ANNALES

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Márcio Henrique BATISTA DA SILVA

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Tome 61, nº 4 (2011), p. 1299-1322.

<http://aif.cedram.org/item?id=AIF_2011__61_4_1299_0>

Ann. Inst. Fourier, Grenoble
61, 4 (2011) 1299-1322

SIMONS TYPE EQUATION IN $\mathbb{S}^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$ AND APPLICATIONS

by Márcio Henrique BATISTA DA SILVA

Abstract. — Let Σ^2 be an immersed surface in $M^2(c) \times \mathbb{R}$ with constant mean curvature. We consider the traceless Weingarten operator ϕ associated to the second fundamental form of the surface, and we introduce a tensor S, related to the Abresch-Rosenberg quadratic differential form. We establish equations of Simons type for both ϕ and S. By using these equations, we characterize some immersions for which $|\phi|$ or $|S|$ is appropriately bounded.

Résumé. — Soit Σ^2 une surface immergée dans $M^2(c) \times \mathbb{R}$ avec une courbure moyenne constante. Nous considérons l'opérateur de Weingarten à trace nulle ϕ associé à la seconde forme fondamentale de la surface et nous introduisons un tenseur S, liés à la forme quadratique de Abresch-Rosenberg. Nous établissons les équations de type Simons pour ϕ et S. En utilisant ces équations, nous caractérisons les immersions pour lesquelles $|\phi|$ ou $|S|$ sont bornés.

1. Introduction

In 1994, using the traceless Weingarten operator $\phi = A - HI$ associated to an immersed hypersurface $M^n \looparrowright \mathbb{S}^{n+1}$, H. Alencar and M. do Carmo, see [2], proved that

Theorem. — *Let $M^n \looparrowright \mathbb{S}^{n+1}$ be an immersed hypersurface. If M^n is compact and orientable with constant mean curvature H and*

$$|\phi|^2 \leqslant B_H,$$

where B_H is the square of the positive root of

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2+1).$$

Then:

Keywords: Surface with constant mean curvature, Simons type equation, Codazzi's equation.
Math. classification: 53A10, 53C42.

(a) *Either* $|\phi|^2 = 0$ *(and* M^n *is totally umbilic) or* $|\phi|^2 = B_H$*.*

(b) *The* $H(r)$*-tori* $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ *with* $r^2 \leqslant \frac{n-1}{n}$ *are the only hypersurfaces with constant mean curvature* H *and* $|\phi|^2 = B_H$*.*

Motivated by this result we study this problem for surfaces in $M^2(c) \times \mathbb{R}$ with $c = \pm 1$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$.

We begin by using the traceless Weingarten operator ϕ associated to an immersed surface $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$.

In [1], the authors defined the quadratic differential form

$$Q(X,Y) = 2H\langle AX, Y\rangle - c\langle X, \partial_t\rangle\langle Y, \partial_t\rangle,$$

and its (2,0)-part

$$Q^{(2,0)}(X,Y) = \frac{1}{2}(Q(X,Y) - Q(JX,JY)) - \frac{1}{2}i(Q(JX,Y) + Q(X,JY)),$$

where J is the standard counter-clockwise rotation operator.

Using this notation, Abresch and Rosenberg proved

Theorem. — *(Thm. 1 in* [1]*) Let* $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ *be an immersed surface with constant mean curvature. Then its quadratic differential* $Q^{(2,0)}$ *is holomorphic on the surface* Σ^2*.*

Inspired in the quadratic differential form Q introduced by Abresch and Rosenberg, we study, in section 3, a special tensor S defined by

$$SX = 2HAX - c\langle X, T\rangle T + \frac{c}{2}(1-\nu^2)X - 2H^2X, \tag{1.1}$$

where $X \in T_p\Sigma$, A is the Weingarten operator associated to the second fundamental form, H is the mean curvature, T is the tangential component of the parallel field ∂_t, tangent to $\mathbb{R}$ in $M^2(c) \times \mathbb{R}$, and $\nu = \langle N, \partial_t\rangle$.

The tensor S is the traceless tensor associated with the quadratic differential Q. In fact,

$$\begin{aligned}\langle SX, Y\rangle &= 2H\langle AX, Y\rangle - c\langle X, T\rangle\langle Y, T\rangle + \frac{c}{2}(1-\nu^2)\langle X, Y\rangle - 2H^2\langle X, Y\rangle \\ &= Q(X,Y) - \frac{trQ}{2}\langle X, Y\rangle.\end{aligned}$$

We will prove that this operator satisfies Codazzi's equation, provided H is constant, with vanishing trace. Moreover, we remark that any surface with $|S| = 0$ and constant mean curvature is very interesting, because the $Q^{(2,0)}$ of these surfaces vanishes.

In [1], Theorem 3, p. 143, the authors described four distinct classes of complete, possibly immersed, constant mean curvature surfaces $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with vanishing of their quadratic differential $Q^{(2,0)}$.

More precisely, the four classes are

(*i*) Σ^2 is an embedded rotationally invariant constant mean curvature sphere S_H^2;

(*ii*) Σ^2 is a convex rotationally invariant constant mean curvature graph D_H^2 over the horizontal leaf $M^2(c) \times \{t_0\}$;

(*iii*) Σ^2 is an embedded annulus, rotationally invariant constant mean curvature surface C_H^2 with two asymptotically conical ends;

(*iv*) Σ^2 is the embedded constant mean curvature surface P_H^2; it is an orbit under some two dimensional solvable subgroup of ambient isometries.

The surface in (i) was known to W.T. Hsiang and W.Y. Hsiang, in [6], and to R. Pedrosa and M. Ritoré, in [7]. We shall refer to S_H^2 as the embedded rotationally invariant constant mean curvature spheres. In this paper we will call the surfaces described in [1] by Abresch-Rosenberg surfaces.

Remark. 1. — In $\mathbb{S}^2 \times \mathbb{R}$ only the spheres S_H^2 occur.

We obtain an equation of Simons type for S and apply it in some particular cases:

THEOREM 1.1. — *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature H and S as defined in* (1.1). *Then,*

$$\langle (\nabla^2 S)x, y\rangle = 2c\nu^2\langle Sx, y\rangle + 2H\langle Ax, Sy\rangle - \langle A^2x, Sy\rangle + \\ + \langle Ay, SAx\rangle - \langle Ax, y\rangle tr(AS)$$

and

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T\rangle\right) + \\ + c|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

Let us consider the polynomial $p_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{4H^2-1}{2}\right)$. When H is greater than one half there is a positive root for p_H. Let L_H be this positive root. One has:

THEOREM 1.2. — *Let $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature H greater than one half. If*

$$\Sigma^2 \textit{ is complete and } \sup_{\Sigma} |S| < L_H,$$

or

$$\Sigma^2 \textit{ is closed and } |S| \leqslant L_H,$$

then $\Sigma^2 = S^2_H$*, i.e,* Σ^2 *is an embedded rotationally invariant constant mean curvature sphere.*

Remark. 2. — The number L_H is $\dfrac{\sqrt{2}H(4H^2-1)}{\sqrt{16H^4-4H^2+1}+1}$.

