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Andrea ALTOMANI, C. Denson HILL,
Mauro NACINOVICH & Egmont PORTEN

Complex vector fields and hypoelliptic partial differential operators

COMPLEX VECTOR FIELDS AND HYPOELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

by Andrea ALTOMANI, C. Denson HILL,
Mauro NACINOVICH & Egmont PORTEN

ABSTRACT. — We prove a subelliptic estimate for systems of complex vector fields under some assumptions that generalize the essential pseudoconcavity for CR manifolds, that was first introduced by two of the authors, and the Hörmander's bracket condition for real vector fields.

Applications are given to prove the hypoellipticity of first order systems and second order partial differential operators.

Finally we describe a class of compact homogeneous CR manifolds for which the distribution of $(0,1)$ vector fields satisfies a subelliptic estimate.

RÉSUMÉ. — On prouve une estimation subelliptique pour les systèmes de champs vectoriels complexes sous certaines hypothèses, qui généralisent la condition de pseudoconcavité essentielle pour les variétés CR, introduite pour la première fois par deux des auteurs, et la condition de commutation d'Hörmander pour des champs vectoriels réels.

On donne des applications afin de démontrer l'hypoellipticité de systèmes de premier ordre et d'opérateurs aux dérivées partielles de second ordre.

Finalement, on décrit une classe de variétés CR compactes homogènes pour lesquelles la distribution des champs vectoriels de type $(0,1)$ satisfait une estimation subelliptique.

Introduction

In this paper we prove a subelliptic estimate for sums of squares of complex vector fields. Namely, given a distribution $\mathfrak{Z}(M)$ of complex vector fields on an m dimensional real smooth manifold M, we find conditions for the subellipticity of $\mathfrak{Z}(M)$, i.e. the validity of the estimate

$$(0.1) \qquad \|u\|_\epsilon^2 \leqslant C \left(\|u\|_0^2 + \sum_{j=1}^n \|L_j(u)\|_0^2 \right) \quad \forall u \in \mathcal{C}_0^\infty(U),$$

where U is a relatively compact open subset of M and $\epsilon > 0$, $C > 0$ are positive constants, depending on U and on $L_1, \ldots, L_n \in \mathfrak{Z}(M)$, and $\mathcal{C}_0^\infty(U)$ is the space of smooth complex valued functions on M, with compact support contained in U. It is known that this estimate implies the $\mathcal{C}^\infty$-hypoellipticity in U of $\sum_{j=1}^n L_j^* L_j$ (see e.g. [16, 12]).

Our result directly applies to the study of the overdetermined systems of first order partial differential operators on M, that are canonically associated to $\mathfrak{Z}(M)$ and to a $\mathbb{C}$-linear connection on a complex vector bundle $E \xrightarrow{\pi} M$.

We also investigate the $\mathcal{C}^\infty$-hypoellipticity of more general second order partial differential operators on M, of the form

$$(0.2) \qquad P(u) = \sum_{j=1}^n \bar{L}_j L_j(u) + L_0(u) + a\, u,$$

where a is a complex valued smooth function in $\mathcal{C}^\infty(M)$, $L_0, L_1, \ldots, L_n$ are smooth complex vector fields on M, and only the imaginary part of L_0 is required to belong to the $\mathcal{C}^\infty(M)$-linear span of $L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n$.

Our original motivation, and also our main applications, involve CR manifolds. However, the study of (0.1) and of the operators (0.2) is of independent interest, and related questions have been considered recently in [18, 7, 6, 22, 23]. These papers show that the consideration of complex L_j's brings completely new phenomena, compared with the real case (see e.g. [14, 8]).

Condition (1.21) for the subellipticity of $\mathfrak{Z}(M)$ can be viewed as a generalization, at the same time, of both the *essential pseudoconcavity* of [12] in CR geometry and Hörmander's condition of [14] for the generalized Kolmogorov equation. In fact, in §2 we prove that, at points that are *generic* for $\mathfrak{Z}(M)$ (in a sense made precise in Definition 2.15), it is equivalent to a condition (2.32), that involves semidefinite *generalized Levi forms* attached to $\mathfrak{Z}(M)$ and their kernels. This quite explicit formulation was suggested by specific examples from [1]. However, conditions (1.21) and (2.32) apply to more general contexts than CR geometry. At the beginning, we assume neither that $\mathfrak{Z}(M)$ is a distribution of constant rank, nor that it satisfies any formal integrability condition, nor anything special about the intersection $\mathfrak{Z}(M) \cap \overline{\mathfrak{Z}(M)}$; but conditions of this type need to be imposed in §2 to obtain the equivalence of (1.21) and (2.32).

The work of [18, 7, 6] shows that the condition that the Lie algebra generated by $\mathfrak{Z}(M)$ spans the full complexified tangent space is in general not sufficient either for subellipticity or for the $\mathcal{C}^\infty$-hypoellipticity of the sum

of squares. The present work is complementary, inasmuch as our sharpest results hold away from singularities and in the case where $\mathfrak{Z}(M)$ is a Lie algebra.

We reduce the question of the subellipticity of $\mathfrak{Z}(M)$ to one involving *real* vector fields. Indeed, our task is to find all real vector fields X that are *enthralled* by $\mathfrak{Z}(M)$, i.e. satisfy

$$(0.3) \qquad \|X(u)\|_{\epsilon-1}^2 \leqslant C \left(\sum_{j=1}^{n} \|L_j(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U)$$

for some $L_1, \ldots, L_n \in \mathfrak{Z}(M)$ and $\epsilon > 0, C > 0$ depending on X and U. In §1 we observe that this set is a module over the Lie algebra $\mathfrak{A}_{\mathfrak{Z}}(M)$ generated by those real vector fields X for which (0.3) holds with $\epsilon = 1$. This was essentially shown in [16]. By an argument of J.J.Kohn (see [18] and Lemma 1.8 below), we know that $(\text{Re}\, Z)$ satisfies (0.3) with $\epsilon = \frac{1}{2}$ for all $Z \in \mathfrak{Z}(M)$. On the other hand, on an open dense subset M' of M that excludes some *singular* points for $\mathfrak{Z}(M)$, one can check that a real vector field X satisfying (0.3) with $\epsilon > \frac{1}{2}$ is necessarily the real part of some $Z \in \mathfrak{Z}(M)$. Hence we conjecture that the real vector fields X, satisfying (0.3), coincide on a dense open subset of M with the elements of the $\mathfrak{A}_{\mathfrak{Z}}(M)$-module generated by the real parts of the elements of $\mathfrak{Z}(M)$. In §2 we characterize, outside the singular set, the Lie algebra $\mathfrak{A}_{\mathfrak{Z}}(M)$. Then equality (2.31), together with Lemma 1.8, would give a complete and explicit description of the set of real vector fields enthralled by $\mathfrak{Z}(M)$.

A motivation for [12] was to understand the structure of a large class of CR manifolds, of higher CR codimension, that are of finite type in the sense of [4], and are not pseudoconcave in the sense of [11], because of the vanishing of their scalar Levi forms in some characteristic codirections. To prove also in this case a subelliptic estimate for the tangential Cauchy-Riemann operator on functions, two of the authors invented in [12] the notion of *essential pseudoconcavity*. Compared with the usual more restrictive notion of pseudoconcavity, it allows some of the scalar Levi forms to be zero. The subelliptic estimate is then obtained for weaker Sobolev norms than the $\frac{1}{2}$-norm in [11]. On essentially pseudoconcave almost CR manifolds the maximum modulus principle and the weak unique continuation principle for CR functions ([12, 13]) are valid. Under the additional assumption of formal integrability of the CR structure, it was also possible to prove, in case the CR manifold is either compact or real analytic, finiteness and vanishing theorems for the highest cohomology groups of the $\bar{\partial}_M$-complexes ([21]).

Our new condition is more general than the weak pseudoconcavity of [12], as here we allow the scalar Levi forms in some characteristic codirections to be semidefinite. As a heuristic motivation, consider a CR submanifold M of a complex manifold F. If M is contained in a strictly pseudoconvex real hypersurface of F, then it is easy to find L^2_{loc} germs of CR functions that are not smooth. We can expect, vice versa, that all CR distributions on M are smooth when all germs of hypersurfaces through M in F are strictly pseudoconcave. This is indeed the case when our condition (2.32) is verified. It is discussed in §5.

