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ACCELERO-SUMMATION OF THE FORMAL SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

by Geertrui Klara IMMINK

Abstract. — In 1996, Braaksma and Faber established the multi-summability, on suitable multi-intervals, of formal power series solutions of locally analytic, nonlinear difference equations, in the absence of “level 1^+”. Combining their approach, which is based on the study of corresponding convolution equations, with recent results on the existence of flat (quasi-function) solutions in a particular type of domains, we prove that, under very general conditions, the formal solution is *accelero-summable*. Its sum is an analytic solution of the equation, represented asymptotically by the formal solution in a certain unbounded domain.

Résumé. — En 1996, Braaksma et Faber ont établi la multi-sommabilité, sur des multi-intervalles convenables, des solutions formelles d’équations aux différences nonlinéaires, localement analytiques, sous la condition que le niveau 1^+ ne se présente pas. En combinant leurs résultats avec d’autres récents pour le cas des deux niveaux 1 et 1^+, on démontre, pour une classe très générale d’équations, l’accéléro-sommabilité de la solution formelle. L’accéléro-somme est solution analytique de l’équation, admettant la solution formelle comme développement asymptotique à l’infini.

1. Introduction

We consider nonlinear difference equations of the form

$$y(z+1) = z^{\lambda/p} F(z^{1/p}, y(z)) \tag{1.1}$$

where $p \in \mathbb{N}$ (the set of positive integers), $\lambda \in \mathbb{Z}$ and F is a $\mathbb{C}^n$-valued function, analytic in a neighbourhood of (∞, y_0) for some $y_0 \in \mathbb{C}^n$. We assume that the equation possesses a formal power series solution of the

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form $\hat{f}(z)=\sum_{m=0}^{\infty} a_m z^{-m/p}$ where $a_m \in \mathbb{C}^n$ for all $m \in \mathbb{N}$, $a_0 = y_0$, and, furthermore (identifying D_2F with its Taylor series at (∞, y_0)), that

$$\widehat{A} := D_2F(z^{1/p}, \hat{f}(z)) \in Gl(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}]) \tag{1.2}$$

It is the purpose of this paper to lift the, usually divergent, formal solution $\hat{f}$ to actual, analytic solutions, represented asymptotically by $\hat{f}$ in certain unbounded domains of the complex plane and characterized by their asymptotic properties in some way. To this end we use the powerful tool of *accelero-summation* developed by Jean Ecalle.

In [3] Braaksma and Faber prove that, under some additional conditions and on appropriate multi-intervals, $\hat{f}$ is *multi-summable* and its multi-sum satisfies the equation (1.1). In particular, they assume that the *levels* of a linear difference operator Δ, associated with (1.1), are $\leqslant 1$ (in general, the levels of a difference operator are nonnegative rational numbers $\leqslant 1$, or the so-called level 1^+). Their approach is based on the study of corresponding convolution equations, one for each positive level k_j of Δ, obtained by applying a Borel transformation of order k_j to the original equation.

The present paper is concerned with the case that Δ possesses an additional level 1^+. In that case, formal power series solutions are generally not multi-summable on any multi-interval. Combining some of the results and techniques from [3] with theorems on the existence of flat (quasi-function) solutions of nonlinear difference equations in [12, 14], we establish the *accelero-summability* of $\hat{f}$ on appropriate multi-intervals. We restrict ourselves to domains of the complex plane that are invariant under a "forward shift" $z \mapsto z+1$. However, if condition (1.2) is satisfied, analogous results can be proved for domains that are invariant under a "backward shift" $z \mapsto z-1$.

The paper is arranged as follows. In section 2 we introduce notations and general definitions and recall a number of basic results. Furthermore, we present two simple examples of nonlinear difference equations with three distinct levels, including the level 1^+.

Section 3 deals with the relatively simple case that 0 is not a singular direction of level 1, or, equivalently, $-\frac{\pi}{2}$ is not a Stokes direction of level 1. In that case, the formal solution is shown to be accelero-summable in the sense of [4], where the corresponding result for *linear* difference equations is proved. In this section we closely follow the method used in [3], except for the very last step in the summation procedure. The main result of this section is Theorem 3.8.

In section 4 we introduce a somewhat weaker notion of accelero-summability and prove that, according to this new definition, the formal solution of (1.1) is accelero-summable, even if 0 is a singular direction of level 1. The main result of this paper is stated in § 4.2, Theorem 4.12.

2. Preliminaries

2.1. Levels and Stokes directions

We use the symbol τ to denote both the automorphism of $\mathbb{C}[[z^{-1/p}]][z^{1/p}]$ defined by $\tau(z^{1/p}) = z^{1/p}\sum_{h=0}^{\infty}\binom{1/p}{h}z^{-h}$ and the shift operator $\tau y(z) := y(z{+}1)$. Two *formal* difference operators $\widehat{\Delta}_1{:=}\widehat{B}_1\tau{-}\widehat{A}_1$ and $\widehat{\Delta}_2 := \widehat{B}_2\tau{-}\widehat{A}_2$, where $\widehat{A}_i$ and $\widehat{B}_i \in Gl(n;\mathbb{C}[[z^{-1/p}]][z^{1/p}])$, for $i = 1, 2$, will be called *equivalent* if there exists $\widehat{F} \in Gl(n;\mathbb{C}[[z^{-1/p}]][z^{1/p}])$ such that $(\tau\widehat{F})^{-1}\widehat{B}_1^{-1}\widehat{\Delta}_1\widehat{F} = \widehat{B}_2^{-1}\widehat{\Delta}_2$, or equivalently, $(\tau\widehat{F})^{-1}\widehat{B}_1^{-1}\widehat{A}_1\widehat{F} = \widehat{B}_2^{-1}\widehat{A}_2$. Any formal difference operator $\widehat{\Delta} := \widehat{B}\tau{-}\widehat{A}$, $\widehat{A}$ and $\widehat{B} \in Gl(n;\mathbb{C}[[z^{-1/p}]][z^{1/p}])$, is known to be equivalent to a *canonical* operator Δ^c:

$$(\tau\widehat{F})^{-1}\widehat{B}^{-1}\widehat{\Delta}\widehat{F} = \Delta^c = \tau - A^c \tag{2.1}$$

where $A^c \in Gl(n;\mathbb{C}\{z^{-1/p}\}[z^{1/p}])$(1) and A^c is a block-diagonal matrix of a particularly simple form (*cf.* [17, 8]). If the convergent series A^c is identified with its sum, the canonical operator Δ^c can be viewed as an *analytic* difference operator, and the homogeneous equation $\Delta^c y = 0$ has a fundamental system of analytic solutions $\{y_j : j = 1,\dots,n\}$ of the following form

$$y_j^c(z) = z^{d_j z}e^{q_j(z)}z^{\lambda_j}g_j(z),\ j = 1,\dots,n, \tag{2.2}$$

where $d_j \in \mathbb{Q}$, $q_j(z)$ is a polynomial in $z^{1/p}$ of degree at most p and with vanishing constant term, $\lambda_j \in \mathbb{C}$ and $g_j \in \mathbb{C}^n[\log z]$, $j = 1,\dots,n$. If $d_j = 0$, we denote the leading term of q_j by $\mu_j z^{\kappa_j}$. If $d_j \neq 0$ we define $\kappa_j = 1$ and denote the term of order 1 by $\mu_j z$. When $\kappa_j = 1$, the number μ_j is determined up to a multiple of $2\pi i$. The numbers κ_j with $j \in \{1,\dots,n\}$ such that $d_j = 0$, are called *levels* of $\widehat{\Delta}$. If there is a $j \in \{1,\dots,n\}$ such that $d_j \neq 0$, $\widehat{\Delta}$ is said to possess a *level* 1^+.

Remark 2.1.

(i) If we replace $\widehat{F}$ in (2.1) by $z^{-\mu}\widehat{F}$, the right-hand side of (2.1) continues to be a canonical difference operator, but with λ_j in (2.2) replaced by $\lambda_j{+}\mu$

(1) In fact, it may be necessary to increase p to some multiple pq with $q \in \mathbb{N}$, but here we assume that p has been chosen sufficiently large at the outset.

for $j = 1, \dots, n$. Thus, the numbers λ_j are determined by $\widehat{\Delta}$ up to a multiple of $1/p$.

(ii) If $\widehat{\Delta}$ has a level < 1, then it is considered to have a level 1 as well, even if there is no $j \in \{1, \dots, n\}$ with the property that $d_j = 0$ and $\kappa_j = 1$. This is related to the fact that $y_j^c(z)e^{2l\pi iz}$ is a solution of the homogeneous equation $\Delta^c y(z) = 0$ for any $l \in \mathbb{Z}$, with exponential growth or decay of order 1 at ∞ if $\kappa_j < 1$ and $l \neq 0$.

Obviously, equivalent formal difference operators have the same canonical forms.

Definition 2.2 (Stokes directions). — *Let $0 < \kappa < 1$ be one of the positive levels of $\widehat{\Delta}$. We shall call singular directions of $\widehat{\Delta}$, of level κ, the directions $\frac{1}{\kappa}(\pi - \arg \mu_j)$, where $j \in \{1, \dots, n\}$ is such that $\kappa_j = \kappa$ and $\arg \mu_j$ is determined up to a multiple of 2π. The singular directions of level 1 are the directions $\pi - \arg(\mu_j + 2l\pi i)$, where $l \in \mathbb{Z}$, $j \in \{1, \dots, n\}$ such that $d_j = 0$ and $\kappa_j = 1$, and the directions $\frac{\pi}{2} \bmod \pi$ if there is a $j \in \{1, \dots, n\}$ such that $d_j = 0$ and $\kappa_j < 1$. If α is a singular direction of level $\kappa \in (0, 1]$, $\alpha - \pi/(2\kappa)$ is called a Stokes direction and the pair $\{\alpha - \pi/(2\kappa), \alpha + \pi/(2\kappa)\}$ a Stokes pair of $\widehat{\Delta}$, of level κ.*
The numbers $\theta \in \mathbb{R}$ such that $d_j \neq 0$ and $d_j\theta = \operatorname{Im} \mu_j$ for some value of $\operatorname{Im} \mu_j$ (determined up to a multiple of 2π) will be called pseudo-Stokes directions of $\widehat{\Delta}$, of level 1^+. The set of all pseudo-Stokes directions of level 1^+ is denoted by $\Theta(\widehat{\Delta})$.

Remark 2.3.
(i) Note that, for $j \in \{1, \dots, n\}$ such that $d_j = 0$ and $\kappa_j > 0$, y_j^c decreases exponentially of order κ_j as $z \to \infty$ in a sector of the form: $\frac{1}{\kappa_j}(\frac{\pi}{2} - \arg \mu_j) < \arg z < \frac{1}{\kappa_j}(\frac{\pi}{2} - \arg \mu_j + \pi)$ (bounded by a Stokes pair), uniformly on closed subsectors. The singular directions of $\widehat{\Delta}$ are the directions of maximal decrease at ∞, of $y_j^c(z)e^{2l\pi iz}$, where $j \in \{1, .., n\}$ is such that $d_j = 0$ and $l \in \mathbb{Z}$, $l \neq 0$ if $\kappa_j = 0$. (In some texts, *cf.* for instance [14], the singular directions have the opposite sign.)

(ii) If $\widehat{\Delta}$ has levels other than 1^+, $\frac{\pi}{2} \bmod \pi$ is either a singular direction or an accumulation point of singular directions of level 1.

With equation (1.1) we associate the formal difference operator $\widehat{\Delta} = \tau - \widehat{A}$, where $\widehat{A}$ is defined by (1.2). We shall sometimes refer to the levels and Stokes directions of this operator as the levels and Stokes directions of the equation.

2.2. Asymptotic behaviour on sectors and multi-summability

In this section, we define classes of analytic functions with different types of asymptotic behaviour on *sectors* of the Riemann surface of the logarithm, to be denoted by $\widetilde{\mathbb{C}^*}$. We recall the definitions of multi-summability and k-precise quasi-function.

DEFINITION 2.4. — *Let I be an interval of $\mathbb{R}$. By $|I|$ we denote the length of I, defined by $|I| := \sup I - \inf I$. We use the following notations*

$$S(I) = \{z \in \widetilde{\mathbb{C}^*} \colon \arg z \in I\}, \quad S(I,R) = \{z \in S(I) \colon |z| > R\}$$

We write $I' \prec I$ if I' is a relatively compact subinterval of I (i.e. I' is bounded and $\bar{I}' \subset I$).

By $\mathcal{A}_0^{\leqslant 0}(I)$ we denote the set of functions $f \colon S(I) \to \mathbb{C}$, with the property that, for every $I' \prec I$, there exists a positive number r such that f is holomorphic and bounded on $\{z \in S(I') \colon |z| < r\}$. (More precisely, we consider equivalence classes of functions: two such functions are identified if, for every $I' \prec I$, they coincide on $\{z \in S(I') \colon |z| < r'\}$ for some $r' > 0$.)

By $\mathcal{A}_0(I)$ we denote the set of functions $f \in \mathcal{A}_0^{\leqslant 0}(I)$ admitting an asymptotic power series expansion of the form $\sum_{m=0}^{\infty} a_m z^{m/p}$, with $p \in \mathbb{N}$, such that

$$\sup_{z \in S(I') \colon |z| < r} |z^{-N/p}(f(z) - \sum_{m=0}^{N-1} a_m z^{m/p})| < \infty$$

for any $I' \prec I$ and some sufficiently small, positive r (depending on I').

By $\mathcal{A}^{\leqslant 0}(I)$ we denote the set of (equivalence classes of) functions $f : S(I) \to \mathbb{C}$, with the property that, for every $I' \prec I$, there exists a positive number R such that f is holomorphic and bounded on $S(I', R)$.
By $\mathcal{A}(I)$ we denote the set of functions $f \in \mathcal{A}^{\leqslant 0}(I)$ admitting an asymptotic expansion $\hat{f}(z) = \sum_{m=0}^{\infty} a_m z^{-m/p}$, with $p \in \mathbb{N}$, such that

$$\sup_{z \in S(I',R)} |z^{N/p}(f(z) - \sum_{m=0}^{N-1} a_m z^{-m/p})| < \infty$$

for any $I' \prec I$ and some sufficiently large R (depending on I'). For all $N \in \mathbb{N}$, $R_N(f; z)$ will denote the remainder: $f(z) - \sum_{m=0}^{N-1} a_m z^{-m/p}$.
By $\mathcal{A}^{\leqslant -k}(I)$ we denote the set of $f \in \mathcal{A}(I)$ with the property that, for any $I' \prec I$, there exist positive constants R and c such that

$$\sup_{z \in S(I',R)} e^{c|z|^k} |f(z)| < \infty$$

(so $\hat{f} = 0$).

By $\mathcal{A}^{\leqslant -1^+}(I)$ *we denote the set of* $f \in \mathcal{A}(I)$ *with the property that, for any* $I' \prec I$*, there exist positive constants* R *and* c *such that*

$$\sup_{z \in S(I',R)} e^{c|z|\log|z|}|f(z)| < \infty$$

$\mathcal{A}_0^{\leqslant 0}$ and $\mathcal{A}^{\leqslant 0}$ are sheaves on $\mathbb{R}$. $\mathcal{A}^{\leqslant -k}$ is a subsheaf of $\mathcal{A}$ for every $k > 0$.

Definition 2.5 (multi-summability). — *Let* $I_0 = \mathbb{R}$ *and* $I_h,\ h = 1,\ldots,q$*, be open intervals of* $\mathbb{R}$ *such that*

- $I_q \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$ *for* $h = 1,\ldots,q$.

$\hat{f} \in \mathbb{C}[[z^{-1/p}]]$ *is multi-summable on the multi-interval* $(I_1,\ldots,I_q)$*, with multi-sum* $f_q \in \mathcal{A}(I_q)$ *if there exist* $f_h \in (\mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}})(I_h)$*,* $h = 0,\ldots,q-1$ *with asymptotic expansion* $\hat{f}$*, satisfying the following conditions:*

- $f_0(ze^{2p\pi i}) = f_0(z)$,
- $f_{h-1}|_{I_h} = f_h \bmod \mathcal{A}^{\leqslant -k_h},\quad h = 1,\ldots,q.$

Any element of $\mathcal{A}^{\leqslant 0}/\mathcal{A}^{\leqslant -k}(I)$ can be represented by a collection of functions $\{\phi_\nu\colon \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}^{\leqslant 0}(\mathcal{I}_\nu)$, $\{\mathcal{I}_\nu\colon \nu \in \mathcal{N}\}$ is an open covering of I and $\phi_\nu - \phi_\mu \in \mathcal{A}^{\leqslant -k}(\mathcal{I}_\nu \cap \mathcal{I}_\mu)$ for all μ and $\nu \in \mathcal{N}$. $\{\phi_\nu\colon \nu \in \mathcal{N}\}$ is called a *k-precise quasi-function* (*cf.* [18]).

Suppose that the formal difference operator $\widehat{\Delta}$ associated with (1.1) has the positive levels $0 < k_1 < \cdots < k_q = 1$. With the formal power series solution $\hat{f}$ of (1.1) one can associate a unique global section f_0 of $(\mathcal{A}/\mathcal{A}^{\leqslant -k_1})^n$ with the property that $f_0(ze^{2p\pi i}) = f_0(z)$ (this is a consequence of the Gevrey order of $\hat{f}$: $\hat{f} \in \mathbb{C}[[z^{-1/p}]]^n_{pk_1}$, *cf.* [18, 12]). f_0 can be represented by a k_1-precise quasi-function $\{\phi_\nu\colon \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}(\mathcal{I}_\nu)$, $\{\mathcal{I}_\nu\colon \nu \in \mathcal{N}\}$ is an open covering of $\mathbb{R}$, $\phi_\nu - \phi_\mu \in \mathcal{A}^{\leqslant -k}(\mathcal{I}_\nu \cap \mathcal{I}_\mu)$ for all μ and $\nu \in \mathcal{N}$ and ϕ_ν is represented asymptotically by $\hat{f}$ as $z \to \infty$, $\arg z \in \mathcal{I}_\nu$, for all $\nu \in \mathcal{N}$. Let $I_0 = \mathbb{R}$ and I_h, $h = 1,\ldots,q$, be open intervals of $\mathbb{R}$ with the following properties:

- $(-\frac{\pi}{2},\frac{\pi}{2}) \subset I_q \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$.
- I_h does not contain a Stokes pair of level k_h.

In [3] it is proved that, under these conditions, (1.1) has solutions $f_h \in (\mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}})^n(I_h)$, $h = 1,\ldots,q-1$ with the property that

$$f_{h-1}|_{I_h} = f_h \bmod (\mathcal{A}^{\leqslant -k_h})^n$$

Moreover, if $\widehat{\Delta}$ doesn't possess a level 1^+, then (1.1) has a solution $f_q \in \mathcal{A}(I_q)^n$ with the property that $f_{q-1}|_{I_q} = f_q \bmod (\mathcal{A}^{\leqslant -1})^n$. This implies that $\hat{f}$ is *multi-summable* on the multi-interval $(I_1,\dots,I_q)$, with multi-sum f_q.

2.3. A particular type of domains

In the case of difference equations without level 1^+, the study of the asymptotic behaviour of solutions on sectors suffices. This is no longer true for difference equations possessing a level 1^+, due to the complicated asymptotic behaviour of y_j^c if $d_j \neq 0$ (*cf.* (2.2)). For example, for all $j \in \{1,\dots,n\}$ such that $d_j < 0$, $y_j^c \in (\mathcal{A}^{\leqslant -1^+}(-\frac{\pi}{2},\frac{\pi}{2}))^n$ and $y_j^c(z)$ increases supra-exponentially as $z \to \infty$ in any direction $\in (\frac{\pi}{2},\frac{3\pi}{2}) \bmod 2\pi$, regardless of the value of μ_j. As μ_j is determined up to a multiple of $2\pi i$, this implies that, in some sense, $\{-\frac{\pi}{2},\frac{\pi}{2}\}$ may be viewed as a Stokes pair of level 1^+, of "infinite multiplicity". However, looking more carefully into the asymptotic behaviour of y_j^c and noting that

$$|e^{d_j z\log z+\mu_j z}| = e^{d_j(\mathrm{Re}(z(\log z+i\frac{\mathrm{Im}\mu_j}{d_j}))}e^{\mathrm{Re}\mu_j\mathrm{Re}z}$$

and

$$\mathrm{Re}\Big(z\Big(\log z+i\frac{\mathrm{Im}\,\mu_j}{d_j}\Big)\Big) = \mathrm{Re}(z(\log z+i\theta)) - \Big(\frac{\mathrm{Im}\,\mu_j}{d_j}-\theta\Big)\,\mathrm{Im}\,z$$

where θ is a real number, we find that y_j^c decreases exponentially of order 1 as $\mathrm{Im}\,z \to \infty$ on any curve of the form $\mathrm{Re}(z(\log z+i\theta)) = c$ with $c \in \mathbb{R}$ and $\theta > \frac{\mathrm{Im}\,\mu_j}{d_j}$ and increases exponentially if $\theta < \frac{\mathrm{Im}\,\mu_j}{d_j}$. Similarly, y_j^c decreases exponentially of order 1 as $\mathrm{Im}\,z \to -\infty$ on a curve of this form if $\theta < \frac{\mathrm{Im}\,\mu_j}{d_j}$ and increases exponentially if $\theta > \frac{\mathrm{Im}\,\mu_j}{d_j}$. This is why, in order to *characterize* solutions of (1.1) by their asymptotic properties in the presence of a level 1^+, we need to consider domains bounded by curves of the type $\mathrm{Re}(z(\log z+i\theta)) = c$.

Definition 2.6. — *By S_+ we denote the sector $S(-\pi,\pi)$. Let $\theta \in \mathbb{R}$, $z \in S_+$ and*

$$\psi_\theta(z) := z(\log z+i\theta).$$

By $C_\theta(z)$ we denote the level set of $\mathrm{Re}\,\psi_\theta$ containing z:

$$C_\theta(z) = \big\{\zeta \in S_+\colon\ \mathrm{Re}\,\psi_\theta(\zeta) = \mathrm{Re}\,\psi_\theta(z)\big\}$$

$C_\theta^+(z)$ and $C_\theta^-(z)$ are defined by

$$C_\theta^\pm(z) = \big\{\zeta \in C_\theta(z)\colon\ \pm\,\mathrm{Im}(\zeta - z) > 0\big\}.$$

In previous papers we introduced two types of domains, denoted by $D_I(z)$ and $\widetilde{D}_I(z)$ and arising rather naturally in the study of difference equations possessing a level 1^+ (*cf.* [11], [12], [14] and the appendix below). $D_I(z)$ is an *intersection* of domains of the form $\{\zeta \in S_+ \colon \operatorname{Re}\psi_\theta(\zeta) \geqslant \operatorname{Re}\psi_\theta(z)\}$, whereas $\widetilde{D}_I(z)$ is a *union* of such domains. In problems involving the level 1^+, these domains play a role similar to that of sectors of aperture $\leqslant \pi$ and $\geqslant \pi$, respectively, in problems of level 1. For notational convenience, we combine both types of domain into one, more general type of domain, somewhat similarly to the example of sectors. Roughly speaking, we 'label' $C_\theta^-(z)$ by a negative number and $C_\theta^+(z)$ by a positive number and define a domain $\widehat{D}_{(a,b)}(R)$, bounded by two such 'rays', with 'labels' a and b, respectively, and part of the 'circle' $|z| = R$. By providing a unified notation for the domains to be considered, this somewhat artificial construction considerably simplifies the study of asymptotic behaviour on these domains. In particular, it allows us to identify a class of "1^+-precise" quasi-functions with the representatives of sections of a (quotient-) sheaf on $\mathbb{R}$.

