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Karol PALKA

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EXCEPTIONAL SINGULAR \mathbb{Q} -HOMOLOGY PLANES

by Karol PALKA (*)

Dedicated to the memory of my colleague and friend, Marcin Hauzer

ABSTRACT. — We consider singular \mathbb{Q} -acyclic surfaces with smooth locus of non-general type. We prove that if the singularities are topologically rational then the smooth locus is \mathbb{C}^1 - or \mathbb{C}^* -ruled or the surface is up to isomorphism one of two exceptional surfaces of Kodaira dimension zero. For both exceptional surfaces the Kodaira dimension of the smooth locus is zero and the singular locus consists of a unique point of type A_1 and A_2 respectively.

RÉSUMÉ. — On considère des surfaces \mathbb{Q} -acycliques singulières dont la partie lisse n'est pas de type général. On démontre que si les singularités sont topologiquement rationnelles, alors soit la partie lisse est réglée par \mathbb{C}^1 ou \mathbb{C}^* , soit la surface est l'une de deux surfaces exceptionnelles de dimension de Kodaira zéro. Pour les deux surfaces exceptionnelles, la dimension de Kodaira de la partie lisse est zéro, il n'y a qu'un seul point singulier et la singularité est de type A_1 ou A_2 , respectivement.

We consider complex algebraic varieties.

1. Main result

Because of their homological similarity to \mathbb{C}^2 smooth \mathbb{Q} -acyclic surfaces serve as a class of test examples for working hypotheses as well for conjectures like cancellation problem or the Jacobian Conjecture, they appear naturally also when studying exotic structures on \mathbb{C}^n 's (see [15, §3.4] for what is known about them).

DEFINITION 1.1. — *A surface is a \mathbb{Q} -homology plane if it is normal and \mathbb{Q} -acyclic, i.e. $H^*(-, \mathbb{Q}) \cong \mathbb{Q}$.*

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A singular \mathbb{Q} -homology plane is *logarithmic* if and only if it has at most quotient singularities, i.e. analytically it is locally of type \mathbb{C}^2/G for some finite subgroup $G < GL(2, \mathbb{C})$. Note that logarithmic \mathbb{Q} -homology planes are rational by [7, 21]. Singular \mathbb{Q} -homology planes appear for example as quotients of smooth ones by the actions of finite groups or as two-dimensional quotients of \mathbb{C}^n by the actions of reductive groups (cf. [13], [5]). Let S' be a \mathbb{Q} -homology plane and let S_0 be its smooth locus ($S' = S_0$ if S' is smooth). Assume that S_0 is not of general type, i.e. its Kodaira dimension $\bar{\kappa}(S_0)$ is smaller than two. The description of these surfaces divides into three main cases depending on the properties of S_0 : (a) S_0 is \mathbb{C}^1 -ruled, (b) S_0 is \mathbb{C}^* -ruled, (c) S_0 is neither \mathbb{C}^1 - nor \mathbb{C}^* -ruled.

DEFINITION 1.2. — *A \mathbb{Q} -homology plane whose smooth locus is not of general type and is neither \mathbb{C}^1 - nor \mathbb{C}^* -ruled is exceptional.*

For non-exceptional \mathbb{Q} -homology planes the analysis reduces to the description of singular fibers of respective rulings using the \mathbb{Q} -acyclicity. Case (a) and part of case (b) (when S' is logarithmic and the \mathbb{C}^* -ruling of S_0 extends to a \mathbb{C}^* -ruling of S') have been done in [16]. The precise classification and the rest of part (b) will be done in our forthcoming paper. By general structure theorems for open surfaces an exceptional \mathbb{Q} -homology plane necessarily has $\bar{\kappa}(S_0) = 0$ (cf. [15, 2.1.1], [10, 2.3]). Up to isomorphism there exist exactly three smooth exceptional \mathbb{Q} -homology planes called $Y\{3, 3, 3\}$, $Y\{2, 4, 4\}$ and $Y\{2, 3, 6\}$ (cf. [4, §8]). The classification of non-smooth exceptional \mathbb{Q} -homology planes is the main goal of this paper. We will do this under some mild assumption on singularities.

DEFINITION 1.3. — *A singular point on a normal surface is a topologically rational singularity if and only if there exists a resolution of this surface with a rational tree as an exceptional divisor.*

Notice that the singularity is topologically rational if and only if it is *quasirational* (cf. [1]) and the dual graph of the respective exceptional locus contains no loops. The class of topologically rational singularities includes the class of rational singularities and is much broader than the class of the quotient ones. Our main result is:

THEOREM 1.4. — *Up to isomorphism there are exactly two exceptional singular \mathbb{Q} -homology planes with at most topologically rational singularities. Both have Kodaira dimension zero and have unique singular points of type A_1 and A_2 respectively.*

One of the above surfaces comes from the famous *dual Hesse configuration* $(12_3, 9_4)$ of points and lines on \mathbb{P}^2 (not realizable in $\mathbb{R}\mathbb{P}^2$) and the

second one from a configuration which contains one line more than the *complete quadrangle* $(4_3, 6_2)$ (see 5.5 and 5.8). We want to emphasize that having some topological results about general singular \mathbb{Q} -homology planes (which we obtain in a forthcoming paper) one can easily show that in the above situation the assumption about topological rationality can be omitted with no change for the thesis. However, it is not true that all singular \mathbb{Q} -homology planes have topologically rational singularities.

As for now there is no description of \mathbb{Q} -homology planes with smooth locus of general type. There are some partial results (see [22], [23, 24], [17], [6], [13]).

The outline of the proof of the theorem is as follows. First with the help of Bogomolov-Miyaoka-Yau inequality we show in section 3 that each smooth rational curve contained in the snc-minimal smooth completion of S_0 has at least two common points with some connected component of the boundary (i.e. it is not *simple*), which in particular shows that S_0 is minimal in the open sense (see [15, 2.3.11]). Let us write the boundary divisor as $D + \widehat{E}$, where \widehat{E} is the reduced exceptional divisor of the resolution of S' . Using the fact that $\bar{\kappa}(S_0) = 0$ and that the intersection matrix of D is not negative definite we get some restrictions on the shape of D following from [4, 8.8]. In fact for smooth exceptional surfaces this would be enough to get the description of them. However, in the singular case we need to obtain more restrictions on D because we do not have much control over \widehat{E} . We do this in section 4. In remaining ten cases we are able to find 0-curves inside D , which give \mathbb{P}^1 -rulings of the completion having nice properties. We analyze singular fibers and sections of these rulings and we eliminate all but two cases. Having enough information on the latter two rulings in section 5 we are able to construct two exceptional singular \mathbb{Q} -homology planes and prove their uniqueness. We compute their automorphism groups, the orders of the first homology groups and show that they came from special line arrangements on \mathbb{P}^2 .

Having the theorem 1.4 one can infer that both singular exceptional \mathbb{Q} -homology planes can be described as quotients of the smooth exceptional ones, namely they are isomorphic to $Y\{3, 3, 3\}/\text{Aut } Y\{3, 3, 3\}$ and $Y\{2, 4, 4\}/\text{Aut } Y\{2, 4, 4\}$ respectively, see [20] for details.

2. Preliminaries

For convenience we recall some facts from the theory of open algebraic surfaces that we use more often, partially to fix the notation. The reader is referred to [15] for details.

2.1. Divisors

Let $D = \sum_{i=1}^n m_i D_i$ with D_i distinct, irreducible and $m_i \in \mathbb{Q} \setminus \{0\}$ be a simple normal crossing (snc-) divisor on a smooth complete surface. Put

$$d(D) = \det(-Q(D)),$$

where $Q(D)$ is the intersection matrix of D , i.e. $Q(D)_{i,j} = m_i m_j D_i \cdot D_j$. We define the reduction of D as $\underline{D} = \sum D_i$ and denote the number of components of D by $\#D$. By a component we always mean an irreducible component. The numerical equivalence of divisors is denoted by \equiv . We write $D \geq 0$ for effective divisors and for \mathbb{Z} -divisors linearly equivalent to effective divisors. Two \mathbb{Q} -divisors A, B are linearly equivalent if rA and rB are linearly equivalent \mathbb{Z} -divisors for some nonzero integer r . For a \mathbb{Q} -divisor D linearly equivalent to some effective \mathbb{Q} -divisor we write $D \geq_{\mathbb{Q}} 0$.

The dual graph of D is a weighted one-dimensional simplicial complex with one vertex v_i of weight D_i^2 for each irreducible component D_i of D and one edge between v_i and v_j for each point of intersection of D_i with D_j . We say that D is a *forest* (*tree*) if $\text{Supp } D$ is simply connected (and connected). It is *rational* if all its components are rational. D is a *chain* if it is connected and each component D_i of D is non-branching, i.e. it has the *branching number* $\beta_D(D_i) = D_i \cdot (D - D_i)$ not greater than two. A *tip* is a component with $\beta_D \leq 1$. A chain D is *admissible* if it is rational and $D_i^2 \leq -2$ for every i . A curve L is a (*b*)-*curve* if and only if $L \cong \mathbb{P}^1$ and $L^2 = b$. We say that D is a *fork* if it is a tree with a unique branching component B and $\beta_D(B) = 3$. Suppose R is a rational chain with some tip R_1 chosen. We write

$$R = [-R_1^2, -R_2^2, \dots, -R_r^2],$$

where R_i 's are components of R ordered in such a way that $R_i \cdot R_{i+1} = 1$ for $i = 1, \dots, r-1$ and we define $d'(R) = d(R - R_1)$ with $d(0) := 1$. R^t is the same chain as R but considered with a reversed order. If R is a (-2) -chain, i.e. $R = [2, 2, \dots, 2]$, then we write $R = [(r)]$, where $r = \#R$. If R is admissible we define

$$\delta(R) = \frac{1}{d(R)}, \quad e(R) = \frac{d'(R)}{d(R)}, \quad \tilde{e}(R) = e(R^t).$$

If D is not a chain we define its *maximal twigs* as the rational chains of maximal length with support contained in $\text{Supp } D$, which do not contain branching components of D and contain a tip of D . Each twig is considered with a natural linear order on the set of components for which its tip is

the first component. If D is not an admissible chain we define its *maximal admissible twigs*, say T_1, \dots, T_s , analogously and put

$$\delta(D) = \sum_{i=1}^s \delta(T_i), \quad e(D) = \sum_{i=1}^s e(T_i), \quad \tilde{e}(D) = \sum_{i=1}^s \tilde{e}(T_i).$$

Smooth pair (X, D) consists of a smooth complete (hence projective by the result of Zariski) surface and a reduced snc-divisor on it. In this case we write $X - D$ for $X \setminus \text{Supp } D$. The divisor D is *snc-minimal* if after a contraction of any (-1) -curve in D the direct image of D is not an snc-divisor. A smooth pair (X, D) is snc-minimal if D is snc-minimal. The pair (X, D) is a *smooth completion* of an open surface U if $X - D = U$.

If $\pi : X' \rightarrow X$ is a birational morphism then we write $\pi^{-1}(D)$ for the *preimage* of D , which we define as π^*D , the reduced total transform of D . A blowup with a center on an snc-divisor D is *subdivisional* for D if the center belongs to two components of D , otherwise it is *sprouting* for D . The sequence of blowups over D (i.e. with centers on D and on its successive preimages) is subdivisional if all blowups are subdivisional for the respective preimages of D . The composable sequence of blowups is *connected* if the exceptional divisor of the composition contains a unique (-1) -curve.

2.2. Rulings

We say that a surface X is \mathbb{P}^1 -ruled (respectively \mathbb{C}^1 -ruled, \mathbb{C}^* -ruled, $\mathbb{C}^{(n^*)}$ -ruled) if there exists a curve B and a surjective morphism $p : X \rightarrow B$ with a general fiber isomorphic to \mathbb{P}^1 (respectively to \mathbb{C}^1 , \mathbb{C}^1 with one, \mathbb{C}^1 with n points deleted). We call also the \mathbb{C}^1 -ruling an *affine* ruling. Clearly, if X is normal then B can be assumed to be smooth.

