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CONTACT GEOMETRY OF MULTIDIMENSIONAL MONGE-AMPÈRE EQUATIONS: CHARACTERISTICS, INTERMEDIATE INTEGRALS AND SOLUTIONS

by Dmitri V. ALEKSEEVSKY, Ricardo ALONSO-BLANCO, Gianni MANNO & Fabrizio PUGLIESE

Abstract. — We study the geometry of multidimensional scalar 2^{nd} order PDEs (*i.e.* PDEs with n independent variables), viewed as hypersurfaces $\mathcal{E}$ in the Lagrangian Grassmann bundle $M^{(1)}$ over a $(2n+1)$-dimensional contact manifold $(M, \mathcal{C})$. We develop the theory of characteristics of $\mathcal{E}$ in terms of contact geometry and of the geometry of Lagrangian Grassmannian and study their relationship with intermediate integrals of $\mathcal{E}$. After specializing such results to general Monge-Ampère equations (MAEs), we focus our attention to MAEs of type introduced by Goursat in 1899:

$$\det\left\| \frac{\partial^2 f}{\partial x^i \partial x^j} - b_{ij}\,(x, f, \nabla f) \right\| = 0.$$

We show that any MAE of this class is associated with an n-dimensional subdistribution $\mathcal{D}$ of the contact distribution $\mathcal{C}$, and viceversa. We characterize these Goursat-type equations together with their intermediate integrals in terms of their characteristics and give a criterion of local contact equivalence. Finally, we develop a method to solve Cauchy problems for this kind of equations.

Keywords: Hypersurfaces of Lagrangian Grassmannians, contact geometry, subdistributions of a contact distribution, Monge-Ampère equations, characteristics, intermediate integrals.
Math. classification: 53D10, 35A30, 58A30, 58A17.

Résumé. — Nous étudions la géométrie des équations aux dérivées partielles scalaires du deuxième ordre multidimensionnelles (c'est-à-dire, EDP avec n variables indépendantes), considérées comme hypersurfaces $\mathcal{E}$ dans le fibré Grassmannien Lagrangien $M^{(1)}$ sur une variété de contact $(2n+1)$-dimensionnelle $(M,\mathcal{C})$. Nous développons la théorie des caractéristiques de $\mathcal{E}$ en termes de la géométrie de contact et de la géométrie du fibré Grassmannien Lagrangien et étudions leur relation avec les intégrales intermédiaires de $\mathcal{E}$. Après avoir appliqué tels résultats aux équations de Monge-Ampère générales (EMA), nous concentrons notre attention sur les EMA du type introduit par Goursat en 1899 :

$$\det\left\|\frac{\partial^2 f}{\partial x^i \partial x^j} - b_{ij}\,(x,f,\nabla f)\right\| = 0.$$

Nous montrons que toutes les EMA de cette classe sont associées à une sous-distribution n-dimensionnelle $\mathcal{D}$ de la distribution de contact $\mathcal{C}$ et vice-versa. Nous caractérisons les équations du type de Goursat avec leurs intégrales intermédiaires en fonction de leurs caractéristiques et donnons un critère d'équivalence locale de contact. Enfin, nous développons une méthode pour résoudre les problèmes de Cauchy pour ce genre d'équations.

Introduction

Characteristics of PDEs are a classic subject ([9, 10, 19, 21]), as they are related to the local existence and uniqueness of solutions of Cauchy problems. Consider the scalar second order partial differential equation with one unknown function (2^{nd} order PDE)

$$F(x^1,\dots,x^n,z,p_1,\dots,p_n,p_{11},p_{12},\dots,p_{nn}) = 0 \tag{0.1}$$

where $z = z(x^1,\dots,x^n)$, $p_i = \partial z/\partial x^i$, $p_{ij} = \partial^2 z/\partial x^i \partial x^j$; the Cauchy problem consists in finding a solution $z = f(x^1,\dots,x^n)$ of (0.1) such that

$$f|_{(X^1(\mathbf{t}),\dots,X^n(\mathbf{t}))} = Z(\mathbf{t}), \quad \left.\frac{\partial f}{\partial x^i}\right|_{(X^1(\mathbf{t}),\dots,X^n(\mathbf{t}))} = P_i(\mathbf{t}), \tag{0.2}$$

where

$$\Phi(\mathbf{t}) = (X^1(\mathbf{t}),\dots,X^n(\mathbf{t}),Z(\mathbf{t}),P_1(\mathbf{t}),\dots,P_n(\mathbf{t})), \qquad \mathbf{t} = (t_1,\dots,t_{n-1}) \tag{0.3}$$

is a given $(n-1)$-dimensional manifold, *i.e.* a *Cauchy datum* (obviously, the particular choice of the parametrization is irrelevant). If submanifold (0.3) is *non-characteristic*, then, in the C^∞ case, Cauchy problem (0.1)–(0.2) admits a unique formal solution; if, moreover, F is real analytic, then such solution is, in fact, an ordinary one, analytical and locally unique.

As a well known example, take $n=2$. In this case, $\Phi(t)$ is a curve in the space (x^1,x^2,z,p_1,p_2); given a point $\overline{m} = \Phi(0) = (\overline{x}^1,\overline{x}^2,\overline{z},\overline{p}_1,\overline{p}_2)$ on this curve and a point $\overline{m}^1 = (\overline{x}^1, \overline{x}^2, \overline{z}, \overline{p}_1, \overline{p}_2, \overline{p}_{11}, \overline{p}_{12}, \overline{p}_{22})$ satisfying (0.1), the tangent vector $v = \dot{\Phi}(0)$ is non-characteristic for (0.1) at $\overline{m}^1$ if

$$\left.\frac{\partial F}{\partial p_{11}}\right|_{\overline{m}^1} (v^2)^2 - \left.\frac{\partial F}{\partial p_{12}}\right|_{\overline{m}^1} v^1 v^2 + \left.\frac{\partial F}{\partial p_{22}}\right|_{\overline{m}^1} (v^1)^2 \neq 0 \tag{0.4}$$

where $v = v^1(\partial_{x^1} + \overline{p}_1\partial_z + \overline{p}_{11}\partial_{p_1} + \overline{p}_{12}\partial_{p_2}) + v^2(\partial_{x^2} + \overline{p}_2\partial_z + \overline{p}_{12}\partial_{p_1} + \overline{p}_{22}\partial_{p_2})$. Vector v can be considered as an "infinitesimal Cauchy datum". From equation (0.4) it is clear that one can associate with any point m^1 satisfying (0.1) two (possibly imaginary) directions in the space (x^1, x^2, z, p_1, p_2), namely, those annihilating (0.4) ("characteristic lines"); if we let m^1 vary on (0.1) keeping m fixed, these two directions form, in general, two distinct cones at m. It is proved that the only PDEs for which these cones degenerate into two 2-dimensional planes are classical Monge-Ampère equations (MAEs) (see for instance [3, 2]).

One of the aims of this paper is to see whether a similar phenomenon occurs also in the case of MAEs with an arbitrary number of independent variables, which, of course, is considerably more complicated.

In fact, MAEs for $n = 2$ have been intensely studied since the second half of XIX century by many *géomètres*, among them Darboux, Lie, Goursat (a systematic account of such investigations can be found in [8] and [9]); later, this classical approach was put aside in favor of more "hard analysis" techniques. The last 40 years have witnessed a renewed interest in the differential-geometric approach to MAEs, mainly due to Lychagin and his school (see [12] and [13] for an exhaustive bibliography). However, such results are focused on the classical case ($n = 2$).

Up to now, no serious effort has been made to extend the classical theory to the general multidimensional case (only very special cases have been studied). In fact, the main achievements so far obtained in this direction are due to Boillat and Lychagin.

Boillat [4] noticed that MAEs with two independent variables are the only 2^{nd} order PDEs exceptional in the sense of Lax [14]. This property was used in [20] to find the general form of a MAE in three independent variables, and in [5] for the case of arbitrary independent variables. Such general form is

$$M_n + M_{n-1} + \cdots + M_0 = 0 \tag{0.5}$$

where M_k is a linear combination (with functions of x^i, z, p_i as coefficients) of all $k \times k$ minors of the Hessian matrix $\|z_{x^i x^j}\|$.

Lychagin [15], by introducing a new approach based on contact geometry, defined multidimensional MAEs as the kernel of a differential operator associated with a class of n-differential forms on a contact manifold. Locally, such PDEs are described by (0.5). In the rest of the paper, when we write “general MAEs” we mean “multidimensional MAEs in the sense of Lychagin”. In [6, 7] an interpretation of MAEs with constant coefficients is given in terms of Lagrangian Grassmannians.

As far as we know, the oldest paper about the multidimensional generalization of classical MAEs dates back to 1899. In [10] it was noticed that classical MAEs ($n = 2$) can be obtained by substituting $dp_1 = p_{11}dx^1 + p_{12}dx^2$ and $dp_2 = p_{12}dx^1 + p_{22}dx^2$ into the following pfaffian system

$$\begin{cases} dp_1 - b_{11}dx^1 - b_{12}dx^2 = 0 \\ dp_2 - b_{21}dx^1 - b_{22}dx^2 = 0, \qquad b_{ij} = b_{ij}(x^1, x^2, z, p_1, p_2) \end{cases}$$

and by requiring the linear dependence of the obtained 1-forms. Obviously, such a procedure can be extended to any number n of independent variables; namely, one can consider the system

$$dp_i - \sum_{j=1}^{n} b_{ij}dx^j = 0, \;\; i = 1, \dots, n, \;\; b_{ij} = b_{ij}(x^1, \dots, x^n, z, p_1, \dots, p_n)$$

thus getting MAE

$$\det ||p_{ij} - b_{ij}|| = 0. \tag{0.6}$$

It turns out that the class of PDEs considered by Goursat is a subclass of those considered by Lychagin.

The above analytical procedure has a natural geometrical meaning, tightly connected with the fundamental notion of characteristics of a PDE. Such a connection, which was already studied in [3, 2] for $n = 2$, will be extended below to the case of any number of independent variables. As we shall see, for $n > 2$ the complexity of the problem dramatically increases. To this purpose, as a first step we develop a coordinate free setting of the theory of characteristics of 2^{nd} order PDEs (with n independent variables) in terms of contact manifolds and Lagrangian Grassmannians. Then, we focus our attention to MAEs of type (0.5) and (0.6), describe them in terms of their characteristics, study their intermediate integrals and the problem of solutions for a given Cauchy datum.

Notations and conventions

In the rest of the paper we work in the C^∞ case: the term “smooth” means C^∞. Latin indices will run from 1 to n, unless otherwise specified.

We will use Einstein convention. We denote by $X \cdot \varrho$ the Lie derivative of the differential form ϱ along the vector field X and by $\vee$ the symmetric tensor product, *i.e.* $A \vee B = \frac{1}{2}(A \otimes B + B \otimes A)$; $S^2(V)$ is the symmetric square of V. The annihilator of a vector subspace U will be denoted by U^0. We denote by $\langle v_i \rangle$ the linear span of vectors $v_1, \dots, v_n$.

