in W_2$. This implies, as in the proof of Lemma 7.12, that $\mathfrak{g}_u(v)$ has dimension greater than one, a contradiction. Hence $\mathfrak{g}_u$ is abelian. If R_p has multiplicity two or more, then it follows from Lemma 7.12 that there is a copy of R_p which sends v to 0 implying that this copy of R_p acts trivially on V, a contradiction. Hence $\mathfrak{g}_u$ is multiplicity free. □

Proposition 7.16. — *For all $i \geqslant 0$, $A^i(v)$ is generic in W_i.*

Proof. — Since v is generic in V and $\mathfrak{g}_u$ is H-stable, W_1 is generated by the H-orbit of $\mathfrak{g}_u(v)$. Hence Av is generic in W_1. Then the same argument shows that the H-orbit of $A^2(v)$ spans W_2, etc. □

We say that a vector $w \in R_l$ has *height k* if $w = a_0x^l + \cdots + a_kx^{l-k}y^k$ where $a_k \neq 0$. A vector in $Z := \sum_i m_iR_i$ has *height at least k* (resp. *height at most k*) if it is generic in Z and when written as a sum $\sum_i v_{i,1}+\cdots+v_{i,m_i}$ where $v_{i,j}$ is in the jth copy of R_i, each $v_{i,j}$ has height at least k (resp. at most k).

Proposition 7.17. — *The H-modules V_i are multiplicity free.*

Proof. — The vector A^jv is generic in W_j, $j \geqslant 0$, and the projection of A^jv to any R_l in W_j cannot be zero. Thus the projection of v to any $R_l \subset W_j$ has height at least j. We have $v + W_j = v_0 + v_1 + \cdots + v_{j-1} + W_j$ where $A^jv \in W_j$. It follows that $A^jv_i = 0$ for $i < j$, hence any v_i is a sum of vectors of height at most i. Since $v_j \in W_j$ it is a sum of vectors of height at least j. Thus A^jv_j is a sum of highest weight vectors and it is generic in W_j. Hence any R_l can occur in W_j with multiplicity at most one. □

Write $v = v_0 + \cdots + v_k$ where $v_i \in V_i$. Then each v_i is a sum $\sum_{l\in F_i} v_{i,l}$ where $F_i \subset \mathbb{N}$ and $v_{i,l}$ lies in the copy of $R_l \subset V_i$.

Corollary 7.18. — *Each vector* $v_{i,l}$ *has height* i.

Lemma 7.19. — *Let* $\varphi\colon R_p\otimes R_m\to R_l$ *be equivariant and nonzero.*

(1) *Necessarily* $m=l+p-2i$ *for some* i *with* $0\leqslant i\leqslant\min\{l,p\}$.
(2) *Suppose that* $w\in R_m$ *has height* $n\leqslant l-i$. *Then* $\varphi(y^p\otimes w)$ *has height* $n+i$.

Proof. — Since representations of H are self-dual, R_l appears in $R_p\otimes R_m$ if and only if R_m appears in $R_p\otimes R_l$. Then Clebsch-Gordan implies (1). Now consider $z:=\varphi(y^p\otimes x^{m-n}y^n)$ where $m=l+p-2i$. Then Remark 7.10 shows that $z\neq 0$ if the weight of $y^p\otimes x^{m-n}y^n$ is at least $-l$. This is equivalent to $n\leqslant l-i$, hence we have (2). □

As above, we have $v_i=\sum_{l\in F_i}v_{i,l}$ where $v_{i,l}\in R_l\subset V_i$. For any $s\geqslant 0$, we have $W_1=V_1\oplus V_2\oplus\cdots\oplus V_{s+1}\oplus W_{s+2}$, hence we have an H-equivariant projection of W_1 to V_{s+1}. Let τ denote σ on $R_p\otimes(V_0+\cdots+V_s)$ followed by projection onto $R_l\subset V_{s+1}$. Since we have $\sigma(y^p\otimes A^r(v))=A^{r+1}(v)$, $r\geqslant 0$, for every $v_{s+1,l}\in R_l\subset V_{s+1}$, $A^{r+1}(v_{s+1,l})$ must be a multiple of $\tau(y^p\otimes A^r(v_0+\cdots+v_s))$ for $r\geqslant 0$. Note that τ vanishes on $R_p\otimes v_{i,t}$ unless $t=l+p-2j$ where $0\leqslant j\leqslant\min\{p,s\}$.

Proposition 7.20. — *Let* s *and* l *be as above. Let* $R_{l+p-2j}\subset V_i$, $j\leqslant\min\{p,s\}$. *If* $i+j>s$, *then* $\tau(R_p\otimes R_{l+p-2j})=0$.