Let $\phi_- \colon \mathbb{R} \to (-\infty, 0)$ and $\phi_+ \colon \mathbb{R} \to (0, \infty)$ be continuous, monotone decreasing and onto. By $\vartheta \colon \mathbb{R}^* \to \mathbb{R}$ we denote the mapping defined by

$$\vartheta(a) = \begin{cases} \phi_-^{-1}(a) & \text{if } a < 0 \\ \phi_+^{-1}(a) & \text{if } a > 0 \end{cases}.$$

For example, one might choose

$$\phi_-(\theta) := -e^\theta \text{ and } \phi_+(\theta) := e^{-\theta}.$$

In that case,

$$\vartheta(a) = \begin{cases} \log(-a) & \text{if } a < 0 \\ -\log a & \text{if } a > 0 \end{cases}.$$

Definition 2.7 (domains). — *Let a, $b \in \mathbb{R}$, $a < b$, and $R > 1$. If $a < 0 < b$,*

$$\widehat{D}_{(a,0)}(R) = \{z \in S_+ \colon \arg z \leqslant 0, |z| \geqslant R,\ \operatorname{Re}\psi_{\vartheta(a)}(z) \geqslant 0\},$$

$$\widehat{D}_{(0,b)}(R) = \{z \in S_+ \colon \arg z \geqslant 0, |z| \geqslant R,\ \operatorname{Re}\psi_{\vartheta(b)}(z) \geqslant 0\}$$

and

$$\widehat{D}_{(a,b)}(R) = \widehat{D}_{(a,0)}(R) \cup \widehat{D}_{(0,b)}(R).$$

If $a < b < 0$,

$$\widehat{D}_{(a,b)}(R) = \{z \in S_+ \colon |z| \geqslant R,\ \operatorname{Re}\psi_{\vartheta(a)}(z) \geqslant 0 \text{ and } \operatorname{Re}\psi_{\vartheta(b)}(z) \leqslant 0\}$$

and if $0 < a < b$,

$$\widehat{D}_{(a,b)}(R) = \{z \in S_+ \colon |z| \geqslant R,\ \operatorname{Re}\psi_{\vartheta(a)}(z) \leqslant 0 \text{ and } \operatorname{Re}\psi_{\vartheta(b)}(z) \geqslant 0\}.$$

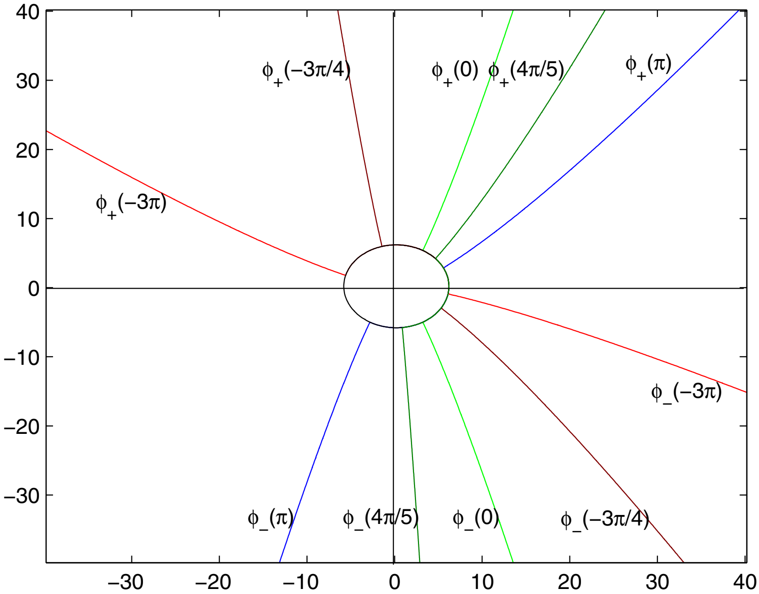

Figure 2.1. This figure shows the curves $\widehat{C}_a(6)$ for various values of a.

For any finite interval $(a,b) \subset \mathbb{R}$ such that $a \neq 0 \neq b$, $\widehat{D}_{(a,b)}(R)$ is bounded by an arc of the 'circle' $|z| = R$, and the level curves of $\operatorname{Re}\psi_{\vartheta(a)}$ and $\operatorname{Re}\psi_{\vartheta(b)}$ with height 0. In order to determine the points of intersection of these curves with the circle $|z| = R$, we note that $\operatorname{Re}\psi_\theta(z) = 0$ iff $\log|z| \cos\arg z = (\arg z + \theta)\sin\arg z$. For any $R > 1$ and $\theta \in \mathbb{R}$, the equation $G(\phi,\theta) := \log R\cos\phi - (\phi+\theta)\sin\phi = 0$ has unique solutions for ϕ in $(-\pi,0)$ and in $(0,\pi)$. Noting that, for all $(\phi,\theta) \in G^{-1}(0)$, $\frac{\partial G/\partial\theta}{\partial G/\partial\phi}(\phi,\theta) = \frac{\sin^2\phi}{\log R + \sin^2\phi}$, one easily verifies that both solutions are continuous, monotone decreasing functions of θ, mapping $\mathbb{R}$ onto $(-\pi,0)$ and $(0,\pi)$, respectively. Now, let $R > 1$ and $a \in \mathbb{R}$. For $a > 0$, let $\phi_R(a)$ denote the unique $\phi \in (0,\pi)$ such that

$$\log R\cos\phi = (\phi + \vartheta(a))\sin\phi \tag{2.3}$$

and, for $a < 0$, the unique $\phi \in (-\pi,0)$ such that (2.3) holds. Putting $\phi_R(0) := 0$, we obtain a continuous, monotone increasing mapping ϕ_R from $\mathbb{R}$ onto $(-\pi,\pi)$.

Definition 2.8. — *Let $R>1$. For all $a\in\mathbb{R}$ we define*

$$z_a(R) := \mathrm{Re}^{i\phi_R(a)},$$

$$\widehat{C}_a(R) := \left\{z\in S_+ \colon\ \operatorname{Re}\psi_{\vartheta(a)}(z)=0, \operatorname{Im}(z-z_a(R))\leqslant 0\right\} \text{ if } a<0,$$

$$\widehat{C}_a(R) := \left\{z\in S_+ \colon\ \operatorname{Re}\psi_{\vartheta(a)}(z)=0, \operatorname{Im}(z-z_a(R))\geqslant 0\right\} \text{ if } a>0,$$

and

$$\widehat{C}_0(R) := (R,\infty) \text{ (cf. Figure 2.1).}$$

If $a\neq 0$, $\operatorname{Re}\psi_{\vartheta(a)}(z_a(R))=0$. Obviously, $\widehat{C}_a(R)=C^-_{\vartheta(a)}(z_a(R))$ if $a<0$, $\widehat{C}_a(R)=C^+_{\vartheta(a)}(z_a(R))$ if $a>0$. From (2.3) we deduce that $\phi_R(a)=\pm\frac{\pi}{2}$ iff $\vartheta(a)=\mp\frac{\pi}{2}$, so $\widehat{C}_{\phi_-(\frac{\pi}{2})}(R)$ is the half line $\{z\in S_+\colon\ \arg z=-\frac{\pi}{2}, |z|\geqslant R\}$ and $\widehat{C}_{\phi_+(-\frac{\pi}{2})}(R)$ is the half line $\{z\in S_+\colon\ \arg z=\frac{\pi}{2}, |z|\geqslant R\}$.

Thus, for any finite interval $(a,b)\subset\mathbb{R}$, $\widehat{D}_{(a,b)}(R)$ is the closed domain in S_+ bounded by $\widehat{C}_a(R)$, $\widehat{C}_b(R)$ and the arc of the circle $|z|=R$ between $z_a(R)$ and $z_b(R)$. In particular, $\widehat{D}_{(\phi_-(\frac{\pi}{2}),0)}(R)=\overline{S((-\frac{\pi}{2},0),R)}$, $\widehat{D}_{(0,\phi_+(-\frac{\pi}{2}))}(R)=\overline{S((0,\frac{\pi}{2}),R)}$ and $\widehat{D}_{(\phi_-(\frac{\pi}{2}),\phi_+(-\frac{\pi}{2}))}(R)=\overline{S((-\frac{\pi}{2},\frac{\pi}{2}),R)}$. For every bounded interval $I=(a,b)$, we note

$$\theta_-(I) := \begin{cases}\infty & \text{if } a=0\\ \vartheta(a) & \text{otherwise}\end{cases},\quad \theta_+(I) := \begin{cases}-\infty & \text{if } b=0\\ \vartheta(b) & \text{otherwise}\end{cases}.$$

For every open interval $I\subset\mathbb{R}$ not containing 0, we define

$$\widetilde{I} := \vartheta(I)$$

and for every interval $I=(a,b)$ containing 0, such that $\vartheta(a)\neq\vartheta(b)$,

$$\widetilde{I} := \big(\min\{\vartheta(a),\vartheta(b)\}, \max\{\vartheta(a),\vartheta(b)\}\big).$$

We call $I=(a,b)$ a *large interval* if $0\in I$ and $\theta_+(I)<\theta_-(I)$. If I is a large interval, then $\widetilde{I}=(\vartheta(b),\vartheta(a))=(\theta_+(I),\theta_-(I))$. If $\theta_-(I)=\theta_+(I)=\theta$, then $I=(\phi_-(\theta),\phi_+(\theta))$. In this case, $\widetilde{I}$ is not defined. If $\theta_-(I)<\theta_+(I)$, then $\widetilde{I}=(\vartheta(a),\vartheta(b))=(\theta_-(I),\theta_+(I))$. Note that $\theta_-(I)<\theta_+(I)$ implies that $0\in I$. If I is a large interval, we have $a=\phi_-(\vartheta(a))<\phi_-(\vartheta(b))<0<\phi_+(\vartheta(a))<\phi_+(\vartheta(b))=b$. Hence $I':=(\phi_-(\vartheta(b)),\phi_+(\vartheta(a)))\prec I$ and $\widetilde{I'}=\widetilde{I}$. $\widehat{D}_I(R)$ and $\widehat{D}_{I'}(R)$ are bounded by different parts of the same level curves: $\operatorname{Re}\psi_{\vartheta(a)}(z)=0$ and $\operatorname{Re}\psi_{\vartheta(b)}(z)=0$ (*cf.* Figure 2.2). For each open interval (θ_1,θ_2) there exist two intervals I_1 and I_2 such that $0\in I_1\cap I_2$ and $\widetilde{I_1}=\widetilde{I_2}=(\theta_1,\theta_2)$: the large interval $I_1=(\phi_-(\theta_2),\phi_+(\theta_1))$ and the interval $I_2=(\phi_-(\theta_1),\phi_+(\theta_2))$.

We end this subsection with some properties of the domains $\widehat{D}_I(R)$ that will be needed later on. In this paper, we are mainly interested in the

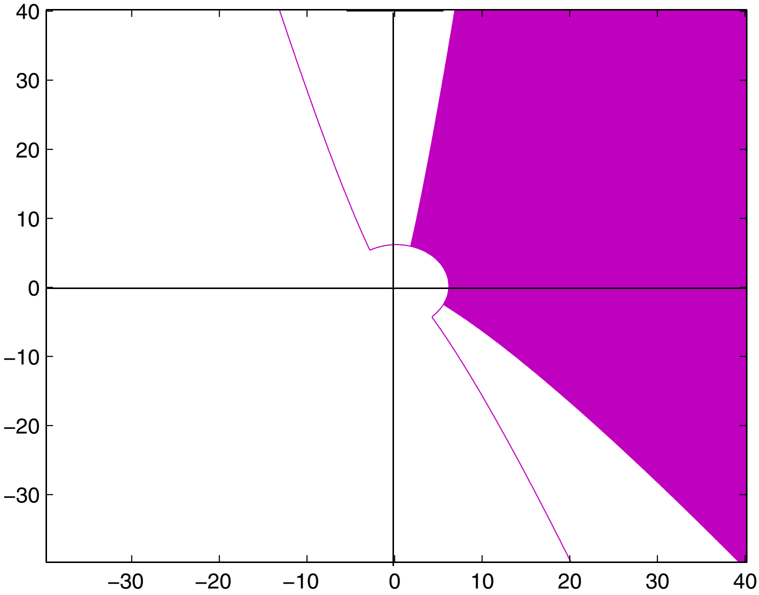

Figure 2.2. The dark region is $\widehat{D}_{(\phi_-(-\pi),\phi_+(-\frac{\pi}{4}))}(6)$*, the larger domain is* $\widehat{D}_{(\phi_-(-\frac{\pi}{4}),\phi_+(-\pi))}(6)$*. The interval* $(\phi_-(-\frac{\pi}{4}),\phi_+(-\pi))$ *is a large interval, whereas* $(\phi_-(-\pi),\phi_+(-\frac{\pi}{4}))$ *is not. In both cases,* $\widehat{D}_I(R)$ *is bounded by the level curves* $\operatorname{Re}\psi_{-\pi}(z)=0$ *and* $\operatorname{Re}\psi_{-\frac{\pi}{4}}(z)=0$*, and* $\widetilde{I}=(-\pi,-\frac{\pi}{4})$*.*

case that $0\in I$. From (2.3) it easily follows that $\arg z\to\pm\frac{\pi}{2}$ as $z\to\infty$ on $\widehat{C}_a(R)$ if $\pm a>0$. Hence, if I' is an open interval containing $[-\frac{\pi}{2},\frac{\pi}{2}]$, $\widehat{D}_I(R)\subset S(I')$ for any bounded, open interval I and all sufficiently large R. On the other hand, if $I'\prec(-\frac{\pi}{2},\frac{\pi}{2})$, I is an interval containing 0 and $R>1$, then $S(I',R')\subset\widehat{D}_I(R)$ for all sufficiently large R'.

Obviously, $I'\subset I$ implies $\widehat{D}_{I'}(R)\subset\widehat{D}_I(R)$. In particular, $I\subset(\phi_-(\frac{\pi}{2}),\phi_+(-\frac{\pi}{2}))$ implies that $\widehat{D}_I(R$ is contained in the right half plane $\overline{S((-\frac{\pi}{2},\frac{\pi}{2}),R)}$. It can be shown that, for any interval $I\prec(\phi_-(\frac{\pi}{2}),\phi_+(-\frac{\pi}{2}))$,

$$\operatorname{Re}z\geqslant\delta\frac{|z|}{\log|z|}\text{ for all }z\in\widehat{D}_I(R)\tag{2.4}$$

where $\delta>0$, provided R is sufficiently large (this is a particular case of (4.5) below).

Let I be an open interval such that $\theta_1:=\theta_-(I)<\theta_2:=\theta_+(I)$. Noting that, for $i\in\{1,2\}$, $\theta\in(\theta_1,\theta_2)$ and all $z\in\widehat{D}_I(R)$, $\operatorname{Re}\psi_\theta(z)=\operatorname{Re}\psi_{\theta_i}(z)+$

$(\theta_i - \theta)\operatorname{Im} z \geqslant (\theta_i - \theta)\operatorname{Im} z$, one easily verifies that

$$\operatorname{Re}\psi_\theta(z) \geqslant c|z| \text{ for all } z \in \widehat{D}_I(R), \tag{2.5}$$

where $c > 0$.

2.4. Sheaves of functions with particular asymptotic properties in the domains $\widehat{D}_I(R)$

We shall now introduce sheaves on $\mathbb{R}$ of functions with different types of asymptotic behaviour in the domains $\widehat{D}_I(R)$.

Definition 2.9. — *Let I be an interval of $\mathbb{R}$. By $\widehat{\mathcal{A}}(I)$ we denote the set of (equivalence classes of) continuous functions $f : S_+ \to \mathbb{C}$, holomorphic in int $\widehat{D}_{I'}(R)$ and admitting an asymptotic expansion $\hat{f} = \sum_{m=0}^{\infty} a_m z^{-m/p}$, with $p \in \mathbb{N}$, uniformly on $\widehat{D}_{I'}(R)$, for any open interval $I' \prec I$ and some sufficiently large R (depending on I'). By $\widehat{\mathcal{A}}^{\leqslant -1}(I)$ we denote the set of all $f \in \widehat{\mathcal{A}}(I)$ with the property that, for any open interval $I' \prec I$, there exist positive constants R and c such that*

$$\sup_{z \in \widehat{D}_{I'}(R)} e^{c\frac{|z|}{\log|z|}} |f(z)| < \infty.$$

By $\widehat{\mathcal{A}}^{\leqslant -1^+}(I)$ we denote the set of all $f \in \widehat{\mathcal{A}}(I)$ with the property that, for any open interval $I' \prec I$, there exist positive constants R and c such that

$$\sup_{z \in \widehat{D}_{I'}(R)} e^{c|z|}|f(z)| < \infty.$$

$\widehat{\mathcal{A}}$ is a sheaf on $\mathbb{R}$ and $\widehat{\mathcal{A}}^{\leqslant -1}$ and $\widehat{\mathcal{A}}^{\leqslant -1^+}$ are subsheaves. From the properties of the domains $\widehat{D}_I(R)$ discussed at the end of § 2.3 it follows, with a slight abuse of notation, that $\mathcal{A}([-\frac{\pi}{2}, \frac{\pi}{2}]) \subset \widehat{\mathcal{A}}(\mathbb{R})$ (strictly speaking, the elements of $\mathcal{A}([-\frac{\pi}{2}, \frac{\pi}{2}])$ should first be extended to continuous functions on S_+) and, for any open interval I containing 0, $\widehat{\mathcal{A}}(I) \subset \mathcal{A}(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, it can be shown that $\widehat{\mathcal{A}}^{\leqslant -1}(I) \subset \mathcal{A}^{\leqslant -1}(-\frac{\pi}{2}, \frac{\pi}{2})$ (*cf.* Lemma 4.9 and remark 4.10) and from Lemma 2.12 below it follows that $\widehat{\mathcal{A}}^{\leqslant -1^+}(I) \subset \mathcal{A}^{\leqslant -1^+}(-\frac{\pi}{2}, \frac{\pi}{2})$, for any open interval I containing 0.

Example 2.10.

(i) (2.4) implies that, for any $\mu < 0$, $e^{\mu z} \in \widehat{\mathcal{A}}^{\leqslant -1}(I)$ iff $I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$.

(ii) Let $d < 0$ and $\mu \in \mathbb{R}$. From (2.5) we deduce that $e^{dz\log z + \mu z} \in \widehat{\mathcal{A}}^{\leqslant -1+}(I)$ iff $I \subset (\phi_-(\theta), \phi_+(\theta))$ with $\theta = \operatorname{Im}\mu/d$.

In § 3, where we consider (finite) Laplace transforms of functions admitting an asymptotic power series expansion in $t^{1/p}$ at the origin, we shall need the following lemma.

LEMMA 2.11. — *Let $r > 0$ and let u be a continuous function on $(0, r)$, admitting an asymptotic power series expansion in $t^{1/p}$ as $t \to 0$. Then* $\int_0^r u(t)e^{-tz}dt \in \widehat{\mathcal{A}}\big(\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})\big)$.

Proof. — Let $y_r(z)=\int_0^r u(t)e^{-tz}dt$. It is a known fact that $y_r \in \mathcal{A}\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. More precisely, there exist positive numbers M_N, $N \in \mathbb{N}$, such that, for all $z \in \mathbb{C}$ with the property that $\operatorname{Re} z \geqslant C > 0$ and all $N \in \mathbb{N}$, $|R_N(y_r; z)| \leqslant M_N(\operatorname{Re} z)^{-N/p}$. Let $I \prec \big(\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})\big)$. In view of (2.4) there exist positive numbers R and δ such that, for all $z \in \widehat{D}_I(R)$, $\operatorname{Re} z \geqslant \delta \frac{|z|}{\log|z|}$ and thus,

$$|R_N(y_r; z)| \leqslant M_N \delta^{-N/p} \Big(\frac{|z|}{\log|z|}\Big)^{-N/p}, \quad N \in \mathbb{N}.$$

The proof is completed by observing that

$$|R_N(y_r; z)| \leqslant |R_{N+1}(y_r; z)| + C_N|z|^{-N/p}$$

where $C_N > 0$ and, for all $z \in \widehat{D}_I(R)$ and $h > 0$,

$$(\log|z|)^h \leqslant (\frac{h}{e})^h|z|.$$

□

LEMMA 2.12.

1. *Let I be an open interval such that $\theta_1 := \theta_-(I) \leqslant \theta_2 := \theta_+(I)$ and let $\theta \in [\theta_1, \theta_2]$. $f \in \widehat{\mathcal{A}}^{\leqslant -1^+}(I)$ iff, for every interval $I' \prec I$ and some sufficiently large R, there exists a positive number t such that*

$$\sup_{z \in \widehat{D}_{I'}(R)} |e^{t\psi_\theta(z)} f(z)| < \infty.$$

2. *For every large interval I, $\widehat{\mathcal{A}}^{\leqslant -1^+}(I) = \{0\}$.*

This lemma is easily deduced from [12, Lemma 0.15], with the aid of Lemma 5.3 in the appendix (§ 5).

We end this section with an important preliminary result. Two difference operators $B_1\tau - A_1$ and $B_2\tau - A_2$, where A_i and $B_i \in Gl(n, \widehat{\mathcal{A}}(I)[z^{1/p}])$, admitting asymptotic expansions $\widehat{A}_i$ and $\widehat{B}_i$, for $i = 1, 2$, will be called *formally equivalent* if the formal operators $\widehat{B}_1\tau - \widehat{A}_1$ and $\widehat{B}_2\tau - \widehat{A}_2$ are equivalent in the sense of § 2.1.

THEOREM 2.13. — *Let I be an open interval such that $\theta_-(I) \leqslant \theta_+(I)$ and let $A \in Gl(n, \widehat{\mathcal{A}}(I)[z^{1/p}])$.*

(i) *The difference operator* $\Delta := \tau - A$ *is formally equivalent to an operator* Δ^c *of the form (2.1), and the homogeneous linear difference equation* $\Delta y(z) = 0$ *has a fundamental system of solutions of the form*

$$y_j(z) = z^{d_j z} e^{q_j(z)} z^{\lambda_j} g_j(z), \quad j = 1, \dots, n \tag{2.6}$$

where $g_j \in (\widehat{\mathcal{A}}(I))^n[\log z]$, d_j, q_j *and* λ_j *are defined as in (2.2).*

(ii) $\mathrm{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n)$ *is a linear space over* $\mathbb{C}$, *spanned by all solutions of the form* $y_j(z)e^{2l\pi i z}$ *with* $j \in \{1, \dots, n\}$ *and* $l \in \mathbb{Z}$ *such that* $d_j < 0$ *and* $\frac{\mathrm{Im}\,\mu_j + 2l\pi}{d_j} \in [\theta_-(I), \theta_+(I)]$.