Suppose that X is smooth and has a ruling as above. Then for some smooth completion (\bar{X}, D) this ruling can be extended to a \mathbb{P}^1 -ruling $\bar{p} : \bar{X} \rightarrow \bar{B}$, where \bar{B} is a smooth completion of B . Let F be a fiber of \bar{p} . An irreducible curve $C \subseteq \bar{X}$ is called an *n-section* if $F \cdot C = n$. We will say just *section* for a 1-section. C is *horizontal* if $n > 0$, otherwise it is *vertical*. If C is vertical then it is called a *D-component* if $C \subseteq D$, otherwise it is called an *X-component*. If the ruling is fixed we denote the divisor consisting of horizontal components of D by D_h . The divisor is horizontal (vertical) if all its components are horizontal (vertical). The completion (\bar{X}, D) is *p-minimal* if it is smooth and minimal with respect to the property that the extension of p from X to \bar{X} exists (the partial order is induced by morphisms of pairs).

For a smooth pair (\bar{X}, D) put $X = \bar{X} - D$. Let π be a \mathbb{P}^1 -ruling of \bar{X} . Following [4] we define some characteristic numbers of the triple $\tau = (\bar{X}, D, \pi)$: h_τ is the number of horizontal D -components, $\sigma_\tau(F)$ is the number of X -components contained in F , ν_τ is the number of fibers contained in D . Put

$$\Sigma_\tau = \sum_{F \not\subseteq D} (\sigma_\tau(F) - 1).$$

If there is no danger of confusion we omit indices writing Σ (or Σ_X) for Σ_τ , h for h_τ , etc. If one contracts a vertical (-1) -curve and simultaneously changes \bar{X} and D for their images then the numbers $b_2(\bar{X}) - b_2(D) - \Sigma + \nu$ and h do not change ($b_i(X) = \dim H_i(X; \mathbb{Q})$). This leads to the following equation (cf. [4, 4.16]):

$$(2.1) \quad \Sigma = h + \nu + b_2(\bar{X}) - b_2(D) - 2.$$

Clearly, $b_2(\bar{X}) - b_2(D)$ depends only on X .

We now summarize some information about singular fibers of \mathbb{P}^1 -rulings (cf. [4, §4]). For a given ruling π and a vertical component C the multiplicity $\mu(C)$ is the coefficient of C in $\pi^*(\pi(C))$.

LEMMA 2.1. — *Let F be a singular fiber of a \mathbb{P}^1 -ruling of a smooth complete surface. Then F is a rational snc-tree containing a (-1) -curve. Each (-1) -curve of F intersects at most two other components of F . Successive contractions of (-1) -curves contract F to a smooth 0-curve. In this process the number of (-1) -curves can increase only in the last but one step, when $[2, 1, 2]$ contracts to $[1, 1]$.*

Suppose that F as above contains a unique (-1) -curve C . The sequence of blowups recovering F from a smooth (0) -curve is connected. Let B_1, \dots, B_n be the branching components of F written in order in which they are produced in the sequence of blowups recovering F from a smooth (0) -curve and let $B_{n+1} = C$. We can write \underline{F} as $\underline{F} = T_1 + T_2 + \dots + T_{n+1}$, where the divisors T_i are chains consisting of all components of $\underline{F} - T_1 - \dots - T_{i-1}$ created not later than B_i . We call T_i the i -th branch of F and say that F is branched if $i > 1$.

Remark 2.2. — Let F and C be as above. Then $\mu(C) > 1$ and there are exactly two components of F having multiplicity one. They are tips of the fiber and belong to the first branch. The connected component of $\underline{F} - C$ not containing curves of multiplicity one is a chain. If $\mu(C) = 2$ then either $F = [2, 1, 2]$ or C is a tip of F and then $\underline{F} - C$ is either a (-2) -chain or a (-2) -fork with two tips as maximal twigs.

2.3. Zariski decomposition

Let (X, D) be a smooth pair. If it is almost minimal (cf. [15, 2.3.11]) and $\bar{\kappa}(X - D) \geq 0$ then the Zariski decomposition of $K_X + D$, where K_X stands for the canonical divisor on X , can be computed explicitly using a *bark* of D . For non-connected D bark is a sum of barks of its connected components, so we will assume D is connected. If D is an snc-minimal resolution of a quotient singularity (i.e. D is an admissible chain or an admissible fork, cf. [15, 2.3.4]) then we define $\text{Bk } D$ as a unique \mathbb{Q} -divisor with $\text{Supp Bk } D \subseteq D$, such that

$$(K_X + D - \text{Bk } D) \cdot D_i = 0 \text{ for each component } D_i \subseteq D.$$

In other case let T_1, \dots, T_s be all the maximal admissible twigs of D . (If $\bar{\kappa}(X - D) \geq 0$ and D is snc-minimal then all maximal twigs of D are admissible, cf. [4, 6.13]). We define $\text{Bk } D$ as a unique \mathbb{Q} -divisor with $\text{Supp Bk } D \subseteq \bigcup T_j$, such that

$$(K_X + D - \text{Bk } D) \cdot D_i = 0 \text{ for each component } D_i \subseteq \bigcup_{j=1}^s T_j.$$

Suppose R is an admissible chain with some tip R_1 chosen. Then we define $\text{Bk}(R, R_1)$ as a unique \mathbb{Q} -divisor with support contained in R , such that

$$R_1 \cdot \text{Bk}(R, R_1) = -1 \text{ and } R_i \cdot \text{Bk}(R, R_1) = 0 \text{ for each component } R_i \subseteq R - R_1.$$

If there is no need to mention the tip explicitly (for example if R is an admissible twig of some fixed divisor then its tip will be a default choice for R_1) we write $\text{Bk}' R$ instead of $\text{Bk}(R, R_1)$.

(This notation does not occur in standard references, but we find it useful). Now we can write $\text{Bk } D = \text{Bk}' T_1 + \dots + \text{Bk}' T_s$. We recall here the properties of $\text{Bk } D$ which we use later and refer the reader to [15, §2.3] for details. We put $D^\# = D - \text{Bk } D$.

LEMMA 2.3. — *Let (X, D) be a smooth pair. Write $D = \sum D_i$ with D_i distinct irreducible and $\text{Bk } D = \sum d_i D_i$. One has:*

- (i) $0 \leq d_i \leq 1$ for each i , $\text{Bk } D$ is rational and $Q(\text{Bk } D)$ is negative definite, unless $\text{Bk } D = 0$,
- (ii) if $d_i = 1$ for some i and D' is a connected component of D containing D_i then $\text{Bk } D' = D'$ and D' consists of (-2) -curves,
- (iii) $\text{Supp Bk } D$ consists of the supports of all maximal admissible twigs of D and of all connected components of D which are either admissible chains or admissible forks (see [15, 2.3.5]),

- (iv) $(K_X + D^\#) \cdot Z = 0$ for every $Z \subseteq \text{Supp Bk } D$,
- (v) if (X, D) is almost minimal and $\bar{\kappa}(X - D) \geq 0$ then $(K_X + D)^- = \text{Bk } D$.

We now state a version of Bogomolov-Miyaoka-Yau inequality proved by Langer ([14, Corollary 5.2]), which generalizes the inequalities of Miyaoka [18, Theorem 1.1] and Kobayashi [11, Theorem 2]. See [14, 3.4, §9] for a definition of the orbifold Euler number $\chi_{orb}(X, D)$ and for computations in special cases.

PROPOSITION 2.4. — *Let (X, D) be a normal projective surface together with a \mathbb{Q} -divisor $D = \sum m_i D_i$ with $0 \leq m_i \leq 1$. Assume that the pair is log-canonical and $K_X + D$ is pseudoeffective. Then*

$$3\chi_{orb}(X, D) + \frac{1}{4}((K_X + D)^-)^2 \geq (K_X + D)^2.$$

COROLLARY 2.5. — *Let (X, D) be a smooth pair with $\kappa(K_X + D) \geq 0$. Then:*

- (i) $3\chi(X - D) + \frac{1}{4}((K_X + D)^-)^2 \geq (K_X + D)^2$.
- (ii) *For each connected component of D , which is a connected component of $\text{Bk } D$ (hence contractible to a quotient singularity) denote by G_P the local fundamental group of the respective singular point P . Then*

$$\chi(X - D) + \sum_P \frac{1}{|G_P|} \geq \frac{1}{3}(K_X + D^\#)^2.$$

Proof. — According to [14, 7.6] if (X, D) is a pair as in 2.4 and D is reduced then for a point $P \in D$ the local orbifold numbers $\chi_{orb}(P; X, D)$ vanish, hence

$$\chi_{orb}(X, D) = \chi(X - \text{Sing } X - D) + \sum_{P \in \text{Sing } X} \chi_{orb}(P; X, D).$$

This already proves (i), where X is smooth. Let $\pi : (X, D) \rightarrow (X', D')$ be a morphism contracting the connected components of $\text{Bk } D$ to quotient singularities. Then by [15, 2.3.14.1] $K_X + D^\# \equiv \pi^*(K_{X'} + D')$ and $K_{X'} + D' = \pi_*(K_X + D^\#)$, in particular $K_{X'} + D'$ is pseudoeffective because $(K_X + D)^- - \text{Bk } D$ is effective by 2.3(iv) and the properties of the Zariski decomposition of $K_X + D$. We need to know $\chi_{orb}(P; X', D')$. If $P \notin D'$ then the preimage of P is a connected component of D (and of $\text{Bk } D$) and by [14, 7.1] we have $\chi_{orb}(P; X', D') = \frac{1}{|G_P|}$. We have also $\chi(X' - \text{Sing } X' - D') = \chi(X - D)$. Since $((K'_{X'} + D')^-)^2 \leq 0$, (ii) follows from 2.4 applied to (X', D') . □

Remark. — Part (ii) generalizes the Kobayashi inequality for the case $\bar{\kappa}(X - D) = 0, 1$, it is stronger than the original Miyaoka inequality (there is no $\frac{1}{4}N^2$ term, using the notation of [18, Theorem 1.1]). If $\bar{\kappa}(X - D) = 2$ then to get the original Kobayashi inequality one applies 2.4 to the *strongly minimal model* of (X, D) (cf. [15, 2.4.12, 2.6.6]).

2.4. Other useful results

As a consequence of elementary properties of determinants one gets the following result.

LEMMA 2.6. — ([13, 2.1.1]). *Let D be a reduced snc-tree.*

- (i) *Let C be a component of D and let D_1, D_2, \dots, D_k be the connected components of $D - C$. If C_i is the component of D_i meeting C then*

$$d(D) = -C^2 \prod_i d(D_i) - \sum_i d(D_i - C_i) \prod_{i \neq j} d(D_j).$$

- (ii) *Let $D = D_1 + D_2$, where D_1, D_2 are connected and intersect in one point. Let $C_1 \subseteq D_1, C_2 \subseteq D_2$ be the intersecting components, then*

$$d(D) = d(D_1)d(D_2) - d(D_1 - C_1)d(D_2 - C_2).$$

Remark. — If D is an snc-divisor then $d(D)$ is invariant under blowup, i.e. if (X, D) is a smooth pair and $\sigma : X' \rightarrow X$ is a blowup, then $d(\sigma^{-1}(D)) = d(D)$. For trees this follows from 2.6 by induction on $\#D$.