1. Preliminaries and description of the main results

Let $(M, \mathcal{C})$ be a $(2n+1)$-*dimensional contact manifold*, *i.e.* $\mathcal{C}$ is a completely non-integrable distribution on M of codimension 1. Locally, $\mathcal{C}$ is the kernel of a 1-form θ, determined up to a non vanishing factor, with $\theta \wedge d\theta \wedge \overset{n}{\cdots} \wedge d\theta \neq 0$. The restriction $\omega := d\theta|_{\mathcal{C}}$ defines on each hyperplane $\mathcal{C}_m$, $m \in M$, a conformal symplectic structure. Lagrangian planes of $\mathcal{C}_m$ are tangent to maximal integral submanifolds of $\mathcal{C}$; for this reason, such submanifolds are called *Lagrangian* (or also *Legendrian*). We denote by $\mathcal{L}(\mathcal{C}_m)$ the *Grassmannian of Lagrangian planes* of $\mathcal{C}_m$ and by

$$\pi\colon M^{(1)} = \bigcup_{m \in M} \mathcal{L}(\mathcal{C}_m) \to M$$

the bundle of Lagrangian planes. Since points of $M^{(1)}$ are Lagrangian planes, throughout the paper we will consider the identification $m^1 \equiv L_{m^1} \in M^{(1)}$, so that the *tautological bundle* $\mathcal{T}(M^{(1)}) := \bigcup_{m^1 \in M^{(1)}} L_{m^1} \to M^{(1)}$ is well defined.

A *scalar* 1^{st} *order PDE with one unknown function and* n *independent variables* (1^{st} order PDE) is a hypersurface $\mathcal{F}$ of M and its *solutions* are integral manifolds of $\mathcal{C}$ contained in $\mathcal{F}$. A *scalar* 2^{nd} *order PDE with one unknown function and* n *independent variables* (2^{nd} order PDE) is a hypersurface $\mathcal{E}$ of $M^{(1)}$ and its *solutions* are Lagrangian submanifolds $\Sigma \subset M$ such that $T\Sigma \subset \mathcal{E}$. A *Cauchy datum* for $\mathcal{E}$ is defined as an $(n-1)$-dimensional integral submanifold of $\mathcal{C}$. The restriction to $\mathcal{E}$ of fibre bundle π is a bundle over M whose fibre at m is the hypersurface of $\mathcal{L}(\mathcal{C}_m)$

$$\mathcal{E}_m := \mathcal{E} \cap \mathcal{L}(\mathcal{C}_m).$$

A *characteristic subspace for* $\mathcal{E}$ *at* m^1 is a hyperplane $U \subset L_{m^1}$ such that the curve $U^{(1)} \subset \mathcal{L}(\mathcal{C}_m)$ of Lagrangian planes containing U is tangent to $\mathcal{E}_m$ at m^1. The tangent space $T_{m^1}U^{(1)}$ is called a *characteristic direction for* $\mathcal{E}$ *at* m^1. When $U^{(1)} \subset \mathcal{E}_m$, hyperplane U is said to be *strongly characteristic*.

By means of previous geometric concepts, we are able to give an intrinsic definition of MAEs of form (0.5) and (0.6). Of these, the former describes, locally, hypersurfaces $\mathcal{E}_\Omega$ of $M^{(1)}$ formed by Lagrangian planes which annihilate an n-form Ω on M (to avoid trivial equations, this form can be chosen in $\Lambda^n(M)\smallsetminus\mathcal{I}(\theta)$, where $\mathcal{I}(\theta)\subset\Lambda^*(M)$ denotes the differential ideal generated by a contact form θ):

$$\mathcal{E}_\Omega=\left\{m^1\in M^{(1)}\mid \Omega|_{L_{m^1}}=0\right\}. \tag{1.1}$$

As to (0.6), it locally describes hypersurfaces $\mathcal{E}_\mathcal{D}$ of $M^{(1)}$ whose points are Lagrangian planes which non trivially intersect an n-dimensional subdistribution $\mathcal{D}$ of $\mathcal{C}$:

$$\mathcal{E}_\mathcal{D}=\left\{m^1\in M^{(1)}\mid L_{m^1}\cap\mathcal{D}_{\pi(m^1)}\neq 0\right\}. \tag{1.2}$$

One of the main geometric objects associated with a 2^{nd} order PDE $\mathcal{E}$ is its *conformal metric* $g_\mathcal{E}$, which is defined by means of the canonical isomorphism $g_{m^1}\colon T^*_{m^1}\mathcal{L}(\mathcal{C}_m)\xrightarrow{\sim} L_{m^1}\vee L_{m^1}$, $\rho\mapsto g_\rho$, where $m^1\in M^{(1)}$ (see Section 2 for details), and defining $g_\mathcal{E}$ as $(g_\mathcal{E})_{m^1}=[g_{(dF)_{m^1}}]$, where $\mathcal{E}=\{F=0\}$.

Now, we are in the position to formulate the main result of the paper.

Theorem 1.1. — *Let $\mathcal{E}\subset M^{(1)}$ be a 2^{nd} order PDE. Then $\mathcal{E}$ is locally of the form $\mathcal{E}_\mathcal{D}$ for some n-dimensional distribution $\mathcal{D}\subset\mathcal{C}$ iff the following properties are satisfied:*

1. *Its conformal metric is decomposable: $(g_\mathcal{E})_{m^1}=\ell_{m^1}\vee\ell'_{m^1}$, where $\ell_{m^1},\ell'_{m^1}\subset L_{m^1}$ are lines;*
2. *if we let vary the point m^1 along the fibre $\mathcal{E}_m$, the lines ℓ_{m^1},ℓ'_{m^1} fill two n-dimensional spaces $\mathcal{D}_{1m}$, $\mathcal{D}_{2m}$ of $\mathcal{C}_m$.*

Furthermore, $\mathcal{D}_1$ and $\mathcal{D}_2$ are mutually orthogonal w.r.t. $\omega=d\theta$ and $\mathcal{E}=\mathcal{E}_{\mathcal{D}_1}=\mathcal{E}_{\mathcal{D}_2}$.

Essentially, we find necessary and sufficient conditions for a scalar 2^{nd} order PDE to be of $\mathcal{E}_\mathcal{D}$ type. In [10] the author found sufficient conditions in terms of the existence of a suitable number of intermediate integrals: we give a geometrical interpretation of this result in Corollary 6.6. Also, we would like to underline that the above theorem is the natural generalization of a well known result for $n=2$: a 2^{nd} order PDE $\mathcal{E}\subset M^{(1)}$ with two independent variables is a non-elliptic MAE iff the characteristic lines fill two 2-dimensional subdistributions $\mathcal{D}_1$, $\mathcal{D}_2\subset\mathcal{C}$ which turn out to be mutually orthogonal w.r.t. $\omega=d\theta$. The equation is parabolic if $\mathcal{D}_1=\mathcal{D}_2$ and hyperbolic otherwise.

Then, we describe some procedures for integrating equations $\mathcal{E}_\mathcal{D}$, based on the existence of classical or nonholonomic intermediate integrals (this notion is a generalization of the ordinary one, see Section 6.4). For this kind of equations, we get an easy generalization of the Monge method stated in Theorem 6.12. As an application, in Section 7.1 we prove that MAEs of type $\mathcal{E}_{\mathcal{D}}$ (possibly, with no ordinary intermediate integral) admitting a special nonholonomic intermediate integral have (smooth) solutions. In Section 7.2 we prove that when a MAE of type $\mathcal{E}_{\mathcal{D}}$ admits a suitable number of independent intermediate integrals the Cauchy problem can be solved. In Section 7.2.1 we work out all details and computations for an explicit equation by using our results, including main Theorem 1.1.

2. Geometry of the tangent and cotangent bundle of the Lagrangian Grassmannian $\mathcal{L}(V)$

Lagrangian Grassmannian $\mathcal{L}(V)$ and its tautological bundle $\mathcal{T}(\mathcal{L}(V))$. Let (V,ω) be a symplectic $2n$-dimensional vector space. A *Lagrangian plane* is an isotropic subspace $L\subset V$ of maximal dimension (*i.e.* $\omega|_L=0$ and $\dim L=n$). We shall denote by $\mathcal{L}(V)$ the *Grassmannian of Lagrangian planes* in V and by $\mathcal{T}(\mathcal{L}(V))$ its *tautological bundle*, *i.e.* the fiber at point $L\in\mathcal{L}(V)$ is L. Fixed a symplectic basis $\{e_i,e^i\}$ (*i.e.* $\omega(e_i,e^j)=\delta_i^j$), each n-plane $L\in\mathcal{L}(V)$ transversal to $\langle e^1,\dots,e^n\rangle$ is uniquely determined by a symmetric real matrix $P=\|p_{ij}\|$: $L=L_P=\langle e_i+p_{ij}e^j\rangle$. If U is a subspace of V, we shall denote by $U^\perp$ the orthogonal complement of U w.r.t. ω.

The *Plücker embedding* $\iota\colon L=\langle v_1,v_2,\dots,v_n\rangle\in\mathcal{L}(V)\mapsto[\mathrm{vol}_L]\in\mathbb{P}\Lambda^n(V)$, where $\mathrm{vol}_L:=v_1\wedge v_2\wedge\cdots\wedge v_n\in\bigwedge^n(V)$, allows to identify $\mathcal{L}(V)$ with its image into the projective space $\mathbb{P}\Lambda^n(V)$. A straight line of $\mathbb{P}\Lambda^n(V)$ which is included in $\iota(\mathcal{L}(V))$ is called a *line* of $\mathcal{L}(V)$. We will denote by $\ell(L,\dot L)$ the line of $\mathbb{P}\Lambda^n(V)$ passing at L with direction $\dot L\in T_L\mathcal{L}(V)$.

Metrics associated with tangent and cotangent vectors of $\mathcal{L}(V)$. It is well known that there is a canonical isomorphism

$$g\colon T_L\mathcal{L}(V)\xrightarrow{\sim}S^2(L^*),\quad \dot L\longmapsto g^{\dot L}. \tag{2.1}$$

In this way, a vector field X on $\mathcal{L}(V)$ defines a section g^X of $S^2\big(\mathcal{T}^*(\mathcal{L}(V))\big)$ which we will call a metric on $\mathcal{T}(\mathcal{L}(V))$ (note that it can be *degenerate*). By duality, we also get a canonical isomorphism $g\colon T^*_L\mathcal{L}(V)\xrightarrow{\sim}S^2(L)$, $\rho\mapsto g_\rho$ (the use of super and subscripts eliminates the ambiguity on maps "g").

