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COMPATIBLE COMPLEX STRUCTURES ON TWISTOR SPACE

by Guillaume DESCHAMPS (*)

Abstract. — Let M be a Riemannian 4-manifold. The associated twistor space is a bundle whose total space Z admits a natural metric. The aim of this article is to study properties of complex structures on Z which are compatible with the fibration and the metric. The results obtained enable us to translate some metric properties on M (scalar flat, scalar-flat Kähler...) in terms of complex properties of its twistor space Z.

Résumé. — Soit M une 4-variété riemannienne. L'espace de twisteur associé est un fibré qui admet une métrique naturelle. Le but de cet article est d'étudier les structures complexes sur Z qui sont compatibles avec la fibration et la métrique. Les résultats obtenu permettent d'exprimer des propriétés métriques sur M (courbure scalaire nulle, Kähler à courbure scalaire nulle...) en termes de propriétés des structures complexes de l'espace de twisteur Z.

Let (M, g) be a Riemannian 4-manifold. The twistor space $Z \to M$ is a $\mathbb{C}P^1$-bundle whose total space Z admits a natural metric $\tilde{g}$. The aim of this article is to study properties of complex structures on $(Z, \tilde{g})$ which are compatible with the $\mathbb{C}P^1$-fibration and the metric $\tilde{g}$. The results obtained enable us to translate some metric properties on M in terms of complex properties on its twistor space Z.

Introduction

Let (M, g) be an oriented 4-dimensional Riemannian manifold (not necessarily compact). Due to the Hodge-star operator $\star$, we have a decomposition

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of the bivector bundle $\bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^-$. Here $\bigwedge^\pm$ is the eigen-subbundle for the eigenvalue ± 1 of $\star$. The metric g on M induces a metric, denoted by $< \ , \ >$, on the bundle $\bigwedge^2 TM$. Let $\pi : \ Z = \mathbb{S}(\bigwedge^+) \longrightarrow M$ be the sphere bundle; the fiber over a point $m \in M$ parameterizes the complex structures on the tangent space $T_m M$ compatible with the orientation and the metric g. It is the twistor space of the manifold (M, g). Since the structural group of Z is $SO(3) \subset \mathrm{Aut}(\mathbb{C}P^1)$, we can thus put the complex structure of $\mathbb{C}P^1$ on each fiber. On the other hand, the Levi-Civita connection on (M, g) induces a splitting of the tangent bundle TZ into the direct sum of the horizontal and vertical distributions: $TZ = H \oplus V$. Therefore, the twistor space Z admits a natural metric $\tilde{g}$ defined by its restrictions to H and V: we endow V with the Fubini-Study metric and $H \simeq \pi^\star TM$ with the pullback of the metric g.

In this article we study some aspects of almost complex structures on $(Z, \tilde{g})$ which are Hermitian and extend the complex structure of the fibers. These structures will be called *compatible almost complex structures* on $(Z, \tilde{g})$. In particular, the integrability of two such structures means that the metric $\tilde{g}$ is bihermitian [33], [4].

To each morphism respecting the twistor fibration

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

we associate a compatible almost complex structure $\mathbb{J}_f$ on $(Z, \tilde{g})$ in the following way. Let $z \in Z$ with $\pi(z) = m \in M$, and write $T_z Z = H_z \oplus V_z$. Here, V_z is the tangent space to the fiber $\pi^{-1}(m) \simeq \mathbb{C}P^1$ and is therefore equipped with a complex structure. On the other hand, we endow $H_z \simeq T_m M$ with the complex structure associated to the point $f(z)$. Conversely, any compatible almost complex structure $\mathbb{J}$ on $(Z, \tilde{g})$ defines a unique morphism $f : Z \longrightarrow Z$ respecting the fibration such that $\mathbb{J}_f = \mathbb{J}$.

The almost complex structure $\mathbb{J}_{Id}$ associated to the identity is the canonical twistor almost complex structure [6]. If σ is the morphism of Z whose restriction to each fiber of π is the antipodal map of $\mathbb{S}^2$, we denote by $\mathbb{J}_\sigma$ the almost complex structure associated to σ. That is the opposite of the almost complex structure J_2 defined in [17] which is know to be never integrable. Now, an almost complex manifold (M, g, J_M) such that J_M is compatible with the orientation and the metric g defines a tautological section of $Z \longrightarrow M$. This section can be taken as the infinity section and we can therefore consider the constant morphism $f = \infty$. The associated almost complex structure will be denoted by $\mathbb{J}_\infty$. Let $\lambda \in \mathbb{C}^\star$ and consider the morphism $f = \lambda Id$ acting as λId in each fiber minus infinity (i.e. $\mathbb{C}P^1 - \{\infty\} \simeq \mathbb{C}$) and preserving infinity. We denote by $\mathbb{J}_{\lambda Id}$ the corresponding almost complex structure on Z.

The integrability of the structures $\mathbb{J}_{Id}$, $\mathbb{J}_\infty$, $\mathbb{J}_{\lambda Id}$ are related to the curvature of the metric g on M. Let $R : \bigwedge^2 TM \longrightarrow \bigwedge^2 TM$ be the curvature operator. The decomposition $\bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^-$ allows us to write R in block matrix form as follows

$$R = \begin{pmatrix} A & {}^tB \\ B & C \end{pmatrix},$$

where $A = W^+ + \frac{s}{12} Id$, $C = W^- + \frac{s}{12} Id$, W^+ (*resp.* W^-) is the selfdual (*resp.* anti-selfdual) Weyl tensor, s is the scalar curvature and B the tracefree Ricci curvature [11].

The main result of this article is the following:

Theorem 1. — *Let (M, g) be an oriented Riemannian 4-manifold.*

A) *The complex structure $\mathbb{J}_{Id}$ is integrable if, and only if, g anti-selfdual (i.e. A is a homothety)* [6].

B) *Let J_M be an almost complex structure on M compatible with the metric g and the orientation. The complex structure $\mathbb{J}_\infty$ is integrable if, and only if:*
   i) *J_M is integrable;*
   ii) *the kernel of A contains the plane $J_M^\perp \subset \bigwedge^+$ orthogonal to the line generated by J_M.*

C) *Let (M, g, J_M) be a Kählerian surface. If $\lambda \notin \{0, 1\}$, the complex structure $\mathbb{J}_{\lambda Id}$ is integrable if, and only if, (M, g, J_M) is scalar-flat Kähler (i.e. A=0).*

D) *Let (M, g) be an anti-selfdual Riemannian manifold. Its scalar curvature is zero if, and only if, any $m \in M$ has an open neighborhood $\mathcal{U}$ such that, over $\mathcal{U}$, $(Z, \tilde{g})$ admits a compatible complex structure different from $\mathbb{J}_{Id}$.*

The conditions i) & ii) of part B in the previous theorem are satisfied as soon as (M, g, J_M) is Kähler. We show in section B that this Kählerian property is equivalent to the integrability of $\mathbb{J}_\infty$ in the compact case. For a scalar-flat Kähler surface (M, g, J_M), the complex structures $\mathbb{J}_{Id}$ [19], $\mathbb{J}_\infty$ and $\mathbb{J}_{\lambda Id}$ are integrable and compatible with the metric $\tilde{g}$ on Z. This gives us a huge family of real 6-dimensional manifolds admitting a bihermitian metric.

Recall that the Penrose correspondence gives a dictionary between holomorphic properties of the twistor space Z and properties of the Riemannian manifold (M,g). The above result can be viewed as a new paragraph of that dictionary. In particular, we deduce from it some new characterizations of Kähler metrics, anti-selfdual scalar-flat metrics and scalar-flat Kähler metrics, in terms of twistor spaces.

The proof of Theorem 1 is split into four theorems, Theorem A,..., D, the proof of each being given in the corresponding labelled section.

In section E we study more precisely the set of all compatible complex structures on the twistor space of a locally conformally Kähler surfaces. Whereas on section F we will study the case of bielliptic surfaces.

We conclude the paper by giving a generalisation of this theorem to quaternionic Kähler manifolds of dimension $4n$ for $n > 1$.

Notation

We will use Einstein summation convention over repeated indices. The fiber of $\pi : Z \longrightarrow M$ over $m \in M$ will be freely identified with $\mathbb{S}^2$, $\mathbb{C}P^1$ or $SO(4)/U(2)$, the set of all complex structure on T_mM. The bundle of bivectors $\bigwedge^2 TM$ will be identified with the bundle of skew-symmetric endomorphisms of TM, or to the bundle of 2-forms.

Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be an oriented g-orthonormal frame defined over an open set $\mathcal{U}$ of (M,g). Define three linear operators $I, J, K \in \mathrm{End}(TM)$, over $\mathcal{U}$, by their matrix in the basis $(\theta_1, \ldots, \theta_4)$:

$$I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \; K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, (I, J, K) gives an oriented orthonormal basis over $\mathcal{U}$ of $\bigwedge^+$ and therefore defines a trivialization of the twistor space $\pi : Z \longrightarrow M$ over $\mathcal{U}$:

$$\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times SO(4)/U(2).$$

Let $(\theta_1^*, \ldots, \theta_4^*)$ be the local coframe dual to $(\theta_1, \ldots, \theta_4)$. Locally, the covariant derivative ∇ (on M) defined by the Levi-Civita connection of the metric g writes $\nabla\theta_j = \Gamma_{ij}^k \theta_i^* \otimes \theta_k$. The Γ_{ij}^k are the Christoffel symbols of the connection ∇; they satisfy $\Gamma_{ij}^k = -\Gamma_{ik}^j$.

Let $z \simeq (m, Q) \in \pi^{-1}(\mathcal{U})$ be a point of Z and write the tangent space as the direct sum of the horizontal and vertical tangent spaces: $T_zZ = V_z \oplus H_z$.

Denote by $\hat{\theta} \in H_z \simeq T_m M$ the horizontal lift of $\theta \in T_m M$. We then have [8]:

$$\begin{cases} V_z = \left\{ X \frac{\partial}{\partial Q} \mid X \in \mathrm{End}(T_m M),\ {}^t X = -X \text{ et } QX = -XQ \right\} \\ H_z = \mathrm{Vect}\left(\hat{\theta}_1(z), \ldots, \hat{\theta}_4(z)\right) \end{cases}$$

with $\begin{cases} \hat{\theta}_i(z) = \theta_i(m) - [\Gamma^{\cdot}_{i\cdot}(m), Q] \frac{\partial}{\partial Q} \\ [\Gamma^{\cdot}_{i\cdot}(m), Q] \frac{\partial}{\partial Q} = \left(\Gamma^{\cdot}_{i\cdot}(m) Q - Q \Gamma^{\cdot}_{i\cdot}(m)\right) \frac{\partial}{\partial Q} \in V_z. \end{cases}$

Remark. — The complex structure of rational curves on the fiber $\pi^{-1}(m) \simeq \mathbb{S}^2$ at a point $z = (m, Q)$ is given by the application [8]:

$$\begin{array}{ccc} V_z \simeq T_Q \mathbb{S}^2 & \longrightarrow & V_z \simeq T_Q \mathbb{S}^2 \\ X \frac{\partial}{\partial Q} & \longmapsto & QX \frac{\partial}{\partial Q}. \end{array}$$

For all $A \in so(4) = \{A \in \mathrm{End}(TM) \mid {}^t A = -A\}$ we can define the vertical vector field $\tilde{A} = [A, Q] \frac{\partial}{\partial Q}$. These vector fields will be called *basic*.

