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THE WEIL ALGEBRA AND THE VAN EST ISOMORPHISM

by Camilo ARIAS ABAD & Marius CRAINIC (*)

ABSTRACT. — This paper belongs to a series of papers devoted to the study of the cohomology of classifying spaces. Generalizing the Weil algebra of a Lie algebra and Kalkman’s BRST model, here we introduce the Weil algebra $W(A)$ associated to any Lie algebroid A . We then show that this Weil algebra is related to the Bott-Shulman complex (computing the cohomology of the classifying space) via a Van Est map and we prove a Van Est isomorphism theorem. As application, we generalize and find a simpler more conceptual proof of the main result of [6] on the reconstructions of multiplicative forms and of a result of [21, 9] on the reconstruction of connection 1-forms. This reveals the relevance of the Weil algebra and Van Est maps to the integration and the pre-quantization of Poisson (and Dirac) manifolds.

RÉSUMÉ. — Cet article fait partie d’une série consacrée à l’étude de la cohomologie des espaces classifiants. En généralisant l’algèbre de Weil d’une algèbre de Lie et le modèle BRST de Kalkman, nous introduisons l’algèbre de Weil $W(A)$ associée à une algébroïde de Lie A . Nous montrons ensuite que cette algèbre de Weil est liée au complexe de Bott-Shulman (calculant la cohomologie de l’espace classifiant) via une application de Van Est et nous prouvons un théorème d’isomorphisme de type Van Est. Une application de ces méthodes conduit à généraliser de façon plus conceptuelle des reconstitutions de formes multiplicatives et de 1-formes de connexion.

Introduction

This paper belongs to a series devoted to the study of the cohomology of classifying spaces. Here we extend the construction of the Weil algebra of a Lie algebra to the setting of Lie algebroids and we show that one of the standard complexes computing the cohomology of classifying spaces (the

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“Bott-Shulman complex”) is related to the Weil algebra via a Van Est map. This Van Est map is new even in the case of Lie groups. As application, we generalize and we find a simpler (and conceptual) proof of the main result of [6] on the integration of Poisson and related structures.

We will be working in the framework of groupoids [15]. Lie groupoids and Lie algebroids are a generalization of Lie groups and Lie algebras, generalization which arises in various geometric contexts such as foliation theory, Poisson geometry, equivariant geometry. Probably the best known example of a Lie groupoid is the homotopy groupoid of a manifold M , which consists of homotopy classes of paths in M , each such path being viewed as an “arrow” from the initial point to the ending point. In the presence of a foliation on M , by restricting to paths inside leaves and leafwise homotopy (or holonomy), one arrives at the groupoids which are central to foliation theory [13]. When G is a Lie group of symmetries of a manifold M , it is important to retain not only the group structure of G but also at the way that the points of M are affected by the action. More precisely, one considers pairs $(g, x) \in G \times M$ and one views such a pair as an “arrow” from x to gx ; one obtains a groupoid $G \times M$, called the action groupoid. In Poisson geometry, the relevance of Lie groupoids [20] is a bit more subtle—they arise through their infinitesimal counterpart (Lie algebroids) after an integration process. They also arise as the canonical solution to the problem of finding symplectic realizations of Poisson manifolds. Indeed, an important feature of the resulting groupoids is that they carry a canonical symplectic structure (for instance, if M is endowed with the zero-Poisson structure, the resulting groupoid is T^*M with the canonical symplectic structure). Conceptually, such symplectic forms can be seen as arising after integrating certain infinitesimal data, called IM forms (see Example 2.8). This integration step is based on the main result of [6] mentioned above, result that will be generalized and proved more conceptually in this paper. The key remark is that the rather mysterious equations that IM forms have to satisfy are nothing but cocycle equations in the Weil algebra of the associated Lie algebroid.

The classifying space BG [19] of a Lie groupoid G has the same defining property as in the case of Lie groups: it is the base of an universal principal G -bundle $EG \rightarrow BG$. It is unique up to homotopy, and there are several known constructions of BG as a topological space and of EG as topological G -bundle - e.g., via a Milnor-type construction [13] or via simplicial methods [4, 5]. Due to the universality of EG , the cohomology of BG is the algebra of universal characteristic classes for G -bundles. For

instance, if G is a compact Lie group, then $H^*(BG) = S(\mathfrak{g}^*)^G$ is the space of G -invariant polynomials on the Lie algebra \mathfrak{g} of G , which is the source of the Chern-Weil construction of characteristic classes via connections. The cohomology of classifying spaces is interesting also from the point of view of equivariant cohomology; for instance, for the Lie groupoid $G \ltimes M$ associated to an action of a Lie group G on a manifold M , the cohomology of $B(G \ltimes M)$ coincides with the equivariant cohomology of M . For general Lie groupoids G , simplicial techniques [19] provides us with a huge but explicit complex computing $H^*(BG)$, known as the Bott-Shulman complex [5], which is a double complex suggestively denoted

$$\Omega^\bullet(G_\bullet).$$

We can now explain the words in the title and our main results. First of all, the classical Weil algebra $W(\mathfrak{g})$ associated to a Lie group G (or, better, to its Lie algebra \mathfrak{g}) arises as the algebraic model for the “DeRham complex” of the total space EG [7] (see also [12, 3]). From the point of view of characteristic classes, the role of $W(\mathfrak{g})$ is to provide an explicit and geometric construction of such classes (the Chern-Weil construction). >From the point of view of equivariant cohomology, it is useful in constructing explicit geometric models for equivariant cohomology (such as the Cartan model). As we have already mentioned, the first aim of this paper is to extend the construction of the Weil algebra from Lie algebras to Lie algebroid- for any Lie algebroid A , its Weil algebra, denoted

$$W^{\bullet,\bullet}(A),$$

will be a differential bi-graded algebra.

Next, the classical Van Est map [11] relates the differentiable cohomology of a Lie group G to its infinitesimal counterpart, *i.e.*, to the Lie algebra cohomology of the Lie algebra \mathfrak{g} of G . It is an isomorphism up to degree k provided G is k -connected. The Van Est isomorphism extends to Lie groupoids and Lie algebroids [21, 8] without much trouble. What is interesting to point out here is that the complex computing the differentiable cohomology of G is just the first line $\Omega^0(G_\bullet)$ of the Bott-Shulman complex, while the complex computing the Lie algebroid cohomology is just the first line $W^{\bullet,0}(A)$ of the Weil algebra. With these in mind, the second aim of the paper is to extend the classical Van Est map to a map of bi-graded differential algebras

$$V : \Omega^\bullet(G_\bullet) \longrightarrow W^{\bullet,\bullet}(A),$$

for any Lie groupoid G with associated Lie algebroid A . Topologically, this map is just an explicit model for the map induced by the pull-back along

the projection $\pi : EG \rightarrow BG$. However, what is more interesting is that the Van Est isomorphism holds not only along the first line, but along all lines. More precisely, we will prove the following:

THEOREM 0.1. — *Let G be a Lie groupoid with Lie algebroid A and k -connected source fibers. When restricted to any q -line (q arbitrary), the Van Est map induces an isomorphism in cohomology*

$$V : H^p(\hat{\Omega}^q(G_\bullet)) \rightarrow H^p(W^{\bullet,q}(A)),$$

for all $p \leq k$, while for $p = k + 1$ this map is injective.

As a consequence, we prove the following generalization of the reconstruction result for multiplicative 2-forms which appears in [6].

THEOREM 0.2. — *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\phi \in \Omega^{k+1}(M)$ be a closed form. Then there is a one to one correspondence between:*

- *multiplicative forms $\omega \in \Omega^k(G)$ which are ϕ -relatively closed.*
- *$C^\infty(M)$ -linear maps $\tau : \Gamma(A) \rightarrow \Omega^{k-1}(M)$ satisfying the equations:*

$$i_{\rho(\beta)}(\tau(\alpha)) = -i_{\rho(\alpha)}(\tau(\beta)),$$

$$\tau([\alpha, \beta]) = L_\alpha(\tau(\beta)) - L_\beta(\tau(\alpha)) + d_{DR}(i_{\rho(\beta)}\tau(\alpha)) + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi).$$

for all $\alpha, \beta \in \Gamma(A)$.

This theorem reveals the relationship between the Van Est map V and the integrability of Poisson and Dirac structures. This relationship can be summarized as follows: the Lie algebroid associated to a Poisson (or Dirac) structure comes together with a tautological cocycle living in the Weil algebra of the associated Lie algebroid. Integrating the Lie algebroid to a Lie groupoid G , the Van Est isomorphism tells us that the tautological cocycle integrates to a cocycle on the Bott-Shulman complex of the groupoid which, in this case, is a multiplicative two-form on the groupoid, making G a symplectic (or presymplectic [6]) groupoid.

Another application of our Van Est isomorphism is a generalization (and another proof) of the result of [9] on the construction of connection 1-forms on prequantizations. Conceptually, it answers the following question: given $\omega \in \Omega^k(G)$ multiplicative and closed, when can one write $\omega = d\theta$ with $\theta \in \Omega^{k-1}(G)$ multiplicative? We prove the following result which, when $k = 2$, coincides with the result of [9] that we have mentioned.

THEOREM 0.3. — *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\omega \in \Omega^k(G)$ be a closed multiplicative k -form. Then there is a 1-1 correspondence between:*

- $\theta \in \Omega^{k-1}(G)$ multiplicative satisfying $d(\theta) = \omega$.
- $C^\infty(M)$ -linear maps $l : \Gamma(A) \rightarrow \Omega^{k-2}(M)$ satisfying

$$(0.1) \quad i_{\rho(\beta)}(l(\alpha)) = -i_{\rho(\alpha)}(l(\beta)),$$

$$(0.2) \quad c_\omega(\alpha, \beta) = -l([\alpha, \beta]) + L_{\rho(\alpha)}(l(\beta)) - L_{\rho(\beta)}(l(\alpha)) + d_{DR}(i_{\rho(\beta)}l(\alpha)).$$

where $c_\omega(\alpha, \beta) = i_{\rho(\alpha)\wedge\rho(\beta)}(\omega)|_M$. The correspondence is given by

$$l(\alpha) = -i_\alpha(\theta)|_M.$$

We should now point out what this paper does not achieve. In the case of a compact Lie group G , the cohomology of BG is isomorphic to $W(\mathfrak{g})_{\text{bas}} \cong S(\mathfrak{g}^*)$ where “bas” refers to the basic sub-complex, which consists of elements that are horizontal and invariant with respect to G . More generally, for an action groupoid $G \ltimes M$, the cohomology of the classifying space is the equivariant cohomology $H_G^*(M)$, and the Weil algebra associated to the Lie algebroid $\mathfrak{g} \ltimes M$ of $G \ltimes M$ is $W(\mathfrak{g} \ltimes M) = W(\mathfrak{g}) \otimes \Omega(M)$. In this situation one can still define the basic subcomplex $W(\mathfrak{g} \ltimes M)_{\text{bas}}$, and it is isomorphic to Cartan’s equivariant De-Rham complex of M . Hence $W(\mathfrak{g} \ltimes M)_{\text{bas}}$ computes the equivariant cohomology of the action, provided G is compact. We discuss this construction in Subsection 1.4, more details can be found in [12, 14]. Back to a general Lie groupoid G with Lie algebroid A , it is natural to expect a construction of a “basic subcomplex” of $W(A)$ which computes the cohomology of BG , at least under some compactness assumptions. However, that does not seem to work in an obvious way. On the other hand, we would like to mention here that in [1], we did find a generalization of Bott’s spectral sequence using the notion of “representations up to homotopy” to define the coadjoint representation of a Lie groupoid. It seems possible that these two approaches can be combined to obtain a general Cartan type model for the cohomology of classifying spaces of Lie groupoids, but at the moment we do not know how to do it.

This paper is organized as follows. In the first two sections we recall some standard facts about classifying spaces of Lie groups, the classical Weil algebra, equivariant cohomology, Lie groupoids and their classifying spaces, the Bott-Shulman complex. In Section 3 we introduce the Weil algebra of a Lie algebroid (Definition 3.1) by giving an explicit choice free description. Then we point out the local formulas and the relationship with the adjoint representation and the algebra described in [2] using representations up to homotopy. Section 4 contains the definition of the Van Est map $V : \hat{\Omega}^\bullet(G_\bullet) \rightarrow W(A)$ (Theorem 4.1). In Section 5 we prove the isomorphism theorem for the homomorphism induced in cohomology by the Van Est map (Theorem 5.1). In Section 6 we explain the relation between the Van Est

map and the integration of IM forms to multiplicative forms (Theorems 6.1 and 6.4). Finally, Section 7 is an appendix where we describe an infinite dimensional version of Kalkman's BRST algebra which is used throughout the paper.

At this point we would also like to mention that, while this work has been carried out, various people used the supermanifold language to construct the Weil algebra of Lie algebroids. We found out about such descriptions from D. Roytenberg and P. Severa (unpublished); A Van Est map using supermanifolds was constructed also by A. Weinstein (unpublished notes). This construction appears in the PhD thesis of R. Mehta [17], where he describes the Weil algebra of a Lie algebroid in terms of supergeometry. That algebra is isomorphic to the one we present here.

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1. Lie groups: reminder on the classical Weil algebra and classifying spaces

In this section we recall some standard facts about classifying spaces of Lie groups, Weil algebras and equivariant cohomology. As references, we mention here [4, 7, 4, 3, 14].

1.1. The universal principal bundle

Associated to any Lie group G there is a classifying space BG and an universal principal G -bundle $EG \rightarrow BG$. These have the following universal property. For any space M there is a bijective correspondence

$$[M; BG] \xrightarrow{\sim} \text{Bun}_G(M)$$

between homotopy classes of maps $f : M \rightarrow BG$ and isomorphism classes of principal G -bundles over a M . This correspondence sends a function f to the pull-back bundle $f^*(EG)$. The universal property determines $EG \rightarrow BG$ uniquely up to homotopy. Another property that determines EG , hence also BG , uniquely, is that EG is a free, contractible G -space. There are explicit combinatorial constructions for the classifying bundle of a group, we will say more about this in a moment.

