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JET SCHEMES OF COMPLEX PLANE BRANCHES AND EQUISINGULARITY

by Hussein MOURTADA (*)

Abstract. — For $m \in \mathbb{N}$, we determine the irreducible components of the $m-$th Jet Scheme of a complex branch C and we give formulas for their number $N(m)$ and for their codimensions, in terms of m and the generators of the semigroup of C. This structure of the Jet Schemes determines and is determined by the topological type of C.

Résumé. — Pour $m \in \mathbb{N}$, nous déterminons les composantes irréductibles des $m-$èmes espaces des jets d'une branche plane complexe C et nous donnons des formules pour leur nombre $N(m)$ et leurs codimensions, en fonction de m et des générateurs du semigroupe de C. Cette structure des espaces des jets détermine et elle est déterminée par le type topologique de C.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field. The space of arcs X_∞ of an algebraic $\mathbb{K}-$variety X is a non-noetherian scheme in general. It has been introduced by Nash in [10]. Nash has initiated its study by looking at its image by the truncation maps $X_\infty \longrightarrow X_m$ in the jet schemes of X. The $m^{th}-$jet scheme X_m of X is a $\mathbb{K}-$ scheme of finite type which parmametizes morphisms $Spec\, \mathbb{K}[t]/(t)^{m+1} \longrightarrow X$. From now on, we assume $char\, \mathbb{K} = 0$. In [10], Nash has derived from the existence of a resolution of singularities of X, that the number of irreducible components of the Zariski closure of the set of the $m-$truncations of arcs on X that send 0 into the singular locus of X is constant for m large enough. Besides a theorem of Kolchin asserts that if X is irreducible, then X_∞ is also irreducible. More recently,

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the jet schemes have attracted attention from various viewpoints. In [9], Mustata has characterized the locally complete intersection varieties having irreducible X_m for $m \geqslant 0$. In [2], a formula comparing the codimensions of Y_m in X_m with the log canonical threshold of a pair (X,Y) is given. In this work, we consider a curve C in the complex plane $\mathbb{C}^2$ with a singularity at 0 at which it is analytically irreducible (i.e. the formal neighborhood$(C,0)$ of C at 0 is a branch). We determine the irreducible components of the space $C_m^0 := \pi_m^{-1}(0)$ where $\pi_m : C_m \longrightarrow C$ is the canonical projection, and we show that their number is not bounded as m grows. More precisely, let x be a transversal parameter in the local ring $O_{\mathbb{C}^2,0}$, i.e. the line $x=0$ is transversal to C at 0 and following [2], for $e \in \mathbb{N}$, let

$$Cont^e(x)_m(resp.Cont^{>e}(x)_m) := \{\gamma \in C_m \mid ord_t x \circ \gamma = e(resp. > e)\},$$

where $Cont$ stands for contact locus. Let $\Gamma(C) =< \overline{\beta}_0, \cdots, \overline{\beta}_g >$ be the semigroup of the branch $(C,0)$ and let $e_i = gcd(\overline{\beta}_0, \cdots, \overline{\beta}_i)$, $0 \leqslant i \leqslant g$. Recall that $\Gamma(C)$ and the topological type of C near 0 are equivalent data and characterize the equisingularity class of $(C,0)$ as defined by Zariski in [13]. We show in theorem 4.9 that the irreducible components of C_m^0 are

$$C_{m\kappa I} = \overline{Cont^{\kappa\bar{\beta}_0}(x)_m},$$

for $1 \leqslant \kappa$ and $\kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leqslant m$,

$$C^j_{m\kappa v} = \overline{Cont^{\frac{\kappa\bar{\beta}_0}{e_{j-1}}}(x)_m}$$

for $2 \leqslant j \leqslant g, 1 \leqslant \kappa, \kappa \not\equiv 0 \ mod\ \frac{e_{j-1}}{e_j}$ and $\kappa\frac{\bar{\beta}_0\bar{\beta}_1}{e_{j-1}} + e_1 \leqslant m < \kappa\bar{\beta}_j$,

$$B_m = Cont^{>\frac{\bar{\beta}_0}{e_1}q}(x)_m,$$

if $q\frac{\bar{\beta}_0}{e_1}\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$.
These irreducible components give rise to infinite and finite inverse systems represented by a tree. We recover $< \overline{\beta}_0, \cdots, \overline{\beta}_g >$ from the tree and the multiplicity $\overline{\beta}_0$ in corollary 4.13, and we give formulas for the number of irreducible components of C_m^0 and their codimensions in terms of m and $(\overline{\beta}_0, \cdots, \overline{\beta}_g)$ in proposition 4.7 and corollary 4.10. We recover the fact coming from [2] and [6] that

$$min_m \frac{codim(C_m^0, \mathbb{C}_m^2)}{m+1} = \frac{1}{\overline{\beta}_0} + \frac{1}{\overline{\beta}_1}.$$

The structure of the paper is as follows: The basics about Jet schemes and the results that we will need are presented in section 2. In section 3 we present the definitions and the reults we will need about branches. The last section is devoted to the proof of the main result and corollaries.

2. Jet schemes

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let X be a $\mathbb{K}-$scheme of finite type over k and let $m \in \mathbb{N}$. The functor F_m : $\mathbb{K} - Schemes \longrightarrow Sets$ which to an affine scheme defined by a $\mathbb{K}-$algebra A associates

$$F_m(Spec(A)) = Hom_{\mathbb{K}}(SpecA[t]/(t^{m+1}), X)$$

is representable by a $\mathbb{K}-$scheme X_m [12]. X_m is the $m-$th jet scheme of X, and F_m is isomorphic to its functor of points. In particular the closed points of X_m are in bijection with the $\mathbb{K}[t]/(t^{m+1})$ points of X.
For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$.
Note that $X_0 = X$. We denote the canonical projection $\pi_{m,0} : X_m \longrightarrow X_0$ by π_m.

Example 2.1. — Let $X = Spec \frac{\mathbb{K}[x_0,\cdots,x_n]}{(f_1,\cdots,f_r)}$ be an affine $\mathbb{K}-$scheme. For a $\mathbb{K}$-algebra A, to give a A-point of X_m is equivalent to give a $\mathbb{K}-$algebra homomorphism

$$\varphi : \frac{\mathbb{K}[x_0, \cdots, x_n]}{(f_1, \cdots, f_r)} \longrightarrow A[t]/(t^{m+1}).$$

The map φ is completely determined by the image of $x_i, i = 0, \cdots, n$

$$x_i \longmapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m$$

such that $f_l(\phi(x_0), \cdots, \phi(x_n)) \in (t^{m+1})$, $l = 1, \cdots, r$.
If we write

$$f_l(\phi(x_0), \cdots, \phi(x_n)) = \sum_{j=0}^{m} F_l^{(j)}(\underline{x}^{(0)}, \cdots, \underline{x}^{(j)})\ t^j \ mod\ (t^{m+1})$$

where $\underline{x}^{(j)} = (x_0^{(j)}, \cdots, x_n^{(j)})$, then

$$X_m = Spec \frac{\mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]}{(F_l^{(j)})_{l=1,\cdots,r}^{j=0,\cdots,m}}$$

Example 2.2. — From the above example, we see that the m-th jet scheme of the affine space $\mathbb{A}^n_{\mathbb{K}}$ is isomorphic to $\mathbb{A}_k^{(m+1)n}$ and that the projection $\pi_{m,m-1} : (\mathbb{A}^n_{\mathbb{K}})_m \longrightarrow (\mathbb{A}^n_{\mathbb{K}})_{m-1}$ is the map that forgets the last n coordinates.

Let $char(\mathbb{K}) = 0$, $S = \mathbb{K}[x_0, \cdots, x_n]$ and $S_m = \mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]$. Let D be the $\mathbb{K}-$derivation on S_m defined by $D(x_i^{(j)}) = x_i^{(j+1)}$ if $0 \leqslant j < m$, and $D(x_i^{(m)}) = 0$. For $f \in S$ let $f^{(1)} := D(f)$ and we recursively define $f^{(m)} = D(f^{(m-1)})$.

Proposition 2.3. — *Let* $X = Spec(S/(f_1, \cdots, f_r)) = Spec(R)$ *and* $R_m = \Gamma(X_m)$. *Then*

$$R_m = Spec(\frac{\mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]}{(f_i^{(j)})_{i=1,\cdots,r}^{j=0,\cdots,m}}.$$

Proof. — For a $\mathbb{K}-$algebra A, to give an $A-$point of X_m is equivalent to give an homomorphism

$$\phi : \mathbb{K}[x_0, \cdots, x_n] \longrightarrow A[t]/(t^{m+1})$$

which can be given by

$$x_i \longrightarrow \frac{x_i^{(0)}}{0!} + \frac{x_i^{(1)}}{1!}t + \cdots + \frac{x_i^{(m)}}{m!}t^m.$$

Then for a polynomial $f \in S$, we have

$$\phi(f) = \sum_{j=0}^{m} \frac{f^{(j)}(\underline{x}^{(0)}, \cdots, \underline{x}^{(j)})}{j!}\ t^j.$$

To see this, it is sufficient to remark that it is true for $f = x_i$, and that both sides of the equality are additive and multiplicative in f, and the proposition follows. □

Remark 2.4. — Note that the proposition shows the linearity of the equations $F_i^j(\underline{x}^{(0)}, \cdots, \underline{x}^{(j)})$ defining X_m with respect to the new variables i.e $\underline{x}^{(j)}$. We can deduce from this that if X is a nonsingular $\mathbb{K}-$variety of dimension n, then the projections $\pi_{m,m-1} : X_m \longrightarrow X_{m-1}$ are locally trivial fibrations with fiber $\mathbb{A}^n_{\mathbb{K}}$. In particular, X_m is a non singular variety of dimension $(m+1)n$.