General results

In this section (M, g) will be an oriented Riemannian 4-manifold. Results – and proofs – given here in dimension 4, can be easily adapted to quaternionic Kähler $4n$-manifolds and will be used in the last section of the paper.

To study the integrability of the almost complex structure $\mathbb{J}_f$ we need to compute the Nijenhuis tensor N of $\mathbb{J}_f$ [28]:

$$N(X, Y) = [\mathbb{J}_f X, \mathbb{J}_f Y] - \mathbb{J}_f[\mathbb{J}_f X, Y] - \mathbb{J}_f[X, \mathbb{J}_f Y] - [X, Y] \quad \forall (X, Y) \in T_z Z.$$

The first necessary condition for the integrability of $\mathbb{J}_f$ appears in the next proposition.

PROPOSITION 1. — *For any morphism f we have:*

i) $N(X, Y) = 0$ *for all* $X, Y \in V_z$*;*

ii) *let* $X, \theta \in V_z \times H_z$*, then*
   - *the vertical component of* $N(X, \theta)$ *is zero*
   - *the horizontal component of* $N(X, \theta)$ *is zero if and only if the restriction of* f *to each fiber is holomorphic.*

As σ is an anti-holomorphic involution on fibers we easily recover the result from [17]:

COROLLARY 1. — *The almost complex structure* $\mathbb{J}_\sigma$ *is never integrable.*

Proof of Proposition 1. For any morphism f, each fiber of $\pi : Z \longrightarrow M$ has the structure of $\mathbb{C}P^1$. It follows immediately from [28] that $N(X,Y) = 0$ for all $X, Y \in V_z$.

Let $\tilde{X}$ be a basic vertical vector field and $\pi^{-1}(m)$ be a fixed fiber. The restriction to that fiber of the application f is:

$$f|_{\pi^{-1}(m)} : \begin{array}{ccc} \mathbb{S}^2 \simeq \pi^{-1}(m) & \longrightarrow & \mathbb{S}^2 \simeq \pi^{-1}(m) \\ Q & \longmapsto & f(Q) \end{array}$$

Observe that $[\tilde{X}, \hat{\theta}_i]$ is vertical when $\tilde{X}$ is. Since the action of the complex structure $\mathbb{J}_f$ on the fiber is equal to the rational curve structure, it does not depend on the fiber. We then have: $[\mathbb{J}_f \tilde{X}, \hat{\theta}_i] = [Q\tilde{X}, \hat{\theta}_i] = Q[\tilde{X}, \hat{\theta}_i] = \mathbb{J}_f[\tilde{X}, \hat{\theta}_i]$. This implies that, for $i \in \{1, \dots, 4\}$:

$$\begin{aligned} N(\tilde{X}, \hat{\theta}_i) &= [Q\tilde{X}, f(Q)\hat{\theta}_i] - Q[Q\tilde{X}, \hat{\theta}_i] + \mathbb{J}_f[\tilde{X}, f(Q)\hat{\theta}_i] - [\tilde{X}, \hat{\theta}_i] \\ &= \Big((Q\tilde{X}).f(Q) \ \ - f(Q)(\tilde{X}.f(Q))\Big)\, \hat{\theta}_i \\ &= \Big(d_Q f(Q\tilde{X}) \ \ - f(Q) d_Q f(\tilde{X})\Big)\, \hat{\theta}_i \end{aligned}$$

where $d_Q f$ is the differential of f at $Q \in \mathbb{S}^2$. The horizontal component of $N(X,\theta)$ vanishes for all $(X,\theta) \in V_z \times H_z$ if and only if the restrictions of f to the fibers are holomorphic. $\square$

In the trivialization of $Z \longrightarrow M$ over an open set $\mathcal{U}$, the morphism f can be written:

$$f|_{\pi^{-1}(\mathcal{U})} : \begin{array}{ccc} \mathcal{U} \times \mathbb{S}^2 & \longrightarrow & \mathcal{U} \times \mathbb{S}^2 \\ (x, Q) & \longmapsto & \Big(x, f(x,Q)\Big). \end{array}$$

In order to simplify the notation we set $P = f(x,Q)$ and $[P_i^j]$ denotes the matrix, in the basis $(\theta_1, \dots, \theta_4)$, of the operator P viewed as an endomorphism of TM.

Proposition 2. — *Let f be any morphism and $(m,Q) \in Z$. Then, for all $i, j \in \{1, \dots, 4\}$ one has:*

i) *the horizontal component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $\widehat{E(\theta_i, \theta_j)} + F_{ij}$*

ii) *the vertical component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $G(\theta_i, \theta_j)\frac{\partial}{\partial Q}$,* *where*

$$\begin{cases} E(\theta_i,\theta_j) \text{ is the Nijenhuis tensor of the almost complex structure } P_0 \\ \text{on } TM \text{ defined by } f(\boldsymbol{.},Q) \text{ over the open set } \mathcal{U} \text{ (where } Q \text{ is fixed);} \\ \\ F_{ij} = -P_i^r[\Gamma^{\boldsymbol{.}}_{r\boldsymbol{.}},Q]\frac{\partial}{\partial Q}P_j^r \ \ \hat\theta_r + P_j^r[\Gamma^{\boldsymbol{.}}_{r\boldsymbol{.}},Q]\frac{\partial}{\partial Q}P_i^l \ \ \hat\theta_l \\ \qquad\quad -P\Big([\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}},Q]\frac{\partial}{\partial Q}P_i^l \ \ \hat\theta_l - [\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},Q]\frac{\partial}{\partial Q}P_j^r \ \ \hat\theta_r\Big); \\ \\ G(\theta_i,\theta_j) = \Big[R\Big(\theta_i\wedge\theta_j - P\theta_i\wedge P\theta_j\Big) + QR\Big(P\theta_i\wedge\theta_j + \theta_i\wedge P\theta_j\Big),Q\Big]. \end{cases}$$

Proof. — The curvature tensor is

$$R(\theta_i,\theta_j) = \nabla_{\theta_i}\nabla_{\theta_j} - \nabla_{\theta_j}\nabla_{\theta_i} - \nabla_{[\theta_i,\theta_j]} = R^l_{kij}\theta^\star_k\otimes\theta_l,$$

with $R^l_{kij} = g\Big(R(\theta_i,\theta_j)\theta_k,\theta_l\Big)$. Hence,

$$R(\theta_i,\theta_j)\theta_k = \nabla_{\theta_i}(\Gamma^m_{jk}\theta_m) - \nabla_{\theta_j}(\Gamma^m_{ik}\theta_m) - \nabla_{(\Gamma^m_{ij}-\Gamma^m_{ji})\theta_m}\theta_k$$

yields

$$R^l_{ijk} = \theta_i(\Gamma^l_{jk}) - \theta_j(\Gamma^l_{ik}) + [\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}}]^l_k - (\Gamma^{\boldsymbol{.}}_{ij} - \Gamma^{\boldsymbol{.}}_{ji})\Gamma^l_{\boldsymbol{.}k}.$$

To finish the proof of the proposition we need the following lemma.

Lemma 1. — *The Lie bracket of $\hat\theta_i$ with $\hat\theta_j$ satisfies:*

$$[\hat\theta_i,\hat\theta_j] = \widehat{[\theta_i,\theta_j]} - \big[R^{\boldsymbol{.}}_{\boldsymbol{.}ij},Q\big]\frac{\partial}{\partial Q}.$$

Proof of Lemma 1. From $\hat\theta_i = \theta_i - [\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},Q]\frac{\partial}{\partial Q}$ we can deduce that:

$$\begin{aligned}
[\hat\theta_i,\hat\theta_j] &= \Big[\theta_i - [\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},Q]\tfrac{\partial}{\partial Q}\ ,\ \theta_j - [\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}},Q]\tfrac{\partial}{\partial Q}\Big] \\
&= [\theta_i,\theta_j] - [\theta_i(\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}}),Q]\tfrac{\partial}{\partial Q} + [\theta_j(\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}}),Q]\tfrac{\partial}{\partial Q} - \Big[[\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}}],Q\Big]\tfrac{\partial}{\partial Q} \\
&= \Big(\Gamma^m_{ij} - \Gamma^m_{ji}\Big)\theta_m - \Big[[\theta_i(\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}}) - \theta_j(\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}}) + [\Gamma^{\boldsymbol{.}}_{i\boldsymbol{.}},\Gamma^{\boldsymbol{.}}_{j\boldsymbol{.}}],Q\Big]\tfrac{\partial}{\partial Q} \\
&= \Big(\Gamma^m_{ij} - \Gamma^m_{ji}\Big)\theta_m - \Big(\Big[R^{\boldsymbol{.}}_{\boldsymbol{.}ij},Q\Big] + (\Gamma^m_{ij} - \Gamma^m_{ji})[\Gamma^{\boldsymbol{.}}_{m\boldsymbol{.}},Q]\Big)\tfrac{\partial}{\partial Q} \\
&= (\Gamma^m_{ij} - \Gamma^m_{ji})\hat\theta_m - \big[R^{\boldsymbol{.}}_{\boldsymbol{.}ij},Q\big]\tfrac{\partial}{\partial Q} \\
&= \widehat{[\theta_i,\theta_j]} - \big[R^{\boldsymbol{.}}_{\boldsymbol{.}ij},Q\big]\tfrac{\partial}{\partial Q}.
\end{aligned}$$

□

We can now complete the proof of Proposition 1. The Nijenhuis tensor is given by

$$N(\hat\theta_i,\hat\theta_j) = [\mathbb{J}_f\hat\theta_i,\mathbb{J}_f\hat\theta_j] - \mathbb{J}_f\Big([\mathbb{J}_f\hat\theta_i,\hat\theta_j] + [\hat\theta_i,\mathbb{J}_f\hat\theta_j]\Big) - [\hat\theta_i,\hat\theta_j],$$

where:

$$\begin{aligned}
[\mathbb{J}_f\hat\theta_i,\mathbb{J}_f\hat\theta_j] &= [P_i^l\hat\theta_l,P_j^r\hat\theta_r]\\
&= \widehat{P\theta_i}.(P_j^r)\;\hat\theta_r-\widehat{P\theta_j}.(P_i^l)\;\hat\theta_l+P_i^lP_j^r[\hat\theta_l,\hat\theta_r]\\
[\mathbb{J}_f\hat\theta_i,\hat\theta_j]+[\hat\theta_i,\mathbb{J}_f\hat\theta_j] &= [P_i^l\hat\theta_l,\hat\theta_j]+[\hat\theta_i,P_j^r\hat\theta_r]\\
&= -\hat\theta_j.(P_i^l)\;\hat\theta_l+P_i^l[\hat\theta_l,\hat\theta_j]+\hat\theta_i.(P_j^r)\;\hat\theta_r+P_j^r[\hat\theta_i,\hat\theta_r].
\end{aligned}$$