Let us consider the polynomial

$$q_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left(\frac{8H^4-12H^2-1}{4H^2}\right).$$

When H is greater than $\sqrt{\dfrac{3+\sqrt{11}}{4}}$, there is a positive root for q_H. Let M_H be this positive root.

Theorem 1.3. — *Let* $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ *be an immersed surface with constant mean curvature* H *greater than* $\sqrt{\dfrac{3+\sqrt{11}}{4}} \approx 1.25664$*. If*

$$\Sigma^2 \text{ is complete and } \sup_\Sigma |S| < M_H,$$

or

$$\Sigma^2 \text{ is closed and } |S| \leqslant M_H,$$

then $\Sigma^2 = S^2_H$*, i.e,* Σ^2 *is an embedded rotationally invariant constant mean curvature sphere.*

Remark. 3. — The number M_H is $\dfrac{8H^4-12H^2-1}{\sqrt{2}H(\sqrt{16H^4-24H^2-1}+1)}$.

Remark. 4. — Besides Theorems 1.2 and 1.3, we obtain in section 4 further applications of Simons equation of Theorem 1.1.

Acknowledgements. I would like to thank Professors M. do Carmo and H. Alencar for encouragement and for many helpful suggestions. I also want to thank the referee and H. Rosenberg for useful suggestions.

2. Preliminaries

Let $\Sigma^2 \looparrowright M^3$ be an immersed surface. Let $\overline{\nabla}$ denote the Levi-Civitá connection on M^3 and let ∇ denote the Levi-Civitá connection on Σ for the induced metric.

Generally speaking, objects defined on M^3 will be denoted by the same symbols as the corresponding objects defined on Σ plus a bar over the symbol.

The Riemannian metric extends to natural inner products on spaces of tensors and the above connections induce natural covariant derivatives of tensor fields. For example, for $\{e_1, e_2\}$ a geodesic frame at $p \in \Sigma^2$ and a tensor ψ on Σ^2, we have

$$\nabla^2\psi(p) = \sum_{i=1}^{2}(\nabla_{e_i}\nabla_{e_i}\psi)(p).$$

For more details about covariant derivatives of tensor fields see [8], sections 1 and 2.

We adopt the following convention for the curvature tensor: if $x, y, z \in T_p\Sigma$, we define $R_{x,y}z$ by

$$R_{x,y}z = R(X,Y)Z(p) = (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)(p),$$

for any local vector fields which extend the given vectors x, y, z.

The second fundamental form is defined by $\alpha(X,Y) = (\overline{\nabla}_X Y)^\perp$ and the associated Weingarten operator is given by $Av = -(\overline{\nabla}_v N)^T$, where N is a unit normal field on Σ^2. We use the Weingarten operator to define the following operators

$$\begin{aligned}(2.1)\quad \langle \bar{R}(A)x, y\rangle := \sum_{i=1}^{2}(-\langle Ax, \bar{R}_{e_i,y}e_i\rangle - \langle Ay, \bar{R}_{e_i,x}e_i\rangle +\\ + \langle Ay, x\rangle\langle N, \bar{R}_{e_i,N}e_i\rangle - 2\langle Ae_i, \bar{R}_{e_i,x}y\rangle)\end{aligned}$$

and

$$\langle \bar{R}'x, y\rangle := \sum_{i=1}^{2}\{\langle(\bar{\nabla}_x\bar{R})_{e_i,y}e_i, N\rangle + \langle(\bar{\nabla}_{e_i}\bar{R})_{e_i,x}y, N\rangle\},$$

where $\{e_1, e_2\}$ is a orthonormal basis of $T_p\Sigma$.

With this notation we have the following result:

Theorem 2.1. — *Let $\Sigma^2 \looparrowright M^3$ be an immersed surface with constant mean curvature H. For any $x, y \in T_p\Sigma$ we have*

$$\begin{aligned}(2.2)\quad \langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax, y\rangle + \langle \bar{R}(A)x, y\rangle +\\ + \langle \bar{R}'x, y\rangle + 2H\langle \bar{R}_{N,x}y, N\rangle + 2H\langle Ax, Ay\rangle.\end{aligned}$$

Proof. — See Theorem 2 in [3] and observe that the codimension is one. □

We will also use the result known as the Omori-Yau Maximum Principle whose proof can be found in [10], Theorem 1.

Theorem 2.2 (Omori-Yau Maximum Principle). — *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is bounded from above, then there exists a sequence of points $\{p_j\} \in M$ such that*

$$\lim_{j\to\infty} u(p_j) = \sup_M u, \quad |\nabla u|(p_j) < \frac{1}{j}, \text{ and } \Delta u(p_j) < \frac{1}{j}.$$

Let us recall Gauss' equation for Σ^2 in $M^2(c)\times\mathbb{R}$:

$$\begin{aligned} R(Y,X)Z = \langle AX,Z\rangle AY - \langle AY,Z\rangle AX + c(\langle X,Z\rangle Y - \langle Y,Z\rangle X + \\ - \langle Y,T\rangle\langle X,Z\rangle T - \langle X,T\rangle\langle Z,T\rangle Y + \\ + \langle X,T\rangle\langle Y,Z\rangle T + \langle Y,T\rangle\langle Z,T\rangle X), \end{aligned} \tag{2.3}$$

where X,Y,Z in $T_p\Sigma$, N is a unitary normal field on Σ^2 and T is the tangential component of the parallel field ∂_t. For more details see [5].

3. Simons' equation in $M^2(c)\times\mathbb{R}$

In this section we will obtain an equation of Simons type for the traceless Weingarten operator ϕ and for the tensor S defined in (1.1).

Let $M^2(c)\times\mathbb{R}$, where $M^2(-1)=\mathbb{H}^2$ and $M^2(1)=\mathbb{S}^2$. In this case we have that $\bar{R}'=0$, because $M^2(c)\times\mathbb{R}$ is locally symmetric.

In Lemmas 3.1 and 3.2 we will consider an immersed surface $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ with constant mean curvature H where A is the Weingarten operator associated to the second fundamental form on Σ^2.