3. Semigroup of complex branches

The main references for this section are [14],[8],[1],[11],[5],[4],[7]. Let $f \in \mathbb{C}[[x, y]]$ be an irreducible power series, which is y-regular (i.e $f(0, y) =$

$y^{\beta_0}u(y)$ where u is invertible in $\mathbb{C}[[y]]$) and such that $mult_0 f = \beta_o$ and let C be the analytically irreducible plane curve(branch for short) defined by f in $Spec\ \mathbb{C}[[x,y]]$. By the Newton-Puiseux theorem, the roots of f are

$$y = \sum_{i=0}^{\infty} a_i w^i x^{\frac{i}{\beta_o}} \tag{1}$$

where w runs over the $\beta_0 - th$-roots of unity in $\mathbb{C}$.This is equivalent to the existence of a parametrization of C of the form

$$x(t) = t^{\beta_0}$$
$$y(t) = \sum_{i \geqslant \beta_0} a_i t^i.$$

We recursively define $\beta_i = min\{i, a_i \neq 0,\ gcd(\beta_0, \cdots, \beta_{i-1})$ is not a divisor of $i\}$.
Let $e_0 = \beta_0$ and $e_i = gcd(e_{i-1}, \beta_i), i \geqslant 1$. Since the sequence of positive integers

$$e_0 > e_1 > \cdots > e_i > \cdots$$

is strictly decreasing, there exists $g \in \mathbb{N}$, sucht that $e_g = 1$. The sequence $(\beta_1, \cdots ., \beta_g)$ is the sequence of Puiseux exponents of C. We set

$$n_i := \frac{e_{i-1}}{e_i}, m_i := \frac{\beta_i}{e_i}, i = 1, \cdots, g$$

and by convention, we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$.
On the other hand, for $h \in \mathbb{C}[[x,y]]$, we define the intersection number

$$(f,h)_0 = (C, C_h)_0 := dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f,h)} = ord_t\ h(x(t), y(t))$$

where C_h is the Cartier divisor defined by h and $\{x(t)), y(t)\}$ is as above. The mapping $v_f : \frac{\mathbb{C}[[x,y]]}{(f)} \longrightarrow \mathbb{N}$, $h \longmapsto (f,h)_0$ defines a divisorial valuation. We define the semigroup of C to be the semigroup of v_f i.e $\Gamma(C) = \Gamma(v_f) = \{(f,h)_0 \in \mathbb{N}, h \not\equiv 0\ mod(f)\}$.
The following propositions and theorem from [14] characterize the structure of $\Gamma(C)$.

Proposition 3.1. — *There exists a unique sequence of $g+1$ positive integers $(\bar{\beta}_0, \cdots, \bar{\beta}_g)$ such that:*
i) $\bar{\beta}_0 = \beta_0$,
ii) $\bar{\beta}_i = min\{\Gamma(C) \setminus < \bar{\beta}_0, \cdots, \bar{\beta}_{i-1} >\}, 1 \leqslant i \leqslant g$,
iii) $\Gamma(C) =< \bar{\beta}_0, \cdots, \bar{\beta}_g >$,
where for $i = 1, \cdots, g+1, < \bar{\beta}_0, \cdots, \bar{\beta}_{i-1} >$ is the semigroup generated by $\bar{\beta}_0, \cdots, \bar{\beta}_{i-1}$. By convention, we set $\bar{\beta}_{g+1} = +\infty$.

Proposition 3.2. — *The sequence* $(\bar{\beta}_0, \cdots, \bar{\beta}_g)$ *verifies:*
i) $e_i = gcd(\bar{\beta}_0, \cdots, \bar{\beta}_i), 0 \leqslant i \leqslant g,$
ii) $\bar{\beta}_0 = \beta_0, \bar{\beta}_1 = \beta_1$ *and* $\bar{\beta}_i = n_{i-1}\overline{\beta_{i-1}} + \beta_i - \beta_{i-1}$. *In particular* $n_i\bar{\beta}_i < \overline{\beta}_{i+1}$, *for* $i = 2, \cdots, g$.

Theorem 3.3. — *The sequence* $(\bar{\beta}_0, \cdots, \bar{\beta}_g)$ *and the sequence* $(\beta_0, \cdots ., \beta_g)$ *are equivalent data. They determine and are determined by the topological type of* C.

Then from the appendix of [14], [1] or [11], we can choose a system of approximate roots (or a minimal generating sequence) $\{x_0, \cdots, x_{g+1}\}$ of the divisorial valuation v_f. We set $x = x_0, y = x_1$; for $i = 2, \cdots, g+1, x_i \in \mathbb{C}[[x, y]]$ is irreducible; for $1 \leqslant i \leqslant g$, the analytically irreducible curve $C_i = \{x_i = 0\}$ has $i-1$ Puiseux exponents and $C_{g+1} = C$. This sequence also verifies
i) $v_f(x_i) = \bar{\beta}_i$, $0 \leqslant i \leqslant g$,
ii) $\Gamma(C_i) = < \frac{\bar{\beta}_0}{e_{i-1}}, \cdots, \frac{\bar{\beta}_{i-1}}{e_{i-1}} >$ and the Puiseux sequence of C_i is $(\frac{\beta_1}{e_{i-1}}, \cdots, \frac{\beta_{i-1}}{e_{i-1}})$,$2 \leqslant i \leqslant g+1$.
iii) for $1 \leqslant i \leqslant g$, there exists a unique system of nonnegative integers b_{ij}, $0 \leqslant j < i$ such that for $1 \leqslant j < i$, $b_{ij} < n_j$ and $n_i\bar{\beta}_i = \Sigma_{0 \leqslant j < i} b_{ij}\bar{\beta}_j$. Furthermore, for $1 \leqslant i \leqslant g$, one can choose x_i such that they satisfy identities of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0,\cdots,\gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i}, (\star)$$

with, $0 \leqslant \gamma_j < n_j$, for $1 \leqslant j \leqslant i$, and $\Sigma_j \gamma_j \bar{\beta}_j > n_i\bar{\beta}_i$ and with $c_{i,\gamma}, c_i \in \mathbb{C}$ and $c_i \neq 0$. These last equations $(\star)$ let us realize C as a complete intersection in $\mathbb{C}^{g+1} = Spec\ \mathbb{C}\ [[x_0, \cdots, x_g]]$ defined by the equations

$$f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0,\cdots,\gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i})$$

for $1 \leqslant i \leqslant g$, with $x_{g+1} = 0$ by convention.

Let $h \in \mathbb{C}[[x, y]]$ be a y-regular irreducible power series with multiplicity $p = ord_y h(0, y)$. Let $y(x^{\frac{1}{\beta_0}})$ and $z(x^{\frac{1}{p}})$ be respectively roots of f and h as in (1). We call contact order of f and h in their Puiseux series the following rational number

$$o_f(h) := max\{ord_x(y(wx^{\frac{1}{\beta_0}}) - z(\lambda x^{\frac{1}{p}})); w^{\beta_0} = 1, \lambda^p = 1\} =$$
$$max\{ord_x(y(wx^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{p}}); w^{\beta_0} = 1\} =$$
$$max\{ord_x(y(x^{\frac{1}{\beta_0}}) - z(\lambda x^{\frac{1}{p}}); \lambda^p = 1\} = o_h(f).$$

The following formula is from [8], see also [5] .

PROPOSITION 3.4. — *Assume that f and h are as above; let $(\beta_1,\cdots,\beta_g)$ the sequence of Puiseux exponents of f and let $i\leqslant g+1$ be the smallest strictly positive integer such that $o_f(h)\leqslant\frac{\beta_i}{\beta_0}$. Then*

$$\begin{aligned}\frac{(f,h)_0}{p}&=\sum_{k=1}^{i-1}\frac{e_{k-1}-e_k}{\beta_0}\beta_k+e_{i-1}o_f(h)\\&=(\bar\beta_{i-1}e_{i-2}+(\beta_0o_f(h)-\beta_{i-1})e_{i-1})\frac{1}{\beta_0}.\end{aligned}$$

COROLLARY 3.5. — [1][5] *Let $i>0$ be an integer.Then $o_f(h)\leqslant\frac{\beta_i}{\beta_0}$ iff $\frac{(f,h)_0}{p}\leqslant e_{i-1}\frac{\bar\beta_i}{\beta_0}$. Moreover $o_f(h)=\frac{\beta_i}{\beta_0}$ iff $\frac{(f,h)_0}{p}=e_{i-1}\frac{\bar\beta_i}{\beta_0}$. In particular $o_f(x_i)=\frac{\beta_i}{\beta_0},1\leqslant i\leqslant g$. We say that $C_ix_i=0$ has maximal contact with C.*

4. Jet schemes of complex branches

We keep the notations of sections 2 and 3. We consider a curve $C\subset\mathbb{C}^2$ with a branch of multiplicity $\beta_0>1$ at 0, defined by f. Note that in suitable coordinates we can write

$$f(x_0,x_1)=(x_1^{n_1}-cx_0^{m_1})^{e_1}+\sum_{a\beta_0+b\beta_1>\beta_0\beta_1}c_{ab}x_0^ax_1^b;c\in\mathbb{C}^\star\ \textit{and}\ c_{ab}\in\mathbb{C}.\quad(\diamond)$$

We look for the irreducible components of $C_m^0:=(\pi_m^{-1}(0))$ for every $m\in\mathbb{N}$, where $\pi_m:C_m\to C$ is the canonical projection. Let J_m^0 be the radical of the ideal defining $(\pi_m^{-1}(0))$ in $\mathbb{C}_m^2$.
In the sequel, we will denote the integral part of a rational number r by $[r]$.

PROPOSITION 4.1. — *For $0<m<n_1\bar\beta_1$, we have that*

$$(C_m^0)_{red}=(\pi_m^{-1}(0))_{red}=Spec\ \frac{\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},x_1^{(0)},\cdots,x_1^{(m)}]}{(x_0^{(0)},\cdots,x_0^{([\frac{m}{\bar\beta_1}])},x_1^{(0)},\cdots,x_1^{([\frac{m}{\beta_0}])})},$$

and

$$\begin{aligned}(C^0_{n_1\bar\beta_1})_{red}&=(\pi^{-1}_{n_1\bar\beta_1}(0))_{red}\\&=Spec\frac{\mathbb{C}[x_0^{(0)},\cdots,x_0^{(n_1\bar\beta_1)},x_1^{(0)},\cdots,x_1^{(n_1\bar\beta_1)}]}{(x_0^{(0)},\cdots,x_0^{(n_1-1)},x_1^{(0)},\cdots,x_1^{(m_1-1)},x_1^{(m_1)^{n_1}}-cx_0^{(n_1)^{m_1}})}.\end{aligned}$$

Proof. — We write $f=\Sigma_{(a,b)}c_{ab}f_{ab}$ where $(a,b)\in\mathbb{N}^2,\ f_{ab}=x_0^ax_1^b,\ c_{ab}\in\mathbb{C}$ and $a\beta_0+b\bar\beta_1\geqslant\beta_0\bar\beta_1$(the segment $[(0,\beta_0)(\bar\beta_1,0)]$ is the Newton Polygon of f). Let $supp(f)=\{(a,b)\in\mathbb{N}^2;c_{ab}\neq0\}$.