As in [12], the homogeneous examples have strongly inspired our investigation. After considering general homogeneous CR manifolds in §6, in §7 we classify all minimal orbits of real forms $\mathbf{G}$ of $\mathbf{G}^{\mathbb{C}}$-homogeneous flag manifolds (see e.g. [24, 1, 2]) that enjoy condition (1.21). In [1, §13], together with Prof. Medori, two of the authors gave the complete classification of the essentially pseudoconcave minimal orbits. Here we show that those satisfying (1.21) form a strictly larger class of CR manifolds, on which the local CR distributions are smooth.

We collected our results on hypoellipticity in §4. Given a complex vector bundle $E \xrightarrow{\pi} M$, endowed with a $\mathbb{C}$-linear connection ∇, for each complex vector field Z on M we obtain a differential operator ∇_Z, acting on the sections of E. We prove that the subellipticity of $\mathfrak{Z}(M)$ implies the $\mathcal{C}^{\infty}$-hypoellipticity of $(\nabla_Z)_{Z \in \mathfrak{Z}(M)}$. This yields, for a compact M, finite dimensionality of the space of global solutions of $(\nabla_Z(u))_{Z \in \mathfrak{Z}(M)} = 0$, and closedness of the range of $(\nabla_Z)_{Z \in \mathfrak{Z}(M)}$. For CR manifolds, this implies that the cohomology groups $H^{p,1}_{\bar{\partial}_M}(M)$ of the tangential Cauchy-Riemann complex are Hausdorff.

The subellipticity of $\mathfrak{Z}(M)$ gives $\mathcal{C}^{\infty}$-hypoellipticity for the sum of squares and also for slightly more general operators (see Theorem 4.3). The question of generalizing the Kolmogorov equation to the case of complex vector fields is more complicated. We obtained two different formulations in Theorems 4.4 and 4.8. In the former, we obtain $\mathcal{C}^{\infty}$-hypoellipticity under the condition that the $\mathfrak{A}_{\mathfrak{Z}}(M)$-module generated by the real parts of the vector fields in $\mathfrak{Z}(M)$ and the "time" vector field span the whole tangent space at a point. This distinction between "space" and "time" vector fields makes this result weaker than the one of [14] for the case $\mathfrak{Z}(M) = \overline{\mathfrak{Z}(M)}$. This is the reason we prove in Theorem 4.8 a result generalizing [14], under the condition that the real parts of the vector fields in $\mathfrak{Z}(M)$ satisfy (0.3) with $\epsilon = 1$.

1. Subelliptic systems of complex vector fields

Let M be a smooth *real* manifold of dimension m. For U open in M, we denote by $\mathcal{C}^\infty(U)$ (resp. $\mathcal{C}_0^\infty(U)$) the space of *complex valued* smooth functions (resp. with compact support) in U, and by $\mathfrak{X}(U)$ (resp. $\mathfrak{X}^\mathbb{C}(U)$) the Lie algebra of the smooth real (resp. complex) vector fields in U.

DEFINITION 1.1. — *A distribution of complex vector fields $\mathfrak{Z}(M)$ in M is any $\mathcal{C}^\infty(M)$-left submodule of $\mathfrak{X}^\mathbb{C}(M)$. This means that $Z_1 + Z_2$ and fZ belong to $\mathfrak{Z}(M)$ if $Z_1, Z_2, Z \in \mathfrak{Z}(M)$ and $f \in \mathcal{C}^\infty(M)$.*
For each point $p \in M$ we set

$$(1.1) \qquad Z_p M = \{Z(p) \mid Z \in \mathfrak{Z}(M)\} \subset T_p^\mathbb{C} M = \mathbb{C} \otimes_\mathbb{R} T_p M.$$

The dimension of the complex vector space $Z_p M$ is called the rank *of $\mathfrak{Z}(M)$ at p. We do not assume in the definition that $\mathfrak{Z}(M)$ has a constant rank. The points where the rank of $\mathfrak{Z}(M)$ is not constant on a neighborhood are the* singularities *of $\mathfrak{Z}(M)$. If $\mathfrak{Z}(M) = \{Z \in \mathfrak{X}^\mathbb{C}(M) \mid Z(p) \in Z_p M, \ \forall p \in M\}$, we say that $\mathfrak{Z}(M)$ has at most simple singularities.*
We say that $\mathfrak{Z}(M)$ is formally integrable *if*

$$(1.2) \qquad\qquad\qquad [\mathfrak{Z}(M), \mathfrak{Z}(M)] \subset \mathfrak{Z}(M).$$

Distributions of real *vector fields and their singular points are defined likewise.*

DEFINITION 1.2. — *The distribution $\mathfrak{Z}(M)$ is said to be* subelliptic *at $p \in M$ if there is an open neighborhood U of p in M, a real $\epsilon > 0$, a constant $C > 0$, and a finite set $L_1, \ldots, L_n$ of vector fields from $\mathfrak{Z}(M)$, such that (0.1) is satisfied, where $\| \cdot \|_\epsilon$ and $\| \cdot \|_0$ are the ϵ-Sobolev norms and the L^2-norm with respect to some Riemannian metric on M (see e.g. [9]).*

Remark 1.3. — When $\mathfrak{Z}(M)$ is the complexification of a distribution of *real* vector fields $\mathfrak{Y}(M) \subset \mathfrak{X}(M)$, then (0.1), at a generic point $p \in M$, is equivalent to the fact that $\mathfrak{Z}(M)$ and its higher order commutators span the whole complexified tangent space $T_p^\mathbb{C} M$ (see, e.g. [14, 8]). However this condition is neither necessary nor sufficient, and does not imply $\mathcal{C}^\infty$-hypoellipticity of the associated sum of squares operators when the vector fields are complex (cf. [18, 7, 6, 23]).

DEFINITION 1.4. — *We say that $\mathfrak{Z}(M)$ enthralls a vector field $Z \in \mathfrak{X}^{\mathbb{C}}(M)$ if*

(1.3)
$$\begin{cases} \forall U^{\text{open}} \Subset M, \ \exists L_1, \ldots, L_n \in \mathfrak{Z}(M), \ \exists \epsilon > 0, \ C > 0 \ s.t. \\ \|Z(u)\|_{\epsilon-1}^2 \leqslant C \left(\sum_{j=1}^n \|L_j(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^{\infty}(U). \end{cases}$$

Set

(1.4)
$$\mathbb{S}_{\mathfrak{Z}}(M) = \{ Z \in \mathfrak{X}^{\mathbb{C}}(M) \mid \mathfrak{Z}(M) \text{ enthralls } Z \}.$$

We notice that $\mathbb{S}_{\mathfrak{Z}}(M)$ is a distribution of complex vector fields containing $\mathfrak{Z}(M)$, and $\mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$ is a distribution of real vector fields, and that both are spaces of global sections of fine sheaves of left modules, the first over the sheaf of germs of *complex* valued smooth functions, the second over the sheaf of germs of *real* valued smooth functions on M.

By the real Frobenius theorem we obtain (see e.g. [14])

PROPOSITION 1.5. — *Let $\mathfrak{W}_{\mathfrak{Z}}(M)$ be the Lie subalgebra of $\mathfrak{X}^{\mathbb{C}}(M)$ generated by $\mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$. Then, if $\mathfrak{W}_{\mathfrak{Z}}(M)$ is a distribution of constant rank on M, we have*

(1.5)
$$\mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M) \subset \mathbb{S}_{\mathfrak{Z}}(M) \subset \mathfrak{W}_{\mathfrak{Z}}(M).$$

Remark 1.6. — The complex distribution $\mathfrak{Z}(M)$ is subelliptic at $p \in M$ if and only if

(1.6)
$$\{ X(p) \mid X \in \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M) \} = T_p M.$$

Remark 1.7. — If $Z \in \mathbb{S}_{\mathfrak{Z}}(M)$ and Λ_{-1} is a properly supported pseudo-differential operator of order (-1) having its symbol in the class $S_{1,0}^{-1}$, then $\Lambda_{-1} \circ Z$ is a *subelliptic multiplier* for $\mathfrak{Z}(M)$ in the sense of J.J.Kohn (see e.g. [17]).

Pseudodifferential operators will be an important tool in the following. For their definition and properties we refer to [15, Chapter XVIII].