By Lemma 1 the horizontal component of the Nijenhuis tensor is:

$$\begin{aligned}
\mathcal{H}\,N(\hat\theta_i,\hat\theta_j) = {} & \widehat{P\theta_i}.(P_j^r)\;\hat\theta_r-\widehat{P\theta_j}.(P_i^l)\;\hat\theta_l+P_i^lP_j^r\widehat{[\theta_l,\theta_r]}\\
& -P\Big(-\hat\theta_j.(P_i^l)\;\hat\theta_l+P_i^l\widehat{[\theta_l,\theta_j]}+\hat\theta_i.(P_j^r)\;\hat\theta_r+P_j^r\widehat{[\theta_r,\theta_i]}\Big)\;-\;\widehat{[\theta_i,\theta_j]}.
\end{aligned}$$

Fix Q and denote by P_0 the almost complex structure on TM, over $\mathcal{U}$, defined by $P_0(m)=f(m,Q)$. Then:

$$\begin{aligned}
\mathcal{H}\,N(\hat\theta_i,\hat\theta_j) &= [P_0\widehat{\theta_i,P_0}\theta_j]-P_0\Big([\widehat{P_0\theta_i,\theta_j}]+[\widehat{\theta_i,P_0\theta_j}]\Big)-\widehat{[\theta_i,\theta_j]}\\
&\quad -P_i^r[\Gamma^{\bullet}_{r\bullet},Q]\tfrac{\partial}{\partial Q}P_j^r\;\;\hat\theta_r+P_j^r[\Gamma^{\bullet}_{r\bullet},Q]\tfrac{\partial}{\partial Q}P_i^l\;\;\hat\theta_l\\
&\quad -P\Big([\Gamma^{\bullet}_{j\bullet},Q]\tfrac{\partial}{\partial Q}P_i^l\;\;\hat\theta_l-[\Gamma^{\bullet}_{i\bullet},Q]\tfrac{\partial}{\partial Q}P_j^r\;\;\hat\theta_r\Big)\\
&= \widehat{E(\theta_i,\theta_j)}+F_{ij}.
\end{aligned}$$

The vertical component of the Nijenhuis tensor is:

$$\begin{aligned}
\mathcal{V}\,N(\hat\theta_i,\hat\theta_j) &= \Big([R^{\bullet}_{\bullet ij},Q]-P_i^lP_j^r[R^{\bullet}_{\bullet lr},Q]-Q\Big(-P_i^l[R^{\bullet}_{\bullet lj},Q]-P_j^r[R^{\bullet}_{\bullet ir},Q]\Big)\Big)\,\tfrac{\partial}{\partial Q}\\
&= \Big[R\Big(\theta_i\wedge\theta_j-P\theta_i\wedge P\theta_j\Big)+QR\Big(P\theta_i\wedge\theta_j+\theta_i\wedge P\theta_j\Big),Q\Big]\,\tfrac{\partial}{\partial Q}\\
&= G(\theta_i,\theta_j)\tfrac{\partial}{\partial Q}.
\end{aligned}$$

$\square$

In order to prove Theorem 1 we need to study the tensor G and we set:

$$\begin{cases} G_1(\theta_i,\theta_j,P)=\theta_i\wedge\theta_j-P\theta_i\wedge P\theta_j\\ G_2(\theta_i,\theta_j,P)=P\theta_i\wedge\theta_j+\theta_i\wedge P\theta_j.\end{cases}$$

An easy computation gives the following lemma.

Lemma 2. — *Let $(\theta_1,\dots,\theta_4)$ be an oriented orthonormal frame over an open set $\mathcal{U}$ and (I,J,K) be the associated basis of $\bigwedge^+$. Then we have:*

$$\begin{aligned}
&I=G_1(\theta_1,\theta_2,J)=G_1(\theta_1,\theta_2,K)\\
&J=G_1(\theta_1,\theta_3,I)=G_1(\theta_1,\theta_3,K)\\
&K=G_1(\theta_1,\theta_4,I)=G_1(\theta_1,\theta_4,J)\\
&0=G_1(\theta_1,\theta_2,I)=G_1(\theta_1,\theta_3,J)=G_1(\theta_1,\theta_4,K)\\
&G_1(\theta_1,\theta_2,aI+bJ+cK)=(1-a^2)I-abJ-acK\\
&G_2(\theta_i,\theta_j,P)=PG_1(\theta_i,\theta_j,P).
\end{aligned}$$

A) The case where f is the identity

In this section we give a proof of (the well known) part A of Theorem 1:

THEOREM A [6]. — *The complex structure $\mathbb{J}_{Id}$ is integrable if and only if A is a homothety.*

The fact that A is a homothety is equivalent to saying that the selfdual Weyl tensor W^+ is zero. In that case the metric is said to be *anti-selfdual.*

Proof. — In the local trivialization $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{C}P^1$ of the previous section the morphism $f = Id$ when restricted to fibers is a holomorphic map, which only depends on the second variable. By Proposition 1 we know that it is sufficient to study $N(\hat\theta_i, \hat\theta_j)$. We have:

$$\begin{aligned}
F_{ij} &= -Q_i^r[\Gamma^{\boldsymbol{\cdot}}_{r\boldsymbol{\cdot}}, Q]\tfrac{\partial}{\partial Q}Q_j^r \;\; \hat\theta_r + Q_j^r[\Gamma^{\boldsymbol{\cdot}}_{r\boldsymbol{\cdot}}, Q]\tfrac{\partial}{\partial Q}Q_i^l \;\; \hat\theta_l \\
&\quad -Q\Big([\Gamma^{\boldsymbol{\cdot}}_{j\boldsymbol{\cdot}}, Q]\tfrac{\partial}{\partial Q}Q_i^l \;\; \hat\theta_l - [\Gamma^{\boldsymbol{\cdot}}_{i\boldsymbol{\cdot}}, Q]\tfrac{\partial}{\partial Q}Q_j^r \;\; \hat\theta_r\Big) \\
&= -Q_i^r[\Gamma^{\boldsymbol{\cdot}}_{r\boldsymbol{\cdot}}, Q]\hat\theta_j + Q_j^r[\Gamma^{\boldsymbol{\cdot}}_{r\boldsymbol{\cdot}}, Q]\hat\theta_i - Q\Big([\Gamma^{\boldsymbol{\cdot}}_{j\boldsymbol{\cdot}}, Q]\hat\theta_i - [\Gamma^{\boldsymbol{\cdot}}_{i\boldsymbol{\cdot}}, Q]\hat\theta_j\Big).
\end{aligned}$$

Using $[\Gamma^{\boldsymbol{\cdot}}_{i\boldsymbol{\cdot}}, Q] = [\nabla_{\theta_i}\boldsymbol{\cdot}, Q]$ one gets:

$$\begin{aligned}
d\pi(F_{ij}) &= -[\nabla_{Q\theta_i}\boldsymbol{\cdot}, Q]\theta_j + [\nabla_{Q\theta_j}\boldsymbol{\cdot}, Q]\theta_i - Q\Big([\nabla_{\theta_j}\boldsymbol{\cdot}, Q]\theta_i - [\nabla_{\theta_i}\boldsymbol{\cdot}\, Q]\theta_j\Big) \\
&= -\nabla_{Q\theta_i}\, Q\theta_j + Q\nabla_{Q\theta_i}\, \theta_j + \nabla_{Q\theta_j}\, Q\theta_i - Q\nabla_{Q\theta_j}\, \theta_i \\
&\quad -Q\nabla_{\theta_j}\, Q\theta_i - \nabla_{\theta_j}\theta_i + Q\nabla_{\theta_i}\, Q\theta_j + \nabla_{\theta_i}\theta_j \\
&= -E(\theta_i, \theta_j).
\end{aligned}$$

The horizontal component of $N(\hat\theta_i, \hat\theta_j)$ is then zero. The vertical component is:

$$G(\theta_i, \theta_j) = \Big[R\big(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j\big) + QR\big(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j\big), Q\Big].$$

But Q preserves the orientation, hence:

$$\begin{cases} \theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j & \in \quad \bigwedge^+ T_mM \\ \theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j & \in \quad \bigwedge^+ T_mM. \end{cases}$$

Recall that the matrix of the curvature operator R has the following splitting:

$$R = \begin{pmatrix} A & {}^tB \\ B & C \end{pmatrix}$$

Since the elements of $\bigwedge^+$ of $\bigwedge^-$ commute [6], the component A in the matrix R is the only one which matters in the computation of $G(\theta_i, \theta_j)$. By Lemma 2, one has the equality:

$$\big(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j\big) + Q\big(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j\big) = 0, \quad \forall \theta_i, \theta_j \in T_mM.$$

Therefore, if the matrix A is a homothety the Nijenhuis tensor of $\mathbb{J}_{Id}$ is zero.

Conversely, assume that $\mathbb{J}_{Id}$ is integrable. We have noticed that the orthonormal frame $(\theta_1, \dots, \theta_4)$ over $\mathcal{U}$ defines an oriented orthonormal basis (I, J, K) of $\bigwedge^+$ over $\mathcal{U}$. Since $G(\theta_i, \theta_j) = 0$ for all $i, j \in \{1, \dots, 4\}$, Lemma 2 implies:

$$\begin{array}{llll}
\text{at the point } (m, I), & G(\theta_1, \theta_3) = & [A(J) + IA(K), I] & = 0 \\
\text{at the point } (m, J), & G(\theta_1, \theta_2) = & [A(I) + JA(-K), J] & = 0 \\
\text{at the point } (m, K), & G(\theta_1, \theta_2) = & [A(I) + KA(J), K] & = 0.
\end{array}$$

Since (I, J, K) is an oriented orthonormal basis, it follows from $IJ = -JI = K$ that relations of the following type hold:

$$[A(J), I] = 2 < A(J), K > J - 2 < A(J), J > \ K.$$

From the previous system we then deduce the following one:

$$\left\{ \begin{array}{lllll}
< A(J), J > & = & - < IA(K), J > & = & < A(K), K > \\
< A(J), K > & = & - < IA(K), K > & = & - < A(K), J > \\
< A(I), I > & = & - < JA(-K), I > & = & < A(K), K > \\
< A(I), K > & = & - < JA(-K), K > & = & - < A(K), I > \\
< A(I), I > & = & - < KA(J), I > & = & < A(J), J > \\
< A(I), J > & = & - < KA(J), J > & = & - < A(J), I >
\end{array} \right.$$

But the matrix A in the basis (I, J, K) is symmetric, thus A is a homothety. □

B) The case where f is constant

Integrability theorem

In this section we give a proof of part B of Theorem 1.