In terms of coordinates, the metric $g^{\dot{L}}$ on $L = \langle w_i := e_i + p_{ij} e^j \rangle$ associated with $\dot{L} \sim ||\dot{p}_{ij}||$ is given by

$$g^{\dot{L}} = \dot{p}_{ij}\, e^i \otimes e^j.$$

In the same way, the metric g_ρ on L^* associated with 1-form $\rho = \rho^{ij} dp_{ij}$ is $g_\rho = \rho^{ij}\, w_i \otimes w_j$, In particular, a function $F \in C^\infty(\mathcal{L}(V))$ defines a metric on L^*:

$$g_{(dF)_L} = \sum_{i \leqslant j} \frac{\partial F}{\partial p_{ij}}\, w_i \vee w_j. \tag{2.2}$$

Rank of tangent vectors of $\mathcal{L}(V)$. Isomorphism (2.1) allows to define the *rank* of a tangent vector $\dot{L} \in T_L \mathcal{L}(V)$ as that of the corresponding symmetric bilinear form $g^{\dot{L}} \in S^2(L^*)$. We call the set $T^1\mathcal{L}(V)$ of vectors of rank 1 the *characteristic cone* or *Segre variety* (see [1]). If $\dot{L} \in T^1_L \mathcal{L}(V)$, then, up to a sign,

$$\dot{L} \simeq g^{\dot{L}} = \eta \otimes \eta, \ \text{ for some } \eta \in L^*. \tag{2.3}$$

From now on, we identify $\dot{L}$ with $g^{\dot{L}}$.

Proposition 2.1. — *The straight line $\ell(L, \dot{L})$ of $\mathbb{P}\Lambda^n(V)$ is a line of $\mathcal{L}(V)$ iff* $\mathrm{rank}(\dot{L}) = 1$.

Proof. — Assume that $\dot{L} \in T^1_L \mathcal{L}(V)$. Take coordinates $P = ||p_{ij}||$ with $P(L) = 0$ and $P(\dot{L}) = \mathrm{diag}(1, 0, \dots, 0)$. Then,

$$\ell(L, \dot{L}) = [(e_1 + te^1) \wedge e_2 \cdots \wedge e_n] = [e_1 \wedge \cdots \wedge e_n + te^1 \wedge e_2 \wedge \cdots \wedge e_n] \subset \mathcal{L}(V).$$

The converse is derived from the following property: if $a, a' \in \Lambda^k(W)$ are two k-vectors such that $ta + sa'$ is decomposable for any $t, s \in \mathbb{R}$, then there exists a decomposable $(k-1)$-vector $b \in \Lambda^{k-1}(W)$ and vectors v, v' such that $a = v \wedge b$ and $a' = v' \wedge b$. Indeed, a k-vector c is decomposable iff it satisfies the Plücker relation $(\gamma \lrcorner c) \wedge c = 0$ for any $\gamma \in \Lambda^{k-1}(W^*)$ (see, for example [11]). By hypothesis, these relations hold for $c = a$, $c = a'$ and $c = a + a'$. Then we get

$$0 = (\gamma \lrcorner a) \wedge a' + (\gamma \lrcorner a') \wedge a, \ \ \forall\, \gamma \in \Lambda^{k-1}(W^*).$$

We choose γ such that $v' := \gamma \lrcorner a \neq 0$ and $v := -\gamma \lrcorner a' \neq 0$. Then $v' \wedge\, a = v \wedge a'$, so that $a = v \wedge b$, $a' = v' \wedge b$ for some $b \in \Lambda^{k-1}(W)$. $\square$

3. Hypersurfaces of the Lagrangian Grassmannian

3.1. Characteristic cone and characteristic subspaces of a hypersurface E of $\mathcal{L}(V)$ and its conformal metric g_{E}

Let $\mathrm{E} = \{F = 0\}$ with $F \in C^\infty(\mathcal{L}(V))$ such that $dF \neq 0$ be a hypersurface of $\mathcal{L}(V)$. We denote by $g_{\mathrm{E}} := [g_{dF}|_{\mathrm{E}}]$ the conformal class of the restriction of g_{dF} to E; we call it the *conformal metric* on E. It is independent of the choice of F and its local expression is given by (2.2).

Definition 3.1. — *The set* $\mathrm{Ch}_L(\mathrm{E}) = T_L\mathrm{E} \cap T^1_L\mathcal{L}(V)$ *of rank* 1 *tangent vectors to* E *at point* L *is called the* characteristic cone *of* E *at* L *and its elements are called* characteristic vectors *for* E *at* L*. The* 1*-dimensional vector space generated by a characteristic vector is called a* characteristic direction*. A characteristic vector* $\dot{L}$ *for* E *at* L *is called* strongly characteristic *if the line* $\ell(L, \dot{L})$ *is contained in* E*.*

Lemma 3.2. — *Characteristic vectors* $\dot{L} \in \mathrm{Ch}_L(\mathrm{E})$ *are, up to sign, the tensor square* $\dot{L} = \eta \otimes \eta$ *of* g_{E}*-isotropic covectors* $\eta \in L^*$*.*

Proof. — By definition, $\dot{L}$ is characteristic for $\mathrm{E} = \{F = 0\}$ if, besides being of the form $\pm\eta \otimes \eta$ (rank 1), it is tangent to E; in other words, if $\dot{L}$ kills $(dF)_L$. So,

$$0 = \langle \dot{L}, (dF)_L \rangle = \langle g^{\dot{L}}, g_{(dF)_L} \rangle = \langle \pm\eta \otimes \eta, g_{(dF)_L} \rangle = \pm g_{(dF)_L}(\eta, \eta)$$

and the result follows because $(g_{\mathrm{E}})_L = g_{(dF)_L}$. □

We define the *prolongation* $U^{(1)} \subset \mathcal{L}(V)$ of a subspace $U \subset V$ by:

$$U^{(1)} := \begin{cases} L \in \mathcal{L}(V) \mid L \supseteq U, & \text{if } \dim(U) \leqslant n \\ L \in \mathcal{L}(V) \mid L \subseteq U, & \text{if } \dim(U) \geqslant n. \end{cases} \tag{3.1}$$

Since $L = L^\perp$, then $U \subset W \Longrightarrow U^{(1)} \supset W^{(1)}$ and also $U^{(1)} = \big(U^\perp\big)^{(1)}$. If U is isotropic, then

$$U^{(1)} \simeq U \oplus \mathcal{L}(W) := \big\{U \oplus L' \mid L' \in \mathcal{L}(W)\big\}, \tag{3.2}$$

where $W := (U \oplus U')^\perp$ with U' satisfying $\dim U' = \dim U$ and $\omega|_{U\oplus U'}$ nondegenerate. In coordinates, let $\{e_i\}$ be a basis of L such that $U = \langle e_a \rangle_{1\leqslant a\leqslant k}$; let also $\{e_i, e^i\}$ be its extension to a symplectic basis of V and consider $U' := \{e^a\}_{1\leqslant a\leqslant k}$. So,

$$U^{(1)} = \Big\{L = \langle e_a, e_i + p_{ij}e^j \rangle \;\Big|\; 1 \leqslant a \leqslant k\,,\ ||p_{ij}|| \in S^2(\mathbb{R}^{n-k})\Big\},$$

and its tangent space at L is given by

$$T_L U^{(1)} = \langle e^i \vee e^j,\ i,j = k+1,\dots,n\rangle \simeq S^2(U^0) \subset S^2(L^*), \tag{3.3}$$

where $U^0 \subset L^*$ denotes the annihilator of U.

Definition 3.3. — *An isotropic subspace U is called* characteristic *for a covector $\rho \in T^*_L\mathcal{L}(V)$ if $U \subset L$ and $\rho|_{T_L U^{(1)}} = 0$. It is called characteristic for a hypersurface $\mathrm{E} = \{F = 0\}$ of $\mathcal{L}(V)$ at a point $L \in \mathrm{E}$ if it is characteristic for the covector $(dF)_L$. It is called* strongly characteristic *if $U^{(1)} \subset \mathrm{E}$. A covector $\eta \in L^*$ is called characteristic for ρ if the hyperplane* $\mathrm{Ker}(\eta)$ *is characteristic for ρ.*

This definition extends Definition 3.1 in the following sense. Let $\eta \in L^*$ and $U := \mathrm{Ker}(\eta) \subset L$ be its associated hyperplane; equation (3.3) gives $T_L U^{(1)} = \langle \eta \otimes \eta\rangle$, so that U is characteristic for $(dF)_L$ in the sense of Definition 3.3 iff the (one-dimensional) tangent direction to $U^{(1)}$ is generated by one characteristic vector for $(dF)_L$ in the sense of Definition 3.1.

By using again identification (3.3), and by arguing as in the proof of Lemma 3.2, we can determine the "characteristicness" of a subspace in terms of the conformal metric: this is the content of the following

Lemma 3.4. — *Let $U \subset L \in \mathcal{L}(V)$ and $\rho \in T^*_L\mathcal{L}(V)$. Then U is characteristic for ρ iff its annihilator $U^0 \subset L^*$ is g_ρ-isotropic.*

The following proposition relates the decomposability of g_ρ with the behavior of the set of characteristic hyperplanes for ρ.

Proposition 3.5. — *Let $\rho \in T^*_L\mathcal{L}(V)$. Then g_ρ is decomposable iff characteristic hyperplanes for ρ form two $(n-2)$-parametric families $\mathcal{H}$ and $\mathcal{H}'$ such that*

$$\dim \bigcap_{U\in\mathcal{H}} U = \dim \bigcap_{U\in\mathcal{H}'} U = 1.$$

Proof. — Let $g_\rho = v \vee w$ for some $v, w \in L$. By Lemma 3.4, a hyperplane $U = \mathrm{Ker}(\eta)$ of L is characteristic iff $g_\rho(\eta,\eta) = \eta(v)\eta(w) = 0$. This means that $v \in U$ or $w \in U$. So we get two families of characteristic hyperplanes $\mathcal{H} := \{U \subset L \mid\ v \in U\}$, $\mathcal{H}' := \{U \subset L \mid w \in U\}$ such that $\bigcap_{U\in\mathcal{H}} U = \langle v\rangle$ and $\bigcap_{U\in\mathcal{H}'} U = \langle w\rangle$.

Viceversa, let $\mathcal{H}$ be a $(n-2)$-parametric family of characteristic hyperplanes for ρ which contain a common line $\langle v\rangle$. By dimensional reasons, the set $\bigcup_{U\in\mathcal{H}} U^0 = \{\eta \in L^* \mid \eta|_U = 0 \text{ for some } U \in \mathcal{H}\}$ contains a conic convex open subset $\mathcal{O}$ of the annihilator $v^0 \subset L^*$. So $\eta, \eta' \in \mathcal{O}$ implies that $\eta+\eta' \in \mathcal{O}$. Lemma 3.4 shows that $g_\rho(\eta,\eta) = g_\rho(\eta',\eta') = g_\rho(\eta+\eta',\eta+\eta') = 0$

which implies $g_\rho(\eta,\eta')=0$, $\forall\eta,\eta'\in\mathcal{O}$. Since v^0 is spanned by $\mathcal{O}$, it is g_ρ-isotropic. Thus, $g_\rho=v\vee w$ for some $w\in L$. □

3.2. Hypersurfaces of $\mathcal{L}(V)$ associated with n-forms and their characteristics

Any n-form $\Omega\in\Lambda^n(V^*)$ defines the hypersurface

$$\mathrm{E}_\Omega=\big\{L\in\mathcal{L}(V)\ \mid\ \Omega|_L=0\big\}. \tag{3.4}$$

For each $\sigma\in\Lambda^{n-2}(V^*)$, the n-form $\Omega^\sigma:=\Omega+\sigma\wedge\omega$ defines the same hypersurface.