The cohomology of BG is the universal algebra of characteristic classes for principal G -bundles. Indeed, from the universal property, any such bundle $P \rightarrow M$ is classified by a map $f_P : M \rightarrow BG$. Although f_P is unique only up to homotopy, the map induced in cohomology

$$f_P^* : H^\bullet(BG) \rightarrow H^\bullet(M)$$

only depends on P and is called the characteristic map of P . Any element $c \in H^\bullet(BG)$ will induce the c -characteristic class of P , $c(P) := (f_P)^*(c) \in H^\bullet(M)$. This is one of the reasons one is often interested in more explicit models for the cohomology of BG . When G is compact, a theorem of Borel asserts that

$$H^\bullet(BG) \cong S(\mathfrak{g}^*)^G.$$

Moreover, the map f_P^* can be described geometrically. This is the Chern-Weil construction of characteristic classes in $H^\bullet(M)$ out of invariant polynomials on \mathfrak{g} , viewed as a map

$$S(\mathfrak{g}^*)^G \rightarrow H(M).$$

1.2. The Weil algebra

Regarding the cohomology of BG and the construction of characteristic classes, the full picture is achieved only after finding a related model for the *De Rham cohomology* of EG . This is the Weil algebra $W(\mathfrak{g})$ of the Lie algebra of G [7, 12, 14]. As a graded algebra, it is defined as

$$W^n(\mathfrak{g}) = \bigoplus_{2p+q=n} S^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}^*).$$

We interpret its elements as polynomials P on \mathfrak{g} with values in $\Lambda(\mathfrak{g}^*)$, but keep in mind that the polynomial degree counts twice. The Weil algebra can also be made into bi-graded algebra, with

$$W^{p,q}(\mathfrak{g}) = S^p(\mathfrak{g}^*) \otimes \Lambda^{p-q}(\mathfrak{g}^*),$$

and its differential d_W can be decomposed into two components:

$$d_W = d_W^h + d_W^v.$$

Here, d_W^v increases q and is given by

$$d_W^v(P)(v) = i_v(P(v)),$$

while d_W^h increases p and is defined as a Koszul differential of \mathfrak{g} with coefficients in $S^q(\mathfrak{g}^*)$ - the symmetric powers of the coadjoint representation. To be more explicit, it is customary to use coordinates. A basis e^1, \dots, e^n for

\mathfrak{g} gives structure constants c_{jk}^i . With this choice, $W(\mathfrak{g})$ can be described as the free graded commutative algebra generated by elements $\theta^1, \dots, \theta^n$ of degree 1 and μ^1, \dots, μ^n of degree 2, with differential:

$$\begin{aligned} d_W^v(\theta^i) &= \mu^i, \\ d_W^v(\mu^i) &= 0, \\ d_W^h(\theta^i) &= -\frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \theta^k, \\ d_W^h(\mu^i) &= -\sum_{j,k} c_{jk}^i \theta^j \mu^k. \end{aligned}$$

Of course, $\theta^1, \dots, \theta^n$ is just the induced basis of $\Lambda^1(\mathfrak{g}^*)$, while μ^1, \dots, μ^n is the one of $S^1(\mathfrak{g}^*)$. The cohomology of $W(\mathfrak{g})$ is \mathbb{R} concentrated in degree zero, as one should expect from the fact that EG is a contractible space.

1.3. \mathfrak{g} -DG algebras

To understand why $W(\mathfrak{g})$ is a model for the De Rham complex of EG , one has to look at the structure present in the De Rham algebras of principal G -bundles. This brings us to the notion of \mathfrak{g} -DG algebras (cf. e.g., [12, 14]). A \mathfrak{g} -DG algebra is a differential graded algebra (\mathcal{A}, d) (for us \mathcal{A} lives in positive degrees and d increases the degree by one), together with

- Degree zero derivations L_v on the DG-algebra \mathcal{A} , depending linearly on $v \in \mathfrak{g}$, which induce an action of \mathfrak{g} on \mathcal{A} .
- Degree -1 derivations i_v on the DG-algebra \mathcal{A} such that for all $v, w \in \mathfrak{g}$

$$[i_v, i_w] = 0, \quad [L_v, i_w] = i_{[v,w]}$$

and such that they determine the Lie derivatives by Cartan's formula

$$di_v + i_v d = L_v.$$

The basic subcomplex of a \mathfrak{g} -DG algebra \mathcal{A} is defined as

$$\mathcal{A}_{\text{bas}} := \{\omega \in \mathcal{A} : i_v \omega = 0, L_v \omega = 0, \forall v \in \mathfrak{g}\}.$$

Of course, the De Rham complexes $\Omega(P)$ of G -manifolds P are the basic examples of \mathfrak{g} -DG algebras. In this case L_v and i_v are just the usual Lie derivative and interior product with respect to the vector field $\rho(v)$ on P induced from v via the action of G . If P is a principal G -bundle over M , then $\Omega(P)_{\text{bas}}$ is canonically isomorphic to $\Omega(M)$.

The Weil algebra $W(\mathfrak{g})$ is a model for the De Rham complex of EG . Indeed, $W(\mathfrak{g})$ is canonically a \mathfrak{g} -DG algebra. The operators L_v are the unique derivations which, on $S^1(\mathfrak{g}^*)$ and on $\Lambda^1(\mathfrak{g}^*)$, are just the coadjoint action. The operators i_v are just the standard interior products on the exterior powers hence they act trivially on $S(\mathfrak{g}^*)$.

1.4. Equivariant cohomology

The Weil algebra is useful because it provides explicit models that compute equivariant cohomology (cf. e.g., [12, 3]). Given a G -space M , the pathological quotient M/G is often replaced by the homotopy quotient

$$M_G = (EG \times M)/G.$$

Here, EG should be thought of as a replacement of the one-point space pt with a free G -space which has the same homotopy as pt . The equivariant cohomology of M is defined as

$$H_G^\bullet(M) := H^\bullet(M_G).$$

When M is a manifold, one would like to have a more geometric De Rham model computing this cohomology. This brings us at Cartan’s model for equivariant cohomology. One defines the equivariant De Rham complex of a G -manifold M as

$$\Omega_G(M) = (S(\mathfrak{g}) \otimes \Omega(M))^G,$$

the space of G -invariant polynomials on \mathfrak{g} with values in $\Omega(M)$. The differential d_G on $\Omega_G(M)$ is very similar to the one of $W(\mathfrak{g})$:

$$d_G(P)(v) = d_{DR}(P(v)) + i_v(P(v)).$$

Let us recall how the Weil algebra leads naturally to the Cartan model. The idea is quite simple. $W(\mathfrak{g})$ is a model for the De Rham complex of EG , the similar model for $EG \times M$ is $W(\mathfrak{g}) \otimes \Omega(M)$ - viewed as a \mathfrak{g} -DG algebra with operators

$$i_v = i_v \otimes 1 + 1 \otimes i_v, \quad L_v = L_v \otimes 1 + 1 \otimes L_v,$$

and with differential

$$d = d_W \otimes 1 + 1 \otimes d_{DR}.$$

The resulting basic subcomplex should provide a model for the cohomology of the homotopy quotient. Indeed, there is an isomorphism:

$$(W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}} \cong \Omega_G(M).$$

This is best seen using Kalkman's BRST model, which is a perturbation of $W(\mathfrak{g}) \otimes \Omega(M)$. As a \mathfrak{g} -DG algebra, it has

$$i_v^K = i_v \otimes 1, \quad L_v^K = L_v \otimes 1.$$

To describe its differential, we use a basis for \mathfrak{g} as above and set:

$$d^K = d + \theta^a \otimes L_{e^a} - \omega^a \otimes i_{e^a}.$$

The resulting basic subcomplex is $\Omega_G(M)$. In fact, there is an explicit automorphism Φ of $W(\mathfrak{g}) \otimes \Omega(M)$ (the Mathai-Quillen isomorphism [16]) which transforms i_v, L_v and d into Kalkman's i_v^K, L_v^K, d^K .

2. Lie groupoids: definitions, classifying spaces and the Bott-Shulman complex

In this section we recall some basic definitions on Lie groupoids and the construction of the Bott-Shulman complex. As references, we use [5, 15, 18].

2.1. Lie groupoids

A groupoid is a category in which all arrows are isomorphisms. A Lie groupoid is a groupoid in which the space of objects G_0 and the space of arrows G_1 are smooth manifolds and all the structure maps are smooth. More explicitly, a Lie groupoid is given by a manifold of objects G_0 and a manifold of arrows G_1 together with smooth maps $s, t : G_1 \rightarrow G_0$ the source and target map, a composition map $m : G_1 \times_{G_0} G_1 \rightarrow G_1$, an inversion map $\iota : G \rightarrow G$ and an identity map $\epsilon : G_0 \rightarrow G_1$ that sends an object to the corresponding identity. These structure maps should satisfy the usual identities for a category. The source and target maps are required to be surjective submersions and therefore the domain of the composition map is a manifold. We will usually denote the space of objects of a Lie groupoid by M and say that G is a groupoid over M . We say that a groupoid is source k -connected if the fibers of the source map are k -connected.

Example 2.1. — A Lie group G can be seen as a Lie groupoid in which the space of objects is a point. Associated to any manifold M there is the pair groupoid $M \times M$ over M for which there is exactly one arrow between each pair of points. If a Lie group G acts on a manifold M there is an associated action groupoid over M denoted $G \ltimes M$ whose space of

objects is $G \times M$. Other important examples of groupoids are the holonomy and monodromy groupoids of foliations, the symplectic groupoids of Poisson geometry- some of which arise via their infinitesimal counterparts, Lie algebroids.

2.2. Lie algebroids

A Lie algebroid over a manifold M is a vector bundle $\pi : A \rightarrow M$ together with a bundle map $\rho : A \rightarrow TM$, called the anchor map and a Lie bracket in the space $\Gamma(A)$ of sections of A satisfying Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for every $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$. It follows that ρ induces a Lie algebra map at the level of sections. Examples of Lie algebroids are Lie algebras, tangent bundles, Poisson manifolds, foliations and Lie algebra actions. Given a Lie groupoid G , its Lie algebroid $A = A(G)$ is defined as follows. As a vector bundle, it is the restriction of the kernel of the differential of the source map to M . Hence, its fiber at $x \in M$ is the tangent space at the identity arrow 1_x of the source fiber $s^{-1}(x)$. The anchor map is the differential of the target map. To describe the bracket, we need to discuss invariant vector fields. A right invariant vector field on a Lie groupoid G is a vector field α which is tangent to the fibers of s and such that, if g, h are two composable arrows and we denote by R^h the right multiplication by h , then

$$\alpha(gh) = D_g(R^h)(\alpha(g)).$$

The space of right invariant vector fields is closed under the Lie bracket of vector fields and is isomorphic to $\Gamma(A)$. Thus, we get the desired Lie bracket on $\Gamma(A)$.

Unlike the case of Lie algebras, Lie's third theorem does not hold in general. Not every Lie algebroid can be integrated to a Lie groupoid. The precise conditions for the integrability are described in [10]. However, Lie's first and second theorem do hold. Due to the first one- which says that if a Lie algebroid is integrable then it admits a canonical source simply connected integration- one may often assume that the Lie groupoids under discussion satisfy this simply-connectedness assumption.

2.3. Actions

A left action of a groupoid G on a space $P \xrightarrow{\nu} M$ over M is a map $G_1 \times_M P \rightarrow P$ defined on the space $G_1 \times_M P$ of pairs (g, p) with $s(g) = \nu(p)$, which satisfies $\nu(gp) = t(g)$ and the usual conditions for actions. Associated to the action of G on $P \rightarrow M$ there is the action groupoid, denoted $G \ltimes P$. The base space is P , the space of arrows is $G_1 \times_M P$, the source map is the second projection and the target map is the action. The multiplication in this groupoid is $(g, p)(h, q) = (gh, q)$.

Example 2.2. — For a Lie groupoid G , we denote by G_k the space of strings of k composable arrows of G . When we write a string of k composable arrows (g_1, \dots, g_k) we mean that $t(g_i) = s(g_{i-1})$. Since the source and target maps are submersions, all the G_k are manifolds. Each of the G_k 's carries a natural left action. First of all, we view G_k over M via the map

$$t : G_k \rightarrow M, \quad (g_1, \dots, g_k) \mapsto t(g_1).$$

The left action of G on $G_k \xrightarrow{t} M$ is just

$$g(g_1, g_2, \dots, g_k) = (gg_1, g_2, \dots, g_k).$$

We denote by $P_{k-1}(G)$ the corresponding action groupoid.

Analogous to actions of Lie groupoids, there is the notion of infinitesimal actions. An action of a Lie algebroid A on a space $P \xrightarrow{\nu} M$ over M is a Lie algebra map $\rho_P : \Gamma(A) \rightarrow \mathfrak{X}(P)$, into the Lie algebra of vector fields on P , which is $C^\infty(M)$ -linear in the sense that

$$\rho_P(f\alpha) = (f \circ \nu)\rho_P(\alpha),$$

for all $\alpha \in \Gamma(A)$, $f \in C^\infty(M)$. Note that this last condition is equivalent to the fact that ρ_P is induced by a bundle map $\nu^*A \rightarrow TP$.

As in the case of Lie groupoids, associated to an action of A on P there is an action algebroid $A \ltimes P$ over P . As a vector bundle, it is just the pull-back of A via μ . The anchor is just the infinitesimal action ρ_P . Finally, the bracket is uniquely determined by the Leibniz identity and

$$[\mu^*(\alpha), \mu^*(\beta)] = \mu^*([\alpha, \beta]),$$

for all $\alpha, \beta \in \Gamma(A)$.