(iii) $\mathrm{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1}(I))^n) = \mathrm{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n)$ *if* $I \not\subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$. *If* $I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$, *then* $\mathrm{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1}(I))^n)$ *is the linear space over* $\mathbb{C}$, *spanned by all solutions of the form* $y_j(z)e^{2l\pi i z}$ *with* $j \in \{1, \dots, n\}$ *and* $l \in \mathbb{Z}$, *such that* $d_j < 0$ *and* $\frac{\mathrm{Im}\,\mu_j + 2l\pi}{d_j} \in [\theta_-(I), \theta_+(I)]$, *or* $d_j = 0$, $k_j = 1$, $\arg \mu_j = \pi$ *and* $l = 0$.

Proof.

(i) A has an asymptotic expansion $\widehat{A} \in Gl(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])$ and there exists $\widehat{F} \in Gl(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])$ such that $(\tau\widehat{F})^{-1}\widehat{A}\widehat{F} = A^c$. Hence the difference equation

$$Y(z+1) = A(z)Y(z)A^c(z)^{-1}$$

has the formal solution $\widehat{F}$. Consequently, it has an analytic solution $F \in Gl(n, \widehat{\mathcal{A}}(I)[z^{1/p}])$ with asymptotic expansion $\widehat{F}$ and $\{Fy_j^c : j = 1, \dots, n\}$ is a fundamental system of solutions of the difference equation $\Delta y = 0$ (*cf.* [11, Theorem 1.2] and Remark 5.2 below).

(ii) Obviously,
$y \in \mathrm{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n)$ iff $u := F^{-1}y \in \mathrm{Ker}(\Delta^c, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n)$. Let $\theta \in [\theta_-(I), \theta_+(I)]$. In view of Lemma 2.12, $u \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$ iff, for every open interval $I' \prec I$ and some sufficiently large R, there exists a positive number t such that $\sup_{z \in \widehat{D}_{I'}(R)} |e^{t\psi_\theta(z)}u(z)| < \infty$. Let $I = (a, b)$, $I' = (a', b')$ such that $a < a' < 0 < b' < b$ and $\widetilde{I'} = (\theta_1, \theta_2)$. Then $\theta_1 = \vartheta(a') < \vartheta(a) = \theta_-(I)$ and $\theta_2 = \vartheta(b') > \vartheta(b) = \theta_+(I)$. Without loss of generality, we may assume that $\widetilde{I'} \cap \Theta(\Delta^c) = [\theta_-(I), \theta_+(I)] \cap \Theta(\Delta^c)$ and that conditions (1) - (4) of Proposition 1.5 in [12] are satisfied (note that there a slightly different definition of y_j^c is used and *cf.* Lemma 5.3 below for the relation between the domains $D_I(R)$ and $\widehat{D}_I(R)$). By that proposition, with $k = 1^+$, Δ^c has a

right inverse Λ^c with the property that $\Lambda^c\Delta^c u(z) = u(z) - \sum_{j=1}^n y_j^c(z)p_j(z)$, where $p_j \equiv 0$ unless $d_j < 0$ and

$$p_j(z) = \sum_{l=s_j}^{l_j} p_{jl} e^{2l\pi iz}$$

if $d_j < 0$. Here, $p_{jl} \in \mathbb{C}$, $s_j = \inf\{l \in \mathbb{Z} : \frac{\operatorname{Im}\mu_j + 2l\pi}{d_j} \in (\theta_1, \theta_2)\}$ and $l_j = \sup\{l \in \mathbb{Z} : \frac{\operatorname{Im}\mu_j + 2l\pi}{d_j} \in (\theta_1, \theta_2)\}$. Consequently, $y = Fu = \sum_{j=1}^n y_j p_j$, where $y_j = Fy_j^c$. Conversely, it is easily verified that every function of this form belongs to $\operatorname{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n)$.

(iii) Similarly,
$y \in \operatorname{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1}(I))^n)$ iff $u := F^{-1}y \in \operatorname{Ker}(\Delta^c, (\widehat{\mathcal{A}}^{\leqslant -1}(I))^n)$. The last two statements follow again from Proposition 1.5 in [12] by observing that $I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ iff $\theta_-(I) \leqslant \frac{\pi}{2}$ and $\theta_+(I) \geqslant -\frac{\pi}{2}$. □

Corollary 2.14. — *Let I be an open interval such that $\theta_-(I) \leqslant \theta_+(I)$, $A \in Gl(n, \widehat{\mathcal{A}}(I)[z^{1/p}])$ and $\Delta = \tau - A$. If $[\theta_-(I), \theta_+(I)] \cap \Theta(\Delta^c) = \emptyset$, then $\operatorname{Ker}(\Delta, (\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n) = \{0\}$.*

2.5. A prepared form of (1.1)

Following Braaksma and Faber in [3], we first transform (1.1) into a convenient ‘prepared form’, consisting of a linear part in canonical form plus a “perturbation” (which may also contain linear terms).

Let $\widehat{\Delta} = \tau - \widehat{A}$ denote the formal difference operator associated with (1.1), where $\widehat{A}$ is defined by (1.2) and let Δ^c be a canonical form of $\widehat{\Delta}$. Thus, there exists $\widehat{F} \in Gl(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])$ such that $(\tau\widehat{F})^{-1}\widehat{\Delta}\widehat{F} = \Delta^c = \tau - A^c$ (*cf.* (2.1) and (2.2)). A^c is a block-diagonal matrix of the form

$$A^c(z) = \bigoplus_{h=0}^{r} \Big\{ I_{n_h} + z^{k_h - 1}(A_h + z^{-1/p}B_h(z)) \Big\}.$$

Here, r is a positive integer and, for each $h \in \{0, \dots, r\}$, n_h is a nonnegative integer and k_h is a nonnegative multiple of $1/p$, such that $0 = k_0 < k_1 < \cdots < k_q = 1 \leqslant k_{q+1} \leqslant \cdots \leqslant k_r$, where $1 \leqslant q \leqslant r$. Moreover, $n_h > 0$ if $h \notin \{0, q\}$ and $k_h > 1$ if $h > q+1$. The rational numbers $k_1, \dots, k_q$ correspond to the positive levels of $\widehat{\Delta}$: for each $h \in \{0, \dots, q\}$ such that $n_h > 0$, there exists an integer $j \in \{1, \dots, n\}$ with the property that $d_j = 0$ and $\kappa_j = k_h$. (The exceptional case that $q = 1$ and $n_0 = n_q = 0$ will not concern us here as it has been dealt with in [10].) A_h is a constant

$n_h \times n_h$ matrix in Jordan normal form, nonsingular and diagonal if $h > 0$, and $B_h \in \mathrm{End}(n_h; \mathbb{C}\{z^{-1/p}\})$. Here, we shall *assume* that $n_0 + n_1 > 0$, that $r > q$, that $A_q + I_{n_q}$ is nonsingular if $n_q > 0$, that $k_{q+1} = 1$ and $A_{q+1} = -I_{n_{q+1}}$. In this case the equation (1.1) possesses *both a level* $\leqslant 1$ *and the level* 1^+. More precisely, $n_{q+1} = \#\{j \in \{1, \dots, n\} : d_j < 0\}$ and $\sum_{q+1<h\leqslant r} n_h = \#\{j \in \{1, \dots, n\} : d_j > 0\}$. If $n_q > 0$, the eigenvalues of $A_q + I_{n_q}$ are the numbers e^{μ_j} with $j \in \{1, \dots, n\}$ such that $d_j = 0$ and $\kappa_j = 1$. Note that, if $n_0 > 0$ or $q > 1$, $\widehat{\Delta}$ has a level 1, even if $n_q = 0$ (*cf.* remark 2.1 (ii)).

Now, let $S \in Gl(n; \mathbb{C}\{z^{-1/p}\}[z^{1/p}])$ and $P \in \mathbb{C}^n[z^{-1/p}]$ be obtained by truncating $\widehat{F}$ and the formal solution $\hat{f}$ of (1.1), respectively, at some sufficiently large power M of $z^{-1/p}$. By the substitution

$$y \mapsto Sy + P$$

(1.1) is transformed into an equation of the form:

$$\Delta y(z) = \varphi_0(z^{1/p}) + E(z^{1/p}, y(z)), \tag{2.7}$$

with formal solution $\hat{f}_N := S^{-1}(\hat{f} - P) \in z^{-N/p}\mathbb{C}^n[[z^{-1/p}]]$ for some $N \in \mathbb{N}$. Δ is a linear difference operator of the form

$$\Delta = \bigoplus_{h=0}^{r} z^{1-k_h} I_{n_h}(\tau - 1) - \bigoplus_{h=0}^{r} A_h - z^{-1/p}B(z) \tag{2.8}$$

where $B \in \mathrm{End}(n; \mathbb{C}\{z^{-1/p}\})$. φ_0 is an analytic function in a neighbourhood of ∞, which is $O(z^{-N'})$ as $z \to \infty$ for some positive integer N', and E is analytic in a neighbourhood U of $(\infty, 0)$. Furthermore, $E(z^{1/p}, 0) = 0$ and $D_2E(z^{1/p}, 0) = 0$ for all sufficiently large z. If $\widehat{F} \in z^{-\mu}Gl(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])$ for some sufficiently large $\mu \in \mathbb{N}$, then $E(\infty, y) = 0$ as well. We shall assume that this is the case (*cf.* Remark 2.1 (i)). Moreover,

$$\bigoplus_{h=0}^{r} z^{k_h-1} I_{n_h}\Delta - \Delta^c \in z^{-M'/p}\,\mathrm{End}(n; \mathbb{C}\{z^{-1/p}\}) \tag{2.9}$$

where $M' > 0$. N, N' and M' can be made arbitrarily large by choosing a sufficiently large M. Therefore, since $A^c \in Gl(n; \mathbb{C}\{z^{-1/p}\}[z^{1/p}])$,

$$\bigoplus_{h=0}^{r}(z^{1-k_h} I_{n_h} + A_h) + z^{-1/p}B(z) \in Gl(n; \mathbb{C}\{z^{-1/p}\}[z^{1/p}])$$

if M is sufficiently large. We shall refer to (2.7) as a “prepared form of (1.1)”. It is easily seen that Δ and Δ^c are formally equivalent. Therefore, we also call Δ^c a canonical form of Δ.

The following result is essentially due to Sibuya, who used a similar idea in the theory of differential equations (*cf.* [19]).

Proposition 2.15. — *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be open intervals and, for $i = 1, 2$, let $y_i \in (\widehat{\mathcal{A}}(\mathcal{I}_i))^n$ be a solution of the nonlinear difference equation (2.7). Then $y_1 - y_2$ satisfies a homogeneous linear difference equation of the form*

$$\widetilde{\Delta} y(z) := \Delta y(z) - H(z)y(z) = 0 \tag{2.10}$$

where $H \in \mathrm{End}(n; \widehat{\mathcal{A}}(\mathcal{I}_1 \cap \mathcal{I}_2))$. There exists a positive constant K, independent of y_1 and y_2, such that, for any $I' \prec \mathcal{I}_1 \cap \mathcal{I}_2$ and some sufficiently large number R, depending on I', and for all $z \in \widehat{D}_{I'}(R)$,

$$\left|E(z^{1/p}, y_1(z)) - E(z^{1/p}, y_2(z))\right| \leqslant K \max\{|y_1(z)|, |y_2(z)|\}\left|y_1(z) - y_2(z)\right| \tag{2.11}$$

and

$$|H(z)| \leqslant K \max\{|y_1(z)|, |y_2(z)|\}. \tag{2.12}$$

Moreover, if N is sufficiently large, $\widetilde{\Delta}$ and Δ have the same canonical form Δ^c.

Proof. — As both y_1 and y_2 are solutions of (2.7),

$$\begin{aligned}\Delta(y_1 - y_2)(z) &= E(z^{1/p}, y_1(z)) - E(z^{1/p}, y_2(z)) \\ &= \int_0^1 D_2 E(z^{1/p}, t y_1(z) + (1-t) y_2(z)) dt (y_1 - y_2)(z).\end{aligned}$$

Thus, $y_1 - y_2$ satisfies (2.10), with $H(z) = \int_0^1 D_2E(z^{1/p}, ty_1(z) + (1 - t)y_2(z))dt$. Obviously, $ty_1 + (1 - t)y_2 \in \widehat{\mathcal{A}}(\mathcal{I}_1 \cap \mathcal{I}_2)^n$ for all $t \in [0, 1]$ and $(z^{1/p}, ty_1(z) + (1-t)y_2(z)) \in U$ for all $z \in \widehat{D}_{I'}(R)$, provided $I' \prec \mathcal{I}_1 \cap \mathcal{I}_2$ and R is sufficiently large. E is analytic in U, hence $H \in \mathrm{End}(n; \widehat{\mathcal{A}}(\mathcal{I}_1 \cap \mathcal{I}_2))$. As $D_2E(z^{1/p}, 0) = 0$, there exists a positive constant K, independent of z and y, such that $|D_2E(z^{1/p}, y)| \leqslant K|y|$ for all $(z^{1/p}, y) \in U$ and thus, (2.11) and (2.12) hold for all z such that $(z^{1/p}, ty_1(z) + (1 - t)y_2(z)) \in U$ and, consequently, for all $z \in \widehat{D}_{I'}(R)$, provided $I' \prec \mathcal{I}_1 \cap \mathcal{I}_2$ and R is sufficiently large. Due to the fact that $\hat{f}_N \in z^{-N/p}\mathbb{C}[[z^{-1/p}]]^n$, $y_i \in z^{-N/p}(\widehat{\mathcal{A}}(\mathcal{I}_i))^n$ for $i = 1, 2$. This implies that $|H(z)| \leqslant K'|z|^{-N/p}$ for all $z \in \widehat{D}_{I'}(R)$ and hence it can be deduced that $\widetilde{\Delta}$ and Δ are formally equivalent if N is sufficiently large and thus have the same canonical form Δ^c. □

2.6. Examples

In [14] we discuss in some detail the trivial, but instructive example of a system of two uncoupled difference equations, of level 1 and 1^+, respectively. Below, we give two simple examples of equations with three distinct levels: $\frac{1}{2}$, 1 and 1^+, both having analytic coefficients at ∞.

Example 2.16.

$$y(z+1) = \begin{pmatrix} 1 & 1 & 0 \\ \frac{a_1}{z} & 1+\frac{a_1}{z} & 0 \\ 0 & 0 & \frac{a_2}{z} \end{pmatrix} y(z) + z^{-2} f(z, y(z)) \tag{2.13}$$

where a_1 and $a_2 \in \mathbb{C}^*$ and f is a 3-dimensional vector function, polynomial in y and analytic at ∞ in z. The substitution

$$y \mapsto Sy$$

where

$$S(z) = \begin{pmatrix} 1 & 1 & 0 \\ a_1^{1/2} z^{-1/2} & -a_1^{1/2} z^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

changes (2.13) into an equation of the form:

$$\begin{aligned} \operatorname{diag}\{z^{1/2}, z^{1/2}, 1\}&(y(z+1) - y(z)) \\ &= \operatorname{diag}\{a_1^{1/2}, -a_1^{1/2}, -1\} y(z) + z^{-1/2} g(z^{1/2}, y(z)) \end{aligned} \tag{2.14}$$

where g is a 3-dimensional vector function, analytic at $(\infty, 0)$. In this example, $r = 3$, $q = 2$, $k_1 = 1/2$, $n_0 = n_2 = 0$, $n_1 = 2$ and $n_3 = 1$. From [13, Theorem 2.7] one easily deduces the existence of a formal solution $\widetilde{f} \in z^{-1/2}\mathbb{C}[[z^{-1/2}]]^3$ of (2.14) and hence the existence of a formal solution $\hat{f} = S^{-1}\widetilde{f} \in \mathbb{C}[[z^{-1/2}]]^3$ of (2.13). The formal difference operator $\widehat{\Delta}$ associated with (2.13) has a diagonal canonical form Δ^c and the homogeneous equation $\Delta^c y = 0$ has solutions of the form $y_1^c(z) = e^{2a_1^{1/2} z^{1/2}} z^{1/4} \mathbf{e}_1$, $y_2^c(z) = e^{-2a_1^{1/2} z^{1/2}} z^{1/4} \mathbf{e}_2$ and $y_3^c(z) = z^{-z}(a_2 e)^z z^{1/2} \mathbf{e}_3$, where $\mathbf{e}_i$ denotes the i-th unit vector of $\mathbb{C}^3$. Thus, the singular directions of level $\frac{1}{2}$ are given by $-\arg a_1 \bmod 2\pi$, those of level 1 by $\frac{\pi}{2} \bmod \pi$ and the pseudo-Stokes directions of level 1^+ correspond to the different determinations of $-\arg a_2$.

Example 2.17.

$$y(z+1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \frac{a_1}{z} & 1+\frac{a_1}{z} & 0 & 0 \\ 0 & 0 & 1+a_2 & 0 \\ 0 & 0 & 0 & \frac{a_3}{z} \end{pmatrix} y(z) + z^{-2} f(z, y(z)) \tag{2.15}$$

where a_1,a_2 and $a_3 \in \mathbb{C}^*$, $a_2 \neq -1$ and f is a 4-dimensional vector function, polynomial in y and analytic at ∞ in z. Similarly to (2.13), (2.15) can be transformed into an equation of the form:

$$\begin{aligned}\operatorname{diag}\{z^{1/2}, z^{1/2}, 1, 1\}\big(y(z+1) - y(z)\big) \\ = \operatorname{diag}\{a_1^{1/2}, -a_1^{1/2}, a_2, -1\}y(z) + z^{-1/2}g\big(z^{1/2}, y(z)\big)\end{aligned}$$

where g is a 4-dimensional vector function, analytic at $(\infty, 0)$. In this example, $r = 3$, $q = 2$, $k_1 = 1/2$, $n_0 = 0$, $n_1 = 2$, and $n_2 = n_3 = 1$. (2.15) has a formal power series solution $\in z^{-1/2}\mathbb{C}[[z^{-1/2}]]^4$. In this case, the singular directions of level 1 are $\frac{\pi}{2} \bmod \pi$ and $\pi - \arg\{\log(1 + a_2) + 2l\pi i\}$, where $l \in \mathbb{Z}$. 0 is a singular direction of level 1 iff $a_2 \in (-1, 0)$.

3. The case that 0 is not a singular direction of level 1

Throughout most of this section (with the exception of Theorem 3.8), we assume that the difference equation already is in a prepared form (2.7), where Δ has the form (2.8), and (2.9) is satisfied for some sufficiently large M. Thus, $\hat{f}$ will denote the formal solution of (2.7) and, consequently, $\hat{f} \in z^{-N/p}\mathbb{C}^n[[z^{-1/p}]]$, where N is some sufficiently large integer, depending on M. Let $I_0 = \mathbb{R}$, let f_0 denote the unique global section of $(\mathcal{A}/\mathcal{A}^{\leqslant -k_1})^n$ with the property that $f_0(ze^{2p\pi i}) = f_0(z)$, associated with $\hat{f}$ (*cf.* §2.2), and let I_h, $h = 1, \ldots, q$, be open intervals of $\mathbb{R}$ with the following properties:

- $(-\frac{\pi}{2}, \frac{\pi}{2}) \subset I_q \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$.
- I_h does not contain a Stokes pair of level k_h.

From [3] we know that (2.7) has solutions $f_h \in z^{-N/p}(\mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}})^n(I_h)$ such that

$$f_{h-1}|_{I_h} = f_h \bmod (\mathcal{A}^{\leqslant -k_h})^n, \quad h = 1, \ldots, q-1. \tag{3.1}$$

The approach taken in [3] is based on a study of convolution equations. With (2.7) one can associate, for every $h \in \{1, \ldots, q\}$, a convolution equation of the form $T_h\eta = \eta$, obtained by applying a formal Borel transformation of order k_h to (2.7). The formal Borel transform of order k_1 of $\hat{f}$ is a convergent power series, defining an analytic solution u_1 of $T_1\eta = \eta$. In [3] it is shown that u_1 is analytic in an infinite sector of the form $S(\widehat{I}_1)$, where $-\widehat{I}_1$ is an open interval not containing any singular direction of level k_1, and that u_1 satisfies a specific growth condition in this sector. By means of a so-called acceleration operator (an extension of a Laplace transformation of order k_1 followed by a Borel transformation of order k_2) u_1 can be transformed into a solution u_2 of $T_2\eta = \eta$. u_2 is analytic in a sector of the form $S(\widehat{I_2})$, where $-\widehat{I_2}$ is an open interval not containing any singular direction of level k_2, and, if $q > 2$, it can be transformed into a solution u_3 of $T_3\eta = \eta$, etc. Moreover, for $h = 1, \dots, q$, u_h coincides with a Borel transform of order k_h of f_{h-1}. If $r = q$, then u_q has at most exponential growth of order 1 and its Laplace transform f_q is the $(k_1, \dots, k_q)$-sum of $\hat{f}$ on $(I_1, \dots, I_q)$. In the case that $r > q$, one has to deal with an additional convolution equation, corresponding to the level 1^+, which, in general, doesn't admit an analytic solution in a sector of the form $S(I)$ for any open interval I.

In this section it is assumed that 0 is not a singular direction of level 1, or, equivalently, that $-\frac{\pi}{2}$ is not a Stokes direction of level 1. Similarly to the case without level 1^+, u_q can be analytically continued to a sector of the form $S(I_q^*)$, where $-I_q^*$ is an open interval not containing any singular direction of level 1, but it may have supra-exponential growth. We show that it satisfies a particular growth condition, making it *accelerable* from level 1 to level 1^+, by means of a so-called *weak* acceleration operator. From the growth property of u_q we can deduce the existence of a particular representative of $f_{q-1}|_{I_q}$, which also represents a solution $f_q \in (\mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q))^n$ of (2.7) (*cf.* proposition 3.9 below). In proposition 3.11 it is shown that f_q has a representative defining a solution $f_{q+1} \in \widehat{\mathcal{A}}(I_{q+1})^n$, for every large interval $I_{q+1} \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ such that $\widetilde{I_{q+1}}$ doesn't contain any pseudo-Stokes directions. f_{q+1} is the $(k_1, \dots, k_q, 1^+)$-sum or *accelero-sum* of $\hat{f}$ on $(I_1, \dots, I_{q+1})$.

3.1. Growth properties of u_q

Definition 3.1 (Laplace and Borel transformations). — *For any open interval $I = (\alpha, \beta)$, with $\beta - \alpha > \pi$, I^* is defined by*

$$I^* := \Big(-\beta + \frac{\pi}{2}, -\alpha - \frac{\pi}{2}\Big).$$

By

$$\mathcal{L} : t^{1/p-1}\mathcal{A}_0^{\leqslant 0}(I^*) \to z^{-1/p}\mathcal{A}^{\leqslant 0}/\mathcal{A}^{\leqslant -1}(I)$$

we denote the finite (or incomplete) Laplace transformation, defined as follows. Let $u \in t^{1/p-1}\mathcal{A}_0^{\leqslant 0}(I^*)$. Let $\{\alpha_\nu : \nu \in \mathcal{N}\} \subset I^*$ such that $\{(-\alpha_\nu - \frac{\pi}{2}, -\alpha_\nu + \frac{\pi}{2}) : \nu \in \mathcal{N}\}$ is a covering of I, and, for each $\nu \in \mathcal{N}$, let $r_\nu > 0$ such that u is continuous on $(0, r_\nu e^{i\alpha_\nu}]$. Then $\mathcal{L}(u)$ is the element of $z^{-1/p}\mathcal{A}^{\leqslant 0}/\mathcal{A}^{\leqslant -1}(I)$ represented by $\{\int_0^{r_\nu e^{i\alpha_\nu}} u(t)e^{-tz}dt : \nu \in \mathcal{N}\}$. $\mathcal{L}$ is a bijection and its inverse $\mathcal{B}$ is the "ordinary" Borel transformation.