DEFINITION 3.6. — *Let $\Omega\in\Lambda^n(V^*)$. A k-dimensional subspace $U=\langle e_1,\cdots,e_k\rangle\subset V$ is called* Ω-isotropic *if $(e_1\wedge\cdots\wedge e_k)\lrcorner\,\Omega=0$.*

THEOREM 3.7. — *Let $L\in\mathrm{E}_\Omega$ and H be a hyperplane of L. Then the following equivalences hold:*

(1) *H is characteristic for E_Ω at L;*
(2) *H is strongly characteristic;*
(3) *H is Ω^σ-isotropic for some $\sigma\in\Lambda^{n-2}(V^*)$.*

Proof. — Implications $2\Rightarrow1$ and $3\Rightarrow1$ are trivial.

$1\Rightarrow2$. Below we will adopt the following notation: if $W=\langle v_i\rangle$, then $\mathrm{vol}_W:=v_1\wedge v_2\wedge\cdots\wedge v_n$. Let $\{e_i,e^i\}$ be a symplectic basis of V such that $H=\langle e_1,\dots,e_{n-1}\rangle\subset\langle e_1,\dots,e_{n-1},e_n\rangle=L$, so that $H^{(1)}=\{L_t=\langle e_1,\dots,e_{n-1},e_n+te^n\rangle\}$. Any Lagrangian plane in a neighborhood of $L=L_0$ is of the form $\widetilde{L}=\langle e_i+p_{ij}e^j\rangle$. Let $\mathrm{vol}_t:=\mathrm{vol}_{L_t}$, so that $\mathrm{vol}_t=\mathrm{vol}_L+t\ \mathrm{vol}_{L'}$ where $L'=\langle e_1,\dots,e_{n-1},e^n\rangle$. In this way the tangent vector to $H^{(1)}$ at L is defined by the derivative along $\mathrm{vol}_{L'}$. Also, let $F(\widetilde{L})=\mathrm{vol}_{\widetilde{L}}\lrcorner\,\Omega$, so that E_Ω is locally described by $\{F=0\}$. The derivative of F at L along $\mathrm{vol}_{L'}$ is

$$\begin{aligned}\lim_{t\to0}\frac{F(L_t)-F(L)}{t}&=\lim_{t\to0}\frac{\mathrm{vol}_t\lrcorner\,\Omega-\mathrm{vol}_L\lrcorner\,\Omega}{t}\\&=\lim_{t\to0}\frac{(\mathrm{vol}_L+t\ \mathrm{vol}_{L'})\lrcorner\,\Omega-\mathrm{vol}_L\lrcorner\,\Omega}{t}\\&=\mathrm{vol}_{L'}\lrcorner\,\Omega=F(L')\end{aligned}$$

which vanishes iff L' belongs to E_Ω. Hence, $H^{(1)}$ is included in E_Ω.

$1 \Rightarrow 3$. By using the above results, we have that

$$
\begin{aligned}
H \text{ is characteristic } &\iff H^{(1)} \subset \mathrm{E}_\Omega \\
&\iff \mathrm{vol}_t \lrcorner\, \Omega = 0 \iff \Omega_a(e_n) = \Omega_a(e^n) = 0
\end{aligned}
$$

where $\Omega_a := a \lrcorner\, \Omega$, $a = e_1 \wedge \cdots \wedge e_{n-1}$. For any $\sigma \in \Lambda^{n-2}(V^*)$, we have that

$$
a \lrcorner\, \Omega^\sigma = \Omega_a + \sum_{j=1}^{n-1} (-1)^j \sigma(e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n-1})(e_j \lrcorner\, \omega).
$$

Thus, $(a \lrcorner\, \Omega^\sigma)|_{L'} = 0$ and $(a \lrcorner\, \Omega^\sigma)(e^i)$ vanishes if $\sigma(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n-1}) = (-1)^{i+1}\Omega_a(e^i)$. For such a σ, $a \lrcorner\, \Omega^\sigma = 0$, *i.e.* H is isotropic for Ω^σ. □

3.3. Hypersurfaces E_D associated with an n-plane D and their characteristics

We associate with an n-dimensional subspace $D \subset V$ the following subset of $\mathcal{L}(V)$:

$$
\mathrm{E}_D = \{L \in \mathcal{L}(V) \mid L \cap D \neq 0\}. \tag{3.5}
$$

If $D = \{\varrho_1 = \varrho_2 = \cdots = \varrho_n = 0\}$, then

$$
\mathrm{E}_D = \mathrm{E}_{\Omega_D} \quad \text{where} \quad \Omega_D := \varrho_1 \wedge \cdots \wedge \varrho_n. \tag{3.6}
$$

If $D = \langle e_i + b_{ij} e^j \rangle$ (for some symplectic basis $\{e_i, e^i\}$), then $\mathrm{E}_D = \{L = L_P \mid \det(P - B) = 0\}$, where $P = ||p_{ij}||$ and $B = ||b_{ij}||$. In particular, E_D is an algebraic hypersurface of $\mathcal{L}(V)$. Below we describe the conformal metric g_{E_D} in coordinates.

Proposition 3.8. — *Let E_D be the hypersurface of $\mathcal{L}(V)$ associated with n-plane $D = \langle e_i + b_{ij} e^j \rangle$ and $L = L_P = \langle w_i = e_i + p_{ij} e^j \rangle \in \mathrm{E}_D$. Then the conformal metric g_{E_D} on L^* is given by*

$$
g_{\mathrm{E}_D} = A^{ij}\, w_i \vee w_j \tag{3.7}
$$

where $A = ||A^{ij}||$ and A^{ij} is the algebraic complement of the (i,j)-entry in matrix $(P - B)$. Moreover

1. *$A = 0$ if $\mathrm{rank}(P - B) < n-1$;*
2. *$A = ||a^i b^j||$ if $\mathrm{rank}(P - B) = n-1$ where $(P - B) \cdot a = 0$ and $(P - B^t) \cdot b = 0$. In particular*
   (a) *$g_{\mathrm{E}_D} = a \vee b$, $a = a^i w_i$, $b = b^i w_i$;*

(b) *matrix* $\frac{1}{2}(A+A^t)$ *has rank equal to* 1 *if* $B=B^t$ *and rank equal to* 2 *if* $B\neq B^t$.

Proof. — Since $\frac{\partial}{\partial p_{ij}}(\det(P-B))$ equals A^{ii} if $i=j$, and $A^{ij}+A^{ji}$ if $i\neq j$, then, for each $\eta\in L^*$,

$$\begin{aligned} g_{\mathrm{E}_D}(\eta,\eta) &= \sum_{i\leqslant j}\frac{\partial}{\partial p_{ij}}\big(\det(P-B)\big)\eta_i\eta_j \\ &= \sum_{i,j}A^{ij}\eta_i\eta_j = \frac{1}{2}\sum_{i,j}(A^{ij}+A^{ji})\eta_i\eta_j, \end{aligned}$$

where $\eta_i=\eta(w_i)$. This proves (3.7). The second part of the lemma follows from elementary properties of adjoint matrices. □

Definition 3.9. — *A point* $L\in\mathrm{E}_D$ *is called* singular *if* $\dim(L\cap D)\geqslant 2$ *and* regular *otherwise. The set of regular points of* E_D *will be denoted by* $\mathrm{E}_D^{\mathrm{reg}}$.

Now we give a criterion to distinguish singular points.

Proposition 3.10. — *A point* $L_P\in\mathrm{E}_D$ *is singular iff the differential of* $\det(P-B)$ *at* L *vanishes, that is iff the metric* g_{E_D} *vanishes at* L.

Proof. — Since $L\cap D=\mathrm{Ker}(P-B)$, we derive the equivalence $\dim(L\cap D)=k \iff \mathrm{rank}(P-B)=n-k$. If $k\geqslant 2$, then $\mathrm{rank}(P-B)\leqslant n-2$, which implies that its adjoint matrix vanishes. Then $\frac{\partial}{\partial p_{ij}}\big(\det(P-B)\big)=0$ at the point L and $(g_{\mathrm{E}_D})_L=0$. □

The following key proposition states that, given an n-dimensional subspace $D\subset V$, the only other subspace defining the same E_D is the skew-orthogonal complement $D^\perp$.

Proposition 3.11. — *Let* (V,ω) *be a* $2n$*-dimensional symplectic vector space. Let* D *and* $\widetilde{D}$ *be* n*-dimensional planes of* V*. Then*

$$\mathrm{E}_{\widetilde{D}}=\mathrm{E}_D \iff \widetilde{D}=D \ \text{ or } \ \widetilde{D}=D^\perp.$$

Proof. — One implication will be proved if $\dim(L\cap D)=\dim(L\cap D^\perp)$ for any Lagrangian plane L. But this easily follows from identities

$$L\cap D^\perp=L^\perp\cap D^\perp=(L\cup D)^\perp=(L+D)^\perp.$$

As to the inverse implication, we must prove that for any $e\in V\setminus(D\cup D^\perp)$ there exists a Lagrangian subspace $L\ni e$ such that $L\cap D=0$, so that $L\notin\mathrm{E}_D$; in this way, if $e\in\widetilde{D}$, then $\mathrm{E}_{\widetilde{D}}\neq\mathrm{E}_D$. In order to get such an L, let us consider the $(n-1)$-dimensional subspace $D\cap e^\perp$ and the quotient map $\Pi\colon e^\perp\to e^\perp/\langle e\rangle$ (symplectic reduction). The projection $\Pi(D\cap e^\perp)$ is a half dimensional space in the symplectic space $e^{\perp}/\langle e\rangle$ and it is elementary the existence of a Lagrangian subspace $\widetilde{L}\subset e^{\perp}/\langle e\rangle$ such that $\Pi(D\cap e^{\perp})\cap\widetilde{L}=0$. Then, $L:=\Pi^{-1}(\widetilde{L})$ is a Lagrangian subspace of V with $e\in L$ and $L\cap D=0$, as required. □

If we translate previous proposition in terms of n-forms, we get the following

Corollary 3.12. — *Up to a factor, at most two different decomposable n-forms give equation E_{Ω}.*

The proposition below describes characteristic hyperplanes for hypersurfaces E_D.