As expected, an action of a groupoid G on a space $P \xrightarrow{\nu} M$ over M induces an action of the Lie algebroid A of G on P . As a bundle map $\rho_P : \nu^*A \rightarrow TP$ it is defined fiberwise as the differential at the identity of the map

$$s^{-1}(\nu(p)) \rightarrow P, \quad g \mapsto gp.$$

Moreover, the Lie algebroid of $G \ltimes P$ is equal to $A \ltimes P$.

Example 2.3. — For the action of G on G_k (Example 2.2), the resulting algebroid $A \ltimes G_k$ is just the foliation \mathcal{F}_k of G_k by the fibers of the map $d_0 : G_k \rightarrow G_{k-1}$, which deletes g_1 from (g_1, \dots, g_k) .

2.4. Classifying spaces

We now recall the construction of the classifying space of a Lie groupoid, as the geometric realization of its nerve [19]. First of all, the nerve of G , denoted $N(G)$, is the simplicial manifold whose k -th component is $N_k(G) = G_k$, with the simplicial structure given by the face maps:

$$d_i(g_1, \dots, g_k) = \begin{cases} (g_2, \dots, g_k) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k) & \text{if } 0 < i < k, \\ (g_1, \dots, g_{k-1}) & \text{if } i = k, \end{cases}$$

and the degeneracy maps:

$$s_i(g_1, \dots, g_k) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_k)$$

for $0 \leq i \leq k$.

The thick geometric realization of a simplicial manifold X_\bullet , is defined as the quotient space

$$\|X_\bullet\| = \left(\prod_{k \geq 0} X_k \times \Delta^k \right) / \sim,$$

obtained by identifying $(d_i(p), v) \in X_k \times \Delta^k$ with $(p, \delta_i(v)) \in X_{k+1} \times \Delta^{k+1}$ for any $p \in X_{k+1}$ and any $v \in \Delta^k$. Here Δ^k denotes the standard topological k -simplex and $\delta_i : \Delta^k \rightarrow \Delta^{k+1}$ is the inclusion as the i -th face. The classifying space of a Lie groupoid G is defined as

$$BG = \|N(G)\|.$$

DEFINITION 2.4. — *The universal G -bundle of a Lie groupoid G is defined as*

$$EG = B(P_0(G)),$$

the classifying space of the groupoid associated to the action of G on itself.

The nerve of $P_0(G)$ satisfies $(P_0(G))_k = G_{k+1}$ which, for each k , is a (principal) G -space over G_k . Moreover, each face map is G -equivariant

with respect to the right action, see Example 2.2. It follows that EG is a principal G -bundle over BG

$$\begin{array}{ccc}
 G_1 & & EG \\
 \Downarrow & \nearrow \mu & \downarrow \pi \\
 G_0 & & BG
 \end{array}$$

Example 2.5. — When G is a Lie group, one recovers (up to homotopy) the usual classifying space of G and the universal principal G -bundle $EG \rightarrow BG$. More generally, for the groupoid $G \ltimes M$ associated to an action of G on M , $B(G \ltimes M)$ is a model for the homotopy quotient

$$B(G \ltimes M) \cong M_G = (EG \times M)/G.$$

2.5. The Bott-Shulman complex

In general, the geometric realization of a simplicial manifold X_\bullet is infinite dimensional and in particular, it is not a manifold. However, there is a De Rham theory that allows one to compute the cohomology of the geometric realization $\|X_\bullet\|$ with real coefficients using differential forms. Given a simplicial manifold X_\bullet the Bott-Shulman complex [5], denoted $\Omega(X_\bullet)$, is the double complex

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \uparrow & & \uparrow d & & \uparrow d & & \uparrow d \\
 \Omega^2(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots \\
 \uparrow d & & \uparrow d & & \uparrow d & & \\
 \Omega^1(X_0) & \xrightarrow{\delta} & \Omega^1(X_1) & \xrightarrow{\delta} & \Omega^1(X_2) & \xrightarrow{\delta} & \dots \\
 \uparrow d & & \uparrow d & & \uparrow d & & \\
 \Omega^0(X_0) & \xrightarrow{\delta} & \Omega^0(X_1) & \xrightarrow{\delta} & \Omega^0(X_2) & \xrightarrow{\delta} & \dots
 \end{array}$$

where the vertical differential is just the De Rham differential and the horizontal differential δ is given by the simplicial structure,

$$\delta = \sum_{i=0}^{p+1} (-1)^i d_i^*$$

The total complex of $\Omega(X_\bullet)$ is the De Rham model for the cohomology of $\|X_\bullet\|$. We will also consider the normalized Bott-Shulman complex of X_\bullet , denoted $\hat{\Omega}(X)$, which is the subcomplex of $\Omega(X_\bullet)$ that consists of forms $\eta \in \Omega^q(X_p)$ such that $s_i^*(\eta) = 0$ for all $i = 0, \dots, p - 1$. The inclusion $\hat{\Omega}(X_\bullet) \rightarrow \Omega(X_\bullet)$ induces an isomorphism in cohomology.

THEOREM 2.6 (Dupont, Bott, Shulman, Stasheff). — *There is a natural isomorphism*

$$H(\text{Tot}(\Omega(X_\bullet))) \cong H(\|X_\bullet\|).$$

For a Lie groupoid G we will write $\Omega(G_\bullet)$ instead of $\Omega(N(G))$. Note that the Bott-Shulman complex $\Omega(G_\bullet)$ provides us with an explicit model computing $H^\bullet(BG)$. However, it is rather big and unsatisfactory compared with the infinitesimal models available for Lie groups. We would like to emphasize another aspect of the Bott-Shulman complex. It is the natural place on which several geometric structures live. The best example is probably that of multiplicative forms. We first recall the definition (see, for instance, [6]).

DEFINITION 2.7. — *A multiplicative k -form on a Lie groupoid G is a k -form $\omega \in \Omega^k(G)$ satisfying*

$$d_1^*(\omega) = d_0^*(\omega) + d_2^*(\omega).$$

Given $\phi \in \Omega^{k+1}(M)$ closed, we say that ω is relatively ϕ -closed if $d\omega = s^\phi - t^*\phi$.*

In terms of the Bott-Shulman complex, the conditions appearing in the previous definition can be put together into just one: $\omega + \phi$ is a cocycle in the Bott-Shulman complex of G .

Example 2.8. — With this terminology, a symplectic groupoid is a Lie groupoid G endowed with a symplectic form ω which is multiplicative. This corresponds to the case $k = 2, \phi = 0$ in the previous definition. Symplectic groupoids arise in Poisson geometry, the global geometry of a Poisson manifold is encoded in a topological groupoid which is a symplectic groupoid provided it is smooth. In turn, smoothness holds under relatively mild topological conditions. The case $k = 2$ and ϕ -arbitrary arises from various generalizations of Poisson geometry which, in turn, show up in the study of Lie-group valued momentum maps. With these motivations, relatively closed multiplicative two forms have been intensively studied in [6] culminating with their infinitesimal description which we now recall. Given a Lie algebroid A over M and a closed 3-form ϕ on M , an IM

(infinitesimally multiplicative) form on A relative to ϕ is, by definition, a bundle map

$$\sigma : A \longrightarrow T^*M,$$

satisfying

$$\begin{aligned} \langle \sigma(\alpha), \rho(\beta) \rangle &= -\langle \sigma(\beta), \rho(\alpha) \rangle, \\ \sigma([\alpha, \beta]) &= L_{\rho(\alpha)}(\sigma(\beta)) - L_{\rho(\beta)}(\sigma(\alpha)) + d\langle \sigma(\alpha), \rho(\beta) \rangle + i_{\rho(\alpha \wedge \beta)}(\phi), \end{aligned}$$

for all $\alpha, \beta \in \Gamma(A)$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual. If A is the Lie algebroid of a Lie groupoid G , then any multiplicative 2-form ω on G which is closed relative to ϕ induces such a σ :

$$\sigma(\alpha) = i_\alpha(\omega)|_M.$$

The main result of [6] says that, if the s -fibers of G are 1-connected, then the correspondence $\omega \mapsto \sigma$ is a bijection. The basic example of this situation comes from Poisson geometry. The cotangent bundle T^*M of a Poisson manifold M carries an induced Lie algebroid structure and the identity map is an IM form. If T^*M is integrable and $\Sigma(M)$ is the (unique) Lie groupoid with 1-connected s -fibers integrating it, the corresponding multiplicative two form ω is precisely the one that makes $\Sigma(M)$ a symplectic groupoid.

3. The Weil algebra

In this section we introduce and discuss Weil algebras in the context of Lie algebroids.

Throughout the section, A is a fixed Lie algebroid over M . The Weil algebra of A , denoted $W(A)$, will be a bi-graded differential algebra. An element $c \in W^{p,q}(A)$ is a sequence $c = (c_0, c_1, \dots)$ of operators that satisfy some compatibility relation. Before explaining what each c_i is, we want to emphasize that c_0 should be viewed as the leading term of c , while the remaining terms c_1, c_2, \dots should be viewed as correction terms for c_0 . The leading term c_0 is just an antisymmetric \mathbb{R} -multilinear map

$$c_0 : \underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{p \text{ times}} \rightarrow \Omega^q(M).$$

As a general principle, the role of the higher order terms is to measure the lack of $C^\infty(M)$ -linearity of c_0 . With this in mind, one can often compute the higher terms from c_0 .

DEFINITION 3.1. — A element in $W^{p,q}(A)$ is a sequence of operators $c = (c_0, c_1, \dots)$ with

$$c_i : \underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{p-i \text{ times}} \rightarrow \Omega^{q-i}(M; S^i(A^*)),$$

satisfying

$$c_i(\alpha_1, \dots, f\alpha_{p-i}) = fc_i(\alpha_1, \dots, \alpha_{p-i}) - df \wedge \partial_{\alpha_{p-i}}(c_{i+1}(\alpha_1, \dots, \alpha_{p-i-1})),$$

for all $f \in C^\infty(M)$, $\alpha_i \in \Gamma(A)$.

Here we use the notation from the Appendix. In particular, for $\alpha \in \Gamma(A)$, $\partial_\alpha : S^k(A^*) \rightarrow S^{k-1}(A^*)$ is the partial derivative along α . Also, viewing elements of $\Omega(M; S(A^*))$ as polynomial functions on A with values in ΛT^*M , we use the notation:

$$c_i(\alpha_1, \dots, \alpha_{p-i}|\alpha) := c_i(\alpha_1, \dots, \alpha_{p-i})(\alpha) \in \Omega(M) \quad (\text{for } \alpha \in \Gamma(A)).$$

Remark 3.2. — Suppose that c, c' are elements of $W^{p,q}(A)$. If $c_0 = c'_0$ then $c = c'$ provided $q \leq \dim(M)$.

3.1. The DGA structure

We now discuss the differential graded algebra structure on $W(A)$. First of all, as in the case of the Weil algebra of a Lie algebra, the differential d of $W(A)$ is a sum of two differentials

$$d = d^v + d^h.$$

The vertical differential d^v increases q and it is induced by the De Rham differential on M in the following sense. Given $c \in W^{p,q}(A)$, the leading term of $d^v(c)$ is, up to a sign, just the De Rham differential of the leading term of c :

$$(d^v c)_0(\alpha_1, \dots, \alpha_p|\alpha) = (-1)^p d_{DR}(c_0(\alpha_1, \dots, \alpha_p|\alpha)).$$

The other components $(d^v c)_k$ ($k \geq 1$) can be found by applying the general principle mentioned above, by looking at the failure of $C^\infty(M)$ -linearity. For instance, replacing α_p with $f\alpha_p$ in the previous formula, one finds the following formula for the next component of $d^v c$:

$$(d^v c)_1(\alpha_1, \dots, \alpha_{p-1}|\alpha) = (-1)^{p-1}(d_{DR}(c_1(\alpha_1, \dots, \alpha_{p-1}|\alpha)) + c_0(\alpha_1, \dots, \alpha_{p-1}, \alpha|\alpha)).$$

Proceeding inductively, one can find the explicit formulas for all the other components. The final result, which will be taken as the complete definition of $(d^v c)$, is:

$$(d^v c)_k(\alpha_1, \dots, \alpha_{p-k} | \alpha) = (-1)^{p-k} (d_{DR}(c_k(\alpha_1, \dots, \alpha_{p-k} | \alpha)) + c_{k-1}(\alpha_1, \dots, \alpha_{p-k}, \alpha | \alpha)).$$

The horizontal differential d^h increases p and, as above, it is induced by the Koszul differential in the following sense. Given $c \in W^{p,q}(A)$, the leading term of $d^h(c)$ is given by the Koszul differential of the leading term of c , where we use $\Omega(M; SA^*)$ as a representation of the Lie algebra $\Gamma(A)$ (see the Appendix):

$$(d^h c)_0(\alpha_1, \dots, \alpha_{p+1}) = \sum_{i < j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_{p+1}) + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1})).$$

As above, replacing α_{p+1} with $f\alpha_{p+1}$ and applying the general principle, one finds the formula for the next component of $d^h c$:

$$(d^h c)_1(\alpha_1, \dots, \alpha_p | \alpha) = \delta(c_1)(\alpha_1, \dots, \alpha_p | \alpha) + (-1)^{p-1} i_{\rho(\alpha)} c_0(\alpha_1, \dots, \alpha_p).$$

Proceeding inductively, one finds the explicit formulas for all the other components. The final result, which will be taken as the complete definition of $(d^h c)$, is:

$$(d^h c)_k(\alpha_1, \dots, \alpha_{p-k+1} | \alpha) = \delta(c_k)(\alpha_1, \dots, \alpha_{p-k+1} | \alpha) + (-1)^{p-k} i_{\rho(\alpha)} c_{k-1}(\alpha_1, \dots, \alpha_{p-k+1} | \alpha).$$

Remark 3.3. — Our signs were chosen so that they coincide with the standard ones for Lie algebra actions. Admittedly, they do not look very natural.