Lemma 3.1. — *Denoting the identity by I, we have that*

$$\bar{R}(A) = c(5\nu^2-1)A - 4cH\nu^2 I.$$

Proof. — Consider an orthonormal basis $\{e_1,e_2\}$ in $T_p\Sigma^2$ such that $Ae_i = k_ie_i$, $i=1,2$. Consider $x,y\in T_p\Sigma$. We have

$$x = x_1e_1 + x_2e_2 \text{ e } y = y_1e_1 + y_2e_2.$$

Computing the first sum in (2.1)

$$\begin{aligned} \sum_{i=1}^{2}\langle \bar{R}_{e_i,y}e_i, Ax\rangle &= k_2x_2y_2\langle \bar{R}_{e_1,e_2}e_1,e_2\rangle + k_1x_1y_1\langle \bar{R}_{e_2,e_1}e_2,e_1\rangle \\ &= -\bar{K}_\Sigma\,(k_2x_2y_2 + k_1x_1y_1) = -\bar{K}_\Sigma\langle Ax,y\rangle, \end{aligned}$$

where $\bar{K}_\Sigma = \langle \bar{R}_{e_1,e_2}e_2,e_1\rangle$.

Hence,

$$\sum_{i=1}^{2}\langle \bar{R}_{e_i,y}e_i, Ax\rangle = -\bar{K}_\Sigma\langle Ax, y\rangle. \tag{3.1}$$

It's simple see that

$$\sum_{i=1}^{2}\langle \bar{R}_{e_i,x}e_i, Ay\rangle = -\bar{K}_\Sigma\langle Ax, y\rangle. \tag{3.2}$$

In the third sum in (2.1) we have

$$\begin{aligned}\langle \bar{R}_{e_i,N}e_i, N\rangle &= -c\{(1-\langle e_i,\partial_t\rangle^2)(1-\nu^2) - \nu^2\langle e_i,\partial_t\rangle^2\}\\ &= -c\{1-\nu^2-\langle e_i,\partial_t\rangle^2\}.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{2}\langle \bar{R}_{e_i,N}e_i, N\rangle = -c(1-\nu^2). \tag{3.3}$$

To finish, we computing the fourth sum.

$$\begin{aligned}\sum_{i=1}^{2}\langle \bar{R}_{e_i,x}y, Ae_i\rangle &= \bar{K}_\Sigma\left(k_1x_2y_2 + k_2x_1y_1\right)\\ &= \bar{K}_\Sigma\left([2H-k_2]x_2y_2 + [2H-k_1]x_1y_1\right)\\ &= \bar{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right),\end{aligned}$$

where we used that $2H = k_1 + k_2$.

Thus,

$$\sum_{i=1}^{2}\langle \bar{R}_{e_i,x}y, Ae_i\rangle = \bar{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right). \tag{3.4}$$

Now, we need computing $\bar{K}_\Sigma$. Using the tensor of curvature in $M^2(c)\times\mathbb{R}$ we have:

$$\bar{K}_\Sigma = \langle \bar{R}_{e_1,e_2}e_2, e_1\rangle = c\left(1-\langle e_1,T\rangle^2 - \langle e_2,T\rangle^2\right) = c(1-|T|^2).$$

Therefore,

$$\bar{K}_\Sigma = c\nu^2. \tag{3.5}$$

Substituting (3.1), (3.2),(3.3) and (3.4) into (2.1), obtain

$$\langle \bar{R}(A)x,y\rangle = 2\bar{K}_\Sigma\langle Ax,y\rangle - c(1-\nu^2)\langle Ax,y\rangle - 2\bar{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right).$$

Using (3.5) we obtain

$$\langle \bar{R}(A)x,y\rangle = 5c\nu^2\langle Ax,y\rangle - c\langle Ax,y\rangle - 4c\nu^2H\langle x,y\rangle.$$

Thus,

$$\bar{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2 I.$$

□

Lemma 3.2. — $\langle \bar{R}_{N,x}y, N\rangle = -c\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\}$.

Proof. — We observe that

$$\langle x^*, y^*\rangle = \langle x,y\rangle - \langle x,T\rangle\langle y,T\rangle,$$

$$\langle x^*, N^*\rangle = \nu\langle x,T\rangle$$

and

$$\langle N^*, N^*\rangle = 1 - \nu^2,$$

where we have used $v^* = v - \langle v, \partial_t\rangle\partial_t$ for any $v \in T_p(M^2(c)\times\mathbb{R})$.
It follows that

$$\begin{aligned}\langle \bar{R}_{N,x}y, N\rangle &= -c\{\langle N^*, x^*\rangle\langle N^*, y^*\rangle - \langle N^*, N^*\rangle\langle x^*, y^*\rangle\}\\ &= -c\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\}.\end{aligned}$$

This concludes the proof. □

Proposition 3.3. — *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature H and let A be the Weingarten operator associated to the second fundamental form on Σ^2. Then,*

$$\begin{aligned}\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax, y\rangle + c(5\nu^2-1)\langle Ax,y\rangle - 4cH\nu^2\langle x,y\rangle +\\ - 2cH\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\} + 2H\langle Ax, Ay\rangle,\end{aligned}$$

where $\nu = \langle N, \partial_t\rangle$.

Proof. — Consider equation (2.2)

$$\begin{aligned}\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax, y\rangle + \langle \bar{R}(A)x, y\rangle\\ + \langle \bar{R}'x, y\rangle + 2H\langle \overline{R}_{N,x}y, N\rangle + 2H\langle Ax, Ay\rangle.\end{aligned}$$

Now, we use Lemmas 3.1 and 3.2 and the fact that $\bar{R}' = 0$ to obtain

$$\begin{aligned}\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax, y\rangle + c(5\nu^2-1)\langle Ax,y\rangle - 4cH\nu^2\langle x,y\rangle +\\ - 2Hc\{\langle x,T\rangle\rangle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\} + 2H\langle Ax, Ay\rangle.\end{aligned}$$

□

Consider two tensors V, W on Σ^2. We define the inner product $\langle V, W\rangle$ at $p \in \Sigma^2$ as

$$\langle V, W\rangle = \sum_{i=1}^{2}\langle Ve_i, We_i\rangle,$$

where $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$.

Corollary 3.4. — *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature and let A be the Weingarten operator associated to the second fundamental form on Σ^2. Then,*

(a) $\langle \nabla^2 A, I\rangle = 0.$

(b) $\langle \nabla^2 A, A\rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T\rangle + +4cH^2|T|^2 + 2Htr(A^3).$

Proof. — Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p\Sigma$. We use the definition of the inner product between tensors and the expression in Proposition 3.3 to obtain

$$\begin{aligned}\langle \nabla^2 A, A\rangle &= \sum_{i=1}^{2}\langle(\nabla^2 A)e_i, Ae_i\rangle = -|A|^2\sum_{i=1}^{2}\langle Ae_i, Ae_i\rangle +\\ &+ c(5\nu^2-1)\sum_{i=1}^{2}\langle Ae_i, Ae_i\rangle - 4cH\nu^2\sum_{i=1}^{2}\langle Ae_i, e_i\rangle - 2cH\{\sum_{i=1}^{2}\langle AT, e_i\rangle\langle e_i, T\rangle +\\ &- \langle T, T\rangle\sum_{i=1}^{2}\langle Ae_i, e_i\rangle\} + 2H\sum_{i=1}^{2}\langle A^2 e_i, Ae_i\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \nabla^2 A, A\rangle = -|A|^4 + c(5\nu^2-1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T\rangle\\ + 4cH^2|T|^2 + 2Htr(A^3).\end{aligned}$$