Proposition 3.13. — *Let D and Ω_D be as in (3.6). Let also $H\subset V$ be an $(n-1)$-dimensional isotropic subspace and $H^{(1)}=\{L_t\}$. Then the following conditions are equivalent:*

(1) *$H\subset L_0$ is characteristic for E_D at $L_0\in\mathrm{E}_D$;*
(2) *$H^{(1)}\subset\mathrm{E}_D$;*
(3) *$\langle\Omega_D,\mathrm{vol}_t\rangle=0$, where vol_t denotes an arbitrary element in $\bigwedge^n(L_t)$ different from zero;*
(4) *$L_t\cap D\neq 0$ for all t;*
(5) *H has non trivial intersection with D or $D^{\perp}$.*

Proof. — Equivalence $1\Leftrightarrow 2$ is Theorem 3.7, taking into account (3.6). Properties 3 and 4 are, by definition, alternative ways to write property 2.

Let us passe to equivalence $2\Leftrightarrow 5$. Let H be characteristic for E_D at L, (so, it is also strongly characteristic and, hence, any Lagrangian plane containing H intersects D non trivially). We want to prove that H has a non trivial intersection with either D or $D^{\perp}$. Let $H\cap D=0$. Let $\{e_i,e^i\}$ be a symplectic basis such that $H=\langle e_1,\dots,e_{n-1}\rangle$ and $L=\langle e_1,\dots,e_{n-1},e_n\rangle$. By assumption, $L\cap D\neq 0$, so that the unique possibility is that $L\cap D$ is generated by a vector $e_n+\sum_{i=1}^{n-1}\alpha_ie_i$. Up to a change of basis, we can assume such generator to be e_n (in particular, $e_n\in D$). Now, the Lagrangian planes $L_t:=\langle e_1,\dots,e_{n-1},e_n+te^n\rangle$ have non trivial intersections with D. In fact, by the same reasoning as above, $L_t\cap D$, $t\neq 0$, is generated by a vector of the form $e_n+te^n+\sum_{i=1}^{n-1}\alpha_i(t)e_i=e_n+t\left(e^n+\sum_{i=1}^{n-1}t^{-1}\alpha_i(t)e_i\right)$. Taking into account that $e_n\in D$, we get $v_n:=e^n+\sum_{i=1}^{n-1}t^{-1}\alpha_i(t)\,e_i\in D$. If we take two different values $t,\bar t$ we have that $\sum_{i=1}^{n-1}\left(t^{-1}\alpha_i(t)-(\,\bar t\,)^{-1}\alpha_i(\bar t)\right)e_i\in D\cap H=0$ which implies that v_n is independent of t. A new change of basis allows to take $e^n=v_n$, so that $L_t\cap D=\langle e_n+te^n\rangle$; in particular,

$D \supset \langle e_n, e^n \rangle$ and $D^\perp \subset \langle e_n, e^n \rangle^\perp$. Also, $H \subset \langle e_n, e^n \rangle^\perp$ and a computation gives us

$$\dim D^\perp \cap H = \dim D^\perp + \dim H - \dim(D^\perp + H) \geqslant n + (n-1) - (2n-2) = 1,$$

because $D^\perp + H \subset \langle e_n, e^n \rangle^\perp$. As a consequence, $H \cap D^\perp \neq 0$, as we wanted. $\square$

Remark 3.14. — Claims 1, 2, 3 of the above theorem remain equivalent also for hypersurfaces E_Ω.

Bringing together Propositions 3.5, 3.8, 3.11, 3.13, in the theorem below we summarize the main results regarding the hypersurfaces of type E_D by pointing out how to describe them in terms of their characteristics.

THEOREM 3.15. — *Let $\mathrm{E}_D^{\mathrm{reg}}$ be the set of regular points of E_D. Then*

- *A hyperplane H of $L \in \mathrm{E}_D^{\mathrm{reg}}$ is characteristic for E_D at L iff it contains one of the following straight lines:*

  $$\ell_L := L \cap D \quad \textit{or} \quad \ell'_L := L \cap D^\perp.$$

  Then, if $\ell_L \neq \ell'_L$, there are two $(n-2)$-parametric families $H(t_1, \dots, t_{n-2})$ and $H'(t_1, \dots, t_{n-2})$ of characteristic hyperplanes in L: one contains $\ell_L = \bigcap_{t_1,\dots,t_{n-2}} H(t_1, \dots, t_{n-2})$ and another contains $\ell'_L = \bigcap_{t_1,\dots,t_{n-2}} H'(t_1, \dots, t_{n-2})$. If $\ell_L = \ell'_L$, these two families coincide.
- *The conformal metric of $\mathrm{E}_D^{\mathrm{reg}}$ is decomposable and is given by $(g_{\mathrm{E}_D^{\mathrm{reg}}})_L = \ell_L \vee \ell'_L$.*
- *For any line $\ell \subset D$ there exists $L \in \mathrm{E}_D^{\mathrm{reg}}$ such that $\ell = \ell_L = L \cap D$. Hence $D = \bigcup_{L \in \mathrm{E}_D} \ell_L$ and $D^\perp = \bigcup_{L \in \mathrm{E}_D} \ell'_L$.*

4. Local description of PDEs and MAEs

In this section we refer to definitions given in Section 1. From now on, for simplicity, we will assume that the contact form θ is globally defined. A diffeomorphism of M which preserves $\mathcal{C}$ is called a *contact transformation*. There exist coordinates (x^i, z, p_i) on M, $i = 1, \dots, n$, such that $\theta = dz - p_i dx^i$. Such coordinates are called *contact (or Darboux) coordinates*. Locally defined vector fields

$$\widehat{\partial}_{x^i} := \partial_{x^i} + p_i \partial_z, \ \ \partial_{p_i}, \quad i = 1, \dots, n,$$

span distribution $\mathcal{C}$. A system of contact coordinates (x^i, z, p_i) on M induces coordinates $(x^i, z, p_i, p_{ij} = p_{ji},\ 1 \leqslant i, j \leqslant n)$ on $M^{(1)}$ as follows: a point $m^1 \equiv L_{m^1} \in M^{(1)}$ has these coordinates iff $\pi(m^1) = (x^i, z, p_i)$

and the corresponding Lagrangian plane is given by $L_{m^1} = L_P := \langle \widehat{\partial}_{x^i} + p_{ij}\partial_{p_j}\rangle \subset \mathcal{C}_{\pi(m_1)}$.

A 1^{st} order PDE is locally described as zero level set $M_f := \{f(x^i, z, p_i) = 0\}$ of a function $f \in C^\infty(M)$, whereas a 2^{nd} order PDE $\mathcal{E}$ is locally described by $\mathcal{E} = \{F(x^i, z, p_i, p_{ij}) = 0\}$, with $F \in C^\infty(M^{(1)})$.

MAEs of type $\mathcal{E}_\Omega$ are, taking into account the beginning of Section 3.2, the zero locus of the following n-form on the tautological bundle $\mathcal{T}(M^{(1)})\colon m^1 \mapsto \Omega|_{L_{m^1}}$. It is straightforward to check that, locally, such MAEs are described by (0.5). For a given n-dimensional subdistribution $\mathcal{D}$ of $\mathcal{C}$, we have

$$\mathcal{E}_\mathcal{D} = \mathcal{E}_{\Omega_\mathcal{D}}, \ \text{ with } \ \Omega_\mathcal{D} := Y_1 \cdot \theta \wedge \cdots \wedge Y_n \cdot \theta,$$

(see also [16]) where Y_i are vector fields generating the orthogonal distribution $\mathcal{D}^\perp$ (w.r.t. $\omega = d\theta$). Indeed the distribution $\mathcal{D}$ is defined by the system of equations $\{\theta = 0,\, Y_i \cdot \theta = 0\}$, so that the result follows from (3.6). On the other hand, it is always possible to choose a contact chart (x^i, z, p_i) such that

$$\mathcal{D} = \langle X_1, X_2, \ldots, X_n\rangle, \quad X_i = \widehat{\partial}_{x^i} + b_{ij}\partial_{p_j},$$

for some functions b_{ij} (it is sufficient that $\mathcal{D} \cap \langle \partial_{p_1}, \ldots, \partial_{p_n}\rangle = 0$). In this case,

$$\mathcal{E}_\mathcal{D} = \left\{ L_P = \langle \widehat{\partial}_{x^i} + p_{ij}\partial_{p_j}\rangle \;\middle|\; \det \|p_{ij} - b_{ij}\| = 0 \right\}.$$

Remark 4.1. — Even if $\mathcal{D}^\perp$ and $\mathcal{D}$ define the same equation, they are not necessarily contactomorphic.

5. Characteristics of PDEs, of MAEs and proof of Theorem 1.1

We define the *prolongation* $N^{(1)} \subset M^{(1)}$ of a submanifold N of a contact manifold M as follows (see (3.1)):

$$N^{(1)} := \begin{cases} m^1 \in M^{(1)} \mid L_{m^1} \supseteq T_m N \cap \mathcal{C}_m, & \text{if } \dim(N) \leqslant n \\ m^1 \in M^{(1)} \mid L_{m^1} \subseteq T_m N \cap \mathcal{C}_m, & \text{if } \dim(N) \geqslant n. \end{cases}$$

If N is an integral submanifold of the contact manifold $(M, \mathcal{C})$, then the natural projection $\pi_N\colon N^{(1)} \to N$ is a fibre bundle whose typical fibre is $U \oplus \mathcal{L}(W) \simeq \mathcal{L}(\mathbb{R}^{2n-2k})$ where U and W are as in the identification (3.2), with $U = T_m N$ and $V = \mathcal{C}_m$.

Definitions given in Section 3.1 can be immediately reformulated in the language of PDEs by replacing V with $\mathcal{C}_m$ and E with $\mathcal{E}_m$. In particular, a direction in $T_{m^1}\mathcal{E}_m$ is called characteristic for $\mathcal{E}$ if it is generated by a rank 1 tangent vector (in $T_{m^1}\mathcal{L}(\mathcal{C}_m)$). In the same way, a subspace $U \subset T_mM$ is said to be *characteristic* for the equation $\mathcal{E}$ at m^1 if $U^{(1)}$ is tangent to $\mathcal{E}$ at m^1. If in addition $U^{(1)} \subset \mathcal{E}$, U is said to be *strongly characteristic*. Also, we can introduce a conformal metric $(g_{\mathcal{E}})_{m^1} = g_{\mathcal{E}_{\pi(m^1)}} \in S^2(L^*_{m^1})$ at each point $m^1 \equiv L_{m^1} \in \mathcal{E}$ and Lemma 3.4 is still valid mutatis mutandis. In coordinates, a tangent vector to $\mathcal{E}_m$ at m^1 having $\dot{P} = ||\dot{p}_{ij}||$ as matrix of coordinates is of rank 1 iff $\dot{p}_{ij} = \eta_i\eta_j$ up to a sign (see also (2.3)). Furthermore, it is characteristics for $\mathcal{E} = \{F = 0\}$ if it satisfies

$$\sum_{i\leqslant j} \frac{\partial F}{\partial p_{ij}}\, \dot{p}_{ij} = \sum_{i\leqslant j} \frac{\partial F}{\partial p_{ij}}\, \eta_i\eta_j = 0 \tag{5.1}$$

i.e. covector η is isotropic for $g_{\mathcal{E}}$. In view of Proposition 3.5, $(g_{\mathcal{E}})_{m^1}$ is decomposable iff characteristic hyperplanes of L_{m^1} are divided in two $(n-2)$-parametric families $\mathcal{H}_{m^1}$ and $\mathcal{H}'_{m^1}$ such that

$$\dim \bigcap_{U\in\mathcal{H}_{m^1}} U = \dim \bigcap_{U\in\mathcal{H}'_{m^1}} U = 1.$$