If U is an open subset of M, and s a real number, we shall denote by $\Psi^s(U)$ the space of properly supported pseudodifferential operators in U, of order less or equal than s, with symbol in $S_{1,0}^s$. For each coordinate neighborhood $V \subset U$, a $\Lambda \in \Psi^s(U)$ is defined, in the local coordinates, by

(1.7)
$$\Lambda(u) = \iint_{U \times \mathbb{R}^m} e^{i\langle x-y, \xi \rangle} a(x, \xi) u(y) dy d\xi, \quad \text{for } u \in \mathcal{C}_0^{\infty}(V),$$

with $a \in \mathcal{C}^\infty(V \times \mathbb{R}^m)$ and

$$(1.8) \quad \begin{cases} \forall K \Subset V, \ \forall \alpha, \beta \in \mathbb{N}^m, \ \exists C = C(K, \alpha, \beta) > 0 \text{ s.t.} \\ \left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| \leqslant C \left(1 + |\xi|\right)^{s - |\beta|}, \ \forall (x, \xi) \in K \times \mathbb{R}^m. \end{cases}$$

The fact that $\Lambda \in \Psi^s(U)$ is *properly supported* means that for every $K \Subset U$ there is $K' \Subset U$ such that

$$(1.9) \qquad \qquad \operatorname{supp}(u) \subset K \Rightarrow \operatorname{supp}(\Lambda(u)) \subset K'.$$

The following Lemma is essentially contained in [18, p.949].

LEMMA 1.8. — *Let $Z \in \mathfrak{X}^\mathbb{C}(M)$ be any complex vector field. For every relatively compact open subset $U \Subset M$ there is a constant $C > 0$ such that*

$$(1.10) \qquad \|\bar{Z}(u)\|^2_{-\frac{1}{2}} \leqslant C \left(\|Z(u)\|_0^2 + \|u\|_0^2 \right) \quad \forall u \in \mathcal{C}_0^\infty(U).$$

Hence

$$(1.11) \qquad \qquad \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)} \subset \mathbb{S}_{\mathfrak{Z}}(M),$$

$$(1.12) \qquad \qquad \{Z + \bar{Z} \mid Z \in \mathfrak{Z}(M)\} \subset \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M).$$

Proof. — Let $Z \in \mathfrak{Z}(M)$ and $U^{\text{open}} \Subset M$. Then, with $\Lambda_0 \in \Psi^0(U)$ and constants $C_1, C_2 > 0$, we obtain

$$\|\bar{Z}(u)\|^2_{-\frac{1}{2}} = (\bar{Z}(u)|\Lambda_0(u))_0 \leqslant |(\Lambda_0^*(u)|Z(u))_0| + C_1 \|u\|_0^2$$
$$\leqslant C_2 \left(\|Z(u)\|_0 + \|u\|_0 \right) \|u\|_0, \quad \forall u \in \mathcal{C}^\infty(U).$$

This yields (1.10), and hence also (1.11) and (1.12). $\qquad \square$

In particular, if $\mathfrak{Z}(M) \neq \{0\}$, the distribution $\mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$ of *real* vector fields that are enthralled by $\mathfrak{Z}(M)$ is not trivial.

DEFINITION 1.9. — *We denote by $\mathbb{E}_{\mathfrak{Z}}(M)$ the set of $Z \in \mathfrak{X}^\mathbb{C}(M)$ such that*

$$(1.13) \quad \begin{cases} \forall U^{\text{open}} \Subset M, \ \exists L_1, \ldots, L_n \in \mathfrak{Z}(M), \ \exists C > 0 \text{ s.t} \\ \|Z(u)\|_0^2 \leqslant C \left(\sum_{j=1}^n \|L_j(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

Also $\mathbb{E}_{\mathfrak{Z}}(M)$ is a distribution of complex vector fields, with

$$(1.14) \qquad \qquad \mathfrak{Z}(M) \subset \mathbb{E}_{\mathfrak{Z}}(M) \subset \mathbb{S}_{\mathfrak{Z}}(M).$$

As a consequence of Lemma 1.8, we get

LEMMA 1.10. —

$$(1.15) \qquad \qquad \mathbb{E}_{\mathfrak{Z}}(M) + \overline{\mathbb{E}_{\mathfrak{Z}}(M)} \subset \mathbb{S}_{\mathfrak{Z}}(M).$$

DEFINITION 1.11. — *Set*

(1.16) $\mathbb{A}_3(M) = \mathbb{E}_3(M) \cap \mathfrak{X}(M),$

(1.17) $\mathfrak{A}_3(M) =$ *the Lie subalgebra of* $\mathfrak{X}(M)$ *generated by* $\mathbb{A}_3(M),$

(1.18) $\mathbb{T}_3^{(0)}(M) = \{Z + \bar{Z} \mid Z \in \mathfrak{z}(M)\},$

(1.19) $\mathbb{T}_3^{(h)}(M) = \left\langle [X,Y] \mid X \in \mathbb{A}_3(M), \ Y \in \mathbb{T}_3^{(h-1)}(M)\} \right\rangle, \quad$ *for* $h \geqslant 1,$

(1.20) $\mathfrak{T}_3(M) = \sum_{h=0}^{\infty} \mathbb{T}_3^{(h)}(M).$

We shall consider the condition at $p \in M$*:*

(1.21) $\{X(p) \mid X \in \mathfrak{T}_3(M)\} = T_p M.$

Remark 1.12. — If condition (1.21) is satisfied at a point $p_0 \in M$, then it is also satisfied at all points p in an open neighborhood U of p_0.

It is convenient to introduce the notation $[Z_1, \ldots, Z_m]$ for the *higher order commutator* of smooth real or complex vector fields. It is recursively defined by

(1.22) $\begin{cases} [Z_1] = Z_1, \\ [Z_1, Z_2] = Z_1 Z_2 - Z_2 Z_1, \\ [Z_1, Z_2, \ldots, Z_m] = [Z_1, [Z_2, \ldots, Z_m]] \quad \text{for } m \geqslant 3. \end{cases}$

PROPOSITION 1.13. — *The distribution of real vector fields* $\mathfrak{T}_3(M)$ *is an* $\mathfrak{A}_3(M)$*-Lie-module.*

Proof. — We prove by recurrence on $r \geqslant 1$ that, if $X_1, \ldots, X_r \in \mathbb{A}_3(M)$ and $Y \in \mathfrak{T}_3(M)$, then also $[[X_1, \ldots, X_r], Y] \in \mathfrak{T}_3(M)$. This follows from the definition of $\mathfrak{T}_3(M)$ for $r = 1$. Assume now that $r > 1$ and that $\mathfrak{T}_3(M)$ contains all commutators $[[X_1, \ldots, X_{r-1}], Y]$ with $X_1, \ldots, X_{r-1} \in \mathbb{A}_3(M)$ and $Y \in \mathfrak{T}_3(M)$. If $X_1, \ldots, X_r \in \mathbb{A}_3(M)$ and $Y \in \mathfrak{T}_3(M)$, we obtain

$$[[X_1, \ldots, X_r], Y] = -[[X_2, \ldots, X_r], [X_1, Y]] + [X_1, [[X_2, \ldots, X_r], Y]].$$

Since $[X_1, Y] \in \mathfrak{T}_3(M)$, by our inductive assumption the first summand on the right hand side also belongs to $\mathfrak{T}_3(M)$. By the inductive assumption, the commutator $[[X_2, \ldots, X_r], Y]$ belongs to $\mathfrak{T}_3(M)$, and hence also the second summand in the right hand side belongs to $\mathfrak{T}_3(M)$. The proof is complete.   □

PROPOSITION 1.14. — *The Lie algebra of real vector fields $\mathfrak{A}_{\mathfrak{Z}}(M)$ is contained in $\mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$, and $\mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$ is an $\mathfrak{A}_{\mathfrak{Z}}(M)$-Lie-submodule of $\mathfrak{X}(M)$. In particular, we have the inclusion*

$$(1.23) \qquad \mathfrak{T}_{\mathfrak{Z}}(M) \subset \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M).$$