LEMMA 2.7. — *Let A and B be some \mathbb{Q} -divisors, such that $A + B$ is effective and $Q(B)$ is negative definite. If $A \cdot B_i = 0$ for each irreducible component B_i of B then A is effective.*

Proof. — We can assume that A and B are \mathbb{Z} -divisors and B is effective and nonzero. Write $B = \sum b_i B_i$ for some positive integers b_i and irreducible components B_i of B . Choose $b'_i \in \mathbb{N}$, such that the sum $\sum b'_i$ is the smallest possible among divisors $\sum b'_i B_i$, such that $A + \sum b'_i B_i$ is effective. If $b'_i > 0$ for some i then $(A + \sum b'_i B_i) \cdot (\sum b'_i B_i) = (\sum b'_i B_i)^2 < 0$ by the assumptions. Hence $\text{Supp}(A + \sum b'_i B_i)$ contains some B_i , a contradiction with the definition of b'_i . Thus A is effective. \square

LEMMA 2.8. — *Let X_0 be a smooth part of $X' \setminus D'$, where D' is a divisor on an affine surface X' . Let (X_m, D_m) be the almost minimal model of some smooth completion of X_0 . Then the almost minimal model $X_m - D_m$ of X_0 is an open subset of X_0 and $\chi(X_m - D_m) \leq \chi(X_0)$.*

Proof. — Let $\epsilon : X \rightarrow X'$ be a resolution with snc-minimal exceptional divisor and let (\bar{X}, D) be a smooth completion of X . Since X' is affine, D is connected and $Q(D)$ is not negative definite. Let $D'' \subseteq \bar{X}$ be the closure of $\epsilon^{-1}(D')$ and let E be the part of the exceptional divisor with support equal to $\epsilon^{-1}(\text{Sing}(X' - D'))$. Blowing on D'' if necessary we can assume that $(\bar{X}, D + D'' + E)$ is a smooth completion of X_0 . Moreover, $D + D''$ is connected. Consider the process of producing an almost minimal model (X_m, D_m) of $(\bar{X}, D + D'' + E)$, it goes by contractions of special (-1) -curves, so-called *log-exceptional curves of the first kind* (cf. [15, 2.4.3]). Notice that in the process the divisor $D' + D''$ cannot be contracted, because $Q(D)$ is not negative definite. By the properties of a log-exceptional curve not contained in the boundary its contraction causes a subtraction of a curve with $\chi = 1$ or $\chi = 0$ from X_0 . Contractions of (-1) -curves contained in the boundary divisor do not affect X_0 , unless some connected component of the boundary is eventually contracted to a smooth point which does not belong to the proper image of the boundary divisor. Then this point adds to the almost minimal model of X_0 . Affineness of X' implies that a log-exceptional curve not contained in E intersects the image of D , so the above cannot happen for connected components of E . \square

3. Basic properties of S'

We now fix the notation for the rest of the paper. Let S' be an exceptional singular \mathbb{Q} -homology plane, i.e. its smooth locus S_0 has $\bar{\kappa}(S_0) \neq 2$ and is neither \mathbb{C}^1 - nor \mathbb{C}^* -ruled. As was explained in section 1, this implies $\bar{\kappa}(S_0) = 0$. Let $\epsilon : S \rightarrow S'$ be a resolution having an snc-divisor as the exceptional locus and let (\bar{S}, D) be a smooth completion of S . By the definition of the logarithmic Kodaira dimension $\bar{\kappa}(S') = \bar{\kappa}(S) = \kappa(K_{\bar{S}} + D)$, where $K_{\bar{S}}$ stands for the canonical divisor on \bar{S} . Let $\{p_1, \dots, p_q\}$ be the singular locus of S' and let $\hat{E}_i = \epsilon^{-1}(p_i)$. We assume that $\hat{E} = \hat{E}_1 + \hat{E}_2 + \dots + \hat{E}_q$ is snc-minimal. The intersection matrix $Q(\hat{E})$ is negative definite. We write $H_i(X, A)$ for $H_i(X, A; \mathbb{Q})$ and $b_i(X, A)$ for $\dim H_i(X, A; \mathbb{Q})$.

LEMMA 3.1. — *Let $i : D \cup \hat{E} \rightarrow \bar{S}$ be the inclusion. The following properties hold:*

- (i) $H_2(i) : H_2(D \cup \hat{E}) \rightarrow H_2(\bar{S})$ is an isomorphism,
- (ii) S' is rational,
- (iii) D is a rational tree,
- (iv) $\Sigma_{S_0} = h + \nu - 2$ and $\nu \leq 1$,
- (v) S' is affine.

Proof. — (i) Let $Tub(\widehat{E})$ be a sum of tubular neighborhoods of \widehat{E}_i 's in S (see [19] for the construction) and let M be the boundary of the closure of $Tub(\widehat{E})$. We can assume that M is a disjoint sum of closed oriented 3-manifolds. There exists a deformation retraction $Tub(\widehat{E}) \rightarrow \widehat{E}$, so by excision $H^j(S_0, M) = H^j(S, Tub(\widehat{E})) = H^j(S, \widehat{E})$ and since for $j > 1$ we have $H^j(S, \widehat{E}) = H^j(S') = 0$, we get $b_j(S_0) = b_j(M)$ for $j > 1$. In fact $b_1(S_0)$ also equals $b_1(M)$ because $H^1(S_0, M) = H^1(S, \widehat{E}) = \mathbb{Q}^{q-1}$ and then $H^0(M) \rightarrow H^1(S_0, M)$ is an epimorphism. By [19] $b_1(M) = b_1(\widehat{E}) = 0$, so each connected component of M is a \mathbb{Q} -homology sphere by the Poincaré duality. We conclude that $b_j(S_0) = 0$ for $j = 1, 2$. Now by the Lefschetz duality $H_j(\overline{S}, D \cup \widehat{E}) = H^{4-j}(S_0)$, hence $H_2(\overline{S}, D \cup \widehat{E}) = H_3(\overline{S}, D \cup \widehat{E}) = 0$. It follows from the exact sequence of the pair $(\overline{S}, D \cup \widehat{E})$ that $H_2(i)$ is an isomorphism.

(ii) Since $H_2(i)$ is an isomorphism, the exact sequence of the pair $(\overline{S}, D \cup \widehat{E})$ gives that $H_3(\overline{S}) \rightarrow H_3(\overline{S}, D \cup \widehat{E})$ is an isomorphism. Therefore by the Lefschetz duality $b_1(\overline{S}) = b_3(\overline{S}) = b_3(\overline{S}, D \cup \widehat{E}) = b_1(S_0) = 0$. Now if $\kappa(\overline{S}) = -\infty$ then \overline{S} is birational to a \mathbb{P}^1 -fibration over some complete curve B . From the homotopy exact sequence of a fibration we know that $b_1(B) = b_1(\overline{S})$, so $B \cong \mathbb{P}^1$, hence \overline{S} is rational. Suppose $\kappa(\overline{S}) \geq 0$. Since $\kappa(\overline{S}) \leq \overline{\kappa}(S_0) = 0$, we see that $\kappa(\overline{S}) = \overline{\kappa}(S) = \overline{\kappa}(S_0) = 0$. We now prove that $Q(D)$ is negative definite. We can assume that (\overline{S}, D) is almost minimal. Then by 2.3(v) $K_{\overline{S}} + D^\# = (K_{\overline{S}} + D)^\# \equiv 0$ and $K_{\overline{S}} \geq_{\mathbb{Q}} 0$, so $D^\# = 0$ because $D^\#$ is effective. Thus $D = \text{Bk } D$, and we are done by 2.3(i). By (i) we get a contradiction with the Hodge index theorem.

(iii) Since $H_2(\overline{S}, D \cup \widehat{E}) = 0$, the exact sequence of the pair $(\overline{S}, D \cup \widehat{E})$ gives the injectivity of $H_1(D \cup \widehat{E}) \rightarrow H_1(\overline{S})$, so $b_1(D) = 0$ by (ii). In the proof of (i) we have shown that $b_1(\overline{S}, D \cup \widehat{E}) = b_3(M)$, so since M is a disjoint sum of $b_0(\widehat{E})$ three-dimensional manifolds, we get $b_1(\overline{S}, D \cup \widehat{E}) = b_0(\widehat{E})$. Now the exact sequence of a pair $(\overline{S}, D \cup \widehat{E})$ gives $b_0(D \cup \widehat{E}) = b_1(\overline{S}, D \cup \widehat{E}) + b_0(\overline{S}) = b_0(\widehat{E}) + 1$, hence D is connected.

(iv) The first equation is a consequence of (2.1) and (i). If $\nu > 1$ then the numerical equivalence of fibers of a \mathbb{P}^1 -ruling gives a numerical dependence of components of $D + \widehat{E}$ in $NS(\overline{S}) \otimes \mathbb{Q}$, where $NS(\overline{S})$ is the Neron-Severi group of \overline{S} . This contradicts (i).

(v) Since $H_2(i)$ is an epimorphism by (i) and since D is connected by (iii), Fujita's argument from the proof of [4, 2.4(3)] works.

□

Remark. — From 3.1(i) and the Hodge index theorem we get $d(D + \widehat{E}) < 0$, so $d(D) < 0$.

LEMMA 3.2. — *Every irreducible curve $L \not\subseteq D \cup \widehat{E}$ satisfies $\bar{\kappa}(S_0 - L) = 2$.*

Proof. — Suppose $\bar{\kappa}(S_0 - L) = 1$. Since S_0 does not contain complete curves, [10, 2.3] implies that $S_0 - L$ is \mathbb{C}^* -ruled. S_0 is not \mathbb{C}^* -ruled, so it is affine-ruled and we get $\bar{\kappa}(S_0) = -\infty$ by the easy addition theorem ([8, Theorem 10.4]), a contradiction. Suppose $\bar{\kappa}(S_0 - L) = 0$. By 3.1(i) $H_2(\bar{S}, \mathbb{Q})$ is generated by cycles contained in $D \cup \widehat{E}$, hence $NS(\bar{S}) \otimes \mathbb{Q}$ is generated by the components of $D + \widehat{E}$. Since \bar{S} is rational, we get $Pic(S_0) \otimes \mathbb{Q} = 0$, so there exists a rational function f on S_0 , such that $(f) = kL$ for some $k > 0$. We get a morphism $f : S_0 - L \rightarrow \mathbb{C}^*$. If $S_0 - L \rightarrow B \rightarrow \mathbb{C}^*$ is its Stein factorization then $\bar{\kappa}(B) \geq \bar{\kappa}(\mathbb{C}^*) = 0$ and $0 \geq \bar{\kappa}(F_b) + \bar{\kappa}(B)$ for a fiber F_b over a generic $b \in B$ by Kawamata addition theorem ([9]). Since $S_0 - L$ is not affine ruled, we get $\bar{\kappa}(F_b) = 0$, i.e. f is a \mathbb{C}^* -ruling, a contradiction. □

DEFINITION 3.3. — *Let (X, B) be a smooth pair. A curve $C \subseteq X$ is a simple curve on (X, B) if and only if $C \cong \mathbb{P}^1$ and C has at most one common point with each connected component of B .*

COROLLARY 3.4. — *There is no simple curve on $(\bar{S}, D + \widehat{E})$. If D is snc-minimal then the pair $(\bar{S}, D + \widehat{E})$ is almost minimal.*

Proof. — Let L be a simple curve on $(\bar{S}, D + \widehat{E})$. Since S' is affine, $L \cap D \neq \emptyset$. Let (X_m, B_m) be the almost minimal model of some smooth completion of $S_0 - L$ and let $(X_m, B_m) \rightarrow (X_r, B_r)$ be the morphism contracting the connected components of $\text{Bk } B_m$. Denote the local fundamental group of a singular point $P \in \text{Sing}(X_r - B_r)$ by G_P . By 2.8 $X_m - B_m$ is an open subset of $S_0 - L$ satisfying $\chi(X_m - B_m) \leq \chi(S_0 - L)$. Since (X_m, B_m) is almost minimal, by 2.3(v) $(K_{X_m} + B_m)^+ \equiv K_{X_m} + B_m^\#$, so by 2.5(ii) and 3.2 $\chi(X_m - B_m) + \sum \frac{1}{|G_P|} > 0$. Put $s = |L \cap \widehat{E}|$. The matrix $Q(D)$ is not negative definite, so $|\text{Sing}(X_r - B_r)| \leq q - s$. This gives $\sum \frac{1}{|G_P|} \leq \frac{q-s}{2}$, so $\chi(S_0 - L) \geq \chi(X_m - B_m) > -\sum \frac{1}{|G_P|} \geq \frac{s-q}{2}$. We compute $\chi(S_0 - L) = \chi(S_0) - \chi(L) + |L \cap D| + s = 1 - q + s - 2 + |L \cap D|$, hence $|L \cap D| = \chi(S_0 - L) + 1 + q - s > \frac{q-s}{2} + 1$, so $|L \cap D| > 1$, a contradiction. Since log-exceptional curves of the first kind not contained in $D \cup \widehat{E}$ are simple, $(\bar{S}, D + \widehat{E})$ is almost minimal. □

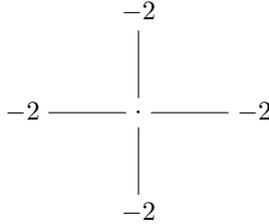
By 3.1(iii) D is a rational tree and since $Q(D)$ is not negative definite, if it is snc-minimal then by [4, 8.8] it is of one of the following types:

(Y): a fork with three maximal admissible twigs and $\delta(D) = 1$,

(H): has dual graph



(X): has dual graph



We will frequently use the fact that, as a consequence of the Riemann-Roch theorem, on a complete rational surface a (0) -curve (and hence any rational tree which contracts to a (0) -curve) induces a \mathbb{P}^1 -ruling with this curve as one of the fibers.

4. Rulings of S_0 with $\nu > 0$

From now on we assume that D is snc-minimal.