Proof. — Consider the pairs (i',j') where $0\leqslant i'\leqslant s$, $0\leqslant j'\leqslant\min\{p,s\}$, $i'+j'>s$ and $v_{i',l+p-2j'}\neq 0$. Assume that i is the maximal i' that occurs and that j is the maximal j' that occurs in a pair (i,j'). Consider $A^i(v_{i,l+p-2j})$. It is a highest weight vector of weight $l+p-2j$. Suppose that $\tau(y^p\otimes A^i(v_{i,l+p-2j}))$ is nonzero. Then it has height $j>s-i$. Moreover, by the choice of i and j, $\tau(y^p\otimes A^i(v_{i,l+p-2j}))$ is the nonzero $\tau(y^p\otimes A^i(v_{i,l+p-2j'}))$ of largest height (equivalently, of lowest weight). But $\tau(y^p\otimes A^i(v))=A^{i+1}(v_{s+1,l})$ where $A^{i+1}(v_{s+1,l})$ has height $s-i$. Thus $\tau(y^p\otimes A^i(v_{i,l+p-2j}))$ must be zero. Now for $0<m\leqslant p$ we have that $\sigma(x^my^{p-m}\otimes A^i(v))=0$. Again, by height considerations, one sees that $\tau(x^my^{p-m}\otimes A^i(v_{i,l+p-2j}))$ must vanish. Hence $\tau(R_p\otimes A^i(v_{i,l+p-2j}))=0$ which shows that $\tau(R_p\otimes R_{l+p-2j})=0$. Now the proof can be completed by downward induction on i and j. □

For $0\leqslant j\leqslant\min\{p,s\}$ and $R_{l+p-2j}\subset V_i$, the restriction of τ to $R_p\otimes R_{l+p-2j}$ is a multiple $t_{i,s-j}\tau_{s-j}$ of τ_{s-j} where $\tau_{s-j}\colon R_p\otimes R_{l+p-2j}\to R_l$ is equivariant and normalized so that $\tau_{s-j}(y^p\otimes x^{l+p-2j})=x^{l-j}y^j$. We have that

$$\sum_{j=0}^{\min\{p,s\}} \sum_{i=0}^{s-j} t_{i,s-j}\tau_{s-j}(y^p \otimes A^r(v_{i,l+p-2j})) = A^{r+1}(v_{s+1,l}) \tag{7.1}$$

for all $r \geqslant 0$ where we set $v_{i,q} = 0$ if $q \notin F_i$.

Let a_i and $b_{i,m}^{s-j}$ be scalars such that $v_{s+1,l} = a_0 x^l + \cdots + a_{s+1}x^{l-s-1}y^{s+1}$ and $v_{i,l+p-2j} = b_{i,0}^{s-j}x^{l+p-2j} + \cdots + b_{i,i}^{s-j}x^{l+p-2j-i}y^i$ whenever $l+p-2j \in F_i$ and $0 \leqslant j \leqslant \min\{p,s\}$. We use t_j and b_j as shorthand for $t_{j,j}$ and $b_{j,j}^j$, respectively. For now assume that $l+p-2j \in F_{s-j}$, $j = 0,\dots,\min\{p,s\}$.

Theorem 7.21. — *For $m = 0,\dots,s$, consider the equation* (7.1) *with $r = m$ in weight $l-2s+2m$. This gives us $s+1$ equations in the unknowns $t_{s-j}b_{s-j}$ for $0 \leqslant j \leqslant \min\{p,s\}$. The unique solutions are*

$$t_{s-j}b_{s-j} = \frac{\binom{s}{j}\binom{p}{j}}{\binom{l+p-j+1}{j}}(s+1)a_{s+1},\ 0 \leqslant j \leqslant \min\{p,s\}.$$

Proof. — First assume that $p \geqslant s$. Since $\tau_{s-j}(y^p \otimes x^{l+p-2j}) = x^{l-j}y^j$, it follows that

$$\tau_{s-j}(y^p \otimes x^{l+p-2j-k}y^k) = \frac{\binom{l-j}{k}}{\binom{l+p-2j}{k}}x^{l-j-k}y^{j+k},\ k \leqslant l-j.$$

Now the 0th equation ($m = 0$) is

$$t_0b_0 + t_1b_1\tfrac{l-s+1}{l+p-2s+2} + \cdots + t_jb_j\frac{\binom{l-s+j}{j}}{\binom{l+p-2s+2j}{j}} + \cdots + t_sb_s\frac{\binom{l}{s}}{\binom{l+p}{s}} = (s+1)a_{s+1}.$$