Remark 3.2. — It is well-known that $\mathcal{L}(t^{N/p-1}\mathcal{A}_0(I^*)) = z^{-N/p}\mathcal{A}/\mathcal{A}^{\leqslant -1}(I)$ for every $N \in \mathbb{N}$. Note that $-I^*$ doesn't contain a singular direction of level 1 iff I doesn't contain a Stokes pair of level 1.

If M is sufficiently large, the function $u_q = \mathcal{B}(f_{q-1})$ satisfies the convolution equation $T\eta = \eta$, obtained by applying a formal Borel transformation to (2.7). Let $\widetilde{E}(z, y(z)) = z^{-1/p}B(z)y(z) + \varphi_0(z^{1/p}) + E(z^{1/p}, y(z))$ and $y = \bigoplus_{h=0}^{r} y_h$, in the partition used in (2.8). Then we have

$$\begin{aligned} &y_h(z+1) - y_h(z) = z^{k_h-1}\{A_h y_h(z) + \widetilde{E}(z, y(z))_h\} \text{ for } h < q,\\ &y_q(z+1) - (I_{n_q} + A_q)y_q(z) = \widetilde{E}(z, y(z))_q \text{ (if } n_q > 0),\\ &y_{q+1}(z+1) = \widetilde{E}(z, y(z))_{q+1},\\ &A_h y_h(z) = z^{1-k_h}(y_h(z+1) - y_h(z)) - \widetilde{E}(z, y(z))_h \text{ for } h > q+1. \end{aligned} \tag{3.2}$$

Hence we deduce the following form for $T\eta$:

$$\begin{aligned} &(T\eta)_h = (e^{-t} - 1)^{-1}\{\frac{t^{-k_h}}{\Gamma(1-k_h)} * (A_h\eta_h + \mathcal{E}(t, \eta)_h)\} \text{ for } h < q,\\ &(T\eta)_q = \{(e^{-t} - 1)I_{n_q} - A_q\}^{-1}\mathcal{E}(t, \eta)_q \text{ (if } n_q > 0),\\ &(T\eta)_{q+1} = e^t\mathcal{E}(t, \eta)_{q+1},\\ &(T\eta)_h = A_h^{-1}\{\frac{t^{k_h-2}}{\Gamma(k_h-1)} * (e^{-t} - 1)\eta_h - \mathcal{E}(t, \eta)_h\} \text{ for } h > q+1. \end{aligned} \tag{3.3}$$

where $*$ denotes the convolution product: $u * v(t) = \int_0^t u(s)v(t-s)ds$ and $\mathcal{E}(t, \mathcal{B}y(t)) = \mathcal{B}(\widetilde{E}(., y(.)))(t)$. Thus, $\mathcal{E}(t, \eta) = \sum_{m\in\mathbb{N}_0^n} \mathcal{E}_m * \eta^{*m}(t)$, where $\mathbb{N}_0$ denotes the set of nonnegative integers and each $\mathcal{E}_m$ is analytic on the Riemann surface of $\log t$, satisfying a condition of the form

$$|\mathcal{E}_m(t)| \leqslant Kb^{|m|}|t|^{1/p-1}e^{c_0|t|} \tag{3.4}$$

uniformly on $S(I_q^*)$, and $\mathcal{E}_0(t)_h = O(t^{N/p-1})$ for all h. K, b and c_0 are positive constants. $|m|$ denotes the 1-norm of the n-vector m: $|m| := \sum_{i=1}^n m_i$.

Lemma 3.3. — *Assume that $I_q^* \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ and $-I_q^*$ does not contain a singular direction of level 1. Then u_q can be analytically continued to $S(I_q^*)$, and its analytic continuation, also denoted by u_q, has the following property: for every interval $I' \prec I_q^*$ there exist positive numbers B and C (depending on I') such that, for all $t \in S(I')$,*

$$|u_q(t)| \leqslant Ce^{e^{B|t|}}. \tag{3.5}$$

Proof. — The proof of the first statement is analogous to the proof in the case without level 1^+, sketched in [3] (*cf.* also [1]). In order to prove the growth property, proceeding as in [3], we introduce an operator $\overline{T}$: $C([0,\infty)) \to C([0,\infty))$, defined by

$$\overline{T}\psi(t) = Me^t(t^{1/p-1}e^{c_0't}) * \sum_{m=0}^{\infty} b^m \psi^{*m}(t) \tag{3.6}$$

where M is a sufficiently large positive number and $c_0' > c_0$. Let $I' \prec I_q^*$ and let $v \in C([0,\infty))$ be defined by

$$v(t) = \sup\bigl\{|u_q(s)| : \arg s \in I', |s| = t\bigr\} \text{ if } t > 0,\ v(0) = 0$$

(v is continuous, due to the fact that $u_q \in t^{N/p-1}(\mathcal{A}_0(I_q^*))^n$.) As $I' \prec (-\frac{\pi}{2}, \frac{\pi}{2})$, e^{-t} and $(e^{-t}-1)^{-1}$ are bounded on $S(I',1)$. Due to the assumption that $-I_q^*$ does not contain a singular direction of level 1, $((e^{-t}-1)I_{n_q} - A_q)^{-1}$ is bounded on $S(I',1)$ as well (the eigenvalues of $(e^{-t}-1)I_{n_q} - A_q$ are $e^{-t} - e^{\mu_j}$ with $j \in \{1,\dots,n\}$ such that $d_j = 0$ and $\kappa_j = 1$, and $-\arg t \neq \pi - \arg(\mu_j + 2l\pi i)$ for all such j and all $l \in \mathbb{Z}$). Furthermore, for all $t \in (0,\infty)$,

$$t^{-k} * (t^{1/p-1}e^{c_0 t}) \leqslant B(1-k,1/p)t^{1/p-k}e^{c_0 t} \leqslant Ct^{1/p-1}e^{c_0' t}$$

and $t^{-k} \leqslant Ct^{1/p-1}e^{c_0't}$ if $0 \leqslant k \leqslant 1-1/p$, and $t^{k-2} \leqslant Ct^{1/p-1}e^{c_0't}$ for all $t \in (0,\infty)$ if $k \geqslant 1/p+1$. Here, B denotes the Euler beta-function and C is a positive constant. Hence it follows that $|u_q(t)| = |Tu_q(t)| \leqslant \overline{T}v(|t|)$ for all $t \in S(I',1)$, provided M is sufficiently large. Let $M_0 > \max\{v(t) : t \leqslant 1\}$ and $\chi_0(t) = e^{Bt+e^{B't}}$, where B and $B' > 0$. Then $|u_q(t)| < M_0\chi_0(|t|)$ for all $t \in S(I')$ such that $|t| \leqslant 1$. Suppose there exists $t_0 \in (0,\infty)$ such that

$$v(t) < M_0\chi_0(t) \text{ for all } t < t_0 \text{ and } v(t_0) = M_0\chi_0(t_0).$$

Obviously, $t_0 > 1$. Consequently, $v(t_0) = \sup\{|u_q(s)|\colon \arg s{\in}I', |s|{=}t_0\} \leqslant \overline{T}v(t_0) < \overline{T}(M_0\chi_0)(t_0)$, $\overline{T}$ being a monotone operator. From Lemma 3.4(iii) below we deduce that $\overline{T}M_0\chi_0(t) \leqslant M_0\chi_0(t)$ for all $t \geqslant 1$, sufficiently large B' and suitable values of B. This implies $v(t_0) < M_0\chi_0(t_0)$, contradictory to the assumption and thus we conclude that $|u_q(t)| < M_0\chi_0(|t|)$ for all $t \in S(I')$, provided B' and M are sufficiently large. Hence the result follows. □

Lemma 3.4. — *Let $p \in \mathbb{N}$, $B' > 0$ and $0 < B < eB'$, and let*

$$\chi_0(t) = e^{Bt+e^{B't}}.$$

(i) $\chi_0^{*m}(t) \leqslant (\frac{e^{2e}}{B'})^{m-1}\chi_0(t)$ *for all $t \in (0,\infty)$ and all $m \in \mathbb{N}$.*

(ii) *Let $1/p \leqslant a \leqslant 1$, $c \geqslant 0$ and $B \geqslant B' + c$. Then*

$$(t^{a-1}e^{ct}) * \chi_0(t) \leqslant (p+1)e^{-B'at}\chi_0(t)$$

for all $t \geqslant 1$, provided B' is sufficiently large.

(iii) *For all sufficiently large values of B' and $B' + c_0' < B < eB'$, there exists a positive constant $K(B')$ with the property that $K(B') \to 0$ as $B' \to \infty$, and*

$$\overline{T}(M_0\chi_0)(t) \leqslant bMK(B')M_0\chi_0(t)$$

for all $t \geqslant 1$.

Proof.
(i) We have

$$(\chi_0 * \chi_0)(t) = e^{Bt} \int_0^t e^{e^{B'(t-\tau)} + e^{B'\tau}} d\tau = 2te^{Bt} \int_0^{1/2} e^{e^{B't(1-s)} + e^{B'ts}} ds.$$

Due to the convexity of $e^{B'ts}$ w.r.t. s,

$$e^{B't(1-s)} + e^{B'ts} \leqslant e^{B't} + 1 + 2s(2e^{B't/2} - e^{B't} - 1)$$

for all $s \in [0, 1/2]$, and thus

$$\begin{aligned}(\chi_0 * \chi_0)(t) &\leqslant 2te^{Bt + e^{B't} + 1} \int_0^{1/2} e^{2s(2e^{B't/2} - e^{B't} - 1)} ds \\ &= 2te^{Bt + e^{B't} + 1} \int_0^{1/2} e^{-2(e^{B't/2} - 1)^2 s} ds \\ &= \frac{te(1 - e^{-(e^{B't/2} - 1)^2})}{(e^{B't/2} - 1)^2} \chi_0(t).\end{aligned}$$

Hence we deduce that

$$(\chi_0 * \chi_0)(t) \leqslant \frac{te}{(e^{B't/2} - 1)^2} \chi_0(t) \leqslant \frac{4e}{B'^2 t} \chi_0(t) \leqslant \frac{4e}{B'} \chi_0(t)$$

for all $t \geqslant 1/B'$. Furthermore, for all $t \leqslant 1/B'$, we have

$$(\chi_0 * \chi_0)(t) \leqslant \chi_0(1/B')(1 * \chi_0)(t) \leqslant \frac{e^{B/B' + e}}{B'} \chi_0(t) \leqslant \frac{e^{2e}}{B'} \chi_0(t)$$

provided $B \leqslant eB'$. The first statement of the lemma follows easily by means of an inductive argument.

(ii) For all $t \geqslant 1$, $B' \geqslant 1$ and $B \geqslant B' + c$ we have

$$(t^{a-1}e^{ct}) * \chi_0(t) = e^{ct}\int_0^{t-e^{-B't}} (t-\tau)^{a-1}e^{(B-c)\tau+e^{B'\tau}}d\tau + e^{ct}\int_{t-e^{-B't}}^{t} (t-\tau)^{a-1}e^{(B-c)\tau+e^{B'\tau}}d\tau$$
$$\leqslant e^{(B-B')t}e^{(1-a)B't}\int_0^t e^{B'\tau+e^{B'\tau}}d\tau + e^{Bt+e^{B't}}\int_{t-e^{-B't}}^{t}(t-\tau)^{a-1}d\tau$$
$$\leqslant \chi_0(t)(\frac{1}{B'}+\frac{1}{a})e^{-B'at} \leqslant (p+1)e^{-B'at}\chi_0(t).$$

(iii) From (i) we deduce that, for all $t > 0$,

$$e^{-t}\overline{T}(M_0\chi_0)(t) \leqslant M\Big((t^{1/p-1}e^{c_0't}) * 1 + (t^{1/p-1}e^{c_0't}) * \sum_{m=1}^{\infty} b(\frac{e^{2e}bM_0}{B'})^{m-1}M_0\chi_0(t)\Big)$$
$$\leqslant pMt^{1/p}e^{c_0't} + 2Mb(t^{1/p-1}e^{c_0't}) * M_0\chi_0(t)$$

if $B' \geqslant 2e^{2e}bM_0$. In view of (ii) this implies that

$$e^{-t}\overline{T}(M_0\chi_0)(t) \leqslant pMt^{1/p}e^{c_0't} + 2M(p+1)be^{-\frac{B'}{p}t}M_0\chi_0(t)$$

for all $t \geqslant 1$, provided $B' + c_0' < B < eB'$ and hence the result follows, with $K(B') = 3(p+1)e^{1-B'/p}$, provided $bM_0 \geqslant 1$, $B' > p$ and $B' + c_0' < B < eB'$. □

3.2. Accelero-summability of $\hat{f}$

We begin by giving the definition of accelero-summability used in [4], which suits our present purpose. The accelero-sum of the formal solution of (1.1) will be an element of $(\widehat{\mathcal{A}}(I_{q+1}))^n$, where I_{q+1} is an appropriate interval. The main result for the case that 0 is not a singular direction of level 1 is stated in Theorem 3.8.

Definition 3.5 (accelero-summability, first version). — *Let $0 = k_0 < k_1 < \cdots < k_q = 1$, $I_0 = \mathbb{R}$ and let I_h, $h = 1, \ldots, q+1$, be open intervals of $\mathbb{R}$ with the following properties:*

- $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$ *for $h = 1, \ldots, q$ and I_{q+1} is a large interval.*

$\hat{f} \in \mathbb{C}[[z^{-1/p}]]$ *is called* $(k_1, \ldots, k_q, 1^+)$*-summable on* $(I_1, \ldots, I_{q+1})$ *with* $(k_1, \ldots, k_q, 1^+)$*-sum* f_{q+1}*, if there exist* $f_h \in \mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}}(I_h)$*,* $h = 0, \ldots, q-1$*,* $f_q \in \mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q)$ *and* $f_{q+1} \in \widehat{\mathcal{A}}(I_{q+1})$*, with asymptotic expansion* $\hat{f}$*, such that*

- $f_0(ze^{2p\pi i}) = f_0(z)$,
- $f_{h-1}|_{I_h} = f_h \bmod \mathcal{A}^{\leqslant -k_h}, \quad h = 1, \ldots, q$

and any representative $\{\phi_\nu : \nu \in \mathcal{N}\}$ *of* f_q*, where* $\phi_\nu \in \mathcal{A}(I_{q,\nu})$ *and* $\{I_{q,\nu} : \nu \in \mathcal{N}\}$ *is an open covering of* I_q*, has the following property: for any interval* $I'_{q,\nu} \prec I_{q,\nu}$ *and any interval* $I' \prec I_{q+1}$*, there exist positive constants* R*,* c *and* C *such that*

$$|f_{q+1}(z) - \phi_\nu(z)| \leqslant Ce^{-c|z|} \text{ for all } z \in \widehat{D}_{I'}(R) \cap S(I'_{q,\nu}). \tag{3.7}$$

Remark 3.6.

(i) If f_q has one representative $\{\phi_\nu : \nu \in \mathcal{N}\}$ such that (3.7) holds, then all representatives have this property.

(ii) In § 4.1 (Lemma 4.6) it is shown that any $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1^+}([-\frac{\pi}{2}, \frac{\pi}{2}])$ defines an element $\widetilde{f}^+ \in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathbb{R})$. Thus, (3.7) can be replaced by the condition

$$\widetilde{f_q}^+|_{I_{q+1}} = f_{q+1} \bmod \widehat{\mathcal{A}}^{\leqslant -1^+}.$$

Lemma 2.12 shows that f_{q+1} is determined uniquely by f_q. The uniqueness of f_q can be deduced from the following lemma:

Lemma 3.7 ("relative Watson Lemma" I). — *Let* I *be an open interval of* $\mathbb{R}$ *such that* $|I| > \pi$*. Then* $\mathcal{A}^{\leqslant -1}/\mathcal{A}^{\leqslant -1^+}(I) = \{0\}$*.*

The proof of this lemma is analogous to the one given in [15, 16], *cf.* also Lemma 4.14 below.

Theorem 3.8. — *Let* F *be a* $\mathbb{C}^n$*-valued function, analytic in a neighbourhood of* (∞, y_0) *for some* $y_0 \in \mathbb{C}^n$*. Suppose that (1.1) has a formal solution* $\hat{f} \in \mathbb{C}^n[[z^{-1/p}]]$*, with constant term* y_0*, such that (1.2) holds, and that the corresponding difference operator* $\widehat{\Delta} = \tau - \widehat{A}$ *has positive levels* $k_1 < \cdots < k_q = 1$ *and a level* 1^+*. Let* I_h*,* $h = 1, \ldots, q+1$*, be open intervals of* $\mathbb{R}$ *with the following properties:*

- $\widetilde{I_{q+1}} \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$ *for* $h = 1, \ldots, q$ *and* I_{q+1} *is a large interval.*
- I_h *does not contain a Stokes pair of level* k_h *for* $h = 1, \ldots, q$*.*
- $\widetilde{I_{q+1}} \cap \Theta(\widehat{\Delta}) = \emptyset$.

Then $\hat{f}$ is $(k_1, \ldots, k_q, 1^+)$-summable on $(I_1, \ldots, I_q, I_{q+1})$ and its sum is a solution of (1.1).

This theorem can be derived from propositions 3.9 and 3.11 below. The condition $\widetilde{I_{q+1}} \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, implying that f_{q+1} is defined on a subset of $S((-\frac{\pi}{2}, \frac{\pi}{2}))$, will be lifted in section 4, Theorem 4.12, I.

As before, let us assume that (1.1) is in prepared form (2.7), let $f_h \in z^{-N/p}(\mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}})^n(I_h)$ be solutions of (2.7), satisfying (3.1) for $h = 1, \ldots, q-1$, and let $u_q = \mathcal{B}(f_{q-1})$. The conditions on I_q imply that $0 \in I_q^*$ and $-I_q^*$ does not contain a singular direction of level 1 (*cf.* Remark 3.2). Hence, in view of Remark 2.3(ii), $I_q^* \subset (-\frac{\pi}{2}, \frac{\pi}{2})$.

For all $\alpha \in I_q^*$, $\theta \in \mathbb{R}$ and $z \in S((-\pi, \pi))$ such that $\alpha + \arg\log(ze^{i\theta}) \in I_q^*$, we define

$$\phi_\alpha^\theta(z) := \int_0^{e^{i\alpha}\log(ze^{i\theta})/B'_\alpha} u_q(s)e^{-sz}ds.$$

(The idea of using finite Laplace integrals of the form $\int_0^{r(z)} u(s)e^{-sz}ds$, where $r(z) \to \infty$ as $z \to \infty$ is due to Braaksma, *cf.* [2].)

Proposition 3.9. — *Let $(-\frac{\pi}{2}, \frac{\pi}{2}) \subset I_q \subset I_{q-1}$ such that $|I_q| > \pi$, $I_q^* \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ and $-I_q^*$ does not contain a singular direction of level* 1. *Let $\mathcal{I} \subset \mathbb{R}$. Then $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ represents a solution f_q of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q))^n$ with the property that $f_q \bmod (\mathcal{A}^{\leqslant -1})^n = f_{q-1}|_{I_q}$.*

Proof. — By Lemma 3.3, for every $\alpha \in I_q^*$ and $\delta_\alpha > 0$ such that $[\alpha - \delta_\alpha, \alpha + \delta_\alpha] \subset I_q^*$, there exist positive numbers B_α and C_α, such that $|u_q(s)| \leqslant C_\alpha e^{e^{B_\alpha |s|}}$ for all $s \in S[\alpha - \delta_\alpha, \alpha + \delta_\alpha]$. Let $\alpha \in I_q^*$, $B'_\alpha > B_\alpha$, $\beta \in (0, \frac{\pi}{2})$, $0 < \delta_\alpha^\beta < \min\{\delta_\alpha, \frac{\pi}{2} - \beta\}$, $\theta \in \mathbb{R}$ and let R be a sufficiently large number such that $\arg\log(ze^{i\theta}) \in [-\delta_\alpha^\beta, \delta_\alpha^\beta]$ for all $z \in S([-\alpha-\beta, -\alpha+\beta], R)$. It is easily seen that, for all $z \in S([-\alpha - \beta, -\alpha + \beta], R)$,

$$|\phi_\alpha^\theta(z) - \int_0^{re^{i\alpha}} u_q(s)e^{-sz}ds| \leqslant Ce^{-\delta|z|}$$

provided R is sufficiently large, where C and $\delta > 0$. Hence it follows that $\phi_\alpha^\theta \in (\mathcal{A}(-\alpha - \frac{\pi}{2}, -\alpha + \frac{\pi}{2}))^n$ and $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ is a representative of $\mathcal{L}(u_q)|_{I_q} = f_{q-1}|_{I_q}$. From Lemma 3.10(ii) below we deduce that $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ also represents an element $f_q \in (\mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q))^n$. Obviously, $f_q \bmod (\mathcal{A}^{\leqslant -1})^n = f_{q-1}|_{I_q}$. The fact that $f_{q-1}|_{I_q}$ is a solution of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leqslant -1}(I_q))^n$ implies that $\{\Delta\phi_\alpha^\theta - \varphi_0 - E(z^{1/p}, \phi_\alpha^\theta) : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ represents an element of $(\mathcal{A}^{\leqslant -1}/\mathcal{A}^{\leqslant -1^+}(I_q))^n$. As $|I_q| > \pi$, by Lemma 3.7, f_q is a solution of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q))^n$. □

Note that Proposition 3.9 also holds if 0 is a singular direction of level 1, provided $0 \notin I_q^*$ (*cf.* Remark 4.13 below).

Lemma 3.10. — *Let I be an open interval of $\mathbb{R}$ and $u : S(I) \to \mathbb{C}$ a holomorphic function, satisfying a growth condition of the form*

$$|u(s)| \leqslant Ce^{e^{B|s|}},$$

where B and C are positive constants.

(i) *Assume that $0 \in I$ and let $I' \prec (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ be an open interval with the property that $\theta_-(I') < \theta_+(I')$. Let $r > 0$, $B' > B$, and $\theta \in \widetilde{I}'$. There exist positive constants C' and δ such that*

$$\Big|\int_r^{\log(ze^{i\theta})/B'} u(s)e^{-sz}ds\Big| \leqslant C'e^{-\delta\frac{|z|}{\log|z|}}$$

for all $z \in \widehat{D}_{I'}(R)$, provided R is sufficiently large.
Let θ_1, $\theta_2 \in \widetilde{I}'$. There exist positive constants C' and c such that

$$\Big|\int_{\log(ze^{i\theta_1})/B'}^{\log(ze^{i\theta_2})/B'} u(s)e^{-sz}ds\Big| \leqslant C'e^{-c|z|}$$

for all $z \in \widehat{D}_{I'}(R)$, provided R is sufficiently large.

(ii) *For $j = 1, 2$, let $B_j > B$, $\theta_j \in \mathbb{R}$, $\alpha_j \in I$ and let $\mathcal{I}_j \prec (-\alpha_j - \frac{\pi}{2}, -\alpha_j + \frac{\pi}{2})$. There exist positive constants C' and δ such that*

$$\Big|\int_0^{e^{i\alpha_1}\log(ze^{i\theta_1})/B_1} u(s)e^{-sz}ds - \int_0^{e^{i\alpha_2}\log(ze^{i\theta_2})/B_2} u(s)e^{-sz}ds\Big| \leqslant C'e^{-\delta|z|\log|z|}$$

for all $z \in S(\mathcal{I}_1 \cap \mathcal{I}_2, R)$, provided R is sufficiently large.