For $0 < m < n_1\bar{\beta}_1$, the proof is by induction on m. For $m = 1$,we have that

$$F^{(1)} = \Sigma_{(a,b)\in supp(f)} c_{ab} F^{(1)}_{ab}$$

where $(F^{(0)}, \cdots, F^{(i)})$ (resp.$(F^{(0)}_{ab}, \cdots, F^{(i)}_{ab})$) is the ideal defining the i-th jet scheme C_i of C(resp. C^{ab}_i the i-th jet scheme of $C^{ab} = \{f_{ab} = 0\}$) in $\mathbb{C}^2_i$ .Then we have

$$F^{(1)}_{ab} = \sum_{\sum i_k=1} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+1})} \cdots x_1^{(i_{a+b})}$$

where $\bar{\beta}_1(a+b) \geqslant a\beta_0 + b\bar{\beta}_1 \geqslant \beta_0\bar{\beta}_1$ so $a+b \geqslant \beta_0 > 1$. Then for every $(a,b) \in supp(f)$ and every $(i_1, \cdots, i_a, \cdots, i_{a+b}) \in \mathbb{N}^{a+b}$ such that $\sum_{k=1}^{a+b} i_k = 1$ there exists $1 \leqslant k \leqslant a+b$ such that $i_k \neq 0$, this means that $F^{(1)}_{ab} \in (x_0^{(0)}, x_1^{(0)})$ and since we are looking over the origin, we have that $(x_0^{(0)}, x_1^{(0)}) \subseteq J^0_1$ therefore $(\pi_1^{-1}(0))_{red} = Spec \frac{\mathbb{C}[x_0^{(0)}, x_0^{(1)}, x_1^{(0)}, x_1^{(1)}]}{(x_0^{(0)}, x_1^{(0)})}$ (In fact this is nothing but the Zariski tangent space of C at 0).
Suppose that the lemma holds until $m-1$ i.e.

$$(\pi^{-1}_{m-1}(0))_{red} = Spec \frac{\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m-1)}, x_1^{(0)}, \cdots, x_1^{(m-1)}]}{(x_0^{(0)}, \cdots, x_0^{([\frac{m-1}{\bar{\beta}_1}])}, x_1^{(0)}, \cdots, x_1^{([\frac{m-1}{\beta_0}])})}.$$

First case: If $[\frac{m-1}{\bar{\beta}_1}] = [\frac{m}{\bar{\beta}_1}]$ and $[\frac{m-1}{\beta_0}] = [\frac{m}{\beta_0}]$. We have

$$F^{(m)} = \sum_{(a,b)\in supp(f)} c_{ab} \sum_{\sum i_k = m} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+1})} \cdots x_1^{(i_{a+b})}$$

Let $(a,b) \in supp(f)$; if for every $k = 1, \cdots, a$, we had $i_k \geqslant [\frac{m}{\bar{\beta}_1}] + 1$, and for every $k = a+1, \cdots, a+b$, we had $i_k \geqslant [\frac{m}{\beta_0}] + 1$, then

$$m \geqslant a([\frac{m}{\bar{\beta}_1}] + 1) + b([\frac{m}{\beta_0}] + 1) > \frac{m}{\bar{\beta}_1}a + \frac{m}{\beta_0}b = m\frac{a\beta_0 + b\bar{\beta}_1}{\beta_0\bar{\beta}_1} \geqslant m.$$

The contradiction means that there exists $1 \leqslant k \leqslant a$ such that $i_k \leqslant [\frac{m}{\bar{\beta}_1}]$ or there exists $a+1 \leqslant k \leqslant a+b$ such that $i_k \leqslant [\frac{m}{\beta_0}]$. So $F^{(m)}$ lies in the ideal generated by J^0_{m-1} in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]$ and $J^0_m = J^0_{m-1}.\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]$.
Second case: If $[\frac{m-1}{\bar{\beta}_1}] = [\frac{m}{\bar{\beta}_1}]$ and $[\frac{m-1}{\beta_0}] + 1 = [\frac{m}{\beta_0}]$ (*i.e.* β_0 divides m). We have that

$$F^{(m)} = F^{(m)}_{0\beta_0} + \sum_{(a,b)\in supp(f);(a,b)\neq(0,\beta_0)} F^{(m)}_{ab}, \qquad (\star\star)$$

where

$$F^{(m)}_{0\beta_0} = \sum_{\sum i_k = m} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})}$$
$$= x_1^{(\frac{m}{\beta_0})^{\beta_0}} + \sum_{\sum i_k=m;(i_1,\cdots,i_{\beta_0})\neq(\frac{m}{\beta_0},\cdots,\frac{m}{\beta_0})} x_1^{(i_1)}\cdots x_1^{(i_{\beta_0})};$$

but $\sum i_k = m$ and $(i_1,\cdots,i_{\beta_0}) \neq (\frac{m}{\beta_0},\cdots,\frac{m}{\beta_0})$ implies that there exists $1 \leqslant k \leqslant \beta_0$ such that $i_k < \frac{m}{\beta_0}$, so

$$\sum_{\sum i_k=m;(i_1,\cdots,i_{\beta_0})\neq(\frac{m}{\beta_0},\cdots,\frac{m}{\beta_0})} x_1^{(i_1)}\cdots x_1^{(i_{\beta_0})} \in J^0_{m-1}.\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},x_1^{(0)},\cdots,x_1^{(m)}].$$

For the same reason as above, we have that

$$\sum_{(a,b)\in supp(f);(a,b)\neq(0,\beta_0)} F^{(m)}_{ab} \in J^0_{m-1}.\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},x_1^{(0)},\cdots,x_1^{(m)}].$$

From $(\star\star)$ we deduce that $x_1^{(\frac{m}{\beta_0})} \in J^0_m$ and

$$F^{(m)} \in (x_0^{(0)},\cdots,x_0^{([\frac{m}{\bar\beta_1}])},x_1^{(0)},\cdots,x_1^{(\frac{m}{\beta_0})}).$$

Then $J^0_m = (x_0^{(0)},\cdots,x_0^{([\frac{m}{\bar\beta_1}])},x_1^{(0)},\cdots,x_1^{(\frac{m}{\beta_0})})$.
The third case i.e. if $[\frac{m-1}{\bar\beta_1}]+1 = [\frac{m}{\bar\beta_1}]$ and $[\frac{m-1}{\beta_0}] = [\frac{m}{\beta_0}]$ is discussed as the second one. Note that these are the only three possible cases since $m < n_1\bar\beta_1 = lcm(\beta_0,\bar\beta_1)$(here lcm stands for the least common multiple).
For $m = n_1\bar\beta_1$, we have that $F^{(m)}$ is the coefficient of t^m in the expansion of

$$f(x_0^{(0)} + x_0^{(1)}t + \cdots + x_0^{(m)}t^m, x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m).$$

But since we are interested in the radical of the ideal defining the m-th jet scheme, and we have found that $x_0^{(0)},\cdots,x_0^{(n_1-1)},x_1^{(0)},\cdots,x_1^{(m_1-1)} \in J^0_{m-1} \subseteq J^0_m$, we can annihilate $x_0^{(0)},\cdots,x_0^{(n_1-1)},x_1^{(0)},\cdots,x_1^{(m_1-1)}$ in the above expansion. Using $(\diamond)$, we see that the coefficient of t^m is $({x_1^{(m_1)}}^{n_1} - c{x_0^{(n_1)}}^{m_1})^{e_1}$. □

In the sequel if A is a ring, $I \subseteq A$ an ideal and $f \in A$, we denote by $V(I)$ the subvariety of *Spec A* defined by I and by $D(f)$ the open set in *SpecA*, $D(f) := SpecA_f$.
The proof of the following corollary is analogous to that of proposition 4.1.

Corollary 4.2. — *Let $m \in \mathbb{N}$; let $k \geqslant 1$ be such that $m = kn_1\bar{\beta}_1 + i; 1 \leqslant i \leqslant n_1\bar{\beta}_1$. Then if $i < n_1\bar{\beta}_1$, we have that*

$$Cont^{>kn_1}(x_0)_m = (\pi^{-1}_{m,kn_1\bar{\beta}_1}(V(x_0^{(0)},\cdots,x_0^{(kn_1)})))_{red} =$$

$$Spec\ \frac{\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},x_1^{(0)},\cdots,x_1^{(m)}]}{(x_0^{(0)},\cdots,x_0^{(kn_1)},\cdots,x_0^{(kn_1+[\frac{i}{\bar{\beta}_1}])},x_1^{(0)},\cdots,x_1^{(km_1)},\cdots,x_1^{(km_1+[\frac{i}{\bar{\beta}_0}])})}$$

and if $i = n_1\bar{\beta}_1$

$$(\pi^{-1}_{m,kn_1\bar{\beta}_1}(V(x_0^{(0)},\cdots,x_0^{(kn_1)})))_{red} =$$

$$Spec\ \frac{\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},x_1^{(0)},\cdots,x_1^{(m)}]}{(x_0^{(0)},\cdots,x_0^{((k+1)n_1-1)},x_1^{(0)},\cdots,x_1^{((k+1)m_1-1)},{x_1^{((k+1)m_1)}}^{n_1}-c{x_0^{((k+1)n_1)}}^{m_1})}.$$

We now consider the case of a plane branch with one Puiseux exponent.

Lemma 4.3. — *Let C be a plane branch with one Puiseux exponent. Let $m,k \in \mathbb{N}$, such that $k \neq 0$ and $m \geqslant kn_1\bar{\beta}_1+1$, and let $\pi_{m,kn_1\bar{\beta}_1} : C_m \to C_{kn_1\bar{\beta}_1}$ be the canonical projection. Then*

$$C_m^k := \pi^{-1}_{m,kn_1\bar{\beta}_1}(V(x_0^{(0)},\cdots,x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red}$$

is irreducible of codimension $k(m_1+n_1)+1+(m-kn_1\bar{\beta}_1)$ in $\mathbb{C}^2_m$.

Proof. — First note that since $e_1 = 1$, we have $m_1 = \frac{\bar{\beta}_1}{e_1} = \bar{\beta}_1$. Let I_m^{0k} be the ideal defining C_m^k in $\mathbb{C}^2_m \cap D(x_0^{(kn_1)})$. Since $m \geqslant kn_1\bar{\beta}_1$, by corollary 4.2, $x_1^{(0)},\cdots,x_1^{(km_1-1)} \in I_m^{0k}$.So I_m^{0k} is the radical of the ideal $I_m^{*0k} := (x_0^{(0)},\cdots,x_0^{(kn_1-1)},x_1^{(0)},\cdots,x_1^{(km_1-1)},F^{(0)},\cdots,F^{(m)})$. Now it follows from $\diamond$ and proposition 2.3 that

$$F^{(l)} \in (x_0^{(0)},\cdots,x_0^{(kn_1-1)},x_1^{(0)},\cdots,x_1^{(km_1-1)})\ for\ 0 \leqslant l < kn_1m_1,$$

$$F^{(kn_1m_1)} \equiv {x_1^{(km_1)}}^{n_1} - c{x_0^{(kn_1)}}^{m_1}\ mod\ (x_0^{(0)},\cdots,x_0^{(kn_1-1)},x_1^{(0)},\cdots,x_1^{(km_1-1)}),$$

$$F^{(kn_1m_1+l)} \equiv n_1{x_1^{(km_1)}}^{n_1-1}x_1^{(km_1+l)} - m_1c{x_0^{(kn_1)}}^{m_1-1}x_0^{(kn_1+l)}$$
$$+H_l(x_0^{(0)},\cdots,x_0^{(kn_1+l-1)},x_1^{(0)},\cdots,x_1^{(km_1+l-1)})\ mod\ (x_0^{(0)},\cdots,x_0^{(kn_1-1)},x_1^{(0)},\cdots,x_1^{(km_1-1)}),$$

for $1 \leqslant l \leqslant m - kn_1m_1$.

