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ON THE BURNS-EPSTEIN INVARIANTS OF SPHERICAL CR 3-MANIFOLDS

by Khoi The VU (*)

Abstract. — In this paper we develop a method to compute the Burns-Epstein invariant of a spherical CR homology sphere, up to an integer, from its holonomy representation. As an application, we give a formula for the Burns-Epstein invariant, modulo an integer, of a spherical CR structure on a Seifert fibered homology sphere in terms of its holonomy representation.

Résumé. — Dans cet article nous développons une méthode pour calculer l'invariant de Burns-Epstein d'une sphère d'homologie CR sphérique, à un nombre entier près, de sa représentation d'holonomie. Comme application, nous donnons une formule pour l'invariant de Burns-Epstein, modulo un nombre entier, d'une structure CR sphérique sur une sphère d'homologie fibrée de Seifert en termes de sa représentation d'holonomie.

Dedicated to Professor Ha Huy Vui on the occasion of his sixtieth birthday.

1. Introduction

In [3], Burns and Epstein define a global, biholomorphic, $\mathbf{R}$-valued invariant μ of a compact, strictly pseudoconvex 3-dimensional CR 3-manifold M whose holomorphic tangent bundle is trivial. As the Burns-Epstein invariant is defined through the transgression form, it depends on the Cartan connection of the CR structure in a delicate way. Therefore it is not easy to compute the Burns-Epstein invariant for a general CR 3-manifold. In

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Burns-Epstein's work [3], they compute the Burns-Epstein invariant for tangent circle bundles over Riemann surfaces and Reinhardt domains in $\mathbf{C}^2$. In [4], Burns and Epstein raise the following question: "An interesting question is left open here about the relationship of these invariants to the Kähler geometry of the interior manifold, and the behavior of developing maps for CR manifolds which are locally CR equivalent to the standard sphere".

This paper is an attempt to answer the second part of this question. Namely, we show that for a spherical CR homology sphere, the Burns-Epstein invariant, modulo an integer, is basically a "topological" invariant. More precisely, it coincides with minus the Chern-Simons invariant of the holonomy representation. The main result of our is the development of a cut-and-paste method, inspired from the works of P. Kirk and E. Klassen in gauge theory (see [15, 16]), to compute the Burns-Epstein invariant, modulo an integer, of spherical CR homology spheres. We first define the normal form of a flat connection near the torus boundary. Next, we prove a result (Theorem 5.1) that expresses the change of the Chern-Simons invariant of a path of normal form flat connections on a manifold with boundary in terms of the boundary holonomy. To compute the Chern-Simons invariant on a closed manifold M, we decompose it as an union of manifolds with torus boundary. On each manifold with boundary, we try to connect our original connection to a connection whose Chern-Simons invariant is already known and then use Theorem 5.1 to compute its Chern-Simons invariant. This method has been applied successfully to find the Chern-Simons invariant of representations into $\mathrm{SU}(2), \mathrm{SL}(2,\mathbf{C}), \mathrm{SU}(n)$ (see [15, 16, 17]) and the Godbillon-Vey invariant of foliations [13].

The rest of this paper is organized as follows. In the next section, we recall some preliminaries about the universal covering group of $\mathrm{U}(2,1)$ and the Burns-Epstein invariant of a CR 3-manifold. In section 3, we show that, up to an integer, the Burns-Epstein invariant of a spherical CR homology sphere equals minus the Chern-Simons invariant of its holonomy representation. Section 4 contains technical results that allow us to define a normal form of a flat connection on a manifold with boundary. In section 5, We prove Theorem 5.1 that expresses the change of the Chern-Simons invariant of a path of flat connections in terms of the boundary holonomy. This theorem are our tool to compute the Chern-Simons invariant in section 6. There, we give an explicit formula for the Chern-Simons invariant of a Seifert fibered homology sphere. As an illustration, we carry out an explicit computation of the Burns-Epstein invariants, modulo an integer, of the homology sphere $\Sigma(2,3,11)$.

2. Preliminaries

We first introduce some notations. We denote by $diag(a,b,c)$ a 3×3 diagonal matrix whose main diagonal is (a,b,c). For a matrix A, we will use the notation $A^\dagger$ to denote its complex conjugate. Next, we recall the definition of the unitary group $\mathrm{U}(2,1)$. Let's define

$$\mathrm{U}(2,1) := \{A=(a_{ij})_{i,j=1,\dots,3} |\ a_{ij}\in\mathbf{C},\ JA^\dagger J = A^{-1}\},$$

where $J = diag(1,1,-1)$. Note that $\mathrm{U}(2,1)$ acts on the open unit ball in $\mathbf{C}^2$, a model for the complex hyperbolic space $H^2_{\mathbf{C}}$, by

$$(z,w)\longmapsto \left(\frac{a_{11}z+a_{12}w+a_{13}}{a_{31}z+a_{32}w+a_{33}}, \frac{a_{21}z+a_{22}w+a_{23}}{a_{31}z+a_{32}w+a_{33}}\right).$$

This action is transitive and the stabilizer of the origin is isomorphic to $\mathrm{U}(2)\times\mathrm{U}(1)$. It follows that the fundamental group of $\mathrm{U}(2,1)$ is isomorphic to $\mathbf{Z}\oplus\mathbf{Z}$.

In the following, we will construct the universal covering group G of $\mathrm{U}(2,1)$. Let's define

$G := \{(A,\theta_1,\theta_2)\in \mathrm{U}(2,1)\times\mathbf{R}\times\mathbf{R} \mid \ \theta_1\equiv \arg(\det(A)) \mod 2\pi, \theta_2\equiv \arg(a_{33}) \mod 2\pi\}$.

Since $A\in\mathrm{U}(2,1)$, we find that $|a_{31}|^2+|a_{32}|^2-|a_{33}|^2=-1$. Therefore $a_{33}\neq 0$, and the definition makes sense.

Let $A=(a_{ij})$ and $B=(b_{ij})$ be two matrices lying in $\mathrm{U}(2,1)$. The multiplication on G is defined by

$$(A,\theta_1,\theta_2)(B,\phi_1,\phi_2) = \left(AB, \theta_1+\phi_1, \theta_2+\phi_2+\arg\left(1+\frac{a_{31}b_{13}+a_{32}b_{23}}{a_{33}b_{33}}\right)\right).$$

Here, arg is the principal argument which takes value in $(-\pi,\pi]$.

Note that $|a_{31}b_{13}+a_{32}b_{23}|^2\leqslant (|a_{31}|^2+|a_{32}|^2)(|b_{13}|^2+|b_{23}|^2) < |a_{33}|^2|b_{33}|^2$. So we get:

$$\left|\frac{a_{31}b_{13}+a_{32}b_{23}}{a_{33}b_{33}}\right| < 1.$$

It follows that in our formula $\arg(1+\frac{a_{31}b_{13}+a_{32}b_{23}}{a_{33}b_{33}})\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Using the same argument as in [10], we see that the multiplication is well-defined and that G is indeed a covering group of $\mathrm{U}(2,1)$. To check that G is simply connected, we consider its action on the open unit ball in $\mathbf{C}^2$ through the action of the first component $A\in\mathrm{U}(2,1)$. It is not hard to see that the action is transitive and the stabilizer of the origin is homeomorphic to $\mathrm{SU}(2) \times \mathbf{R} \times \mathbf{R}$. This implies that, homotopically, G is the same as $\mathrm{SU}(2)$ therefore it is simply connected. We will also identify the Lie algebra of G with $\mathfrak{u}(2,1)-$ the Lie algebra of $\mathrm{U}(2,1)$.

Elements of $\mathrm{U}(2,1)$ can be divided into 3 types according to their action on the complex hyperbolic space $H^2_{\mathbf{C}}$ (see [5]). Namely, a matrix is called *elliptic* if it has a fixed point in $H^2_{\mathbf{C}}$. It is called *parabolic* if it has a unique fixed point in $\overline{H^2_{\mathbf{C}}}$ and this lies on $\partial H^2_{\mathbf{C}}$. And finally, a matrix is called *loxodromic* if it has exactly two fixed points in $\overline{H^2_{\mathbf{C}}}$ which lie on $\partial H^2_{\mathbf{C}}$. We will call an element of the universal covering group G elliptic, parabolic or loxodromic if its image under the projection map $G \to \mathrm{U}(2,1)$ is of the corresponding type.