LEMMA 4.1. — *Let $D_0 \subseteq D$ be a component of D meeting some maximal twig of D and such that $D_0^2 \geq 0$. Let $\sigma : (\tilde{S}, \tilde{D}) \rightarrow (\bar{S}, D)$ be modification over D obtained by blowing up successively in the point of intersection of D_0 with the preimage of this maximal twig until $D_0^2 = 0$. Let $\pi : \tilde{S} \rightarrow \mathbb{P}^1$ be the induced \mathbb{P}^1 -ruling with D_0 as a fiber. Then a component of a fiber is an S_0 -component if and only if it is exceptional.*

Proof. — Denote the maximal twig of D as above by T . Let L be an S_0 -component of some fiber. We have $\bar{\kappa}(S_0) = 0$, so $(K_{\bar{S}} + D + \widehat{E})^+ \equiv 0$ by [4, 6.11] and then $K_{\bar{S}} + D + \widehat{E} \equiv \text{Bk } D + \text{Bk } \widehat{E}$ by 3.4 and 2.3(v). The sequence of blowups defining σ is subdivisioal for D , so $K_{\tilde{S}} + \tilde{D} + \widehat{E} \equiv \sigma^* \text{Bk } D + \text{Bk } \widehat{E}$ and $L^2 = -2 - L \cdot K_{\tilde{S}} = -2 + L \cdot (\tilde{D} - \sigma^* \text{Bk } D) + L \cdot (\widehat{E} - \text{Bk } \widehat{E}) \geq -2 + L \cdot (\tilde{D} - \sigma^* \text{Bk } D)$. Since $D_0 \not\subseteq \text{Supp } \text{Bk } D$ by 2.3(i) and each of the blowups is sprouting for the respective preimages of T , by 2.3(ii) the coefficients of components of \tilde{D} in $\sigma^* \text{Bk } D$ are smaller than one. Thus $L^2 > -2$ because $L \cdot \tilde{D} > 0$, so we are done. \square

Remark. — Notice that no fiber of a \mathbb{P}^1 -ruling of \tilde{S} can be contained in \hat{E} , otherwise \tilde{D} would be vertical, so S' would contain complete curves.

The following lemma, which is a generalization of arguments from [12, 6.2] allows to bound from below the self-intersection of one of the branching components of D having four maximal twigs.

LEMMA 4.2. — *Let T be an snc-minimal divisor with two branching components B, B' and such that $\beta_T(B) = \beta_T(B') = 3$. Let T_1, T_2 and T_3, T_4 be the maximal twigs of T intersecting B and B' respectively. If $Q(T - B - B')$ is negative definite, $\tilde{e}(T_1) + \tilde{e}(T_2) \leq -B^2 - 1$ and $\tilde{e}(T_3) + \tilde{e}(T_4) \leq -B'^2 - 1$ then either $Q(T)$ is negative definite or $d(T) = 0$ and then $T - T_1 - T_2 - T_3 - T_4$ is a (-2) -chain and $\tilde{e}(T_1) + \tilde{e}(T_2) = \tilde{e}(T_3) + \tilde{e}(T_4) = 1$.*

Proof. — Write $T - T_1 - T_2 - T_3 - T_4 = B_1 + B_2 + \dots + B_t$ with $B_1 = B$ and $B_t = B'$. Define $T_0 = B_2 + \dots + B_{t-1}$ and $d_i = d(D^{(i)})$, where $D^{(i)} = T_3 + T_4 + B_t + B_{t-1} + \dots + B_i$. By 2.6(i) $d_2 = d(T_3)d(T_4)d(T_0)(-B_t^2 - \tilde{e}(T_3) - \tilde{e}(T_4) - \tilde{e}(T_0)) \geq d(T_3)d(T_4)d(T_0)(1 - \tilde{e}(T_0)) > 0$, so $D^{(2)}$ is negative definite by Sylvester's theorem. We now prove that $d_2 \geq d_3$. Notice that $B_i^2 \leq -2$ for each i , so by 2.6 for $i = 2, \dots, t-1$ we get $d_i - d_{i+1} = (-B_i^2 - 2)d_{i+1} + d_{i+1} - d_{i+2} \geq d_{i+1} - d_{i+2}$. We have $d_t - d_{t+1} = d(T_3)d(T_4)(-B_t^2 - \tilde{e}(T_3) - \tilde{e}(T_4) - 1) \geq 0$, so we are done. By 2.6(i) $d(T) = d_2d(T_1)d(T_2)(-B_1^2 - \frac{d_3}{d_2} - \tilde{e}(T_1) - \tilde{e}(T_2)) \geq d_2d(T_1)d(T_2)(-B_1^2 - \tilde{e}(T_1) - \tilde{e}(T_2) - 1) \geq 0$. Hence by Sylvester's theorem if $d(T) \neq 0$ then $Q(T)$ is negative definite. On the other hand, if $d(T) = 0$ then all the weak inequalities above become equalities and the thesis follows. □

From now on we denote the maximal twigs of D by T_1, \dots, T_s . If D has only one branching component we denote it by B .

LEMMA 4.3. — *D can be only of type (X) or (Y). If it is of type (X) then $-1 \leq B^2 \leq 0$. If it is of type (Y) then $B^2 = -1$ and the triple $(d(T_1), d(T_2), d(T_3))$ is up to permutation one of the following: $(3, 3, 3)$, $(2, 3, 6)$, $(2, 4, 4)$.*

Proof. — In case (H) let B, B' be the branching components of D . The chain $D - T_1 - T_2 - T_3 - T_4 - B - B'$ is admissible, otherwise after some subdivisional modification of D it contains a (0) -curve, hence gives a \mathbb{C}^* -ruling of S_0 , which contradicts our assumptions about S_0 . Since $Q(D)$ is not negative definite and $d(D) \neq 0$, by 4.2 we can assume that $B^2 \geq -1$. Assume T_1 and T_2 meet B . Blow up on the intersection of B with $D - T_1 - T_2 - B$ until $B^2 = -1$. We have $T_1^2 = T_2^2 = -2$, so $T_1 + 2B + T_2$ gives a \mathbb{C}^* -ruling of S_0 , a contradiction. Thus only types (X) and (Y)

remain. We have $d(D) < 0$, so by 2.6 $-B^2 + \delta(D) < \sum_{i=1}^s \frac{d'(T_i^t)+1}{d(T_i)} \leq s$. For both types we obtain $B^2 \geq -1$.

In case (Y) we have $\delta(D) = 1$ by definition, so we need only to prove that $B^2 = -1$. Suppose $B^2 > 0$ in case (X) or $B^2 \geq 0$ in case (Y). Let $\sigma : (\tilde{S}, \tilde{D}) \rightarrow (\bar{S}, D)$ be the modification obtained by blowing up the point of intersection of T_1 with B until $B^2 = 0$. Consider the \mathbb{P}^1 -ruling of \tilde{S} given by B . We see that \tilde{D} contains no vertical (-1) -curves. The divisor D_h consists of three or four sections of the ruling. Put $D_v = \tilde{D} - D_h - B$. Notice that if some section intersects a vertical component V then $\mu(V) = 1$ and it does not intersect any other component lying in the fiber containing V . By 4.1 the S_0 -components of singular fibers are exceptional.

Let F be a fiber containing some connected component of \hat{E} (\hat{E} is vertical, because $B \cdot \hat{E} = 0$). If F contains some \tilde{D} -components then there exists a chain of S_0 -components in F connecting $\hat{E} \cap F$ with some \tilde{D} -component of F . In fact this chain consists of a unique (-1) -curve L , since all S_0 -components are (-1) -curves and two of them cannot meet. By 3.4 $D_h \cdot L > 0$, so $\mu(L) = 1$, a contradiction. Therefore there are no \tilde{D} -components in F , hence each S_0 -component intersects D_h , so it has $\mu = 1$. We have $\#D_h \leq 4$, so from 3.4 it follows that there are exactly two S_0 -components in F , each intersecting two components of D_h . This eliminates the case (Y). Notice that it follows also that these two (-1) -curves are tips of F , which by 2.1 implies that $\hat{E} \cap F$ is a (-2) -chain between them.

Consider the case (X). We have $D_v \neq 0$, because $B^2 > 0$. The divisor D_v is a chain and by the definition of σ can be written as $D_v = D_0 + D_1 + \dots + D_n$, where $D_0^2 = -3$, $n \geq 0$ and $D_i^2 = -2$ for every $1 \leq i \leq n$. Let F' be a fiber containing D_v . By 3.4 the connectedness of D_v implies that each (-1) -curve of F' intersects D_h . In particular, the (-1) -curves, and hence all components of F' have $\mu = 1$. It follows that $\hat{E} \cap F' = \emptyset$. We have $K \cdot D_v = 1$ and $K \cdot F' = -2$, so there are exactly three (-1) -curves in F' , call them L_2, L_3 and L_4 . We have $\sigma(F') = 3$, $\sigma(F) = 2$ and $\Sigma = 3$ by 3.1(iv), so any other singular fiber has $\sigma = 1$. However, the unique (-1) -curve of such a fiber has $\mu > 1$, so cannot intersect D_h , hence cannot intersect \tilde{D} , which is impossible. Thus F and F' are the only singular fibers, which implies that \hat{E} is connected. Since $\mu(L_i) = 1$ and F' cannot contain a (0) -curve as a proper subdivisor, we get that one of the L_i 's, say L_4 , intersects D_n and two others intersect D_0 (it is possible that $n = 0$). Each L_i intersects exactly one T_j , so by renaming L_i 's we can assume that for $i = 2, 3, 4$ we have $L_i \cdot T_i = 1$. The remaining section contained in D_h , call it T'_1 , is a (-1) -curve and intersects D_n . Let M_2 be the (-1) -curve of F

intersected by T_4 . Denote the second (-1) -curve of F by M_1 . If $T'_1 \cdot M_2 > 0$ then the contraction of $F - M_2 + F' - L_4$ does not touch T_4 and touches T'_1 once, so the images of T_4 and T'_1 are disjoint sections of a \mathbb{P}^1 -ruling of a Hirzebruch surface and have self-intersections -2 and 0 . This is impossible, so we infer that $T'_1 \cdot M_2 = 0$ and $T'_1 \cdot M_1 = 1$. Now by symmetry we can assume that T_2 intersects M_2 and T_3 intersects M_1 . The contraction of $F - M_1 + F' - L_3$ does not touch T_3 and touches T'_1 exactly $n + 1$ times. Thus as above we get a \mathbb{P}^1 -ruling of a Hirzebruch surface with two disjoint sections having self-intersections -2 and n . It follows from the properties of a Hirzebruch surface that $n = 2$. Now observe that $T_4 + 2L_4 + D_2$ and $T_3 + 2L_3 + D_0 + L_2$ are disjoint (0) -divisors, so they are fibers of the same \mathbb{P}^1 -ruling of \tilde{S} . This contradicts the fact that T_2 intersects the second one and not the first one. \square

PROPOSITION 4.4. — *Let S_0 be the smooth locus of an exceptional singular \mathbb{Q} -homology plane S' . If S' has at most topologically rational singularities then $\bar{\kappa}(S') = \bar{\kappa}(S_0) = 0$ and S' has a unique singular point. Moreover, either*

- (i) *S' (hence S_0) is \mathbb{C}^{**} -ruled, its singularity is of type A_1 and its snc-minimal boundary D is a fork with branching (-1) -curve and three maximal twigs: $[2]$, $[2, 2, 2]$ and $[2, 2, 2]$ (cf. 5.2) or*
- (ii) *S' (hence S_0) is \mathbb{C}^{***} -ruled, its singularity is of type A_2 and its snc-minimal boundary D is a fork with branching (-1) -curve and three maximal twigs: $[2, 2]$, $[2, 2]$ and $[2, 2]$. (cf. 5.4).*

Proof. — We check easily that admissible chains with $d(-)$ equal to $2, 3$ or 6 have only one component or consist of (-2) -curves, so by 4.3 we have only thirteen cases to consider:

(X0) $T_i = [2]$ for $i = 1, 2, 3, 4$ and $B^2 = 0$,

(X1) $T_i = [2]$ for $i = 1, 2, 3, 4$ and $B^2 = -1$,

D is of type (Y) with $B^2 = -1$ and:

(Y1a) $T_1 = [3], T_2 = [3], T_3 = [3]$,

(Y1b) $T_1 = [3], T_2 = [3], T_3 = [2, 2]$,

(Y1c) $T_1 = [3], T_2 = [2, 2], T_3 = [2, 2]$,

(Y1d) $T_1 = [2, 2], T_2 = [2, 2], T_3 = [2, 2]$,

(Y2a) $T_1 = [2], T_2 = [4], T_3 = [4]$,

(Y2b) $T_1 = [2], T_2 = [4], T_3 = [2, 2, 2]$,

$$(Y2c) \quad T_1 = [2], T_2 = [2, 2, 2], T_3 = [2, 2, 2],$$

$$(Y3a) \quad T_1 = [2], T_2 = [3], T_3 = [6],$$

$$(Y3b) \quad T_1 = [2], T_2 = [3], T_3 = [2, 2, 2, 2, 2],$$

$$(Y3c) \quad T_1 = [2], T_2 = [2, 2], T_3 = [6],$$

$$(Y3d) \quad T_1 = [2], T_2 = [2, 2], T_3 = [2, 2, 2, 2, 2].$$

Write each T_i as $T_i = T_{i,1} + T_{i,2} + \dots + T_{i,k_i}$, where $T_{i,1}$ is a tip of D . In cases (Y1a), (Y2a) and (Y3a) we compute $d(D) = 0$, which contradicts 3.1(i). In each other case we specify a \mathbb{P}^1 -ruling $\pi : \bar{S} \rightarrow \mathbb{P}^1$ with $\nu > 0$ defined by some (0)-divisor F_∞ with support in D . By 3.1(iv) we have $\Sigma = \#D_h - 1$. Below we list the quadruples $(F_\infty, F \cdot D, \Sigma, D_v)$, where F is the generic fiber and $D_v = D - \underline{F}_\infty - D_h$.

$$(X0) \quad (B, 4, 3, 0),$$

$$(X1) \quad (T_1 + 2B + T_2, 4, 1, 0),$$

$$(Y1b) \quad (T_1 + 3B + 2T_{3,2} + T_{3,1}, 3, 0, 0),$$

$$(Y1c) \quad (T_1 + 3B + 2T_{3,2} + T_{3,1}, 3, 0, T_{2,1}),$$

$$(Y1d) \quad (T_{1,2} + 2B + T_{3,2}, 4, 2, T_{2,1}),$$

$$(Y2b) \quad (T_1 + 2B + T_{3,3}, 3, 1, T_{3,1}),$$

$$(Y2c) \quad (T_1 + 2B + T_{3,3}, 3, 1, T_{3,1} + T_{2,1} + T_{2,2}),$$

$$(Y3b) \quad (T_1 + 2B + T_{3,5}, 3, 1, T_{3,1} + T_{3,2} + T_{3,3}),$$

$$(Y3c) \quad (T_1 + 2B + T_{2,2}, 3, 1, 0),$$

$$(Y3d) \quad (T_1 + 2B + T_{3,5}, 3, 1, T_{2,1} + T_{3,1} + T_{3,2} + T_{3,3}).$$

Notice that D_v has at most two connected components and each of them is a chain of (-2) -curves. Let F be some singular fiber of π . The S_0 -components of F are (-1) -curves by 4.1, denote them by L_i , $i = 1, \dots, \sigma(F)$. We use 3.4 repeatedly.

Claim 1. — Every S_0 -component intersects D_h .

Suppose L is an S_0 -component, such that $L \cdot D_h = 0$. Then L intersects two D -components by 3.4 and these are (-2) -curves, so $F = [2, 1, 2]$. Both these D -components must be tips of D . Since $L \cdot D_h = 0$ and $\nu > 0$, we obtain $F \cdot D = 2$, otherwise D would contain a loop. This is a contradiction.

Claim 2. — If $\mu(L_i) > 1$ for some i then $\sigma(F) = 1$ and $\mu(L_1) = 2$.

Suppose $\sigma(F) \geq 2$ and $\mu(L_1) > 1$. L_1 intersects some D -component of F , otherwise $D_h \cdot L_1 \geq 2$ and $D_h \cdot F \geq D_h \cdot (\mu(L_1)L_1 + L_2) > 4$, which is impossible. Thus $D_v \cap F \neq \emptyset$ and we get $4 \geq D_h \cdot F \geq D_h \cdot (\mu(L_1)L_1 + D_v \cap F + \mu(L_2)L_2) \geq 2 + D_h \cdot (D_v \cap F) + D_h \cdot \mu(L_2)L_2$, so by (1) $\mu(L_2) = D_h \cdot L_2 = 1$ and $D_v \cap F$ is connected. Moreover, $\sigma(F) = 2$. We get $L_2 \cdot D_v > 0$, because L_2 cannot be simple. Since D_v is a (-2) -chain and $\mu(L_2) = 1$, L_2 intersects D_v in a tip, so $F = [1, (k), 1]$ for some $k > 0$ (recall that $[(k)]$ is a chain consisting of k (-2) -curves). This contradicts $\mu(L_1) > 1$.

Suppose $\sigma(F) = 1$ and $\mu(L_1) > 2$. Since $D_h \cdot L_1 > 0$, D_h contains an n -section with $n > 2$, which is possible only for (Y1b) or (Y1c). Then $FD = 3$, so $D_h(F - L_1) = 0$, hence there are no D -components in F . Thus L_1 is simple, a contradiction.

Claim 3. — If $\sigma(F) > 1$ then $F = [1, (k), 1]$ for some $k \geq 0$. If $\sigma(F) = 1$ then in cases other than (X1) $F = [2, 1, 2]$ and F contains a D -component.

If $\sigma(F) > 1$ then all L_i 's are tips of F by (2). Suppose $\sigma(F) > 2$. Then there are some D -components in F , otherwise $F \cdot D \geq 6$ by 3.4. The divisor $F - \sum_i L_i$ is connected and contains a D -component, so there are no \widehat{E} -components in F . Since D_v consists of (-2) -curves, we get $-2 = K_{\overline{S}} \cdot F = \sum_i K_{\overline{S}} \cdot L_i = -\sigma(F)$, a contradiction. Thus $\sigma(F) = 2$ and both (-1) -curves have multiplicities one by (2), so $F = [1, (k), 1]$ for some $k \geq 0$.

Assume $\sigma(F) = 1$ and consider cases different from (X1). We have $\mu(L_1) = 2$ by (2). There are some D -components in F , otherwise by 3.4 L would meet two 2-sections contained in D_h , which is possible in case (X1) only. Suppose F is branched. Then by 2.2 L_1 is a tip of F and $F - L_1$ is one of the connected components of D_v , hence it must be $[2, 2, 2]$, which is possible for (Y3b) only. In this case D_v is connected, $F \cdot D = 3$ and $\Sigma = 1$. In particular, there exists a fiber F' with $\sigma(F') = 2$ and it does not have any D -components, so both S_0 -components of F' meet D_h at least twice, which contradicts $F \cdot D = 3$. Thus F is a chain, so $F = [2, 1, 2]$.

Claim 4. — $\overline{\kappa}(S) = 0$ and $K_{\overline{S}} + D^\# \equiv 0$.

By (2), (3) and 2.2 every singular fiber consists of (-1) - and (-2) -curves. \widehat{E} is vertical, hence consist of (-2) -curves, so by 2.7 $\overline{\kappa}(S) = \overline{\kappa}(S_0) = 0$. The pair $(\overline{S}, D + \widehat{E})$ is almost minimal, so by 2.3(v) $K_{\overline{S}} + D^\# + \widehat{E}^\# \equiv 0$. By 2.7 $K_{\overline{S}} + D^\# \geq_{\mathbb{Q}} 0$, so $\widehat{E} = \text{Bk } \widehat{E}$ and $K_{\overline{S}} + D^\# \equiv 0$.

Claim 5. — Cases other than (X0), (X1), (Y1d) and (Y2c) are impossible. $\#\widehat{E} = 8 - B^2 - \#D$.

By (4) we have $K_{\bar{S}} \cdot \text{Bk } D = K_{\bar{S}}^2 + K_{\bar{S}} \cdot D$, so $K_{\bar{S}} \cdot \text{Bk } D \in \mathbb{Z}$. This excludes (Y1b), (Y1c), (Y2b), (Y3b) and (Y3c). In the remaining cases (X0), (X1), (Y1d), (Y2c) and (Y3d) the maximal twigs of D are (-2) -chains, so by (4) $K_{\bar{S}} \cdot (K_{\bar{S}} + B) = 0$. Since \bar{S} is rational, we have $\chi(\bar{S}) = 2 + \#D + \#\widehat{E}$ by 3.1(i) and then the Noether formula gives $12 = K_{\bar{S}}^2 + 2 + \#D + \#\widehat{E}$, so $\#\widehat{E} = 8 - B^2 - \#D$. For (Y3d) we get $\#\widehat{E} = 0$, a contradiction.

Claim 6. — \widehat{E} is connected. Case (X0) is impossible.

Notice that 2.5(ii) gives $0 \leq \chi(S_0) + \sum_P \frac{1}{|G_P|} \leq 1 - q + \frac{g}{2}$, so if \widehat{E} is not connected then $q = 2$ and $|G_{P_1}| = |G_{P_2}| = 2$, hence \widehat{E}_1 and \widehat{E}_2 are (-2) -curves. In cases (X1) and (X0) we have $\#\widehat{E} = 3 - B^2 \geq 3$ by (5) and in case (Y2c) $\#\widehat{E} = 1$, so \widehat{E} is connected. Consider the case (Y1d). Suppose there exists a singular fiber F with $\sigma(F) = 1$. By (3) $F = [2, 1, 2]$ and there is a D -component in F , so $D_v = T_{2,1} \subseteq F$ and F contains an \widehat{E} -component. It follows that the sections $T_{1,1}$ and $T_{3,1}$ intersect L_1 , a contradiction with $F \cdot D \leq 4$. Since $\Sigma = 2$, by (3) there are only two singular fibers and they are of type $[1, (k), 1]$, so \widehat{E} is connected because $D_v \neq 0$.

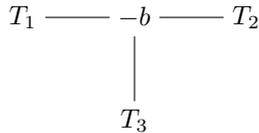
Suppose that the case (X0) occurs. Since $\Sigma = 3$, there is a singular fiber F with $\sigma(F) > 1$, hence by (3) $F = [1, (k), 1]$ for some $k \geq 0$. It is easy to see that for every such fiber $k > 0$. Indeed, we know that $D_v = 0$ and L_1, L_2 are not simple, so each is intersected by precisely two sections from D_h , so if $H_1, H_2 \subseteq D_h$ intersect L_1 then $k = 0$ implies that $H_1 + 2L_1 + H_2$ gives a \mathbb{C}^* -ruling of S_0 , a contradiction. Since \widehat{E} is connected, we see that there is only one fiber with $\sigma > 1$. This contradicts $\Sigma = 3$.

Claim 7. — Case (X1) is impossible.