This implies that

$$I_m^{*0k} = (x_0^{(0)},\cdots,x_0^{(kn_1-1)},x_1^{(0)},\cdots,x_1^{(km_1-1)},F^{(kn_1m_1)},\cdots,F^{(m)}).$$

Moreover the subscheme of $\mathbb{C}^2_m \cap D(x_0^{(kn_1)})$ defined by I_m^{*0k} is isomorphic to the product of $\mathbb{C}^*$($\mathbb{C}^*$ is isomorphic to the regular locus of ${x_1^{(km_1)}}^{n_1} - c{x_0^{(kn_1)}}^{m_1}$) by an affine space and its codimension is $k(m_1+n_1)+1+(m-kn_1m_1)$; so it is reduced and irreducible, and it is nothing but C_m^k, or equivalently $I_m^{0k} = I_m^{*0k}$. □

Corollary 4.4. — *Let C be a plane branch with one Puiseux exponent. Let $m \in \mathbb{N}, m \neq 0$. let $q \in \mathbb{N}$ be such that $m = qn_1\bar{\beta}_1 + i; 0 < i \leqslant n_1\bar{\beta}_1$. Then $C_m^0 = \pi_m^{-1}(0)$ has $q+1$ irreducible components which are:*

$$C_{mkI} = \overline{C_m^k}, 1 \leqslant k \leqslant q,$$

$$\textit{and } B_m = Cont^{>qn_1}(x)_m = \pi^{-1}_{m,qn_1\bar{\beta}_1}(V(x_0^{(0)}, \cdots, x_0^{(qn_1)})).$$

We have that

$$codim(C_{mkI}, \mathbb{C}^2_m) = k(m_1+n_1)+1+(m-kn_1m_1)$$

and

$$codim(B_m, \mathbb{C}^2_m) = q(m_1+n_1)+[\frac{i}{\beta_0}]+[\frac{i}{\bar{\beta}_1}]+2 = [\frac{m}{\beta_0}]+[\frac{m}{\bar{\beta}_1}]+2 \; if \; i < n_1\bar{\beta}_1$$

$$codim(B_m, \mathbb{C}^2_m) = (q+1)(m_1+n_1)+1 \; if \; i = n_1\bar{\beta}_1.$$

Proof. — The codimensions and the irreducibility of B_m and C_{mkI} follow from corollary 4.2 and lemma 4.3. This shows that if $1 \leqslant k < k' \leqslant q$, we have $codim(C_{mk'I}, \mathbb{C}^2_m) < codim(C_{mkI}, \mathbb{C}^2_m)$, then $C_{mk'I} \nsubseteq C_{mkI}$. On the other hand, since $C_{mk'I} \subseteq V(x_0^{(kn_1)})$ and $C_{mkI} \nsubseteq V(x_0^{(kn_1)})$, we have that $C_{mkI} \nsubseteq C_{mk'I}$. This also shows that $dim\ B_m \geqslant dim\ C_{mkI}$ for $1 \leqslant k \leqslant q$, therefore $B_m \nsubseteq C_{mkI}, 1 \leqslant k \leqslant q$.But $C_{mkI} \nsubseteq B_m$ because $B_m \subseteq V(x_0^{(qn_1)})$ and $C_{mkI} \nsubseteq V(x_0^{(qn_1)})$ for $1 \leqslant k \leqslant q$. We thus have that $C_{mkI} \nsubseteq B^m$ and $B^m \nsubseteq C_{mkI}$. We conclude the corollary from the fact that by construction $C_m^0 = \cup_{k=1}^q C_{mkI} \cup B_m$. □

To understand the general case, i.e. to find the irreducible components of C_m^0 where C has a branch with g Puiseux exponents at 0, since for $kn_1\bar{\beta}_1 < m \leqslant (k+1)n_1\bar{\beta}_1, m, k \in \mathbb{N}$ we know by corollary 4.2 the structure of the m-jets that project to $V(x_0^{(0)}, \cdots, x_0^{(kn_1)}) \cap C^0_{kn_1\bar{\beta}_1}$, we have to understand for $m > kn_1\bar{\beta}_1$ the m-jets that projects to $V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)})$, i.e. $C_m^k := \pi^{-1}_{m,kn_1\bar{\beta}_1}(V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red}$.
Let $m, k \in \mathbb{N}$ be such that $m \geqslant kn_1\bar{\beta}_1$. Let $j = max\{l, n_2 \cdots n_{l-1}$ divides $k\}$(we set $j = 2$ if the greatest common divisor $(k, n_2) = 1$ or if $g = 1$). Set κ such that $k = \kappa n_2 \cdots n_{j-1}$, then we have $kn_1 = \kappa \frac{\beta_0}{n_j \cdots n_g}$.

Proposition 4.5. — *Let $2 \leqslant j \leqslant g+1$; for $i = 2,..,g$, and $kn_1\bar{\beta}_1 < m < \kappa e_{i-1}\frac{\bar{\beta}_i}{e_{j-1}}$, we have that*

$$C_m^k = \bar{\pi}^{-1}_{m,[\frac{m}{n_i\cdots n_g}]}(C^k_{i,[\frac{m}{n_i\cdots n_g}]}),$$

where $\bar{\pi}_{m,[\frac{m}{n_i\cdots n_g}]} : \mathbb{C}^2_m \longrightarrow \mathbb{C}^2_{[\frac{m}{n_i\cdots n_g}]}$ is the canonical map. For $j < g+1$ and $m \geqslant \kappa\bar{\beta}_j$,we have that

$$C_m^k = \emptyset$$

Proof. — Let $\phi \in C_m^k$. Let $\tilde{\phi} : Spec\ \mathbb{C}[[t]] \longrightarrow (\mathbb{C}^2, 0)$ be such that $\phi = \tilde{\phi}$ mod t^{m+1}. Let $\tilde{f} \in \mathbb{C}[[x,y]]$ be a function that defines the branch $\tilde{C}$ image of $\tilde{\phi}$. we may assume that the map $Spec\mathbb{C}[[t]] \longrightarrow \tilde{C}$ induced by $\tilde{\phi}$ is the normalization of $\tilde{C}$. Since $ord_t x_0 \circ \tilde{\phi} = kn_1, ord_t x_1 \circ \tilde{\phi} = km_1$ the multiplicity $m(\tilde{f})$ of $\tilde{C}$ at the origin is $ord_{x_1}\tilde{f}(0,x_1) = kn_1 = \kappa\frac{\beta_0}{n_j\cdots n_g}$.

Claim: If $(f,\tilde{f})_0 < \kappa e_{i-1}\frac{\bar{\beta}_i}{e_{j-1}}$ then $(f,\tilde{f})_0 = n_i\cdots n_g(x_i,\tilde{f})_0$.

Indeed, we have that $\frac{(f,\tilde{f})_0}{ord_y\tilde{f}(0,y)} < e_{i-1}\frac{\bar{\beta}_i}{\beta_0}$, therefore by corollary 3.5 we have that

$$o_f(\tilde{f}) < \frac{\beta_i}{\beta_0} = o_f(x_i).$$

We will prove that $o_f(\tilde{f}) = o_{x_i}(\tilde{f})$. (It was pointed by the referee that this follows from [1]. For the convenience of the reader we give a detailed proof below.)

Let $y(x^{\frac{1}{\beta_0}})$, $z(x^{\frac{1}{n_1\cdots n_{i-1}}})$ and $u(x^{\frac{1}{m(\tilde{f})}})$ be respectively Puiseux-roots of f,x_i and $\tilde{f}$. There exist w, $\lambda \in \mathbb{C}$ such that $w^{\frac{\beta_0}{n_i\cdots n_g}} = 1, \lambda^{m(\tilde{f})} = 1$ and

$$o_f(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}}))$$

and

$$o_f(x_i) = ord_x(y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1\cdots n_{i-1}}})).$$

Since $o_f(\tilde{f}) < o_f(x_i)$, we have that

$$\begin{aligned} o_f(\tilde{f}) &= ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}}) + y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1\cdots n_{i-1}}})) \\ &= ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(wx^{\frac{1}{n_1\cdots n_{i-1}}})) \leqslant o_{x_i}(\tilde{f}). \end{aligned}$$

On the other hand, there exist λ and $\delta \in \mathbb{C}$, such that $\lambda^{m(\tilde{f})} = 1, \delta^{\beta_0} = 1$ and such that

$$o_{x_i}(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(x^{\frac{1}{n_1\cdots n_{i-1}}}))$$

and

$$o_f(x_i) = ord_x(y(\delta x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{n_1\cdots n_{i-1}}})).$$

We have then that

$$o_{x_i}(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}}) + y(\delta x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1\cdots n_{i-1}}})).$$

Now

$$ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}})) \leqslant o_f(\tilde{f}) < o_f(x_i) = ord_x(y(\delta x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1\cdots n_{i-1}}})).$$

So

$$o_{x_i}(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}})) \leqslant o_f(\tilde{f}).$$

We conclude that $o_f(\tilde{f}) = o_{x_i}(\tilde{f})$, and since the sequence of Puiseux exponents of C_i is $(\frac{\beta_0}{n_i\cdots n_g}, \cdots, \frac{\beta_{i-1}}{n_i\cdots n_g})$, applying proposition 3.4 to C and C_i, we find that $(f,\tilde{f})_0 = n_i \cdots n_g(x_i,\tilde{f})_0$ and claim follows.
On the other hand by the corollary 3.5 applied to f and $\tilde{f}$,$(f,\tilde{f})_0 \geqslant \kappa e_{i-1}\frac{\bar{\beta}_i}{e_{j-1}}$ if and only if $o_f(\tilde{f}) \geqslant \frac{\beta_i}{\beta_0} = o_{x_i}(f) = o_f(x_i)$ so $o_f(\tilde{f}) \geqslant \frac{\beta_i}{\beta_0}$ if and only if $o_{x_i}(\tilde{f}) \geqslant \frac{\beta_i}{\beta_0}$, therefore $(x_i,\tilde{f})_0 \geqslant \kappa\frac{\bar{\beta}_i}{e_{j-1}}$. This proves the first assertion.
The second assertion is a direct consequence of lemma 5.1 in [5]. □

To further analyse the C_m^k's, we realize, as in section 3, C as a complete intersection in $\mathbb{C}^{g+1} = Spec\ \mathbb{C}[x_0,\cdots,x_g]$ defined by the ideal $(f_1,\cdots,f_g)$ where

$$f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0,\cdots,\gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i})$$

for $1 \leqslant i \leqslant g$ and $x_{g+1} = 0$. This will let us see the C_m^k's as fibrations over some reduced scheme that we understand well.
We keep the notations above and let I_m^0 be the radical of the ideal defining C_m^0 in $\mathbb{C}_m^{g+1}$ and let I_m^{0k} be the ideal defining

$$C_m^k = (V(I_m^0, x_0^{(0)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red} \quad \text{in} \quad D(x_0^{(kn_1)}).$$