The algebra structure on $W(A)$ is the following. Given $c \in W^{p,q}(A)$, $c' \in W^{p',q'}(A)$ we describe $cc' \in W^{p+p',q+q'}(A)$. The leading term is

$$(cc')_0(\alpha_1, \dots, \alpha_{p+p'} | \alpha) = (-1)^{qp'} \sum \text{sgn}(\sigma) c_0(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)} | \alpha) c'_0(\alpha_{\sigma(p+1)}, \dots, \alpha_{\sigma(p+p')} | \alpha),$$

where the sum is over all (p, p') -shuffles. The other components can be deduced, again, by applying the general principle we have already used. The general formula also follows from the relation with the Kalkman’s BRST algebra (Proposition 3.5 below).

THEOREM 3.4. — d^v, d^h and the product are well defined, d^v, d^h are derivations and

$$d^v d^v = 0, d^h d^h = 0, d^v d^h + d^h d^v = 0.$$

In conclusion, $W(A)$ becomes a bigraded bidifferential algebra.

Proof. — The statement is a consequence of Proposition 3.5. □

In order to shed some light into the formulas, we point out the relationship with the infinite dimensional version of Kalkman’s BRST algebra (see the Appendix). We will consider $W(\mathfrak{g}; \Omega(M))$ applied to the Lie algebra

$$\mathfrak{g}_A := \Gamma(A)$$

acting on M via the anchor map. We will use the canonical inclusion

$$W(A) \hookrightarrow W(\mathfrak{g}_A; \Omega(M))$$

to realize $W(A)$ as a subspace of Kalkman’s complex. From the explicit formulas, we deduce the following:

PROPOSITION 3.5. — *The algebra $W(A)$ is a sub-algebra of $W(\mathfrak{g}_A; \Omega(M))$. Moreover, the horizontal and the vertical differentials of $W(\mathfrak{g}_A; \Omega(M))$ restrict to $W(A)$ and coincide with the ones defined above.*

Proof. — As a vector space $W(A)$ lives inside of $W(\mathfrak{g}_A; \Omega(M))$ and a simple computation shows that it is closed under the product. The explicit formulas for the differentials in the two algebras clearly coincide on $W(A)$. Thus one only needs to show that the differentials preserve the $W(A)$. For the vertical differential we compute:

$$\begin{aligned} (d^v c)_k(\alpha_1, \dots, f\alpha_{p-k}|\alpha) &= (-1)^{p-k} (d_{DR}(c_k(\alpha_1, \dots, \alpha_{p-k}|\alpha)) \\ &\quad + c_{k-1}(\alpha_1, \dots, f\alpha_{p-k}, \alpha|\alpha)) \\ &= (-1)^{p-k} (fd_{DR}(c_k(\alpha_1, \dots, \alpha_{p-k}|\alpha)) \\ &\quad + df \wedge (c_k(\alpha_1, \dots, \alpha_{p-k}|\alpha)) \\ &\quad + df \wedge d_{DR}(\partial_{\alpha_{p-k}} c_{k+1}(\alpha_1, \dots, \alpha_{p-k-1}|\alpha)) \\ &\quad + fc_{k-1}(\alpha_1, \dots, \alpha_{p-k}, \alpha|\alpha) \\ &\quad + df \wedge \partial_{\alpha_{p-k}}(c_k(\alpha_1, \dots, \alpha_{p-k-1}, \alpha|\alpha))) \\ &= f(d^v c)_k(\alpha_1, \dots, f\alpha_{p-k}|\alpha) \\ &\quad - df \wedge (\partial_{\alpha_{p-k}}(d^v c)_{k+1}(\alpha_1, \dots, \alpha_{p-k-1}|\alpha)). \end{aligned}$$

The fact that the horizontal differentials preserves $W(A)$ follows by a similar computation once we observe that

$$\begin{aligned} \delta(c_k)(\alpha_1, \dots, f\alpha_{p-k+1}|\alpha) &= f\delta(c_k)(\alpha_1, \dots, \alpha_{p-k+1}|\alpha) \\ &\quad - df \wedge \partial_{\alpha_{p-k+1}}\delta(c_{k+1})(\alpha_1, \dots, \alpha_{p-k}|\alpha) \\ &\quad + (-1)^{p-k+1}df \wedge i_{\rho(\alpha_{p-k+1})}c_k(\alpha_1, \dots, \alpha_{p-k}|\alpha). \end{aligned}$$

□

Remark 3.6. — The previous proposition can be taken as a definition of the differentials and the product on $W(A)$. The converse is more interesting, Kalkman’s formulas can be recovered from the De Rham and Koszul differentials by computing the higher order terms.

Example 3.7. — When $A = \mathfrak{g}$ is a Lie algebra one recovers the usual Weil algebra. Also, when $A = \mathfrak{g} \ltimes M$, one recovers Kalkman’s differentials.

3.2. The Weil algebra in local coordinates

Since all the operators involved are local, it is possible to describe $W(A)$ in coordinates.

DEFINITION 3.8. — *Let (x_a) be local coordinates in a chart for M on which there is a trivialization (e_i) of the vector bundle A . Over this chart, we obtain the following algebra $W_{\text{flat}}(A)$. As a bigraded algebra, it is the commutative bigraded algebra over the space of smooth functions generated by elements ∂^a of bidegree $(0, 1)$, elements θ^i of bi-degree $(1, 0)$ and elements μ^i of bi-degree $(1, 1)$.*

There is an isomorphism between $W_{\text{flat}}(A)$ and $W(A)$, over the trivializing chars, given by:

- (1) ∂^a to $dx_a \in \Omega^1(M) = W^{0,1}(A)$.
- (2) θ^i to the duals of e_i , viewed as elements in $\Gamma(\Lambda^1 A^*) = W^{1,0}(A)$.
- (3) μ^i to the elements $\widehat{\mu}^i \in W^{1,1}(A)$, where $\widehat{\mu}^i$ is determined by the fact that $\widehat{\mu}_0^i$ vanishes on the e_i ’s, while $\widehat{\mu}_1^i$ is the dual of e_i , viewed as an element of $\Gamma(S^1 A^*)$.

The map $W_{\text{flat}}(A) \rightarrow W(A)$ is an isomorphism. The differentials can now be computed explicitly on generators and one finds (compare with [12]):

$$\begin{aligned} d_{\text{flat}}^v(\partial^a) &= 0, \\ d_{\text{flat}}^v(\theta^i) &= \mu^i, \\ d_{\text{flat}}^v(\mu^i) &= 0, \\ d_{\text{flat}}^h(\partial^a) &= -\rho_i^a \mu^i + \frac{\partial \rho_i^a}{\partial x_b} \theta^i \partial^b, \\ d_{\text{flat}}^h(\theta^i) &= -\frac{1}{2} c_{jk}^i \theta^j \theta^k, \\ d_{\text{flat}}^h(\mu^i) &= -c_{jk}^i \theta^j \mu^k + \frac{1}{2} \frac{\partial c_{jk}^i}{\partial x_a} \theta^j \theta^k \partial^a, \end{aligned}$$

where we use the Einstein summation convention, ρ_i^a are the coefficients of ρ and c_{jk}^i are the structure functions of A . Namely,

$$\rho(e_i) = \sum \rho_i^a \partial_a, \quad [e_j, e_k] = \sum c_{jk}^i e_i.$$

Note that, on smooth functions:

$$d_{\text{flat}}^v(f) = \partial_a(f) \partial^a, \quad d_{\text{flat}}^h(f) = \partial_a(f) \rho_i^a \theta^i.$$

3.3. The Weil algebra using a connection

A global version of the previous remark is possible with the help of a connection ∇ on the vector bundle A and produces a version $W_\nabla(A)$ of $W(A)$ depending on ∇ . As a bigraded algebra it is just:

$$W_\nabla^{p,q}(A) = \bigoplus_k \Gamma(\Lambda^{q-k} T^* M \otimes S^k(A^*) \otimes \Lambda^{p-k}(A^*)).$$

However, the associated operators d_∇^h, d_∇^v acting on $W_\nabla(A)$ are more involved and are computed in [2]. Working with the global ∇ , one can write down the explicit local formulas for d_∇^h, d_∇^v on generators. The resulting equations will be similar to the ones for $W_{\text{flat}}(A)$, but they have rather non-trivial extra-terms which involve the coefficients of the connection and two types of curvature tensors. The explicit map

$$I_\nabla : W_\nabla(A) \rightarrow W(A),$$

is defined as follows. It is the unique algebra map which is $C^\infty(M)$ -linear and has the properties:

- on $\Omega(M)$ and $\Gamma(\Lambda A^*)$, which are subspaces of both $W_\nabla(A)$ and $W(A)$, I_∇ is the identity.

- for $\xi \in \Gamma(S^1A^*)$, $I_{\nabla}(\xi) = \widehat{\xi}$, where

$$\widehat{\xi}_0(\alpha) = -\xi(\nabla(\alpha)), \widehat{\xi}_1 = \xi.$$

PROPOSITION 3.9. — I_{∇} is an isomorphism of bigraded algebras.

Proof. — We view I_{∇} as a map of sheaves. It suffices to show that I_{∇} is an isomorphism locally. We then use the generators ∂^a , θ^i and μ^i as above. These elements also belong to the Kalkman algebra $W(\mathfrak{g}_A, \Omega(M))$ and the map I_{∇} can be seen as a map from $W_{\nabla}(A)$ into $W(\mathfrak{g}_A, \Omega(M))$ which leaves ∂^a and θ^i invariant, but which sends μ^i into $\widehat{\mu}^i$. Since the Kalkman algebra is free commutative and the map is injective on the generators, we conclude that I_{∇} is injective. Surjectivity is a consequence of the fact that $W(A)$ is generated by the elements in $W^{0,0}(A)$, $W^{1,0}(A)$, $W^{0,1}(A)$ and the map I_{∇} is clearly a bijection in those degrees. Finally, since the differentials are derivations, it is enough to prove that they coincide in low degree, and this is a simple check. □

3.4. The Weil algebra and the adjoint representation

Let us give now a short summary of our paper [2] and explain the connection with the Weil algebra. In order to be able to talk about the adjoint representation of a Lie algebroid, one has to enlarge the category $\text{Rep}(A)$ of (standard) representations and work in the category $\text{Rep}^{\infty}(A)$ of representations up to homotopy. Such representations, by their nature, serve as coefficients for the cohomology of A . Underlying any object of $\text{Rep}^{\infty}(A)$ there is a cochain complex (E, ∂) of vector bundles over A ; the extra-structure present on (E, ∂) is a linear operation of A on E , which is not quite an action- but the failure is precisely measured and there are higher and higher coherence conditions. For instance, for the adjoint representation, the underlying complex is:

$$(3.1) \quad A \xrightarrow{\rho} TM$$

with A in degree zero, TM in degree one and zero otherwise. However, to give this complex the structures of a representation up to homotopy, one needs to use a connection ∇ on the vector bundle A . The resulting object $\text{Ad}_{\nabla} \in \text{Rep}^{\infty}(A)$ does not depend on ∇ up to isomorphism. Its isomorphism class is denoted by Ad . This indicates in particular that the resulting cohomologies with coefficients in Ad_{∇} or other associated representations (such as symmetric powers, duals etc) do not depend on ∇ and can be computed by an intrinsic complex. The rows $(W^{\bullet, q}(A), d^{\text{th}})$ of the

Weil algebra are the intrinsic complexes computing the cohomology of A with coefficient in $S^q(\text{Ad}^*)$:

$$(3.2) \quad H(W^{\bullet,q}(A)) \cong H(A; S^q(\text{Ad}^*)).$$

From this description it immediately follows that the cohomology of $W(A)$ is isomorphic to the cohomology of M - which should be expected because the fibers of the map $EG \rightarrow M$ are contractible.

Example 3.10 (Multiplicative forms). — Closed multiplicative forms on groupoids are related to homogeneous cocycles of the Weil algebra. To illustrate this, let A be the Lie algebroid of a Lie groupoid G over M . Then, any 2-form $\omega \in \Omega^2(G)$ induces an element $c \in W^{1,2}(A)$ with leading term

$$c_0 : \Gamma(A) \rightarrow \Omega^2(M), c_0(|\alpha) = L_\alpha(\omega)|_M,$$

where $\alpha \in \Gamma(A)$ is identified with the induced right invariant vector field on G and we use the inclusion $M \hookrightarrow G$ as units. The other component, $c_1 \in \Omega^1(M; S^1 A^*)$, is given by

$$c_1(|\alpha) = -i_\alpha(\omega)|_M.$$

When ω is closed c is d^v closed and when ω is multiplicative c is d^h -closed. This is an instance of the Van Est map that will be explained in the next section.

Example 3.11 (IM forms). — In turn, $(1, 2)$ cocycles on the Weil algebra of a Lie algebroid A can be identified with the IM forms on A (see Example 2.8 in the case when $\phi = 0$). To see this, we first remark that an element $c \in W^{1,2}(A)$ which is d^v -closed is uniquely determined by its component c_1 , which we interpret as a bundle map $A \rightarrow T^*M$ as before and denote it by σ . Indeed, the condition $(d^v c)_1 = 0$ gives us

$$c_0(|\alpha) = -d_{DR}(\sigma(\alpha)).$$

If c is also d^h -closed one has in particular that $(d^h c)_2 = 0$ and $(d^h c)_1 = 0$. These two conditions coincide with the conditions for σ to be an IM form (see Example 2.8). One can check directly that, conversely, these conditions also imply $(d^h c)_0 = 0$.

As a conclusion of the last two examples, the correspondence between multiplicative two-forms on groupoids and *IM*-forms on algebroids described in Example (2.8) factors through the Weil algebra. This will be generalized to arbitrary forms on the nerve of G in the next section. We will show that the main result of [6] can be derived as a consequence of a general Van Est isomorphism theorem.