For $m = 1$ the equation is

$$t_1b_1 + 2t_2b_2\tfrac{l-s+2}{l+p-2s+4} + \cdots + jt_jb_j\frac{\binom{l-s+j}{j-1}}{\binom{l+p-2s+2j}{j-1}} + \cdots + st_sb_s\frac{\binom{l}{s-1}}{\binom{l+p}{s-1}} = s(s+1)a_{s+1}$$

and the mth equation is

$$m!t_mb_m + \cdots + j!/(j-m)!t_jb_j\frac{\binom{l-s+j}{j-m}}{\binom{l+p-2s+2j}{j-m}} + \cdots + \frac{s!}{(s-m)!}t_sb_s\frac{\binom{l}{s-m}}{\binom{l+p}{s-m}} = \frac{(s+1)!}{(s-m)!}a_{s+1}.$$

Thus our system of equations is equivalent to

$$\sum_{j=0}^{s}\binom{j}{m}c_j\frac{\binom{l-s+j}{j-m}}{\binom{l+p-2s+2j}{j-m}} = \binom{s}{m},\ m = 0,\dots,s \tag{7.2}$$

where $c_j = t_j b_j/((s+1)a_{s+1})$. Since the equations are in triangular form, there is a unique solution. Now the theorem will be proved if we can show that a solution to (7.2) is

$$c_{s-j} = \frac{\binom{s}{j}\binom{p}{j}}{\binom{l+p-j+1}{j}}.$$

But one can prove this using the WZ method [28]. (See [23] for a brief introduction.) We used the implementation of the WZ method in MAPLE.

Now suppose that $p < s$. We may still consider the system of equations (7.2). The solutions remain the same, but note that for $j > p$, the formula for c_{s-j} gives zero. Hence the theorem is true even when $p < s$. □

Remark 7.22. — Let $0 \leqslant j \leqslant \min\{p,s\}$. We assumed that R_{l+p-2j} occurred in V_{s-j}. But the equations force $t_{s-j}b_{s-j}$ to be nonzero. Thus, in fact, R_{l+p-2j} must occur in V_{s-j} for there to be a solution of (7.1) for all $r \geqslant 0$.

Since we are guaranteed to have vectors $v_{s-j,l+p-2j}$ in our solution of (7.1), what role do the vectors $v_{i,l+p-2j}$ play for $i < s-j$? It is easy to see that the term involving $v_{i,l+p-2j}$ may be eliminated if we change $v_{s-j,l+p-2j}$ to $v_{s-j,l+p-2j} + \frac{t_{i,s-j}}{t_{s-j}} v_{i,l+p-2j}$. Let us say that $v'' = \sum_j v'_{s-j,l+p-2j}$ is obtained from $v' = \sum_j v_{s-j,l+p-2j}$ by an *admissible modification* if each $v'_{s-j,l+p-2j}$ differs from $v_{s-j,l+p-2j}$ by a linear combination of the $v_{i,l+p-2j}$ for $i < s-j$. Thus we have the following

Remark 7.23. — We have a solution of (7.1) if and only if, up to an admissible modification of the $v_{s-j,l+p-2j}$, we have a solution of

$$\sum_{j=0}^{\min\{p,s\}} t_{s-j}\tau_{s-j}(y^p \otimes A^r(v_{s-j,l+p-2j})) = A^{r+1}(v_{s+1,l}),\ r \geqslant 0. \tag{7.3}$$

Proposition 7.24. — *Let $v_{s+1,l} \in V_{s+1}$ and $v_{s-j,l+p-2j} \in V_{s-j}$ have coefficients a_j and $b^{s-j}_{i,m}$ as above. Fix the b_{s-j}, $0 \leqslant j \leqslant \min\{p,s\}$. Then there are unique values of the t_{s-j} and $b^{s-j}_{s-j,m}$ for $m < s-j$ such that there is a solution of* (7.3).

Proof. — We know that the t_{s-j} are uniquely determined. We only need to show that the $b^{s-j}_{s-j,m}$ for $m < s-j$ are unique satisfying (7.3). This is easy because of the triangular form of the equations. For $r = 0$, the equation in weight l reads $t_s b^s_{s,0} = a_1$. For arbitrary $r \leqslant s$, the equation in weight l is $r! t_s b^s_{s,r} = (r+1)! a_{r+1}$. For $r = s$ this is one of the equations we considered in Theorem 7.21 and we have $b^s_{s,r} = \frac{r+1}{t_s} a_{r+1}$ for $r < s$. Now suppose that we have determined the $b^{s-q}_{s-q,j}$ for $0 \leqslant q < m$. Consider (7.3) in weight $l-2m$ with $r=0$. It gives an expression for $t_{s-m}b^{s-m}_{s-m,0}$ in terms of the a_i and $b^{s'}_{s',s'-j'}$ for $s' > s-m$. Thus we may solve for $b^{s-m}_{s-m,0}$. For $0 < r \leqslant s-m$ we obtain an equation that we can solve for $b^{s-m}_{s-m,r}$. The equation that we get for b_{s-m} is one of the equations that we considered in Theorem 7.21. Hence given $a_1, \dots, a_{s+1}$ and the b_{s-j}, there are unique t_{s-j} and $b^{s-j}_{s-j,m}$ solving (7.3). □