Using the definition of the inner product and Proposition 3.3 we obtain

$$\begin{aligned}\langle \nabla^2 A, I\rangle &= \sum_{i=1}^{2}\langle(\nabla^2 A)e_i, e_i\rangle = -|A|^2\sum_{i=1}^{2}\langle Ae_i, e_i\rangle +\\ &+ c(5\nu^2-1)\sum_{i=1}^{2}\langle Ae_i, e_i\rangle - 8cH\nu^2 - 2cH\{\sum_{i=1}^{2}\langle T, e_i\rangle\langle e_i, T\rangle +\\ &- 2\langle T, T\rangle\} + 2H\sum_{i=1}^{2}\langle A^2 e_i, e_i\rangle.\end{aligned}$$

Therefore,

$$\langle \nabla^2 A, I\rangle = -2H|A|^2 + c(5\nu^2-1)2H - 8cH\nu^2 + 2cH\langle T,T\rangle + 2H|A|^2 = 0,$$

where we have used that $\nu^2 + |T|^2 = 1$. □

PROPOSITION 3.5. — *Let $\Sigma \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature H and let ϕ be the traceless Weingarten operator, then*

(*a*) $|\phi|^2 = |A|^2 - 2H^2$.
(*b*) $\nabla\phi = \nabla A$.
(*c*) $trA^3 = 3H|\phi|^2 + 2H^3$.

Proof. — The proof of item (a) is:

$$\begin{aligned}|\phi|^2 = \langle\phi,\phi\rangle = \langle A-HI, A-HI\rangle &= \langle A,A\rangle - 2H\langle A,I\rangle + H^2\langle I,I\rangle\\ &= |A|^2 - 4H^2 + 2H^2 = |A|^2 - 2H^2,\end{aligned}$$

where $\langle A,I\rangle = 2H$ and $\langle I,I\rangle = 2$.
To prove item (b), we consider tangent fields X, Y. Then,

$$\begin{aligned}(\nabla_X\phi)Y = (\nabla_X A)Y - (\nabla_X(HI))Y &= (\nabla_X A)Y - \nabla_X HI(Y) + H\nabla_X Y\\ &= (\nabla_X A)Y - H\nabla_X Y - X(H)Y + H\nabla_X Y = (\nabla_X A)Y,\end{aligned}$$

because H is constant.
Finally, the proof of item(c) is:

$$\begin{aligned}tr(A^3) &= \sum_{i=1}^{2}\langle A^3 e_i, e_i\rangle = \sum_{i=1}^{2}\langle(\phi+HI)^3 e_i, e_i\rangle\\ &= \sum_{i=1}^{2}\langle(\phi^3 + 3H\phi^2 + 3H^2\phi + H^3 I)e_i, e_i\rangle = 3H|\phi|^2 + 2H^3,\end{aligned}$$

because $tr\phi = tr\phi^3 = 0$. □

Next we shall derive an equation of Simons type for the traceless Weingarten operator ϕ:

THEOREM 3.6. — *Let $\Sigma \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature H and let ϕ be the traceless Weingarten operator. Then*

$$\langle\nabla^2\phi,\phi\rangle = -|\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle\phi T, T\rangle$$

and

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle\phi T, T\rangle.$$

Proof. — We use Proposition 3.5 to show that

$$\langle \nabla^2\phi, \phi\rangle = \langle \nabla^2 A, A - HI\rangle = \langle \nabla^2 A, A\rangle - H\langle \nabla^2 A, I\rangle.$$

Now, we use Corolarry 3.4 to obtain

$$\langle \nabla^2\phi, \phi\rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 + 2cH\langle AT, T\rangle + \\ + 4cH^2|T|^2 + 2Htr(A^3).$$

Therefore,

$$\langle \nabla^2\phi, \phi\rangle = -(|\phi|^2 + 2H^2)^2 + c(5\nu^2 - 1)(|\phi|^2 + 2H^2) - 8cH^2\nu^2 + \\ - 2cH\langle(\phi + HI)T, T\rangle + 4cH^2|T|^2 + 2H(3H|\phi|^2 + 2H^3),$$

which brings us to

$$\langle \nabla^2\phi, \phi\rangle = -|\phi|^4 + 2H^2|\phi|^2 + c(5\nu^2 - 1)|\phi|^2 - 2cH\langle \phi T, T\rangle.$$

To finish, we use that $\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \langle\nabla^2\phi, \phi\rangle$. □

Now we evaluate the Laplacian of $|S|^2$ where S is defined by (1.1), i.e,

$$S = 2HA - c\langle T, \cdot\rangle T + \frac{c}{2}(1 - \nu^2)I - 2H^2 I.$$

We observe the fact that S is a traceless operator, i.e,

$$tr(S) = 2Htr(A) - c|T|^2 + c(1 - \nu^2) - 4H^2 = 0,$$

where we used that $|T|^2 + \nu^2 = 1$ and $tr(A) = 2H$.

PROPOSITION 3.7 (Codazzi's Equation). — *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature and the S be the tensor defined in* (1.1)*. Then*

$$(\nabla_X S)Y = (\nabla_Y S)X,$$

for all tangent fields X, Y on Σ^2.

Proof. — We consider (u, v) isothermal parameters of the surface Σ^2. Now, we consider the complex parameter, $z = u + iv$. Let us set

$$T_S(X, Y) := (\nabla_X S)Y - (\nabla_Y S)X = \nabla_X(SY) - \nabla_Y(SX) - S[X, Y].$$

We will prove that T_S is null. For this, consider the derivatives

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

We will compute T_S in the basis $\{\partial_z, \partial_{\overline{z}}\}$. First note that,

$$\begin{aligned}\langle T_S(\partial_z, \partial_{\overline{z}}), \partial_z\rangle &= \partial_z\langle S\partial_{\overline{z}}, \partial_z\rangle - \langle S\partial_{\overline{z}}, \nabla_{\partial_z}\partial_z\rangle + \\ &\qquad - \partial_{\overline{z}}\langle S\partial_z, \partial_z\rangle + \langle S\partial_z, \nabla_{\partial_{\overline{z}}}\partial_z\rangle \\ &= -Q^{(2,0)}_{\overline{z}} = 0,\end{aligned}$$

because $Q^{(2,0)}$ is holomorphic, Theorem 1 in [1], and using the fact that $\nabla_{\partial_z}\partial_{\overline{z}} = 0$, $\nabla_{\partial_z}\partial_z = \dfrac{\lambda_z}{\lambda}\partial_z$, $\langle S\partial_z, \partial_z\rangle = Q^{(2,0)}$ and $\langle S\partial_z, \partial_{\overline{z}}\rangle = 0$, where $\lambda = \langle \partial_z, \partial_{\overline{z}}\rangle$.