(iii) *Assume that $0 \in I$ and let $I' \prec (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ be an open interval with the property that $\theta_-(I') < \theta_+(I')$. Let $B' > B$, $\theta \in \widetilde{I}'$, $\alpha \in I$ and $I'' \prec (-\alpha - \frac{\pi}{2}, -\alpha + \frac{\pi}{2})$. There exist positive constants C' and c such that*

$$\Big|\int_{\log(ze^{i\theta})/B'}^{e^{i\alpha}\log(ze^{i\theta})/B'} u(s)e^{-sz}ds\Big| \leqslant C'e^{-c|z|}$$

for all $z \in \widehat{D}_{I'}(R) \cap S(I'')$, provided R is sufficiently large.

Proof.
(i) As $\widehat{D}_{I'}(R) \subset S((-\frac{\pi}{2}, \frac{\pi}{2}), R)$, we have, for all $z \in \widehat{D}_{I'}(R)$,

$$\Big|\int_r^{|\log(ze^{i\theta})|/B'} u(s)e^{-sz}ds\Big| \leqslant C(\operatorname{Re} z)^{-1}e^{e^{B|\log(ze^{i\theta})|/B'} - r\operatorname{Re} z}.$$

With the aid of (2.4) it follows that, for all $z \in \widehat{D}_{I'}(R)$,

$$\Big|\int_r^{|\log(ze^{i\theta})|/B'} u(s)e^{-sz}ds\Big| \leqslant C'e^{K_\theta|z|^{\frac{B}{B'}}-\delta'\frac{|z|}{\log|z|}}$$

provided R is sufficiently large, where C', K_θ and $\delta' > 0$. As $\arg\log(ze^{i\theta}) \to 0$ as $|z| \to \infty$, $\log(ze^{i\theta})/B' \in S(I)$ for all $z \in \widehat{D}_{I'}(R)$ if R is sufficiently large. Let $\epsilon \in (0, \frac{\pi}{2})$ such that $(-\epsilon, \epsilon) \subset I$, and take R so large that $|\arg\log(ze^{i\theta})| < \epsilon/2$ for all $z \in \widehat{D}_{I'}(R)$. Let $C_{12}(z)$ denote the arc of the circle $|s| = |\log ze^{i\theta}|/B'$ between $\arg s = 0$ and $\arg s = \arg\log(ze^{i\theta})$. For all $z \in \widehat{D}_{I'}(R)$ such that $|\arg z| \leqslant \frac{\pi}{2} - \epsilon$ we have

$$\begin{aligned}\Big|\int_r^{\log(ze^{i\theta})/B'} u(s)e^{-sz}ds - \int_r^{|\log ze^{i\theta}|/B'} u(s)e^{-sz}ds\Big| \\ &= \Big|\int_{C_{12}(z)} u(s)e^{-sz}ds\Big| \\ &\leqslant C\int_{C_{12}(z)} e^{e^{B|s|}-|s||z|\cos((\pi-\epsilon)/2))}|ds| \\ &\leqslant C'e^{e^{B|\log(ze^{i\theta})|/B'}-|\log ze^{i\theta}||z|\sin(\epsilon/2)/B'},\end{aligned}$$

whereas, for all $z \in \widehat{D}_{I'}(R)$ such that $|\arg z| \geqslant \frac{\pi}{2} - \epsilon$ and $s \in C_{12}(z)$, $|\arg z + \arg s| \leqslant |\arg(z\log ze^{i\theta})|$, hence, with (2.5),

$$\begin{aligned}\Big|\int_{C_{12}(z)} u(s)e^{-sz}ds\Big| &\leqslant C\int_{C_{12}(z)} e^{e^{B|s|}-|s||z|\cos(\arg(z\log(ze^{i\theta})))}|ds| \\ &\leqslant C'e^{e^{B|\log(ze^{i\theta})|/B'}-\mathrm{Re}(\psi_\theta(z))/B'} \\ &\leqslant C'e^{K_\theta|z|^{\frac{B}{B'}}-c'|z|}\end{aligned}$$

where $c' > 0$. From the above estimates the first statement of the lemma follows. Similarly, supposing that $\theta_1 < \theta_2$, we find

$$\begin{aligned}\Big|\int_{\log(ze^{i\theta_1})/B'}^{\log(ze^{i\theta_2})/B'} u(s)e^{-sz}ds\Big| &\leqslant C'e^{K|z|^{\frac{B}{B'}}}\int_{\theta_1}^{\theta_2} e^{-\mathrm{Re}(\psi_\theta(z))/B'}d\theta \\ &\leqslant C''e^{K|z|^{\frac{B}{B'}}-c''|z|}\end{aligned}$$

where C'', K and $c'' > 0$.

(ii) Without loss of generality we may take $\mathcal{I}_j = (-\alpha_j - \beta, -\alpha_j + \beta)$ for $j = 1, 2$, where $0 < \beta < \frac{\pi}{2}$. Let $\epsilon \in (0, \frac{\pi}{2} - \beta)$ such that, for $j = 1, 2$, $(\alpha_j - \epsilon, \alpha_j + \epsilon) \subset I$, and take R so large that $|\arg\log(ze^{i\theta_j})| < \epsilon/2$ for all $z \in S(\mathcal{I}_1 \cap \mathcal{I}_2, R)$. Then, for $j = 1$ and 2, and all $z \in S(\mathcal{I}_1 \cap \mathcal{I}_2, R)$, $\cos(\arg z\log(ze^{i\theta_j}) + \alpha_j) \geqslant \sin(\epsilon/2)$. Let $s_j := e^{i\alpha_j}\log(ze^{i\theta_1})/B_1$, $j = 1, 2$

and let $C_{12}(z)$ denote the arc of the circle $|s|=|\log(ze^{i\theta_1})|/B_1$ between s_1 and s_2. For all $z\in S(\mathcal{I}_1\cap\mathcal{I}_2,R)$, $C_{12}(z)\subset S(I)$ and $\cos(\arg z+\arg s)\geqslant \sin(\epsilon/2)$ for all $s\in C_{12}(z)$, hence

$$\begin{aligned}\Big|\int_0^{s_1} u(s)e^{-sz}ds-\int_0^{s_2}u(s)e^{-sz}ds\Big|&\\ &\leqslant C\int_{C_{12}(z)} e^{e^{B|s|}-|s||z|\cos(\arg z+\arg s)}|ds|\\ &\leqslant C|\alpha_2-\alpha_1||\log z|e^{e^{B|\log(ze^{i\theta_1})|/B_1}-|z\log(ze^{i\theta_1})|\sin(\epsilon/2)/B_1}\\ &=e^{-\sin(\epsilon/2)/B_1|z|\log|z|(1+o(1))}\text{ as } z\to\infty \text{ in } S(\mathcal{I}_1\cap\mathcal{I}_2,R).\end{aligned}$$

Furthermore, supposing that $B_1\leqslant B_2$, we have for all $z\in S(\mathcal{I}_1\cap\mathcal{I}_2,R)$,

$$\begin{aligned}\Big|\int_0^{s_2} u(s)e^{-sz}ds-\int_0^{e^{i\alpha_2}\log(ze^{i\theta_2})/B_2}u(s)e^{-sz}ds\Big|&\\ &\leqslant C\int_{|\log(ze^{i\theta_2})|/B_2}^{|\log(ze^{i\theta_1})|/B_1} e^{e^{B|s|}-\epsilon'|s||z|}|ds|\\ &\leqslant C/(\epsilon'|z|)e^{K|z|^{\frac{B}{B_1}}-\epsilon'|z|\log|z|/B_2}\end{aligned}$$

where $K>0$ and $\epsilon'=\sin(\epsilon/2)$. The statement of the lemma now follows immediately.

(iii) Suppose $0<\alpha<\pi$ and let $C_{12}(z)$ denote the arc of the circle $|s|=|\log(ze^{i\theta})|/B'$ between $\log(ze^{i\theta})/B'$ and $e^{i\alpha}\log(ze^{i\theta})/B'$. In view of (ii) it suffices to consider $z\in\widehat{D}_{I'}(R)$ with the property that $\arg z\leqslant -\alpha/2-\delta$, where $0<\delta<(\pi-\alpha)/2$. Then $\arg(z\log(ze^{i\theta}))\leqslant\arg z+\arg s\leqslant\alpha/2-\delta+\arg\log(ze^{i\theta})$. As $-\pi/2<\arg(z\log(ze^{i\theta}))\leqslant-\alpha/2-\delta+\arg\log(ze^{i\theta})<\alpha/2+\delta-\arg\log(ze^{i\theta})$ if R is sufficiently large, $\cos(\arg z+\arg s)\geqslant\cos\arg(z\log(ze^{i\theta}))$ for all $s\in C_{12}(z)$, provided R is sufficiently large. Then we have, with (2.5),

$$\begin{aligned}\Big|\int_{C_{12}(z)}u(s)e^{-sz}ds\Big|&\\ &\leqslant C\int_{C_{12}(z)} e^{e^{B|\log(ze^{i\theta})|/B'}-|z\log(ze^{i\theta})|\cos(\arg z+\arg s)/B'}|ds|\\ &\leqslant C'e^{e^{B|\log(ze^{i\theta})|/B'}-\mathrm{Re}(\psi_\theta(z))/B'}\\ &\leqslant C'e^{K_\theta|z|^{\frac{B}{B'}}-c'|z|}\end{aligned}$$

where C', K_θ and $c'>0$. The proof for the case that $-\pi<\alpha<0$ is similar. If $|\alpha|\geqslant\pi$, then $\widehat{D}_{I'}(R)\cap S(I'')=\emptyset$ for all $R>1$. □

In order to establish the accelero-summability of the formal solution of (2.7) in the case that 0 is not a singular direction of level 1, it remains to prove the existence of a solution $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$ with the properties mentioned in definition 3.5. This will be done by suitably modifying some of the functions ϕ_0^θ defined above.

Proposition 3.11. — *Let I_{q+1} be a large interval such that $\widetilde{I_{q+1}} \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q$ and $\widetilde{I_{q+1}} \cap \Theta(\Delta^c) = \emptyset$. Assume that $-I_q^*$ does not contain a singular direction of level 1. Then equation (2.7) has a solution $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$ with the properties mentioned in definition 3.5.*

Proof. — Let $\theta \in \widetilde{I_{q+1}}$ and $y = w + \phi_0^\theta$. Then $(\phi_-(\theta), \phi_+(\theta)) \subset I_{q+1} \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ and y is a solution of the equation (2.7) if and only if w satisfies the equation

$$\Delta w(z) = G(z, w(z)) := E(z^{1/p}, w(z) + \phi_0^\theta(z)) + \varphi_0(z^{1/p}) - \Delta\phi_0^\theta(z). \tag{3.8}$$

By Lemma 2.11, the function ϕ_r defined by $\phi_r(z) = \int_0^r u_q(s)e^{-sz}ds$, is an element of $(\widehat{\mathcal{A}}(\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})))^n \subset (\widehat{\mathcal{A}}(\phi_-(\theta), \phi_+(\theta)))^n$. From Lemma 3.10 (i) we infer that $\phi_0^\theta - \phi_r \in (\widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta), \phi_+(\theta)))^n$, hence $\phi_0^\theta \in (\widehat{\mathcal{A}}(\phi_-(\theta), \phi_+(\theta)))^n$. With proposition 3.9, (2.11) and Lemma 3.10 (iii) it follows that $G(z, 0) = E(z^{1/p}, \phi_0^\theta(z)) + \varphi_0(z^{1/p}) - \Delta\phi_0^\theta(z) \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta), \phi_+(\theta)))^n$. According to [12, Theorem 1.2], with $I = \{\theta\}$, $k = 1^+$ and $l = 0$, the equation (3.8) has a solution $w_\theta \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta), \phi_+(\theta)))^n$. Now, let $\theta_1, \theta_2 \in \widetilde{I_{q+1}}$, $\theta_1 < \theta_2$, and let $y_i = w_{\theta_i} + \phi_0^{\theta_i}$, $i = 1, 2$. As $w_{\theta_i} \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_i), \phi_+(\theta_i)))^n$ for $i = 1, 2$, and, by Lemma 3.10(i), $\phi_0^{\theta_1} - \phi_0^{\theta_2} \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1), \phi_+(\theta_2)))^n$, $y_1 - y_2 \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1), \phi_+(\theta_2)))^n$ as well. Both y_1 and y_2 are solutions of the nonlinear difference equation (2.7), so the difference $y_1 - y_2$ satisfies a homogeneous linear equation of the form (2.10), with $H \in \mathrm{End}(n; (\widehat{\mathcal{A}}(\phi_-(\theta_1), \phi_+(\theta_2)))$. By proposition 2.15, $\widetilde{\Delta}$ and Δ have a common canonical form Δ^c if N is sufficiently large and so, by Corollary 2.14, $\mathrm{Ker}(\widetilde{\Delta}, (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1), \phi_+(\theta_2)))^n) = \{0\}$. It follows that the solutions $w_\theta + \phi_0^\theta$, with $\theta \in \widetilde{I_{q+1}}$, can be glued together, to define an analytic function $f_{q+1} \in \cap_{\theta \in \widetilde{I_{q+1}}} (\widehat{\mathcal{A}}(\phi_-(\theta), \phi_+(\theta)))^n = \widehat{\mathcal{A}}(I_{q+1})^n$. □

Remark 3.12. — Let $I \prec I_{q+1}$ be a large interval, let $\theta \in \widetilde{I}$ and let $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$ be the unique solution of (1.1) with the properties mentioned in definition 3.5. Then the function $u_{q+1,\theta}$ defined by

$$u_{q+1,\theta}(t) = \frac{1}{2\pi i} \int_{\delta \widehat{D}_I(R)} f_{q+1}(z) e^{t\psi_\theta(z)} d\psi_\theta(z), \quad \arg t = 0$$

where R is a sufficiently large positive number and $\delta\widehat{D}_I(R)$ is described in the direction of increasing imaginary part, is *quasi-analytic* on the half line $\arg t = 0$. $u_{q+1,\theta}$ is a so-called *weak accelerate* of $u_q = \mathcal{B}(f_{q-1})$. $u_{q+1,\theta}$ has exponential growth as $t \to \infty$ and f_{q+1} can be represented by the Laplace integral

$$f_{q+1}(z) = y_0 + \int_0^\infty u_{q+1,\theta}(t) e^{-t\psi_\theta(z)} dt, \quad \operatorname{Re}\psi_\theta(z) \geqslant c_\theta$$

where $c_\theta > 0$.

For a very general discussion of weak acceleration operators and their properties we refer the reader to [5, 7].

4. The general case

Let I_{q-1} be an open interval of $\mathbb{R}$ such that $|I_{q-1}| > \pi/k_{q-1}$ and $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_{q-1}$, and $f_{q-1} \in (\mathcal{A}/\mathcal{A}^{\leqslant -1}(I_{q-1}))^n$. In the case that 0 is a singular direction of Δ^c, of level 1, the matrix $((e^{-t}-1)I_{n_q} - A_q)^{-1}$ in (3.3) has a singularity on the half line $\arg t = 0$ and, consequently, the Borel transform u_q of f_{q-1} cannot be continued analytically to this half line. In order to "bypass" in some sense, possible singularities of u_q on $\arg t = 0$, we introduce a variable $r_\theta(z)$, *equivalent* to z in the sense that $\lim_{z\to\infty} r_\theta(z) z^{-1} = 1$.

Definition 4.1. — *For all $z \in S((-\pi,\pi),1)$ and $\theta \in \mathbb{R}$ we define*

$$r_\theta(z) = \frac{\psi_\theta(z)}{\log z} \text{ and } \rho_\theta(z) = \operatorname{Re} r_\theta(z).$$

We can illustrate the "bypassing" of a singularity on the half line $\arg t = 0$ with the following, very simple example.

Example 4.2. — For any $\theta \in \mathbb{R}$ and $R > 1$, the function

$$\phi_\theta(t) = \int_R^\infty e^{-z + t r_\theta(z)} dr_\theta(z)$$

is analytic in the half plane $\operatorname{Re} t < 1$. For $\theta = 0$ we have

$$\phi_0(t) = \int_R^\infty e^{(t-1)z} dz = -\frac{e^{R(t-1)}}{t-1},$$

so ϕ_0 has a simple pole at 1. For any $\theta \neq 0$, however, ϕ_θ can be continued to a quasi-analytic function on the positive real axis. Let us consider the case that $\theta > 0$. By deformation of the path of integration we get, if $\operatorname{Im} t > 0$,

$$\phi_\theta(t) = \int_R^{R+i\infty} e^{-z + t r_\theta(z)} dr_\theta(z).$$

Noting that, for all z on the line $\operatorname{Re} z = R$ and all $t \in \mathbb{R}$,

$$\operatorname{Re}(-z + tr_\theta(z)) = (t - 1 + \frac{\theta t \arg z}{|\log z|^2})R - \frac{\theta t \operatorname{Im} z \log |z|}{|\log z|^2}$$

one easily verifies that the function defined by the right-hand side is continuous on the half plane $\operatorname{Im} t \geqslant 0$ and C^∞ on $\operatorname{Im} t = 0$. Moreover, for any closed interval $[a, b] \subset (0, \infty)$, there exist positive constants K and A such that

$$\begin{aligned}|\phi_\theta^{(m)}(t)| &\leqslant K \int_R^{R+i\infty} |r_\theta(z)|^m e^{-\frac{\theta t \operatorname{Im} z \log |z|}{|\log z|^2}} |dr_\theta(z)| \\ &\leqslant KA^m (m \log m)^m \text{ for all } m \in \mathbb{N} \text{ and all } t \in [a, b].\end{aligned}$$

This implies that ϕ_θ belongs to the Denjoy class ${}^1D[a, b]$ and thus is quasi-analytic on the positive real axis.

For every $\theta \in \mathbb{R}$, the function ϕ_θ has exponential growth of order 1 as $t \to \infty$ on the positive real axis. The function f_θ, defined by the Laplace integral

$$f_\theta(z) := \int_0^\infty \phi_\theta(t) e^{-tr_\theta(z)} dt$$

is analytic in a domain of the form $\rho_\theta(z) > K' > 0$. In the case that $\theta = 0$ this obviously is a right half plane. In general, as we shall see, it contains a domain of the form $\widehat{D}_I(R)$ for every interval $I \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$. If $\theta \neq 0$ it does not contain a half plane, but is slightly 'tilted' and contains a part of either the positive (if $\theta < 0$) or negative (if $\theta > 0$) imaginary axis (*cf.* Figures 4.1 and 4.2).

Remark 4.3. — There is a certain amount of freedom in the choice of the variable $r_\theta(z)$. However, the "perturbation" $z - r_\theta(z)$ shouldn't be too small. If, in example 4.2, $r_\theta(z)$ is defined as $r_\theta(z) = z + \frac{i\theta z}{(\log z)^2}$, we obtain the estimate

$$|\phi_\theta^{(m)}(1)| = m^m (\log m)^{2m} (\theta e)^{-m+o(m)} \text{ as } m \to \infty$$

which implies that ϕ_θ is not quasi-analytic on any interval of the positive real axis containing 1. Moreover, in order to deal with the level 1^+, the set $\operatorname{Re} r_\theta(z) \geqslant 0$ (with $\theta \neq 0$) should contain a domain of the form $\widehat{D}_I(R)$, where I is a large interval. This rules out larger perturbations of z like $r_\theta(z) = z + \frac{i\theta z}{\log \log z}$. On the other hand, an alternative definition of the type $r_\theta(z) = z + (i\theta - 1)\frac{z}{\log z}$ would yield completely analogous results to those obtained with Definition 4.1, provided $\theta \neq 0$. The case $\theta = 0$ corresponds to a "pseudodeceleration" in the terminology used by Ecalle, serving to regularize the singularities of u_q (*cf.* [6]).

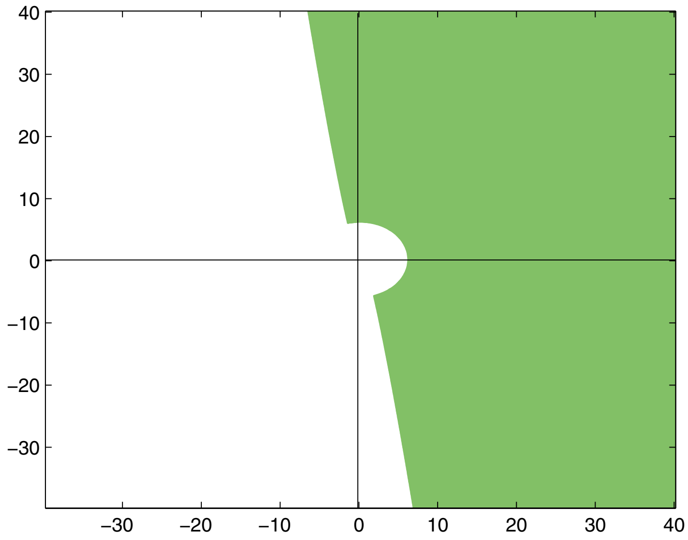

Figure 4.1. The domain $\widehat{D}_{(\phi_-(\theta+\frac{\pi}{2}),\phi_+(\theta-\frac{\pi}{2}))}(6)$ *with* $\theta=-\frac{\pi}{4}$

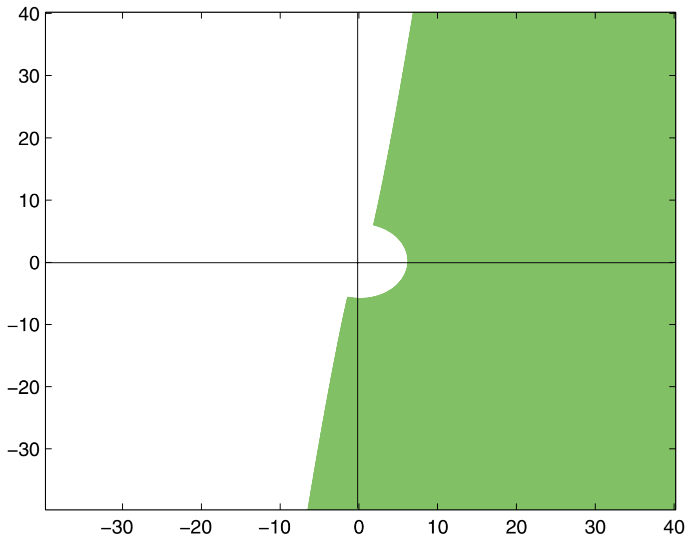

Figure 4.2. The domain $\widehat{D}_{(\phi_-(\theta+\frac{\pi}{2}),\phi_+(\theta-\frac{\pi}{2}))}(6)$ *with* $\theta=\frac{\pi}{4}$.