Remark 7.25. — Suppose that $l+p-2j \in F_i$ for all $i+j \leqslant s$, $0 \leqslant j \leqslant \min\{p,s\}$. Then we may modify the $v_{s-j,l+p-2j}$ admissibly so that the $b^{s-j}_{s-j,m}$, $m < s-j$, are arbitrary. Hence there are t_{s-j} giving solutions of (7.3) (after admissible modifications) and giving solutions of (7.1) (without changing any vectors).

Let us formulate the conditions that need to be satisfied to have $R_p \subset \mathfrak{g}_u$, $p > 0$.

Definition 7.26. — *Let $v \in V$ be generic. We say that v satisfies $(*_p)$ if*

1. *We have a decomposition $V = \oplus_{i=0}^k V_i$ where the V_i are multiplicity free H-modules. Let $F_i \subset \mathbb{N}$ such that $V_i = \oplus_{l\in F_i} R_l$.*
2. *Possibly replacing v by hv for some $h \in H$, we have that $v = \sum_i \sum_{l\in F_i} v_{i,l}$ where $v_{i,l} \in R_l \subset V_i$ has height i.*
3. *For every $v_{s+1,l} \in V_{s+1}$, $s \geqslant 0$, we have that $l+p-2j \in F_{s-j}$ for $0 \leqslant j \leqslant \min\{p,s\}$. Let the t_{s-j} be given by Theorem 7.21. Then the vectors $v_{s-j,l+p-2j}$, perhaps after admissible modification, are solutions of* (7.3).

Theorem 7.27. — *Let $v \in V$ be generic. Then $R_p \subset \mathfrak{g}_u$, $p > 0$, if and only if v satisfies $(*_p)$.*

Proof. — We have shown that $R_p \subset \mathfrak{g}_u$ implies that $(*_p)$ holds. Conversely, if $(*_p)$ holds, then we have constants $t_{i,s-j}$ such that (7.1) is satisfied. Let τ denote the corresponding map

$$R_p \otimes \Big(\bigoplus_{j=0}^{\min\{p,s\}} \bigoplus_{i=0}^{s-j} \bigoplus_{l+p-2j\in F_i} R_{l+p-2j} \subset V_i \Big) \to R_l \subset V_{s+1}.$$

The various mappings τ combine to give us an equivariant map $\sigma\colon R_p \otimes V \to V$. It follows from Remark 7.14 that $\sigma(x^i y^{p-i} \otimes v) = 0$ for $i > 0$. By construction, $\sigma(R_p \otimes v)$ is one-dimensional and generated by $\sigma(y^p \otimes v) = A(v)$. If S denotes the copy of $R_p \subset \operatorname{End}(V)$ corresponding to σ, then $[S,S](v) =$

0 which implies that $[S,S]$ acts trivially on V, *i.e.*, $[S,S]=0$. By construction, S consists of nilpotent transformations. Now by Proposition 7.6 we have $S\subset\mathfrak{g}_u$. □

Corollary 7.28. — *Suppose that there is a generic $v\in V$ such that $\mathfrak{g}_u$ is not zero or the trivial H-module. Then there are subsets $F_0,\dots,F_k\subset\mathbb{N}$ such that $V=\bigoplus_{i=0}^k\bigoplus_{l\in F_i}R_l$ and $p>0$ such that for every $l\in F_{s+1}$, $s\geqslant 0$ we have $l+p-2j\in F_{s-j}$ for $0\leqslant j\leqslant\min\{p,s\}$.*

Corollary 7.29. — *The group G normalizes H if V does not satisfy the condition of Corollary 7.28. In particular, G normalizes H in the following cases.*

1. *V is an isotypic H-module.*
2. *The multiplicity of R_l is at least two, where l is maximal such that $R_l\subset V$.*

Proof. — Part (1) is clear. In case (2), there has to be a vector $v_{s+1,l}$ where $s\geqslant 0$. Thus we must have $R_{p+l}\subset V_s$, which obviously fails. □

Using Remark 7.25 it is clear that one can have extremely complicated situations where $R_p\subset\mathfrak{g}_u$. Here is a modestly complicated case.