Suppose the case (X1) occurs. We have $\Sigma = 1$, so by (3) there is a fiber $F_1 = [1, (k), 1]$ with $k \geq 0$. Suppose $k > 0$. We have $D_v = 0$, so $\widehat{E} \subseteq F_1$ by (6) and F_∞ and F_1 are the only singular fibers. By (5) we can write $F_1 = L_1 + E_1 + E_2 + E_3 + E_4 + L_2$. Notice that D_h consists of two 2-sections, T_3 and T_4 , and by 3.4 D_h intersects $F_1 - \widehat{E}$ in four points. If L_1 intersects both 2-sections then the contraction of $\underline{F}_\infty - T_2 + F_1 - L_1$ touches T_3 seven times, so the image of T_3 is a smooth 2-section on a Hirzebruch surface with self-intersection 5, a contradiction. Thus L_1 intersects only one component of D_h , say T_3 , hence L_2 intersects T_4 . After the contraction of $\underline{F}_\infty - T_1 + F_1 - L_1$ the surface becomes a Hirzebruch surface and the images of the 2-sections, T'_3 and T'_4 , satisfy $T'_3 \cdot T'_4 = 2, T'^2_3 = 0$ and $T'^2_4 = 20$. However, $T'_3 - T'_4 \equiv \alpha F$ for some $\alpha \in \mathbb{Z}$ and a generic fiber F , because T'_3 and T'_4 are 2-sections. Thus $(T'_3 - T'_4)^2 = 0$, which is a contradiction. Thus $k = 0$

and $\widehat{E} \subseteq F_0$, where F_0 is a singular fiber with $\sigma(F_0) = 1$. By (5) and (1) \widehat{E} is a (-2) -fork with four components. Let M be the (-1) -curve of F_0 . Denote the \widehat{E} -component intersecting M by E_0 and the branching component of \widehat{E} by E_1 . Consider a new \mathbb{P}^1 -ruling of \overline{S} given by the (0) -divisor $T_3 + 2M + T_4$. For this ruling we have $\Sigma = 0$. Let F' be a fiber containing $\widehat{E} - E_0$. There is exactly one (-1) -curve $U \subseteq F'$, which is the unique S_0 -component of F' . Notice that now the only possible D -components of F' are T_1 and T_2 , which are (-2) -curves. Since U intersects some \widehat{E} -component of F' , which is also a (-2) -curve, U cannot intersect other (-2) -curves than F' , otherwise $F' = [2, 1, 2]$, which is not the case. We conclude that F' has no D -components, hence U intersects E_1 and $\mu(E_1) = \mu(U) = 2$. It follows that E_0 intersects F' only in E_1 and B intersects U in one point. Thus U is a simple curve on $(\overline{S}, D + \widehat{E})$, a contradiction. \square

Remark 4.5. — Let us notice that up to isomorphism there exist exactly three smooth exceptional \mathbb{Q} -homology planes (see [4, 8.64] or [15, 4.4.4] for their construction). They are called: $Y\{3, 3, 3\}$, $Y\{2, 4, 4\}$ and $Y\{2, 3, 6\}$, where the boundary of $Y\{d_1, d_2, d_3\}$ is a fork



with $d(T_i) = d_i$ (surfaces of type $H[-k, k]$ considered in the above references are \mathbb{C}^* -ruled). More precisely, (b, T_1, T_2, T_3) is equal to $(1, [2, 2], [2, 2], [2, 2])$ in case $Y\{3, 3, 3\}$, to $(0, [2], [2, 2, 2], [2, 2, 2])$ in case $Y\{2, 4, 4\}$ and to $(1, [2], [2, 2], [2, 2, 2, 2, 2])$ in case $Y\{2, 3, 6\}$.

Trying to follow this notation we will denote a singular exceptional \mathbb{Q} -homology plane admitting a boundary as above by $SY\{d_1, d_2, d_2\}$. It follows from the above proposition that the only possibilities are $SY\{3, 3, 3\}$ (ruling (Y1d)) with $(b, T_1, T_2, T_3) = (1, [2, 2], [2, 2], [2, 2])$ and $\widehat{E} = [2, 2]$ and $SY\{2, 4, 4\}$ (ruling (Y2c)) with $(b, T_1, T_2, T_3) = (1, [2], [2, 2, 2], [2, 2, 2])$ and $\widehat{E} = [2]$.

5. Constructions

We now find a more precise description of rulings of type (Y2c) and (Y1d) and then use it to construct exceptional singular \mathbb{Q} -homology planes of type $SY\{2, 4, 4\}$ and $SY\{3, 3, 3\}$ respectively (cf. 4.5). We produce $\text{Aut}(\overline{S}, D + \widehat{E})$ -equivariant contractions $\theta: \overline{S} \rightarrow \mathbb{P}^2$.

LEMMA 5.1. — *In the case (Y2c) there are three singular fibers (see Fig. 5.1): $F_\infty = T_1 + 2B + T_{3,3}$, $F_1 = L_1 + T_{2,2} + T_{2,1} + L_2$ and $F_0 = T_{3,1} + M + \widehat{E}$, where $\widehat{E} = [2]$ and L_1, L_2, M are (-1) -curves. The 2-section $T_{2,3}$ meets L_2 and $L_1 \cdot T_{3,2} = 1$. There is a morphism $\theta : \overline{S} \rightarrow \mathbb{P}^1$ contracting $B + T_1 + M + T_{3,1} + T_{3,2} + M' + T_{2,1} + T_{2,2}$, where M' is some (-1) -curve, such that $\theta(T_{3,3})$ and $\theta(T_{2,3})$ are smooth conics tangent at $\theta(B)$, meeting at $\theta(T_{2,1})$ and $\theta(T_{3,1})$ and $\theta(\widehat{E})$ is a smooth conic, such that for $i = 1, 2$ $\theta(\widehat{E})$ intersects $\theta(T_{i,3})$ in $\theta(T_{i,1})$ with multiplicity three (see Fig. 5.3).*

Proof. — We use the facts showed in the proof of 4.4. We have $\Sigma = 1$, so by (3) there exists a fiber $F_1 = [1, (k), 1]$ with $k \geq 0$ and this is a unique fiber with $\sigma > 1$. Since $D_v \neq 0$, F_1 cannot be the only singular fiber, so there exists a singular fiber F_0 with $\sigma(F_0) = 1$. We have $F_0 = [2, 1, 2]$ by (3). We have $\#\widehat{E} = 1$, $\#D_v = 3$ and D_v has two connected components, so $k = 2$ and F_0 contains \widehat{E} and one D -component. Besides F_∞ there are no more singular fibers. Notice that $T_{2,3}$ is a 2-section intersecting the unique (-1) -curve of F_0 , call it M , in a branching point of $\pi|_{T_{2,3}}$. Let $L_1 \subset F_1$ be the (-1) -curve meeting $T_{2,2}$. Suppose L_1 meets the 2-section $T_{2,3}$ too. Then L_2 , the second (-1) -curve of F_1 , meets $T_{2,1}$ and $T_{3,2}$. The contraction of $F_\infty - T_{3,3} + F_1 - T_{2,2} + F_0 - T_{3,1}$ touches $T_{2,3}$ five times, so the image of $T_{2,3}$ is a 2-section on a Hirzebruch surface having self-intersection 3, which is impossible. Hence L_1 meets $T_{3,2}$. Let M' be an exceptional component of a \mathbb{P}^1 -ruling of \overline{S} induced by $T_1 + 2B + T_{2,3}$, such that $M' \cdot \widehat{E} > 0$. Since the structure of fibers and sections is analogous, $M' \cdot T_{2,1} = M' \cdot T_{3,3} = M \cdot \widehat{E} = 1$ and M' does not intersect any other component of D (see Fig. 5.2). Thus the chains $M + T_{3,1} + T_{3,2}$, $B + T_1$, $M' + T_{2,1} + T_{2,2}$ are disjoint and we can define $\theta : \overline{S} \rightarrow \mathbb{P}^2$ as their contraction. Then $\theta(T_{2,3})$, $\theta(T_{3,3})$ and $\theta(\widehat{E})$ are smooth conics with prescribed properties. \square

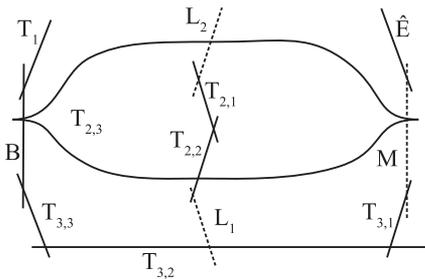


Figure 5.1. (Y2c), ruling

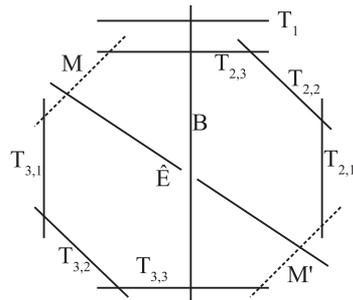


Figure 5.2. (Y2c), contraction

CONSTRUCTION 5.2. — Let $T_{2,3} \subseteq \mathbb{P}^2$ be a smooth conic and (P_1, P_2, P_3) a triple of distinct points on it. This choice is unique up to an automorphism of \mathbb{P}^2 . There is a unique pair of smooth conics $(\widehat{E}, T_{3,3})$, such that $P_2, P_3 \in T_{3,3} \cap \widehat{E}$, $T_{3,3}$ is tangent to $T_{2,3}$ at P_1 and \widehat{E} intersects $T_{i,3}$ with multiplicity three at P_i for $i = 2, 3$ (see Fig. 5.3). (This can be seen as follows. Suppose $T_{2,3} = \{2yz = y^2 - x^2\}$, $P_1 = (0, 0, 1)$, $P_2 = (1, -1, 0)$ and $P_3 = (1, 1, 0)$. Then the family of conics $T_{3,3}(u)$ through P_2, P_3 and tangent to $T_{2,3}$ at P_1 is one-dimensional: $T_{3,3}(u) = \{u y z = y^2 - x^2\}$. The family of conics $\widehat{E}(v)$ through P_2 and P_3 , intersecting $T_{2,3}$ at P_2 with multiplicity three is one-dimensional too: $\widehat{E}(v) = \{v(y^2 - x^2 - 2yz) = z^2 - yz - xz\}$. The condition for intersection at P_3 implies $(u, v) = (-2, \frac{1}{2})$.) We use the same names for divisors and their birational transforms. Blow up three times over P_2 on the intersection of $T_{2,3}$ with \widehat{E} and denote the subsequent exceptional curves by $T_{3,2}, T_{3,1}$ and M , similarly blow up three times over P_3 on the intersection of $T_{3,3}$ with \widehat{E} and denote the subsequent exceptional curves by $T_{2,2}, T_{2,1}$ and M' . Then blow up twice over P_1 so that the birational transforms of $T_{3,3}$ and $T_{2,3}$ do not meet, denote the exceptional curves by T_1 and B . Denote the resulting complete surface by \overline{S} . Define $D = T_{3,1} + T_{3,2} + T_{3,3} + T_{2,1} + T_{2,2} + T_{2,3} + T_1 + B$, $S = \overline{S} - D$ and $S' = S/\widehat{E}$. Clearly, D is a fork with $\delta(D) = 1$, $B^2 = -1$ and other components of D are (-2) -curves.

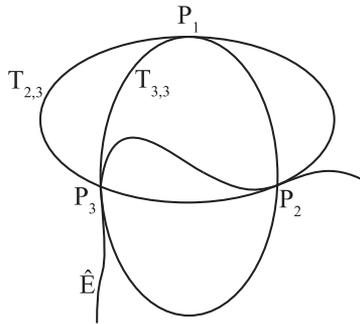


Figure 5.3. (Y2c), after contraction

LEMMA 5.3. — In the case (Y1d) there are three singular fibers (see Fig. 5.4): $F_\infty = T_{1,2} + 2B + T_{3,2}$, $F_1 = L_1 + E_1 + E_2 + L_2$ and $F_2 = M + T_{2,1} + L_3$, where $\widehat{E} = E_1 + E_2 = [2, 2]$ and L_1, L_2, L_3, M are (-1) -curves. $T_{3,1} \cdot M = T_{3,1} \cdot L_1 = 1$, $T_{1,1} \cdot L_2 = T_{1,1} \cdot M = 1$, $T_{2,2} \cdot L_1 = T_{2,2} \cdot L_2 = T_{2,2} \cdot L_3 = 1$ and $T_{2,2} \cap T_{2,1} \neq T_{2,2} \cap L_3$. There exists a morphism $\theta : \overline{S} \rightarrow \mathbb{P}^2$

contracting the divisor $B + M + L_1 + L_2 + L'_1 + L'_2 + L''_1 + L''_2$ consisting of disjoint (-1) -curves, such that the image of $T_{1,2} + T_{2,2} + T_{3,2}$ is a triple of lines intersecting in $\theta(B)$ and the image of $T_{1,1} + T_{2,1} + T_{3,1}$ is a triple of lines intersecting in $\theta(M)$ (see Fig. 5.6). Moreover, $\theta(T_{1,2}) \cap \theta(T_{2,1})$, $\theta(T_{2,2}) \cap \theta(T_{3,1})$, $\theta(T_{3,2}) \cap \theta(T_{1,1})$ lie on a line $\theta(E_1)$ and $\theta(T_{1,2}) \cap \theta(T_{3,1})$, $\theta(T_{2,2}) \cap \theta(T_{1,1})$, $\theta(T_{3,2}) \cap \theta(T_{2,1})$ lie on a line $\theta(E_2)$.