Lemma 4.6. — *Let $k \neq 0$, j and κ as above. For $1 \leqslant i < j \leqslant g$ (resp.$1 \leqslant i < j-1 = g$) and for $\kappa n_i \cdots n_{j-1}\bar{\beta}_i \leqslant m < \kappa n_{i+1}\cdots n_{j-1}\overline{\beta}_{i+1}$, we have*

$$I_m^{0k} = (x_0^{(0)}, \cdots, x_0^{(\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}-1)},$$

$$x_l^{(0)}, \cdots, x_l^{(\frac{\kappa\bar{\beta}_l}{n_j\cdots n_g}-1)}, F_l^{(\kappa\frac{n_l\bar{\beta}_l}{n_j\cdots n_g})}, \cdots, F_l^{(m)}, 1 \leqslant l \leqslant i,$$

$$x_{i+1}^{(0)}, \cdots, x_{i+1}^{([\frac{m}{n_{i+1}\cdots n_g}])},$$

$$F_l^{(0)}, \cdots, F_l^{(m)}, i+1 \leqslant l \leqslant g-1).$$

Moreover for $1 \leqslant l \leqslant i$,

$$F_l^{(\kappa \frac{n_l \bar{\beta}_l}{n_j \cdots n_g})} \equiv -(x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \cdots n_g})^{n_l}} - c_l x_0^{(\kappa \frac{\bar{\beta}_0}{n_j \cdots n_g})^{b_{l0}}} \cdots . x_{l-1}^{(\kappa \frac{\overline{\beta}_{l-1}}{n_j \cdots n_g})^{b_{l(l-1)}}})$$

$$mod\ ((x_l^{(0)}, \cdots, x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \cdots n_g} - 1)})_{0 \leqslant l \leqslant i}, x_{i+1}^{(0)}, \cdots, x_{i+1}^{([\frac{m}{n_{i+1} \cdots n_g}])}),$$

for $1 \leqslant l < i$ *and* $\kappa \frac{n_l \bar{\beta}_l}{n_j \cdots n_g} < n < \kappa \frac{\overline{\beta}_{l+1}}{n_j \cdots n_g}$ *(resp.* $l = i$ *and* $\kappa \frac{n_i \bar{\beta}_i}{n_j \cdots n_g} < n \leqslant [\frac{m}{n_{i+1} \cdots n_g}]$*)*

$$F_l^{(n)} \equiv -(n_l x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \cdots n_g})^{n_l - 1}} x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \cdots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \cdots n_g})} -$$

$$c_l \sum_{0 \leqslant h \leqslant l-1} b_{lh} x_0^{(\kappa \frac{\bar{\beta}_0}{n_j \cdots n_g})^{b_{l0}}} \cdots x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \cdots n_g})^{b_{lh} - 1}} x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \cdots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \cdots n_g})} \cdots x_{l-1}^{(\kappa \frac{\overline{\beta_{l-1}}}{n_j \cdots n_g})^{b_{l(l-1)}}} +$$

$$H_l(\cdots, x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \cdots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \cdots n_g} - 1)}, \cdots))$$

$$mod\ ((x_l^{(0)}, \cdots, x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \cdots n_g} - 1)})_{0 \leqslant l \leqslant i}, x_{i+1}^{(0)}, \cdots, x_{i+1}^{([\frac{m}{n_{i+1} \cdots n_g}])}),$$

for $1 \leqslant l < i$ *and* $\kappa \frac{\overline{\beta_{l+1}}}{n_j \cdots n_g} \leqslant n \leqslant m$*(resp.* $l = i$ *and* $[\frac{m}{n_{i+1} \cdots n_g}] < n \leqslant m$*), or* $i+1 \leqslant l \leqslant g-1$ *and* $0 \leqslant n \leqslant m$,

$$F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \cdots, x_0^{(n)}, \cdots, x_l^{(0)}, \cdots, x_l^{(n)}).$$

For $i = j - 1 = g$ *and* $m \geqslant \kappa n_g \bar{\beta}_g$,

$$I_m^{0k} = (x_0^{(0)}, \cdots, x_0^{(\kappa \bar{\beta}_0 - 1)},$$

$$x_l^{(0)}, \cdots, x_l^{(\kappa \bar{\beta}_l - 1)}, F_l^{(\kappa n_l \bar{\beta}_l)}, \cdots, F_l^{(m)}), 1 \leqslant l \leqslant g,$$

where for $1 \leqslant l < g$ *and* $\kappa n_l \bar{\beta}_l \leqslant n \leqslant m$*, the above formula for* $F_l^{(n)}$ *remains valid,*

$$F_g^{(\kappa n_g \bar{\beta}_g)} \equiv -(x_g^{(\kappa \bar{\beta}_g)^{n_g}} - c_g x_0^{(\kappa \bar{\beta}_0)^{b_{g0}}} \cdots . x_{g-1}^{(\kappa \overline{\beta_{g-1}})^{b_{g(g-1)}}})$$

$$mod\ ((x_l^{(0)}, \cdots, x_l^{(\kappa \bar{\beta}_l - 1)}))_{0 \leqslant l \leqslant g}$$

and for $\kappa n_g \bar{\beta}_g < n \leqslant m$,

$$F_g^{(n)} \equiv -(n_g x_g^{(\kappa \bar{\beta}_g)^{n_g - 1}} x_g^{(\kappa \bar{\beta}_g + n - \kappa n_g \bar{\beta}_g)} -$$

$$c_g \sum_{0 \leqslant h \leqslant g-1} b_{g0} x_0^{(\kappa \bar{\beta}_0)^{b_{gh}}} \cdots x_h^{(\kappa \bar{\beta}_h)^{b_{gh} - 1}} x_h^{(\kappa \bar{\beta}_h + n - \kappa n_h \bar{\beta}_h)} \cdots x_{g-1}^{(\kappa \overline{\beta_{g-1}})^{b_{g(g-1)}}} +$$

$$H_g(\cdots, x_h^{(\kappa \bar{\beta}_h + n - \kappa n_h \bar{\beta}_h)}, \cdots))$$

$$mod\ ((x_l^{(0)}, \cdots, x_l^{(\kappa \bar{\beta}_l - 1)}))_{0 \leqslant l \leqslant g}$$

Proof. — First assume that $\kappa n_i \cdots n_{j-1}\bar{\beta}_i \leqslant m < \kappa n_{i+1} \cdots n_{j-1}\bar{\beta}_{i+1}$ for $1 \leqslant i < j \leqslant g$ (resp. $1 \leqslant i < j-1 = g$). By proposition 4.5, we have that $C^k_m = \bar{\pi}^{-1}_{m,[\frac{m}{n_{i+1}\cdots n_g}]}(C^k_{i+1,[\frac{m}{n_{i+1}\cdots n_g}]})$ where $\bar{\pi}_{m,[\frac{m}{n_{i+1}\cdots n_g}]} : \mathbb{C}^2_m \longrightarrow \mathbb{C}^2_{[\frac{m}{n_{i+1}\cdots n_g}]}$ is the canonical map. Now $\mathbb{C}^2 = Spec\, \mathbb{C}[x_0, x_1](resp.\ C_{i+1} = V(x_{i+1}))$ is realized as the complete intersection in $\mathbb{C}^{g+1} = Spec\, \mathbb{C}[x_0, \cdots, x_g]$ defined by the ideal $(f_1, \cdots, f_{g-1})$(resp. $(f_1, \cdots, f_{g-1}, x_{i+1})$). So since $m \geqslant kn_1\bar{\beta}_1$, I^{0k}_m is the radical of the ideal $I^{*0k}_m =$

$$(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}, x_1^{(0)}, \cdots, x_1^{(km_1-1)}, F_1^{(0)}, \cdots, F_1^{(m)},$$

$$\cdots, F_{g-1}^{(0)}, \cdots, F_{g-1}^{(m)}, x_{i+1}^{(0)}, \cdots, x_{i+1}^{([\frac{m}{n_{i+1}\cdots n_g}])}).$$

We first observe that $F_1^{(n)} \equiv x_2^{(n)}\, mod\, (x_0^{(0)}, \cdots, x_0^{(kn_1-1)}, x_1^{(0)}, \cdots, x_1^{(km_1-1)})$ for $0 \leqslant n < kn_1\bar{\beta}_1$. Now since $\frac{m}{n_2\cdots n_g} \geqslant [\frac{m}{n_2\cdots n_g}] \geqslant kn_1m_1$, we have

$$F_1^{(kn_1m_1)} \equiv -(x_1^{(km_1)^{n_1}} - c_1 x_0^{(kn_1)^{m_1}})$$

$$mod\ (x_0^{(0)}, \cdots, x_0^{(kn_1-1)}, x_1^{(0)}, \cdots, x_1^{(km_1-1)}, x_2^{(0)}, \cdots, x_2^{([\frac{m}{n_2\cdots n_g}])})$$

and

$$F_1^{(n)} \equiv -(n_1 x_1^{(km_1)^{n_1-1}} x_1^{(km_1+n-kn_1m_1)} - m_1c_1x_0^{(kn_1)^{m_1-1}} x_0^{(kn_1+n-kn_1m_1)})$$

$$+H_1(x_0^{(0)}, \cdots, x_0^{(kn_1+n-kn_1m_1-1)}, x_1^{(0)}, \cdots, x_1^{(km_1+n-kn_1m_1-1)})$$

$$mod\ (x_0^{(0)}, \cdots, x_0^{(kn_1-1)}, x_1^{(0)}, \cdots, x_1^{(km_1-1)}, x_2^{(0)}, \cdots, x_2^{([\frac{m}{n_2\cdots n_g}])})$$

for $kn_1\bar{\beta}_1 < n \leqslant [\frac{m}{n_2\cdots n_g}]$. Finally, for $l = 1$ and $[\frac{m}{n_2\cdots n_g}] < n \leqslant m$, or $2 \leqslant l \leqslant g-1$ and $0 \leqslant n \leqslant m$, we have

$$F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \cdots, x_0^{(n)}, \cdots, x_l^{(0)}, \cdots, x_l^{(n)}).$$

As a consequence for $i = 1$, the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(kn_1)})$ defined by I^{*0k}_m is isomorphic to the product of $\mathbb{C}^*$ by an affine space, so it is reduced and irreducible and $I^{*0k}_m = I^{0k}_m$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(kn_1)}}$, generated by a regular sequence, i.e the proposition holds for $i = 1$.