THEOREM B. — *Let (M, g, J_M) be an almost complex manifold such that J_M is compatible with the orientation and the metric. The complex structure $\mathbb{J}_\infty$ is integrable if and only if:*

i) *J_M is integrable;*

ii) *the kernel of A contains the subspace $J_M^\perp \subset \bigwedge^+$ orthogonal to the line generated by J_M (i.e. $J_M^\perp \subset \ker(A)$).*

Notice that the integrability condition is not conformal on g. Moreover, when $\mathbb{J}_\infty$ is integrable, it gives to the twistor projection $\pi : (Z, \mathbb{J}_\infty) \longrightarrow (M, J_M)$ the structure of a holomorphic $\mathbb{C}P^1$-bundle.

For a complex manifold (M, g, J_M) we have a decomposition $\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1}$ into $\pm i$ eigenspaces of J_M. We then obtain:

$$\begin{cases} \mathbb{C} \otimes \bigwedge^+ = \mathbb{C}J_M \oplus^\perp (\bigwedge^{2,0} \oplus \overline{\bigwedge^{2,0}}) \\ \mathbb{C} \otimes \bigwedge^- = \{\psi \in \bigwedge^{1,1} \mid <\psi, J_M>= 0\} \end{cases} \text{where} \begin{cases} \bigwedge^{2,0} = T^{1,0} \wedge T^{1,0} \\ \bigwedge^{1,1} = T^{1,0} \wedge T^{0,1} \end{cases}$$

Condition ii) says that $(\bigwedge^{2,0} \oplus \overline{\bigwedge^{2,0}}) \subset \ker(A)$. For a Kählerian manifold the curvature R may be viewed as a symmetric endomorphism of $\bigwedge^{1,1}$, so in some orthonormal basis compatible with these decompositions we have $A = \begin{bmatrix} \frac{s}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $W^+ = \begin{bmatrix} \frac{s}{6} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & -\frac{s}{12} \end{bmatrix}$. We then have the following result:

Proposition 3. — *For any Kählerian surface (M, g, J_M) the complex structure $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ is integrable. Furthermore, if (M, g, J_M) is Kähler and the scalar curvature of g is never zero, then $\mathbb{J}_\infty$ and $\mathbb{J}_{-\infty}$ (the compatible complex structure on $(Z, \tilde{g})$ associated to $-J_M$) are the only compatible complex structures on $(Z, \tilde{g})$.*

In other terms, for a Kählerian manifold whose scalar curvature is non zero there are, even locally, only two compatible complex structures on its twistor space.

Proof. — The first part being a consequence of Theorem B, we only need to prove the second part of the proposition. Let $\mathbb{J}_f$ be a compatible complex structure on $(Z, \tilde{g})$ and assume that the scalar curvature of (M, g, J_M) is never zero. One can build an orthonormal basis (I, J, K) of $\bigwedge^+$ over an open set $\mathcal{U}$ as follows. Setting $I = J_M$, pick any unitary vector J orthonormal to I and define $K = IJ$. For any $m \in \mathcal{U}$, there exists $(a, b, c) \in \mathbb{S}^2$ such that $f(m, J) = aI + bJ + cK$. But, as (M, g, J_M) is Kähler, in this basis we have $A = \begin{bmatrix} \frac{s}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let θ_1 be a unitary vector field defined over $\mathcal{U}$; set $\theta_2 = I\theta_1$. As $\mathbb{J}_f$ is integrable, $G(\theta_1, \theta_2)$ is identically zero on $\mathcal{U}$. In particular, at the point (m, J) we obtain:

$$\begin{aligned} G(\theta_1, \theta_2) &= 0 \\ &= [A\Big((1 - a^2)I - abJ - acK\Big) + JA(cJ - bK), J] = 0 \\ &= [(1 - a^2)\tfrac{s}{4}I, J] = (1 - a^2)\tfrac{s}{2}K. \end{aligned}$$

Therefore $a = \pm 1$, that is $f(m, J) = \pm I$ for all J orthonormal to I. Since f must be holomorphic in the fibers we get that f is constant, equal to I or $-I$. $\square$

Proof of Theorem B. — By Proposition 1, it is sufficient to check that $N(\hat{\theta}_i, \hat{\theta}_j) = 0$. As f is constant on fibers we always have $F_{ij} = 0$. Therefore: $\mathbb{J}_\infty$ integrable $\iff$ $E(\theta_i, \theta_j) = G(\theta_i, \theta_j) = 0$ $\iff$ $\{J_M$ integrable and $G(\theta_i, \theta_j) = 0\}$. But for all $\theta_i, \theta_j \in TM$ we have

$$\begin{cases} \theta_i \wedge \theta_j - J_M\theta_i \wedge J_M\theta_j \in J_M^\perp \\ J_M\theta_i \wedge \theta_j + \theta_i \wedge J_M\theta_i \in J_M^\perp \end{cases}.$$

Consequently, if $J_M^\perp \subset \ker(A)$ we obtain $G(\theta_i, \theta_j) = 0$ for all $\theta_i, \theta_j \in TM$.

Conversely, suppose that $\mathbb{J}_\infty$ is integrable. Set $J_0 = J_M$. Locally over an open set $\mathcal{U}$ one can complete $\{J_0\}$ to get an oriented orthonormal basis (I_0, J_0, K_0) of $\bigwedge^+$. Let θ_1 be a unitary vector field defined over $\mathcal{U}$; set $\theta_2 = I_0\theta_1$. If $G = 0$, then, for all $m \in \mathcal{U}$ and $Q \in \pi^{-1}(m)$, Lemma 2 implies that at the point (m, Q):

$$G(\theta_1, \theta_2) = \quad [A(I_0) + QA(-K_0), Q] \quad = 0.$$

In particular, for $Q = A(K_0)$, we have $[A(I_0), A(K_0)] = 0$ and it follows that $A(K_0) = cA(I_0)$ for some constant c. The former equation yields:

$$\begin{aligned} \forall Q \in \pi^{-1}(m), \quad 0 &= [A(I_0) + QA(-K_0), Q] \\ &= (Id - cQ)\,[A(I_0), Q] \Longrightarrow A(I_0) = 0. \end{aligned}$$

Therefore $J_0^\perp = \mathrm{Vect}(I_0, K_0) \subset \ker A$. $\square$

Recall that we have a characterization of an integrable almost complex structure J_M on M in terms of the twistor space and one of the Kählerian complex structures.

Proposition (see, for example, [37, 15]). — *Let J_M be a Hermitian almost complex structure on (M, g). Then:*

- *J_M is integrable if and only if the associated section of the twistor space, $s : (M, J_M) \longrightarrow (Z, \mathbb{J}_{Id})$, is almost holomorphic, that is: the differential ds satisfies $ds \circ J_M = \mathbb{J}_{Id} \circ ds$;*
- *J_M is Kähler if and only if s is an horizontal section, that is to say: the tangent space of the submanifold $s(M) \subset Z$ is included in the horizontal distribution.*

It is well known that the existence of a Kähler metric on a compact complex surface (M, J_M) is equivalent for the first Betti number b_1 to be even [27, 39, 25]. Theorem B gives a new characterization of compact Kählerian surfaces in terms of compatible complex structures on the associated twistor spaces.

Proposition 4. — *A compact almost Hermitian 4-dimensional manifold (M, g, J_M) is Kähler if and only if $\mathbb{J}_\infty$ is integrable.*

In section D we will deduce from that proposition a characterisation of compact scalar-flat Kähler manifolds in terms of compatible complex structures on $(Z, \tilde{g})$ (cf. Proposition 8).

Proof. — Let θ be the Lee form of (M, g, J_M) defined by $dJ_M = -2\theta \wedge J_M$, where $J \in \bigwedge^+$ is viewed as a 2-form. Denote by κ the conformal scalar curvature, which is related to the scalar curvature s by $\kappa = s + 6(\delta\theta - |\theta|^2)$. The condition $J_M^\perp \subset \ker A$ is equivalent to the following: the selfdual Weyl tensor W^+ is degenerate (meaning that, in every point, two of the eigenvalues coincident) and the scalar curvature of (M, g) is equal to the conformal scalar curvature [3]. This is also equivalent to $\delta\theta = |\theta|^2$. Integrating this expression over M gives $\theta = 0$ by the Brochner-Grenn theorem. But (M, g, J_M) is Kähler if and only if θ vanishes identically. $\square$

Corollary 2. — *Assume that a compact 4-dimensional manifold (M, g) admits two almost complex structures $J_1 \neq \pm J_2$ compatible with the metric and the orientation. Then the associated compatible almost complex structures $\mathbb{J}_{\infty 1}, \mathbb{J}_{\infty 2}$ on $(Z, \tilde{g})$ are integrable if and only if $\{J_1, J_2\}$ spans a hyperkähler structure on (M, g).*

Proof. — By Proposition 4, $\mathbb{J}_{\infty 1}$ and $\mathbb{J}_{\infty 2}$ are integrable if and only if J_1 and J_2 are Kähler. As $J_1 \neq \pm J_2$, then J_1 is different from $\pm J_2$ everywhere. The holonomy of g reduces to $U(2)$ by J_1 and further to $SU(2)$ by J_2. This says that g is hyperkähler. $\square$

Study of the manifold $(Z, \mathbb{J}_\infty)$

Any scalar-flat Kähler surfaces (M, g, J_M) is automatically anti-selfdual [19]. For such a manifold we can put two natural complex structures on its twistor space: $\mathbb{J}_{Id}$ and $\mathbb{J}_\infty$. The next proposition shows that these complex structures are never deformation of each other.