All results of Section 3.2 can be applied to fibers $\mathcal{E}_{\Omega m}$ just by replacing Ω with Ω_m and $\mathcal{E}_{\Omega m}$ with E_{Ω}, $m \in M$. In fact, in view of (1.1) and (3.4), we have that $\mathcal{E}_\Omega = \bigcup_{m\in M} \mathrm{E}_{\Omega_m}$. For the sake of completeness, we reformulate the results of Theorem 3.7 in the language of MAEs:

Theorem 5.1. — *Let $m^1 \in \mathcal{E}_\Omega$. Then a hyperplane of L_{m^1} is characteristic for $\mathcal{E}_\Omega$ iff it is strongly characteristic. Moreover, characteristic hyperplanes are those hyperplanes which are isotropic with respect to some n-form $\Omega^\sigma := \Omega + \sigma \wedge d\theta$, where $\sigma \in \Lambda^{n-2}(M)$.*

Furthermore, all results of Section 3.3 can be applied to fibers $\mathcal{E}_{\mathcal{D}_m}$ just by replacing $\mathcal{D}_m$ with D and $\mathcal{E}_{\mathcal{D}_m}$ with E_D, $m \in M$. In fact, in view of (1.2) and (3.5), we have that $\mathcal{E}_{\mathcal{D}} = \bigcup_{m\in M} \mathrm{E}_{\mathcal{D}_m}$. The following statement is a reformulation of Theorem 3.15.

Theorem 5.2. — *Let $m^1 \in \mathcal{E}_{\mathcal{D}m}$ be a regular point. Then $(g_{\mathcal{E}_{\mathcal{D}}})_{m^1} = \ell_{m^1} \vee \ell'_{m^1}$, where $\ell_{m^1} = L_{m^1} \cap \mathcal{D}_m$ and $\ell'_{m^1} = L_{m^1} \cap \mathcal{D}_m^\perp$ are lines. Thus there exist only two $(n-2)$-parametric families of characteristic hyperplanes of L_{m^1}: one rotates around ℓ_{m^1}, the other around ℓ'_{m^1}. Moreover, $Ch_{m^1}(\mathcal{E}_{\mathcal{D}}) = \{\pm\eta\otimes\eta,\ \eta \in \ell^0_{m^1} \cup \ell'^0_{m^1}\}$ where $\ell^0_{m^1}, \ell'^0_{m^1} \subset L^*_{m^1}$ are, respectively, the annihilators of ℓ_{m^1} and ℓ'_{m^1}. Covectors $\eta \in L^*_{m^1}$ corresponding to characteristic directions and belonging to $\ell^0_{m^1}$ (resp., $\ell'^0_{m^1}$) define hyperplanes $\{\eta = 0\}$ which contain ℓ_{m^1} (resp., ℓ'_{m^1}). If one let the point m^1 vary* *on $\mathcal{E}_{\mathcal{D}m}$, the line ℓ_{m^1} (resp., ℓ'_{m^1}) fills the n-dimensional space $\mathcal{D}_m$ (resp. $\mathcal{D}_m^\perp$).*

Conversely, let us assume that the PDE $\mathcal{E} \subset M^{(1)}$ has the following property: there exists a subdistribution $\mathcal{D}$ such that $L_{m^1} \cap \mathcal{D}_m \neq 0$ for all $m^1 \in \mathcal{E} \mapsto m \in M$. Obviously, in this situation we have that $\mathcal{E} \subseteq \mathcal{E}_{\mathcal{D}}$. As $\dim \mathcal{E} = \dim \mathcal{E}_{\mathcal{D}}$, these submanifolds, locally, coincide. But, in order to have a converse of Theorem 5.2, one must find $\mathcal{D}$ (if possible) by following the steps indicated in the statement of the above theorem; in this way, Theorem 1.1 is proved.

Example 5.3. — Consider the PDE $\mathcal{E}\colon \{p_{12} = f\}$, $f \in C^\infty(M)$. Equation of characteristics (5.1) of $\mathcal{E}$ is $\eta_1\eta_2 = 0$, so that the conformal metric of $\mathcal{E}$ at a point m^1 is equal to $(g_{\mathcal{E}})_{m^1} = \ell_{m^1} \vee \ell'_{m^1}$ where

$$\ell_{m^1} = \left\langle \widehat{\partial}_{x^1} + p_{11}\partial_{p_1} + f\partial_{p_2} + p_{13}\partial_{p_3} \right\rangle,$$
$$\ell'_{m^1} = \left\langle \widehat{\partial}_{x^2} + f\partial_{p_1} + p_{22}\partial_{p_2} + p_{23}\partial_{p_3} \right\rangle$$

If we let vary the point m^1 on the fibre $\mathcal{E}_m$, $m = \pi(m^1)$, lines ℓ_{m^1} and ℓ'_{m^1} fill, respectively, the following mutually orthogonal 3-dimensional planes at m

$$\mathcal{D}_m = \left\langle \widehat{\partial}_{x^1} + f\partial_{p_2}, \partial_{p_1}, \partial_{p_3} \right\rangle, \quad \mathcal{D}_m^\perp = \left\langle \widehat{\partial}_{x^2} + f\partial_{p_1}, \partial_{p_2}, \partial_{p_3} \right\rangle,$$

so that we obtain distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M. Thus, in view of Theorem 1.1, $\mathcal{E} = \mathcal{E}_{\mathcal{D}}$.

6. Intermediate integrals of MAEs and Monge method

We prove that the existence of an intermediate integral of a 2^{nd} order PDE is equivalent to the existence of a special vector field (Hamiltonian vector field) whose directions are strongly characteristic (Theorem 6.5). By applying this result to MAEs of type $\mathcal{E}_{\mathcal{D}}$, we see that their intermediate integrals coincide with the first integrals of the distribution $\mathcal{D}$ or $\mathcal{D}^\perp$ (Theorem 6.7), which will be useful, among other things, to prove Theorem 6.12.

6.1. Cartan and Hamiltonian vector fields

Definition 6.1. — *Sections of the contact distribution $\mathcal{C}$ are called* Cartan vector fields. *The* type *of a Cartan field Y is defined as the rank of the sequence θ, $Y \cdot \theta$, $Y \cdot (Y \cdot \theta)$,…*

Any 1-form $\alpha \in \Lambda^1(M)$ determines a Cartan vector field $Y_\alpha \in \mathcal{C}$ by the relation

$$Y_\alpha \cdot \theta = Y_\alpha \lrcorner d\theta = \alpha - \alpha(Z)\theta$$

where Z is the Reeb vector field (associated with θ) defined by conditions $\theta(Z) = 1$, $Z \lrcorner d\theta = 0$. In particular $Y = Y_{(Y\cdot\theta)}$ for any Cartan field Y. Although Y_α depends on the choice of θ, its direction does not change.

Definition 6.2. — *A vector field $Y_f := Y_{df}$ is called a* Hamiltonian vector field.

It is easy to check that Y_f is of type 2 (the minimum possible).

In addition, Y_f is a characteristic symmetry of the distribution $\{\theta = 0, df = 0\}$. In other words, Y_f coincides with the classical characteristic vector field of the 1^{st} order PDE $f(x^i, z, p_i) = 0$.

Two functions f and g on M are said to be in *involution* if $\omega(Y_f, Y_g) = 0$. This condition is equivalent to the integrability of the distribution $\langle Y_f, Y_g\rangle$. By using this fact, it can be proved the following theorem which we extracted from [18] and comes from Jacobi.

Theorem 6.3. — *Any set $(f_1, \ldots, f_k)$ of $k \leqslant n$ independent functions on the contact manifold M which are in involution can be extended to a contact chart.*

6.2. Intermediate integrals of 2^{nd} order PDEs

Recall that $M_f = \{m \in M \mid f(m) = 0\}$ denotes the zero level set of a function $f \in C^\infty(M)$.

Definition 6.4. — *Let $\mathcal{E} \subset M^{(1)}$ be a 2^{nd} order PDE. A function $f \in C^\infty(M)$ is called an* intermediate integral *of $\mathcal{E}$ if all solutions of the family $\{M_{f-c}\}_{c\in\mathbb{R}}$ of 1^{st} order PDEs, are also solutions of $\mathcal{E}$.*

Theorem 6.5. — *The following conditions are equivalent:*

(1) *A function $f \in C^\infty(M)$ is an intermediate integral of $\mathcal{E}$;*
(2) $M^{(1)}_{f-c} \subset \mathcal{E}$, $\forall\, c \in \mathbb{R}$;
(3) *Integral curves of Y_f are strongly characteristic for $\mathcal{E}$.*

Proof.

$1 \Rightarrow 2$. Assume that f is an intermediate integral. Let $m^1 \equiv L_{m^1} \in M^{(1)}_{f-c}$ for some $c \in \mathbb{R}$. The plane L_{m^1} is always tangent to some solution Σ of PDE $f = c$ which, by hypothesis, is also a solution of $\mathcal{E}$. This means that $m^1 \in \Sigma^{(1)} \subset \mathcal{E}$.

$2 \Rightarrow 1$. We have just to use that $\Sigma \subset M$ is solution of the 1^{st} order PDE $f = c$ iff $\Sigma^{(1)} \subset M^{(1)}_{f-c}$.

$2 \Leftrightarrow 3$. Recall that $Y_f = Y_{df} = Y_{f-c}$. Also, $\langle (Y_f)_m \rangle^\perp = \mathcal{C}_m \cap T_m M_{f-f(m)}$. Then $(Y_f)^{(1)}_m = (T_m M_{f-f(m)})^{(1)}$ and the equivalence follows. □

As an application of previous results we are able to characterize 2^{nd} order PDEs which have a large number of intermediate integrals. Such PDEs are described in the following corollary whose statement was known by Goursat [10]. We give a simple and clear geometric proof of it.