Next,

$$\begin{aligned}\langle T_S(\partial_z, \partial_{\overline{z}}), \partial_{\overline{z}}\rangle &= -\partial_{\overline{z}}\langle \partial_{\overline{z}}, S\partial_z\rangle + \langle S\partial_z, \nabla_{\partial_{\overline{z}}}\partial_{\overline{z}}\rangle + \\ &\qquad + \partial_z\langle S\partial_{\overline{z}}, \partial_{\overline{z}}\rangle - \langle S\partial_{\overline{z}}, \nabla_{\partial_z}\partial_{\overline{z}}\rangle \\ &= \overline{Q^{(2,0)}_{\overline{z}}} = 0,\end{aligned}$$

where we have used that $\nabla_{\partial_{\overline{z}}}\partial_{\overline{z}} = \dfrac{\lambda_{\overline{z}}}{\lambda}\partial_{\overline{z}}$ and $\overline{Q^{(2,0)}}_z = \overline{Q^{(2,0)}_{\overline{z}}}$. It follows that $T_S = 0$. □

Lemma 3.8. — *Let Z be a symmetric operator satisfying Codazzi's equation and $tr(Z) = 0$, then*

$$\langle(\nabla^2 Z)x, y\rangle = \sum_{i=1}^{2}\{-\langle Zy, R_{e_i,x}e_i\rangle - \langle Ze_i, R_{e_i,x}y\rangle\}, \tag{3.6}$$

where $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$.

Proof. — See *Lemma a.* in [8], p. 81, adapted for codimension 1. □

Let us evaluate each summand in expression (3.6).

Lemma 3.9. — *Let Z be an operator as in Lemma 3.8. Then,*

$$i)\ \sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c\nu^2\langle Zx, y\rangle - 2H\langle Ax, Zy\rangle + \langle A^2x, Zy\rangle.$$

and

$$ii)\ \sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\nu^2\langle Zx, y\rangle - \langle Ay, ZAx\rangle + \langle Ax, y\rangle tr(AZ).$$

Proof. — Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p\Sigma$. Using Gauss' equation (2.3) we find

$$\begin{aligned}\langle Zy, R_{e_i,x}e_i\rangle = -c\{\langle x, Zy\rangle - \langle x, e_i\rangle\langle Zy, e_i\rangle - \langle x, T\rangle\langle Zy, T\rangle + \\ - \langle e_i, T\rangle^2\langle x, Zy\rangle + \langle e_i, T\rangle\langle x, e_i\rangle\langle Zy, T\rangle + \\ + \langle x, T\rangle\langle e_i, T\rangle\langle e_i, Zy\rangle\} - \langle Ae_i, e_i\rangle\langle Ax, Zy\rangle + \\ + \langle Ax, e_i\rangle\langle Ae_i, Zy\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c\{2\langle x, Zy\rangle - \sum_{i=1}^{2}\langle x, e_i\rangle\langle Zy, e_i\rangle + \cdots \\ \cdots - 2\langle x, T\rangle\langle Zy, T\rangle - \langle x, Zy\rangle\sum_{i=1}^{2}\langle e_i, T\rangle^2 + \\ + \langle Zy, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle x, e_i\rangle + \langle x, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle e_i, Zy\rangle\} + \\ - \langle Ax, Zy\rangle\sum_{i=1}^{2}\langle Ae_i, e_i\rangle + \sum_{i=1}^{2}\langle Ax, e_i\rangle\langle Ae_i, Zy\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c\{2\langle x, Zy\rangle - \langle Zx, y\rangle - 2\langle x, T\rangle\langle Zy, T\rangle + \\ - \langle x, Zy\rangle|T|^2 + \langle Zy, T\rangle\langle x, T\rangle + \langle x, T\rangle\langle T, Zy\rangle\} + \\ - \langle Ax, Zy\rangle 2H + \langle Ax, AZy\rangle.\end{aligned}$$

Hence,

$$\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c(1 - |T|^2)\langle Zx, y\rangle - 2H\langle Ax, Zy\rangle + \langle A^2x, Zy\rangle,$$

which shows the validity of (i). Now, one may verify that

$$\begin{aligned}\langle Ze_i, R_{e_i,x}y\rangle = -c\{\langle e_i, y\rangle\langle Ze_i, x\rangle - \langle x, y\rangle\langle Ze_i, e_i\rangle + \\ - \langle x, T\rangle\langle Ze_i, T\rangle\langle e_i, y\rangle - \langle e_i, T\rangle\langle y, T\rangle\langle x, Ze_i\rangle + \\ + \langle e_i, T\rangle\langle x, y\rangle\langle Ze_i, T\rangle + \langle x, T\rangle\langle y, T\rangle\langle e_i, Ze_i\rangle\} + \\ - \langle Ae_i, y\rangle\langle Ax, Ze_i\rangle + \langle Ax, y\rangle\langle Ae_i, Ze_i\rangle.\end{aligned}$$

Therefore

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\{\sum_{i=1}^{2}\langle e_i, y\rangle\langle Ze_i, x\rangle - \langle x, y\rangle\sum_{i=1}^{2}\langle Ze_i, e_i\rangle +$$
$$- \langle x, T\rangle\sum_{i=1}^{2}\langle Ze_i, T\rangle\langle e_i, y\rangle - \langle y, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle x, Ze_i\rangle + \cdots$$
$$\cdots + \langle x, y\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle Ze_i, T\rangle + \langle x, T\rangle\langle y, T\rangle\sum_{i=1}^{2}\langle e_i, Ze_i\rangle\} +$$
$$- \sum_{i=1}^{2}\langle Ae_i, y\rangle\langle Ax, Ze_i\rangle + \langle Ax, y\rangle\sum_{i=1}^{2}\langle Ae_i, Ze_i\rangle.$$

Therefore

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\{\langle Zx, y\rangle - \langle x, T\rangle\langle Zy, T\rangle - \langle y, T\rangle\langle Zx, T\rangle +$$
$$+ \langle ZT, T\rangle\langle x, y\rangle\} - \langle Ay, ZAx\rangle + \langle Ax, y\rangle tr(AZ),$$

noting that $trZ = 0$.