Proposition 5. — *If (M, g, J_M) is a scalar-flat Kähler surface, the complex structure $\mathbb{J}_\infty$ on Z is never a deformation of the complex structure $\mathbb{J}_{Id}$.*

Proof. — It is sufficient to show that $(Z, \mathbb{J}_{Id})$ and $(Z, \mathbb{J}_\infty)$ do not have the same Chern classes. Let h be the generator of the second cohomology group $H^2(\mathbb{C}P^1, \mathbb{Z}) \simeq \mathbb{Z}$. By Leray-Hirsch theorem's [12] the cohomology ring of Z is a $H^\star(M, \mathbb{R})$-module generated by h with relation $4h^2 = 3\tau + 2\chi$, where

τ and χ are the signature and the Euler characteristic of M. Denote by $c_1(J_M)$ the first Chern class of the manifold (M, J_M). Under this notation we have :

$$\begin{aligned} c(\mathbb{J}_{Id}) &= 1 + 4h + 3\tau + 3\chi + 2h\chi \quad [22] \\ c(\mathbb{J}_\infty) &= (1+2h)(1 + c_1(J_M) + \chi) \\ &= 1 + 2h + c_1(J_M) + 2hc_1(J_M) + \chi + 2h\chi. \end{aligned}$$

If the complex structures were deformations of each other, they would have the same Chern numbers: $c_1(\mathbb{J}_{Id})^3 = 16(3\tau+2\chi)h = c(\mathbb{J}_\infty)^3 = 8(3\tau+2\chi)h$. This forces $3\tau + 2\chi = 0$. Let μ_g be the volume form on M associated to the metric g; by the Gauss-Bonnet formula [2], [20]:

$$3\tau + 2\chi = \frac{1}{4\pi^2}\int_M 2\|W^+\| + \frac{1}{24}s^2 - 2\|B\|^2\mu_g = -\frac{1}{2\pi^2}\int_M \|B\|^2\mu_g.$$

Thus, $3\tau + 2\chi = 0$ implies $B = 0$. As the scalar curvature of (M, g) is supposed to be zero, the manifold (M, g, J_M) would be Ricci-flat, hence $c_1(J_M) = 0$. Therefore the first Chern classes of $(Z, \mathbb{J}_{Id})$ and of $(Z, \mathbb{J}_\infty)$ are different and these two manifolds are never deformations of each other. □

When (M, g, J_M) is a complex spin surface, Hitchin has shown that there exists a holomorphic line bundle $L \longrightarrow M$ such that $L \otimes L = K_M$ is the canonical line bundle [21]. Then, the twistor space Z can be identified, in a $\mathcal{C}^\infty$-way, to the projectivization bundle $\mathbb{P}(L \oplus L^\star)$ [36]. By this construction we see that the manifold $Z \simeq \mathbb{P}(L \oplus L^\star)$ admits a natural complex structure denoted by $\mathbb{I}$. When (M, g, J_M) is not spin, but only complex, the bundle $L \oplus L^\star$ exists only locally. Nevertheless, the projectivization $\mathbb{P}(L \oplus L^\star)$ still exists globally, due to the fact that the transition functions on $L \oplus L^\star$ are well defined holomorphic maps up to sign. In general $\mathbb{I}$ is not a compatible complex structure on $(Z, \tilde{g})$.

Now, if (M, g, J_M) satisfies the conditions of Theorem B, we can put another complex structure on its twistor space, namely $\mathbb{J}_\infty$. The question is then to determine the relationship between the manifolds $(Z, \mathbb{I})$ and $(Z, \mathbb{J}_\infty)$. In that direction we have the following result.

Proposition 6. — *Let (M, g, J_M) be a manifold satisfying conditions of Theorem B (i.e. $\mathbb{J}_\infty$ integrable). The complex structures $\mathbb{I}$ and $\mathbb{J}_\infty$ on Z are deformations of each other: there exists on Z a path of integrable complex structures $\mathbb{J}_t$, $t \in [0, 1]$, connecting $\mathbb{I}$ to $\mathbb{J}_\infty$.*

By combining this result and [41, Theorem 4.1] we obtain another proof of Proposition 5.

Proof. — In an appropriate local trivialization of the bundle $Z \longrightarrow M$, the almost complex structure $\mathbb{I}$ on $\mathcal{U} \times \mathbb{S}^2$ can be identified with the product structure $J_M \times J_{\mathbb{C}P^1}$. Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be an oriented orthonormal frame defined over $\mathcal{U}$ providing this trivialization. Set $\hat{\theta}_{i,t} = \theta_i - t[\Gamma^{\cdot}_{i\cdot}, Q]\frac{\partial}{\partial Q}$ for $t \in [0,1]$. The subspace $H_t = \mathrm{Vect}(\hat{\theta}_{1,t}, \ldots, \hat{\theta}_{4,t})$ is in direct sum with the vertical distribution V_z and can be glued into a global distribution over Z. Define the almost complex structure $\mathbb{J}_t$ on $\pi^{-1}\mathcal{U}$ as follows: endow V_z with the complex structure of the fibers (complex projective lines) and pull back on $H_t \simeq T_m M$ the complex structure J_M. Then, $\mathbb{J}_t$ is a path of almost complex structures from $\mathbb{I}$ to $\mathbb{J}_\infty$. The integrability of $\mathbb{J}_t$ is shown in the same way as that of $\mathbb{J}_\infty$. □

C) The case where f is a homothety

Integrability theorem

In this section we prove part C of Theorem 1.

THEOREM C. — *Let (M, g, J_M) be a Kähler surface. For all complex $\lambda \notin \{0,1\}$ the almost complex structure $\mathbb{J}_{\lambda Id}$ is integrable if and only if the scalar curvature of g is zero.*

The condition $A = 0$ is equivalent to saying that the metric g is Hermitian scalar-flat and anti-selfdual. These metrics are called *optimal* by LeBrun because they are absolute minimizers of the functional $\mathcal{K}(g) = \int_M |R|^2 d\mathrm{vol}$ [26]. Let (M, g, J_M) be a compact scalar-flat Kähler surface and $c_1(M)$ be the real first Chern class of (M, J_M). Two possibilities may occur [24]. Either $c_1(M) = 0$ and (M, g, J_M) is then finitely covered by a hyperkähler surface, i.e. a flat torus or a $K3$-surface with Ricci-flat Kähler metric [13], [29]. Or $c_1(M) < 0$, in which case (M, g) is obtained by blowing up a ruled surface [23], i.e. (M, g) is obtained by blowing up m points on a $\mathbb{C}P^1$-bundle over a Riemann surface of genus γ. The condition $c_1(M) < 0$ gives a lower bound on the number of points m to be blown up: namely $m \geqslant 9$ when $\gamma = 0$, $m \geqslant 1$ when $\gamma = 1$ and there is no restriction for $\gamma > 1$. Conversely we have:

THEOREM [23]. — *A ruled surface M has a blow-up $\tilde{M}$ which is a scalar-flat Kähler surface. Moreover, any further blow up of $\tilde{M}$ admits a scalar-flat Kähler metric.*

For simply connected manifold we have the following result:

THEOREM [34]. — *A smooth compact simply connected 4-manifold M admits a scalar-flat Kähler structure if, and only if, M is diffeomorphic to a $K3$-surface or to the connected sum $\mathbb{C}P^2\sharp k\overline{\mathbb{C}P^2}$ for some $k \geqslant 10$.*

Proof of Theorem C. — By Propositions 1 & 2, if $A = 0$ it is enough to show that $\widehat{E(\theta_i, \theta_j)} + F_{ij} = 0$ to get the integrability of $\mathbb{J}_{\lambda Id}$. Let (m, Q) be a point of Z. There exists an orthonormal basis $(\theta_1, \dots, \theta_4)$ over an open set $\mathcal{U}$ such that $I = J_M$ and $Q \simeq aI + bJ$, for some $(a, b) \in \mathbb{S}^1$. As J_M is Kähler, we know that $\Gamma^{\bullet}_{i\bullet} = \nabla_{\theta_i \bullet}$ belongs to the commutator of I, for all i. Hence, $[\Gamma^{\bullet}_{i\bullet}, Q]\frac{\partial}{\partial Q} = [\nabla_{\theta_i \bullet}, bJ]\frac{\partial}{\partial Q}$ is in the subspace of $T_Q\mathbb{S}^2$ generated by K. Viewing $\mathbb{S}^2$ as a subset of $\mathbb{R} \times \mathbb{C}$, with coordinates (a, z), the application $f = \lambda Id$ has the following form:

$$\begin{array}{cccc} f: & \mathcal{U} \times \mathbb{S}^2 & \longrightarrow & \mathcal{U} \times \mathbb{S}^2 \\ & \Big(m, \big(a, z\big)\Big) & \longmapsto & \Big(m, \big(f_1(a), f_2(a)\lambda z\big)\Big) \end{array}$$

Where f_1, f_2 only depend on $|\lambda|$. Thus $df(K) = f_2(a)\lambda\ K$. According to these notations we have at the point (m, Q):

$$\begin{aligned} d\pi(F_{ij}) &= -df([\nabla_{P\theta_i \bullet}, Q])\ \theta_j + df([\nabla_{P\theta_j \bullet}, Q])\ \theta_i - P\Big(df([\nabla_{\theta_j \bullet}, Q])\ \theta_i - \\ &\qquad df([\nabla_{\theta_i \bullet}, Q])\ \theta_j\Big) \\ &= f_2(a)\lambda\ \Big(-[\nabla_{P\theta_i \bullet}, bJ]\ \theta_j + [\nabla_{P\theta_j \bullet}, bJ]\ \theta_i - P\big([\nabla_{\theta_j \bullet}, bJ]\ \theta_i - \\ &\qquad [\nabla_{\theta_i \bullet}, bJ]\ \theta_j\big)\Big) \\ &= -[\nabla_{P\theta_i \bullet}, f_2(a)\lambda bJ]\ \theta_j + [\nabla_{P\theta_j \bullet}, f_2(a)\lambda bJ]\ \theta_i - P\Big([\nabla_{\theta_j \bullet}, f_2(a)\lambda bJ]\ \theta_i - \\ &\qquad [\nabla_{\theta_i \bullet}, f_2(a)\lambda bJ]\ \theta_j\Big) \\ &= -[\nabla_{P\theta_i \bullet}, P]\ \theta_j + [\nabla_{P\theta_j \bullet}, P]\ \theta_i - P\Big([\nabla_{\theta_j \bullet}, P]\ \theta_i - [\nabla_{\theta_i \bullet}, P]\ \theta_j\Big). \end{aligned}$$

One can conclude as in section A that $d\pi(F_{ij}) = -E(\theta_i, \theta_j)$ and $\mathbb{J}_{\lambda Id}$ is integrable.

Conversely, assume that $\mathbb{J}_{\lambda Id}$ is integrable. Proposition 3 implies that the scalar curvature is zero, hence $A = 0$. □

Study of the manifold $(Z, \mathbb{J}_{\lambda Id})$

When (M, g, J_M) is Kähler, the tangent bundle admits a $\mathbb{C}$-action which commutes with the holonomy group of the metric g. The action of any $\lambda \in \mathbb{S}^1$ lifts naturally to a smooth action on the total space Z inducing the identity on the base manifold M. This lift coincides with the homothety $\lambda^2 Id$. Therefore, $(Z, \mathbb{J}_{\lambda Id})$ is isomorphic to $(Z, \mathbb{J}_{Id})$ for each $\lambda \in \mathbb{S}^1$. Using Theorem A&C we recover the result from [19]: for any Kählerian surfaces (M, g, J_M), the metric g is anti-selfdual if, and only if, g is scalar-flat.

At least two cases may occur.
– Firstly, all the $(Z, \mathbb{J}_{\lambda Id})$ are biholomorphic to $\mathbb{J}_{Id}$. Thus there exists a 1-dimensional family of biholomorphisms of $(Z, \mathbb{J}_{Id})$. We will see in section F that this is the case for any bi-elliptic surface (quotient of a flat torus).
– Secondly, there is no one complex-parameter family of automorphims of $(Z, \mathbb{J}_{Id})$. Then, we have a 1-dimensional family of non isomorphic complex structures on Z. For example, if one blows-up at least 10 points in $\mathbb{C}P^2$, one gets $\mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}$ for some $k \geqslant 10$. This manifold admits a scalar-flat Kähler metric g [34] but there is no non trivial conformal map from $(\mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}, g)$ to itself. Thus, on its twistor space, there does not exist any 1-dimensional family of biholomorphisms. Therefore, the structures $(Z, \mathbb{J}_{\lambda Id})$, $\lambda \in \mathbb{C}^\star$, give a 1-dimensional family of non isomorphic complex structures.