We now explain a version of the Weil algebra with coefficients which will be used in the proof of our main theorem. The coefficients are the generalizations of \mathfrak{g} -DG algebras to the context of Lie algebroids.

DEFINITION 3.12. — Given a Lie algebroid A over M , an A -DG algebra is a DG-algebra (\mathcal{A}, d) together with

- a structure of $\Gamma(A)$ -DG algebra, with Lie derivatives and interior products denoted by L_α and i_α , respectively.
- a graded multiplication $\Omega(M) \otimes \mathcal{A} \rightarrow \mathcal{A}$ which makes (\mathcal{A}, d) a DG-algebra over the De Rham algebra $\Omega(M)$ and which is compatible with L_α and i_α (i.e., it is a map of $\Gamma(A)$ -DG algebras)

such that

$$i_{f\alpha}(a) = fi_\alpha(a), \quad L_{f\alpha}(a) = fL_\alpha(a) + (df)i_\alpha(a),$$

for all $\alpha \in \Gamma(A)$, $f \in C^\infty(M)$, $a \in \mathcal{A}$.

Given such an A -DG algebra, we define $W(A; \mathcal{A})$ as follows. An element $c \in W^{p,q}(A; \mathcal{A})$ is a sequence (c_0, c_1, \dots) where

$$c_i : \underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{p-i \text{ times}} \times \underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{i \text{ times}} \rightarrow \mathcal{A}^{q-i},$$

$$(\alpha_1, \dots, \alpha_{p-i}, \alpha_{p-i+1}, \dots, \alpha_p) \mapsto c_i(\alpha_1, \dots, \alpha_{p-i} | \alpha_{p-i+1}, \dots, \alpha_p)$$

is \mathbb{R} -multilinear and antisymmetric on $\alpha_1, \dots, \alpha_{p-i}$ and is $C^\infty(M)$ -multilinear and symmetric on $\alpha_{p-i+1}, \dots, \alpha_p$; moreover, c_i and c_{i+1} are required to be related as in Definition 3.1. As before, $W(A; \mathcal{A})$ sits inside Kalkman’s $W(\mathfrak{g}_A; \mathcal{A})$ and we use this inclusion to induce the algebra structure and the two differentials on $W(A; \mathcal{A})$.

Example 3.13. — The basic example of an A -DG algebra is the De Rham complex of M , in which case we recover $W(A)$. More generally, if A acts on a space $P \xrightarrow{\mu} M$ over M , $\Omega(P)$ has the structure of an A -DG algebra. In this case L_α and i_α are the usual Lie derivatives and interior products with respect to the vector fields on P induced by α , while the $\Omega(M)$ -module structure is

$$\Phi \cdot \omega = \mu^*(\Phi) \wedge \omega,$$

for $\Phi \in \Omega(M)$ and $\omega \in \Omega(P)$.

Remark 3.14. — Consider an action of A on $P \xrightarrow{\mu} M$ and the induced A -DG algebra structure on $\Omega(P)$. Then, the algebra $W(A; \Omega(P))$ is isomorphic (as a bigraded differential algebra) to $W(A \ltimes P)$, where $A \ltimes P$ is the corresponding action Lie algebroid.

4. The Van Est map

In this section we introduce the Van Est map which relates the Bott-Shulman complex of a groupoid to the Weil algebra of its algebroid.

Throughout this section, G is a Lie groupoid over M and A is its Lie algebroid. Any section $\alpha \in \Gamma(A)$ induces a vector field α^p on each of the spaces G_p of strings of p -composable arrows. Explicitly, for $g = (g_1, \dots, g_p) \in G_p$ with $t(g) = x$, α_g^p is the image of $\alpha_x \in A_x$ (i.e., in the tangent space at 1_x of $s^{-1}(x)$) by the differential of the map

$$R_g : s^{-1}(x) \longrightarrow G_p, \quad a \mapsto ag := (ag_1, g_2, \dots, g_p).$$

The map $\alpha \longrightarrow \alpha^p$ is nothing but the infinitesimal action induced by the canonical right action of G on G_p (see Subsection 2 and in particular Example 2.2). When no confusion arises, we will denote the vector field α^p simply by α . The induced Lie derivative acting on $\Omega(G_p)$, combined with the simplicial face map $s_0 : G_{p-1} \longrightarrow G_p$ (which inserts a unit on the first place) induces a map

$$R_\alpha : \Omega^q(G_p) \longrightarrow \Omega^q(G_{p-1}).$$

Intuitively, $R_\alpha(\omega)$ is the derivative on the first argument along α , at the units.

PROPOSITION 4.1. — *Let G be a Lie groupoid over M with Lie algebroid A . For any normalized form in the Bott-Shulman complex of G , $\omega \in \hat{\Omega}^q(G_p)$, the map*

$$\underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{p \text{ times}} \longrightarrow \Omega^q(M),$$

$$(\alpha_1, \dots, \alpha_p) \mapsto (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma \in S_p} \text{sgn}(\sigma) R_{\alpha_{\sigma(1)}} \dots R_{\alpha_{\sigma(p)}}(\omega)$$

is the leading term of a canonical element $V(\omega) \in W^{p,q}(A)$ induced by ω . Moreover, the resulting map

$$V : \hat{\Omega}^q(G_p) \rightarrow W^{p,q}(A)$$

is compatible with the horizontal and the vertical differentials in the sense that

$$(4.1) \quad Vd = (-1)^p d^v V,$$

$$(4.2) \quad V\delta = d^h V.$$

Remark 4.2 (More standard Van Est maps). — The standard Van Est map for a Lie groupoid G relates the differentiable cohomology $H_d^\bullet(G)$ with the cohomology $H^\bullet(A)$ of the associated Lie algebroid. These cohomologies can be identified in our framework as follows. $H_d^\bullet(G)$ is the cohomology of the first row $\Omega^0(G_\bullet)$ of the Bott-Shulman complex of G . On the other hand $H^\bullet(A)$ is the cohomology of the first row $W^{\bullet,0}(A)$ of the Weil algebra. Our Van Est map extends the ordinary one to a map of double complexes.

As in the discussions in the previous section, one can heuristically derive all the components of $V(\omega)$ out of the formula for the leading term. However, strictly speaking we have to specify the higher order terms for $V(\omega)$ to be well defined. To achieve this, we need an operation similar to R_α , but which uses interior products instead of Lie derivatives:

$$J_\alpha : \Omega^q(G_p) \longrightarrow \Omega^{q-1}(G_{p-1}), \quad J_\alpha(\omega) := s_0^*(i_\alpha(\omega)).$$

The component $V(\omega)_i$ evaluated on sections of $\Gamma(A)$,

$$V(\omega)_i(\alpha_1, \dots, \alpha_{p-i} | \alpha) \in \Omega^{q-i}(M; S^i(A^*))$$

will be a sum in which each term arises by applying the operators R_{α_k} $p-i$ times and J_α i times in all possible ways, with the appropriate sign. The summation is over all permutations $\sigma \in S_p$ such that

$$\sigma^{-1}(p-i+1) < \dots < \sigma^{-1}(p-1) < \sigma^{-1}(p).$$

We denote by $S_p(i)$ the set of all such permutations. For each $\sigma \in S_p(i)$, we consider the expression

$$V(\omega)_i^\sigma(\alpha_1, \dots, \alpha_{p-i} | \alpha) := (-1)^i D_1 \dots D_p(\omega)$$

where the ordered sequence D_1, \dots, D_p is obtained as follows. One starts with the sequence

$$R_{\alpha_{\sigma(1)}}, \dots, R_{\alpha_{\sigma(p)}}$$

and one replaces R_{α_k} by J_α whenever $k \in \{p-i+1, \dots, p\}$. With these conventions we define

$$V(\omega)_i = (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma \in S_p(i)} \text{sgn}(\sigma) V(\omega)_i^\sigma.$$

Proof of Proposition 4.1. — We first point out the following properties of the operators R_α and J_α , which follow immediately from similar properties

of the operators L_α and i_α :

$$(4.3) \quad R_\alpha = J_\alpha d + dJ_\alpha,$$

$$(4.4) \quad R_\alpha(\eta\omega) = R_\alpha(\eta)s_0^*(\omega) + s_0^*(\eta)R_\alpha(\omega),$$

$$(4.5) \quad J_\alpha(\eta\omega) = J_\alpha(\eta)s_0^*(\omega) + (-1)^q s_0^*(\eta)J_\alpha(\omega),$$

$$(4.6) \quad R_{f\alpha}(\eta) = d(f)J_\alpha(\eta) + fR_\alpha(\eta),$$

$$(4.7) \quad J_{f\alpha} = fJ_\alpha.$$

Next, R_α and J_α interact with the degeneracy maps s_i as follows:

$$(4.8) \quad s_j^*J_\alpha = J_\alpha s_{j+1}^*,$$

$$(4.9) \quad s_j^*R_\alpha = R_\alpha s_{j+1}^*.$$

The second equation follows formally from the first one and formula (4.3).

The first one follows from the simplicial relations and the equation

$$(4.10) \quad s_{j+1}^*i_{\alpha^q} = i_{\alpha^{q-1}}s_{j+1}^*.$$

In order to prove this last equation it is enough to evaluate it on a one form $\omega \in \Omega^1(G_q)$. We will use the formula

$$(ds_{j+1})_g(\alpha^{q-1}) = \alpha_{s_{j+1}(g)}^q,$$

which follows from the definition of α^q and the fact that $s_{j+1}R_g = R_{s_{j+1}(g)}$. Indeed, one simply computes:

$$\begin{aligned} i_{\alpha^{q-1}}s_{j+1}^*(\omega)_g &= s_{j+1}^*(\omega)(\alpha_g^{q-1}) \\ &= \omega(ds_{j+1})_g(\alpha_g^{q-1}) \\ &= \omega(\alpha_{s_{j+1}(g)}^q) \\ &= s_{j+1}^*i_{\alpha^q}(\omega)_g. \end{aligned}$$

In particular, the equations above imply that R_α and J_α preserve the normalized subcomplex. We will also use the $\Omega(M)$ -module structure on $\Omega(G_p)$ given by

$$\Phi\omega = t^*(\Phi) \wedge \omega.$$

As a consequence of the previous formulas we have:

$$R_\alpha(\Phi\omega) = \Phi R_\alpha(\omega), \quad J_\alpha(\Phi\omega) = (-1)^{\deg(\Phi)} \Phi J_\alpha(\omega),$$

for all $\Phi \in \Omega(M), \omega \in \hat{\Omega}(G_\bullet)$. From these and (4.6) and (4.7), it immediately follows that the components $V(\omega)_i$ satisfy the desired $C^\infty(M)$ -linearity in the symmetric variables while on the other variables we obtain the equation which expresses the relation between $V(\omega)_i$ and $V(\omega)_{i+1}$. In other words, $V(\omega)$ does belong to $W(A)$.

We will use the following remark on the functoriality of the Van Est map. Given an action of G on a space $P \xrightarrow{\mu} M$, there is an action groupoid $G \ltimes P$ over P with associated Lie algebroid $A \ltimes P$. The pull-back from M to P induces inclusions of the Weil algebra of A and of the Bott-Shulman complex of G , into the ones corresponding to $A \ltimes P$ and $G \ltimes P$, respectively (see also Remark 3.14), which is compatible with the Van Est map:

$$\begin{CD} W(A) @>V>> \hat{\Omega}(G) \\ @V{\text{incl}}VV @VV{\text{incl}}V \\ W(A \ltimes P) @>V>> \hat{\Omega}(P \ltimes G) \end{CD}$$

Also, the inclusion maps are clearly compatible with the vertical and the horizontal differentials. We will use this diagram in order to simplify the proof of the compatibility of V with the differentials. For instance, take $\omega \in \hat{\Omega}^q(G_p)$ and $q \leq \dim(M)$. In order to prove that

$$V(d(\omega)) = (-1)^p d^V(V(\omega)),$$

it suffices to show that their leading terms coincide (see Remark 3.2). However, using a G -space P with the dimension of P big enough (for a fixed ω !), the previous diagram shows that all we have to show is that

$$V(d(\omega))_0 = (-1)^p d^V(V(\omega))_0,$$

for all algebroids and all ω 's. In turn, this formula follows immediately from the definition of $d^V(\omega)_0$ and the fact that the operations R_α commute with De Rham differentials.

For the compatibility of V with the horizontal differentials, we will use the following formulas.

$$(4.11) \quad R_\alpha d_i^* = \begin{cases} d_{i-1}^* R_\alpha & \text{if } i > 1, \\ L_\alpha & \text{if } i = 1, \\ 0 & \text{if } i = 0. \end{cases}$$

$$(4.12) \quad R_\alpha L_\beta - R_\beta L_\alpha = R_{[\alpha, \beta]}.$$

Equations (4.11) follow from the simplicial equations and the following formula, which can be proven in the same way in which (4.10) was proved:

$$i_{\alpha^{q+1}} d_i^* = \begin{cases} d_i^* i_{\alpha^q} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Equation (4.12) follows immediately from $[L_\alpha, L_\beta] = L_{[\alpha, \beta]}$. We now prove the compatibility with the horizontal differentials. As before, it suffices to

show that

$$(4.13) \quad V(\delta(\omega))_0 = d^h(V(\omega))_0.$$

Assume that $\omega \in \Omega^q(G_{p-1})$ and we evaluate the right hand side on $(\alpha_1, \dots, \alpha_p)$. We have two types of terms. The first type is

$$\sum_{i=1}^p (-1)^{i+1} L_{\alpha_i}(V(\omega)(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_p)).$$

Writing out V (for each i fixed) we get a sum over permutations σ_0 of $1, \dots, \hat{i}, \dots, p$. To the pair (i, σ_0) corresponds the permutation $\sigma = (i, \sigma_0(1), \dots) \in S_p$. Note that the number $\tau(\sigma)$ of transpositions of σ equals to $i - 1 + \tau(\sigma_0)$, so the sum above equals to

$$(4.14) \quad \sum_{i=1}^p (-1)^{\frac{p(p-1)}{2}} \text{sgn}(\sigma) L_{\alpha_{\sigma(1)}}(R_{\alpha_{\sigma(2)}} \dots R_{\alpha_{\sigma(p)}} \omega).$$

The other term is

$$\sum_{i < j} (-1)^{i+j} (V\omega)([\alpha_i, \alpha_j], \alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_p).$$

Again, for each i and j , writing out V we get a sum over permutations σ_1 of the list

$$0, 1, \dots, \hat{i}, \dots, \hat{j}, \dots, p,$$

where 0 is used to index the position of $[\alpha_i, \alpha_j]$. To (i, j, σ_1) we associate

- a number $k \in \{1, \dots, p - 1\}$ defined by the condition that 0 is on the $(p - k)^{\text{th}}$ -position of σ_1 .
- a permutation $\sigma \in S_p$ which is obtained from σ_1 by inserting i on the $(p - k)^{\text{th}}$ place, and j on the $(p - k + 1)^{\text{th}}$ (so that $\sigma(p - k) = i$, $\sigma(p - k + 1) = j$ and the ordered sequence $\sigma(1), \sigma(2), \dots$ from which i and j are deleted coincides with ordered sequence $\sigma_1(0), \sigma_1(1), \dots$ from which 0 is deleted).