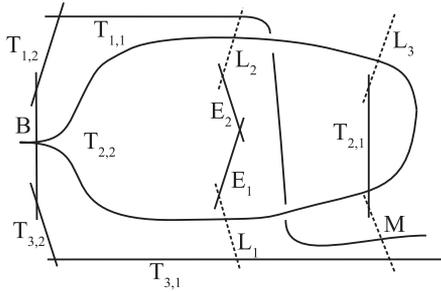


Figure 5.4. (Y1d), ruling

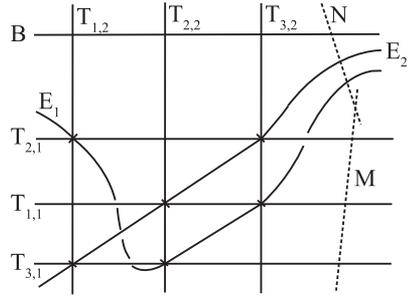


Figure 5.5. (Y1d), contraction

Proof. — We have $\Sigma = 2$, so by (3) there exist fibers $F_1 = [1, (k_1), 1]$, $F_2 = [1, (k_2), 1]$ with $k_1, k_2 \geq 0$. Since $\widehat{E} = [2, 2]$ by (5) and singular fibers with $\sigma = 1$ are of type $[2, 1, 2]$, we can assume that $\widehat{E} \subseteq F_1$ and $k_1 = 2$. Since D_v is connected, there are no more singular fibers besides F_∞ , hence $T_{2,1} \subseteq F_2$ and $k_2 = 1$. Let $M \subseteq F_2$ be the (-1) -curve not intersecting $T_{2,2}$. By 3.4 $T_{1,1} + T_{3,1}$ intersects M , so by symmetry we can assume that $T_{3,1}$ does. Let L_1 be the (-1) -curve of F_1 intersecting $T_{3,1}$. The contraction of $F_\infty - T_{3,2} + F_1 - L_1 + F_2 - M$ does not touch $T_{3,1}$ and the images of $T_{3,1}$ and $T_{1,1}$ are two disjoint sections on a Hirzebruch surface, hence the image of $T_{1,1}$ must have self-intersection 2 and we infer that the contraction touches $T_{1,1}$ exactly four times. Since $k_2 = 2$, it follows that $T_{1,1}$ does not intersect L_1 and intersects M (see Fig. 5.4). Clearly, the analogous rulings of \overline{S} induced by $F'_\infty = T_{1,2} + 2B + T_{2,2}$ or $F''_\infty = T_{2,2} + 2B + T_{3,2}$ have analogous structure of singular fibers and configuration of special horizontal components. Denote the (-1) -curves of the fibers of these rulings containing \widehat{E} as L'_1, L'_2 and L''_1, L''_2 respectively. It is easy to see that $L_1, L'_1, L''_1, L_2, L'_2, L''_2$ are disjoint. For example, for $i = 1, 2$ we have $L_i \cdot F'_\infty = 1$, so $L_i \cdot (L'_1 + L'_2) = 0$. Let $\omega : \overline{S} \rightarrow \widetilde{S}$ be the contraction of all these exceptional curves. For any $i, j, k \in \{1, 2\}$ we have $\omega(T_{i,1}) \cdot \omega(T_{j,2}) = 1$, $\omega(T_{i,j})^2 = 0$ and $\omega(E_k)^2 = 1$. We see also that $\omega(E_k)$ meets each $T_{i,j}$ once and only in points being images of curves contracted by ω (see Fig. 5.5).

Now since $b_2(\tilde{S}) = b_2(\bar{S}) - 6 = 3$, the \mathbb{P}^1 -ruling $\tilde{\rho} : \tilde{S} \rightarrow \mathbb{P}^1$ induced by $\omega(T_{1,2})$ has only one singular fiber $\tilde{F} = [1, 1]$. Furthermore, M is not touched by ω and $\omega(T_{1,2}) \cdot M = 0$, so $\tilde{F} = M + N$, where N is a birational transform of some S_0 -component (see Fig. 5.5). We have $\omega(T_{i,j}) \cdot N = 0$ and $B \cdot N = 1$. If we define θ as the composition of ω with the contraction of $B + M$ then the properties of θ stated in the thesis follow (see Fig. 5.6). \square

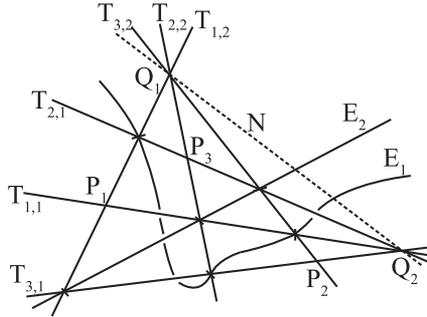


Figure 5.6. (Y1d), after contraction

CONSTRUCTION 5.4. — Let $P_1 = [0, 1, 1], P_2 = [1, 1, 0], Q_1 = [1, 0, 0], Q_2 = [0, 0, 1]$ be points in $\mathbb{P}^2_{(x,y,z)}$. The lines $\overline{Q_1P_1}, \overline{Q_1P_2}, \overline{Q_2P_1}$ and $\overline{Q_2P_2}$ have equations $y = z, z = 0, x = 0$ and $x = y$. Put $P_3 = [1, \epsilon, \epsilon - 1]$, where $\epsilon = -\zeta$ for some primitive third root of unity ζ . Then the points $\overline{Q_1P_1} \cap \overline{Q_2P_2} = [1, 1, 1], \overline{Q_1P_2} \cap \overline{Q_2P_3} = [\epsilon, \epsilon - 1, 0], \overline{Q_1P_3} \cap \overline{Q_2P_1} = [0, 1, \epsilon]$ lie on a line $E_2 = \{(1 - \epsilon)x + \epsilon y = z\}$ and the points $\overline{Q_1P_1} \cap \overline{Q_2P_3} = [1, \epsilon, \epsilon], \overline{Q_1P_2} \cap \overline{Q_2P_1} = [0, 1, 0], \overline{Q_1P_3} \cap \overline{Q_2P_2} = [1, 1, \epsilon]$ lie on a line $E_1 = \{z = \epsilon x\}$. Blow once in Q_1 and Q_2 and denote the exceptional curve of the first blowup by B . Blow once in each of the six points of intersection of lines $\overline{Q_iP_j}$ with $E_1 + E_2$. Let D be the divisor consisting of the proper transforms of B and of lines $\overline{Q_iP_j}$. Denote the resulting surface by \bar{S} and put $S = \bar{S} \setminus D, S' = S/\hat{E}$, where $\hat{E} = E_1 + E_2$. Clearly, D is a fork with $\delta(D) = 1, B^2 = -1$ and $D - B + \hat{E}$ consists of (-2) -curves.

Remark 5.5. — Notice that the points $Q_3 = E_1 \cap E_2 = [1, 1 + \epsilon, \epsilon], P_1 = [0, 1, 1], P_2 = [1, 1, 0]$ and $P_3 = [1, \epsilon, \epsilon - 1]$ lie on a common line $L : y = x + z$. Then the set of twelve points $\bigcup_{i=1}^3 \{Q_i, P_i\} \cup (E_1 \cup E_2) \cap \bigcup_{i,j=1}^3 T_{i,j}$ (with $T_{i,j}$ as on the picture 5.6) and of nine lines $\bigcup_{i,j=1}^2 \{T_{i,j}\} \cup \{E_1, E_2, L\}$ is a famous *dual Hesse configuration* $(12_3, 9_4)$, which is dual to the configuration of nine flexes on a smooth cubic and lines joining them (cf. [2] and [3]).

Recall that (a_b, c_d) -configuration is a configuration of a points and c lines, such that each point lies on b lines and each line contains d points. This configuration has the property that each point belongs to three lines, so by the projective dual of the Sylvester-Gallai theorem, it cannot be realized in \mathbb{RP}^2 .

We now prove the theorem 1.4.

Proof. — It follows from 4.4 (or rather from its proof) that S' is of type $SY\{2, 4, 4\}$ or $SY\{3, 3, 3\}$ (cf. 4.5). If S' is of type $SY\{2, 4, 4\}$ then the analysis of the ruling (Y2c) of \bar{S} done in 5.1 implies that it can be constructed as in 5.2. The construction was determined uniquely by a choice of a smooth conic in \mathbb{P}^2 and an ordered triple of distinct points on it, hence S' of type $SY\{2, 4, 4\}$ is unique up to isomorphism. Clearly, the surfaces S' of type $SY\{2, 4, 4\}$ and of type $SY\{3, 3, 3\}$ are non-isomorphic, because their singularities are of different type. We now prove that if S' is of type $SY\{3, 3, 3\}$ then it can be constructed as in 5.4. Let $\theta : \bar{S} \rightarrow \mathbb{P}^2$ be as in 5.3, put $Q_1 = \theta(B)$, $Q_2 = \theta(M)$, $P_1 = \theta(T_{1,2} \cap T_{1,1})$ and $P_2 = \theta(T_{3,2} \cap T_{3,1})$, we can assume that their coordinates are as in 5.4. Since $P_3 = \theta(T_{2,2} \cap T_{2,1}) \notin \overline{P_1 Q_2}$, we can write $P_3 = [1, \epsilon, u]$ for some $\epsilon, u \in \mathbb{C}$. The condition of collinearity of $\theta(T_{1,2}) \cap \theta(T_{2,1}) = [1, \epsilon, \epsilon]$, $\theta(T_{2,2}) \cap \theta(T_{3,1}) = [\epsilon, \epsilon, u]$, $\theta(T_{3,2}) \cap \theta(T_{1,1}) = [0, 1, 0]$ implies $u = \epsilon^2$ and the condition of collinearity of $\theta(T_{1,2}) \cap \theta(T_{3,1}) = [1, 1, 1]$, $\theta(T_{2,2}) \cap \theta(T_{1,1}) = [0, \epsilon, u]$, $\theta(T_{3,2}) \cap \theta(T_{2,1}) = [1, \epsilon, 0]$ implies $\epsilon^2 - \epsilon + 1 = 0$, hence $-\epsilon$ is a primitive third root of unity. Therefore for a fixed choice of points P_1, P_2, Q_1, Q_2 there are two choices for P_3 , denote them by P_3 and P'_3 . The construction was determined uniquely by a choice of a quadruple of distinct points in \mathbb{P}^2 and a primitive third root of unity, hence up to isomorphism there are at most two surfaces S' of type $SY\{3, 3, 3\}$. For (P_1, P_2, Q_1, Q_2) fixed the collinearity conditions determine the set $\{P_3, P'_3\}$. Moreover, the role of P_1 and P_2 is symmetric, so the quadruples (P_1, P_2, Q_1, Q_2) and (P_2, P_1, Q_1, Q_2) determine the same set $\{P_3, P'_3\}$. The automorphism $\sigma \in \text{Aut } \mathbb{P}^2$ given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

fixes Q_1 and Q_2 and changes P_1 with P_2 . Since σ changes P_3 with P'_3 , we conclude that the choices of P_3 and P'_3 are equivalent.

We now check that constructions 5.4 and 5.2 result with singular \mathbb{Q} -homology planes with prescribed properties. In each case we have $b_1(\bar{S}) = 0$,

$b_2(\overline{S}) = 9$ and since $d(D) \neq 0$, the components of $D + \widehat{E}$ are independent in $NS(\overline{S}) \otimes \mathbb{Q}$, hence $H_2(D + \widehat{E}) \rightarrow H_2(\overline{S})$ is an isomorphism. The homology exact sequence of a pair (\overline{S}, D) and the Lefschetz duality give $b_1(S) = b_3(S) = b_4(S) = 0$ and $b_2(S) = \#\widehat{E}$. We know that $H_2(\widehat{E}) \rightarrow H_2(S)$ is a monomorphism, so the homology exact sequence of a pair (S, \widehat{E}) gives that S' is \mathbb{Q} -acyclic. The exceptional divisors \widehat{E} are resolutions of singular points of type A_1 and A_2 respectively, so the constructed S' 's are normal. We check easily that in both cases $K_{\overline{S}} + D^\#$ intersects trivially with all components of $D + \widehat{E}$, hence $K_{\overline{S}} + D^\# \equiv 0$ by 3.1(i). Then $\overline{\kappa}(S) = \overline{\kappa}(S_0) = 0$ by 2.7.