Let M be a smooth, compact, oriented 3-manifold. A *contact structure* on M is an oriented 2-plane field $V = \ker \alpha$, where α is an 1-form such that $\alpha \wedge d\alpha$ is nowhere zero. A *strictly pseudoconvex CR structure* on M is a contact structure V together with a complex structure J on V. Let $V \otimes \mathbf{C} = v \oplus \bar{v}$, where $v, \bar{v}$ are the i and $-i$ eigenspaces of J respectively. We call v the *holomorphic tangent bundle* of the CR manifold M.

A CR structure, which is locally isomorphic to the standard CR structure on the unit sphere $\mathbf{S}^3 \subset \mathbf{C}^2$, is called a *spherical CR structure.* A spherical CR structure is determined by a pair (D, ρ), where $D : \tilde{M} \to \mathbf{S}^3$ is a local isomorphism and $\rho : \jmath(M) \to \mathrm{PU}(2,1)$ is the holonomy representation such that $D \circ \gamma = \rho(\gamma) \circ D,$ for all $\gamma \in \jmath(M)$. See [8] for more details about spherical CR structures.

We now briefly recall the definition of the Burns-Epstein invariant. The reader is referred to [3] for more details. Let (M,V,J) be a CR 3-manifold with trivial holomorphic tangent bundle and $\pi_Y : Y \to M$ be its CR structure bundle. We denote by π the Cartan connection form, that is an $\mathfrak{su}(2,1)$-valued 1-form on Y. Let $\Pi = d\pi + \pi \wedge \pi$ be its curvature form. Consider the following 3-form on Y :

$$TC_2(\pi) := \frac{1}{8\pi^2} Tr(\pi \wedge \Pi + \frac{1}{3}\pi \wedge \pi \wedge \pi).$$

The main theorem of Burns-Epstein [3] says that there exists a 3-form $\tilde{T}C_2(\pi)$ on M, which is defined up to an exact form, such that $\pi_Y^*(\tilde{T}C_2(\pi)) = TC_2(\pi)$. Moreover, the value of the integral $\int_M \tilde{T}C_2(\pi)$ is a biholomorphic invariant of the CR structure on M. For a given CR 3-manifold (M, V, J), this value is simply denoted by $\mu(M)$ and called the *Burns-Epstein invariant* of M.

Since the Burns-Epstein invariant is only defined when the holomorphic tangent bundle is trivial, for simplicity, we will restrict ourself to the case of homology spheres so that this condition is automatically fulfilled. With some modifications, the reader may extend our results here to the relative version of the Burns-Epstein on a general 3-manifold as defined in [6].

3. The relation between Burns-Epstein invariant and Chern-Simons invariant

Next, we recall the definition of the Chern-Simons invariant of a flat connection associated to a representation. The reader is referred to [7] for general facts about the Chern-Simons invariant. For technical reasons, we will work with the universal covering group G.

Let $\rho : \jmath(M) \longrightarrow G$ be a representation of the fundamental group of M. Consider the flat G-bundle $E_\rho := \tilde{M} \times_\rho G$ associated to ρ. Let A be the connection form of the flat connection on E_ρ, then the Chern-Simons form of A is defined by

$$CS(A) := \frac{1}{8\pi^2} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

As we have shown that homotopically G is the same as $\mathrm{SU}(2)$, standard obstruction theory implies that E_ρ is a trivial bundle. The *Chern-Simons invariant* of ρ is defined by

$$cs(\rho) := \int_M s^*(CS(A)) \mod \mathbf{Z},$$

where s is a section of E_ρ. It is not hard to see that the Chern-Simons invariant is well-defined. The reason is that the difference between two sections is, homologically, a multiple of the fiber. Moreover, the Chern-Simons form, when restricted to the fiber, is the generator of $H^3(G; \mathbf{Z}) \cong \mathbf{Z}$.

We can also define the Chern-Simons invariant for a representation ρ into the groups $\mathrm{U}(2,1), \mathrm{SU}(2,1)$ or $\mathrm{PU}(2,1)$ in the same manner as long as the associated bundle E_ρ is trivial. However we will get nothing new, since in these cases we can lift ρ to a representation $\tilde{\rho}$ into G and $cs(\rho) \equiv cs(\tilde{\rho})$. The equality of the Chern-Simons invariant of ρ and $\tilde{\rho}$ follows from two facts. The first one is that the connection form A is induced from the pullback of the Maurer-Cartan form by the projection map $\tilde{M} \times G \to G$. The second fact is that the Maurer-Cartan forms are preserved under pullback by the covering maps of Lie groups.

Next, we show that the Burns-Epstein invariant of a spherical CR structure, modulo an integer, coincides with minus the Chern-Simons invariant of the holonomy representation. So the Burns-Epstein invariant, modulo an integer, only depends on the holonomy representation.

Proposition 3.1. — *Let* (M, V, J) *be a spherical CR homology sphere whose holonomy representation is* $\rho : \jmath(M) \longrightarrow \mathrm{PU}(2,1)$ *then*

$$\mu(M) \equiv -cs(\rho) \mod \mathbf{Z}.$$

Proof. — Let $D : \tilde{M} \longrightarrow S^3$ be the developing map for the CR structure on M and $\mathcal{D}$ be the induced map on the CR structure bundles. We denote by $Y_{\tilde{M}}$ and $Y_{\mathbf{S}^3}$ the CR structure bundles over $\tilde{M}$ and $\mathbf{S}^3$ respectively. Consider the commutative diagram below

$$\begin{array}{ccc} Y_{\tilde{M}} & \xrightarrow{\mathcal{D}} & Y_{\mathbf{S}^3} \\ {\scriptstyle \pi_{\tilde{M}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbf{S}^3}} \\ \tilde{M} & \xrightarrow{D} & \mathbf{S}^3 \end{array}$$

Let A be the Cartan connection form on $Y_{\mathbf{S}^3}$. By the naturality of Cartan connections, we find that $\mathcal{D}^*(A)$ is the Cartan connection form on $Y_{\tilde{M}}$ which is denoted by $\tilde{\pi}$. As M is a homology sphere, the CR structure bundle over M is trivial. So, there exists an equivariant section $s : \tilde{M} \longrightarrow Y_{\tilde{M}}$. The equivariant 3-form $s^*(TC_2(\mathcal{D}^*(A)))$ on $\tilde{M}$ will descend to the 3-form $\tilde{T}C_2(\pi)$ on M.

As the curvature $\tilde{\Pi} = d\tilde{\pi} + \tilde{\pi} \wedge \tilde{\pi} \equiv 0$, we deduce that $d\tilde{\pi} = -\tilde{\pi} \wedge \tilde{\pi}$. Therefore, we obtain:

$$TC_2(\mathcal{D}^*(A)) = \frac{1}{8\pi^2} Tr(\tilde{\pi} \wedge \tilde{\Pi} + \frac{1}{3} \tilde{\pi} \wedge \tilde{\pi} \wedge \tilde{\pi}) = -\frac{1}{8\pi^2} Tr(\tilde{\pi} \wedge d\tilde{\pi} + \frac{2}{3} \tilde{\pi} \wedge \tilde{\pi} \wedge \tilde{\pi}).$$

The reader may have noticed that the form in the last equality above is nothing but minus the Chern-Simons form associated to $\tilde{\pi}$. This gives us the following:

$$s^*(TC_2(\mathcal{D}^*(A))) = s^*(-CS(\mathcal{D}^*(A))) = -CS((\mathcal{D} \circ s)^*(A)). \tag{3.1}$$

On the other hand, $Y_{\mathbf{S}^3}$ can be identified with the Lie group $\mathrm{SU}(2,1)$ (see [11]). Moreover, the Cartan connection A on $Y_{\mathbf{S}^3}$ is identified with the Maurer-Cartan form on $\mathrm{SU}(2,1)$.

Consider the equivariant map $\mathcal{D} \circ s : \tilde{M} \to Y_{\mathbf{S}^3} \equiv \mathrm{SU}(2,1)$. Under the above identification, it can be regarded as a section of the principal bundle $E_\rho = \tilde{M} \times_\rho \mathrm{SU}(2,1)$ over M. Therefore $(\mathcal{D} \circ s)^*(A)$ is an equivariant 1-form on $\tilde{M}$ that descends to the flat connection form on M. For simplicity, we still use $(\mathcal{D} \circ s)^*(A)$ to denote the flat connection form on M.