Considering that

$$-\langle x, T\rangle\langle Zy, T\rangle - \langle y, T\rangle\langle Zx, T\rangle + \langle ZT, T\rangle\langle x, y\rangle = -(1-\nu^2)\langle Zx, y\rangle,$$

we find

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\nu^2\langle Zx, y\rangle - \langle Ay, ZAx\rangle + \langle Ax, y\rangle tr(AZ),$$

which demonstrates (ii). □

Theorem 3.10. — *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with non zero constant mean curvature H and let Z be an operator on Σ^2 satisfying Codazzi's equation with $tr(Z) = 0$. Then,*

$$\langle(\nabla^2 Z)x, y\rangle = 2c\nu^2\langle Zx, y\rangle + 2H\langle Ax, Zy\rangle - \langle A^2x, Zy\rangle +$$
$$+ \langle Ay, ZAx\rangle - \langle Ax, y\rangle tr(AZ).$$

Proof. — We use the expressions of Lemma 3.9 in equation (3.6) obtained in Lemma 3.8. □

Next we derive an equation of Simons type for the operator S as defined in (1.1).

THEOREM 3.11 (Thm 1.1 in Introduction). — *Let* $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ *be an immersed surface with non zero constant mean curvature* H *and* S *as defined in* (1.1). *Then,*

$$\langle(\nabla^2 S)x,y\rangle = 2c\nu^2\langle Sx,y\rangle + 2H\langle Ax,Sy\rangle - \langle A^2x,Sy\rangle + \\ + \langle Ay,SAx\rangle - \langle Ax,y\rangle tr(AS),$$

and

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST,T\rangle\right) + \\ + c|ST|^2 - \frac{1}{4H^2}\langle ST,T\rangle^2.$$

Proof. — First, since S satisfies Proposition 3.7, we can use the Theorem 3.10 with $Z = S$,

Now, we know that $\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + \langle\nabla^2 S,S\rangle$. Furthermore, we find that

$$\langle\nabla^2 S,S\rangle = 2c\nu^2|S|^2 + 2Htr(AS^2) - [tr(AS)]^2.$$

Now, we need to compute $tr(AS^2)$ and $tr(AS)$, as follows:

$$\begin{aligned} tr(AS^2) &= tr\{S^2(S + \frac{c}{2H}\langle T,\cdot\rangle T - \frac{c}{4H}(1-\nu^2)I + HI)\} \\ &= trS^3 + \frac{c}{2H}tr(\langle T,S^2\cdot\rangle T) - \left(\frac{c}{4H}(1-\nu^2) - H\right)trS^2 \\ &= 0 + \frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1-\nu^2) - H\right)|S|^2 \end{aligned}$$

and

$$\begin{aligned} tr(AS) &= tr\{S(S + \frac{c}{2H}\langle T,\cdot\rangle T - \frac{c}{4H}(1-\nu^2)I - HI)\} \\ &= trS^2 + \frac{c}{2H}tr(\langle T,S\cdot\rangle T) - (\frac{c}{4H}(1-\nu^2) - H)trS \\ &= |S|^2 + \frac{c}{2H}\langle ST,T\rangle - 0, \end{aligned}$$

noting that $trS = trS^3 = 0$, also that

$$tr(\langle T,S\cdot\rangle T) = \sum_{i=1}^{2}\langle T,Se_i\rangle\langle T,e_i\rangle = \langle ST,T\rangle$$

and that

$$tr(\langle T,S^2\cdot\rangle T) = \sum_{i=1}^{2}\langle T,S^2e_i\rangle\langle T,e_i\rangle = \langle S^2T,T\rangle.$$

Therefore,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + 2H\left(\frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1-\nu^2) - H\right)|S|^2\right) + \\ - \left(|S|^2 + \frac{c}{2H}\langle ST, T\rangle\right)^2,$$

in this way,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + c|ST|^2 - \left(\frac{c}{2}(1-\nu^2) - 2H^2\right)|S|^2 + \\ - |S|^4 - \frac{c}{H}\langle ST, T\rangle|S|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

Rearranging terms, we obtain finally

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T\rangle\right) + \\ + c|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

□

4. Applications

In this section, we will apply the results found in section 3 together with the Omori-Yau's Theorem to classify some surfaces in $M^2(c)\times\mathbb{R}$.

THEOREM 4.1. — *Let $\Sigma^2 \looparrowright \mathbb{H}^2\times\mathbb{R}$ be an oriented complete immersed minimal surface. Assume that*

$$\sup_{\Sigma}(|A|^2 + 5\nu^2) < 1.$$

Then Σ^2 is a vertical plane $\gamma\times\mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$.

Proof. — Using Theorem 3.6 with $H = 0$ and $c = -1$, one finds

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4 + (1-5\nu^2)|A|^2 \geqslant |A|^2\left(-|A|^2 + 1 - 5\nu^2\right).$$

Let $\frac{d}{2} := -\sup_{\Sigma}(|A|^2 + 5\nu^2) + 1 > 0$. Therefore,

$$\Delta|A|^2 \geqslant d\cdot|A|^2. \tag{4.1}$$

Using Gauss' equation (2.3) in $\mathbb{H}^2\times\mathbb{R}$ we have

$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|A|^2 + 5\nu^2}{2} + \frac{3\nu^2}{2} \geqslant -\frac{1}{2}.$$

Now we can use Theorem 2.2 with $u = |A|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty} |A|^2(p_j) = \sup_{\Sigma} |A|^2 \ \textit{and} \ \lim_{j\to\infty} \Delta|A|^2(p_j) \leqslant 0.$$

Next, we use inequality (4.1) to conclude that $\sup_{\Sigma} |A|^2 = 0$, i.e, Σ^2 is totally geodesic with $|\nu| < \sqrt{0.2}$.

Since Σ^2 is totally geodesic and $|\nu| < \sqrt{0.2}$ it cannot be a slice, it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$. □

Theorem 4.2. — *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature H. Assume that*

$$\sup_{\Sigma}(|\phi|^2 + 5\nu^2) < 2H^2 + 1 \ \ \textit{and} \ \ \langle \phi T, T\rangle \geqslant 0.$$

Then Σ^2 is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$.

Proof. — We consider the expression in Theorem 3.6 for the particular case $c = -1$:

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2 + 2H\langle \phi T, T\rangle.$$

As $\langle \phi T, T\rangle \geqslant 0$, we find

$$\frac{1}{2}\Delta|\phi|^2 \geqslant -|\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2.$$

Consider $\dfrac{d}{2} := 2H^2 + 1 - \sup_{\Sigma}(|\phi|^2 + 5\nu^2) > 0$. Then

$$\Delta|\phi|^2 \geqslant 2|\phi|^2(2H^2 + 1 - 5\nu^2 - |\phi|^2) \geqslant d|\phi|^2,$$

which implies,

$$\Delta|\phi|^2 \geqslant d|\phi|^2. \tag{4.2}$$

Using Gauss' equation (2.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|\phi|^2 + 5\nu^2 - 2H^2}{2} + \frac{3\nu^2}{2} \geqslant -\frac{1}{2}.$$

Now we can use Theorem 2.2 with $u = |\phi|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty} |\phi|^2(p_j) = \sup_{\Sigma} |\phi|^2 \ \textit{and} \ \lim_{j\to\infty} \Delta|\phi|^2(p_j) \leqslant 0.$$

Furthermore, we use inequality (4.2) to conclude that $\sup_{\Sigma} |\phi|^2 = 0$, i.e, Σ^2 is totally umbilical.