Note that, modulo 2, the number of transpositions of these permutations satisfy:

$$\begin{aligned} \tau(\sigma_1) &= \tau(\sigma(1), \dots, \sigma(p - k - 1), 0, \sigma(p - k + 2), \dots, \sigma(p)) \\ &= p - k + 1 + \tau(\sigma(1), \dots, \sigma(p - k - 1), \sigma(p - k + 2), \dots, \sigma(p)), \end{aligned}$$

$$\begin{aligned}
 \tau(\sigma) &= \tau(\sigma(1), \dots, \sigma(p-k-1), i, j, \sigma(p-k+2), \dots, \sigma(p)) \equiv \\
 &\equiv \tau(i, j, \sigma(1), \dots, \sigma(p-k-1), \sigma(p-k+2), \dots, \sigma(p)) \\
 &\quad + \tau(p-k, p-k+1, 1, 2, \dots) \\
 &= \tau(i, j, \sigma(1), \dots, \sigma(p-k-1), \sigma(p-k+2), \dots, \sigma(p)) \\
 &= (i-1) + (j-2) + \tau(\sigma(1), \dots, \sigma(p-k-1), \sigma(p-k+2), \dots, \sigma(p)).
 \end{aligned}$$

Thus

$$(-1)^{i+j} \text{sgn}(\sigma_1) = (-1)^{p-k} \text{sgn}(\sigma).$$

We conclude that the second term which arises from the right hand side of (4.13) is:

$$(4.15) \quad (-1)^{\frac{p(p+1)}{2}} \sum_k \sum_{\sigma: \sigma(p-k) < \sigma(p-k+1)} (-1)^k \text{sgn}(\sigma) R_{\alpha_{\sigma(1)}} \dots R_{\alpha_{\sigma(p-k-1)}} R_{[\alpha_{\sigma(p-k)}, \alpha_{\sigma(p-k+1)}]} \dots R_{\alpha_{\sigma(p)}}.$$

The left hand side of (4.13) applied to $(\alpha_1, \dots, \alpha_p)$ is

$$(-1)^{\frac{p(p+1)}{2}} \sum_{\sigma} \sum_{k=0}^p \text{sgn}(\sigma) (-1)^k R_{\alpha_{\sigma(1)}} \dots R_{\alpha_{\sigma(p)}} d_k^* \omega.$$

Using (4.11), this is equal to

$$(-1)^{\frac{p(p+1)}{2}} \sum_{k=1}^p \text{sgn}(\sigma) (-1)^k R_{\alpha_{\sigma(1)}} \dots R_{\alpha_{\sigma(p-k)}} L_{\alpha_{\sigma(p-k+1)}} \dots R_{\alpha_{\sigma(p)}}.$$

When $k = p$ we obtain precisely (4.14). It remains to show that the remaining terms give us (4.15). In that sum (over σ and $k \leq p-1$) we distinguish two cases:

- (k, σ) satisfies: $\sigma(p-k) < \sigma(p-k+1)$.
- (k, σ) -satisfies: $\sigma(p-k) > \sigma(p-k+1)$.

Note that, using the transposition $\tau_k := (p-k, p-k+1)$, we have a bijection $(k, \sigma) \mapsto (k, \sigma \circ \tau_k)$ between the first and second cases. Hence, both cases can be indexed by (k, σ) which satisfy $\sigma(p-k) < \sigma(p-k+1)$, but the second case will produce terms of type:

$$(-1)^{\frac{p(p+1)}{2}} (-\text{sgn}(\sigma)) (-1)^k R_{\alpha_{\sigma(1)}} \dots R_{\alpha_{\sigma(p-k+1)}} L_{\alpha_{\sigma(p-k)}} \dots R_{\alpha_{\sigma(p)}},$$

where we used that $\text{sgn}(\sigma \circ \tau_k) = -\text{sgn}(\sigma)$. Putting together the two cases, we obtain:

$$\begin{aligned}
 (-1)^{\frac{p(p+1)}{2}} (-1)^k \text{sgn}(\sigma) R_{\alpha_{\sigma(1)}} \dots (R_{\alpha_{\sigma(p-k)}} L_{\alpha_{\sigma(p-k+1)}} - R_{\alpha_{\sigma(p-k+1)}} L_{\alpha_{\sigma(p-k)}}) \\
 \dots R_{\alpha_{\sigma(p)}},
 \end{aligned}$$

which combined with (4.12) gives us precisely (4.15). □

5. The Van Est isomorphism

In this section we will prove the following Van Est isomorphism theorem.

THEOREM 5.1. — *Let G be a Lie groupoid with Lie algebroid A and k -connected source fibers. The homomorphism induced in cohomology by the Van Est map:*

$$V : H^p(\hat{\Omega}^q(G_\bullet)) \rightarrow H^p(W^{\bullet,q}(A)),$$

is an isomorphism for $p \leq k$ and is injective for $p = k + 1$.

Remark 5.2. — When $q = 0$ one recovers the Van Est isomorphism of [8]. In view of the isomorphism (3.2), the theorem gives an isomorphism between $H^p(\Omega^q(G_\bullet))$ and $H^p(A; S^q(\text{Ad}^*))$. When G is a Lie group and $A = \mathfrak{g}$ is a Lie algebra, this should be compared with the result of Bott [4] which gives an isomorphism between $H^p(\Omega^q(G_\bullet))$ and the differentiable cohomology $H_d^{p-q}(G; S^q \mathfrak{g}^*)$. In our coming paper [1] we will show that the result of Bott holds for arbitrary Lie groupoids.

The proof of the theorem will be divided in two steps. First we will prove that there is a homomorphism in cohomology which is an isomorphism in the required degrees. Then we will prove that this homomorphism coincides with the one induced by the Van Est map.

The first step is organized in the following co-augmented double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d^h & & \uparrow \delta^v & & \uparrow \delta^v \\
 \Omega^q(G_2) & \xrightarrow{d_0^*} & W^{0,q}(\mathcal{F}_2) & \xrightarrow{d^h} & W^{1,q}(\mathcal{F}_2) & \longrightarrow & \dots \\
 \uparrow d^h & & \uparrow \delta^v & & \uparrow \delta^v & & \\
 \Omega^q(G_1) & \xrightarrow{d_0^*} & W^{0,q}(\mathcal{F}_1) & \xrightarrow{d^h} & W^{1,q}(\mathcal{F}_1) & \longrightarrow & \dots \\
 \uparrow d^h & & \uparrow \delta^v & & \uparrow \delta^v & & \\
 \Omega^q(G_0) & \xrightarrow{d_0^*} & W^{0,q}(\mathcal{F}_0) & \xrightarrow{d^h} & W^{1,q}(\mathcal{F}_0) & \longrightarrow & \dots \\
 & & \uparrow \delta^v & & \uparrow \delta^v & & \\
 & & W^{0,q}(A) & \xrightarrow{d^h} & W^{1,q}(A) & \longrightarrow & \dots
 \end{array}$$

Let us explain how this double complex is defined. As before, G_k denotes the space of strings of k -composable arrows. Next, \mathcal{F}_k is the foliation on G_{k+1} given by the fibers of the map $d_0 : G_{k+1} \rightarrow G_k$. We interpret \mathcal{F}_k as an integrable sub-bundle of TG_{k+1} (namely the kernel of the differential of d_0), hence also as a Lie algebroid over G_{k+1} , with the inclusion as anchor. $W(\mathcal{F}_k)$ is the Weil algebra of \mathcal{F}_k and the d^h 's are the corresponding horizontal differentials. We also define $\mathcal{F}_{-1} := A$.

The maps $d_0^* : \Omega^q(G_k) \rightarrow W^{0,q}(\mathcal{F}_k) = \Omega^q(G_{k+1})$ are given by the pull-back by d_0 . To explain δ^v , we view $W(\mathcal{F}_k)$ as follows. First of all, using the action of G on G_{k+1} (Example 2.2) and the induced infinitesimal action of A on G_{k+1} , we have already mentioned that the associated Lie algebroid $A \ltimes G_{k+1}$ can be identified with \mathcal{F}_k (see Example 2.3). Hence, by Remark 3.14, we have a canonical isomorphism

$$W(\mathcal{F}_k) \cong W(A; \Omega(G_{k+1})),$$

where the left hand side is the Weil algebra with coefficients in the A -DG algebra $\Omega(G_{k+1})$ associated to the action of A on G_{k+1} (see Example 3.13). Since the simplicial maps $d_i : G_{k+1} \rightarrow G_k$ are maps of G -spaces for $i \geq 1$, we will have induced maps

$$d_i^* : W(A; \Omega(G_k)) \rightarrow W(A; \Omega(G_{k+1}))$$

which commute with d^h . We define

$$\delta^v = \sum_{i \geq 1} (-1)^i d_i^*.$$

This completes the description of the double complex. Next, we claim that the co-augmented columns of the double complex

$$0 \longrightarrow W^{p,q}(A) \xrightarrow{\delta^v} W^{p,q}(\mathcal{F}_0) \xrightarrow{\delta^v} W^{p,q}(\mathcal{F}_1) \xrightarrow{\delta^v} W^{p,q}(\mathcal{F}_1) \xrightarrow{\delta^v} \dots$$

are exact. This is a rather standard argument. Since δ^v comes from a simplicial structure arising from the nerve of G by deleting the first face map (d_0 is not used in the definition of δ^v), the first degeneracy map s_0 can be used to produce a contraction. More precisely, since $s_0^* : \Omega(G_k) \rightarrow \Omega(G_{k-1})$ respects the $\Omega(M)$ -module structure, it will induce a map:

$$s_0^* : W(A; \Omega(G_k)) \rightarrow W(A; \Omega(G_{k-1})).$$

The fact that s_0^* is a contracting homotopy follows formally from the simplicial identities:

$$\begin{aligned} \sigma_0^* \delta^v + \delta^v \sigma_0^* &= \sum_{i=1}^{j+2} (-1)^{i+1} \sigma_0^* \delta_i^* + \sum_{i=1}^{j+1} (-1)^{i+1} \delta_i^* \sigma_0^* \\ &= \text{Id} + \sum_{i=2}^{j+2} (-1)^{i+1} \sigma_0^* \delta_i^* + \sum_{i=1}^{j+1} (-1)^{i+1} \delta_i^* \sigma_0^* \\ &= \text{Id} - \sum_{i=1}^{j+1} (-1)^{i+1} \delta_i^* \sigma_0^* + \sum_{i=1}^{j+1} (-1)^{i+1} \delta_i^* \sigma_0^* = \text{Id}. \end{aligned}$$

This proves that the co-augmented columns of the double complex are exact. The standard homological algebra of double complexes implies that the map induced by the co-augmentation maps are isomorphisms:

$$a : H(W^{\bullet,q}(A)) \cong H(\text{Tot}(W^{\bullet,q}(\mathcal{F}_\bullet))).$$

Next, we look at the co-augmented rows

$$0 \longrightarrow \Omega^q(G_j) \xrightarrow{d_0^*} W^{0,q}(\mathcal{F}_j) \longrightarrow W^{1,q}(\mathcal{F}_j) \longrightarrow \dots,$$

and we show that, under the assumption in the theorem, these are exact up to degree k . Again, homological algebra implies that the map induced by the co-augmentation of the columns

$$b : H^p(\Omega^q(G_\bullet)) \rightarrow H^p(\text{Tot}(W^{\bullet,q}(\mathcal{F}_\bullet))),$$

is an isomorphism for $p < k + 1$ and is injective for $p = k + 1$. To prove the (partial) acyclicity of the rows, we will use the following lemma.