Assume that it holds for $i < j-1 < g$(resp. $i < j-2 = g-1$). For $\kappa n_{i+1} \cdots n_{j-1}\overline{\beta}_{i+1} \leqslant m < \kappa n_{i+2} \cdots n_{j-1}\overline{\beta}_{i+2}$, the ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(kn_1)}}$ generated by $I^{0k}_{\kappa n_{i+1}\cdots n_{j-1}\overline{\beta_{i+1}}-1}$ is contained in I^{0k}_m. By the inductive hypothesis, $x_l^{(0)}, \cdots, x_l^{(\frac{\kappa\bar{\beta}_l}{n_j\cdots n_g}-1)} \in$

$I^{0k}_{\kappa n_{i+1}\cdots n_{j-1}\overline{\beta}_{i+1}-1}$ for $l=1,\cdots,i+1$. So I^{0k}_m is the radical of

$$I^{*0k}_m=(x_0^{(0)},\cdots,x_0^{(\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}-1)},$$
$$x_l^{(0)},\cdots,x_l^{(\frac{\kappa\bar{\beta}_l}{n_j\cdots n_g}-1)},F_l^{(0)},\cdots,F_l^{(m)},1\leqslant l\leqslant i+1,$$
$$x_{i+2}^{(0)},\cdots,x_{i+2}^{([\frac{m}{n_{i+2}\cdots n_g}])},$$
$$F_l^{(0)},\cdots,F_l^{(m)},i+2\leqslant l\leqslant g-1).$$

Now for $0\leqslant n<\frac{\kappa n_l\bar{\beta}_l}{n_j\cdots n_g}$,we have

$$F_l^{(n)}\equiv x_{l+1}^{(n)}\ mod\ (x_0^{(0)},\cdots,x_l^{(\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}-1)},x_l^{(0)},\cdots,x_l^{(\frac{\kappa\bar{\beta}_l}{n_j\cdots n_g}-1)},$$
$$1\leqslant l\leqslant i+1).$$

Here since $\overline{\beta}_{l+1}>n_l\bar{\beta}_l$, for $1\leqslant l\leqslant i$ and $\frac{m}{n_{i+2}\cdots n_g}\geqslant[\frac{m}{n_{i+2}\cdots n_g}]\geqslant\frac{\kappa n_{i+1}\overline{\beta}_{i+1}}{n_j\cdots n_g}$, we can delete $F_l^{(n)}$, $1\leqslant l\leqslant i+1, 0\leqslant n<\frac{\kappa n_l\bar{\beta}_l}{n_j\cdots n_g}$ from the above generators of I^{*0k}_m. The identities relative to the $F_l^{(n)}$ for $1\leqslant l\leqslant i+1$, $\frac{\kappa n_l\bar{\beta}_l}{n_j\cdots n_g}\leqslant n\leqslant m$ or $i+2\leqslant l\leqslant g-1$ and $0\leqslant n\leqslant m$ follow immediately from $(\diamond)$. Hence the subscheme of $\mathbb{C}^{g+1}\cap D(x_0^{(kn_1)})$ defined by I^{*0k}_m is isomorphic to the product of $\mathbb{C}^*$ by an affine space, so it is reduced and irreducible and $I^{*0k}_m=I^{0k}_m$ is a prime ideal in $\mathbb{C}[x_0^{(0)},\cdots,x_0^{(m)},\cdots,x_g^{(0)},\cdots,x_g^{(m)}]_{x_0^{(kn_1)}}$, generated by a regular sequence, i.e the proposition holds for $i+1$.
The case $i=j-1=g$ and $m\geqslant\kappa n_g\overline{\beta_g}$ follows by similar arguments. □

As an immediate consequence we get

Proposition 4.7. — *Let C be a plane branch with g Puiseux exponents. Let $k\neq 0, j$ and κ as above. For $m\geqslant kn_1\bar{\beta}_1$, let $\pi_{m,kn_1\bar{\beta}_1}:C_m\to C_{kn_1\bar{\beta}_1}$ be the canonical projection and let $C^k_m:=\pi^{-1}_{m,kn_1\bar{\beta}_1}(D(x_0^{(kn_1)})\cap V(x_0^{(0)},\cdots,x_0^{(kn_1-1)}))_{red}$. Then for $1\leqslant i<j\leqslant g$ (resp.$1\leqslant i<j-1=g$) and $\kappa n_i\cdots n_{j-1}\bar{\beta}_i\leqslant m<\kappa n_{i+1}\cdots n_{j-1}\overline{\beta}_{i+1}$, C^k_m is irreducible of codimension*

$$\frac{\kappa}{n_j\cdots n_g}(\bar{\beta}_0+\bar{\beta}_1+\sum_{l=1}^{i-1}(\overline{\beta}_{l+1}-n_l\overline{\beta_l}))+([\frac{m}{n_{i+1}\cdots n_g}]-\frac{\kappa n_i\bar{\beta}_i}{n_j\cdots n_g})+1$$

in $\mathbb{C}^2_m$. (We suppose that the sum in the formula is equal to 0 when $i=1$.) For $j\leqslant g$ and $m\geqslant\kappa\bar{\beta}_j$ (resp.$j=g+1$ and $m\geqslant\kappa n_g\bar{\beta}_g$),

$$C^k_m=\emptyset$$

(resp. C_m^k is of codimension

$$\kappa(\bar{\beta}_0+\bar{\beta}_1+\sum_{l=1}^{g-1}(\overline{\beta}_{l+1}-n_l\overline{\beta_l}))+m-\kappa n_g\bar{\beta}_g+1)$$

in $\mathbb{C}_m^2$.

The referee kindly pointed out that for $m\in\mathbb{N}$ such that $\kappa n_i\cdots n_{j-1}\bar{\beta}_i\leqslant m<\kappa n_{i+1}\cdots n_{j-1}\overline{\beta}_{i+1}$, the codimension of C_m^k can also be written as :

$$\frac{\kappa}{e_{j-1}}(\bar{\beta}_0+\beta_{i+1}-\overline{\beta}_{i+1})+[\frac{m}{e_i}]+1.$$

For $k'\geqslant k$ and $m\geqslant k'n_1\bar{\beta}_1$, we now compare codim($C_m^k$,$\mathbb{C}_m^2$) and codim($C_m^{k'}$,$\mathbb{C}_m^2$).

Corollary 4.8. — *For $k'\geqslant k\geqslant 1$ and $m\geqslant k'n_1\bar{\beta}_1$, if C_m^k and $C_m^{k'}$ are nonempty, we have*

$$codim(C_m^{k'},\mathbb{C}_m^2)\leqslant codim(C_m^k,\mathbb{C}_m^2).$$

Proof. — Let $\gamma^k:[kn_1\bar{\beta}_1,\infty[\longrightarrow[k(n_1+m_1),\infty[$ be the piecewise linear function given by

$$\gamma^k(m)=\frac{k}{e_1}(\bar{\beta}_0+\bar{\beta}_1+\sum_{l=1}^{i-1}(\overline{\beta}_{l+1}-n_l\overline{\beta_l}))+(\frac{m}{e_i}-\frac{kn_i\bar{\beta}_i}{e_1})+1$$

for $1\leqslant i\leqslant g$ and $\frac{k\bar{\beta}_i}{n_2\cdots n_{i-1}}\leqslant m<\frac{k\overline{\beta}_{i+1}}{n_2\cdots n_i}$. (Recall that by convention $\overline{\beta}_{g+1}=\infty$)

In view of proposition 4.7, we have that codim(C_m^k,$\mathbb{C}_m^2$) $=[\gamma^k(m)]$ for $k\equiv 0 \mod n_2\cdots n_{j-1}$ and $k\not\equiv 0 \mod n_2\cdots n_j$ with $2\leqslant j\leqslant g$ and any integer $m\in[kn_1\bar{\beta}_1,\frac{k\overline{\beta}_j}{n_2\cdots n_{j-1}}[$ or for $k\equiv 0 \mod n_2\cdots n_g$and any integer $m\geqslant kn_1\bar{\beta}_1$. Similarly we define $\gamma^{k'}:[k'n_1\bar{\beta}_1,\infty[\longrightarrow[k'(n_1+m_1),\infty[$ by changing k to k'.

Let $\Gamma^k(resp.\Gamma^{k'})$ be the graph of γ^k(resp $\gamma^{k'}$) in $\mathbb{R}^2$.Now let $\tau:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be defined by $\tau(a,b)=(a,b-1)$ and let $\lambda^{k'/k}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be defined by $\lambda^{k'/k}(a,b)=\frac{k'}{k}(a,b)$. We note that $\tau(\Gamma^{k'})=\lambda^{k'/k}(\tau(\Gamma^k))$; we also note that the endpoints of $\tau(\Gamma^k)$ and $\tau(\Gamma^{k'})$ lie on the line through 0 with slope $\frac{\beta_0+\bar{\beta}_1}{e_1n_1\bar{\beta}_1}=\frac{1}{e_1}\frac{n_1+m_1}{n_1m_1}<\frac{1}{e_1}$. Since $\frac{k'}{k}\geqslant 1$, the image of $\tau(\Gamma^k)$ by $\lambda^{k'/k}$ lies in the interior subset of $\mathbb{R}^2_{\geqslant 0}$ whith boundary the union of $\tau(\Gamma^k)$, of the segment joining its endpoint $(kn_1\bar{\beta}_1,\frac{k}{e_1}(\beta_0+\bar{\beta}_1))$ to $(kn_1\bar{\beta}_1,0)$ and of $[kn_1\bar{\beta}_1,\infty[\times 0$. This implies that $\gamma^{k'}(m)\leqslant\gamma^k(m)$ for $m\geqslant k'n_1\bar{\beta}_1$, hence $[\gamma^{k'}(m)]\leqslant[\gamma^k(m)]$ and the claim. □

THEOREM 4.9. — *Let C be a plane branch with $g \geqslant 2$ Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leqslant m < n_1\bar{\beta}_1+e_1$,$C_m^0 = Cont^{>0}(x_0)_m$ is irreducible. For $qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$,with $q \geqslant 1$ in $\mathbb{N}$, the irreducible components of C_m^0 are :*

$$C_{m\kappa I} = \overline{Cont^{\kappa\bar{\beta}_0}(x_0)_m}$$

for $1 \leqslant \kappa$ and $\kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leqslant m$,

$$C^j_{m\kappa v} = \overline{Cont^{\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}}(x_0)_m}$$

for $j = 2, \cdots, g$, $1 \leqslant \kappa$ and $\kappa \not\equiv 0$ mod n_j and such that $\kappa n_1 \cdots n_{j-1}\bar{\beta}_1 + e_1 \leqslant m < \kappa\bar{\beta}_j$,

$$B_m = Cont^{>n_1 q}(x_0)_m.$$

Proof. — We first observe that for any integer $k \neq 0$ and any $m \geqslant kn_1\bar{\beta}_1$,

$$(C_m^0)_{red} = \cup_{1\leqslant h\leqslant k} C_m^h \cup Cont^{>kn_1}(x_0)_m$$

where $C_m^h := Cont^{hn_1}(x_0)_m$. Indeed, for $k = 1$, we have that $(C_m^0)_{red} \subset V(x_0^{(0)}, \cdots, x_0^{(n_1-1)})$ by proposition 4.1. Arguing by induction on k, we may assume that the claim holds for $m \geqslant (k-1)n_1\bar{\beta}_1$.Now by corollary 4.2, we know that for $m \geqslant kn_1\bar{\beta}_1$, $Cont^{>(k-1)n_1}(x_0)_m \subset V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)})$, hence the claim for $m \geqslant kn_1\bar{\beta}_1$.
We thus get that for $qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$,

$$(C_m^0)_{red} = \cup_{1\leqslant k\leqslant q} C_m^k \cup Cont^{>qn_1}(x_0)_m.$$