D) Metric properties on M in terms of compatible complex structures on $(Z, \tilde{g})$

We can use the almost complex structures $\mathbb{J}_f$ to characterize some properties of the metric g on M. Indeed, by (the well known) Theorem A we have that g is anti-selfdual if and only if $\mathbb{J}_{Id}$ is integrable. We showed that a compact almost Hermitian manifold (M, g, J_M) is Kähler if and only if $\mathbb{J}_\infty$ is integrable; furthermore the integrability of $\mathbb{J}_{Id}$ and $\mathbb{J}_\infty$ is equivalent to (M, g, J_M) scalar-flat Kähler (cf. Proposition 8).

When limiting to the case where (M, g) is anti-selfdual, we can give a characterization of metrics which are scalar-flat in terms of compatible complex structures on $(Z, \tilde{g})$. According to the terminology of LeBrun these provide examples of optimal metrics, in compact case [26].

Theorem D. — *Let (M, g) be an anti-selfdual Riemannian manifold. The following are equivalent:*

- *the scalar curvature of g is flat;*
- *every $m \in M$ has an open neighborhood $\mathcal{U}$ such that Z admits, over $\mathcal{U}$, an integrable compatible complex structure $\mathbb{J}_f$ for at least one (and then infinitely many) morphism(s) $f \neq Id$.*

In other words, if (M, g) is an anti-selfdual metric with non zero scalar curvature then, even locally on Z, the only integrable almost complex structure among the $\mathbb{J}_f$'s is $\mathbb{J}_{Id}$. This result should be compared to the following result of Salamon:

Proposition [38] (see also [33]). — *A metric g on M is anti-selfdual if, and only if, locally around each point $m \in M$ there exist infinitely many compatible complex structures on (M, g).*

In a similar direction, Pontecorvo gives a conformal characterization of scalar-flat Kähler surfaces among anti-selfdual Hermitian surfaces. Indeed, let (M, g, J_M) be an anti-selfdual complex Hermitian manifold. The complex structure J_M on M defines a section $s : Z \longrightarrow M$ [15], whose image will be noted $\Sigma = s(M)$. Similarly, the hypersurface $\overline{\Sigma} = \sigma(\Sigma)$ of Z corresponds to the conjugate complex structure $-J_M$. Let X be the divisor $\Sigma + \overline{\Sigma}$ in Z and consider the holomorphic line bundle $[X]$. Denote by K_Z the canonical line bundle of $(Z, \mathbb{J}_{Id})$.

Proposition [30]. — *Let (M, g, J_M) be a Hermitian anti-selfdual manifold. The line bundles $[X]$ and $-\frac{1}{2}K_Z$ are isomorphic if and only if g is conformal to a scalar-flat Kähler metric.*

Notice that Theorem 1 and Proposition 3&4 give a non conformal characterization of compact scalar-flat Kähler surfaces.

Proposition 8. — *Let (M, g, J_M) be a compact almost Hermitian manifold. The following are equivalent:*

- *the metric g is scalar-flat Kähler;*
- *the compatible complex structures $\mathbb{J}_{Id}$ and $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ are integrable;*
- *the compatible complex structures $\mathbb{J}_{\lambda Id}$ and $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ are integrable.*

Proof. — A Kählerian surface (M, g, J_M) is scalar-flat if and only if g is anti-selfdual [19]. Then, it follows from Proposition 3&4 and Theorem 1 that: $\{\mathbb{J}_\infty$ and $\mathbb{J}_{\lambda Id}$ are integrable$\} \iff \{g$ is scalar-flat Kähler$\} \iff \{(M, g, J_M)$ is anti-selfdual Kähler$\} \iff \{\mathbb{J}_\infty$ and $\mathbb{J}_{Id}$ are integrable$\}$. □

Proof of Theorem D. — Let (M, g) be a scalar-flat anti-selfdual manifold, its twistor space is complex and (M, g) admits, locally, at least one complex structure J_M [38]. Then Theorem B ensures that the locally defined almost complex structure $\mathbb{J}_\infty$ on Z is integrable. Actually, as soon as (M, g) is anti-selfdual there are, locally, infinitely many integrable complex structures J_M on M and so, when g is also scalar-flat, there are infinitely many integrable complex structures $\mathbb{J}_\infty$ on Z.

Conversely, let (M, g) be a manifold with an anti-selfdual metric g having non zero scalar curvature. Let $f : Z \longrightarrow Z$ be a morphism such that $\mathbb{J}_f$ is integrable over an open set $\mathcal{U}$. Let (m, Q) be a point in $\pi^{-1}(\mathcal{U})$ and set

$f(m,Q) = P$. According to our notations, if $\mathcal{U}$ is small enough we can build an orthonormal basis $(\theta_1, \ldots, \theta_4)$ of vector fields on M such that $P = J = \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4$. Then there exists $(a,b,c) \in \mathbb{S}^2$ such that $Q = aI + bJ + cK$.

As $\mathbb{J}_f$ is integrable, $G(\theta_1, \theta_2)$ vanishes everywhere. In particular, at the point (m,Q) one obtains:

$$\begin{aligned} G(\theta_1,\theta_2) &= 0 \\ &= \tfrac{s}{12}[I - QK, Q] \\ &= \tfrac{2s}{12}\Big(acI - c(1-b)J + \big(b(1-b) - a^2\big)K\Big) \end{aligned} \Longrightarrow \begin{cases} ac = 0 \\ c(1-b) = 0 \\ b = a^2 + b^2 \end{cases}$$

Therefore we have $Q = J = P$ for every $(m,Q) \in \pi^{-1}(\mathcal{U})$, that is to say $f = Id$. □

E) Compatible complex structure on locally conformally Kähler surfaces

The aim of this section is to give a local description of the set $\mathcal{I}$ of integrable compatible complex structures on the twistor space $(Z, \tilde{g})$ of a compact locally conformally Kähler (abbreviated in *l.c.k.*) surface (M, g, J_M). This condition is equivalent to W^+ being degenerate, which means that at each point of M at least two eigenvalues of W^+ coincide.

We start by recalling the main results about the l.c.k. surfaces.

A result by Tricerri, generalizing the analogous result in the Kähler case, shows that it is enough to understand minimal complex surfaces.

Proposition [40]. — *A complex surfaces (M, g, J_M) is l.c.k if and only if the blow-up of M at a point is l.c.k.*

When the first Betti number b_1 is even, a l.c.k. surface is globally Kähler.

Proposition [42]. — *Every l.c.k. metric on a compact surface (M, J_M) with even first Betti number is globally conformally Kähler.*

When the first Betti number is odd and the Euler characteristic is zero, we have a classification due to Belgun, Gauduchon-Ornea, Tricerri, Vaisman.

Proposition [9]. — *The complete list of compact minimal l.c.k. surfaces with odd first Betti number and zero Euler characteristic is:*

i) *the properly elliptic surfaces (i.e. surfaces with* $\mathrm{Kod}(M) = 1$ *and* b_1 *odd);*
ii) *the Kodaira surfaces (i.e. surfaces with* $\mathrm{Kod}(M) = 0$ *and* b_1 *odd);*

iii) *the Hopf surfaces;*
iv) *the Inoue-Bombieri surfaces different from* $S^-_{n,u}$ *with* $u \notin \mathbb{R}$ [40].

When the first Betti number is odd and the Euler characteristic is non zero, the only other possible case is that of surfaces of class VII with $0 < \chi = b_2$ [7], for which there is (yet) no classification. (For some existence results see [18].)

Let $\mathbb{J}$ be a compatible almost complex structure on $(Z, \tilde{g})$. We say that $\mathbb{J}$ is semi-integrable if the vertical component of the Nijenhuis tensor is zero. Denote by $\mathcal{I}_{\frac{1}{2}}$ (resp. $\mathcal{I}$) the set of semi-integrable (resp. integrable) compatible complex structures on $(Z, \tilde{g})$. Propositions 1 and 2 give a necessary and sufficient condition for $\mathbb{J}$ to be semi-integrable, or integrable. The set $\mathcal{I}$ on a l.c.k. manifold (M, g, J) depends on the spectrum of the operator $A = W^+ + \frac{s}{12}$. Let κ be the conformal curvature defined in the proof of proposition 4. Then on an adapted basis we have:

$$A = W^+ + \frac{s}{12} Id = \begin{bmatrix} \frac{2\kappa}{12} & 0 & 0 \\ 0 & \frac{-\kappa}{12} & 0 \\ 0 & 0 & \frac{-\kappa}{12} \end{bmatrix} + \begin{bmatrix} \frac{s}{12} & 0 & 0 \\ 0 & \frac{s}{12} & 0 \\ 0 & 0 & \frac{s}{12} \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}.$$

Moreover J_M is actually an eigenvector of W^+ for the simple eigenvalue $\frac{\kappa}{6}$.

Theorem 2. — *Let* (M, g, J_M) *be a compact surface l.c.k., if we don't have* $x = y = 0$ *we note* $\frac{x}{y} \in \mathbb{R} \cup \{\infty\}$. *On an open set* $\mathcal{U}$ *of* M *:*

A) *We have* $x = y = 0$ *if, and only if, on* $\mathcal{U}$ *one of the following equivalent conditions hold:*
   i) (M, g, J_M) *is scalar-flat Kähler.*
   ii) g *anti-selfdual scalar-flat.*
   iii) *The compatible complex structures* $\mathbb{J}_{Id}$, $\mathbb{J}_\infty$ *and* $\mathbb{J}_{\lambda Id}$ *are integrable.*
   iv) *The cardinal of* $\mathcal{I}$ *is infiny.*

   This is the case globally if, and only if, (M, g, J_M) *is a flat torus (or a quotient), a* $K3$*-surface with a Calabi-Yau metric (or a quotient), a* $\mathbb{C}P^1$*-bundle over a Riemann surface* Σ_γ *of genus* $\gamma > 1$ *with the conformally flat Kähler metric which locally is a product of the (+1)-curvature metric on* $\mathbb{C}P^1$ *and (-1)-curvature metric on* Σ_γ [14], [31].

B) *We have* $\frac{x}{y} = \infty$ *if, and only if, on* $\mathcal{U}$ *one of the following equivalent conditions hold:*
   i) (M, g, J_M) *is Kähler with* $s \neq 0$.
   ii) $\mathcal{I} = \mathcal{I}_{\frac{1}{2}} = \{\mathbb{J}_{-\infty}, \mathbb{J}_\infty\}$.

This is the case globally on M if (M, g, J_M) is Kähler-Einstein not Ricci-flat (that is a Fano manifolds or a manifold where the canonical line bundle is ample).