Example 7.30. — Let $V_0=R_{l+p-2}\oplus R_{l+2p}$, $V_1=R_{l+p}$ and $V_2=R_l$, where $l>1$, $p>0$. Let $v=v_{0,l+p-2}+v_{0,l+2p}+v_{1,l+p}+v_{2,l}\in V=V_0\oplus V_1\oplus V_2$ where the $v_{r,s}$ are of height r in $R_s\subset V_r$. Then by Remark 7.25, $\mathfrak{g}_u$ contains a copy of R_p. Here we have that $\sigma(y^p\otimes v_{0,l+2p})=A(v_{1,l+p})$ and $\sigma(y^p\otimes A^r(v_{0,l+p-2}+v_{1,l+p}))=A^{r+1}(v_{2,l})$, $r=0$, 1. If we add a copy of R_l to V_1 and a copy of R_l to V_0 (assume $p\neq 2$) with corresponding components $v_{1,l}$ and $v_{0,l}$ in v, then we also have a copy of R_0 in $\mathfrak{g}_u$. If $p=2$, we already have $R_l\subset V_0$ and we only have to add $R_l\subset V_1$ and $v_{1,l}$.

Example 7.31. — Suppose that $V=2R_l\oplus R_{l-1}\oplus R_{l+1}$ where $l\geqslant 2$. Then it is possible to have a generic $v\in V$ such that $R_1\subset\mathfrak{g}_u$. However, one can check that this is not possible if we increase the multiplicity of R_l to 3.

8. Appendix

Here we establish the branching rules which are used in Table 2 and the calculation of V^K in cases 6.3 and 6.4. Recall that if φ is a G-module, then $\mathcal{S}(\varphi)=\oplus_k\mathcal{S}^k(\varphi)$ where $\mathcal{S}^k(\varphi)$ denotes the subspace of $S^k(\varphi)$ obtained using Cartan multiplication of the irreducible subrepresentations of φ.

Let $n = 2m+1$, $m \geqslant 1$. Let X denote the cone in $\varphi_{n-1}(\mathsf{D}_n)$ which is the closure of the orbit of a highest weight vector. Consider the action of $\mathrm{SL}_n \times \mathbb{C}^*$ on $\mathcal{O}(X)$ where $\varphi_1(\mathsf{D}_n) = \mathbb{C}^{2n} = \varphi_1(\mathrm{SL}_n) \oplus \varphi_{n-1}(\mathrm{SL}_n) = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. Here $\mathbb{C}^n$ is the span of the positive weight vectors of D_n, $\mathbb{C}^*$ acts on $\mathbb{C}^n$ with weight 2 and on $(\mathbb{C}^n)^*$ with weight -2. As $\mathrm{SL}_n \times \mathbb{C}^*$ representation, $\varphi_{n-1}(\mathsf{D}_n)$ is $\nu_n \oplus \varphi_{n-2}(\mathrm{SL}_n) \otimes \nu_{n-4} + \varphi_{n-4}(\mathrm{SL}_n) \otimes \nu_{n-8} + \cdots + \varphi_1(\mathrm{SL}_n) \otimes \nu_{-n+2}$ and $\varphi_n(\mathsf{D}_n) = \varphi_{n-1}(\mathsf{D}_n)^* = \varphi_{n-1}(\mathrm{SL}_n) \otimes \nu_{n-2} + \cdots + \varphi_2(\mathrm{SL}_n) \otimes \nu_{-n+4} + \nu_{-n}$.

Theorem 8.1. — *Let X be the closure of the highest weight orbit in $\varphi_{n-1}(\mathsf{D}_n)$. Then, as $(\mathrm{SL}_n \times \mathbb{C}^*)$-module, $\mathcal{O}(X) = \mathcal{S}(\varphi_n(\mathsf{D}_n))$.*

Proof. — It is well-known that X is normal, and every point of X except the origin is smooth. For $x \in \nu_n$, $x \neq 0$, x is a smooth point of X, x is a fixed point of SL_n and the slice representation of SL_n at x is $\theta_1 + \varphi_{n-2}$. Since φ_{n-2} has no invariants, $\mathcal{O}(X)^{\mathrm{SL}_n}$ is generated by a coordinate function z on ν_n. Then z is not a zero divisor in $\mathcal{O}(X)$, hence $\mathcal{O}(X)$ is a free $\mathbb{C}[z]$-module. By Luna's slice theorem [14], $\mathcal{O}(X)$ is a free $\mathbb{C}[z]$-module on $\mathcal{O}(\varphi_{n-2})$. But $\mathcal{O}(\varphi_{n-2}) = S(\varphi_2)$ is just the sum of all representations of the form $\varphi_2^{a_2} \cdot \varphi_4^{a_4} \ldots \varphi_{n-1}^{a_{n-1}}$ each with multiplicity one. It follows that the products of the highest weight vectors of the restriction of $\varphi_{n-1}(\mathsf{D}_n)^*$ to SL_n freely generate the highest weights of $\mathcal{O}(X)$ as an SL_n-module and as an $(\mathrm{SL}_n \times \mathbb{C}^*)$-module. □