By proposition 4.7, for $1 \leqslant k \leqslant q, C_m^k$ is either irreducible or empty. We first note that if $C_m^k \neq \emptyset$, then $\overline{C_m^k} \not\subset Cont^{>qn_1}(x_0)_m$. Similarly, if $1 \leqslant k < k' \leqslant q$ and if C_m^k and $C_m^{k'}$ are nonempty, then $\overline{C_m^k} \not\subset \overline{C_m^{k'}}$. On the other hand by corollary 4.8, we have that $codim(C_m^{k'}, \mathbb{C}_m^2) \leqslant codim(C_m^k, \mathbb{C}_m^2)$. So $\overline{C_m^{k'}} \not\subset \overline{C_m^k}$. Finally we will show that $Cont^{>qn_1}(x_0)_m \not\subset \overline{C_m^k}$ if $C_m^k \neq \emptyset$ for $1 \leqslant k \leqslant q$. To do so, it is enough to check that $codim(C_m^k, \mathbb{C}_m^2) \geqslant codim(Cont^{>qn_1}(x_0)_m, \mathbb{C}_m^2)$. For $m \in [qn_1\bar{\beta}_1 + e_1, (q+1)n_1\bar{\beta}_1[$, we have

$$\begin{aligned} \delta^q(m) &:= codim(Cont^{>qn_1}(x_0)_m, \mathbb{C}_m^2) \\ &= 2 + q(n_1+m_1) + [\frac{m - qn_1\bar{\beta}_1}{\beta_0}] + [\frac{m - qn_1\bar{\beta}_1}{\bar{\beta}_1}] \end{aligned}$$

by corollary 4.2. Let $\lambda^q : [qn_1\bar{\beta}_1 + e_1[\longrightarrow [q(n_1+m_1), \infty[$ be the function given by $\lambda^q(m) = q(n_1+m_1) + \frac{m - qn_1\bar{\beta}_1}{e_1} + 1$. For simplicity, set $i = m - qn_1\bar{\beta}_1$.For any integer i such that $e_1 \leqslant i < n_1\bar{\beta}_1 = n_1m_1e_1$, we have $1 + [\frac{i}{n_1e_1}] + [\frac{i}{m_1e_1}] \leqslant [\frac{i}{e_1}]$. Indeed this is true for $i = e_1$ and it follows by induction on i from the fact that for any pair of integers (b, a), we have $[\frac{b+1}{a}] = [\frac{b}{a}]$ if and only if $b+1 \not\equiv 0 \mod a$ and $[\frac{b+1}{a}] = [\frac{b}{a}] + 1$ otherwise, since $i < n_1m_1e_1$. So $\delta^q(m) \leqslant [\lambda^q(m)]$.
But in the proof of corollary 4.8, we have checked that if $C^k_m \neq \emptyset$, then $\mathrm{codim}(C^k_m, \mathbb{C}^2_m) = [\gamma^k(m)]$. We have also checked that for $q \geqslant k$ and $m \geqslant qn_1\beta,\ \gamma^k(m) \geqslant \gamma^q(m)$. Finally in view of the definitions of γ^q and λ^q, we have $\gamma^q(m) \geqslant \lambda^q(m)$, so $[\gamma^q(m)] \geqslant [\lambda^q(m)] \geqslant \delta^q(m)$.
For $m = (q+1)n_1\bar{\beta}_1$, we have $\delta^q(m) = (q+1)(n_1+m_1)+1$ by corollary 4.2. For $m \in [(q+1)n_1\bar{\beta}_1, (q+1)n_1\bar{\beta}_1 + e_1[$, we have

$$Cont^{>qn_1}(x_0)_m = C^{q+1}_m \cup Cont^{>(q+1)n_1}(x_0)_m$$

and

$$Cont^{>(q+1)n_1}(x_0)_m = V(x_0^{(0)}, \cdots, x_0^{((q+1)n_1)}, x_1^{(0)}, \cdots, x_1^{((q+1)m_1)})$$

again by corollary 4.2. If in addition we have $m < (q+1)\bar{\beta}_2$, then by proposition 4.5 $C^{q+1}_m = V(x_0^{(0)}, \cdots, x_0^{((q+1)n_1-1)},\ x_1^{(0)}, \cdots, x_1^{((q+1)m_1-1)},$ $x_1^{((q+1)m_1)^{n_1}} - c_1x_0^{((q+1)n_1)^{m_1}}) \cap D(x_0^{((q+1)n_1)}$, thus we have $Cont^{>qn_1}(x_0)_m$ $= \overline{C^{q+1}_m}$ and $\delta^q(m) = (q+1)(n_1+m_1)+1$. We have $(q+1)n_1\bar{\beta}_1 + e_1 \leqslant (q+1)\overline{\beta}_2$ if $q+1 \geqslant n_2$, because $\overline{\beta}_2 - n_1\overline{\beta}_1 \equiv 0 \mod (e_2)$. If not , we may have $(q+1)\overline{\beta}_2 < (q+1)n_1\bar{\beta}_1 + e_1$, so for $(q+1)\overline{\beta_2} \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$, we have $C^{q+1}_m = \emptyset, Cont^{>qn_1}(x_0)_m = Cont^{>(q+1)n_1}(x_0)_m$ and $\delta^q(m) = (q+1)(n_1+m_1)+2$.
In both cases, for $m \in [(q+1)n_1\bar{\beta}_1, (q+1)n_1\bar{\beta}_1 + e_1[$, we have $\delta^q(m) \leqslant (q+1)(n_1+m_1)+2$. Since $[\lambda^q(m)] = q(n_1+m_1) + n_1m_1 + 1$, we conclude that $[\lambda^q(m)] \geqslant \delta^q(m)$, so for $1 \leqslant k \leqslant q$, if $C^k_m \neq \emptyset$, we have $[\gamma^k(m)] \geqslant \delta^q(m)$. This proves that the irreducible components of C^0_m are the $\overline{C^k_m}$ for $1 \leqslant k \leqslant q$ and $C^k_m \neq \emptyset$, and $Cont^{>qn_1}(x_0)_m$, hence the claim in view of the characterization of the nonempty $C^{k's}_m$'s given in proposition 4.5. ☐

Corollary 4.10. — *Under the assumption of theorem* 4.9*, let* $q_0 + 1 = min\{\alpha \in \mathbb{N}; \alpha(\overline{\beta}_2 - n_1\overline{\beta}_1) \geqslant e_1\}$*. Then* $0 \leqslant q_0 < n_2$*. For* $1 \leqslant m < (q_0+1)n_1\bar{\beta}_1 + e_1, C^0_m$ *is irreducible and we have* $codim(C^0_m, \mathbb{C}^2_m) =$

$$2 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \ for\ 0 \leqslant q \leqslant q_0\ and\ qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1$$

$$or\ 0 \leqslant q \leqslant q_0\ and\ (q+1)\overline{\beta}_2 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1.$$

$$1+[\frac{m}{\bar\beta_0}]+[\frac{m}{\bar\beta_1}]\ for\ 0\leqslant q<q_0\ and\ (q+1)n_1\bar\beta_1\leqslant m<(q+1)\overline{\beta}_2$$

$$or\ (q_0+1)n_1\bar\beta_1\leqslant m<(q_0+1)n_1\bar\beta_1+e_1.$$

For $q\geqslant q_0+1$ in $\mathbb{N}$ and $qn_1\bar\beta_1+e_1\leqslant m<(q+1)n_1\bar\beta_1+e_1$, the number of irreducible components of C^0_m is:

$$N(m)=q+1-\sum_{j=2}^{g}([\frac{m}{\bar\beta_j}]-[\frac{m}{n_j\bar\beta_j}])$$

and $codim(C^0_m,\mathbb{C}^2_m)=$

$$2+[\frac{m}{\bar\beta_0}]+[\frac{m}{\bar\beta_1}]\ for\ qn_1\bar\beta_1+e_1\leqslant m<(q+1)n_1\bar\beta_1.$$

$$1+[\frac{m}{\bar\beta_0}]+[\frac{m}{\bar\beta_1}]\ for\ (q+1)n_1\bar\beta_1\leqslant m<(q+1)n_1\bar\beta_1+e_1.$$

Proof. — We have already observed that $n_2(\overline{\beta}_2-n_1\overline{\beta}_1)\geqslant e_1$ because $\overline{\beta}_2-n_1\overline{\beta}_1\equiv 0 \mod (e_2)$, so $1\leqslant q_0+1\leqslant n_2$.
For $qn_1\bar\beta_1+e_1\leqslant m<(q+1)n_1\bar\beta_1+e_1$, with $q\geqslant 1$, we have seen in the proof of theorem 4.9 that the irreducible components of C^0_m are the $\overline{C^k_m}$ for $1\leqslant k\leqslant q$ and $C^k_m\neq\emptyset$, and $Cont^{qn_1}(x_0)_m$. We thus have to enumerate the empty C^k_m for $1\leqslant k\leqslant q$. By proposition 4.5, $C^k_m=\emptyset$ if and only if $j:=max\{l;l\geqslant 2$ and $k\equiv 0 \mod n_2\cdots n_{l-1}\}\leqslant g$ and $m\geqslant\frac{k}{n_2\cdots n_{j-1}}\overline{\beta}_j$. Now recall that $\overline{\beta}_{i+1}>n_i\overline{\beta}_i$ for $1\leqslant i\leqslant g-1$ and that $\overline{\beta}_2-n_1\overline{\beta_1}\geqslant e_2$. This implies that for $3\leqslant j\leqslant g$, we have $\overline{\beta}_j-n_1\cdots n_{j-1}\overline{\beta}_1>n_2\cdots n_{j-1}(\overline{\beta}_2-n_1\overline{\beta}_1)\geqslant n_2\cdots n_{j-1}e_2\geqslant e_1$. So if $j\geqslant 3$ and κ is a positive integer such that $m\geqslant\kappa\overline{\beta}_j$, we have $\frac{m-e_1}{n_1\bar\beta_1}>\kappa n_2\cdots n_{j-1}$, hence $q=[\frac{m-e_1}{n_1\bar\beta_1}]\geqslant\kappa n_2\cdots n_{j-1}$. Therefore for $j\geqslant 3$, there are exactly $[\frac{m}{\overline{\beta}_j}]$ integers $\kappa\geqslant 1$ such that $m\geqslant\kappa\overline{\beta}_j$ and $\kappa n_2\cdots n_{j-1}\leqslant q$, among them $[\frac{m}{n_j\overline{\beta}_j}]$ are $\equiv 0 \mod (n_j)$.
Similarly if $(q+1)n_1\bar\beta_1+e_1\leqslant(q+1)\overline{\beta}_2$, or equivalently $q\geqslant q_0$, and if κ is a positive integer such that $m\geqslant\kappa\overline{\beta}_2$, we have $\kappa\leqslant\frac{m}{\overline{\beta}_2}<q+1$. Therefore if $q\geqslant q_0+1$, we conclude that there are $\sum_{j=2}^{g}([\frac{m}{\overline{\beta}_j}]-[\frac{m}{n_j\overline{\beta}_j}])$ empty C^k_m's with $1\leqslant k\leqslant q$. Moreover we have shown in the proof of theorem 4.9 that $codim(C^0_m,\mathbb{C}^2_m)=codim(Cont^{>qn_1}(x_0)_m,\mathbb{C}^2_m)=2+[\frac{m}{\bar\beta_0}]+[\frac{m}{\bar\beta_1}]$ if $m<(q+1)n_1\bar\beta_1(resp.1+(q+1)(n_1+m_1)=1+[\frac{m}{\bar\beta_0}]+[\frac{m}{\bar\beta_1}]$ for $m\geqslant(q+1)n_1\bar\beta_1)$.Also note that $q_0\overline{\beta}_2<q_0n_1\bar\beta_1+e_1<(q_0+1)n_1\bar\beta_1+e_1\leqslant(q_0+1)\overline{\beta}_2\leqslant n_2\overline{\beta}_2<\overline{\beta}_3\cdots$. Therefore for $q_0n_1\bar\beta_1+e_1\leqslant m<(q_0+1)n_1\bar\beta_1+e_1$, we have $[\frac{m}{\overline{\beta}_2}]=q_0,[\frac{m}{n_2\overline{\beta}_2}]=[\frac{m}{\overline{\beta}_3}]=\cdots=0$, so $N(m)=1$, i.e. C^0_m is irreducible.