Suppose that the smooth locus S_0 admits a \mathbb{C}^* -ruling. There exists a modification $(\widetilde{S}, \widetilde{D} + \widetilde{E}) \rightarrow (\overline{S}, D + \widehat{E})$ over $D + \widehat{E}$, such that this ruling extends to a \mathbb{P}^1 -ruling $\pi : \widetilde{S} \rightarrow \mathbb{P}^1$. We can assume that $\widetilde{D} + \widetilde{E}$ is π -minimal. We have $\overline{\kappa}(S') \neq -\infty$, so there are no sections contained in \widetilde{E} , hence $\widetilde{E} = \widehat{E}$. The divisor D does not contain components with non-negative self-intersection, which implies that this property holds for \widetilde{D} too. Suppose $\#\widetilde{D}_h = 1$. We have $\nu = 1$ by 3.1(iv), so there exists a fiber $F_\infty \subseteq \widetilde{D}$. Since \widetilde{D} is simply connected, F_∞ can intersect D_h only in a branching point of $\pi|_{\widetilde{D}_h}$, hence by π -minimality $F_\infty = [2, 1, 2]$. The contractions minimalizing \widetilde{D} cannot contract components of F_∞ , hence D contains two (-2) -tips as maximal twigs, a contradiction. Therefore we can write $\widetilde{D}_h = D_0 + D_\infty$ and we have $\Sigma = \nu \leq 1$ by 3.1(iv). If $\nu > 0$ then $D_0 + D_\infty$ intersects the fiber contained in \widetilde{D} in two different points, so this fiber is smooth by the π -minimality of \widetilde{D} , which contradicts the fact that all components of \widetilde{D} have negative self-intersection. Thus $\Sigma = \nu = 0$. Now $\overline{\kappa}(S_0) = 0$ implies that $F \cdot (K_{\overline{S}} + \widetilde{D} + \widehat{E})^- = F \cdot (K_{\widetilde{S}} + \widetilde{D} + \widehat{E}) = 0$, so D_0 and D_∞ are not contained in maximal twigs of \widetilde{D} , because are not contained in $\text{Supp}(K_{\widetilde{S}} + \widetilde{D} + \widehat{E})^-$. The divisor \widetilde{D} is simply connected, so there exists a unique fiber F_0 , such that $F_0 \cap \widetilde{D}$ is connected. By the π -minimality of \widetilde{D} other singular fibers are chains intersected by D_0 and D_∞ in tips. It follows that there are at least two such fibers, otherwise D_0 and D_∞ would be contained in maximal twigs of \widetilde{D} . This implies that D_0 and D_∞ are branching in \widetilde{D} and since exceptional components of \widetilde{D} can appear only in F_0 , after the snc-minimalization of \widetilde{D} the images of D_0 and D_∞ are branching in D , a contradiction. \square

COROLLARY 5.6. — $\text{Aut } SY\{3, 3, 3\} \cong \mathbb{Z}_3$ and $\text{Aut } SY\{2, 4, 4\} \cong \mathbb{Z}_2$.

Proof. — Let η be an automorphism of a surface $S' = SY\{3, 3, 3\}$ or $SY\{2, 4, 4\}$. Since $D + \widehat{E}$ does not contain curves with non-negative self-intersection, $\eta|_{S_0}$ extends to $\bar{\eta} \in \text{Aut}(\bar{S}, D + \widehat{E})$.

Suppose $S' = SY\{3, 3, 3\}$. We proved that S' can be constructed as in 5.4, so we can assume that $\theta : \bar{S} \rightarrow \mathbb{P}^2$ maps B to Q_1 and M to Q_2 and maps the set of nodes of $D - B$ to the fixed set of three points $\{P_1, P_2, P_3\} \subseteq \mathbb{P}^2$ (we showed in the proof of the main theorem that Q_1, Q_2, P_1, P_2 can be fixed arbitrarily and then up to an automorphism of \mathbb{P}^2 fixing Q_1, Q_2 and $\{P_1, P_2\}$ there is only one choice for P_3). Notice that $\bar{\eta}$ fixes B and M and acts on $\{L_1, L'_1, L_2, L'_2, L_3, L'_3\}$, hence descends to $\tilde{\eta} \in \text{Aut } \mathbb{P}^2 = \theta(\bar{S})$ fixing Q_1, Q_2 and $\{P_1, P_2, P_3\}$. The automorphism of \mathbb{P}^2 is defined uniquely by specifying the images of four points in a general position, so $\text{Aut } S' < S_3$. However, σ defined in the proof of 1.4, which fixes Q_1, Q_2 and exchanges P_1 with P_2 , does not fix P_3 , hence $\text{Aut } S' < \mathbb{Z}_3$. We conclude that $\text{Aut } S' \cong \mathbb{Z}_3$ with the generator in the coordinates as before given by $(x, y, z) \rightarrow (x - y, -\epsilon y, -\epsilon y + z)$, where $\epsilon = -\zeta$ for some primitive third root of unity ζ .

Suppose $S' = SY\{2, 4, 4\}$. We proved that S' can be constructed as in 5.2. Since $\bar{\eta}$ permutes M with M' and $T_{2,i}$ with $T_{3,i}$ for $i = 1, 2, 3$, by the definition of the contraction $\theta : \bar{S} \rightarrow \mathbb{P}^2$ it descends to $\tilde{\eta} \in \text{Aut } \mathbb{P}^2$ fixing $P_1, \{P_2, P_3\}$ and $\{T_{2,3}, T_{3,3}\}$. Notice that if $\tilde{\eta}(T_{2,3}) = T_{2,3}$ then, since $\bar{\eta}$ fixes \widehat{E} and \widehat{E} is tangent to $T_{2,3}$ only at P_2 , $\bar{\eta}$ fixes each P_i , hence is an identity. It follows that if $\bar{\eta}$ is non-trivial then $\tilde{\eta}(T_{2,3}) = T_{3,3}$. Moreover, $\text{Aut } S' < \mathbb{Z}_2$. In fact $\text{Aut } S' \cong \mathbb{Z}_2$, with the generator (for conics and points as in 5.2) given by $(x, y, z) \rightarrow (x, -y, z)$. \square

Remark 5.7. — Let M_D and M be the 3-dimensional manifolds, which are boundaries of closures of tubular neighborhoods of D and \widehat{E} in S . By [19] we compute that $H_1(M_D, \mathbb{Z}) \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_2$, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}_2$ for $SY\{2, 4, 4\}$ and $H_1(M_D, \mathbb{Z}) \cong \mathbb{Z}_9 \oplus \mathbb{Z}_3$, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}_3$ for $SY\{3, 3, 3\}$. Having this it is not difficult to prove that $|H_1(SY\{2, 4, 4\}, \mathbb{Z})| = 4$ and $|H_1(SY\{3, 3, 3\}, \mathbb{Z})| = 3$.

In view of the results of [22] it is an interesting question if the contraction $\theta : \bar{S} \rightarrow \mathbb{P}^2$ can be chosen so that $\theta_*D + \theta_*\widehat{E}$ is a sum of lines. This is clearly so for $SY\{3, 3, 3\}$ (cf. 5.1) and is also possible for $SY\{2, 4, 4\}$. Let \bar{S} be an snc-minimal completion of a resolution of $SY\{2, 4, 4\}$. We denote the twigs of D as before, i.e. $T_1 = [2]$, $T_2 = [2, 2, 2]$, $T_3 = [2, 2, 2]$. Let $\pi' : \bar{S} \rightarrow \mathbb{P}^1$ be a \mathbb{P}^1 -ruling induced by a 0-curve $T_{2,3} + 2B + T_{3,3}$. Let L_1, L_2, M and M' be (-1) -curves on \bar{S} as defined in 5.1.

LEMMA 5.8. — *The ruling π' defined above has three singular fibers besides $F_\infty = T_{2,3} + 2B + T_{3,3}$ (see Fig. 5.7): $F_0 = U_2 + \widehat{E} + U_3$, $F_1 = U_1 + L_1$ and $F_2 = T_{2,1} + U_4 + T_{3,1}$, where $\widehat{E} = [2]$ and U_1, U_2, U_3, U_4 are (-1) -curves. We have $T_1 \cdot U_2 = T_1 \cdot U_3 = T_1 \cdot U_4 = 1$, $T_1 \cdot U_1 = 2$ and $T_{2,2} \cdot U_2 = T_{3,2} \cdot U_3 = 1$. The morphism $\theta' : \overline{S} \rightarrow \mathbb{P}^2$ contracting $B + M + M' + L_1 + U_1 + U_4 + U_2 + T_{3,2} + L_2$ maps $D + \widehat{E}$ into a set of lines. Namely, $\theta'(T_{2,3})$, $\theta'(T_1)$ and $\theta'(T_{3,3})$ are lines intersecting in $\theta'(B)$, $\theta'(T_{2,3})$, $\theta'(\widehat{E})$ and $\theta'(T_{3,1})$ are lines intersecting in $\theta'(M)$ and $\theta'(T_{2,1})$ is a line through $\theta'(T_{3,3}) \cap \theta'(E) = \theta'(M')$ and $\theta'(T_{3,1}) \cap \theta'(T_1) = \theta'(U_4)$ (see Fig. 5.8).*

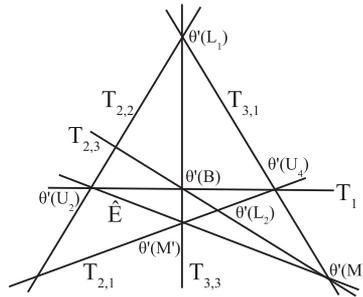
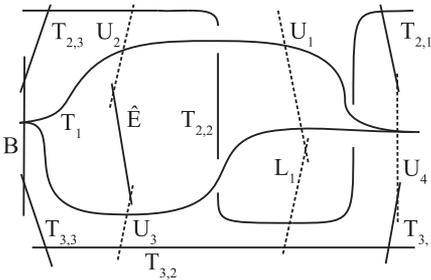


Figure 5.7. $SY\{2, 4, 4\}$, ruling π' Figure 5.8. $SY\{2, 4, 4\}$, image of θ'

Proof. — In the proof of 4.4 we have shown that $K_{\overline{S}} + D^\# \equiv 0$. Let U be an S_0 -component of some singular fiber of π' . Since $U \cdot D^\# > 0$, we have $U \cdot K_{\overline{S}} < 0$, so U is a (-1) -curve. Then $U \cdot D^\# = 1$, so computing $\text{Bk } D$ we get $2U \cdot D_h + U \cdot (T_{2,1} + T_{3,1}) = 4$. Let F_2 be a fiber containing $T_{2,1}$ and let U_4 be the S_0 -component intersecting it. Then, since $U_4 \cdot (T_{2,1} + T_{3,1})$ is even and since F_2 is a tree, we get $U_4 \cdot T_{3,1} > 0$, so in fact $U_4 \cdot T_{2,1} = U_4 \cdot T_{3,1} = 1$. Moreover, $\underline{E}_2 = T_{2,1} + U_4 + T_{3,1}$ and $U_4 \cdot D_h = 2$, which implies that U_4 , having multiplicity 2, intersects the 2-section T_1 . It follows that all remaining S_0 -components U have $U \cdot D_h = 2$. Since $L_1 \cdot \widehat{E} = 0$ (cf. Fig. 5.1) and the fiber F_1 containing L_1 has no D -components, L_1 intersects some S_0 -component U_1 , so $F_1 = L_1 + U_1$. We have $L_1 \cdot T_{3,2} = L_1 \cdot T_{2,2} = 1$, so $U_1 \cdot T_1 = 2$. The fiber F_0 containing \widehat{E} has no D -components, so $F_0 = U_1 + \widehat{E} + U_2$ for some (-1) -curves U_1, U_2 . By 3.1(iv) $\Sigma_{S_0} = 2$ for π' , so there are no more singular fibers. Recall that $U_2 \cdot D_h = U_3 \cdot D_h = 2$. It follows that each of U_2 and U_3 intersects some 1-section contained in D , because if, say, $T_1 \cdot U_2 = 0$ then the contraction of $\underline{F}_\infty - T_{3,3} + F_0 - U_2 + U_1 + \underline{E}_2 - T_{3,1}$ does not touch $T_{3,2}$ and touches $T_{2,2}$ twice, which would result with disjoint 0- and

(-2)-curves as sections on a Hirzebruch surface. One can easily check that the divisors $B + L_1 + U_4 + U_2 + T_{3,2}$ and $M + M' + L_2$ do not intersect, which implies that the contraction of $G = B + L_1 + U_4 + U_2 + T_{3,2} + M + M' + L_2$ defines a morphism $\theta' : \bar{S} \rightarrow \mathbb{P}^2$. Each component of $D + \widehat{E}$ not contained in G has the intersection number with G equal to three, hence each maps to a line in \mathbb{P}^2 and the configuration of lines can be checked to be the one on the picture 5.8. In particular, taking out any of the lines $\theta'(T_{2,1})$, $\theta'(T_{2,2})$ or $\theta'(T_{2,3})$ we get *complete quadrangle* configurations $(4_3, 6_2)$. \square

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Karol PALKA
University of Warsaw
Institute of Mathematics
ul. Banacha 2
02-097 Warsaw (Poland)
Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-956 Warsaw (Poland)
palka@impan.pl