Next, we use that if Σ^2 is totally umbilical with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ then Σ^2 is totally geodesic, which follows from [9] section 4.

Since Σ^2 is totally geodesic and $|\nu| < \sqrt{0.2}$ it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$. This concludes the proof. □

We need the following result:

Lemma 4.3. — *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be a complete immersed surface with non zero constant mean curvature H. Then $|S|=0$ if and only if Σ^2 is an Abresch-Rosenberg surface.*

Proof. — We consider (u,v) isothermal parameters on the surface Σ^2. Now, we consider the complex parameter, $z=u+iv$ and the (2,0)-part of the Abresch-Rosenberg differential

$$Q(x,y)=2H\langle Ax,y\rangle - c\langle x,T\rangle\langle y,T\rangle.$$

We can rewrite Q as

$$Q(x,y)=\langle Sx,y\rangle - \frac{c}{2}(1-\nu^2)\langle x,y\rangle + 2H^2\langle x,y\rangle.$$

Next we evaluate $Q(\partial_z,\partial_z)$ noting that $\langle\partial_z,\partial_z\rangle=0$:

$$Q(\partial_z,\partial_z)=\langle S\partial_z,\partial_z\rangle=\left(\frac{\tilde e-\tilde g}{4}\right)-i\frac{\tilde f}{2},$$

where $\tilde e=\langle S\partial_u,\partial_u\rangle=-\langle S\partial_v,\partial_v\rangle=-\tilde g$ and $\tilde f=\langle S\partial_u,\partial_v\rangle$. Therefore

$$|Q^{(2,0)}|=\sqrt{\left(\frac{\tilde e-\tilde g}{4}\right)^2+\frac{\tilde f^2}{4}}=\sqrt{\frac{\tilde e^2}{4}+\frac{\tilde f^2}{4}}=\frac{E^2}{2\sqrt2}|S|,$$

where $E=|\partial_u|>0$. This concludes the proof. □

Let us consider the polynomial $p_H(t)=-t^2-\dfrac{1}{\sqrt2 H}t+\left(\dfrac{4H^2-1}{2}\right)$. When H is greater than one half there is a positive root for p_H. Let L_H be the positive root. One has:

Theorem 4.4 (Thm 1.2 in Introduction). — *Let $\Sigma^2 \looparrowright \mathbb{S}^2\times\mathbb{R}$ be an immersed surface with constant mean curvature H greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_\Sigma |S| < L_H$$

or

$$\Sigma^2 \text{ is closed and } |S|\leqslant L_H,$$

then $\Sigma^2=S^2_H$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Proof. — Let consider two cases. First, Σ is complete and second, Σ is closed.

First Case. Consider the expression in Theorem 3.11 with $c=1$:

$$\frac12\Delta|S|^2=|\nabla S|^2-|S|^4+|S|^2\left(\frac{5\nu^2}{2}-\frac12+2H^2-\frac1H\langle ST,T\rangle\right)+$$

$$+|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

As $|\langle ST, T\rangle| \leqslant |ST| \leqslant \frac{1}{\sqrt{2}}|S|$, we have

$$\frac{1}{2}\Delta|S|^2 \geqslant -|S|^4+|S|^2\left(\frac{5\nu^2}{2}+\frac{4H^2-1}{2}-\frac{1}{\sqrt{2}H}|S|\right)+\left(\frac{4H^2-1}{4H^2}\right)\langle ST, T\rangle^2,$$

hence,

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{4H^2-1}{2}-\frac{1}{\sqrt{2}H}|S|-|S|^2\right)+\frac{5}{2}\nu^2|S|^2, \tag{4.3}$$

because $H > \frac{1}{2}$.

Observe that

$$\frac{4H^2-1}{2}-\frac{1}{\sqrt{2}H}|S|-|S|^2 \geqslant p_H(\sup_{\Sigma}|S|) =: \frac{d}{2} > 0$$

and $\nu^2|S|^2 \geqslant 0$. Therefore

$$\Delta|S|^2 \geqslant d|S|^2. \tag{4.4}$$

Now we estimate $|S|$.

$$|S| \geqslant 2H|A| - |\langle T, \cdot\rangle T| - (1-\nu^2) - 4H^2 \geqslant 2H|A| - 2(1-\nu^2) - 4H^2,$$

that is,

$$L_H \geqslant |S| \geqslant 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (2.3) in $\mathbb{S}^2 \times \mathbb{R}$ we find

$$K_\Sigma = K_{ext} + \nu^2 = -\frac{|A|^2}{2} + 2H^2 + \nu^2 \geqslant -\frac{1}{2}\left(\frac{L_H+2+4H^2}{2H}\right)^2.$$

Now we can use Theorem 2.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty}|S|^2(p_j) = \sup_{\Sigma}|S|^2 \textit{ and } \lim_{j\to\infty}\Delta|S|^2(p_j) \leqslant 0.$$

By means of inequality (4.4) we conclude that $\sup_\Sigma |S|^2 = 0$, i.e, $|S| = 0$ in Σ^2. Using Lemma 4.3 and Remark 1 of the Introduction we conclude the proof.

Second case. Let us consider expression (4.3)

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{4H^2-1}{2}-\frac{1}{\sqrt{2}H}|S|-|S|^2\right)+\frac{5}{2}\nu^2|S|^2.$$

As $|S| \leqslant L_H$, we have $\dfrac{4H^2-1}{2} - \dfrac{1}{\sqrt{2}H}|S| - |S|^2 \geqslant 0$. Hence,

$$\frac{1}{2}\Delta|S|^2 \geqslant \frac{5}{2}\nu^2|S|^2.$$

Integrating and using Stokes' Theorem we find

$$0 \geqslant \frac{5}{2}\int_\Sigma \nu^2|S|^2 d\Sigma \geqslant 0.$$

It follows that

$$|S| \cdot \nu = 0. \tag{4.5}$$

Let $\Theta = \{p \in \Sigma^2 : \nu(p) = 0\} = \nu^{-1}(0)$ be the nodal lines of ν. We know that

$$\Delta\nu + (|A|^2 + Ric(N,N)\nu = 0.$$

Hence, we can apply Theorem 2.5 in [4], p. 49, to conclude that Θ has empty interior. Thus, using (4.5), $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ.