As the convolution equations obtained from (2.7) by applying a Borel transformation with respect to the variable $r_\theta(z)$, in the case that $\theta\neq 0$, appear quite unwieldy, we take a different approach here, more along the lines of the proof given by Ramis and Sibuya in [19]. However, instead of using an existence theorem for ordinary, analytic solutions of nonlinear difference equations, we use an existence theorem for *quasi-function* solutions, which considerably simplifies the argument. In this subsection we introduce 1-precise and 1^+-precise quasi-functions ‘of the second kind’, which, instead of being defined on sectors, are defined on domains of the type $\widehat{D}_I(R)$ and represent sections of the quotient sheaves $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}$ and $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}$. We show that $f_{q-1}|_{(-\frac{\pi}{2},\frac{\pi}{2})}$ has a particular representative, which also represents an element of $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(\mathbb{R}))^n$. Now, let I_q be a large interval such that $|\widetilde{I_q}| > \pi$. On every large subinterval I of I_q such that $|\widetilde{I}| \leqslant \pi$, we can modify this representative by means of exponentially small, 1^+-precise *quasi-function solutions* of an associated difference equation, using a recent existence result for this type of solutions (Theorem 4.16), and obtain a solution of (2.7) in $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$. Moreover, this has the property that its restriction to any large subsector I' of I is represented by a solution of (2.7) in $(\widehat{\mathcal{A}}(I'))^n$, provided $\Theta(\Delta^c) \cap \widetilde{I'} = \emptyset$. Due to the fact that the difference of two solutions of (2.7) satisfies a homogeneous linear difference equation of the form (2.10) and by virtue of Theorem 2.13, these solutions can be glued together, resulting in a solution $f_q \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I_q))^n$, with the property that $f_q|_{I_{q+1}}$ is represented by a solution $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$, provided $\Theta(\Delta^c) \cap \widetilde{I_{q+1}} = \emptyset$. f_{q+1} is an accelero-sum of $\hat{f}$ in a slightly weaker sense than that of Definition 3.5 (*cf.* Definition 4.11 below).

4.1. The quotient sheaves $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}$ and $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}$

Definition 4.4 (1-precise and 1^+-precise quasi-functions). — *Let I be an interval of $\mathbb{R}$. A 1-precise quasi-function (of the second kind) on I is a collection of functions $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \widehat{\mathcal{A}}(\mathcal{I}_\nu)$, $\{\mathcal{I}_\nu : \nu \in \mathcal{N}\}$ is an open covering of I, and $\phi_\nu - \phi_{\nu'} \in \widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_\nu \cap \mathcal{I}_{\nu'})$ for all ν and $\nu' \in \mathcal{N}$. Two 1-precise quasi-functions $\{\phi_\nu \in \widehat{\mathcal{A}}(\mathcal{I}_\nu) : \nu \in \mathcal{N}\}$ and $\{\psi_\mu \in \widehat{\mathcal{A}}(\mathcal{I}'_\mu) : \mu \in \mathcal{M}\}$ on I are considered equivalent if $\phi_\nu - \psi_\mu \in \widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_\nu \cap \mathcal{I}'_\mu)$, for all $\nu \in \mathcal{N}$ and all $\mu \in \mathcal{M}$.*

Similarly, a 1^+-precise quasi-function (of the second kind) on I is a collection of functions $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \widehat{\mathcal{A}}(\mathcal{I}_\nu)$, $\{\mathcal{I}_\nu : \nu \in \mathcal{N}\}$ is an open covering of I, and $\phi_\nu - \phi_{\nu'} \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_\nu \cap \mathcal{I}_{\nu'})$ for all ν and $\nu' \in \mathcal{N}$. Two 1^+-precise quasi-functions $\{\phi_\nu \in \widehat{\mathcal{A}}(\mathcal{I}_\nu) : \nu \in \mathcal{N}\}$ and $\{\psi_\mu \in \widehat{\mathcal{A}}(\mathcal{I}'_\mu) : \mu \in \mathcal{M}\}$ on I are considered equivalent if $\phi_\nu - \psi_\mu \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_\nu \cap \mathcal{I}'_\mu)$, for all $\nu \in \mathcal{N}$ and all $\mu \in \mathcal{M}$.

Obviously, 1-precise and 1^+-precise quasi-functions of the second kind represent sections of the quotient sheaves $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}$ and $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}$, respectively. The following two lemma's provide us with the necessary link between $\mathcal{A}/\mathcal{A}^{\leqslant -1}$ and $\mathcal{A}/\mathcal{A}^{\leqslant -1^+}$ on one hand and $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}$ and $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}$ on the other.

Lemma 4.5. — *For every $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1}([-\frac{\pi}{2}, \frac{\pi}{2}])$ there exists a global section $\widetilde{f}$ of $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1})$, represented by a 1-precise quasi-function $\{\tilde{\varphi}_\theta : \theta \in \mathbb{R}\}$ of the second kind, with the following properties:*

(i) $\tilde{\varphi}_\theta \in \widehat{\mathcal{A}}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$,

(ii) *If $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}(\mathcal{I}_\nu)$ and $\{\mathcal{I}_\nu : \nu \in \mathcal{N}\}$ is an open covering of $[-\frac{\pi}{2}, \frac{\pi}{2}]$, is a representative of f, then, for any open interval $I'_\nu \prec \mathcal{I}_\nu$ and any $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive constants R, c and C such that*

$$|\tilde{\varphi}_\theta(z) - \phi_\nu(z)| \leqslant C e^{-c \frac{|z|}{\log |z|}} \tag{4.1}$$

for all $z \in \widehat{D}_{I'}(R) \cap S(I'_\nu)$.

(iii) $\tilde{\varphi}_{\theta_1} - \tilde{\varphi}_{\theta_2} \in \widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_2 - \frac{\pi}{2}))$ *if* $\theta_1 < \theta_2$.

Lemma 4.6. — *For every $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1^+}([-\frac{\pi}{2}, \frac{\pi}{2}])$ there exists $\widetilde{f}^+ \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+})(\mathbb{R})$, represented by a 1^+-precise quasi-function $\{\tilde{\varphi}^+_\theta : \theta \in \mathbb{R}\}$ of the second kind, with the following properties:*

(i) $\tilde{\varphi}^+_\theta \in \widehat{\mathcal{A}}(\phi_-(\theta), \phi_+(\theta))$,

(ii) *If $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}(\mathcal{I}_\nu)$ and $\{\mathcal{I}_\nu : \nu \in \mathcal{N}\}$ is an open covering of $[-\frac{\pi}{2}, \frac{\pi}{2}]$, is a representative of f, then, for any $I'_\nu \prec \mathcal{I}_\nu$ and any $I' \prec (\phi_-(\theta), \phi_+(\theta))$, there exist positive constants R, c and C such that*

$$|\tilde{\varphi}^+_\theta(z) - \phi_\nu(z)| \leqslant C e^{-c|z|} \tag{4.2}$$

for all $z \in \widehat{D}_{I'}(R) \cap S(I'_\nu)$.

(iii) $\tilde{\varphi}^+_{\theta_1} - \tilde{\varphi}^+_{\theta_2} \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1), \phi_+(\theta_2))$ *if* $\theta_1 < \theta_2$.

Remark 4.7. — Let $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1}([-\frac{\pi}{2}, \frac{\pi}{2}])$ and $g \in \mathcal{A}/\mathcal{A}^{\leqslant -1^+}([-\frac{\pi}{2}, \frac{\pi}{2}])$ such that $f = g \bmod \mathcal{A}^{\leqslant -1}$. Let $\widetilde{f}$ and $\widetilde{g}^+$ denote the corresponding elements of $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1})(\mathbb{R})$ and $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+})(\mathbb{R})$, respectively. Then it is easily seen that $\widetilde{f} = \widetilde{g}^+ \bmod \widehat{\mathcal{A}}^{\leqslant -1}$.

Definition 4.8. — *Let $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1}([-\frac{\pi}{2}, \frac{\pi}{2}])$ or $f \in \mathcal{A}/\mathcal{A}^{\leqslant -1^+}([-\frac{\pi}{2}, \frac{\pi}{2}])$, respectively, and let $\widetilde{f}$ or $\widetilde{f}^+$ denote the corresponding elements of $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1})(\mathbb{R})$ and $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+})(\mathbb{R})$, respectively. Let I be an interval of $\mathbb{R}$.*

Then by $\tilde{f|}_I$ *we denote* $\tilde{f}|_I$ $(\in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(I))$, *or* $\tilde{f}^+|_I$ $(\in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))$, *respectively.*

In order to prove Lemma's 4.5 and 4.6, we first derive some asymptotic properties of ρ_θ. A straightforward computation shows that, for all θ and $\theta' \in \mathbb{R}$,

$$\rho_\theta(z) = \frac{\{(\operatorname{Re}\psi_{\theta'}(z) + (\theta' - \theta + \arg z)\operatorname{Im} z\}\log|z| + \arg z(\arg z + \theta)\operatorname{Re} z}{|\log z|^2}.$$

Hence we deduce the estimates

$$\rho_\theta(z) = \frac{(\theta - \theta' + \frac{1}{2}\pi)|z|}{\log|z|}\Big(1 + O(\frac{1}{\log|z|})\Big) \text{ as } z \to \infty \text{ on } \widehat{C}_{\phi_-(\theta')}(R) \tag{4.3}$$

valid for any real $\theta' \neq \theta + \frac{\pi}{2}$ and all sufficiently large R, and

$$\rho_\theta(z) = \frac{(\theta' - \theta + \frac{1}{2}\pi)|z|}{\log|z|}\Big(1 + O(\frac{1}{\log|z|})\Big) \text{ as } z \to \infty \text{ on } \widehat{C}_{\phi_+(\theta')}(R) \tag{4.4}$$

valid for $\theta' \neq \theta - \frac{\pi}{2}$ and sufficiently large R (*cf.* [12]). Furthermore, for any interval $I \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive numbers R and δ such that

$$\rho_\theta(z) \geqslant \delta\frac{|z|}{\log|z|} \text{ for all } z \in \widehat{D}_I(R). \tag{4.5}$$

If, in addition, $0 \notin \overline{I}$, there exist positive numbers R, δ_1 and δ_2 such that

$$\delta_1\frac{|z|}{\log|z|} \leqslant \rho_\theta(z) \leqslant \delta_2\frac{|z|}{\log|z|} \text{ for all } z \in \widehat{D}_I(R). \tag{4.6}$$

From [12, Lemma 0.13]) and (4.6) we deduce the following result.

Lemma 4.9.

1. *Let* $\theta \in \mathbb{R}$, $I = (a, b)$ *such that* $a \neq 0 \neq b$, $I \prec (\phi_-(\theta + \frac{1}{2}\pi), \phi_+(\theta - \frac{1}{2}\pi))$ *and let* R *be a sufficiently large number. Let* $f : \widehat{D}_I(R) \to \mathbb{C}$ *be a continuous function, holomorphic in int* $\widehat{D}_I(R)$. *Then the following statements are equivalent.*
   (i) *There exist positive numbers* c *and* C, *such that, for all* $z \in \widehat{D}_I(R)$,
   $$|f(z)| \leqslant Ce^{-c\rho_\theta(z)}.$$
   (ii) *There exist positive numbers* δ *and* C, *such that, for all* $z \in \widehat{D}_I(R)$,
   $$|f(z)| \leqslant Ce^{-\delta\frac{|z|}{\log|z|}}.$$
2. *Let* I *be a large, open interval such that* $|\tilde{I}| > \pi$. *Then* $\widehat{\mathcal{A}}^{\leqslant -1}(I) = \{0\}$.

Remark 4.10. — From Lemma's 4.9 and 2.12 we deduce that, for any open interval I containing 0, $\widehat{\mathcal{A}}^{\leqslant -1}(I) \subset \mathcal{A}^{\leqslant -1}(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\widehat{\mathcal{A}}^{\leqslant -1^+}(I) \subset \mathcal{A}^{\leqslant -1^+}(-\frac{\pi}{2}, \frac{\pi}{2})$. Obviously, $e^{-cr_\theta(z)} \in \widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta + \frac{1}{2}\pi), \phi_+(\theta - \frac{1}{2}\pi))$ for all $c > 0$ and $\theta \in \mathbb{R}$. Statement 2 of Lemma 4.9 extends the well-known result that $\mathcal{A}^{\leqslant -1}([-\frac{\pi}{2}, \frac{\pi}{2}]) = \{0\}$.

Proof of Lemma 4.5. — Let $\{\phi_\nu : \nu \in \{1, \dots, N\}\}$ be a representative of f, with respect to a "good" open covering $\{\mathcal{I}_\nu : \nu \in \{1, \dots, N\}\}$ of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (*i.e.* $\mathcal{I}_\nu \cap \mathcal{I}_\mu = \emptyset$ unless $|\nu - \mu| = 1$), such that $\inf \mathcal{I}_\nu < \inf \mathcal{I}_{\nu+1}$ for $\nu = 1, \dots, N-1$. For all $\nu \in \{1, \dots, N\}$ let $I'_\nu \prec \mathcal{I}_\nu$, such that $\{I'_\nu : \nu \in \{1, \dots, N\}\}$ is a good, open covering of $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Choose open subintervals I''_ν of $\mathcal{I}_\nu$, such that $I'_\nu \prec I''_\nu \prec \mathcal{I}_\nu$ for $\nu \in \{1, \dots, N\}$ and let $\alpha_0 = \inf I''_1$, $\alpha_N = \sup I''_N$, $\alpha_\nu \in I''_\nu \cap I''_{\nu+1} \setminus (I'_\nu \cap I'_{\nu+1})$ for $\nu = 1, \dots, N-1$, and let $R > 0$ such that ϕ_ν is analytic and bounded on $S(I''_\nu, R)$ for all $\nu \in \{1, \dots, N\}$. There exist C' and $c' > 0$ such that, for all $z \in S(I''_\nu \cap I''_{\nu+1}, R)$ and $\nu = 1, \dots, N-1$,

$$|\phi_\nu(z) - \phi_{\nu+1}(z)| \leqslant C' e^{-c'|z|}.$$

Let $r < c'$. Then the function $\tilde{\varphi}_\theta$ defined, for all $z \in \cup_{\nu=1,\dots,N} S((\alpha_{\nu-1}, \alpha_\nu), R)$, by

$$\tilde{\varphi}_\theta(z) = \sum_{\nu=1}^{N} \int_{C_\nu} \frac{e^{r(r_\theta(\zeta) - r_\theta(z))} - 1}{2\pi i (r_\theta(\zeta) - r_\theta(z))} \phi_\nu(\zeta) dr_\theta(\zeta) + \sum_{\nu=0}^{N} \int_{\mathrm{Re}^{i\alpha_\nu}}^{\infty e^{i\alpha_\nu}} \frac{e^{r(r_\theta(\zeta) - r_\theta(z))} - 1}{2\pi i (r_\theta(\zeta) - r_\theta(z))} (\phi_\nu - \phi_{\nu+1})(\zeta) dr_\theta(\zeta)$$

where C_ν denotes the arc of the circle $|z| = R$ from $\mathrm{Re}^{i\alpha_{\nu-1}}$ to $\mathrm{Re}^{i\alpha_\nu}$ and $\phi_0 \equiv \phi_{N+1} \equiv 0$, can be analytically continued to $S((\alpha_0, \alpha_N), R)$ by deformation of the paths of integration. Moreover, with Cauchy's theorem it follows that, for all $z \in S(I'_\nu, R)$

$$\tilde{\varphi}_\theta(z) - \phi_\nu(z) = \frac{e^{-rr_\theta(z)}}{2\pi i} \left(\sum_{\mu=1}^{N} \int_{C_\mu} \frac{e^{rr_\theta(\zeta)}}{r_\theta(\zeta) - r_\theta(z)} \phi_\mu(\zeta) dr_\theta(\zeta) + \sum_{\mu=0}^{N} \int_{\mathrm{Re}^{i\alpha_\mu}}^{\infty e^{i\alpha_\mu}} \frac{e^{rr_\theta(\zeta)}}{r_\theta(\zeta) - r_\theta(z)} (\phi_\mu - \phi_{\mu+1})(\zeta) dr_\theta(\zeta) \right).$$

Hence we deduce, with the aid of (4.5), that for any interval $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive constants c and C such that

$$|\tilde{\varphi}_\theta(z) - \phi_\nu(z)| \leqslant C e^{-c\frac{|z|}{\log|z|}}$$

for all $z \in \widehat{D}_{I'}(R) \cap S(I'_\nu)$. It is easily seen that $\tilde{\varphi}_\theta$ is independent of the choice of representative and the covering $\{\mathcal{I}_\nu : \nu \in \{1,\dots,N\}\}$. As $\widehat{D}_{I'}(R) \subset \cup_{\nu\in\{1,\dots,N\}} S(I'_\nu)$ for all sufficiently large R, it follows that $\tilde{\varphi}_\theta \in \widehat{\mathcal{A}}(\phi_-(\theta+\frac{\pi}{2}), \phi_+(\theta-\frac{\pi}{2}))$.

Now, let θ_1, $\theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2$. Using the estimates for $\tilde{\varphi}_{\theta_i} - \phi_\nu$ derived above and varying the I'_ν, one easily shows that $\tilde{\varphi}_{\theta_1} - \tilde{\varphi}_{\theta_2} \in \widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_2 - \frac{\pi}{2}))$. □

Note that, for any $\theta \in \mathbb{R}$, $\tilde{\varphi}_\theta$ is a representative of $f|_{(-\frac{\pi}{2},\frac{\pi}{2})}$. Lemma 4.6 can be proved similarly, with the aid of (2.5).

4.2. Accelero-summability of $\hat{f}$ in the general case

Definition 4.11 (accelero-summability (generalization)). — *Let $0 = k_0 < k_1 < \cdots < k_q = 1$, $I_0 = \mathbb{R}$ and let I_h, $h = 1,\dots,q+1$, be open intervals of $\mathbb{R}$ with the following properties:*

- $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_{q-1} \subset \cdots \subset I_1$ *and* $I_{q+1} \subset I_q$.
- $|I_h| > \frac{\pi}{k_h}$ *for* $h = 1,\dots,q-1$, I_q *and* I_{q+1} *are large intervals and* $|\widetilde{I_q}| > \pi$.

$\hat{f} \in \mathbb{C}[[z^{-1/p}]]$ is called $(k_1,\dots,k_q,1^+)$-summable on $(I_1,\dots,I_{q+1})$ with $(k_1,\dots,k_q,1^+)$-sum f_{q+1}, if there exist $f_h \in \mathcal{A}/\mathcal{A}^{\leqslant -k_{h+1}}(I_h)$, $h = 0,\dots,q-1$, $f_q \in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I_q)$ and $f_{q+1} \in \widehat{\mathcal{A}}(I_{q+1})$, with asymptotic expansion $\hat{f}$, such that

- $f_0(ze^{2p\pi i}) = f_0(z)$,
- $f_{h-1}|_{I_h} = f_h \bmod \mathcal{A}^{\leqslant -k_h}$, $h = 1,\dots,q-1$,
- $f_{q-1}|_{\widetilde{I_q}} = f_q \bmod \widehat{\mathcal{A}}^{\leqslant -1}$,
- $f_q|_{I_{q+1}} = f_{q+1} \bmod \widehat{\mathcal{A}}^{\leqslant -1^+}$.

Note that the role played by I_q in this definition is quite different from that in definition 3.5. Here, we consider sections over I_q of $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}$ and in definition 3.5 sections of $\mathcal{A}/\mathcal{A}^{\leqslant -1^+}$.

From Lemma 2.12, 2 we deduce that f_{q+1} is determined uniquely by f_q. For, suppose $g_{q+1} \in \widehat{\mathcal{A}}(I_{q+1})$ has the same properties as f_{q+1}. Then $f_{q+1} - g_{q+1} \in \widehat{\mathcal{A}}^{\leqslant -1^+}(I_{q+1})$, and thus, by Lemma 2.12, 2, $g_{q+1} \equiv f_{q+1}$. Similarly, it can be deduced from Lemma 4.14 below that f_q is determined uniquely by f_{q-1}. Suppose $g_q \in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I_q)$ has the same properties as f_q. Then $f_q - g_q \in \widehat{\mathcal{A}}^{\leqslant -1}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I_q)$ and from Lemma 4.14 it follows that $f_q = g_q$.

Now let I_h, $h = 1,\dots,q+1$, be open intervals of $\mathbb{R}$, satisfying the conditions of definition 3.5. Assume that $\hat{f}$ is $(k_1,\dots,1,1^+)$-summable on

$(I_1, \ldots, I_q, I_{q+1})$ according to definition 3.5. With the aid of Lemma's 4.5, 4.6 and Remark 4.7, replacing f_q by $\widetilde{f_q}^+$, it is easily seen that $\hat{f}$ is $(k_1, \ldots, 1, 1^+)$-summable on $(I_1, \ldots, I_{q-1}, \mathbb{R}, I_{q+1})$ according to definition 4.11.

The main result of this paper is stated in the following theorem.

Theorem 4.12. — *Let F be a $\mathbb{C}^n$-valued function, analytic in a neighbourhood of (∞, y_0) for some $y_0 \in \mathbb{C}^n$. Suppose that (1.1) has a formal solution $\hat{f} \in \mathbb{C}^n[[z^{-1/p}]]$, with constant term y_0, such that (1.2) holds, and that the corresponding difference operator $\widehat{\Delta} = \tau - \widehat{A}$ has positive levels $k_1 < \cdots < k_q = 1$ and a level 1^+. Let I_h, $h = 1, \ldots, q+1$, be open intervals of $\mathbb{R}$ with the following properties:*

- $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$ *for $h = 1, \ldots, q-1$ and I_{q+1} is a large interval.*
- *I_h does not contain a Stokes pair of level k_h for $h = 1, \ldots, q-1$.*
- $\widetilde{I_{q+1}} \cap \Theta(\widehat{\Delta}) = \emptyset$.