By (3.1) and the above reasoning, we deduce that

$$\mu(M) \equiv \int_M \tilde{T}C_2(\pi) \equiv \int_M -CS((\mathcal{D}\circ s)^*(A)) \equiv -cs(\rho) \mod \mathbf{Z}.$$

□

Remark 3.2. — In the proof of Proposition 3.1 we use the hypothesis that M is a homology sphere only to get the triviality of the CR structure bundle. So this proposition is true anytime the CR structure bundle is trivial.

According to [11], over the standard 3-sphere, the form $\tilde{T}C_2(A)$ coincides with $-\frac{1}{2\pi^2}dVol$, where $dVol$ is the volume form on the unit 3-sphere. So it follows from the proof above that

$$\mu(M) \equiv \int_M s^*(TC_2(\mathcal{D}^*(A))) \equiv \int_M (\mathcal{D}\circ s)^*(TC_2(A))$$
$$\equiv \int_M (\mathcal{D}\circ s)^*(\pi_{\mathbf{S}^3})^*(-\frac{1}{2\pi^2}dVol) \equiv -\frac{1}{2\pi^2}\int_M D^*(dVol).$$

So we deduce that if $\kappa : \tilde{M} \to \mathbf{S}^3$ is any equivariant map, regarded as a section of the associated bundle $\tilde{M}\times_\rho \mathbf{S}^3$, then

$$\mu(M) \equiv -\int_M \kappa^*(\frac{1}{2\pi^2}dVol) \mod \mathbf{Z}.$$

The reason is that two sections D and κ are homologically differed by a multiple of the fiber $\mathbf{S}^3$ and the integral of the form $\frac{1}{2\pi^2}dVol$ over the unit 3-sphere is 1. We get the following corollary:

Corollary 3.3. — *Let M be a spherical CR homology sphere. Suppose that the holonomy representation $\rho : \jmath(M) \longrightarrow \mathrm{PU}(2,1)$ is reducible then $\mu(M) \equiv 0 \mod \mathbf{Z}$.*

Proof. — As the holonomy representation ρ is reducible, we can find a common fixed point $* \in \mathbf{S}^3$ for all elements in its image. So we can take the constant map as a section of the associated bundle $\tilde{M}\times_\rho \mathbf{S}^3$ and the corollary follows. □

4. Normal forms of flat connections near the boundary torus

On a manifold with boundary, the integral of the pullback of the Chern-Simons form may depend on the way we choose the section s. To be able to define the Chern-Simons invariant on the manifold with boundary we will firstly define explicit normal forms of flat connections near the boundary. We then show that every flat connection can be gauge transformed into a normal form. This will be done by finding an explicit form, near the boundary, of the developing map associated to the flat connection.

Let X be a 3-manifold whose boundary ∂X is a torus T. We fix a pair of meridian and longitude μ, λ on T. Choose a coordinate $(e^{2\pi i x}, e^{2\pi i y})$ on T such that the corresponding map:

$$\begin{array}{ccc} \mathbf{R}^2 & \longrightarrow & T \\ (x,y) & \longmapsto & (e^{2\pi i x}, e^{2\pi i y}) \end{array}$$

sends the horizontal line to μ and the vertical line λ. Let $T \times [0,1]$ be the collar neighborhood of T in X. Suppose that $\{dx, dy, dr\}$ is an oriented basis of 1-forms on X near T. Here r is the coordinate on $[0,1]$ such that $T \times \{1\} = \partial X$ and we orient T by the "outward normal last" convention.

Let A be a flat connection form on a principal G-bundle over X with the holonomy ρ. As the bundle is trivial, we could regard A as an $\mathfrak{u}(2,1)$-valued 1-form on X. Recall that the developing map of A is a map $D_A : \tilde{X} \longrightarrow G$ such that $D_A(\alpha \circ \tilde{x}) = \rho(\alpha) \circ D_A(\tilde{x})$ for all $\alpha \in \jmath(X)$ and $\tilde{x} \in \tilde{X}$.

We will follow the scheme in [13] and define the normal form for a flat connection on X by dividing into several cases according to the type of the boundary holonomy.

(I) Elliptic: suppose that the boundary holonomies $\rho(\mu)$ and $\rho(\lambda)$ are elliptic. By conjugation we may assume that

$$\rho(\mu) = (A, 2\pi(\alpha_1 + \alpha_2 + \alpha_3), 2\pi\alpha_3) \text{ and } \rho(\lambda) = (B, 2\pi(\beta_1 + \beta_2 + \beta_3), 2\pi\beta_3).$$

Where α_i, β_i are real numbers and A, B are respectively the following matrices:

$$\begin{pmatrix} e^{2\pi i\alpha_1} & 0 & 0 \\ 0 & e^{2\pi i\alpha_2} & 0 \\ 0 & 0 & e^{2\pi i\alpha_3} \end{pmatrix}, \quad \begin{pmatrix} e^{2\pi i\beta_1} & 0 & 0 \\ 0 & e^{2\pi i\beta_2} & 0 \\ 0 & 0 & e^{2\pi i\beta_3} \end{pmatrix}.$$

We can see that, near the boundary, the developing map $D : \mathbf{R}^2 \times [0,1] \longrightarrow G$ is given by

$$D(x,y,r) = (M, 2\pi(\alpha_1 x + \alpha_2 x + \alpha_3 x + \beta_1 y + \beta_2 y + \beta_3 y), 2\pi(\alpha_3 x + \beta_3 y)).$$

Where M is the following matrix:

$$\begin{pmatrix} e^{2\pi i(\alpha_1 x + \beta_1 y)} & 0 & 0 \\ 0 & e^{2\pi i(\alpha_2 x + \beta_2 y)} & 0 \\ 0 & 0 & e^{2\pi i(\alpha_3 x + \beta_3 y)} \end{pmatrix}.$$

It follows that near $\partial X = T$, we can gauge transform the flat connection A to the form

$$D^{-1}dD = \begin{pmatrix} 2\pi i(\alpha_1 dx + \beta_1 dy) & 0 & 0 \\ 0 & 2\pi i(\alpha_2 dx + \beta_2 dy) & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx + \beta_3 dy) \end{pmatrix}.$$

(II) Loxodromy: it follows from [5, Lemma 3.2.2] that in this case the boundary holonomy can be conjugated to the form

$$\rho(\mu) = (A, 2\pi\theta_1 + 4\pi\theta_2, 2\pi\theta_2) \text{ and } \rho(\lambda) = (B, 2\pi\tau_1 + 4\pi\tau_2, 2\pi\tau_2).$$

Where θ_i, τ_i, u, v in the formula are real numbers and A, B are respectively the following matrices:

$$\begin{pmatrix} e^{2\pi i\theta_1} & 0 & 0 \\ 0 & e^{2\pi i\theta_2}\cosh u & e^{2\pi i\theta_2}\sinh u \\ 0 & e^{2\pi i\theta_2}\sinh u & e^{2\pi i\theta_2}\cosh u \end{pmatrix}, \begin{pmatrix} e^{2\pi i\tau_1} & 0 & 0 \\ 0 & e^{2\pi i\tau_2}\cosh v & e^{2\pi i\tau_2}\sinh v \\ 0 & e^{2\pi i\tau_2}\sinh v & e^{2\pi i\tau_2}\cosh v \end{pmatrix}.$$

The developing map, near the boundary, is given by

$$D(x, y, r) = (M, 2\pi(\theta_1 x + \tau_1 y) + 4\pi(\theta_2 x + \tau_2 y), 2\pi(\theta_2 x + \tau_2 y)).$$

Where M is the matrix

$$\begin{pmatrix} e^{2\pi i(\theta_1 x+\tau_1 y)} & 0 & 0 \\ 0 & e^{2\pi i(\theta_2 x+\tau_2 y)}\cosh(ux + vy) & e^{2\pi i(\theta_2 x+\tau_2 y)}\sinh(ux + vy) \\ 0 & e^{2\pi i(\theta_2 x+\tau_2 y)}\sinh(ux + vy) & e^{2\pi i(\theta_2 x+\tau_2 y)}\cosh(ux + vy) \end{pmatrix}.$$

So, near the boundary, the connection has the form

$$D^{-1}dD = \begin{pmatrix} 2\pi i(\theta_1 dx + \tau_1 dy) & 0 & 0 \\ 0 & 2\pi i(\theta_2 dx + \tau_2 dy) & udx + vdy \\ 0 & udx + vdy & 2\pi i(\theta_2 dx + \tau_2 dy) \end{pmatrix}$$

(III) Parabolic: according to [5, Thm. 3.4.1], a parabolic element $g \in \mathrm{U}(2,1)$ has a unique decomposition $g = pe = ep$, where p is unipotent and e is elliptic. Furthermore, parabolic elements of $\mathrm{U}(2,1)$ can be divided into two types according to whether the minimal polynomial of the unipotent part p is $(x-1)^3$ or $(x-1)^2$.