Now suppose the $n = 2m$, $m \geqslant 2$. Let X denote the closure of the orbit of a highest weight vector of $\varphi_{n-1}(\mathsf{D}_n)$. Consider the action of $\mathrm{SL}_n \times \mathbb{C}^* \subset \mathsf{D}_n$ such that $\varphi_1(\mathsf{D}_n)$ becomes $\mathbb{C}^n \otimes \nu_1 \oplus (\mathbb{C}^n)^* \otimes \nu_{-1}$. Effectively, we have the action of GL_n. Then $\varphi_{n-1}(\mathsf{D}_n)$, as a GL_n-module, is $\nu_m + \varphi_{n-2} \otimes \nu_{m-2} + \cdots + \nu_{-m}$.

Theorem 8.2. — *Let $n = 2m$, $m \geqslant 2$ and let X be the closure of the highest weight orbit in $\varphi_{n-1}(\mathsf{D}_n)$. Then, as GL_n-module, $\mathcal{O}(X) = \mathcal{S}(\varphi_{n-1}(\mathsf{D}_n))$.*

Proof. — Let $z_\pm$ be coordinate functions on the copies of $\nu_{\pm m}$ in $\varphi_{n-1}(\mathsf{D}_n)$. As above, one computes that there is a slice representation $(\varphi_{2m-2} + \theta_1, \mathrm{SL}_{2m})$ for the action of SL_{2m} on X. The slice representation has a quotient of dimension two and principal isotropy group C_m. It follows that the GL_n-invariants have dimension 1, hence they must be generated by z_+z_-. Moreover, the only way that the trivial SL_n-representation can occur in $\mathbb{C}[\varphi_{n-2} \otimes \nu_{m-2} + \cdots + \varphi_2 \otimes \nu_{-m+2}]$ is in products whose $\mathbb{C}^*$-weight is a multiple of $\pm m$ (just count boxes in Young diagrams). Since GL_n is spherical in D_n, each ν_{km}, $k \in \mathbb{Z}$, occurs once in the free $\mathbb{C}[z_+z_-]$-module

$\mathcal{O}(X)$. Thus the SL_n-invariants must be the polynomial ring $\mathbb{C}[z_+, z_-]$ and $\mathcal{O}(X)$ is free over $\mathbb{C}[z_+, z_-]$. For the corresponding map $X \to \mathbb{C}^2$, the general fiber is SL_n/C_m, which gives that the only SL_n-representations that occur are $\varphi_2^{a_2} \dots \varphi_{n-2}^{a_{n-2}}$ for $a_2, \dots, a_{n-2} \geqslant 0$, each with multiplicity one. It follows that $\mathcal{O}(X) = \mathcal{S}(\varphi_{n-1}(\mathsf{D}_n))$. □

Finally, we consider the case where X is the closure of the highest weight vector in $\varphi_n(\mathsf{D}_n)$, $n = 2m \geqslant 4$. As GL_n-module, we have $\varphi_n(\mathsf{D}_n) = \varphi_{n-1} \otimes \nu_{m-1} \oplus \cdots \oplus \varphi_1 \otimes \nu_{-m+1}$.

Theorem 8.3. — *As GL_n-module, $\mathcal{O}(X) = \mathcal{S}(\varphi_n(\mathsf{D}_n))$.*

Proof. — There are no invariants in this case, so we have to proceed a little differently. We first find a general point of X. Let $e_1, \dots, e_n$ be the usual basis of $\mathbb{C}^n$. Let $\omega = e_2 \wedge e_3 + \cdots + e_{2n-2} \wedge e_{2n-1}$ considered as an element of the Lie algebra of D_n. Then the action of $\exp(\omega)$ on e_1 sends it to the sum v of the elements $e_1 \wedge \omega^k \in \varphi_{2k+1}$, $k = 0, \dots, m-1$. The isotropy group H of SL_n acting on v is the semidirect product of C_{m-1} with $\mathrm{Hom}(\mathbb{C} \cdot e_n, \mathbb{C}^{n-1}) \oplus \mathrm{Hom}(\mathbb{C}^{n-2}, \mathbb{C} \cdot e_1)$ where $\mathbb{C}^{n-2}$ here stands for the span of $e_2, \dots, e_{n-1}$ and $\mathbb{C}^{n-1}$ stands for $\mathbb{C}^{n-2} \oplus \mathbb{C} \cdot e_1$. Note that our copy of C_{m-1} acts standardly on $\mathbb{C}^{n-2}$. Now $\dim SL_n/H = \dim X$, so that $\mathrm{SL}_n \cdot v$ is a dense orbit in X. Since X is factorial [26, Theorem 4], any divisor in the complement of the dense orbit must be defined by a semi-invariant of SL_n, hence by an invariant. Thus there are no such divisors, so that the complement of $\mathrm{SL}_n \cdot v$ has codimension 2. It follows that $\mathcal{O}(X) \simeq \mathcal{O}(\mathrm{SL}_n / H)$. But the irreducibles of SL_n with an H-fixed vector are those of the form $\varphi_1^{a_1} \varphi_3^{a_3} \dots \varphi_{n-1}^{a_{n-1}}$ where the a_i are nonnegative, and the fixed point set has dimension one. Thus $\mathcal{O}(X)$ is as claimed. □