Proof. — Let U be a relatively compact open subset of M. Assume that $X \in \mathbb{A}_{\mathfrak{Z}}(M)$, $Y \in \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$ and let $\epsilon > 0$ be such that that $Y = Z$ satisfies the estimate in (1.3). We can assume that $0 < \epsilon \leqslant \frac{1}{2}$. If U' is an open relatively compact subset of U, we have, with some $\Lambda_{\epsilon-1} \in \Psi^{\epsilon-1}(U)$ and suitable positive constants C_0, C_1,

$$
\begin{aligned}
\|[X,Y](u)\|_{\frac{\epsilon}{2}-1}^2 &\leqslant |([X,Y](u)|\Lambda_{\epsilon-1}(u))_0| \\
&\leqslant |(XY(u)|\Lambda_{\epsilon-1}(u))_0| + |(YX(u)|\Lambda_{\epsilon-1}(u))_0| \\
&\leqslant \left|(\Lambda_{\epsilon-1}^*(Y(u))|X(u))_0\right| + |(X(u)|\Lambda_{\epsilon-1}Y(u))| \\
&\qquad + C_0\|u\| \left(\|u\|_0 + \|Y(u)\|_{\epsilon-1} + \|X(u)\|_0\right) \\
&\leqslant C_1 \left(\|X(u)\|_0^2 + \|Y(u)\|_{\epsilon-1}^2 + \|u\|_0^2\right), \qquad \forall u \in \mathcal{C}^\infty(U')
\end{aligned}
$$

The last term of this chain of inequalities is bounded by a constant times $\left(\sum_{j=1}^n \|L_j(u)\|^2 + \|u\|_0^2\right)$, for a suitable choice of $L_1, \ldots, L_n \in \mathfrak{Z}(M)$. This shows that $[X,Y] \in \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$.

Since $\mathbb{A}_{\mathfrak{Z}}(M) \subset \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$, also $\mathfrak{A}_{\mathfrak{Z}}(M) \subset \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$. The argument in the proof of Proposition 1.13 shows that, since $[\mathbb{A}_{\mathfrak{Z}}(M), \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)] \subset \mathbb{S}_{\mathfrak{Z}}(M) \cap \mathfrak{X}(M)$, this distribution is an $\mathfrak{A}_{\mathfrak{Z}}(M)$-Lie-submodule of $\mathfrak{X}(M)$. Then the inclusion (1.23) is a consequence of the inclusion (1.12). $\square$

By using Proposition 1.14 we obtain

COROLLARY 1.15. — *Let $\mathfrak{Z}(M)$ be a smooth distribution of complex vector fields on M. Then $\mathfrak{Z}(M)$ is subelliptic at all points $p \in M$ at which condition (1.21) of Definition 1.11 is satisfied.*

Corollary 1.15 is a trivial consequence of the inclusion (1.23). However, in §2 we will show that actually we are able, in case $\mathfrak{Z}(M)$ is formally integrable, to compute explicitly the left hand side of (1.21) at the points of an open dense subset of M, where $\mathfrak{Z}(M)$ satisfies some genericity assumptions.

Remark 1.16. — When $\mathfrak{Z}(M)$ is the complexification of a distribution of *real* vector fields, then $\mathfrak{A}_{\mathfrak{Z}}(M) = \mathfrak{T}_{\mathfrak{Z}}(M)$, and condition (1.21) is equivalent to Hörmander's condition in [14]. Thus Corollary 1.15 can be viewed as a generalization of the analogous result for distributions of real vector fields.

Remark 1.17. — If $\mathfrak{Z}(M)$ is the distribution of $(0,1)$-vector fields of an almost CR manifold M, it follows from §2 below that the *essential pseudoconcavity* condition of [12] implies that $\mathbb{E}_\mathfrak{Z}(M) = \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$ and that M is of finite type in the sense of [4]. Therefore, Corollary 1.15 also generalizes [12, Theorem 4.1].

2. The distributions $\mathbb{K}_\mathfrak{Z}(M)$ and $\Theta_\mathfrak{Z}(M)$

As pointed out after the statement of Corollary 1.15, condition (1.21) becomes an effective criterion for subellipticity when it is possible to give an explicit description of $\mathbb{E}_\mathfrak{Z}(M)$, or of some nontrivial part of it. We begin by giving an upper bound for $\mathbb{E}_\mathfrak{Z}(M)$.

LEMMA 2.1. — *Let $\mathfrak{Z}(M)$ be a distribution of complex vector fields. If $(\mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)})$ has at most simple singularities, and in particular if $(\mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)})$ has constant rank, then*

$$(2.1) \qquad\qquad \mathbb{E}_\mathfrak{Z}(M) \subset \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}.$$

Proof. — Since $\mathbb{E}_\mathfrak{Z}(M) \subset \mathbb{E}_{\mathfrak{Z}+\overline{\mathfrak{Z}}}(M)$, we can reduce the proof to the case where $\mathfrak{Z}(M) = \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$ is the complexification of a distribution of real vector fields and $Z \in \mathbb{E}_\mathfrak{Z}(M)$ is real. Fix $p \in M$ and take a coordinate patch U of p in M, centered at p, for which (1.13) holds, for real $L_1, \ldots, L_n \in \mathfrak{X}(M) \cap \mathfrak{Z}(M)$ such that $L_1(p), \ldots, L_n(p)$ generate $Z_p M$. We apply the inequality in (1.13) to the test function $u_\tau(x) = \tau^{\frac{m-4}{4}} \chi(x) e^{i\tau\langle x, \xi\rangle - (\tau/2)|x|^2}$, where $\chi(x) \in \mathcal{C}_0^\infty(U)$ satisfies $\chi(x) = 1$ for x in a neighborhood of 0. Denote by $z(x, \xi)$ and $\ell_j(x, \xi)$ the symbols of Z and L_j, respectively. We obtain

$$L_j(u_\tau) = \tau^{\frac{m-4}{4}} \left(\tau \ell_j(x, \xi + ix) + L_j(\chi)\right) e^{i\tau\langle x, \xi\rangle - (\tau/2)|x|^2}.$$

Computing the integral by the change of variables $y = x\sqrt{\tau}$, we obtain

$$\|L_j(u_\tau)\|_0^2 = \int \chi^2(y/\sqrt{\tau}) |\ell_j(y/\sqrt{\tau}, i\xi + y/\sqrt{\tau})|^2 e^{-|y|^2} dy + O(\tau^{-\infty}).$$

Likewise, we have

$$\|Z(u_\tau)\|_0^2 = \int \chi^2(y/\sqrt{\tau}) |z(y/\sqrt{\tau}, i\xi + y/\sqrt{\tau})|^2 e^{-|y|^2} dy + O(\tau^{-\infty}).$$

By letting τ tend to ∞ in the estimate (1.13), we obtain that

$$|z(0, \xi)|^2 \leqslant C \sum_{j=1}^n |\ell_j(0, \xi)|^2, \quad \forall \xi \in \mathbb{R}^m.$$

Since $L_1, \ldots, L_n$ are real, the above inequality implies that $Z(p) \in Z_p M$. Since p was an arbitrary point of M, this implies that $Z \in \mathfrak{Z}(M)$.   □

Remark 2.2. — The proof of Lemma 2.1 yields a stronger statement:

Let $\mathfrak{Z}(M)$ be a distribution of complex vector fields and assume that $\mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$ has at most simple singularities. If $Z \in \mathbb{S}_{\mathfrak{Z}}(M)$ and

$$(2.2) \quad \begin{cases} \forall U^{\mathrm{open}} \Subset M, \quad \exists \epsilon > \dfrac{1}{2}, \ \exists L_1, \dots, L_n \in \mathfrak{Z}(M), \ \exists C > 0 \ \text{s.t.} \\[2mm] \|Z(u)\|_{\epsilon-1}^2 \leqslant C \left(\displaystyle\sum_{j=1}^n \|L_j(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U), \end{cases}$$

then $Z \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$.

It suffices indeed to apply (2.2), in a coordinate patch U, as in the proof of Lemma 2.1, to the test functions $v_\tau = \tau^{(1-\epsilon')/2} u_\tau$, with $\frac{1}{2} < \epsilon' < \epsilon$, and let $\tau \to +\infty$. Then, if $\ell_j(0, \xi) = 0$ for $j = 1, \dots, n$, the right hand side of (2.2) stays bounded, while the left hand side tends to $+\infty$, unless $z(p, \xi) = 0$.