If the minimal polynomial of p is $(x-1)^3$ then g can be conjugated to the form $\begin{pmatrix} 1-s & \bar{a} & s \\ -a & 1 & a \\ -s & \bar{a} & 1+s \end{pmatrix} e^{2\pi i\alpha}$ with $\Re(s) = \frac{|a|^2}{2}$. Note that using a further conjugation by an appropriate diagonal matrix, we may assume that a is real.

In case g has the unipotent part p with the minimal polynomial $(x-1)^2$, it can be conjugated to the following form:

$$\begin{pmatrix} e^{2\pi i\theta_1}(1-ip) & 0 & ipe^{2\pi i\theta_1} \\ 0 & e^{2\pi i\theta_2} & 0 \\ -ipe^{2\pi i\theta_1} & 0 & e^{2\pi i\theta_1}(1+ip) \end{pmatrix}.$$

So we consider two cases.
Case 1: by conjugation, we may assume that

$$\rho(\mu) = (A, 6\pi\alpha, \arctan(\frac{p}{1+a^2/2}) + 2\pi\alpha),$$

$$\text{where } A = \begin{pmatrix} 1-\frac{a^2}{2}-ip & a & \frac{a^2}{2}+ip \\ -a & 1 & a \\ -\frac{a^2}{2}-ip & a & 1+\frac{a^2}{2}+ip \end{pmatrix} e^{2\pi i\alpha}.$$

As $\rho(\lambda)$ is of the same type and commutes with $\rho(\mu)$, it is not hard to check that $\rho(\lambda)$ must have the following form:

$$\rho(\lambda) = (B, 6\pi\beta, \arctan(\frac{q}{1+b^2/2}) + 2\pi\beta).$$

Here the matrix B has a similar form:

$$B = \begin{pmatrix} 1-\frac{b^2}{2}-iq & b & \frac{b^2}{2}+iq \\ -b & 1 & b \\ -\frac{b^2}{2}-iq & b & 1+\frac{b^2}{2}+iq \end{pmatrix} e^{2\pi i\beta}.$$

Note that all the parameters $a, b, p, q, \alpha, \beta$ are real numbers.

We then can choose the developing map to be

$$D(x,y,r) = (M, 6\pi(\alpha x+\beta y), \arctan(\frac{px+qy}{1+(ax+by)^2/2}) + 2\pi(\alpha x+\beta y)),$$

$$\text{where } M = \begin{pmatrix} 1-\frac{(ax+by)^2}{2}- i(px+qy) & (ax+by) & \frac{(ax+by)^2}{2}+ i(px+qy) \\ -(ax+by) & 1 & (ax+by) \\ -\frac{(ax+by)^2}{2} -i(px+qy) & (ax+by) & 1+\frac{(ax+by)^2}{2}+ i(px+qy) \end{pmatrix} e^{2\pi i(\alpha x+\beta y)}.$$

By straightforward computations, we deduce that, near the boundary, the connection form $D^{-1}dD$ is the following:

$$\begin{pmatrix} -i(pdx+qdy) \\ 2\pi i(\alpha dx+\beta dy) & (adx+bdy) & i(pdx+qdy) \\ -(adx+bdy) & 2\pi i(\alpha dx+\beta dy) & (adx+bdy) \\ -i(pdx+qdy) & (adx+bdy) & i(pdx+qdy)+ \\ & & 2\pi i(\alpha dx+\beta dy) \end{pmatrix}.$$

Case 2: after conjugation, we may assume that

$$\rho(\mu) = (A, 4\pi\theta_1 + 2\pi\theta_2, 2\pi\theta_1 + \arctan(p))$$

$$\text{and } \rho(\lambda) = (B, 4\pi\tau_1 + 2\pi\tau_2, 2\pi\tau_1 + \arctan(q)).$$

Where the parameters θ_i, τ_i, p, q are real numbers and

$$A = \begin{pmatrix} e^{2\pi i\theta_1}(1-ip) & 0 & ipe^{2\pi i\theta_1} \\ 0 & e^{2\pi i\theta_2} & 0 \\ -ipe^{2\pi i\theta_1} & 0 & e^{2\pi i\theta_1}(1+ip) \end{pmatrix},$$

$$B = \begin{pmatrix} e^{2\pi i\tau_1}(1-iq) & 0 & iqe^{2\pi i\tau_1} \\ 0 & e^{2\pi i\tau_2} & 0 \\ -iqe^{2\pi i\tau_1} & 0 & e^{2\pi i\tau_1}(1+iq) \end{pmatrix}.$$

So the developing map is of the form

$$D(x,y,r)$$
$$= (M, 4\pi(\theta_1 x + \tau_1 y) + 2\pi(\theta_2 x + \tau_2 y), 2\pi(\theta_1 x + \tau_1 y) + \arctan(px+qy)).$$

Where M is the following matrix:

$$\begin{pmatrix} e^{2\pi i(\theta_1 x+\tau_1 y)} - \\ i(px+qy)e^{2\pi i(\theta_1 x+\tau_1 y)} & 0 & i(px+qy)e^{2\pi i(\theta_1 x+\tau_1 y)} \\ 0 & e^{2\pi i(\theta_2 x+\tau_2 y)} & 0 \\ -i(px+qy)e^{2\pi i(\theta_1 x+\tau_1 y)} & 0 & e^{2\pi i(\theta_1 x+\tau_1 y)} + \\ & & i(px+qy)e^{2\pi i(\theta_1 x+\tau_1 y)} \end{pmatrix}.$$

After some lengthy computations, we find the connection form $D^{-1}dD$ to be

$$\begin{pmatrix} 2\pi i(\theta_1 dx + \tau_1 dy) - i(pdx+qdy) & 0 & i(pdx+qdy) \\ 0 & 2\pi i(\theta_2 dx + \tau_2 dy) & 0 \\ -i(pdx+qdy) & 0 & 2\pi i(\theta_1 dx + \tau_1 dy) + i(pdx+qdy) \end{pmatrix}.$$

Definition 4.1. — *We say that a flat G-connection form on a manifold with torus boundary X is in normal form if, near the boundary, it has one of the forms as in the cases (I), (II) and (III) above.*

Note that one connection may have different normal forms as we can add integers to the exponential parameters in the holonomy matrices without changing the matrices themselves.

For the computation of the Burns-Epstein invariant, we need to bring connections to the normal forms. The following proposition shows that we can always do so.

Proposition 4.2. — *a) Let A be a flat connection form on X. Then there exists a gauge transformation that brings A into a normal form.*
b) Let A_t be a path of flat connection 1-forms on X. Then there exists a path of gauge transformations g_t such that $g_t \cdot A_t$ is in normal form for all t.

Proof. — We have shown above that, near the torus boundary, we can always gauge transform the flat connection into a normal form. By using standard obstruction theory argument as in [16, Prop. 2.3], we can extend the gauge transformation to all of X and therefore prove the proposition. □

The next lemma tells us that the integral of the Chern-Simons form of a normal form flat connection on a manifold with boundary is gauge invariant.

Lemma 4.3. — *Let A and B be two normal form flat connections on a manifold with torus boundary X. Suppose that:*

- *A and B are gauge equivalent.*
- *A and B are in normal form and equal near the boundary.*

Then we get the equality $\int_X CS(A) \equiv \int_X CS(B) \mod \mathbf{Z}$.