C) *We have $|\frac{x}{y}| \leqslant 1$ if, and only if, on $\mathcal{U}$: $\mathcal{I}_{\frac{1}{2}} = \{\mathbb{J}_{e^{\pm i\theta}Id}\}$ where $\cos\theta = \frac{x}{y}$.*

D) *We have $\infty \neq |\frac{x}{y}| \geqslant 1$ if, and only if, on $\mathcal{U}$: $\mathcal{I}_{\frac{1}{2}} = \{\mathbb{J}_{u_1 Id}, \mathbb{J}_{u_2 Id}\}$ where $u_1 = \frac{1+\sin\theta}{\cos\theta}$, $u_2 = \frac{1-\sin\theta}{\cos\theta}$ and $\cos\theta = (\frac{x}{y})^{-1}$.*

Remark. — We have $\frac{x}{y} = 1$ if, and only if, (M, g, J_M) is anti-selfdual with $s \neq 0$. If it is the case globally then (M, J_M) must be in class VII [14]. We can find some global example of manifolds (M, g, J_M) with arbitrary $\frac{x}{y}$ in [5].

Proof of A. — The multiplicity of the eigenvalue 0 of A is equal to 3 $\iff$ $\kappa = s = 0$ $\iff$ (M, J_M, g) scalar-flat Kähler $\iff$ (M, J_M, g) anti-selfdual scalar-flat [14] $\iff$ $\mathbb{J}_{Id}$, $\mathbb{J}_\infty$ and $\mathbb{J}_{\lambda Id}$ integrable by proposition 8. The equivalence with condition iv) will be a consequence of (the rest of the proof of) the theorem. $\square$

Proof of B. — The multiplicity of the eigenvalue 0 of A is equal to 2 $\iff$ $\kappa = s \neq 0$ $\iff$ (M, J_M, g) Kähler with $s \neq 0$ $\iff$ $\mathcal{I} = \mathcal{I}_{\frac{1}{2}} = \{\mathbb{J}_\infty, \mathbb{J}_{-\infty}\}$ by Proposition 3. $\square$

Proof of C & D. — In those cases the matrix of A in a basis adapted to the decompostion $\mathbb{C} \otimes \bigwedge^+ = \mathbb{C}J_M \oplus^\perp (\bigwedge^{1,0} \oplus \bigwedge^{0,1})$ is $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}$ with $y \neq 0$. Let f such that $\mathbb{J}_f \in \mathcal{I}_{\frac{1}{2}}$, (m, Q) be any point of Z and $(\theta_1, ..., \theta_4)$ be a local frame such that $\begin{cases} J_M = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 \\ Q \in Vect(I, J) \end{cases}$. So there exist $(a, b), (\alpha, \beta, \gamma) \in \mathbb{S}^2$ such that $Q = aI + bJ$ and $P = f(Q) = \alpha I + \beta J + \gamma K$. In that case at the point (m, Q) we have :

$$\begin{aligned} G(\theta_1, \theta_2) &= 0 \\ &= [(1-\alpha^2)xI - \alpha\beta yJ - \alpha\gamma yK + Q(\gamma yJ - \beta yK), Q] \\ &= \Big[\Big((1-\alpha^2)x - b\beta y\Big)I + (a-\alpha)\beta yJ + (a-\alpha)\gamma yK, Q] \end{aligned}$$

$$\iff \begin{cases} (a-\alpha)\gamma y a = 0 \\ b\Big((1-\alpha^2)x - b\beta y\Big) = a(a-\alpha)\beta y \end{cases}$$

$$\iff \begin{cases} \gamma = 0 \\ \beta b x = y(1 - a\alpha) \\ \alpha^2 + \beta^2 = 1 \end{cases} \quad \text{ou} \quad \begin{cases} \alpha = a \\ \beta = \frac{x}{y} b \\ \beta^2 + \gamma^2 = b^2 \end{cases}$$

The resolution of $G(\theta_1,\theta_3)=0$ or $G(\theta_1,\theta_4)=0$ gives the same system. Two cases can happen first $|\frac{x}{y}|>1$ then the second system doesn't have any solution and the first one has two solutions. An easy computation enable us to verify that they correspond to $f_1=u_1 Id$ or $f_2=u_2 Id$.

On the other hand if $|\frac{x}{y}|<1$ then the second system gives two solutions which correspond to $f=e^{\pm i\theta}Id$, whereas the first system doesn't have any solution:

$$\begin{array}{ll} & 1-\alpha^2=\beta^2=\frac{y^2}{b^2x^2}(1-a\alpha)^2>\frac{(1-a\alpha)^2}{b^2} \\ \Longrightarrow & b^2-b^2\alpha^2>1+a^2\alpha^2-2a\alpha \\ \Longrightarrow & 0>(\alpha-a)^2. \end{array}$$

When $|\frac{x}{y}|=1$ both system give the same solutions. □

F) Example

Let $\mathbb{T}$ be a torus which is a quotient of $\mathbb{C}$ by the lattice $\mathbb{Z}\oplus i\mathbb{Z}$. Define (M,g,I) to be the quotient of the complex flat torus $\mathbb{T}^2=\mathbb{T}\times\mathbb{T}$ by the group $H=\mathbb{Z}/2\mathbb{Z}$ generated by an element h. If $(z_1,z_2)=(x_1+ix_2,x_3+ix_4)$ are the canonical coordinates on $\mathbb{C}\times\mathbb{C}$, we have:

$$h(z_1,z_2)=\Big(z_1+\frac{1}{2},-z_2\Big).$$

The manifold (M,g,I) is a bi-elliptic surface which is scalar-flat Kähler; denote by $Z\longrightarrow M$ its twistor space. In this section we will study in details this example, especially the integrability of $\mathbb{J}_f$. Thanks to Theorem 1, one knows that $\mathbb{J}_{Id}$, $\mathbb{J}_\infty$ and $\mathbb{J}_{\lambda Id}$ are integrable.

Let $(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4})$ be the canonical basis of $\mathbb{C}^2$ identified with $\mathbb{R}^4$. This furnishes a basis of vector fields on $\mathbb{T}^2$ and, consequently, a global trivialisation of its twistor space $Z_0\simeq\mathbb{T}^2\times\mathbb{S}^2$. Define another basis (on $\mathbb{T}^2$) by:

$$\theta_1+i\theta_2=\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\qquad\text{and}\qquad\theta_3+i\theta_4=e^{2i\pi x_1}(\frac{\partial}{\partial x_3}+i\frac{\partial}{\partial x_4}).$$

Then, $(\theta_1,\theta_2,\theta_3,\theta_4)$ is a global basis on $\mathbb{T}^2$ which goes down to a basis of M. This defines a new trivialisation of Z_0, denoted by $\tilde{M}\times\mathbb{S}^2$. The manifold Z is the quotient of $\tilde{M}\times\mathbb{S}^2$ by the group $\tilde{H}\simeq\mathbb{Z}/2\mathbb{Z}$, generated by $\tilde{h}$ acting as follows:

$$\begin{array}{rccc}\tilde{h}: & \tilde{M}\times\mathbb{S}^2 & \longrightarrow & \tilde{M}\times\mathbb{S}^2 \\ & \Big(m,Q\Big) & \longmapsto & \Big(h(m),Q\Big).\end{array}$$

Viewing $\mathbb{S}^2$ as a subset of $\mathbb{R} \times \mathbb{C}$ with coordinates (a, z), the identity map Ψ of Z_0 has the following form in these trivialisations:

$$\begin{array}{rccc} \Psi : & Z_0 \simeq \mathbb{T}^2 \times \mathbb{S}^2 & \longrightarrow & Z_0 \simeq \tilde{M} \times \mathbb{S}^2 \\ & \xi \simeq \Big(m, (a, z)\Big) & \longmapsto & \xi \simeq \Big(m, (a, e^{-2i\pi x_1} z)\Big). \end{array}$$

The matrix, in both basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$ and $(\theta_1, \theta_2, \theta_3, \theta_4)$, of the natural complex structure I on $\mathbb{T}^2$ is equal to $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. According to our notation, this is the infinity section.

Endow Z_0 with the complex structure of twistor space $\mathbb{J}_{Id}$. As $(\mathbb{T}^2, I)$ is hyperkähler, the projection $pr_2 : Z_0 \simeq \mathbb{T}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{C}P^1$ is a holomorphic submersion [14]. For $n \in \mathbb{Z}^*$ and $\lambda \in \mathbb{C}^*$, consider the application $f_n : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ equal to λz^n. Then there exist two applications f_1, f_2 depending only on $|\lambda|$ such that:

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{f_n} & \mathbb{S}^2 \\ (a, z) & \longrightarrow & \Big(f_1(a), \lambda f_2(a) z^n\Big) \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{C} \cup \{\infty\} & \xrightarrow{f_n} & \mathbb{C} \cup \{\infty\} \\ U = \frac{z}{1-a} & \longrightarrow & \lambda U^n \end{array}$$

Introduce now the pull back $Z_n = f_n^\star Z_0$:

$$\begin{array}{ccc} Z_n & \longrightarrow & Z_0 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle pr_2} \\ \mathbb{C}P^1 & \xrightarrow{f_n} & \mathbb{C}P^1 \end{array}$$

Since the fibration $Z_0 \longrightarrow \mathbb{C}P^1$ is topologically trivial, this is also the case for $Z_n \longrightarrow \mathbb{C}P^1$. Therefore one can identify the manifold Z_n with $\mathbb{T}^2 \times \mathbb{S}^2$ equipped with a complex structure denoted by J_n. If one considers the morphism $\tilde{f}_n = Id \times f_n : \mathbb{T}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{T}^2 \times \mathbb{S}^2$, which respects the fibration, one has $J_n = \mathbb{J}_{\tilde{f}_n}$.