2.1. The distribution $\Theta_{\mathfrak{Z}}(M)$

Lemma 2.1 suggests that, in order to find non trivial elements of $\mathbb{E}_{\mathfrak{Z}}(M)$, one should search in $\overline{\mathfrak{Z}(M)}$. To this aim, we introduce the following

DEFINITION 2.3. — *Given a distribution $\mathfrak{Z}(M)$ of complex vector fields, we set*

$$(2.3) \quad \Theta_{\mathfrak{Z}}(M) = \left\{ Z \in \mathfrak{Z}(M) \, \middle| \, \begin{array}{l} \exists r \geqslant 0, \ \exists Z_1, \dots, Z_r \in \mathfrak{Z}(M), \ \text{s.t.} \\ [Z, \bar{Z}] + \sum_{j=1}^r [Z_j, \bar{Z}_j] \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)} \end{array} \right\}.$$

LEMMA 2.4. — *The set $\Theta_{\mathfrak{Z}}(M)$ is a distribution of complex vector fields.*

Proof. — Clearly, if $Z \in \Theta_{\mathfrak{Z}}(M)$ and $\phi \in \mathcal{E}(M)$, the product ϕZ also belongs to $\Theta_{\mathfrak{Z}}(M)$. To prove that $\Theta_{\mathfrak{Z}}(M)$ contains the finite sums of its elements, it suffices to show that, if, for some $r \geqslant 1$, $Z_0, \dots, Z_r \in \Theta_{\mathfrak{Z}}(M)$ and $\sum_{j=0}^r [Z_j, \bar{Z}_j] \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$, then also $Z_0 + Z_1 \in \Theta_{\mathfrak{Z}}(M)$. This follows from: $[Z_0 + Z_1, \overline{Z_0 + Z_1}] + [Z_0 - Z_1, \overline{Z_0 - Z_1}] = 2 \left([Z_0, \bar{Z}_0] + [Z_1, \bar{Z}_1] \right)$. □

LEMMA 2.5. — *Let $\mathfrak{Z}(M)$ be a distribution of complex vector fields and let $\Theta_{\mathfrak{Z}}(M)$ be defined by (2.3). Then*

$$(2.4) \quad \overline{\Theta_{\mathfrak{Z}}(M)} \subset \mathbb{E}_{\mathfrak{Z}}(M).$$

In particular, if $Z \in \Theta_{\mathfrak{Z}}(M)$, then $Z + \bar{Z} \in \mathbb{A}_{\mathfrak{Z}}(M)$.

Proof. — The proof closely follows that of [12, Theorem 2.5]. If $Z \in \Theta_{\mathfrak{Z}}(M)$, by (2.3), there are $Z_1, \ldots, Z_r, Z_{r+1} \in \mathfrak{Z}(M)$ such that

$$(2.5) \qquad [Z, \bar{Z}] + \sum_{j=1}^{r} [Z_j, \bar{Z}_j] = Z_{r+1} - \bar{Z}_{r+1}.$$

Set $Z_0 = Z$ and let $Z_j^* = -\bar{Z}_j + a_j$, with $a_j \in \mathcal{C}^\infty(M)$, the L^2 formal adjoint of Z_j, for $0 \leqslant j \leqslant r + 1$. Integrating by parts, and using (2.5) to compute the sum of the commutators, we obtain, for all $u \in \mathcal{C}_0^\infty(M)$,

$$\sum_{j=0}^{r} \|\bar{Z}_j(u)\|_0^2 = -\sum_{j=0}^{r} ((Z_j - \bar{a}_j)(\bar{Z}_j(u))|u)_0$$

$$= -\sum_{j=0}^{r} \left\{ ([Z_j, \bar{Z}_j](u)|u)_0 + (\bar{Z}_j Z_j(u)|u)_0 - (\bar{Z}_j(u)|a_j u)_0 \right\}$$

$$= \sum_{j=0}^{r} \|Z_j(u)\|_0^2 + \sum_{j=0}^{r+1} \operatorname{Re}(Z_j(u)|b_j u) + \operatorname{Re}(u|b'u)_0,$$

where $b_j, b' \in \mathcal{C}^\infty(M)$. Hence the inequality in (1.13) (with $\bar{Z}$ replacing Z) follows, with $n = r + 2$ and $L_j = Z_{j-1}$ for $j = 1, \ldots, r + 2$. □

2.2. The characteristic bundle and the scalar Levi forms

Next we define the characteristic bundle of $\mathfrak{Z}(M)$ and the analogues, for a general distribution $\mathfrak{Z}(M)$, of the scalar Levi forms of CR manifolds.

DEFINITION 2.6. — *The characteristic bundle of $\mathfrak{Z}(M)$ is the set $H^0 M \subset T^* M$, consisting of the real covectors ξ with $\langle L, \xi \rangle = 0$ for all $L \in \mathfrak{Z}(M)$.*

If the set $H_p^0 M$ of characteristic covectors at $p \in M$ is $\{0\}$, we say that $\mathfrak{Z}(M)$ is elliptic at p.

For each $p \in M$, the set $H_p^0 M = H^0 M \cap T_p^* M$ is a vector space. Its dimension $\dim_{\mathbb{R}} H_p^0 M$ is an upper semicontinuous function of $p \in M$. In particular, if $\mathfrak{Z}(M)$ is elliptic at a point $p_0 \in M$, it is elliptic for p in an open neighborhood U of p_0. In this case (0.1) is valid with $\epsilon = 1$ by Gårding's inequality. Hence obstructions to the validity of the subelliptic estimate (0.1) may come only from the characteristic codirections of $\mathfrak{Z}(M)$.

We restate condition (1.21) in terms of the characteristic bundle.

PROPOSITION 2.7. — *Condition (1.21) at $p \in M$ is equivalent to*

(2.6)
$$
\begin{cases}
\forall \xi \in H_p^0 M, \text{ with } \xi \neq 0, \\
\exists Z_0 \in \mathfrak{Z}(M), Z_1, \ldots, Z_r \in \mathbb{E}_{\mathfrak{Z}}(M) \cap \overline{\mathbb{E}_{\mathfrak{Z}}(M)}, \\
\quad s.t. \quad i\xi([Z_1, \ldots, Z_r, \bar{Z}_0]) \neq 0,
\end{cases}
$$

Proof. — This follows because the elements of $\mathfrak{T}_{\mathfrak{Z}}(M)$ can be expressed as linear combinations of the real parts of the $[Z_1, \ldots, Z_r, \bar{Z}_0]$'s, with $Z_0 \in \mathfrak{Z}(M)$ and $Z_1, \ldots, Z_r \in \mathbb{E}_{\mathfrak{Z}}(M) \cap \overline{\mathbb{E}_{\mathfrak{Z}}(M)}$, and vice versa, the real and imaginary parts of these $[Z_1, \ldots, Z_r, \bar{Z}_0]$ belong to $\mathfrak{T}_{\mathfrak{Z}}(M)$. $\qquad\square$

DEFINITION 2.8. — *If $\xi \in H_p^0 M$, we define the* scalar Levi form *of $\mathfrak{Z}(M)$ at ξ as the Hermitian symmetric form*

(2.7)
$$
\mathfrak{L}_\xi(L_1, \bar{L}_2) = i\xi([L_1, \bar{L}_2]) \quad \text{for} \quad L_1, L_2 \in \mathfrak{Z}(M).
$$

The value of the right hand side of (2.7) only depends on the values $L_1(p)$, $L_2(p)$ of the two vector fields L_1, L_2 at the base point $p = \pi(\xi)$. Thus (2.7) is a Hermitian symmetric form on the finite dimensional complex vector space $Z_p M$.

Remark 2.9. — In the case where $\mathfrak{Z}(M)$ is the space of $(0,1)$-vector fields of an abstract CR manifold, it was shown in [11] that the subelliptic estimate (0.1) is valid with $\epsilon = 1/2$ under the assumption that, for every $\xi \in H_p^0 M \setminus \{0\}$, the Levi form $\mathfrak{L}_\xi$ is indefinite; this assumption was weakened to allow $\mathfrak{L}_\xi = 0$ for some nonzero characteristics ξ in [12]. These results suggest that the obstructions to the validity of (0.1) come from the characteristic ξ's for which $\mathfrak{L}_\xi \neq 0$ is semidefinite.

2.3. The distribution $\mathbb{K}_{\mathfrak{Z}}(M)$

Our next aim is to relate $\mathbb{E}_{\mathfrak{Z}}(M)$ and the *Levi forms* associated to $\mathfrak{Z}(M)$.