LEMMA 5.3. — *Let \mathcal{F} be a foliation given by the fibers of a submersion with homologically k -connected fibers. Then, for $0 < p \leq k$,*

$$H^p(W^{\bullet,q}(\mathcal{F})) = 0.$$

Proof. — Here we will use the interpretation of the Weil algebra in terms of representations up to homotopy given in Remark 3.1. The adjoint complex of a foliation is quasi-isomorphic to the complex $\nu[1]$ which consists of the normal bundle $\nu = TM/\mathcal{F}$ concentrated in degree 1. We now use the properties of representations up to homotopy proved in [2]. We observe that the representation up to homotopy $S^q(\text{Ad}^*)$ is quasi-isomorphic to the ordinary representation $\Lambda^q(\nu^*)$. Passing to cohomology and using the isomorphism (3.2), we deduce that

$$H^p(W^{\bullet,q}(\mathcal{F})) \cong H^p(\mathcal{F}, S^q(\nu^*)) = 0$$

where the last equation is a direct application of Theorem 2 from [8]. □

We still have to show that the map

$$a^{-1} \circ b : H^p(\Omega^q(G_\bullet)) \rightarrow H^p(W^{\bullet,q}(A)),$$

which is an isomorphism in the desired degrees, is the same as the Van Est map. More precisely we will show that, in cohomology,

$$V = \pm a^{-1} \circ b \circ e,$$

where e is the isomorphism induced by the inclusion $\hat{\Omega}(G_\bullet) \hookrightarrow \Omega(G_\bullet)$. First we observe that s_0^* , the homotopy operator for the columns of the double complex, gives a formula for the map a^{-1} . Consider an element $c \in W^{j,q}(\mathcal{F}_p)$ such that $\delta^v(c) = d^h(c) = 0$. Then, chasing the diagram we obtain that $a^{-1}(c) = (-1)^p s_0^*(d^h s_0^*)^p$. We will use this formula to compute $a^{-1} \circ b \circ e$. Take an element $\eta \in \hat{\Omega}^q(G_p)$ such that $\delta(\eta) = 0$ and compute:

$$a^{-1} \circ b \circ e(\eta) = (-1)^p s_0^*(d^h s_0^*)^p d_0^*(\eta) = (-1)^p (s_0^* d^h)^p(\eta).$$

We claim that for each $0 \leq l \leq p$:

$$(s_0^* d^h)^l(\eta)_0(\alpha_1, \dots, \alpha_l) = (-1)^{lq} \sum_{\lambda \in S_l} (-1)^{|\lambda|} R_{\alpha_{\lambda_1}} \dots R_{\alpha_{\lambda_l}}(\eta),$$

and also that $s_0^*((s_0^* d^h)^l(\eta)_0(\alpha_1, \dots, \alpha_l)) = 0$. We will prove our claim by induction on l . For $l = 0$ the claim is true because η is normalized. We assume now that the condition holds for $l - 1$ and compute:

$$\begin{aligned} & (s_0^* d^h)^l(\eta)_0(\alpha_1, \dots, \alpha_l) \\ &= s_0^*(d^h((s_0^* d^h)^{l-1}(\eta))_0(\alpha_1, \dots, \alpha_l)) \\ &= s_0^*\left(\sum_{i=1}^l (-1)^{i+q+1} L_{\alpha_i}((s_0^* d^h)^{l-1}(\eta))_0(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_l)\right) \\ &= (-1)^q \left(\sum_{i=1}^l (-1)^{i+1} R_{\alpha_i}((s_0^* d^h)^{l-1}(\eta))_0(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_l)\right) \\ &= (-1)^{lq} \sum_{\lambda \in S_l} (-1)^{|\lambda|} R_{\alpha_{\lambda_1}} \dots R_{\alpha_{\lambda_l}}(\eta). \end{aligned}$$

In particular, for $l = p$ this means that $V = \pm a^{-1} \circ b \circ e$. Since all e, b, a^{-1} are isomorphisms in the required degrees, this completes the proof.

COROLLARY 5.4. — *Let G be a Lie groupoid with Lie algebroid A and k -connected source fibers. Then, the Van Est map $V : H^p(\Omega(G_\bullet)) \rightarrow H^p(\text{Tot}(W(A)))$ is an isomorphism for $p \leq k$ and is injective for $k = p + 1$.*

Remark 5.5. — The isomorphism theorem 5.1 is the result one would expect from a topological point of view. The map V corresponds to the projection $EG \rightarrow BG$, whose fibers are isomorphic to the fibers of the source of G . In case this map is a fibration, the Lerray-Serre spectral sequence gives isomorphisms in cohomology in degrees less than the connectedness of the fiber.

6. Applications

In this section we discuss multiplicative forms from the point of view of the Van Est isomorphism theorem. In particular, we will prove Theorem 0.2 and Theorem 0.3 from the introduction.

We start with the following more precise version of Theorem 0.2, which generalizes the main result of [6] on integration of Dirac structures.

THEOREM 6.1. — *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\phi \in \Omega^{k+1}(M)$ be a closed form. Then there is a one to one correspondence between:*

- *multiplicative forms $\omega \in \Omega^k(G)$ which are ϕ -relatively closed.*
- *$C^\infty(M)$ -linear maps $\tau : \Gamma(A) \rightarrow \Omega^{k-1}(M)$ satisfying the equations:*

$$(6.1) \quad i_{\rho(\beta)}(\tau(\alpha)) = -i_{\rho(\alpha)}(\tau(\beta)),$$

$$(6.2) \quad \tau([\alpha, \beta]) = L_\alpha(\tau(\beta)) - L_\beta(\tau(\alpha)) + d_{DR}(i_{\rho(\beta)}\tau(\alpha)) + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi).$$

for all $\alpha, \beta \in \Gamma(A)$.

The correspondence is given by:

$$\tau(\alpha) = i_\alpha(\omega)|_M,$$

where, on the right hand side, $\alpha \in \Gamma(A)$ is identified with the corresponding right invariant vector field on G and the restriction to M makes use of the inclusion $M \hookrightarrow G$ as units.

First we show that the correspondence $\omega \mapsto \tau$ is well-defined, i.e., τ satisfies the equations (6.1) and (6.2). For that, we first remark that τ is precisely $V(\omega)_1 \in \Omega^{k-1}(M; S^1(A^*))$, viewed as a map $\Gamma(A) \rightarrow \Omega^{k-1}$. Since $\omega + \phi$ is a cocycle in the Bott-Shulman complex (see subsection 2), it follows that $V(\omega) + \phi$ is a cocycle in the Weil algebra. The desired equations for τ will then be implied by the following:

PROPOSITION 6.2. — *Given $\phi \in \Omega^{k+1}(M)$, $\sigma = (\sigma_0, \sigma_1) \in W^{1,k}(A)$, $\sigma + \phi$ is a cocycle in the Weil algebra if and only if:*

- (1) ϕ is a closed form and σ_1 satisfies equations (6.1), (6.2).
- (2) $\sigma_0(\alpha) = i_{\rho(\alpha)}(\phi) - d(\sigma_1(\alpha))$.

Proof. — The condition that $\sigma + \phi$ is a cocycle means that $d^v(\sigma) + d^h(\phi) = 0$. But

$$\begin{aligned} d^v(\sigma)_0(\alpha) &= -d_{DR}(\sigma_0(\alpha)), \quad (d^v\sigma)_1(|\alpha) = d_{DR}(\sigma_1(|\alpha)) + \sigma_0(\alpha), \\ (d^h\phi)_0(\alpha) &= L_{\rho(\alpha)}(\phi), \quad (d^h\phi)_1(|\alpha) = -i_{\rho(\alpha)}(\phi), \end{aligned}$$

hence we obtain the equations:

$$\sigma_0(\alpha) = i_{\rho(\alpha)}(\phi) - d(\sigma_1(\alpha)), \quad d(\sigma_0(\alpha)) = L_{\rho(\alpha)}(\phi).$$

Since the second equation is obtained by applying d_{DR} to the first one and using that ϕ is closed, we only have to keep in mind the first equation.

The other condition for $\sigma + \phi$ to be a cocycle is $d^h(\sigma) = 0$. We write the components:

$$\begin{aligned} (d^h\sigma)_0(\alpha, \beta) &= -\sigma_0([\alpha, \beta]) + L_{\rho(\alpha)}(\sigma_0(\beta)) - L_{\rho(\beta)}(\sigma_0(\alpha)), \\ (d^h\sigma)_1(\alpha|\beta) &= L_{\rho(\alpha)}(\sigma_1(\beta)) - \sigma_1([\alpha, \beta]) + i_{\rho(\beta)}(\sigma_0(\alpha)), \\ (d^h\sigma)_2(|\alpha) &= -i_{\rho(\alpha)}(\sigma_1(\alpha)). \end{aligned}$$

Clearly, $(d^h\sigma)_2 = 0$ is equivalent to (6.1). Also, $(d^h\sigma)_1 = 0$ is equivalent to

$$\sigma_1([\alpha, \beta]) = L_{\rho(\alpha)}(\sigma_1(\beta)) - L_{\rho(\beta)}(\sigma_1(\alpha)) + di_{\rho(\beta)}(\sigma_1(\alpha)) + i_{\rho(\beta)}i_{\rho(\alpha)}(\phi),$$

which is equivalent to (6.2). Finally, a simple computation shows that if $(d^h\sigma)_2 = 0$ and $(d^h\sigma)_1 = 0$ then $(d^h\sigma)_0 = 0$. □

Next, we prove that the correspondence $\omega \mapsto \tau$ in the theorem is injective. Since $\tau = V(\omega)_1$, this part follows from the following:

LEMMA 6.3. — *If G is a Lie groupoid with connected source-fibers and $\omega \in \Omega^k(G)$ is closed and multiplicative, then the following are equivalent:*

- (i) $V(\omega) = 0$.
- (ii) $V(\omega)_1 = 0$.
- (iii) $\omega_x = 0$ for all $x \in M$.
- (iv) $\omega = 0$.

Proof. — Since $V(\omega)$ is a cocycle in the Weil algebra, Proposition 6.2 tells us that $V(\omega)$ is determined by $V(\omega)_1$, hence (i) and (ii) are equivalent. In turn, from the definition of $V(\omega)_1$, (ii) means that

$$\omega(\alpha_x, V_x^2, \dots, V_x^k) = 0$$

for all $x \in M$, $V_x^k \in T_xM$, $\alpha \in \Gamma(A)$, where we identify α with the induced right invariant vector field on G . In other words, ω_x is zero when applied to

one vector tangent to the s -fiber and $(k-1)$ vectors tangent to the base. We have to show that this implies ω_x is zero when applied to all vectors. But T_xG splits as the sum of the tangent space to the s -fiber and the tangent space of M (both at x), hence it remains to show that $\omega|_M = 0$. But this follows immediately from $\omega|_M = s_0^*d^h\omega$. Finally, we have to show that (iii) implies $\omega = 0$. For this we evaluate expressions of type

$$(6.3) \quad \omega_g(\alpha_g, V_g^2, \dots, V_g^k)$$

for $g \in G$, $\alpha \in \Gamma(A)$, $V_g^i \in T_gG$ arbitrary. To make use of the multiplicativity of ω , we write

$$\alpha_g = (dm)_{y,g}(\alpha_y, 0), v_g^i = (dm)_{y,g}((dt)_g(V_g^i), V_g^i)$$

and we find that (6.3) is equal to

$$\omega(\alpha_y, (dt)_g(V_g^2), \dots, (dt)_g(V_g^k)),$$

which, by assumption, is zero. We conclude that $i_\alpha(\omega) = 0$ for all α hence, since ω is also closed, it is basic with respect to the submersion $s : G \rightarrow M$. Since the s -fibers are connected, we find θ on M such that $\omega = s^*\theta$. But $\omega|_M = 0$ implies $\theta = 0$ and then $\omega = 0$. □

End of proof of theorem 6.1. — Finally, we prove that the correspondence $\omega \mapsto \tau$ in the theorem is surjective. Note that the case $k = 2$ was proved in [6] and surjectivity was the most difficult part of the proof. Given τ , take $\sigma \in W^{1,k}(A)$ as in Proposition 6.2 with $\sigma_1 = \tau$. Since σ is d^h -closed, Theorem 5.1 implies that there exist some multiplicative form $\omega' \in \Omega^p(G)$ such that $V(\omega') = d^h(\theta) + \sigma$, for some $\theta \in \Omega^k(M)$. It is then clear that $\omega = \omega' - \delta(\theta)$ is a multiplicative form satisfying $V(\omega) = \sigma$. In particular, $V(d\omega + \delta\phi) = d^v\sigma + d^h\phi = 0$ and $d^v\omega + d^h\phi$ is both multiplicative and closed. Using the previous lemma, we find that ω is ϕ -relatively closed. This concludes the proof of Theorem 6.1. □

Next, we discuss Theorem 0.3 from the introduction. We will prove the following more precise statement.

THEOREM 6.4. — *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\omega \in \Omega^k(G)$ be a closed multiplicative k -form. Then there is a 1-1 correspondence between:*

- $\theta \in \Omega^{k-1}(G)$ multiplicative satisfying $d(\theta) = \omega$.
- $C^\infty(M)$ -linear maps $l : \Gamma(A) \rightarrow \Omega^{k-2}(M)$ satisfying

$$(6.4) \quad i_{\rho(\beta)}(l(\alpha)) = -i_{\rho(\alpha)}(l(\beta)),$$

$$(6.5) \quad c_\omega(\alpha, \beta) = -l([\alpha, \beta]) + L_{\rho(\alpha)}(l(\beta)) - L_{\rho(\beta)}(l(\alpha)) + d_{DR}(i_{\rho(\beta)}l(\alpha)).$$

where $c_\omega(\alpha, \beta) = i_{\rho(\alpha)\wedge\rho(\beta)}(\omega)|_M$. The correspondence is given by

$$l(\alpha) = -i_\alpha(\theta)|_M.$$

From the previous two theorems we immediately deduce what is the infinitesimal data associated to multiplicative forms on groupoids.

Remark 6.5. — When $k = 2$, $\Omega^{k-2}(M) = C^\infty(M)$, so that (6.4) is void while (6.5) simply becomes $c_\omega = \delta(l)$, where δ is the differential of the DeRham complex $(\Omega(A), \delta)$ of A . Hence, in this case, we obtain the following: if $\omega \in \Omega^2(G)$ is multiplicative and closed then $c_\omega \in \Omega^2(A)$ is a cocycle and there is a 1-1 correspondence between $\theta \in \Omega^1(G)$ multiplicative such that $d\theta = \omega$ and $l \in \Omega^1(A)$ satisfying $\delta(l) = \omega$. This is the statement that appears in [9].