Proof. — The proof is similar to the one of [16, Thm. 2.4], so we will leave it to the reader as an exercise. □

By Lemma 4.3, we may define the *Chern-Simons invariant of a normal form flat connection A* as follows:

$$cs(A) := \int_X CS(A) \mod \mathbf{Z}.$$

5. Variations of the Chern-Simons invariant

In this section, we will prove the main technical tool for computing the Chern-Simons invariant. Our result is a formula that expresses the change of the Chern-Simons invariant of a path of normal form flat connections in terms of the boundary holonomy.

THEOREM 5.1. — *Let A_t be a path of normal form flat connections on a manifold with torus boundary X. Suppose that $\rho_t : \jmath(X) \longrightarrow G$ is the corresponding path of holonomy representations. In the following, we consider several cases.*
(I) Elliptic: let $\rho_t(\mu)$ and $\rho_t(\lambda)$ be elliptic elements. Suppose that, near the boundary, A_t has the form

$$\begin{pmatrix} 2\pi i(\alpha_1(t)dx+\beta_1(t)dy) & 0 & 0 \\ 0 & 2\pi i(\alpha_2(t)dx+\beta_2(t)dy) & 0 \\ 0 & 0 & 2\pi i(\alpha_3(t)dx+\beta_3(t)dy) \end{pmatrix}.$$

Then

$$cs(A_1)-cs(A_0) \equiv \frac{1}{2}\int_0^1 [(\alpha_1\dot{\beta}_1-\dot{\alpha}_1\beta_1)+(\alpha_2\dot{\beta}_2-\dot{\alpha}_2\beta_2)+(\alpha_3\dot{\beta}_3-\dot{\alpha}_3\beta_3)]dt.$$

(II) Loxodromy: let $\rho_t(\mu)$ and $\rho_t(\lambda)$ be loxodromy elements. Suppose that, near the boundary, A_t has the form

$$\begin{pmatrix} 2\pi i(\theta_1(t)dx+\tau_1(t)dy) & 0 & 0 \\ 0 & 2\pi i(\theta_2(t)dx+\tau_2(t)dy) & u(t)dx+v(t)dy \\ 0 & u(t)dx+v(t)dy & 2\pi i(\theta_2(t)dx+\tau_2(t)dy) \end{pmatrix}.$$

Then

$$\begin{aligned} cs(A_1)-cs(A_0) \equiv \frac{1}{2}\int_0^1 (\theta_1\dot{\tau}_1-\dot{\theta}_1\tau_1)dt \\ +\int_0^1(\theta_2\dot{\tau}_2-\dot{\theta}_2\tau_2)dt+\frac{1}{4\pi^2}\int_0^1(\dot{u}v-u\dot{v})dt. \end{aligned}$$

(III) Parabolic:

Case 1: let $\rho_t(\mu)$ and $\rho_t(\lambda)$ be parabolic elements whose unipotent parts have minimal polynomial $(x-1)^3$. Suppose that, near the boundary, A_t has the form

$$\begin{pmatrix} \begin{matrix}-i(p(t)dx+q(t)dy)+\\ 2\pi i(\alpha(t)dx+\beta(t)dy)\end{matrix} & a(t)dx+b(t)dy & i(p(t)dx+q(t)dy) \\ -(a(t)dx+b(t)dy) & 2\pi i(\alpha(t)dx+\beta(t)dy) & a(t)dx+b(t)dy \\ -i(p(t)dx+q(t)dy) & a(t)dx+b(t)dy & \begin{matrix}i(p(t)dx+q(t)dy)+\\ 2\pi i(\alpha(t)dx+\beta(t)dy)\end{matrix} \end{pmatrix}.$$

Then

$$cs(A_1)-cs(A_0)\equiv\frac{3}{2}\int_0^1(\alpha\dot{\beta}-\dot{\alpha}\beta)dt.$$

Case 2: let $\rho_t(\mu)$ and $\rho_t(\lambda)$ be parabolic elements whose unipotent parts have minimal polynomial $(x-1)^2$. Suppose that, near the boundary, A_t has the form

$$\begin{pmatrix} \begin{matrix}2\pi i(\theta_1(t)dx+\tau_1(t)dy)-\\ i(p(t)dx+q(t)dy)\end{matrix} & 0 & i(p(t)dx+q(t)dy) \\ 0 & \begin{matrix}2\pi i\theta_2(t)dx+\\ 2\pi i\tau_2(t)dy\end{matrix} & 0 \\ -i(p(t)dx+q(t)dy) & 0 & \begin{matrix}2\pi i(\theta_1(t)dx+\tau_1(t)dy)+\\ i(p(t)dx+q(t)dy)\end{matrix} \end{pmatrix}.$$

Then

$$cs(A_1)-cs(A_0)\equiv\int_0^1(\theta_1\dot{\tau}_1-\dot{\theta}_1\tau_1)dt+\frac{1}{2}\int_0^1(\theta_2\dot{\tau}_2-\dot{\theta}_2\tau_2)dt.$$

Proof. — Consider the path A_t as a connection form on $X\times[0,1]$, then it is a flat connection, that is $F^{A_t}\wedge F^{A_t}\equiv 0$. by Stokes' Theorem

$$\int_X CS(A_1)-\int_X CS(A_0)-\int_{\partial X\times[0,1]}CS(A_t)=\int_{X\times[0,1]}tr(F^{A_t}\wedge F^{A_t})=0.$$

So

$$cs(A_1)-cs(A_0)\equiv\int_{\partial X\times[0,1]}\frac{1}{8\pi^2}tr(A_t\wedge dA_t+\frac{2}{3}A_t\wedge A_t\wedge A_t).$$

We now use this formula and the normal form of the connections to compute the changes of Chern-Simons invariant in each case.

(I) Elliptic: in this case $tr(A_t\wedge dA_t+\frac{2}{3}A_t\wedge A_t\wedge A_t)=4\pi^2[(\alpha_1\dot{\beta}_1-\dot{\alpha}_1\beta_1)+(\alpha_2\dot{\beta}_2-\dot{\alpha}_2\beta_2)+(\alpha_3\dot{\beta}_3-\dot{\alpha}_3\beta_3)]dx\wedge dy\wedge dt$. So the result follows.

(II) Loxodromy: after some computation, we find that
$tr(A_t \wedge dA_t + \frac{2}{3} A_t \wedge A_t \wedge A_t) = 4\pi^2(\theta_1 \dot{\tau_1} - \dot{\theta_1}\tau_1) dx \wedge dy \wedge dt +$
$8\pi^2(\theta_2 \dot{\tau_2} - \dot{\theta_2}\tau_2) dx \wedge dy \wedge dt + 2(\dot{u}v - u\dot{v}) dx \wedge dy \wedge dt.$
So we deduce the required formula.
(III) Parabolic:
Case 1: straightforward computations show that

$$tr(A_t \wedge dA_t + \frac{2}{3} A_t \wedge A_t \wedge A_t) = 12\pi^2(\alpha\dot{\beta} - \dot{\alpha}\beta) dx \wedge dy \wedge dt.$$

So the change in the Chern-Simons invariant is given by the stated formula.
Case 2: in this case, we find that
$tr(A_t \wedge dA_t + \frac{2}{3} A_t \wedge A_t \wedge A_t) = 8\pi^2(\theta_1 \dot{\tau_1} - \dot{\theta_1}\tau_1) dx \wedge dy \wedge dt +$
$4\pi^2(\theta_2 \dot{\tau_2} - \dot{\theta_2}\tau_2) dx \wedge dy \wedge dt.$
So the formula follows. □

In the last part of this section, we will study the difference between the Chern-Simons invariant of two different normal form connections in the elliptic case. This result will be used in the computation of the next section. Our result is similar to Theorem 2.5 of [16].

Consider a manifold with torus boundary X. Let A be a normal form flat connection that has the following form near the boundary ∂X:

$$\begin{pmatrix} 2\pi i(\alpha_1 dx + \beta_1 dy) & 0 & 0 \\ 0 & 2\pi i(\alpha_2 dx + \beta_2 dy) & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx + \beta_3 dy) \end{pmatrix}.$$

We define $h : \mathbf{S}^1 \to G$ by $h(e^{2\pi i\theta}) = (diag(e^{2\pi i\theta}, e^{-2\pi i\theta}, 1), 0, 0)$. It is not hard to see that h is a nullhomotopic map into G. So we may find a path $h_t : \mathbf{S}^1 \to G,\ 0 \leqslant t \leqslant 1$, such that:

- h_t is constant when t is near 0 or t is near 1.
- $h_0 \equiv 1 \in G$ and $h_1 = h$.