I. *Suppose that 0 is not a singular direction of level 1 and $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q \subset I_{q-1}$. Then $\hat{f}$ is $(k_1, \ldots, k_{q-1}, 1, 1^+)$-summable on $(I_1, \ldots, I_{q+1})$ in the sense of definition 3.5, and the sum is a solution of (1.1).*

II. *Suppose that 0 is a singular direction of level 1, $I_q = (\phi_-(\frac{\pi}{2}), \infty)$ or $(-\infty, \phi_+(-\frac{\pi}{2}))$ and $I_{q+1} \subset I_q$. Then $\hat{f}$ is $(k_1, \ldots, k_{q-1}, 1, 1^+)$-summable on $(I_1, \ldots, I_{q+1})$ in the sense of definition 4.11, and the sum is a solution of (1.1).*

Remark 4.13. — Let $I \prec I_{q+1}$ be a large interval, let $\theta' \in \widetilde{I}$ and let $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$ be the unique solution of (1.1) with the properties mentioned in definition 4.11. Then the function $u_{q+1,\theta'}$ defined by

$$u_{q+1,\theta'}(t) = \frac{1}{2\pi i} \int_{\delta \widehat{D}_I(R)} f_{q+1}(z) e^{t\psi_{\theta'}(z)} d\psi_{\theta'}(z), \quad \arg t = 0$$

where R is a sufficiently large positive number and $\delta \widehat{D}_I(R)$ is described in the direction of increasing imaginary part, is quasi-analytic on the half line $\arg t = 0$ (like ϕ_θ in Example 4.2 it belongs to the Denjoy class ${}^1D[a, b]$ for any closed interval $[a, b] \subset (0, \infty)$). Let $\theta \in -\widetilde{I_q}^*$ (i.e. $\theta < 0$ if $I_q = (\phi_-(\frac{\pi}{2}), \infty)$, $\theta > 0$ if $I_q = (-\infty, \phi_+(-\frac{\pi}{2}))$) and let $u_{q,\theta} := \mathcal{B}_{1,\theta}(f_{q-1})$ denote the Borel transform of f_{q-1} with respect to the variable $r_\theta(z)$, defined by

$$u_{q,\theta}(t) = \frac{1}{2\pi i}\left(\sum_{\nu=1}^{N}\int_{C_\nu}\phi_\nu(z)e^{tr_\theta(z)}dr_\theta(z) + \sum_{\nu=0}^{N}\int_{\mathrm{R}e^{i\alpha_\nu}}^{\infty e^{i\alpha_\nu}}(\phi_\nu-\phi_{\nu+1})(z)e^{tr_\theta(z)}dr_\theta(z)\right) \tag{4.7}$$

where $\{\phi_\nu : \nu \in \{1,\dots,N\}\}$ is a representative of $f_{q-1}|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ with respect to a good, open covering $\{\mathcal{I}_\nu : \nu \in \{1,\dots,N\}\}$ of $[-\frac{\pi}{2},\frac{\pi}{2}]$ such that $-\frac{\pi}{2} \in \mathcal{I}_1$ and $\frac{\pi}{2} \in \mathcal{I}_N$, and $\phi_0 \equiv \phi_{N+1} \equiv 0$. For $\nu = 1,\dots,N-1$, $\alpha_\nu \in \mathcal{I}_\nu \cap \mathcal{I}_{\nu+1}$, $\alpha_0 \in \mathcal{I}_1 \cap (-\infty,-\frac{\pi}{2})$ and $\alpha_N \in \mathcal{I}_N \cap (\frac{\pi}{2},\infty)$. C_ν denotes the arc of the circle $|z| = R$ from $\mathrm{R}e^{i\alpha_{\nu-1}}$ to $\mathrm{R}e^{i\alpha_\nu}$. It is easily seen that the function defined by the right-hand side is independent of the choice of representative $\{\phi_\nu : \nu \in \{1,\dots,N\}\}$. It is analytic in a sector of the form $\{t \in S(\frac{\pi}{2}-\alpha_N,-\frac{\pi}{2}-\alpha_0) : |t| < r\}$, where $r > 0$. From the fact that $f_{q-1}|_{\widetilde{I_q}} = f_q \bmod (\widehat{\mathcal{A}}^{\leqslant -1})^n$ it can be derived that $u_{q,\theta}$ can be continued quasi-analytically to the half line $\arg t = 0$. This quasi-analytic continuation can be represented by an expression similar to (4.7), obtained by deforming the paths of integration and replacing $\{\phi_\nu : \nu \in \{1,\dots,N\}\}$ with a representative of f_q. $u_{q+1,\theta'}$ is a *weak accelerate* of $u_{q,\theta}$, provided $|\theta'-\theta| < \frac{\pi}{2}$. (*Cf.* [14, Proposition 4 and Theorem 5] for the case that $q = 1$.)

On the other hand, in view of proposition 3.9, there exist $\alpha < -\frac{\pi}{2}$, $\beta > \frac{\pi}{2}$ and $f_q^\pm \in (\mathcal{A}/\mathcal{A}^{\leqslant -1^+}(I_q^\pm))^n$, where $I_q^- = (\alpha,\frac{\pi}{2})$, $I_q^+ = (-\frac{\pi}{2},\beta)$, with the property that $f_q^\pm \bmod (\mathcal{A}^{\leqslant -1})^n = f_{q-1}|_{I_q^\pm}$. This implies that, for every $\theta \in \mathbb{R}$, $u_{q,\theta}$ can be analytically continued to the sectors $S(\frac{\pi}{2}-\beta,0)$ and $S(0,-\alpha-\frac{\pi}{2})$. One might expect that, for $\arg t = 0$, $\lim_{\epsilon\downarrow 0} u_{q,\theta}(t+i\epsilon) = u_{q,\theta}(t)$ if $\theta > 0$ and $\lim_{\epsilon\downarrow 0} u_{q,\theta}(t-i\epsilon) = u_{q,\theta}(t)$ if $\theta < 0$, but that is as yet an open question.

4.3. Another relative Watson Lemma

Lemma 4.14 ("relative Watson Lemma" II). — *Let I be a large, open interval of $\mathbb{R}$ such that $|\widetilde{I}| > \pi$. Then $\widehat{\mathcal{A}}^{\leqslant -1}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I) = \{0\}$.*

Proof. — We sketch a proof of this lemma, analogous to the one given in [15, 16]. Let $f \in \widehat{\mathcal{A}}^{\leqslant -1}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I)$ and let $I' \prec I$ be a large interval such that $\widetilde{I'} = (\theta_1,\theta_2)$ with $\theta_2-\theta_1 > \pi$. Similarly to the statement of Lemma 4.6, it is easily shown that $f|_{I'}$ admits a representative $\{\phi_1,\phi_2\}$, where $\phi_j \in \widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta_j),\phi_+(\theta_j))$, $j = 1,2$, and $\phi_1-\phi_2 \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1),\phi_+(\theta_2))$.

We shall show that $\phi_j \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_j),\phi_+(\theta_j))$ for $j=1,2$. This implies $f|_{I'}=0$ and, consequently, $f=0$. By remark 4.10, $\phi_j \in \mathcal{A}^{\leqslant -1}(-\frac{\pi}{2},\frac{\pi}{2})$, hence there exist positive numbers c, C and R such that, for $j=1,2$,

$$|\phi_j(z)| \leqslant Ce^{-cz}, \quad z\in(R,\infty).$$

For all $\theta \in -\widetilde{I}'^* = (\theta_1+\frac{\pi}{2},\theta_2-\frac{\pi}{2})$ and $j\in\{1,2\}$ we define

$$\eta_j^\theta(s) = \int_R^\infty \phi_j(z)e^{-sr_\theta(z)}dr_\theta(z), \quad \operatorname{Re} s > -c. \tag{4.8}$$

We begin by proving that, for $j=1,2$, η_j^θ is an entire function, satisfying a specific growth condition. η_j^θ is analytic in the half plane $\operatorname{Re} s > -c$. By rotation of the path of integration, it can be analytically continued to the sector $|\arg(s+c)|<\pi$. It has at most exponential growth of order 1, uniformly on closed subsectors. Note that $\phi_-(\theta+\frac{\pi}{2}) > \phi_-(\theta_2)$ and $\phi_+(\theta-\frac{\pi}{2}) < \phi_+(\theta_1)$. Now choose $I_j' = (a_j,b_j) \prec \mathcal{I}_j := (\phi_-(\theta_j),\phi_+(\theta_j))$, $j=1,2$, such that $a_2 < \phi_-(\theta+\frac{\pi}{2}) < a_1 < 0 < b_2 < \phi_+(\theta-\frac{\pi}{2}) < b_1$. Thus, $\vartheta(a_2) > \theta+\frac{\pi}{2}$ and $\vartheta(b_1) < \theta-\frac{\pi}{2}$. By (4.3) and (4.4), $\rho_\theta(z)$ is bounded above on $\widehat{C}_{a_2}(R)$ and $\widehat{C}_{b_1}(R)$. If R is sufficiently large, there exist C' and $c'>0$, such that

$$|\phi_j(z)| \leqslant C'e^{-c'\frac{|z|}{\log|z|}}, \quad z\in\widehat{D}_{I_j'}(R).$$

Hence we can deform the path of integration in (4.8) into a path γ_j, consisting of the arc of the circle $|z|=R$ between $z=R$ and $z=z_{b_1}(R)$ and $\widehat{C}_{b_1}(R)$ if $j=1$, or the arc of the circle $|z|=R$ between $z=R$ and $z=z_{a_2}(R)$ and $\widehat{C}_{a_2}(R)$ if $j=2$. Noting that $\operatorname{Im} r_\theta(z) = \operatorname{Im} z + \theta\operatorname{Re}(\frac{z}{\log z}) = \operatorname{Im} z(1+o(1))$ as $z\to\infty$ on γ_j, we conclude that η_1^θ is continuous on $-\pi\leqslant\arg(s+c)<\pi$, and has at most exponential growth of order 1, uniformly on sectors of the form $-\pi\leqslant\arg(s+c)\leqslant\pi-\delta$, for any $\delta>0$, whereas η_2^θ is continuous on $-\pi<\arg(s+c)\leqslant\pi$, and has at most exponential growth of order 1, uniformly on sectors of the form $-\pi+\delta\leqslant\arg(s+c)\leqslant\pi$, for any $\delta>0$. Moreover, it is easily seen that η_1^θ and η_2^θ are C^∞ on $\arg s=-\pi$ and $\arg s=\pi$, respectively. For all s such that $\arg s=-\pi$ and all $m\in\mathbb{N}_0$ we have

$$\left|\frac{d^m}{ds^m}\eta_1^\theta(s)\right| \leqslant \widetilde{C}e^{\tilde{c}|s|}\int_{\gamma_1}|r_\theta(z)|^m e^{-c'\frac{|z|}{\log|z|}}|dr_\theta(z)|$$

where $\widetilde{C}$ and $\tilde{c}$ are positive numbers, depending on θ. Applying the method of Laplace to the integral, we obtain estimates of the form $|\frac{d^m}{ds^m}\eta_1^\theta(s)| \leqslant Ke^{\tilde{c}|s|}A^m(m\log m)^m$ for all $m\geqslant 2$, where K and A are positive constants (depending on θ), proving that η_1^θ is quasi-analytic on the half line $\arg s =$

$-\pi$ (*cf.* [9]). Similarly it is shown that, for all $\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})$, η_2^θ is quasi-analytic on $\arg s = \pi$.

Obviously,

$$\eta_1^\theta(s) - \eta_2^\theta(s) = \int_R^\infty (\phi_1(z) - \phi_2(z)) e^{-sr_\theta(z)} dr_\theta(z).$$

The fact that $\phi_1 - \phi_2 \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap \mathcal{I}_2)$ implies the existence of positive constants C'', δ' and R', such that, for all $z \in (R', \infty)$,

$$|\phi_1(z) - \phi_2(z)| \leqslant C'' e^{-\delta' z \log z}.$$

Consequently, if $R > R'$, the integral can be estimated as follows

$$\Big| \int_R^\infty (\phi_1(z) - \phi_2(z)) e^{-sr_\theta(z)} dr_\theta(z) \Big| \leqslant C'' \int_R^\infty e^{-\delta' z \log z + c''|s| z} dz, \quad s \in \mathbb{C}$$

where c'' is a positive constant, depending on θ. The integrand on the right-hand side is maximal when $\log z + 1 = c''|s|/\delta'$ and its maximum value is $e^{\delta' e^{c''|s|/\delta' - 1}}$. Hence we deduce that $\eta_1^\theta - \eta_2^\theta$ is an entire function, satisfying a growth condition of the form

$$|\eta_1^\theta(s) - \eta_2^\theta(s)| \leqslant C''' e^{e^{B|s|}}$$

where C''' and B are positive constants, depending on θ. It now follows that, for $j = 1, 2$ and $\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})$, η_j^θ is analytic in $|\arg(s+c)| < \pi$, continuous on $|\arg(s+c)| \leqslant \pi$ and quasi-analytic on $\arg s = \pm\pi$, hence an entire function, satisfying a growth condition of the form

$$|\eta_j^\theta(s)| \leqslant C'''' e^{e^{B|s|}} \tag{4.9}$$

where C'''' is a positive constant, depending on θ.

Note that $\phi_-(\theta_2) < \phi_-(\theta_1 + \pi) < \phi_-(\theta_1)$ and $\phi_+(\theta_2) < \phi_+(\theta_2 - \pi) < \phi_+(\theta_1)$. Now, let $I_j' = (a_j, b_j) \prec \mathcal{I}_j$ such that $a_2 < \phi_-(\theta_1 + \pi) < a_1 < 0 < b_2 < \phi_+(\theta_2 - \pi) < b_1$. Choose $\theta \in (\vartheta(a_2) - \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})$. Then $\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})$ and $(a_2, 0) \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$. By (4.5), there exists $\delta > 0$ such that

$$\rho_\theta(z) \geqslant \delta \frac{|z|}{\log |z|} \text{ for all } z \in \widehat{D}_{(a_2,0)}(R) \tag{4.10}$$

η_2^θ has at most exponential growth of order 1 as $s \to \infty$, uniformly on $S_\alpha^+ := S[-\pi + \alpha, \pi]$, for any $\alpha > 0$. Let α and $\beta \in (0, \frac{\pi}{2})$. Then, in view of (4.10), $-\frac{\pi}{2} < \arg r_\theta(z) < \frac{\pi}{2} - \alpha$ for all $z \in \widehat{D}_{I_2'}(R') \cap S[-\pi, -\beta] (\subset \widehat{D}_{(a_2,0)}(R'))$, provided R' is sufficiently large. Thus, according to the inversion formula we have, for all $z \in \widehat{D}_{I_2'}(R') \cap S[-\pi, -\beta]$,

$$\phi_2(z) = \frac{1}{2\pi i} \int_{\partial S_\alpha^+} \eta_2^\theta(s) e^{s r_\theta(z)} ds.$$

Replacing the path of integration by the path Γ (in $\mathbb{C}$), consisting of the half line l_1 from $\infty e^{-i(\pi-\alpha)}$ to $\sigma/\cos\alpha e^{-i(\pi-\alpha)}$, where $\sigma := \frac{\cos\alpha}{B}\log\frac{\rho_\theta(z)}{B}$, the segment l_2 from $\sigma/\cos\alpha e^{-i(\pi-\alpha)}$ to $-\sigma$ and the half line l_3 from $-\sigma$ to $-\infty$, we get, with the aid of (4.9),

$$\begin{aligned} |\phi_2(z)| \leqslant C_2 \Big[\int_{l_1} & e^{c_2|s|} e^{-\sigma\rho_\theta(z) - \operatorname{Im} s \operatorname{Im} r_\theta(z)} |ds| \\ & + \int_{l_2} e^{e^{B_2|s|}} e^{-\sigma\rho_\theta(z) - \operatorname{Im} s \operatorname{Im} r_\theta(z)} |ds| + \int_{-\infty}^{-\sigma} e^{(\rho_\theta(z) - c_2)s} ds \Big] \end{aligned}$$

where C_2 and c_2 are positive constants. Using the (in)equalities

$$\begin{aligned} \int_{l_1} e^{c_2|s|} e^{-\sigma\rho_\theta(z) - \operatorname{Im} s \operatorname{Im} r_\theta(z)} |ds| &\leqslant e^{-\sigma\rho_\theta(z)} \int_0^\infty e^{(c_2 + \operatorname{Im} r_\theta(z)\sin\alpha)|s|} d|s| \\ &= \frac{e^{-\frac{\cos\alpha}{B_2}\log(\frac{\rho_\theta(z)}{B_2})\rho_\theta(z)}}{-(\operatorname{Im} r_\theta(z)\sin\alpha + c_2)} \text{ if } \operatorname{Im} r_\theta(z) \\ &< -\frac{c_2}{\sin\alpha}, \\ \int_{l_2} e^{e^{B_2|s|}} e^{-\sigma\rho_\theta(z) - \operatorname{Im} s \operatorname{Im} r_\theta(z)} |ds| &\leqslant \sigma \tan\alpha e^{e^{B_2\sigma/\cos\alpha} - \sigma\rho_\theta(z)} \\ &= \frac{\sin\alpha}{B_2}\log(\frac{\rho_\theta(z)}{B_2}) e^{\frac{\rho_\theta(z)}{B_2}(1 - \cos\alpha\log\frac{\rho_\theta(z)}{B_2})} \end{aligned}$$

and

$$\int_{-\infty}^{-\sigma} e^{(\rho_\theta(z) - c_2)s} ds = \frac{e^{-(\rho_\theta(z) - c_2)\frac{\cos\alpha}{B_2}\log\frac{\rho_\theta(z)}{B_2}}}{\rho_\theta(z) - c_2} \text{ if } \rho_\theta(z) > c_2$$

and noting that both $-\operatorname{Im} r_\theta(z) < -\frac{c_2}{\sin\alpha}$ and $\rho_\theta(z) > c_2$ for all $z \in \widehat{D}_{I_2'}(R') \cap S[-\pi, -\beta]$ if R' is sufficiently large, we obtain an estimate of the form

$$|\phi_2(z)| \leqslant C_2' e^{-\epsilon_2 \rho_\theta(z)\log\rho_\theta(z)}, \ z \in \widehat{D}_{I_2'}(R') \cap S[-\pi, -\beta],$$

where C_2' and $\epsilon_2 > 0$, and, in view of (4.10), this implies $|\phi_2(z)| \leqslant C_2' e^{-c_2'|z|}$ for $z \in \widehat{D}_{I_2'}(R') \cap S[-\pi, -\beta]$, where $c_2' > 0$, provided R' is sufficiently large. A similar estimate can be derived for $\phi_1(z)$ for all $z \in \widehat{D}_{I_1'}(R') \cap S[\beta, \pi]$. Combining this with the fact that $\phi_1 \in \mathcal{A}^{\leqslant -1}(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\phi_1 - \phi_2 \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap \mathcal{I}_2)$, we conclude that $|\phi_1(z)| \leqslant C_1' e^{-c_1'|z|}$ for all $z \in$

$\widehat{D}_{I_1'}(R')$, where C_1' and $c_1' > 0$, provided R' is sufficiently large. Hence $\phi_j \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_j)$ for $j = 1, 2$. □

4.4. The proof of Theorem 4.12

In the remaining part of this section we show that the condition $\widetilde{I_{q+1}} \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ in Theorem 3.8 can be lifted and, in case II, we prove the existence, for appropriate intervals I_q and I_{q+1}, of solutions $f_q \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I_q))^n$ and $f_{q+1} \in (\widehat{\mathcal{A}}(I_{q+1}))^n$ of (1.1) satisfying the conditions of definition 4.11. With f_{q-1} we can associate an element $\widetilde{f_{q-1}}$ of $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(\mathbb{R}))^n$, similarly to the scalar case (*cf.* Definition 4.8). By suitably modifying a representative of $\widetilde{f_{q-1}|_{I_q}} := \widetilde{f_{q-1}}|_{I_q}$ we obtain a representative of f_q. Without loss of generality we may assume that the equation is in the prepared form (2.7) and $\widetilde{f_{q-1}} \in z^{-N/p}(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(\mathbb{R}))^n$ for some sufficiently large $N \in \mathbb{N}$.

Lemma 4.15. — *$\widetilde{f_{q-1}}$ is a solution of (2.7) in $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(\mathbb{R}))^n$. More precisely, it has a representative $\{\tilde{\varphi}_{q,\theta} : \theta \in \mathbb{R}\}$ with the following properties.*

(i) *For all $\theta \in \mathbb{R}$, $\tilde{\varphi}_{q,\theta} \in (\widehat{\mathcal{A}}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$.*

(ii) *$\tilde{\varphi}_{q,\theta_1} - \tilde{\varphi}_{q,\theta_2} \in (\widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_2 - \frac{\pi}{2})))^n$ if $\theta_1 < \theta_2$.*

(iii) *For all $\theta \in \mathbb{R}$,*

$$\Delta\tilde{\varphi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) - \varphi_0(z^{1/p}) \in (\widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$$

If 0 is not a singular direction of level 1, then f_q defines a section $\widetilde{f_q^+} \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathbb{R}))^n$, represented by a 1^+-precise quasi-function $\{\tilde{\varphi}^+_{q,\theta} : \theta \in \mathbb{R}\}$ with the following properties.

(iv) *For all $\theta \in \mathbb{R}$, $\tilde{\varphi}^+_{q,\theta} \in (\widehat{\mathcal{A}}(\phi_-(\theta), \phi_+(\theta)))^n$,*

(v) *$\tilde{\varphi}^+_{q,\theta_1} - \tilde{\varphi}^+_{q,\theta_2} \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta_1), \phi_+(\theta_2)))^n$ if $\theta_1 < \theta_2$.*

(vi) *For all $\theta \in \mathbb{R}$,*

$$\Delta\tilde{\varphi}^+_{q,\theta}(z) - E(z^{1/p}, \tilde{\varphi}^+_{q,\theta}(z)) - \varphi_0(z^{1/p}) \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\phi_-(\theta), \phi_+(\theta)))^n$$

Proof. — For all $\theta \in \mathbb{R}$, $\tilde{\varphi}_{q,\theta}$ is defined similarly to $\tilde{\varphi}_\theta$ in Lemma 4.5 and the first two statements of Lemma 4.15 follow immediately from that lemma.

Now, let $\{\phi_\nu : \nu \in \{1, \dots, N'\}\}$, where $\phi_\nu \in \mathcal{A}(\mathcal{I}_\nu)$ and $\{\mathcal{I}_\nu : \nu \in \{1, \dots, N'\}\}$ is a good, open covering of $[-\frac{\pi}{2}, \frac{\pi}{2}]$, be a representative of $f_{q-1}|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$. As f_{q-1} is a solution of (2.7) in $z^{-N/p}(\mathcal{A}/\mathcal{A}^{\leqslant -1})(I_{q-1})^n$, we have, for all $\nu \in \{1, \dots, N'\}$,

$$\Delta\phi_\nu(z) - E(z^{1/p}, \phi_\nu(z)) - \varphi_0(z^{1/p}) \in (\mathcal{A}^{\leqslant -1}(\mathcal{I}_\nu))^n. \tag{4.11}$$

Let $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$ and $I'_\nu \prec \mathcal{I}_\nu$. From (2.11) and (4.1) we deduce the existence of positive numbers C and c such that

$$|\Delta(\phi_\nu(z) - \tilde{\varphi}_{q,\theta}(z)) - (E(z^{1/p}, \phi_\nu(z)) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)))| \leqslant Ce^{-c\frac{|z|}{\log|z|}}$$

for all $z \in \widehat{D}_{I'}(R) \cap S(I'_\nu)$, if R is sufficiently large. Using (4.11) and varying ν, we conclude that there exist R', C' and $c' > 0$ such that

$$|\Delta\tilde{\varphi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) - \varphi_0(z^{1/p})| \leqslant C'e^{-c'\frac{|z|}{\log|z|}}$$

for all $z \in \widehat{D}_{I'}(R')$ and this implies $\Delta\tilde{\varphi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) - \varphi_0(z^{1/p}) \in (\widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$.

The statements concerning $\widetilde{f}_q^+$ are proved analogously. □

First, consider the case that 0 is not a singular direction of level 1. Let $\{\tilde{\varphi}^+_{q,\theta} : \theta \in \mathbb{R}\}$ be a representative of $\widetilde{f}_q^+$ with the properties mentioned in Lemma 4.15. Analogously to the proof of proposition 3.11, with ϕ_0^θ replaced by $\tilde{\varphi}^+_{q,\theta}$, $\theta \in \widetilde{I_{q+1}}$ (so that $(\phi_-(\theta), \phi_+(\theta)) \subset I_{q+1}$), it can be proved that this proposition continues to hold when $\widetilde{I_{q+1}} \not\subset [-\pi/2, \pi/2]$. Hence the first statement of the theorem follows.

Now, suppose that 0 is a singular direction of level 1 (case II). Let $\theta_1 < \theta_2$ and let $I = (\phi_-(\theta_2 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2}))$. According to Lemma 4.15, $\{\tilde{\varphi}_{q,\theta_1}, \tilde{\varphi}_{q,\theta_2}\}$ represents a solution of (2.7) in $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1}(I))^n$ (viz. $\widetilde{f_{q-1}}|_I$). We shall show that $\tilde{\varphi}_{q,\theta_1}$ and $\tilde{\varphi}_{q,\theta_2}$ can be modified by exponentially small functions in such a manner that the resulting quasi-function is 1^+-precise and represents a solution f_q of (2.7) in $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$. For that purpose we need the following theorem.