Corollary 6.6. — *Let $\mathcal{E}$ be a 2^{nd} order PDE. If there exist n independent functions $f_1, \dots, f_n$ such that $f = \varphi(f_1, \dots, f_n)$ is an intermediate integral for any φ, then $\mathcal{E} = \mathcal{E}_\mathcal{D}$ where $\mathcal{D} = \langle Y_{f_1}, \dots, Y_{f_n} \rangle$.*

Proof. — For each $f = \varphi(f_1, \dots, f_n)$ we have that $Y_f^{(1)} \subset \mathcal{E}$ by Theorem 6.5. Now let us define

$$\mathcal{D}_m = \big\{ (Y_f)_m \mid f = \varphi(f_1, \dots, f_n) \text{ with } \varphi \text{ arbitrary} \big\};$$

it describes an n-dimensional subdistribution of $\mathcal{C}$. In fact, if $\dim \mathcal{D} < n$, then $\{Y_{f_1}, \dots, Y_{f_n}\}$ would be dependent, that would imply that the contact form θ is dependent on $\{df_1, \dots, df_n\}$, which is not possible as θ must depend at least on the exterior differential of $(n+1)$ independent functions. By definition, $\bigcup_{f=\varphi} (Y_f)^{(1)}_m = \mathcal{E}_{\mathcal{D}_m}$. Since $\bigcup_{f=\varphi} (Y_f)^{(1)}_m \subseteq \mathcal{E}_m$, we conclude that $\mathcal{E}_{\mathcal{D}_m} \subseteq \mathcal{E}_m$. □

6.3. Intermediate integrals of MAEs of type $\mathcal{E}_\mathcal{D}$

Below we apply Theorem 6.5 to describe intermediate integrals of equations $\mathcal{E}_\mathcal{D}$ in terms of $\mathcal{D}$. In the rest of the paper we denote by $\mathcal{D}'$ the derived distribution of $\mathcal{D}$, *i.e.* the distribution spanned by vector fields of $\mathcal{D}$ and all their commutators.

Theorem 6.7. — *A function $f \in C^\infty(M)$ is an intermediate integral of $\mathcal{E}_\mathcal{D}$ iff the associated Hamiltonian field Y_f belongs to $\mathcal{D}$ or $\mathcal{D}^\perp$. Equivalently, the intermediate integrals are the first integrals of $\mathcal{D}$ or $\mathcal{D}^\perp$.*

Proof. — According to Theorem 6.5, f is an intermediate integral of $\mathcal{E}_\mathcal{D}$ iff Y_f is strongly characteristic. By arguing as in the proof of Proposition 3.11, we obtain that for equations of type $\mathcal{E}_\mathcal{D}$ this means that $Y_f \in \mathcal{D}$ or $Y_f \in \mathcal{D}^\perp$. □

Some consequences easily follow:

COROLLARY 6.8. — *If $\mathcal{D}$ (or $\mathcal{D}^\perp$) admits a first integral, or equivalently its derived flag $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}'' \subseteq \cdots \subseteq \mathcal{D}^k \subseteq \cdots$ is such that $\mathcal{D}^k \neq TM$ for any k, then $\mathcal{E}_\mathcal{D}$ admits a smooth solution.*

COROLLARY 6.9. — *The set of intermediate integrals of $\mathcal{E}_\mathcal{D}$ is the union of two subrings $\mathcal{R}_1$ and $\mathcal{R}_2$ of $C^\infty(M)$ which are in involution, in the sense that if $f_i \in \mathcal{R}_i$, $i = 1, 2$, then $\{f_1, f_2\} := \omega(Y_{f_1}, Y_{f_2}) = 0$.*

The following corollary characterizes the simplest equation of type $\mathcal{E}_\mathcal{D}$. Such characterization was known by Goursat [10]; we give a proof by using Theorem 6.7 and elementary contact geometry.

COROLLARY 6.10. — *The following conditions are equivalent:*

(1) *$\mathcal{D}$ is an n-dimensional integrable distribution of $\mathcal{C}$;*
(2) *$\mathcal{D}$ is generated by n commuting Hamiltonian vector fields;*
(3) *$\mathcal{E}_\mathcal{D}$ is contact-equivalent to the equation $\det ||p_{ij}|| = \det ||z_{x^i x^j}|| = 0$;*
(4) *$\mathcal{E}_\mathcal{D}$ is contact-equivalent to the equation $p_{11} = z_{x^1 x^1} = 0$;*
(5) *$\mathcal{E}_\mathcal{D}$ admits a ring of intermediate integrals generated by $(n+1)$ independent functions.*

Proof.

$1 \Rightarrow 2$. In fact, since $\mathcal{D}$ is integrable, we can find $n+1$ functions $\{f_i\}_{i=0\dots n}$ such that $\mathcal{D}$ is described by $\mathcal{D} = \{df_0 = df_1 = \cdots = df_n = 0\}$. Since $\mathcal{D} \subset \mathcal{C}$, then (up to a factor) $\theta = df_0 + \sum_{i=1}^n a_i df_i$ for some $a_1, ..., a_n \in C^\infty(M)$. Hence $x^i = f_i$, $z = f_0$, $p_i = -a_i$, are contact coordinates on M and $\mathcal{D}$ can be written as $\mathcal{D} = \{dx^1 = 0,\ dx^2 = 0, \dots, dx^n = 0,\ dz = 0\} = \langle \partial_{p_1}, \dots, \partial_{p_n} \rangle$.

$2 \Rightarrow 1$. It is an easy application of Theorem 6.3.

$1 \Leftrightarrow 3$. In fact, we already proved that condition 1 implies that $\mathcal{D}$ is contact-equivalent to $\langle \partial_{p_1}, \dots, \partial_{p_n} \rangle$. By using the Legendre transformation $x'^i = p_i,\ z' = z - p_j x^j,\ p'_i = -x^i,\ i = 1, \dots, n$, we realize that $\mathcal{D}$ is also contact-equivalent to $\langle \widehat{\partial}_{x^1}, \dots, \widehat{\partial}_{x^n} \rangle$, whose associated $\mathcal{E}_\mathcal{D}$ is $\det ||p_{ij}|| = 0$.

$1 \Leftrightarrow 4$. This equivalence goes as the previous one by using the partial Legendre transformation $z' = z - p_1 x^1$, $x'^1 = p_1$, $p'_1 = -x^1$, $x'^\beta = x^\beta$, $p'_\beta = p_\beta$, $\beta = 2, \dots, n$.

$1 \Rightarrow 5$. In fact, $\mathcal{D}$ is integrable iff there exist $(n+1)$ functions f_i, $i = 0, \dots, n$, such that $\mathcal{D} = \{df_0 = 0, \dots, df_n = 0\}$. This implies that $\varphi(f_0, f_1, \dots, f_n)$ is a first integral of $\mathcal{D}$ for any function φ.

$5 \Rightarrow 1$. If 5 holds, either $\mathcal{D}$ or $\mathcal{D}^\perp$ has $(n+1)$ independent first integrals. This condition is equivalent to their, simultaneous, integrability. □

6.4. Construction of solutions of $\mathcal{E}_{\mathcal{D}}$ by the generalized Monge method

The concept of intermediate integral can be naturally extended (see [2]) as follows.

Definition 6.11. — *A* nonholonomic intermediate integral *of $\mathcal{E}_{\mathcal{D}}$ is a type 2 Cartan field $X \in \mathcal{D}$.*

Next theorem describes a method for constructing solutions of $\mathcal{E}_{\mathcal{D}}$ by generalizing the Monge method of characteristics (see [9, 17]).

Theorem 6.12. — *Let X be a nonholonomic intermediate integral of $\mathcal{E}_{\mathcal{D}}$. Let $N \subset M$ be an $(n-1)$-dimensional integral submanifold of the distribution of $\mathcal{C}$ transversal to X. Then $\Sigma = \bigcup_t \varphi_t(N) \subset M$, where φ_t is the local flow of X, is solution of the equation $\mathcal{E}_{\mathcal{D}}$ iff $\omega(T_m N, X_m) = 0\ \forall\ m \in N$.*

Proof. — Let us recall that Σ is a solution of $\mathcal{E}_{\mathcal{D}}$ if it satisfies the following two conditions: a) $T_m\Sigma \cap \mathcal{D}_m \neq 0$, $\forall\, m \in \Sigma$, and b) $T_m\Sigma \subset \mathcal{C}_m$, $\forall\, m \in \Sigma$.

Condition a) is obviously satisfied. To check condition b) we choose coordinates (t, y^i) on Σ such that (y^i) are local coordinates on N and $X = \partial_t$. Any vector field $Y \in \mathcal{X}(N)$ can be considered as vector field on Σ which does not depend on t, hence commutes with X. It is sufficient to check that the function $F(t, y^i) := \theta_{(t,y^i)}(Y)$ is identically zero. Due to the fact that X is of type 2, the first two derivatives of F w.r.t. t are

$$\begin{aligned}\dot F &= (X\cdot\theta)(Y) = \omega(X, Y),\\ \ddot F &= (X\cdot(X\cdot\theta))Y = \lambda\theta(Y) + \mu(X\cdot\theta)(Y) = \lambda\, F + \mu\,\dot F,\end{aligned}$$

for some functions λ, μ. Hence, F satisfies a linear 2^{nd} order ODE with the initial conditions $F(0, y^i) = 0$, $\dot F(0, y^i) = \omega(X, Y)|_N = 0$. This shows that $F \equiv 0$. □

When $X = Y_f$, the above theorem reduces to the method of characteristics for integrating PDE $f = 0$.

7. Some applications

7.1. On the existence of smooth solutions of MAEs of type $\mathcal{E}_{\mathcal{D}}$

Let us consider a Cartan field of the form

$$X = Y_f + \lambda Y_g, \tag{7.1}$$

where $X(\lambda) = 0$ and f, g are two functions in involution (in particular $X(f) = X(g) = 0$). Then MAEs $\mathcal{E}_{\mathcal{D}}$ for which $X \in \mathcal{D}$ admit smooth solutions.

Such a Cartan field is of type 2 and we can show the existence of a Cauchy datum N such that T_pN is orthogonal (w.r.t. $\omega = d\theta$) to X_p and $X_p \notin T_pN$, for all $p \in N$. In fact, by using Theorem 6.3, we can suppose $f = p_1$ and $g = p_2$, so that $X = \widehat{\partial}_{x^1} + \lambda\widehat{\partial}_{x^2}$. The $(n-1)$-dimensional submanifold N defined by equations

$$x^1 = 0, \quad z = 0, \quad p_i = 0, \quad i = 1, \ldots, n,$$

satisfies $\theta|_N = 0$ and, if $p \in N$, $T_pN = \langle \widehat{\partial}_{x^2}|_p, \ldots, \widehat{\partial}_{x^n}|_p \rangle$. Now, by Theorem 6.12, it is possible to construct a smooth solution of $\mathcal{E}_{\mathcal{D}}$ by expanding the Cauchy datum N using the flow of X.

We would like to underline that there exist MAEs of type $\mathcal{E}_{\mathcal{D}}$ without intermediate integrals but such that $\mathcal{D}$ or $\mathcal{D}^{\perp}$ contains a vector field of type (7.1). Let us consider, for instance, $n = 3$ and the distribution $\mathcal{D} = \mathcal{D}^{\perp} = \langle X, Y, Z \rangle$, where

$$X = \widehat{\partial}_{x^1} + (p_2 + p_3)\widehat{\partial}_{x^2}, \ Y = \partial_{p_2} - (p_2 + p_3)\partial_{p_1}, \ Z = \widehat{\partial}_{x^3} + (x^1 + p_2)\partial_{p_3}.$$

It is easy to check that $\mathcal{D}' = \mathcal{C}$ and, so, $\mathcal{D}''$ equals the complete module of vector fields.