Using Lemma 4.3 and Remark.1 of the Introduction we conclude the proof. □

Theorem 4.5. — *There exists no $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ complete immersed surface with constant mean curvature greater than one half such that $|S| = L_H$.*

Proof. — Suppose that there exist $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (4.3)

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{4H^2-1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2\right) + \frac{5}{2}\nu^2|S|^2,$$

with $|S| = L_H$ one find that

$$0 \geqslant 0 + \frac{5}{2}\nu^2 L_H^2 \geqslant 0.$$

Hence $\nu = 0$, i.e, $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ is a cylinder $\gamma \times \mathbb{R}$ for some $\gamma \in \mathbb{S}^2$ with constant curvature $2H$.

On the other hand, for a cylinder $\gamma \times \mathbb{R}$, where $\gamma \in \mathbb{S}^2$ with constant curvature $2H$, we may write

$$S = \begin{pmatrix} 2H^2 + \frac{1}{2} & 0 \\ 0 & -2H^2 - \frac{1}{2} \end{pmatrix}.$$

As $|S| = \dfrac{\sqrt{2}}{2}(4H^2+1) > L_H$ we have a contradiction. □

In next theorem we need the following result:

Lemma 4.6. — *Any Abresch-Rosenberg surface* $\Sigma^2 \looparrowright \mathbb{H}^2\times\mathbb{R}$ *with* $H > \frac{1}{2}$ *is an embedded rotationally invariant constant mean curvature sphere.*

Proof. — See Proposition 4.3 in [1], p. 159. □

Let us consider the polynomial

$$q_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left(\frac{8H^4 - 12H^2 - 1}{4H^2}\right).$$

When H is greater than a positive root of the polynomial $r(x) = 8x^4 - 12x^2 - 1$, i.e, H is greater than $\sqrt{\frac{3+\sqrt{11}}{4}}$, there is a positive root for q_H. Let M_H be the positive root.

Theorem 4.7 (Thm 1.3 in Introduction). — *Let* $\Sigma^2 \looparrowright \mathbb{H}^2\times\mathbb{R}$ *be an immersed surface with constant mean curvature* H *greater than* $\sqrt{\frac{3+\sqrt{11}}{4}} \approx 1.25664$. *If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma}|S| < M_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leqslant M_H,$$

then $\Sigma^2 = S^2_H$, *i.e,* Σ^2 *is an embedded rotationally invariant constant mean curvature sphere.*

Proof. — Let us consider two cases. First, Σ is complete and second, Σ is closed.

First case. Consider the expression in Theorem 3.11 with $c = -1$

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(-\frac{5\nu^2}{2} + \frac{1}{2} + 2H^2 + \frac{1}{H}\langle ST, T\rangle\right) + \\ - |ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

As $|\langle ST, T\rangle| \leqslant |ST| \leqslant \frac{1}{\sqrt{2}}|S|$, we may write

$$\frac{1}{2}\Delta|S|^2 \geqslant -|S|^4 + |S|^2\left(\frac{4H^2+1-5\nu^2}{2} - \frac{1}{\sqrt{2}H}|S|\right) - \left(\frac{4H^2+1}{4H^2}\right)|S|^2,$$

i.e,

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{4H^2-4+5-5\nu^2}{2} - \frac{1}{\sqrt{2}H}|S| - \frac{4H^2+1}{4H^2} - |S|^2\right).$$

This may be rewritten as,

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{8H^4-12H^2-1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2\right) + \frac{5}{2}(1-\nu^2)|S|^2. \tag{4.6}$$

Observe that

$$\frac{8H^4-12H^2-1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geqslant q_H(\sup_\Sigma |S|) =: \frac{d}{2} > 0$$

and $(1-\nu^2)|S|^2 \geqslant 0$. Therefore,

$$\Delta|S|^2 \geqslant d|S|^2. \tag{4.7}$$

Next we estimate $|S|$.

$$|S| \geqslant 2H|A| - |\langle T,\cdot\rangle T| - (1-\nu^2) - 4H^2 \geqslant 2H|A| - 2(1-\nu^2) - 4H^2,$$

i.e,

$$M_H \geqslant |S| \geqslant 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (2.3) in $\mathbb{H}^2\times\mathbb{R}$ we find

$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|A|^2}{2} + 2H^2 - \nu^2 \geqslant -\frac{1}{2}\left(\frac{M_H+2+4H^2}{2H}\right)^2.$$

Now we can use Theorem 2.2 with $u=|S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty}|S|^2(p_j) = \sup_\Sigma |S|^2 \textit{ and } \lim_{j\to\infty}\Delta|S|^2(p_j) \leqslant 0.$$

Inequality (4.7) allows us conclude that $\sup_\Sigma|S|^2 = 0$, i.e, $|S|=0$ in Σ^2. Then, by using Lemmas 4.3 and 4.6, we conclude the proof.

Second case. Let us consider expression (4.6)

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{8H^4-12H^2-1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2\right) + \frac{5}{2}(1-\nu^2)|S|^2.$$

As $|S| \leqslant M_H$, we have that $\frac{8H^4-12H^2-1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geqslant 0$. Hence,

$$\frac{1}{2}\Delta|S|^2 \geqslant \frac{5}{2}(1-\nu^2)|S|^2.$$

Integrating and using Stokes' Theorem we write

$$0 \geqslant \frac{5}{2}\int_\Sigma (1-\nu^2)|S|^2 d\Sigma \geqslant 0.$$

Moreover

$$(1-\nu^2)\cdot|S|^2 = 0. \tag{4.8}$$

Consider $\Theta = \{p \in \Sigma^2; \nu^2(p) = 1\} \subset \mathbb{H}^2 \times \{t_0\}$, for any t_0. Since H is positive we have that Θ has empty interior. Thus, using (4.8), we conclude that $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ. Using Lemma 4.3 and the fact that the only Abresch-Rosenberg closed surface is S_H^2 we conclude the proof. □

THEOREM 4.8. — *There exists no $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ a complete immersed surface with constant mean curvature greater than* $\sqrt{\dfrac{3+\sqrt{11}}{4}} \approx 1.25664$ *such that $|S| = M_H$.*

Proof. — Suppose that there exists $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (4.6)

$$\frac{1}{2}\Delta|S|^2 \geqslant |S|^2\left(\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2\right) + \frac{5}{2}(1-\nu^2)|S|^2$$

with $|S| = M_H$ we obtain:

$$0 \geqslant 0 + \frac{5}{2}(1-\nu^2)M_H^2 \geqslant 0.$$

Hence $\nu^2 = 1$, i.e, $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ is a slice $\mathbb{H}^2 \times \{t_0\}$. But $\mathbb{H}^2 \times \{t_0\}$ has zero mean curvature, and this is impossible because H is positive. □

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Manuscrit reçu le 14 juillet 2009,
accepté le 27 avril 2010.

Márcio Henrique BATISTA DA SILVA
Universidade Federal de Alagoas
Instituto de Matemática
CEP: 57072-900
Maceió - Alagoas (Brazil)
mhbs28@impa.br