COROLLARY 6.6. — *Let G be a source simply connected Lie groupoid over M with Lie algebroid A . Then there is a one to one correspondence between:*

- *multiplicative forms $\theta \in \Omega^k(G)$.*
- *$C^\infty(M)$ -linear maps $\tau : \Gamma(A) \rightarrow \Omega^k(M)$ and $l : \Gamma(A) \rightarrow \Omega^{k-1}(M)$ satisfying the equations:*

$$\begin{aligned} i_{\rho(\beta)}(l(\alpha)) &= -i_{\rho(\alpha)}(l(\beta)), \\ i_{\rho(\beta)}(\tau(\alpha)) &= -l([\alpha, \beta]) + L_{\rho(\alpha)}(l(\beta)) - L_{\rho(\beta)}(l(\alpha)) + d_{DR}(i_{\rho(\beta)}l(\alpha)). \\ \tau([\alpha, \beta]) &= L_\alpha(\tau(\beta)) - L_\beta(\tau(\alpha)) + d_{DR}(i_{\rho(\beta)}\tau(\alpha)), \end{aligned}$$

for all $\alpha, \beta \in \Gamma(A)$.

Proof. — A multiplicative k -form θ is determined by the following data:

- (1) A closed multiplicative $(k + 1)$ -form ω ,
- (2) A multiplicative k -form θ such that $d\theta = \omega$.

Thus, in order to reconstruct θ it suffices to apply the previous two theorems. □

We now discuss the proof. Let $\sigma = V(\omega)$ and let us first look at solutions $\xi \in W^{1,k-1}(A)$ of the equations:

$$(6.6) \quad d^v(\xi) = \sigma, d^h(\xi) = 0.$$

We use the same formulas as in the proof of Proposition 6.2 (but applied to ξ instead of σ) to write out explicitly the equations. For $d^v(\xi) = \sigma$ we find

$$-d_{DR}(\xi_0(\alpha)) = \sigma_0(\alpha), d_{DR}(\xi_1(\alpha)) + \xi_0(\alpha) = \sigma_1(\alpha).$$

As in the proof of Proposition 6.2, we only have to remember the second one. In other words, $d^V(\xi) = \sigma$ tells us that ξ_0 is determined by ξ_1 :

$$(6.7) \quad \xi_0(\alpha) = \sigma_1(\alpha) - d_{DR}(\xi_1(\alpha)).$$

The condition $d^h(\xi) = 0$ gives three equations, corresponding to the three components. For $(d^h(\xi))_2 = 0$ we find that ξ_1 must satisfy the anti-symmetry condition (6.4). For $(d^h(\xi))_1 = 0$ we find:

$$\xi_1([\alpha, \beta]) = L_{\rho(\alpha)}(\xi_1(\beta)) + i_{\rho(\beta)}(\xi_0(\alpha)).$$

Using the formula for ξ_0 in terms of ξ_1 , we find that ξ_1 must satisfy (6.5).

Next, if θ is as in the theorem, we have $\delta\theta = 0$ and $d\theta = -\omega$, i.e., $\xi := -V(\theta)$ must satisfy (6.6). The previous discussion shows that $l = \xi_1$ must satisfy the equations above. From the definition of the Van Est map it follows that $l(\alpha) = -J_\alpha(\theta) = -i_\alpha(\theta)|_M$.

Assume now that l satisfies the equations from the statement. Let $\xi \in W^{1,k-1}(A)$ with $\xi_1 = l$ and ξ_0 defined by (6.7), so that ξ satisfies (6.6). Using the Van Est isomorphism, we find $\theta' \in \Omega^{k-1}(G)$ multiplicative and $\eta \in \Omega^{k-1}(M)$ such that $V(\theta') = -\xi + d^h(\eta)$. Choose $\theta = \theta' - \delta\eta$. Then

$$V(d(\theta) - \omega) = -d^V(V(\theta)) - V(\omega) = -d^V(-\xi + d^h\eta - V(d^h\eta)) - \sigma = d^V\xi - \sigma = 0.$$

On the other hand, $d(\theta) - \omega$ is both multiplicative and closed and therefore Lemma 6.3 implies that $\omega = d(\theta)$. By construction, the l corresponding to θ is the l we started with, concluding the proof of the surjectivity. For the injectivity, one proceeds exactly as in the proof of Theorem 6.1. If θ and θ' have the same associated l and transgress ω , $\theta - \theta'$ will be multiplicative and closed with $V(\theta - \theta')_1 = l - l = 0$. In this case Lemma 6.3 implies that $\theta = \theta'$.

7. Appendix: Kalkman’s BRST algebra in the infinite dimensional case

In this paper we use some constructions which, although standard in the finite dimensional case, need some clarification in the infinite dimensional setting. Here we make these clarifications and we fix our notations. In particular, we will give an intrinsic description of Kalkman’s BRST algebra which applies also to infinite dimensional Lie algebras and more general coefficients.

7.1. Chevalley-Eilenberg complexes

For a representation V of a Lie algebra \mathfrak{g} , the action $\mathfrak{g} \otimes V \rightarrow V$ is denoted by $(\alpha, v) \mapsto L_\alpha(v)$. The Chevalley-Eilenberg complex with coefficients in V is

$$\Lambda(\mathfrak{g}^*, V) := \text{Hom}_{\mathbb{R}}(\Lambda\mathfrak{g}, V),$$

where “ $\text{Hom}_{\mathbb{R}}$ ” stands for the space of \mathbb{R} -linear maps. In degree k , $\Lambda^k(\mathfrak{g}^*, V)$ consists of antisymmetric multilinear maps depending on k -variables from \mathfrak{g} , with values in V . The Chevalley-Eilenberg differential,

$$\delta : \Lambda^p(\mathfrak{g}^*, V) \rightarrow \Lambda^{p+1}(\mathfrak{g}^*, V)$$

is given by the Koszul formula:

$$\begin{aligned} (\delta(c))(\alpha_1, \dots, \alpha_{p+1}) &= \sum_{i < j} (-1)^{i+j} c([\alpha_i, \alpha_j], \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_{p+1}) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1})). \end{aligned}$$

7.2. Symmetric powers

We now specify our conventions and notations regarding symmetric powers. For any two vector spaces E and V , the space of V -valued polynomials on E is

$$S(E^*, V) := \text{Hom}_{\mathbb{R}}(SE, V).$$

where “ $\text{Hom}_{\mathbb{R}}$ ” stands for the space of \mathbb{R} -linear maps. A polynomial of degree k will be viewed either as a symmetric k -multilinear map

$$P : \underbrace{E \times \dots \times E}_{k \text{ times}} \rightarrow V$$

or as an actual function on E with values in V : $P(\alpha) = P(\alpha, \dots, \alpha)$. We will also use the following operation. For each $\alpha \in E$ there is a partial derivative

$$\begin{aligned} \partial_\alpha : S^k(E^*, V) &\rightarrow S^{k-1}(E^*, V), \\ \partial_\alpha(P)(\alpha_0) &:= \left. \frac{d}{dt} \right|_{t=0} P(\alpha_0 + t\alpha). \end{aligned}$$

Here the derivative should be interpreted formally. In the multilinear notation this operation is:

$$\partial_\alpha(P)(\alpha_0) = kP(\alpha, \alpha_0, \dots, \alpha_0).$$

7.3. Some representations

For any representation V of \mathfrak{g} , $S(\mathfrak{g}^*; V)$ is itself a representation in a canonical way. The action

$$\mathfrak{g} \otimes S(\mathfrak{g}^*; V) \longrightarrow S(\mathfrak{g}^*; V), \quad (\alpha, P) \mapsto L_\alpha(P)$$

is induced from the coadjoint action on \mathfrak{g}^* and the given action on V . These, together with the Leibniz identity for L_α , determine the action uniquely in the finite dimensional case. We take the resulting explicit formula as the definition in the general case:

$$L_\alpha(P)(\alpha_1, \dots, \alpha_p) = L_\alpha(P(\alpha_1, \dots, \alpha_p)) - \sum_i P(\alpha_1, \dots, [\alpha, \alpha_i], \dots, \alpha_p).$$

A related representation arises in the case when $\mathfrak{g} = \Gamma(A)$ is the Lie algebra of sections of a Lie algebroid A over M . It is the algebra $\Omega(M; SA^*)$ of forms on M with values in the symmetric algebra of A . The action $(\alpha, \omega) \mapsto L_\alpha(\omega)$ is uniquely determined by

- the Leibniz derivation identity: for all $\omega, \omega' \in \Omega(M; SA^*)$,

$$L_\alpha(\omega\omega') = L_\alpha(\omega)\omega' + \omega L_\alpha(\omega').$$

- on $\Omega(M)$, L_α coincides with the usual Lie derivative $L_{\rho(\alpha)}$ along the vector field $\rho(\alpha)$.
- on $\Gamma(A^*)$, L_α is given by $L_\alpha(\xi)(\beta) = L_\alpha(\xi(\beta)) - \xi(L_\alpha(\beta))$.

Actually, $\Omega(M; SA^*)$ is just a sub-representation of $S(\mathfrak{g}^*; V)$ with $V = \Omega(M)$.

7.4. The Weil algebra with coefficients

Assume now that \mathfrak{g} is a Lie algebra and \mathcal{A} is a \mathfrak{g} -DG algebra. We define

$$W(\mathfrak{g}; \mathcal{A}) := \Lambda(\mathfrak{g}^*; S(\mathfrak{g}^*, \mathcal{A}))$$

with the following bi-grading:

$$W^{p,q}(\mathfrak{g}; \mathcal{A}) := \bigoplus_k \Lambda^{p-k}(\mathfrak{g}^*; S^k(\mathfrak{g}^*, \mathcal{A}^{q-k})).$$

For an element $c \in \Lambda^{p-k}(\mathfrak{g}^*; S^k(\mathfrak{g}^*, \mathcal{A}^{q-k}))$ we use the notation

$$c(\alpha_1, \dots, \alpha_{p-k} | \alpha) := c(\alpha_1, \dots, \alpha_{p-k})(\alpha) \in \mathcal{A}^{q-k},$$

which is an expression multilinear antisymmetric in its first entries and polynomial in the last entry. $W(\mathfrak{g}; \mathcal{A})$ has a product structure compatible with the bi-grading. For

$$c \in \Lambda^p(\mathfrak{g}^*; S^k(\mathfrak{g}^*, \mathcal{A}^q)), c' \in \Lambda^{p'}(\mathfrak{g}^*; S^{k'}(\mathfrak{g}^*, \mathcal{A}^{q'})),$$

cc' is given by

$$(cc')(\alpha_1, \dots, \alpha_{p+p'}|\alpha) = (-1)^{qp'} \sum \text{sgn}(\sigma) c(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}|\alpha) c'(\alpha_{\sigma(p+1)}, \dots, \alpha_{\sigma(p+p')}|\alpha),$$

where the sum is over all (p, p') -shuffles. The sign in front of the sum comes from the finite dimensional case and the standard sign conventions: in $W(\mathfrak{g}) \otimes \mathcal{A}$,

$$(w \otimes a)(w' \otimes a') = (-1)^{\deg(a)\deg(w')} (ww' \otimes aa').$$

7.5. Kalkman’s BRST differentials

As in the case of the standard Weil algebra, there are two differentials. The first one, d^h , increases p and is given by

$$d^h(c) = \delta(c) + i_{\mathcal{A}}(c).$$

Here δ is the Koszul differential, while

$$i_{\mathcal{A}} : \Lambda^p(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^q)) \longrightarrow \Lambda^p(\mathfrak{g}^*, S^{k+1}(\mathfrak{g}^*, \mathcal{A}^{q-1}))$$

is given by

$$i_{\mathcal{A}}(c)(\alpha_1, \dots, \alpha_p|\alpha) = (-1)^{p+1} i_{\alpha}(c(\alpha_1, \dots, \alpha_p|\alpha)).$$

Both δ and $i_{\mathcal{A}}$ are derivations (and that motivates the sign in $i_{\mathcal{A}}$).

The second differential, d^v , increases q and is given by

$$d^v(c) = d_{\mathcal{A}}(c) + i_{\mathfrak{g}}(c).$$

Here $d_{\mathcal{A}}$ is given by

$$d_{\mathcal{A}}(c)(\alpha_1, \dots, \alpha_p|\alpha) = (-1)^p d_{\mathcal{A}}(c(\alpha_1, \dots, \alpha_p|\alpha)),$$

while

$$i_{\mathfrak{g}} : \Lambda^p(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^q)) \longrightarrow \Lambda^{p-1}(\mathfrak{g}^*, S^{k+1}(\mathfrak{g}^*, \mathcal{A}^q))$$

is given by

$$i_{\mathfrak{g}}(c)(\alpha_1, \dots, \alpha_{p-1}|\alpha) = (-1)^{p+1} c(\alpha_1, \dots, \alpha_{p-1}, \alpha|\alpha).$$

Again, both $d_{\mathcal{A}}$ and $i_{\mathfrak{g}}$ are derivations.

In the case that \mathfrak{g} is finite dimensional, $W(\mathfrak{g}; \mathcal{A}) = W(\mathfrak{g}) \otimes \mathcal{A}$ and we can use a basis e^a of \mathfrak{g} to write the formulas more explicitly. We denote by θ^a the induced basis of $\Lambda^1 \mathfrak{g}^*$, by μ^a the induced basis of $S^1 \mathfrak{g}^*$ and by d_W the differential of $W(\mathfrak{g})$. From the derivation property of all the operators $\delta, i_{\mathcal{A}}, d_{\mathcal{A}}, i_{\mathfrak{g}}$ and after a straightforward checking on generators, we deduce that

$$\begin{aligned} \delta &= d_W^h \otimes 1 + \theta^a \otimes L_{e^a}, \quad i_{\mathcal{A}} = -\mu^a \otimes i_{e^a}, \\ d_{\mathcal{A}} &= 1 \otimes d_{\mathcal{A}}, \quad i_{\mathfrak{g}} = d_W^v \otimes 1. \end{aligned}$$

Hence, in this case, $d^h + d^v$ does coincide with Kalkman's differential on $W(\mathfrak{g}) \otimes \mathcal{A}$.

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