Next we use the map h_t above to define two gauge transformations on X as follows. Recall that we denote by $T \times [0,1]$, with the coordinate $(e^{2\pi ix}, e^{2\pi iy}, r)$, a collar neighborhood of ∂X in X such that $T \times \{1\} = \partial X$. Let $g_x, g_y : T \times [0,1] \to G$ be maps that are defined by

$$g_x(e^{2\pi ix}, e^{2\pi iy}, r) = h_r(e^{2\pi ix}) \text{ and } g_y(e^{2\pi ix}, e^{2\pi iy}, r) = h_r(e^{2\pi iy}).$$

Note that we can extend g_x and g_y to be identical to $1 \in G$ outside the collar neighborhood.

It is not hard to check that $g_x \cdot A$ and $g_y \cdot A$ are also in normal form. In fact, near ∂X, we have

$$g_x \cdot A = \begin{pmatrix} \begin{matrix}2\pi i(\alpha_1+1)dx+\\ 2\pi i\beta_1 dy\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}2\pi i(\alpha_2-1)dx+\\ 2\pi i\beta_2 dy\end{matrix} & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx+\beta_3 dy) \end{pmatrix} \text{ and}$$

$$g_y \cdot A = \begin{pmatrix} \begin{matrix}2\pi i\alpha_1 dx+\\ 2\pi i(\beta_1+1)dy\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}2\pi i\alpha_2 dx+\\ 2\pi i(\beta_2-1)dy\end{matrix} & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx+\beta_3 dy) \end{pmatrix}.$$

We now state the following theorem:

Theorem 5.2. — *Let A and B be two gauge equivalent normal form flat connections on X. Suppose that*

$$A = \begin{pmatrix} 2\pi i(\alpha_1 dx+\beta_1 dy) & 0 & 0 \\ 0 & 2\pi i(\alpha_2 dx+\beta_2 dy) & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx+\beta_3 dy) \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \begin{matrix}2\pi i(\alpha_1+m)dx+\\ 2\pi i(\beta_1+n)dy\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}2\pi i(\alpha_2-m)dx+\\ 2\pi i(\beta_2-n)dy\end{matrix} & 0 \\ 0 & 0 & 2\pi i(\alpha_3 dx+\beta_3 dy) \end{pmatrix}$$

near the boundary ∂X, for some integers m and n. Then

$$cs(B) - cs(A) \equiv m(\beta_1-\beta_2)/2 - n(\alpha_1-\alpha_2)/2 \mod \mathbf{Z}.$$

Proof. — By Lemma 4.3, if g is a gauge transformation such that $g\cdot A \equiv B$ near ∂X then $cs(g\cdot A) \equiv cs(B) \mod \mathbf{Z}$. Therefore, it is enough to prove that $cs(g_x \cdot A) - cs(A) \equiv (\beta_1-\beta_2)/2 \mod \mathbf{Z}$ and $cs(g_y\cdot A) - cs(A) \equiv (\alpha_2-\alpha_1)/2 \mod \mathbf{Z}$.

By Proposition 1.27(e) of [9], the difference between two Chern-Simons forms $CS(g_x\cdot A) - CS(A)$ is equal to

$$\frac{1}{8\pi^2} d(Tr(g_x^{-1}Ag_x \wedge g_x^{-1}dg_x)) - \frac{1}{24\pi^2} Tr(g_x^{-1}dg_x \wedge g_x^{-1}dg_x \wedge g_x^{-1}dg_x).$$

It follows from the definition of g_x that $\frac{\partial g_x}{\partial y} = 0$. So the last term in this formula vanishes. By direct computation, we deduce that

$$cs(g_x \cdot A) - cs(A) \equiv \frac{1}{8\pi^2} \int_T Tr(g_x^{-1} A g_x \wedge g_x^{-1} dg_x) = \frac{1}{2} \int_T (\beta_1 - \beta_2) dxdy \equiv (\beta_1 - \beta_2)/2.$$

By using a similar argument, we can show that the formula for g_y also holds. Thus the theorem follows. □

6. Applications

In this section, we will apply our main theorem to find the Chern-Simons invariant of representations of a Seifert fibered homology sphere. We also give an explicit example where we find all the Chern-Simons invariants of the manifold and deduce result about the Burns-Epstein invariant.

Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere, where $a_i > 1$ are pairwise relatively prime integers. We put $a := a_1 \cdots a_n$. We denote by $\mathcal{R}^*(\Sigma)$ the space of irreducible representations from $\jmath(\Sigma)$ to G.

The fundamental group of Σ is given by

$$\jmath(\Sigma) = \{x_1, \dots, x_n, h \mid h \text{ is central, } x_1^{a_1} h^{b_1} = \cdots = x_n^{a_n} h^{b_n} = x_1 \cdots x_n = 1\}.$$

Here the numbers b_i are chosen so that $\sum_1^n \frac{b_i}{a_i} = \frac{1}{a}$.

Let ρ be an element of $\mathcal{R}^*(\Sigma)$. Since h is in the center of $\jmath(\Sigma)$, $\rho(h)$ is in the center of G. As $x_i^{a_i} h^{b_i} = 1$, we deduce that $\rho(x_i)$ is an elliptic element for all i. Suppose that the representation ρ is given by

$$\rho(h) = (diag(e^{2\pi i p_0}, e^{2\pi i q_0}, e^{2\pi i r_0}), 2\pi(p_0 + q_0 + r_0), 2\pi r_0) \text{ and}$$

$$\rho(x_i) \sim (diag(e^{2\pi i p_i}, e^{2\pi i q_i}, e^{2\pi i r_i}), 2\pi(p_i + q_i + r_i), 2\pi r_i).$$

Here we use the notation $\sim$ to denote the conjugacy relation in G.

As $\rho(h)$ is central, we have $p_0 \equiv q_0 \equiv r_0 \mod \mathbf{Z}$. Since ρ must preserve the relation $x_i^{a_i} h^{b_i} = 1,\ i = 1, \dots, n$, we deduce that

$$s_i := a_i p_i + b_i p_0 = -(a_i q_i + b_i q_0) \text{ are integers for all } i = 1, \dots, n. \tag{6.1}$$

$$a_i r_i + b_i r_0 = 0 \text{ for all } i = 1, \dots, n. \tag{6.2}$$

Theorem 6.1. — *The Chern-Simons invariant of ρ is given by*

$$cs(\rho) \equiv \frac{1}{2} a \left((\sum_{i=1}^{n} p_i)^2 + (\sum_{i=1}^{n} q_i)^2 + (\sum_{i=1}^{n} r_i)^2 \right).$$

Proof. — We write $\Sigma = X \cup (-S)$, where X is the complement of the n^{th} exceptional fiber and S is the solid torus neighborhood of the n^{th} exceptional fiber. Next, we then find paths of representations on X and S that connect ρ to the trivial representation. We then apply Theorem 5.1 to compute the Chern-Simons invariant on each X and S.

Step 1. Computation on X.

We know that $\jmath(X)$ is obtained from $\jmath(\Sigma)$ by eliminating the relation $x_n^{a_n} h^{b_n}$. Moreover, X has a natural meridian $\mu = x_n^{a_n} h^{b_n}$ and longitude $\lambda = x_n^{-a_1 \cdots a_{n-1}} h^c$, where $c = a_1 \cdots a_{n-1} \sum_1^{n-1} \frac{b_i}{a_i}$.