DEFINITION 2.10. — *Define*

(2.8)
$$
H^\oplus M = \left\{ \xi \in H^0 M \mid \mathfrak{L}_\xi \geqslant 0 \right\},
$$

(2.9)
$$
\mathbb{K}_{\mathfrak{Z}}(M) = \{ Z \in \mathfrak{Z}(M) \mid \mathfrak{L}_\xi(Z, \bar{Z}) = 0, \ \forall \xi \in H^\oplus M \}.
$$

PROPOSITION 2.11. — *For every distribution $\mathfrak{Z}(M) \subset \mathfrak{X}^{\mathbb{C}}(M)$, the set $\mathbb{K}_{\mathfrak{Z}}(M)$ is also a distribution.*

Assume in addition that $\mathfrak{Z}(M)$ is formally integrable and that $\mathfrak{Z}(M)$, $\mathfrak{Z}(M) \cap \overline{\mathfrak{Z}(M)}$ are both distributions of constant rank. Then

(2.10)
$$
\mathfrak{Z}(M) \cap \overline{\mathbb{E}_{\mathfrak{Z}}(M)} \subset \mathbb{K}_{\mathfrak{Z}}(M).
$$

Proof. — The first claim is a consequence of the fact that the set of isotropic vectors of a semidefinite Hermitian symmetric form is a complex linear space.

To complete the proof, we need to show that, if $\xi \in H^\oplus M$, we have $\mathcal{L}_\xi(Z, \bar{Z}) = 0$ for all $Z \in \mathfrak{Z}(M) \cap \overline{\mathbb{E}_{\mathfrak{Z}}(M)}$. Having fixed $\xi \in H^\oplus M$, we can argue in a small coordinate patch U about its base point $p = \pi(\xi)$. By the assumption that $\mathfrak{Z}(M)$ is formally integrable, we can choose real coordinates $x_1, \ldots, x_m$, centered at p, such that, for a pair of nonnegative integers h, ℓ with $2h + \ell \leqslant m$, setting $z_j = x_j + i\, x_{h+j}$ for $j = 1, \ldots, h$, a system of generators of $\mathfrak{Z}(M)$ in U is given by the vector fields

$$(2.11) \qquad \begin{cases} L_j = \frac{\partial}{\partial \bar{z}_j} + L'_j & \text{for } j = 1, \ldots, h, \\ L_{h+j} = \frac{\partial}{\partial x_{m+1-j}} & \text{for } j = 1, \ldots, \ell. \end{cases}$$

Here $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial x_{h+j}}\right)$, $L'_j \in \mathfrak{X}^{\mathbb{C}}(U)$ satisfies $L'_j(0) = 0$, and

$$(2.12) \qquad [L_i, L_j] = 0 \quad \text{for} \quad 1 \leqslant i, j \leqslant h + \ell.$$

This is obtained by first noticing that the real vector fields in $\mathfrak{Z}(M)$ are a formally integrable distribution of real vector fields. By the classical Frobenius theorem, we can choose a system of local coordinates in which this real distribution is locally generated by the $L_{h+1}, \ldots, L_{h+\ell}$ above. By linear algebra we can obtain, in a neighborhood of p, from any basis of $\mathfrak{Z}(M)$ that includes $L_{h+1}, \ldots, L_{h+\ell}$, a new one in which the $L_1, \ldots, L_h$ have the property that $L_j - \frac{\partial}{\partial x_j}$ does not contain either $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, h$, or $\frac{\partial}{\partial x_{m+1-i}}$ for $i = 1, \ldots, \ell$. By this choice we obtain (2.12). Let

$$(2.13) \qquad L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{i=1}^{m-\ell} a_j^i \frac{\partial}{\partial x_i}, \quad \text{with } a_j^i(0) = 0.$$

We can assume, by a change of coordinates, that

$$(2.14)$$
$$L_j(a_r^i)(0) = \frac{\partial a_r^i(0)}{\partial \bar{z}_j} = 0 \quad \text{for} \quad j, r = 1, \ldots, h, \ i = 2h+1, \ldots, m - \ell.$$

In fact, by the formal integrability condition, it follows that

$$(2.15) \qquad \frac{\partial}{\partial \bar{z}_j} + \sum_{i=2h+1}^{m-\ell} \left(\sum_{r=1}^{h} \frac{\partial a_j^i(0)}{\partial \bar{z}_r} \bar{z}_r \right) \frac{\partial}{\partial x_i}, \quad \text{for } j = 1, \ldots, h,$$

are commuting vector fields, and, since they also commute with their conjugates, by a change of the coordinates $x_1, \ldots, x_{m-\ell}$ we can obtain a new coordinate system for which (2.14) is also satisfied. Let $\xi \in \mathbb{R}^m$ be such that $\ell_j(0, \xi) = 0$ for $j = 1, \ldots, h + \ell$, where by $\ell_j(x, \xi)$ we indicate the

symbols of the differential operators L_j. This means that the components ξ_i of ξ are zero for $1 \leqslant i \leqslant 2h$ and $(m - \ell) < i \leqslant m$. By the formal integrability condition, there is a second degree homogeneous polynomial $q_\xi(x) \in \mathbb{C}[x_1, \ldots, x_{m-\ell}]$ such that

$$(2.16) \qquad L_j(i\langle x, \xi\rangle + q_\xi(x)) = O(|x|^2) \quad \text{for } x \to 0.$$

Next we observe that, by identifying ξ with the corresponding element in $T_0^* \mathbb{R}^m$, and setting $v_\xi = (i\langle x, \xi\rangle + q_\xi(x))$, we obtain

$$0 = d^2(i\langle x, \xi\rangle + q_\xi(x))(Z, \bar{Z}) = Z\bar{Z}(v_\xi)(0) - \bar{Z}Z(v_\xi)(0) - \mathcal{L}_\xi(Z, \bar{Z}),$$
$$\forall Z \in \mathfrak{X}^{\mathbb{C}}(U).$$

Thus, in particular, $Z\bar{Z}v_\xi(0) = \mathcal{L}_\xi(Z, \bar{Z}) \geqslant 0$ if $Z \in \mathfrak{z}(M)$ and $\xi \in H_0^{\oplus}M$. By (2.14), we have $Z\bar{Z}v_\xi(0) = Z\bar{Z}q_\xi(0)$. Consider the expression for q_ξ as a polynomial in $z_1, \ldots, z_h, \bar{z}_1, \ldots, \bar{z}_h, x_{2h+1}, \ldots, x_{m-\ell}$,

$$(2.17) \qquad \begin{aligned} q_\xi(x) &= q_\xi^{2,0,0}(z, z) + q_\xi^{1,1,0}(z, \bar{z}) + q_\xi^{0,2,0}(\bar{z}, \bar{z}) + q_\xi^{1,0,1}(z, x'') \\ &\quad + q_\xi^{0,1,1}(\bar{z}, x'') + q_\xi^{0,0,2}(x'', x''), \end{aligned}$$

where $x'' = (x_{2h+1}, \ldots, x_{m-\ell})$. The assumption that $\mathcal{L}_\xi \geqslant 0$ means that $q_\xi^{(1,1,0)}(z, \bar{z}) \geqslant 0$. We can add to q_ξ any second degree homogeneous polynomial f in $\mathbb{C}[z, x'']$, since $L_j(f) = O(|x|^2)$ for any such polynomial. In this way, we obtain a new q_ξ, still satisfying (2.16), with the property that

$$(2.18) \qquad \text{Re}(q_\xi)(x) \geqslant 0, \ \forall x \in \mathbb{R}^m.$$

Fix any real valued function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ with $\chi(x) = 1$ for $|x| \leqslant 1$, $0 \leqslant \chi(x) \leqslant 1$ in $\mathbb{R}^m$, and $\chi(x) = 0$ for $|x| \geqslant 2$. For large $\tau > 0$ the function

$$(2.19) \qquad u_\tau = \sqrt[6]{\tau^m}\, \chi(x\sqrt[3]{\tau}) e^{-\tau(i\langle x, \xi\rangle + q_\xi(x))}$$

belongs to $\mathcal{C}_0^\infty(U)$. We have

$$(2.20) \qquad \|u_\tau\|_0^2 = \int_{\mathbb{R}^m} e^{-2\sqrt[3]{\tau}q_\xi(x)} \chi^2(x) dx \leqslant \int_{\mathbb{R}^m} \chi^2(x) dx,$$

because of (2.18). If $Z \in \mathfrak{z}(M)$, we have:

$$|Z(u_\tau)|^2 = \sqrt[3]{\tau^m} \left| -\tau\chi(x\sqrt[3]{\tau}/2)Z(v_\xi) + \sqrt[3]{\tau}[Z(\chi)](x\sqrt[3]{\tau}) \right|^2 e^{-2\tau\text{Re}\,q_\xi(x)}$$
$$\leqslant C_0 \sqrt[3]{\tau^m} \cdot \tau^2 |x|^4 \chi(x\sqrt[3]{\tau}/2) \quad \text{for } \tau \gg 1,$$

with a positive constant C_0. Indeed $\chi(x\sqrt[3]{\tau}/2) = 1$ when $Z(u_\tau) \neq 0$, and we used the fact that $Z(v_\xi) = O(|x|^2)$ and that $Z(\chi) = O(|x|^2)$. Computing $\|Z(u_\tau)\|_0^2$ by making the change of coordinates $y = x\sqrt[3]{\tau}$ shows that, with a constant $C_1 > 0$ independent of τ,

$$(2.21) \qquad \|Z(u_\tau)\|_0^2 \leqslant C_1 \sqrt[3]{\tau^2} \quad \text{for } \tau \gg 1.$$