We now compute the ring of K-invariants in the cases (6.3) and (6.4) of Table 2.

Proposition 8.4. — *Let X (resp. Y) be the closure of the orbit of the highest weight vector of $\varphi_7(\mathsf{D}_8)$ (resp. $\varphi_8(\mathsf{D}_8)$). Consider the action of B_4 on X and Y where $\varphi_1(\mathsf{D}_8)|\mathsf{B}_4 = \varphi_4(\mathsf{B}_4)$. Then $\mathcal{O}(X)^{\mathsf{B}_4} = \mathbb{C}[f_4]$ and $\mathcal{O}(Y)^{\mathsf{B}_4} = \mathbb{C}[f_2, f_3]$ where $\deg f_i = i$.*

Proof. — Using LiE [25, 24] one computes that the Poincaré series of $\mathcal{O}(X)^{\mathsf{B}_4}$ is $1 + t^4 + \dots$ and that the Poincaré series of $\mathcal{O}(Y)^{\mathsf{B}_4}$ is $1 + t^2 + t^3 + t^4 + t^5 + \dots$. Recall that X and Y are normal, hence so are $\mathcal{O}(X)^{\mathsf{B}_4}$ and $\mathcal{O}(Y)^{\mathsf{B}_4}$. Thus $\dim \mathcal{O}(Y)^{\mathsf{B}_4} \geqslant 2$. The restriction of $\varphi_7(\mathsf{D}_8)$ (resp. $\varphi_8(\mathsf{D}_8)$) to B_4 is $\varphi_1\varphi_4$ (resp. $\varphi_1^2 + \varphi_3$). Let P (resp. Q) be the stabilizer of the highest weight line in $\varphi_7(\mathsf{D}_8)$ (resp. $\varphi_8(\mathsf{D}_8)$). Then the Levi components $L(P)$ and

$L(Q)$ of P and Q double cover representatives of the two SO_{16}-conjugacy classes of embeddings of GL_8 in SO_{16}. We have $L(P) \simeq (\mathrm{SL}_8 \times \mathbb{C}^*)/(\mathbb{Z}/4\mathbb{Z})$ and the same for $L(Q)$. Restricted to $L(P)$, $\varphi_7(\mathsf{D}_8)$ becomes the representation $\nu_4 + \wedge^6(\mathbb{C}^8) \otimes \nu_2 + \wedge^4(\mathbb{C}^8) + \wedge^2(\mathbb{C}^8) \otimes \nu_{-2} + \nu_{-4}$. The highest weight space of $\varphi_7(\mathsf{D}_8)$ is ν_4. The tangent space to X at a nonzero point of ν_4 is $\nu_4 + \wedge^6(\mathbb{C}^8) \otimes \nu_2$ so that $\dim X = 29$. The restriction of $\varphi_7(\mathsf{D}_8)$ to $L(Q)$ is $\wedge^7(\mathbb{C}^8) \otimes \nu_3 + \wedge^5(\mathbb{C}^8) \otimes \nu_1 + \wedge^3(\mathbb{C}^8) \otimes \nu_{-1} + \mathbb{C}^8 \otimes \nu_{-3}$. For $\varphi_8(\mathsf{D}_8)$, the decompositions relative to $L(P)$ and $L(Q)$ are reversed, so $\dim Y = 29$, also.