After conjugation, we may assume that

$$\rho(x_n) = (diag(e^{2\pi i p_n}, e^{2\pi i q_n}, e^{2\pi i r_n}), 2\pi(p_n + q_n + r_n), 2\pi r_n).$$

Since each conjugacy class in G is connected, we can deform $\rho|_X(x_i)$, within its conjugacy class, to the diagonal form for all $i = 1, \dots, n-1$. This means that we can find a path of representations $\rho_t : \jmath(X) \longrightarrow G, 0 \leqslant t \leqslant 1$, such that $\rho_0 = \rho|_X$, $\rho_t(h) = \rho(h)$ for all t and

$$\rho_1(x_i) = (diag(e^{2\pi i p_i}, e^{2\pi i q_i}, e^{2\pi i r_i}), 2\pi(p_i + q_i + r_i), 2\pi r_i), i = 1, \dots, n-1.$$

For this path of representations we find that

$$\rho_t(x_n) = (diag(e^{2\pi i p(t)}, e^{2\pi i q(t)}, e^{2\pi i r(t)}), 2\pi(p(t) + q(t) + r(t)), 2\pi r(t)).$$

Where $p(t), q(t), r(t)$ are functions with the following properties:

- $p(0) = p_n, q(0) = q_n, r(0) = r_n$.
- $p(1) = -\sum_1^{n-1} p_i, q(1) = -\sum_1^{n-1} q_i, r(1) = -\sum_1^{n-1} r_i$.

Therefore, we get

$$\rho_t(\mu) = (M(t), 2\pi a_n(p(t) + q(t) + r(t)) + 2\pi b_n(p_0 + q_0 + r_0), 2\pi a_n r(t) + 2\pi b_n r_0),$$

$$\text{where } M(t) = diag(e^{2\pi i(a_n p(t) + b_n p_0)}, e^{2\pi i(a_n q(t) + b_n q_0)}, e^{2\pi i(a_n r(t) + b_n r_0)}).$$

And $\rho_t(\lambda) = (N(t), -2\pi a_1 \cdots a_{n-1}(p(t) + q(t) + r(t)) + 2\pi c(p_0 + q_0 + r_0),$

$$-2\pi a_1 \cdots a_{n-1} r(t) + 2\pi c r_0), \text{ where}$$

$$N(t) = diag(e^{2\pi i(-a_1 \cdots a_{n-1} p(t) + c p_0)}, e^{2\pi i(-a_1 \cdots a_{n-1} q(t) + c q_0)}, e^{2\pi i(-a_1 \cdots a_{n-1} r(t) + c r_0)}).$$

So, by Proposition 4.2, we can find a path of normal form $\mathfrak{u}(2,1)-$valued flat connections on X that is given near ∂X by

$$A_t = diag(2\pi i((a_n p(t) + b_n p_0)dx + (-a_1 \cdots a_{n-1} p(t) + c p_0)dy), 2\pi i((a_n q(t) + b_n q_0)dx + (-a_1 \cdots a_{n-1} q(t) + c q_0)dy), 2\pi i((a_n r(t) + b_n r_0)dx + (-a_1 \cdots a_{n-1} r(t) + c r_0)dy)).$$

We now use Theorem 5.1 to compute the difference of Chern-Simons invariants:

$$cs(A_1) - cs(A_0) \equiv -\frac{1}{2}\int_0^1 (a_n c + a_1 \cdots a_{n-1} b_n)(p_0 \dot{p}(t) + q_0 \dot{q}(t)) + r_0 \dot{r}(t))dt$$
$$= -\frac{1}{2}(a_n c + a_1 \cdots a_{n-1} b_n)(p_0 p(t) + q_0 q(t) + r_0 r(t))|_0^1.$$

So we arrive at the following:

$$cs(A_0) \equiv cs(A_1) - \frac{1}{2}(p_0 \sum_{i=1}^{n} p_i + q_0 \sum_{i=1}^{n} q_i + r_0 \sum_{i=1}^{n} r_i) \tag{6.3}$$

Now, by using (6.1) and (6.2), we can write

$$A_1 = diag(2\pi i(-a_n \sum_{i=1}^{n-1} p_i + b_n p_0)dx + 2\pi i(a_1 \cdots a_{n-1} \sum_{i=1}^{n-1} \frac{s_i}{a_i})dy, 2\pi i(-a_n \sum_{i=1}^{n-1} q_i$$
$$+ b_n q_0)dx - 2\pi i(a_1 \cdots a_{n-1} \sum_{i=1}^{n-1} \frac{s_i}{a_i})dy, 2\pi i(a_n \sum_{i=1}^{n-1} r_i + b_n r_0)dx).$$

Let A_1' be a normal form flat connection on X which is gauge equivalent to A_1. Moreover, we assume that A_1' has the following form near the boundary ∂X :

$$A_1' = diag(2\pi i(-a_n \textstyle\sum_{i=1}^{n-1} p_i + b_n p_0)dx, 2\pi i(-a_n \sum_{i=1}^{n-1} q_i + b_n q_0)dx,$$
$$2\pi i(-a_n \textstyle\sum_{i=1}^{n-1} r_i + b_n r_0)dx).$$

Since $a_1 \cdots a_{n-1} \sum_{i=1}^{n-1} \frac{s_i}{a_i}$ is an integer, by using Theorem 5.2, we find that

$$cs(A_1) \equiv cs(A_1') - \frac{1}{2}(a_1 \cdots a_{n-1} \sum_{i=1}^{n-1} \frac{s_i}{a_i})(-a_n \sum_{i=1}^{n-1} p_i + b_n p_0 + a_n \sum_{i=1}^{n-1} q_i - b_n q_0). \tag{6.4}$$

As the holonomy representation ρ_1' of the connection A_1' is abelian, it factors through the projection map $\jmath(X) \to H_1(X) \equiv \mathbf{Z}$. We find that $\rho_1'(\lambda)$ equals to 1 and $\rho_1'(\mu)$ equals to

$$(diag(e^{2\pi i(-a_n \sum_{i=1}^{n-1} p_i + b_n p_0)}, e^{2\pi i(-a_n \sum_{i=1}^{n-1} q_i + b_n q_0)}, e^{2\pi i(-a_n \sum_{i=1}^{n-1} r_i + b_n r_0)}),$$
$$-2\pi a_n \sum_{i=1}^{n-1}(p_i + q_i + r_i) + 2\pi b_n(p_0 + q_0 + r_0), -2\pi a_n \sum_{i=1}^{n-1} r_i + b_n r_0).$$

So we can deform ρ_1' to the trivial representation. We now get a path of flat connections A_t' linking A_1' to the trivial connection whose Chern-Simons invariant is obviously zero. Applying Theorem 5.1 to the path

$$A'_t = diag(2t\pi i(-a_n \sum_{i=1}^{n-1} p_i + b_n p_0)dx, 2t\pi i(-a_n \sum_{i=1}^{n-1} q_i + b_n q_0)dx,$$

$$2t\pi i(-a_n \textstyle\sum_{i=1}^{n-1} r_i + b_n r_0)dx),\ 0 \leqslant t \leqslant 1,$$

we find that $cs(A'_1) \equiv 0$. Now combining this with (6.3) and (6.4), we get

$$cs(A_0) \equiv -\frac{1}{2}a(\sum_{i=1}^{n-1} \frac{s_i}{a_i})(-\sum_{i=1}^{n-1} p_i + \frac{b_n}{a_n}p_0 + \sum_{i=1}^{n-1} q_i - \frac{b_n}{a_n}q_0) -$$

$$\tfrac{1}{2}(p_0 \textstyle\sum_{i=1}^{n} p_i + q_0 \sum_{i=1}^{n} q_i + r_0 \sum_{i=1}^{n} r_i).$$

Using (6.1), we can rewrite this identity as follows:

$$cs(A_0) \equiv -\frac{1}{2}a(\sum_{i=1}^{n-1} \frac{s_i}{a_i})(-\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i + 2\frac{s_n}{a_n}) - \tag{6.5}$$

$$\tfrac{1}{2}(p_0 \textstyle\sum_{i=1}^{n} p_i + q_0 \sum_{i=1}^{n} q_i + r_0 \sum_{i=1}^{n} r_i).$$

Step 2. Computation on the solid torus.

We denote the connection form corresponding to $\rho|_S$ by B_0. Near the boundary ∂X, B_0 coincides with A_0 and is given by

$$B_0 = diag(2\pi i((a_n p_n + b_n p_0)dx + (-a_1 \cdots a_{n-1} p_n + c p_0)dy)), 2\pi i((a_n q_n +$$

$$b_n q_0)dx + (-a_1 \cdots a_{n-1} q_n + c q_0)dy)), 2\pi i(-a_1 \cdots a_{n-1} r_n + c r_0)dy).$$

Let B_1 be a normal form flat connection on S which is gauge equivalent to B_0. Moreover, we assume that

$B_1 = diag(2\pi i(-a_1 \cdots a_{n-1} p_n + c p_0)dy, 2\pi i(-a_1 \cdots a_{n-1} q_n +$
$c q_0)dy, 2\pi i(-a_1 \cdots a_{n-1} r_n + c r_0)dy)$ near the boundary ∂X.