On the other hand, for $u \in \mathcal{C}_0^\infty(U)$, we have, with a constant $C_2 > 0$ independent of u,

$$(2.22) \qquad \|\bar{Z}(u)\|_0^2 \geqslant \|Z(u)\|_0^2 - \operatorname{Re}([Z, \bar{Z}](u)|u)_0 - C_2\|u\|_0^2.$$

For $u = u_\tau$ and by using, while computing the integral, the change of variables $y = x\sqrt{\tau}$, we obtain

$$(2.23) \qquad -\operatorname{Re}([Z, \bar{Z}](u_\tau)|u_\tau)_0 = \tau \int \mathcal{L}_\xi(Z, \bar{Z}) e^{-2\operatorname{Re} q_\xi(y)} \chi^2(y/\sqrt[6]{\tau}) dy$$
$$+ O(\sqrt[3]{\tau^2}) \qquad \text{for } \tau \gg 1.$$

Since $\chi^2(y/\sqrt[6]{\tau})$ is increasing with τ, we get, with a constant $C_3 \geqslant 0$,

$$(2.24)$$
$$-\operatorname{Re}([Z, \bar{Z}](u_\tau)|u_\tau)_0 \geqslant \tau \int \mathcal{L}_\xi(Z, \bar{Z}) e^{-2\operatorname{Re} q_\xi(y)} \chi^2(y) dy - C_3|\tau|^{\frac{3}{2}}$$
$$\text{for } \tau \gg 1.$$

By (2.20) and (2.21), $(\sum_{j=1}^{h+\ell} \|L_j(u_\tau)\|_0^2 + \|u\|_0^2)$ is $O(\sqrt[3]{\tau^2})$ for $\tau \to \infty$. Thus, by (2.22) and (2.24) we obtain that $\mathcal{L}_\xi(Z, \bar{Z}) = 0$ for $Z \in \mathfrak{Z}(M) \cap \overline{\mathbb{E}\mathfrak{Z}(M)}$. $\qquad \square$

Remark 2.12. — By a slight variant of the proof of Proposition 2.11 we obtain: *Assume that $\mathfrak{Z}(M)$ is formally integrable and that $\mathfrak{Z}(M)$ and $\mathfrak{Z}(M) \cap \overline{\mathfrak{Z}(M)}$ are both distributions of constant rank. Let $p \in M$. If there exists $\xi \in H_p^0 M$ such that $\mathcal{L}_\xi$ is definite on a complement of $Z_p M \cap \overline{Z_p M}$ in $Z_p M$, then $\mathfrak{Z}(M)$ is not subelliptic at p.*

PROPOSITION 2.13. — *For every distribution of complex vector fields $\mathfrak{Z}(M)$, we have*

$$(2.25) \qquad\qquad \Theta\mathfrak{Z}(M) \subset \mathbb{K}\mathfrak{Z}(M).$$

Assume that

$$(2.26) \qquad \mu_0(p) = \dim_\mathbb{R}(Z_p M + \overline{Z_p M})$$

$$(2.27) \qquad \delta_0(p) = \dim_\mathbb{R} \langle H_p^\oplus M \rangle,$$

$$(2.28) \qquad \nu_0(p) = \dim_\mathbb{C}\{Z(p) \in Z_p M \mid Z \in \mathbb{K}\mathfrak{Z}(M)\}$$

are constant in M. Then

$$(2.29) \qquad\qquad \Theta\mathfrak{Z}(M) = \mathbb{K}\mathfrak{Z}(M).$$

Finally, if $\mathfrak{Z}(M)$ is formally integrable and of constant rank,

$$(2.30) \qquad\qquad \mathbb{E}\mathfrak{Z}(M) = \mathfrak{Z}(M) + \overline{\mathbb{K}\mathfrak{Z}(M)},$$

$$(2.31) \qquad\qquad \mathbb{A}\mathfrak{Z}(M) = \{Z + \bar{Z} \mid Z \in \mathbb{K}\mathfrak{Z}(M)\}.$$

Proof. — By the definition of $\Theta_3(M)$, we have $\mathcal{L}_\xi(Z, \bar{Z}) = 0$ for all $\xi \in H^\oplus M$ and $Z \in \Theta_3(M)$. Thus (2.25) is always valid.

To prove that, under the additional assumptions, we have the equality (2.29), we apply an argument similar to the one employed in [12, Theorem 2.5].

By the constancy of $\mu_0(p)$, the characteristic set $H^0 M$ is a smooth real vector bundle on M. Then the assumption that $\delta_0(p)$ is constant implies that $H^\oplus M$ generates a smooth linear subbundle of $H^0 M$, and therefore the quotient $H^0 M / \langle H^\oplus M \rangle$ is a smooth real linear bundle on M.

Since $\nu_0(p)$ is constant,

$$M \ni p \rightarrow K_p M = \{Z(p) \in Z_p M \mid Z \in \mathbb{K}_3(M)\}$$

is a complex vector bundle KM on M. The map $\xi \rightarrow \mathcal{L}_\xi|_{KM}$ is injective from the quotient bundle $H^0 M / \langle H^\oplus M \rangle$ to the bundle $\mathrm{Herm}(KM)$ of Hermitian symmetric forms on KM. We denote by LM the image bundle.

The dual bundle $\mathrm{Herm}^*(KM)$ of $\mathrm{Herm}(KM)$ is the real linear subbundle of $KM \otimes_M \overline{KM}$ generated by the elements of the form $Z(p) \otimes \bar{Z}(p)$, for $p \in M$ and $Z(p) \in K_p M$. The annihilator bundle $L^0 M$ of LM in $\mathrm{Herm}^*(KM)$ contains, for all $p \in M$, positive definite elements of $\mathrm{Herm}^*(KM)$. Since the positive definite elements of $L^0 M$ form an open set in $L^0 M$, it is easy to construct a global section $\mathfrak{z}$ of $L^0 M$ with $\mathfrak{z}(p) > 0$ for all $p \in M$, by first constructing local sections, and then patching them together by a partition of unity. If $p_0 \in M$ and $Z \in \mathbb{K}_3(M)$, with $Z(p_0) \neq 0$, by the standard Gram-Schmidt orthogonalization process, we can find a smooth function $\phi \in \mathcal{E}(M)$ with $\phi(p_0) \neq 0$ and sections $Z_2, \ldots, Z_k \in \mathbb{K}_3(M)$, such that, by setting $Z_1 = \phi \cdot Z$, we have $\mathfrak{z}(p) = \sum_{j=1}^{k} Z_j(p) \otimes \bar{Z}_j(p)$ for p in an open neighborhood U of p_0 in M, where k is the rank of the complex bundle KM. The equality

$$\sum_{j=1}^{k} \mathcal{L}_\xi(Z_j, \bar{Z}_j) = 0 \quad \forall p \in U, \ \forall \xi \in H_p^0 M$$

implies that

$$\sum_{j=1}^{k} [Z_j, \bar{Z}_j] \in \mathfrak{z}(U) + \overline{\mathfrak{z}(U)}.$$

By repeating this argument for a set of elements of $\mathbb{K}_3(M)$ whose values at p_0 give a basis for $K_{p_0} M$, we prove that every $Z \in \mathbb{K}_3(M)$ coincides, on an open neighborhood of p_0 in M, with the restriction of an element of $\Theta_3(M)$. Moreover, the number of elements of $\Theta_3(M)$ that are needed in (2.3) does not exceed $\nu = k^2 - k$. Thus, if $Z \in \mathbb{K}_3(M)$, we can find an

open covering $\{U_a\}$ of M and, for each a, some $Z_2^{(a)}, \ldots, Z_\nu^{(a)} \in \Theta_{\mathfrak{Z}}(M)$, for which

$$[Z, \bar{Z}] + \sum_{j=1}^{\nu} [Z_j^{(a)}, \bar{Z}_j^{(a)}] \in \mathfrak{Z}(U_a) + \overline{\mathfrak{Z}(U_a)}, \quad \forall a.$$

We can assume that the covering $\{U_a\}$ has a finite index. Then, by using a partition of unity $\{\chi_a\}$ subordinated to $\{U_a\}$, and summing together vector fields $\chi_a \cdot Z_j^{(a)}$ with disjoint supports, we end up with a finite subset $Z_1, \ldots, Z_r$ of vector fields in $\mathfrak{Z}(M)$ such that

$$\mathcal{L}_\xi(Z, \bar{Z}) + \sum_{j=1}^{r} \mathcal{L}_\xi(Z_j, \bar{Z}_j) = 0,$$

showing that $Z \in \Theta_{\mathfrak{Z}}(M)$.