Consider the action of $H = \operatorname{Ad}\mathrm{SL}_3$ on $\mathbb{C}^9$ as $\varphi_1\varphi_2 + \theta_1$. Then $\varphi_4(\mathsf{B}_4)|H = 2\varphi_1\varphi_2$. Clearly the image of H in SO_{16} lies in a copy of GL_8. Suppose that this copy of GL_8 is double covered by a conjugate of $L(P)$. Then $X^H \neq (0)$, and the B_4-orbit of a nonzero fixed point is closed since the normalizer of H in B_4 is a finite extension of H [15, 3.1 Corollary 1]. It is easy to check that the isotropy group of a nonzero point of X^H is at most a finite extension of H. Thus the dimension of the corresponding closed B_4-orbit is 28. Hence $\dim \mathcal{O}(X)^{\mathsf{B}_4} \leqslant 1$ and the Poincaré series information gives that $\mathcal{O}(X)^{\mathsf{B}_4} = \mathbb{C}[f_4]$ where $\deg f_4 = 4$. If our copy of GL_8 were double covered by a conjugate of $L(Q)$, then we would see that $\dim \mathcal{O}(Y)^{\mathsf{B}_4} \leqslant 1$, which is a contradiction. Thus $\mathcal{O}(X)^{\mathsf{B}_4}$ is as claimed.

Now consider the group $K = \mathrm{SO}_6 \times \mathrm{SO}_3 \subset \mathrm{SO}_9$. Then the double cover $\tilde{K}$ of K is $(\mathrm{SL}_4 \times \mathrm{SL}_2)/\pm I$ and $\varphi_4(\mathsf{B}_4)$, as $\tilde{K}$-representation, is $\mathbb{C}^4 \otimes \mathbb{C}^2 + (\mathbb{C}^4)^* \otimes \mathbb{C}^2$. Thus $\tilde{K}$ is a subgroup of a copy of GL_8 in SO_{16}. If this GL_8 is double covered by a conjugate of $L(P)$, then one sees that there are no nonzero fixed points of $\tilde{K}$ (actually K) in $\varphi_8(\mathsf{D}_8)$. But $\varphi_8(\mathsf{D}_8)|\mathsf{B}_4 = \varphi_1^2 + \varphi_3$ has K-fixed points of dimension 2. Hence our copy of GL_8 is double covered by a conjugate of $L(Q)$ and the weight space ν_4 of the restriction of $\varphi_8(\mathsf{D}_8)$ to $L(Q)$ lies in Y and is fixed by $\tilde{K}$. The group $N_{\mathsf{B}_4}(\tilde{K})/\tilde{K} \simeq \mathbb{Z}/2$ flips the highest and lowest weight spaces $\nu_{\pm 4}$. Since $\tilde{K}$ is a maximal connected reductive subgroup of B_4, the stabilizer of ν_4 is $\tilde{K}$ and any point of ν_4 lies on a closed orbit. The slice representation of $\tilde{K}$ is $S^2(\mathbb{C}^4) + \theta_1$ which shows that the principal isotropy group H of the action of B_4 (actually SO_9) is $SO_3 \times \mathrm{SO}_3 \times \mathrm{SO}_3$. It follows that $\dim Y/\!\!/\mathsf{B}_4 = 2$. Now $N_{\mathrm{SO}_9}(H)/H \simeq W(\mathsf{D}_3)$, the Weyl group of D_3, where $V := \varphi_8(\mathsf{D}_8)^H$ has dimension 5. One easily computes that the generators of $\mathcal{O}(V)^{W(\mathsf{D}_3)}$ are of degree at most 5. Then by the Luna-Richardson theorem [15, 3.2 Corollary] it follows that the invariants of $\mathcal{O}(Y)^{\mathrm{SO}_9}$ have generators in degree at most 5, and then from our information about the Poincaré series it follows that $\mathcal{O}(Y)^{\mathrm{SO}_9} = \mathbb{C}[f_2, f_3]$ where $\deg f_i = i$, $i = 2, 3$. □

Remark 8.5. — There is no representation of SO_9 with principal isotropy group $H = SO_3 \times SO_3 \times SO_3$ and slice representation $S^2(\mathbb{C}^4) + \theta_1$ of $K = SO_6 \times SO_3$ which has homogeneous invariants f_2 and f_3 of degrees 2 and 3, respectively. The reason is that we would have a slice which is an open K-invariant subset of the linear subspace $V = \mathbb{C} \cdot v + S^2(\mathbb{C}^4)$ where K fixes v, and the restrictions of the f_i to V would have to be functions of $\mathbb{C} \cdot v$ alone since the invariant of $S^2(\mathbb{C}^4)$ is of degree 4. Thus f_2 and f_3 would be algebraically dependent, a contradiction to normality.

Remark 8.6. — The generators f_2 and f_3 form a homogeneous regular sequence in $\mathcal{O}(Y)$, hence $\mathcal{O}(Y)$ is a free graded $\mathbb{C}[f_2, f_3]$-module [22, Lemma 3.3]. It follows that $\mathcal{O}(Y)$ is cofree, *i.e.*, each module of covariants is free over $\mathbb{C}[f_2, f_3]$. Of course, we have the analogous result for $\mathcal{O}(X)$.

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