7.2. MAEs of type $\mathcal{E}_{\mathcal{D}}$ admitting n intermediate integrals

In the case in which a MAE of type $\mathcal{E}_{\mathcal{D}}$ admits n independent intermediate integrals which are first integrals of a family of characteristics (see Theorem 6.7) we can solve the Cauchy problem for $\mathcal{E}_{\mathcal{D}}$.

In fact, let N be a Cauchy datum and $f_1, \ldots, f_n$ be independent first integrals of $\mathcal{D}$ (the same reasoning holds true if $f_1, \ldots, f_n$ are first integrals of $\mathcal{D}^{\perp}$). Let us denote by g_i the restriction of f_i to N. Of course the functions g_i are dependent, so that there exists a non trivial functional relation $\psi(g_1, \ldots, g_n) = 0$. The function $f = \psi(f_1, \ldots, f_n)$ turns out to be an intermediate integral which vanishes on N, so that a solution of $\mathcal{E}_{\mathcal{D}}$ with initial condition N can be constructed.

Also, $\mathcal{E}_{\mathcal{D}}$ can be reconstructed from the set of its intermediate integrals. More precisely we have that

$$\mathcal{E}_{\mathcal{D}} = M_{\mathcal{I}} := \bigcup_{\phi} M^{(1)}_{\phi(f_1, \ldots, f_n)}$$

where ϕ is an arbitrary function of n variables. In fact, on one hand $M_{\mathcal{I}} \subset \mathcal{E}_{\mathcal{D}}$, since, if $L \in M_{\mathcal{I}}$, then $L = T_m \Sigma$ where Σ is a solution of a 1^{st} order PDE M_f for some first integral of the form $f = \varphi(f_1, \dots, f_n)$. But Σ is also a solution of $\mathcal{E}_{\mathcal{D}}$, so that $L \in \mathcal{E}_{\mathcal{D}}$. On the other hand $M_{\mathcal{I}} \supset \mathcal{E}_{\mathcal{D}}$, since, if $L = L_{m^1} \in \mathcal{E}_{\mathcal{D}}$, then $L_{m^1} \cap \mathcal{D}_{\pi(m^1)}$ contains a vector $(Y_f)_{\pi(m^1)}$ for an appropriate first integral f of $\mathcal{D}$. As a consequence, $L \in M_f^{(1)}$.

7.2.1. An example

All the examples of MAEs of type $\mathcal{E}_{\mathcal{D}}$ integrated in [10] have the following property: both the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are such that $\mathcal{D}'$ and $(\mathcal{D}^{\perp})'$ are of dimension $n+1$ and integrable. Of course, this twofold requirement is quite restrictive: for instance, for the evolutionary equation $p_3 = p_{12}$ none of the previous conditions is satisfied (to see this, it is sufficient to consider Example 5.3 and put $f = p_3$); whereas, for the equation $(p_1)^2 = p_{12}$ only one of them is true (again, consider Example 5.3 and put $f = (p_1)^2$). Our method for solving Cauchy problems covers a more general type of equations. In what follows we completely study an example of a MAE of type $\mathcal{E}_{\mathcal{D}}$ in three variables that does *not* fall into any of the types that Goursat integrated in the cited work. To start with, we apply the criterion given by Theorem 1.1 to show, by the algorithmic procedure indicated therein, that it is an equation of type $\mathcal{E}_{\mathcal{D}}$, which is not obvious a priori. Then, for better illustrating our results, we will also solve an explicit Cauchy problem.

Let us consider the equation

$$\mathcal{E}\colon\ p_{11} + (x^1 + p_2)p_{12} + x^1 p_2 p_{22} - p_3 p_{13} - x^1 p_3 p_{23} = 0,$$

where p_i, p_{ij}, denote, as usual, the partial derivatives $\partial z/\partial x^i$, $\partial^2 z/\partial x^i \partial x^j$ of the unknown function z which depends on the independent variables x^1, x^2, x^3.

Computation of characteristics. A straightforward computation shows that conformal metric $g_{\mathcal{E}}$ is of rank 2 and decomposable. We have that

$$(g_{\mathcal{E}})_{m^1} = \ell_{m^1} \vee \ell'_{m^1} = (x^1 w_2 + w_1) \vee (w_1 + p_2 w_2 - p_3 w_3)$$

where $w_1 = \widehat{\partial}_{x^1} - \big((x^1 + p_2)p_{12} + x^1 p_2 p_{22} - p_3 p_{13} - x^1 p_3 p_{23}\big)\,\partial_{p_1} + p_{12}\partial_{p_2} + p_{13}\partial_{p_3}$, $w_2 = \widehat{\partial}_{x^2} + p_{12}\partial_{p_1} + p_{22}\partial_{p_2} + p_{23}\partial_{p_3}$, and $w_3 = \widehat{\partial}_{x^3} + p_{13}\partial_{p_1} + p_{23}\partial_{p_2} + p_{33}\partial_{p_3}$.

Proof that $\mathcal{E}$ is of Goursat type: Application of criterion of Theorem 1.1. By substituting the values of w_1, w_2 and w_3 in ℓ_{m^1} we obtain

$$\ell_{m^1} = \widehat{\partial}_{x^1} + x^1\widehat{\partial}_{x^2} + (p_{13} + x^1 p_{23})(\partial_{p_3} + p_3\partial_{p_1}) + (p_{12} + x^1 p_{22})(\partial_{p_2} - p_2\partial_{p_1})$$

If we let vary the point m^1 along the fibre $\mathcal{E}_m$ of the equation $\mathcal{E}$, we obtain the distribution

$$\mathcal{D} = \langle X_1, X_2, X_3\rangle,\ X_1 = \widehat{\partial}_{x^1} + x^1\widehat{\partial}_{x^2},\ X_2 = \partial_{p_3} + p_3\partial_{p_1},\ X_3 = \partial_{p_2} - p_2\partial_{p_1}.$$

Theorem 1.1 proves that $\mathcal{E} = \mathcal{E}_{\mathcal{D}}$. On the other hand, by taking the orthogonal complement of $\mathcal{D}$ or by performing the analogous calculation for ℓ'_{m^1} we get the distribution

$$\mathcal{D}^{\perp} = \langle Y_1, Y_2, Y_3\rangle,\ Y_1 = \widehat{\partial}_{x^1} + p_2\widehat{\partial}_{x^2} - p_3\widehat{\partial}_{x^3},\ Y_2 = \partial_{p_2} - x^1\partial_{p_1},\ Y_3 = \partial_{p_3}.$$

Intermediate integrals. It is easy to check that $\mathcal{D}'$ is of rank 4 and integrable, and it is generated by $\partial_{x^1} + x^1\partial_{x^2}$, $\partial_{p_3} + p_3\partial_{p_1}$, $\partial_{p_2} - p_2\partial_{p_1}$ and ∂_z. It can be easily seen that $(\mathcal{D}^{\perp})''$ is of rank 5 and then the equation is not of the type studied in [10].

In view of Theorem 6.7, a standard computation gives the following 3 independent intermediate integrals of $\mathcal{E}_{\mathcal{D}}$:

$$\lambda_1 := (x^1)^2 - 2x^2,\ \ \lambda_2 := (p_3)^2 - (p_2)^2 - 2p_1,\ \ \lambda_3 := x^3.$$

A Cauchy problem. In view of Section 7.2, any Cauchy datum can be extended to a solution as we found 3 first integrals of $\mathcal{D}$. For this case, a Cauchy datum N consists of a 2-dimensional integral submanifold of $\mathcal{C}$. If we suppose that this datum can be parameterized by x^1 and x^2, then we can arbitrarily fix z, x^3 and p_3 as functions of x^1 and x^2 and then determine p_1 and p_2 by the contact condition. Let us choose, for instance,

$$N\colon z = (x^1)^2 + x^2,\ x^3 = x^1,\ p_3 = 0.$$

Then, from the contact condition, one easily obtains that $p_1 = 2x^1$ and $p_2 = 1$, which completes the parametrization of N.

The restrictions of λ_1, λ_2 and λ_3 to N are $\overline{\lambda}_1 = (x^1)^2 - 2x^2$, $\overline{\lambda}_2 = -1 - 4x^1$, $\lambda_3 = x^1$. A first integral vanishing on N is $f := \lambda_2 + 4\lambda_3 + 1$, whose associated Hamiltonian field is

$$Y_f = 2p_3 Y_{p_3} - 2p_2 Y_{p_2} - 2Y_{p_1} + 4Y_{x^3} = 2\left(p_3\widehat{\partial}_{x^3} - p_2\widehat{\partial}_{x^2} - \widehat{\partial}_{x^1} - 2\partial_{p_3}\right)$$

which has the following 6 independent first integrals

$$\mu_1 := p_1,\ \ \mu_2 := p_2,\ \ \mu_3 := \frac{1}{2}(p_3)^2 + 2x^3,\ \ \mu_4 := p_2x^1 - x^2,$$
$$\mu_5 := x^1 - \frac{1}{2}p_3,\ \ \mu_6 := \frac{1}{2}((p_2)^2 + p_1)p_3 - \frac{1}{6}(p_3)^3 - z.$$

In order to prolong the Cauchy datum N along the orbits of Y_f, we restrict the above 6 first integrals on N (the bar denotes such a restriction): $\overline{\mu}_1 = 2x^1$, $\overline{\mu}_2 = 1$, $\overline{\mu}_3 = 2x^1$, $\overline{\mu}_4 = x^1 - x^2$, $\overline{\mu}_5 = x^1$, $\overline{\mu}_6 = -((x^1)^2 + x^2)$. By eliminating parameters x^1 and x^2 we obtain 4 independent relations

$$\mu_2 = 1,\quad \mu_3 - \mu_1 = 0,\quad \mu_5 - \frac{1}{2}\mu_1 = 0,\quad \mu_6 + \frac{1}{4}(\mu_1)^2 + \frac{1}{2}\mu_1 - \mu_4 = 0 \tag{7.2}$$

for which the prolongation must hold. If we substitute the μ's in (7.2) we get

$$\begin{cases} p_2 = 1 \\ \frac{1}{2}(p_3)^2 + 2x^3 - p_1 = 0 \\ x^1 - \frac{1}{2}p_3 - \frac{1}{2}p_1 = 0 \\ \frac{1}{2}((p_2)^2 + p_1)p_3 - \frac{1}{6}(p_3)^3 - z + \frac{1}{4}(p_1)^2 + \frac{1}{2}p_1 - p_2x^1 + x^2 = 0. \end{cases} \tag{7.3}$$

Finally, from the first three equations of system (7.3) we can obtain the p's in terms of the x's and then, the fourth equation allows to express z as a function of x^1, x^2, x^3 which is the required solution:

$$z = \frac{1}{6} + x^1 + (x^1)^2 + x^2 - x^3 \mp \sqrt{1 - 4x^3 + 4x^1}\left(\frac{1}{6} + \frac{2}{3}x^1 - \frac{2}{3}x^3\right).$$

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