We were wondering whether this complex structure goes down to Z, i.e.: does it commute with the action of the group $\tilde{H}$? We need to study

$\Psi \circ \tilde{f}_n \circ \Psi^{-1}$:

$$\begin{array}{ccc}
\Big(\mathbb{T}^2 \times \mathbb{S}^2, J_n\Big) & \xrightarrow{\quad \tilde{f}_n \quad} & \Big(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{J}_{Id}\Big) \\
\Big(m, (a, e^{2i\pi x_1} z)\Big) & \xrightarrow{\qquad\quad} & \Big(m, \big(f_1(a), \lambda f_2(a)(e^{2i\pi x_1} z)^n\big)\Big) \\
\Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Psi} \\
\Big(\tilde{M} \times \mathbb{S}^2, J_n\Big) & \xrightarrow{\Psi \circ \tilde{f}_n \circ \Psi^{-1}} & \Big(\tilde{M} \times \mathbb{S}^2, \mathbb{J}_{Id}\Big) \\
\Big(m, (a, z)\Big) & \xrightarrow{\qquad\quad} & \Big(m, \big(f_1(a), \lambda e^{2i\pi(n-1)x_1} f_2(a) z^n\big)\Big)
\end{array}$$

Thus, in the trivialisation of $Z_0 \simeq \tilde{M} \times \mathbb{S}^2$ associated to $(\theta_1, \theta_2, \theta_3, \theta_4)$, the complex structure J_n is $\mathbb{J}_{\Psi \circ \tilde{f}_n \circ \Psi^{-1}} = \mathbb{J}_{\lambda e^{2i\pi(n-1)x_1} z^n}$. It commutes with $\tilde{H}$ if and only if n is odd. Moreover, for n=1, $\tilde{f}_1$ is a biholomorphism. We have proved the following:

Proposition 9. — *For all $\lambda \in \mathbb{C}^\star$ the complex structures $\mathbb{J}_{\lambda z}$ on Z are biholomorphic. Furthermore, the compatible almost complex structures $\mathbb{J}_{\lambda e^{2i\pi(n-1)x_1} z^n}$ are integrable for odd n in $\mathbb{Z}^\star$.*

This proposition can be generalised to other bi-elliptic surfaces. A computation similar to the one made in Proposition 5 enables us to say that, for different integers n, these complex structures are not deformation of each other. This is consequence of the fact that they do not have the same Chern classes. Indeed, the first Chern class satisfies $c_1(\mathbb{J}_{\lambda e^{2i\pi(n-1)x_1} z^n}) = 2(n+1)h$. In [16], following an idea of LeBrun, we showed that for any hypercomplex manifold M there exist infinitely many complex structures on its twistor space $Z \simeq M \times \mathbb{S}^2$ which are not deformation of each other. Recall that the only compact hypercomplex surfaces are the torus, the $K3$-surfaces and the quaternionic Hopf surfaces [14]. The previous proposition can therefore be viewed as a generalisation of this result to bi-elliptic surfaces.

G) Higher dimension

The previous sections have focused on the 4-dimensional case. We now briefly give a generalization of Theorem 1 in higher dimension. Let $n > 1$ and (M, g) be an oriented $4n$-dimensional Riemannian manifold, not necessarily compact. An almost hypercomplex structure on (M, g) is a triple (I, J, K) of almost complex structures compatible with the orientation and the metric, such that $IJ = -JI = K$. When I, J, K are integrable one speaks about a hypercomplex structure. When they are Kähler one says that (M, g) is hyperkähler.

An almost quaternionic structure D on (M, g) is a rank 3 subbundle $D \subset \mathrm{End}(TM)$ which is locally spanned by an almost hypercomplex structure $H = (I, J, K)$; such a triple is called a local admissible basis. For $n > 1$, one says that (M, g, D) is a quaternionic structure if there exists a torsion free connection ∇ on TM preserving D. If one can choose ∇ to be the Levi-Civita connection, (M, g, D) is called quaternionic Kähler. This is equivalent to saying that the holonomy group of g is contained in $Sp(1)Sp(n)$ [11].

A compatible almost complex structure on (M, g, D) is a section J_M of $D \longrightarrow M$ such that $J_M^2 = -Id$.

Let (M, g, D) be a Riemannian almost quaternionic $4n$-manifold. One can define a scalar product on D by saying that a local admissible basis of D is orthonormal. One can then define the twistor space $Z \longrightarrow M$, which is the unit sphere bundle of D. This is a locally trivial bundle over M with fiber $\mathbb{S}^2$ and structure group $SO(3)$. As in the introduction, one can define a natural metric $\tilde{g}$ and look for the compatible almost complex structures on $(Z, \tilde{g})$ which are integrable. When (M, g, D, J_M) is quaternionic Kähler with a compatible almost complex structure J_M, its twistor space $(Z, \tilde{g})$ admits different compatible almost complex structures: $\mathbb{J}_\sigma$, $\mathbb{J}_{Id}$, $\mathbb{J}_\infty$, $\mathbb{J}_{\lambda Id}$, defined as previously. The main result of this section is the following, where no hypothesis of compacity is made.

Theorem 3. — *Let (M, g, D) be a quaternionic Kähler manifold.*

A) *The almost complex structure $\mathbb{J}_\sigma$ is never integrable.*
B) *The almost complex structure $\mathbb{J}_{Id}$ is always integrable* [35].
C) *If (M, g, D, J_M) is a compatible almost complex quaternionic Kähler manifold the almost complex structure $\mathbb{J}_\infty$ is integrable if, and only if:*
   i) *J_M is integrable;*
   ii) *g is scalar-flat.*
D) *If (M, g, D, J_M) is a quaternionic Kähler manifold with a compatible Kählerian complex structure J_M then, for all $\lambda \notin \{0, 1\}$, the complex structure $\mathbb{J}_{\lambda Id}$ is integrable if, and only if, g is scalar-flat.*
E) *Let (M, g, D) be a quaternionic Kähler manifold. Then the scalar curvature is flat if, and only if, one (and then any) $m \in M$ has an open neighborhood $\mathcal{U}$ such that $(Z, \tilde{g})$ admits over $\mathcal{U}$ an integrable compatible complex structure different from $\mathbb{J}_{Id}$.*

Any quaternionic Kähler manifold which is scalar-flat is locally hyperkähler [11]. Thus, part E of the previous theorem yields a characterization of locally hyperkähler manifolds among quaternionic Kähler's in terms of twistor spaces.

It is possible to give a simpler version of that theorem in the compact case because of the following result.

Proposition [32]. — *In the compact case any compatible complex structure J_M on a quaternionic Kähler manifold (M, g, D) is automatically scalar-flat Kähler.*

In particular, in the compact case, Theorem 3 has the following corollary.

Corollary 3. — *Let (M, g, D, J_M) be a compact quaternionic Kähler manifold with a compatible almost complex structure. Then J_M is integrable if, and only if, $\mathbb{J}_\infty$ is integrable. In this case $\mathbb{J}_{\lambda Id}$ is integrable for all $\lambda \in \mathbb{C}^\star$.*

Proof of Theorem 3. — Proposition 1 and Proposition 2 remain true in dimension $4n$. Since σ is an antiholomorphic involution when restricted to the fibers, part A can be easily proved.

The proof of part B is the same as the one given in dimension 4. Notice first that $d\pi F_{ij} = -E(\theta_i, \theta_j)$ for all $(i, j) \in \{1, \dots, 4n\}$. It remains to show that $G(\theta_i, \theta_j) = 0$ for all $i, j \in \{1, \dots, 4n\}$. To get that result we use the following lemma.

Lemma 3 [11]. — *Let $r(.,.)$ be the Ricci tensor. For all $(X, Y) \in TM$ one has:*

$$\begin{aligned} [R(X,Y), I] &= \gamma(X,Y)J - \beta(X,Y)K \\ [R(X,Y), J] &= -\gamma(X,Y)I + \alpha(X,Y)K \\ [R(X,Y), K] &= \beta(X,Y)I - \alpha(X,Y)J \end{aligned} \quad \textit{with} \quad \begin{cases} \alpha(X,Y) = \frac{2}{n+2} r(IX, X) \\ \beta(X,Y) = \frac{2}{n+2} r(JX, X) \\ \gamma(X,Y) = \frac{2}{n+2} r(KX, X) \end{cases}$$

Let $(m, I) \in Z$ and (I, J, K) be a local admissible basis. Then Lemma 3 yields:

$$\begin{aligned} G(\theta_i, \theta_j) &= \Big[R\Big(\theta_i \wedge \theta_j - I\theta_i \wedge I\theta_j\Big) + IR\Big(\theta_i \wedge I\theta_j + I\theta_i \wedge \theta_j\Big), I\Big] \\ &= \gamma(\theta_i, \theta_j)J - \beta(\theta_i, \theta_j)K - \gamma(I\theta_i, I\theta_j)J + \beta(I\theta_i, I\theta_j)K \\ &\quad + \gamma(I\theta_i, \theta_j)K + \beta(I\theta_i, \theta_j)J + \gamma(\theta_i, I\theta_j)K + \beta(\theta_i, I\theta_j)J \end{aligned}$$

But any quaternionic Kähler manifold is Einstein [10], hence $r = \frac{s}{4}g$, where s is the scalar curvature of g. One then has, for all (θ_i, θ_j):

$$\begin{aligned} G(\theta_i,\theta_j) &= \tfrac{2s}{4(n+2)}\Big(\big(2g(K\theta_i,\theta_j) - 2g(K\theta_i,\theta_j)\big)J \\ &\qquad\qquad + \big(2g(J\theta_i,\theta_j) - 2g(J\theta_i,\theta_j)\big)K\Big) \\ &= 0. \end{aligned}$$

To prove part C observe that, as in dimension 4: $\{\mathbb{J}_\infty \text{ integrable}\} \iff \{E(\theta_i,\theta_j) = G(\theta_i,\theta_j) = 0\} \iff \{J_M \text{ integrable and } G(\theta_i,\theta_j) = 0\}$. Since (M, g, Q) is Einstein, (M, g, Q) scalar-flat implies (M, g, Q) Ricci-flat and $G(\theta_i, \theta_j) = 0$. The converse is a consequence of part E: if $\mathbb{J}_\infty$ integrable then $s = 0$.

To get part D we use the technique of dimension 4 to prove that $d\pi(F_{ij}) = -E(\theta_i,\theta_j)$. So $\mathbb{J}_{\lambda Id}$ is integrable as soon as s=0. The converse is again a consequence of part E.

Proof of E: suppose that the scalar curvature s of (M, g, D) is non zero. Let $f : Z \longrightarrow Z$ be a morphism such that $\mathbb{J}_f$ is integrable over an open set $\mathcal{U}$. Let (m, Q) be a point in $\pi^{-1}(\mathcal{U})$ and set $f(m, Q) = P$. If $\mathcal{U}$ is small enough there exists an orthonormal basis $(\theta_1, \ldots, \theta_{4n})$ and a local admissible basis (I, J, K) such that $P = J$. Write $Q = aI + bJ + cK$ with $(a, b, c) \in \mathbb{S}^2$.

As $\mathbb{J}_f$ is integrable we have $G(\theta_1, \theta_2) = 0$ everywhere. In particular at the point (m, Q) :

$$\begin{aligned} G(\theta_1,\theta_2) &= 0 \\ &= \Big[R(\theta_1\wedge\theta_2 + \theta_3\wedge\theta_4) - QR(\theta_1\wedge\theta_4 + \theta_2\wedge\theta_3), Q\Big] \\ &= \tfrac{2s}{4(n+2)}(-2cJ + 2bK) - Q\Big[R(\theta_1\wedge\theta_4 + \theta_2\wedge\theta_3), Q\Big] \\ &= \tfrac{s}{n+2}\Big(-cJ + bK - Q(-bI + aJ)\Big) \\ &= \tfrac{s}{n+2}\Big(acI + c(b-1)J + (b-1)K\Big) \end{aligned}$$

Hence $Q = J = P$ for any $(m, Q) \in \pi^{-1}(\mathcal{U})$, that is $f = Id$.

The converse is the same as the one given in section D.

Indeed, a quaternionic Kähler manifolds (M, g, D) admits, locally, infinitly many compatible complex structures J_M (for example [1]). $\square$

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