Finally, assume that $qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$ with $q \geqslant 1$ and $q \leqslant q_0$. Since $q_0 < n_2$, for $1 \leqslant k \leqslant q$ we have $k \not\equiv 0 \bmod(n_2)$ and $m \geqslant qn_1\bar{\beta}_1 + e_1 > q\overline{\beta}_2$, hence for $1 \leqslant k \leqslant q, C_m^k = \emptyset$ and $C_m^0 = Cont^{qn_1}(x_0)_m$ is irreducible.(The case $q = q_0$ was already known).So for $n_1\bar{\beta}_1 \leqslant m < (q_0+1)n_1\bar{\beta}_1+e_1$, C_m^0 is irreducible.(Recall that for $1 \leqslant m < q_0n_1\bar{\beta}_1+e_1$, the irreducibility of C_m^0 is already known).It only remains to check the codimensions of C_m^0 for $1 \leqslant m \leqslant q_0n_1\bar{\beta}_1+e_1$. Here again we have seen in the proof of Theorem 4.9 that $codim(C_m^0, \mathbb{C}_m^2) = codim(Cont^{>qn_1}(x_0)_m, \mathbb{C}_m^2) =: \delta^q(m)$ for any $q \geqslant 1$ and $qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$ and that

$$\delta^q(m) = 2 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \text{ for any } q \geqslant 1 \text{ and } qn_1\bar{\beta}_1 + e_1 \leqslant m < (q+1)n_1\bar{\beta}_1$$

$$(q+1)(n_1+m_1)+1 = 1 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \text{ for } q < q_0 \text{ and } (q+1)n_1\bar{\beta}_1 \leqslant m < (q+1)\overline{\beta}_2$$

$$(q+1)(n_1+m_1)+2 = 2 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \text{ for } q < q_0 \text{ and } (q+1)\overline{\beta}_2 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1.$$

This completes the proof. □

In [6], Igusa has shown that the log-canonical threshold of the pair $((\mathbb{C}^2,0),(C,0))$ is $\frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1}$. Here $(\mathbb{C}^2,0)$(resp.$(C,0)$)) is the formal neighberhood of $\mathbb{C}^2$ (resp. C) at 0. Corollary 4.10 allows to recover corollary B of [2] in this special case.

Corollary 4.11. — *If the plane curve C has a branch at 0, with multiplicity β_0, and first Puiseux exponent $\bar{\beta}_1$, then*

$$min_m \frac{codim(C_m^0, \mathbb{C}_m^2)}{m+1} = \frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1}.$$

Proof. — For any $m, p \neq 0$ in $\mathbb{N}$, we have $m - p[\frac{m}{p}] \leqslant p-1$ and $m - p[\frac{m}{p}] = p-1$ if and only if $m+1 \equiv 0 \bmod (p)$; so for any $m \in \mathbb{N}, 2 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \geqslant (m+1)(\frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1})$ and we have equality if and only if $m+1 \equiv 0 \bmod (\beta_0)$ and mod $(\bar{\beta}_1)$ or equivalently $m+1 \equiv 0 \bmod (n_1\bar{\beta}_1)$ since $n_1\bar{\beta}_1$ is the least common multiple of β_0 and $\bar{\beta}_1$.If not we have $1 + [\frac{m}{\beta_0}] + [\frac{m}{\bar{\beta}_1}] \geqslant (m+1)(\frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1})$. Now if $(q+1)n_1\bar{\beta}_1 \leqslant m < (q+1)n_1\bar{\beta}_1 + e_1$ with $q \in \mathbb{N}$,we have $(q+1)n_1\bar{\beta}_1 < m+1 \leqslant (q+1)n_1\bar{\beta}_1 + e_1 < (q+2)n_1\bar{\beta}_1$, so $m+1 \not\equiv 0$ mod $(n_1\bar{\beta}_1)$. If $(q+1)n_1\bar{\beta}_1 \leqslant m < (q+1)\overline{\beta}_2$ with $q \in \mathbb{N}$ and $q < q_0$, then $(q+1)n_1\bar{\beta}_1 < m+1 \leqslant (q+1)n_1\bar{\beta}_1 + e_1 < (q+2)n_1\bar{\beta}_1$, so $m+1 \not\equiv 0$ mod

$(n_1\bar{\beta}_1)$. So in both cases, we have $1+[\frac{m}{\beta_0}]+[\frac{m}{\bar{\beta}_1}] \geqslant (m+1)(\frac{1}{\beta_0}+\frac{1}{\bar{\beta}_1})$. The claim follows from corollary 4.10. □

It also follows immediately from corollary 4.10 .

Corollary 4.12. — *Let $q_0 \in \mathbb{N}$ as in corollary 4.10. There exists $n_1\bar{\beta}_1$ linear functions, $L_0,\cdots,L_{n_1\bar{\beta}_1-1}$ such that $dim(C_m^0)=L_i(m)$ for any $m \equiv i \bmod (n_1\bar{\beta}_1)$ such that $m \geqslant q_0 n_1\bar{\beta}_1+e_1$.*

The canonical projections $\pi_{m+1,m}: C^0_{m+1} \longrightarrow C^0_m, m \geqslant 1$, induce infinite inverse systems

$$\cdots B_{m+1} \longrightarrow B_m \cdots \longrightarrow B_1$$

$$\cdots C_{(m+1)\kappa I} \longrightarrow C_{m\kappa I} \cdots \longrightarrow C_{(\kappa\beta_0\bar{\beta}_1+e_1)\kappa I} \longrightarrow B_{\kappa\beta_0\bar{\beta}_1+e_1-1}$$

and finite inverse systems

$$C^j_{(\kappa\overline{\beta}_j-1)\kappa v} \longrightarrow C^j_{m\kappa v} \cdots \longrightarrow C^j_{(\kappa n_1\cdots n_{j-1}\bar{\beta}_1+e_1)\kappa v} \longrightarrow B_{\kappa n_1\cdots n_{j-1}\bar{\beta}_1+e_1-1}$$

for $2 \leqslant j \leqslant g$, and $\kappa \not\equiv 0 \bmod (n_j)$.
We get a tree $T_{C,0}$ by representing each irreducible component of $C^0_m, m \geqslant 1$, by a vertex $v_{i,m}, 1 \leqslant i \leqslant N(m)$, and by joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if $\pi_{m+1,m}$ induces one of the above maps between the corresponding irreducible components.

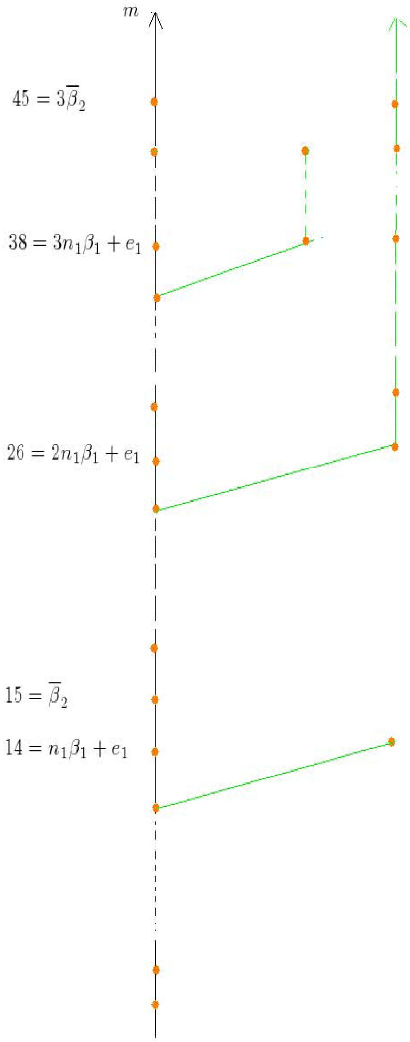
This tree only depends on the semigroup Γ.
Conversely, we recover $\overline{\beta}_0,\cdots,\overline{\beta}_g$ from this tree and $max\{m, codim(B_m,\mathbb{C}^2_m)=2\}=\overline{\beta}_0-1$. Indeed the number of edges joining two vertices from which an infinite branch of the tree starts is $\beta_0\bar{\beta}_1$. We thus recover $\overline{\beta}_1$ and e_1. We recover $\overline{\beta}_2-n_1\overline{\beta}_1,\cdots,\overline{\beta}_j-n_1\cdots n_{j-1}\overline{\beta}_1,\cdots,\overline{\beta}_g-n_1\cdots n_{g-1}\overline{\beta}_1$, hence $\overline{\beta}_2,\cdots,\overline{\beta}_g$ from the number of edges in the finite branches.

Corollary 4.13. — *Let C be a plane branch with $g \geqslant 1$ Puiseux exponents. The tree $T_{C,0}$ described above and $max\{m, dim\ C^0_m=2m\}$ determines the sequence $\overline{\beta}_0,\cdots,\overline{\beta}_g$ or equivalently the equisingularity class of C and conversely.*

We represent below the tree for the branch defined by

$$f(x,y)=(y^2-x^3)^2-4x^6y-x^9=0,$$

whose semigroup is $<\bar{\beta}_0=4,\bar{\beta}_1=6,\bar{\beta}_2=15)>$, and for which we have $e_1=2,\ e_2=1$ and $n_1=n_2=2$.



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