By (6.1), the numbers $a_n p_n + b_n p_0 = s_n$ and $a_n q_n + b_n q_0 = -s_n$ are integers. So by applying Theorem 5.2, we find that

$$cs(B_0) \equiv cs(B_1) + \frac{1}{2}s_n(-a_1 \cdots a_{n-1} p_n + c p_0 + a_1 \cdots a_{n-1} q_n - c q_0).$$

Note that B_1 is a connection form that corresponds to an abelian representation. Moreover, B_1 contains the dy terms only. Carrying out a similar computation as we did for the connection A'_1 in the previous step, we conclude that $cs(B_1) \equiv 0$. Therefore we can write

$$cs(B_0) \equiv \frac{1}{2}a\frac{s_n}{a_n}(-p_n + p_0 \sum_{i=1}^{n-1} \frac{b_i}{a_i} + q_n - q_0 \sum_{i=1}^{n-1} \frac{b_i}{a_i}).$$

Moreover, from (6.1) we deduce that

$$p_0 \sum_{i=1}^{n-1} \frac{b_i}{a_i} = \sum_{i=1}^{n-1} \frac{s_i}{a_i} - \sum_{i=1}^{n-1} p_i$$

and

$$q_0 \sum_{i=1}^{n-1} \frac{b_i}{a_i} = -\sum_{i=1}^{n-1} \frac{s_i}{a_i} - \sum_{i=1}^{n-1} q_i.$$

So we arrive at the following:

$$cs(B_0) \equiv \frac{1}{2} a \frac{s_n}{a_n} \Big(-\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i + 2\sum_{i=1}^{n-1} \frac{s_i}{a_i}\Big). \tag{6.6}$$

Now as $\Sigma = X \cup (-S)$, we see that $cs(\rho) = cs(A_0) - cs(B_0)$. Note that the term $a\frac{s_n}{a_n}\sum_{i=1}^{n-1}\frac{s_i}{a_i}$, which appears in (6.5) and (6.6), is an integer and therefore can be ignored. So we obtain the formula

$$cs(\rho) \equiv \frac{1}{2} a \Big(\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \frac{s_i}{a_i} - \sum_{i=1}^{n} q_i \sum_{i=1}^{n} \frac{s_i}{a_i}\Big)$$
$$-\tfrac{1}{2}\big(p_0 \textstyle\sum_{i=1}^{n} p_i + q_0 \sum_{i=1}^{n} q_i + r_0 \sum_{i=1}^{n} r_i\big).$$

It follows from (6.1) and (6.2) that

$$\sum_{i=1}^{n} \frac{s_i}{a_i} = \sum_{i=1}^{n} p_i + \frac{p_0}{a} = -\sum_{i=1}^{n} q_i - \frac{q_0}{a} \text{ and } r_0 = -a\sum_{i=1}^{n} r_i.$$

So, finally, we arrive at the needed formula:

$$cs(\rho) \equiv \frac{1}{2} a \left((\sum_{i=1}^{n} p_i)^2 + (\sum_{i=1}^{n} q_i)^2 + (\sum_{i=1}^{n} r_i)^2 \right).$$

□

We can use the above theorem to find *all the possible values* of the Chern-Simons invariants if we know the representation space $\mathcal{R}^*(\Sigma)$. For a general manifold, this space is hard to describe in details. Fortunately, for the case of Seifert fibered homology spheres with three singular fibers, the representation spaces have been studied in [14]. So we are able to find all the possible values of the Chern-Simons invariants.

As an illustration, we present an example of the homology sphere $\Sigma(2,3,11)$. Its fundamental group has the following presentation:

$$\jmath(\Sigma(2,3,11)) = \langle x_1, x_2, x_3, h \mid h \text{ central}, \ x_1^2 h^{-1} = x_2^3 h = x_3^{11} h^2 = x_1 x_2 x_3 = 1\rangle.$$

By the computation in [14], we know that $\Sigma(2,3,11)$ has five distinct irreducible representations into $\mathrm{PU}(2,1)$. For homological reason, each $\mathrm{PU}(2,1)$

representation has a unique lift to a representation into the universal covering group G. By further computations, we obtain the following list of representations into G.

1) $\rho(x_1) \sim (diag(1,-1,-1),0,-\pi), \rho(x_2) \sim (diag(1,e^{4\pi i/3},e^{2\pi i/3}),0,\frac{2\pi}{3}),$
$\rho(x_3) \sim (diag(e^{12\pi i/11},e^{6\pi i/11},e^{4\pi i/11}),0,\frac{4\pi}{11}), \quad \rho(h) \sim (I,0,-2\pi).$
2) $\rho(x_1) \sim (diag(1,-1,-1),0,\pi), \rho(x_2) \sim (diag(e^{2\pi i/3},1,e^{4\pi i/3}),0,-\frac{2\pi}{3})),$
$\rho(x_3) \sim (diag(e^{-12\pi i/11},e^{-6\pi i/11},e^{-4\pi i/11}),0,-\frac{4\pi}{11}), \quad \rho(h) \sim (I,0,2\pi).$
3) $\rho(x_1) \sim (diag(-1,-1,1),0,2\pi), \rho(x_2) \sim (diag(1,e^{4\pi i/3},e^{2\pi i/3}),0,-\frac{4\pi}{3})),$
$\rho(x_3) \sim (diag(e^{-4\pi i/11},e^{-10\pi i/11},e^{-8\pi i/11}),0,-\frac{8\pi}{11}), \quad \rho(h) \sim (I,0,4\pi).$
4) $\rho(x_1) \sim (diag(-1,-1,1),0,-2\pi), \rho(x_2) \sim (diag(e^{2\pi i/3},1,e^{4\pi i/3}),0,\frac{4\pi}{3})),$
$\rho(x_3) \sim (diag(e^{4\pi i/11},e^{10\pi i/11},e^{8\pi i/11}),0,\frac{8\pi}{11}), \quad \rho(h) \sim (I,0,-4\pi).$
5) $\rho(x_1) \sim (diag(-1,-1,1),0,0), \rho(x_2) \sim (diag(e^{2\pi i/3},e^{4\pi i/3},1),0,0),$
$\rho(x_3) \sim (diag(e^{-2\pi i/11},e^{2\pi i/11},1),0,0), \quad \rho(h) \sim (I,0,0).$

Using Theorem 6.1, we can find the Chern-Simons invariant of these representations and list them below:

Representation	1	2	3	4	5
$cs(\rho) \mod \mathbf{Z}$	$\frac{13}{66}$	$\frac{13}{66}$	$\frac{7}{66}$	$\frac{7}{66}$	$\frac{25}{66}$

Therefore, we can deduce that *the Burns-Epstein invariant, modulo an integer, of any spherical CR structure on* $\Sigma(2,3,11)$ *with irreducible holonomy representation is one of the values above.*

Remark 6.2. — Little is known about the problem of classification of spherical CR structures on 3-manifolds. On Seifert fibered manifolds, the only work done so far is the classification of the $\mathbf{S}^1$-invariant CR structures by Kamishima and Tsuboi [12]. We do not know any example of a spherical CR structure on $\Sigma(2,3,11)$ whose holonomy is one of the representations listed above.

Recently, Biquard and Herzlich [2] introduce an invariant ν for strictly pseudoconvex 3-dimensional CR manifolds. They show that their invariant agrees with three times the Burns-Epstein invariant up to a constant. Furthermore, by relating the ν invariant to a kind of eta invariant, Biquard, Herzlich and Rumin [1] are able to give an explicit formula for the ν invariant of the transverse $\mathbf{S}^1$-invariant CR structure on a Seifert fibered manifold. It would be interesting to work out the relationship between the ν invariant, modulo an integer, and the metric Chern-Simons invariant.

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Khoi The VU
Institute of Mathematics
18 Hoang Quoc Viet road, 10307
Hanoi (Vietnam)
vtkhoi@math.ac.vn