Theorem 4.16. — *Let $I = (a, b)$ be a large, open interval such that $|\widetilde{I}| \leqslant \pi$. Let Δ^c be a canonical difference operator (cf. (2.1) and (2.2)) and $\varphi : S(-\pi, \pi) \times \mathbb{C}^n \to \mathbb{C}^n$ a function of the form*

$$\varphi(z, y) = \varphi_0(z) + A(z)y + \psi(z, y)$$

where $\varphi_0 \in \widehat{\mathcal{A}}^{\leqslant -1}(I)^n$, $A \in z^{-\nu_0 - 1/p}\,\mathrm{End}(n; \widehat{\mathcal{A}}(I))$ with $\nu_0 = \max\{-d_j : j \in \{1, \ldots, n\}\}$ and ψ has the following properties: for any open interval $I' \prec I$ there exists a positive number R, such that

(i) *ψ is holomorphic on $\widehat{D}_{I'}(R) \times U$, where $U \subset \mathbb{C}^n$ is a neighbourhood of 0,*

(ii) *ψ admits an asymptotic expansion of the form $\sum_{m=m_0}^\infty \psi_m(y)z^{-m/p}$ as $z \to \infty$, uniformly on $\widehat{D}_{I'}(R) \times U$, where $m_0 \in \mathbb{Z}$ and the ψ_m are holomorphic $\mathbb{C}^n$-valued functions,*

(iii) *$\psi'_2(z, 0) = 0$ for all $z \in \widehat{D}_{I'}(R)$.*

Furthermore, we assume: if $-\frac{\pi}{2}$ *is a Stokes direction of* Δ^c *of level* 1*, then either* $a < \phi_-(\frac{\pi}{2})$ *or* $b > \phi_+(-\frac{\pi}{2})$*.*

Let $\mathcal{I}_1 = (a, b_1)$ *and* $\mathcal{I}_2 = (a_2, b)$ *be large, open subintervals of* I *such that* $\vartheta(a_2) \leqslant \vartheta(b_1)$ *(i.e.* $\mathcal{I}_1 \cap \mathcal{I}_2$ *is not a large interval), and* $\widetilde{\mathcal{I}_i} \cap \Theta(\Delta^c) = \emptyset$ *for* $i = 1, 2$*. Then the equation*

$$\Delta^c y(z) = \varphi(z, y(z)) \tag{4.12}$$

has a unique solution $f \in \widehat{\mathcal{A}}^{\leqslant -1}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I)^n$ *with representative* $\{f_1, f_2\}$*, where* f_i *is a solution of (4.12) in* $\widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_i)^n$ *and* $f|_{\mathcal{I}_i} = f_i \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n$*. Moreover, if* $\widetilde{I} \cap \Theta(\Delta^c) = \emptyset$*, then (4.12) has a unique solution* $f \in \widehat{\mathcal{A}}^{\leqslant -1}(I)^n$*.*

In the case that Δ^c has no levels $\kappa_j \in (0, 1)$ and $A(z) \sim 0$ as $z \to \infty$, uniformly on $\widehat{D}_{I'}(R)$ for any open interval $I' \prec I$ and some sufficiently large R, this theorem follows easily from Theorems 4 and 2 in [14]. Using the same type of argument, it can be shown that these results continue to hold when $\kappa_j \in (0, 1)$ for certain $j \in \{1, \ldots, n\}$ and $A \in z^{-\nu_0 - 1/p}\,\mathrm{End}(n; \widehat{\mathcal{A}}(I))$, where $\nu_0 = \max\{-d_j : j \in \{1, \ldots, n\}\}$ (*cf.* also [14, Remark 7]).

From Theorem 4.16 we derive the following result.

Lemma 4.17. — *Let* Δ *and* Δ^c *be the difference operators in (2.7) and (2.9). Let* $I = (a, b)$ *be a large, open interval such that either* $a < \phi_-(\frac{\pi}{2})$ *or* $b > \phi_+(-\frac{\pi}{2})$ *and* $|\widetilde{I}| \leqslant \pi$*. Let* $\mathcal{I}_1 = (a, b_1)$ *and* $\mathcal{I}_2 = (a_2, b)$ *be large subintervals of* I *such that* $\vartheta(a_2) \leqslant \vartheta(b_1)$ *and* $\widetilde{\mathcal{I}_i} \cap \Theta(\Delta^c) = \emptyset$ *for* $i = 1, 2$*. Then (2.7) has a unique solution* f *in* $(\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$ *such that* $f \bmod (\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}|_{\widetilde{I}}$ *and* $f|_{\mathcal{I}_i} = \phi_i \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n$*, where* ϕ_i *is a solution of (2.7) in* $(\widehat{\mathcal{A}}(\mathcal{I}_i))^n$ *for* $i = 1, 2$*.*

Moreover, ϕ_1 *is unique if* $a < \phi_-(\frac{\pi}{2})$*,* ϕ_2 *is unique if* $b > \phi_+(-\frac{\pi}{2})$*.*

Proof. — Let $\widetilde{I} = (\alpha, \beta)$, $\theta \in [\beta - \frac{\pi}{2}, \alpha + \frac{\pi}{2}]$ and $y = w + \tilde{\varphi}_{q,\theta}$. Then $I \subset (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$. y is a solution of the equation (2.7) if and only if w satisfies the equation

$$\Delta w(z) = G(z, w(z)) := E(z^{1/p}, w(z) + \tilde{\varphi}_{q,\theta}(z)) + \varphi_0(z^{1/p}) - \Delta\tilde{\varphi}_{q,\theta}(z). \tag{4.13}$$

According to (2.9), there exists $L \in Gl(n; \mathbb{C}[z^{-1/p}][z^{1/p}])$ such that $L(z)\Delta = \Delta^c + A(z)$, where $A \in z^{-\nu_0 - 1/p}\,\mathrm{End}(n; \mathbb{C}\{z^{-1/p}\})$ provided M is chosen sufficiently large (*cf.* §2.5). Thus, w is a solution of the equation (4.13) if and only if

$$\Delta^c w(z) = \varphi(z, w(z)) := L(z)G(z, w(z)) - A(z)w(z). \tag{4.14}$$

By Lemma 4.15, $\varphi(z, 0) = L(z)(E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) + \varphi_0(z^{1/p}) - \Delta\tilde{\varphi}_{q,\theta}(z)) \in (\widehat{\mathcal{A}}^{\leqslant -1}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$. From the fact that $\tilde{\varphi}_{q,\theta} \in z^{-N/p}(\widehat{\mathcal{A}}(\phi_-(\theta +$

$\frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$, it follows that φ admits an asymptotic expansion in $z^{-1/p}$ as $z \to \infty$, uniformly on $\widehat{D}_{I'}(R) \times V$, for any interval $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, sufficiently large R and some neighbourhood $V \subset \mathbb{C}^n$ of 0 (*cf.* [8, Lemma 14.3] and [10, Lemma 4.6]). Furthermore,

$$\varphi_2'(z, 0) = L(z) D_2 E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) - A(z).$$

As $D_2E(z^{1/p}, 0) = 0$, there exist positive constants K and K' such that $|D_2E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z))| \leqslant K|\tilde{\varphi}_{q,\theta}(z)| \leqslant K'|z|^{-N/p}$ for all $z \in \widehat{D}_{I'}(R)$, where $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$ and R sufficiently large (K' and R depend on I'). Consequently, $\varphi_2'(z, 0) \in z^{-\nu_0 - 1/p} \operatorname{End}(n; \widehat{\mathcal{A}}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))$ if N is sufficiently large. According to Theorem 4.16, (4.14) has a unique solution $w \in \widehat{\mathcal{A}}^{\leqslant -1}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I)^n$ with representative $\{w_1, w_2\}$, where w_i is a solution of (4.14) in $\widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_i)^n$ with the property that $w_i \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n = w|_{\mathcal{I}_i}$, for $i = 1, 2$. Now let $\phi_i = w_i + \tilde{\varphi}_{q,\theta}$, $i = 1, 2$. Then $\phi_i \in \widehat{\mathcal{A}}(\mathcal{I}_i)^n$, ϕ_i is a solution of (2.7) for $i = 1, 2$ and $\phi_1 - \phi_2 = w_1 - w_2 \in \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap \mathcal{I}_2)^n$. Moreover, $\phi_i - \tilde{\varphi}_{q,\theta} = w_i \in (\widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_i))^n$ for $i = 1, 2$. The uniqueness of ϕ_1 or ϕ_2 follows immediately from the last statement of Theorem 4.16. $\{\phi_1, \phi_2\}$ represents an element $f \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$ with the required properties. □

From Lemma 4.17 we deduce the following proposition, which completes the proof of Theorem 4.12.

Proposition 4.18. — *Let I be a large, open interval such that $0 \notin \widetilde{I}^*$ and $|\widetilde{I}| > \pi$. Then (2.7) has a unique solution $f \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I))^n$ with the property that $f_{q-1}\tilde{|}_I = f \bmod (\widehat{\mathcal{A}}^{\leqslant -1})^n$.*

Moreover, if J is a large, open subinterval of I such that $\widetilde{J} \cap \Theta(\Delta^c) = \emptyset$, there exists a solution $g \in (\widehat{\mathcal{A}}(J))^n$ of (2.7) with the property that $f|_J = g \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n$.

Proof. — We shall prove the proposition for the case that I (and hence J) is a bounded interval and leave the reader to extend the proof to the general case (*i.e.* $I \subset (-\infty, \phi_+(-\frac{\pi}{2}))$ or $(\phi_-(\frac{\pi}{2}), \infty)$). Let $-\widetilde{I}^* = (\theta_1, \theta_2)$. Then $\widetilde{I} = (\theta_1 - \frac{\pi}{2}, \theta_2 + \frac{\pi}{2})$ and $I = (\phi_-(\theta_2 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2})) =: (a, b)$. Either $\theta_1 \geqslant 0$ or $\theta_2 \leqslant 0$. Suppose that $\theta_1 \geqslant 0$. The proof of the other case is similar. Let $\mathcal{I}_1 = (a, b_1)$ and $\mathcal{I}_2 = (a_2, b)$ be large subintervals of I such that $\vartheta(a_2) \leqslant \vartheta(b_1)$, $|\widetilde{\mathcal{I}_i}| \leqslant \pi$ and $\widetilde{\mathcal{I}}_i \cap \Theta(\Delta^c) = \emptyset$ for $i = 1, 2$. Let $I' = (a', b) := (\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2}))$ and let $I_1' = (a', b')$ be a large subinterval of I' such that $\widetilde{I_1'} \cap \Theta(\Delta^c) = \emptyset$. Note that $|\widetilde{I'}| = \pi$ and thus $|\widetilde{I_1'}| \leqslant \pi$ and $\mathcal{I}_2 \subset I'$. Without loss of generality we may assume that $b_1 \leqslant b'$. Then $\mathcal{I}_1 \cap I' = \mathcal{I}_1 \cap I_1' = (a', b_1)$. $\theta_1 \geqslant 0$ implies that $a = \phi_-(\theta_2 + \frac{\pi}{2}) < a' = \phi_-(\theta_1 + \frac{\pi}{2}) \leqslant \phi_-(\frac{\pi}{2})$. According to

Lemma 4.17, (2.7) has unique solutions $\phi_1 \in (\widehat{\mathcal{A}}(\mathcal{I}_1))^n$, $\phi_1' \in (\widehat{\mathcal{A}}(I_1'))^n$ and $f' \in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I')^n$, and a solution $\phi_2 \in (\widehat{\mathcal{A}}(\mathcal{I}_2))^n$, such that ϕ_1 mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_{\mathcal{I}_1}$, f' mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_{I'}$, $f'|_{I_1'} = \phi_1'$ mod $(\widehat{\mathcal{A}}^{\leqslant -1^+})^n$ and $f'|_{\mathcal{I}_2} = \phi_2$ mod $(\widehat{\mathcal{A}}^{\leqslant -1^+})^n$. Obviously, ϕ_1' mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_{I_1'}$, so $\phi_1 - \phi_1' \in \widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_1 \cap I_1')^n$. If $\mathcal{I}_1 \cap I_1'$ is a large interval, then, by Lemma 4.17, $\phi_1|_{\mathcal{I}_1 \cap I_1'}$ is the unique solution of (2.7) in $(\widehat{\mathcal{A}}(\mathcal{I}_1 \cap I_1'))^n$ with the property that $\phi_1|_{\mathcal{I}_1 \cap I_1'}$ mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_{\mathcal{I}_1 \cap I_1'}$, so $\phi_1' = \phi_1$. Now assume $\mathcal{I}_1 \cap I_1'$ is not a large interval (*i.e.* $\theta_-(\mathcal{I}_1 \cap I_1') = \theta_1 + \frac{\pi}{2} \leqslant \theta_+(\mathcal{I}_1 \cap I_1') = \vartheta(b_1)$ or, equivalently, $b_1 \geqslant \phi_+(\theta_1 + \frac{\pi}{2})$). $\phi_1 - \phi_1'$ satisfies a homogeneous linear difference equation of the form (2.10) (with $y_1 = \phi_1$ and $y_2 = \phi_1'$). As $a' < \phi_-(\frac{\pi}{2})$, $\mathcal{I}_1 \cap I_1' \not\subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ and thus, according to Theorem 2.13(iii), $\mathrm{Ker}(\widetilde{\Delta}, \widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_1 \cap I_1')^n) = \mathrm{Ker}(\widetilde{\Delta}, \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap I_1')^n) = \mathrm{Ker}(\widetilde{\Delta}, \widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap I')^n)$. Let $f_1 := \phi_1$ mod $(\widehat{\mathcal{A}}^{\leqslant -1^+})^n$. Then $f_1|_{\mathcal{I}_1 \cap I'} = f'|_{\mathcal{I}_1 \cap I'}$ and, consequently, there exists a unique $f \in \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cup I')^n = \widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I)^n$ such that $f|_{\mathcal{I}_1} = f_1$ and $f|_{I'} = f'$. Obviously, f mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_I$. Its uniqueness follows from Lemma 4.14. Moreover, $f|_{\mathcal{I}_2} = f'|_{\mathcal{I}_2} = \phi_2$ mod $(\widehat{\mathcal{A}}^{\leqslant -1^+})^n$.

Now suppose $J = (c, d)$ is a large subinterval of I such that $\widetilde{J} \cap \Theta(\Delta^c) = \emptyset$. To begin with, assume that $|\widetilde{J}| \leqslant \pi$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ as above. If $J \subset \mathcal{I}_1$ or $J \subset \mathcal{I}_2$, the last statement of the theorem immediately follows from the above argument. So suppose that $a < c < a_2$ and $b_1 < d < b$. Without loss of generality we may assume that $\vartheta(c) \leqslant \vartheta(b_1)$ and $\vartheta(a_2) \leqslant \vartheta(d)$ (this can always be achieved by decreasing b_1 and increasing a_2, if necessary, taking care that $\vartheta(b_1)$ remains less than $\vartheta(a)$ and $\vartheta(a_2) \geqslant \vartheta(b)$). Let $I'' := (a, d)$, $I_1'' := \mathcal{I}_1$ and $I_2'' := J$. Then the above argument proves the existence of a unique solution $f'' \in (\widehat{\mathcal{A}}/\widehat{\mathcal{A}}^{\leqslant -1^+}(I''))^n$, of (2.7), with the following properties:

(i) f'' mod $(\widehat{\mathcal{A}}^{\leqslant -1})^n = f_{q-1}\widetilde{\rceil}_{I''}$.

(ii) (2.7) has a solution $\phi_i'' \in \widehat{\mathcal{A}}(I_i'')^n$ such that $f''|_{I_i''} = \phi_i''$ mod $(\widehat{\mathcal{A}}^{\leqslant -1^+})^n$ for $i = 1, 2$.

The uniqueness of ϕ_1 implies that $\phi_1'' = \phi_1$. Furthermore, $\phi_2'' - \phi_2 \in (\widehat{\mathcal{A}}^{\leqslant -1}(I_2'' \cap \mathcal{I}_2))^n$. Thus, $\phi_2'' - \phi_2$ satisfies a homogeneous linear difference equation of the form (2.10), with y_1 and y_2 replaced by ϕ_2'' and ϕ_2 and $H \in \mathrm{End}(n; \widehat{\mathcal{A}}^{\leqslant -1}(I_2'' \cap \mathcal{I}_2))$. Note that $\theta_-(I_2'' \cap \mathcal{I}_2) = \vartheta(a_2) \leqslant \theta_+(I_2'' \cap \mathcal{I}_2) = \vartheta(d)$. According to Theorem 2.13, $\phi_2'' - \phi_2$ can be written in the form

$$\phi_2''(z) - \phi_2(z) = \sum_{j=1}^{n} z^{d_j z} e^{q_j(z)} z^{\lambda_j} g_j(z) p_j(z)$$

where $g_j \in \widehat{\mathcal{A}}(I_2'' \cap \mathcal{I}_2)^n[\log z]$ and the p_j are trigonometric polynomials, $p_j \equiv 0$ unless $d_j < 0$, or $d_j = 0$, $k_j = 1$ and $\arg \mu_j = \pi$. As $\phi_1 - \phi_2 \in (\widehat{\mathcal{A}}^{\leqslant -1}(\mathcal{I}_1 \cap \mathcal{I}_2))^n$, and $\phi_2'' - \phi_1 \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap I_2''))^n \subset (\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap \mathcal{I}_2))^n$, we have $\phi_2'' - \phi_2 \in (\widehat{\mathcal{A}}^{\leqslant -1}(I_2'' \cap \mathcal{I}_2))^n \cap (\widehat{\mathcal{A}}^{\leqslant -1^+}(\mathcal{I}_1 \cap \mathcal{I}_2))^n$. This implies that $p_j \equiv 0$ unless $d_j < 0$. Consequently, $\phi_2'' - \phi_2 \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(I_2'' \cap \mathcal{I}_2))^n$ and thus $f|_J = \phi_2'' \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n$.

Finally, suppose that $|\widetilde{J}| > \pi$. For any pair of large subintervals J_1 and J_2 of J such that $|\widetilde{J}_i| \leqslant \pi$, there exist solutions $\phi_1' \in (\widehat{\mathcal{A}}(J_1))^n$ and $\phi_2' \in (\widehat{\mathcal{A}}(J_2))^n$ of (2.7) such that $f|_{J_i} = \phi_i' \bmod (\widehat{\mathcal{A}}^{\leqslant -1^+})^n$ for $i = 1, 2$. If, in addition, $J_1 \cap J_2$ is again a large interval, then, in view of Lemma 2.12 2, $\phi_1' - \phi_2' \in (\widehat{\mathcal{A}}^{\leqslant -1^+}(J_1 \cap J_2))^n = \{0\}$, so $\phi_1' = \phi_2'$ and the result follows by glueing together all these solutions. □

5. Appendix

In this section we compare the domains $\widehat{D}_I(R)$ to the domains $D_I(R)$ and $\widetilde{D}_I(R)$ used in previous papers ([11, 12, 14]).

Definition 5.1 ("old domains"). — *Suppose that* $\operatorname{Re} \psi_\theta(z) > 1/e + |\theta|$. *By* $D_\theta(z)$ *we denote the domain*

$$D_\theta(z) := \big\{ z \in S_+ : \operatorname{Re} \psi_\theta(\zeta) \geqslant \operatorname{Re} \psi_\theta(z) \big\}.$$

Let I *be a finite interval of* $\mathbb{R}$, *such that* $\overline{I} = [\theta_1, \theta_2]$. *Let* $z \in S_+$ *such that* $\operatorname{Re} \psi_\theta(z) > 1/e + |\theta|$ *for all* $\theta \in I$. *By* $D_I(z)$ *and* $\widetilde{D}_I(z)$ *we denote the domains*

$$D_I(z) = \cap_{\theta \in I} D_\theta(z) = D_{\theta_1}(z) \cap D_{\theta_2}(z)$$

and

$$\widetilde{D}_I(z) = \cup_{\theta \in I} D_\theta(z).$$

Remark 5.2. — Let I be an open interval, containing 0. If $\theta_-(I) < \theta_+(I)$, $\widehat{\mathcal{A}}(I)$, $\widehat{\mathcal{A}}^{\leqslant -1}(I)$ and $\widehat{\mathcal{A}}^{\leqslant -1^+}(I)$ coincide with the sets $\mathcal{A}(\widetilde{I})$, $\mathcal{A}_{1,0}(\widetilde{I})$ and $\mathcal{A}_{1^+,0}(\widetilde{I})$ defined in [14], respectively. If I is a large interval, $\widehat{\mathcal{A}}(I)$ and $\widehat{\mathcal{A}}^{\leqslant -1}(I)$ coincide with the sets $\widetilde{\mathcal{A}}(\widetilde{I})$ and $\widetilde{\mathcal{A}}_{1,0}(\widetilde{I})$ defined in [14], respectively. This is an immediate corollary to the following simple lemma.

Lemma 5.3. — *Let* I *be an open interval, containing* 0, *and* $R > 1$. *If* $\theta_-(I) < \theta_+(I)$ *there exists* $R' > R$ *such that*

$$D_{\widetilde{I}}(R') \subset \widehat{D}_I(R).$$

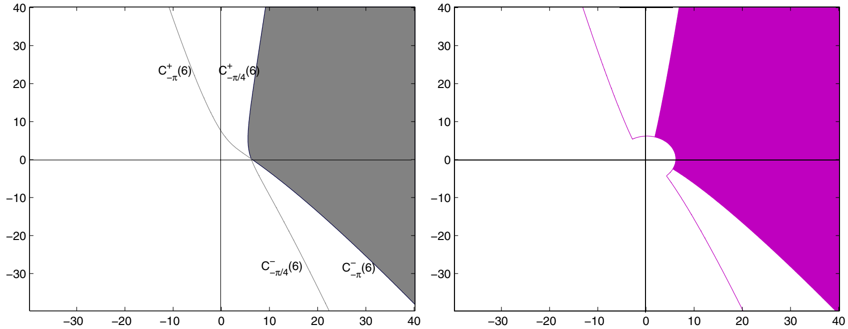

Figure 5.1. The picture on the left shows the "old" domains $D_{[-\pi,-\frac{\pi}{4}]}(6)$ *and* $\widetilde{D}_{[-\pi,-\frac{\pi}{4}]}(6)$ *(the large domain, bounded by* $C^-_{-\frac{\pi}{4}}(6)$ *and* $C^+_{-\pi}(6)$*). The corresponding domains in the picture on the right are* $\widehat{D}_{[\phi_-(-\pi),\phi_+(-\frac{\pi}{4})]}(6)$ *and* $\widehat{D}_{[\phi_-(-\frac{\pi}{4}),\phi_+(-\pi)]}(6)$.

For every interval $I' \prec \widetilde{I}$ *there exists* $R' > R$ *such that*

$$\widehat{D}_I(R') \subset D_{I'}(R).$$

If I *is a large interval, there exists* $R' > R$ *such that*

$$\widetilde{D}_{\widetilde{I}}(R') \subset \widehat{D}_I(R).$$

For every interval I' *such that* $\widetilde{I} \prec I'$*, there exists* $R' > R$ *such that*

$$\widehat{D}_I(R') \subset \widetilde{D}_{I'}(R).$$

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