When moreover $\mathfrak{Z}(M)$ is formally integrable and has constant rank, we obtain (2.30) as a consequence of Lemma 2.5 and Proposition 2.11. $\square$

Definition 2.14. — *We say that a higher Levi form concavity condition is satisfied at the point $p \in M$ if*

$$(2.32) \quad \begin{cases} \forall \xi \in H_p^0 M \setminus \{0\} \text{ with } \mathcal{L}_\xi \geqslant 0, \\ \exists Z_0 \in \mathfrak{Z}(M), Z_1, \ldots, Z_r \in \mathbb{K}_{\mathfrak{Z}}(M) + \overline{\mathbb{K}_{\mathfrak{Z}}(M)} \\ \text{with } i\xi([Z_1, \ldots, Z_r, \bar{Z}_0]) \neq 0. \end{cases}$$

Definition 2.15. — *We say that the distribution $\mathfrak{Z}(M)$ is regular at a point $p_0 \in M$ if its rank, and the functions $\mu_0(p)$ of (2.26), $\delta_0(p)$ of (2.27), $\nu_0(p)$ of (2.28), are all constant in an open neighborhood U of p_0.*

Since the rank of $\mathfrak{Z}(M)$, and the functions μ_0 and ν_0 are all integral valued and semicontinuous, and δ_0 is semicontinuous on the dense open subset where μ_0 is constant, the set of regular points of $\mathfrak{Z}(M)$ is open and dense in M.

Proposition 2.16. — *Let p_0 be a regular point for $\mathfrak{Z}(M)$. Then (2.32) implies (1.21). If in addition we assume that $\mathfrak{Z}(M)$ is formally integrable, then the two conditions (2.32) and (1.21) are equivalent.*

Proof. — The regularity assumption implies that all $\xi \in H_{p_0}^0 M$ with $\ker \mathcal{L}_\xi \supset \Theta_{\mathfrak{Z}}(M)$ belong to the linear span $\langle H_{p_0}^{\oplus} M \rangle$. Thus condition (2.6) holds if $\xi \in H_{p_0}^0 M \setminus \{0\}$ and $\ker \mathcal{L}_\xi \supset \Theta_{\mathfrak{Z}}(M)$. When $\ker \mathcal{L}_\xi \not\supset \Theta_{\mathfrak{Z}}(M)$, the restriction of $\mathcal{L}_\xi$ to $\Theta_{\mathfrak{Z}}(M)$ is semidefinite, and hence there are $Z_1, Z_2 \in \Theta_{\mathfrak{Z}}(M)$ with $\mathcal{L}_\xi(Z_1, \bar{Z}_2) = i\xi([Z_1, \bar{Z}_2]) \neq 0$.

Then because of Propositions 2.7 and 2.13, and Lemma 2.5, we obtain that (2.32) implies condition (1.21) at regular points of $\mathfrak{Z}(M)$.

If in addition $\mathfrak{Z}(M)$ is formally integrable, then by the equality (2.30), the opposite implication is also true. $\qquad\square$

3. Pullbacks of Distributions

Let M, N be smooth real manifolds, and $N \xrightarrow{\varpi} M$ a smooth submersion.

Given a distribution of complex vector fields $\mathfrak{Z}(M)$, its pullback $[\varpi^*\mathfrak{Z}](N)$ consists of all $W \in \mathfrak{X}^{\mathbb{C}}(N)$ with the property that

(3.1)
$$\forall U^{\text{open}} \subset M, \forall \sigma \in \mathcal{C}^{\infty}(U, N), \text{ with } \varpi \circ \sigma = \text{id}_U, \ d\varpi^{\mathbb{C}}(W \circ \sigma) \in \mathfrak{Z}(U).$$

Let

(3.2) $\qquad V^{\varpi}N = \{v \in TN \mid d\varpi(v) = 0\} \qquad$ be the vertical bundle, and

(3.3) $\qquad \mathfrak{V}^{\varpi}(N) = \mathcal{C}^{\infty}(N, V^{\varpi}N) \qquad\qquad$ the vertical distribution.

LEMMA 3.1. — Let $N \xrightarrow{\varpi} M$ be a smooth submersion. Then

(3.4) $\qquad \mathfrak{V}^{\varpi}(N) \subset [\varpi^*\mathfrak{Z}](M) \quad$ and $\quad [\mathfrak{V}^{\varpi}(N), [\varpi^*\mathfrak{Z}](N)] \subset [\varpi^*\mathfrak{Z}](N).$

Proof. — We note that $[\varpi^*\mathfrak{Z}](N)$ is the space of global sections of a fine sheaf of left $\mathcal{C}^{\infty}$-modules. Thus by localization we can reduce the discussion to the case where $N = M \times \Omega$ for an open subset Ω of a Euclidean space $\mathbb{R}^k$, and ϖ is the projection onto the first factor, in which situation the statement is trivial. $\qquad\square$

DEFINITION 3.2. — If $N \xrightarrow{\iota} M$ is a smooth immersion, we define the pullback $[\iota^*\mathfrak{Z}](N)$ of $\mathfrak{Z}(M)$ to N to be the set of complex vector fields $Z' \in \mathfrak{X}^{\mathbb{C}}(N)$ having the following property

(3.5)
$$\begin{cases} \forall q_0 \in N, \ \exists V^{\text{open}} \subset N \text{ with } V \ni p_0, \text{ and } Z \in \mathfrak{Z}(M) \\ \qquad \text{s.t.} \quad d\iota(q)(Z'(q)) = Z(\iota(q)) \ \forall q \in V. \end{cases}$$

Let M, N be smooth manifolds. A smooth map $N \xrightarrow{\phi} M$ is a submersion onto its image if there exists a smooth manifold S, a submersion $N \xrightarrow{\varpi} S$ and an immersion $S \xrightarrow{\iota} M$ that factorize ϕ, i.e. that make the following diagram commute:

(3.6)
$$\begin{array}{ccc} N & \xrightarrow{\phi} & M \\ \varpi \downarrow & & \uparrow \iota \\ S & = \!\!\!= & S \end{array}$$

DEFINITION 3.3. — *The pullback of $\mathfrak{Z}(M)$ by a map $N \xrightarrow{\phi} M$, which is a submersion onto its image, is the distribution of complex vector fields*

$$(3.7) \qquad\qquad [\phi^*\mathfrak{Z}](N) = [\varpi^*[\iota^*\mathfrak{Z}]](N),$$

where ϖ and ι are the maps in (3.6).

Remark 3.4. — If N is an open subset of M and $\iota : N \hookrightarrow M$ is the inclusion, then $\iota^*(\mathfrak{Z})(N) = \mathfrak{Z}(N)$ is the distribution on N that is generated by the restrictions to N of the vector fields $Z \in \mathfrak{Z}(M)$.

More generally, if $N \subset M$ is a smooth submanifold, and $\iota : N \hookrightarrow M$ the embedding map, then $[\iota^*\mathfrak{Z}](N)$ is the distribution generated by the restrictions to N of the vector fields $Z \in \mathfrak{Z}(M)$ with $Z(p) \in T_p^{\mathbb{C}}N$ for every $p \in N$.

DEFINITION 3.5. — *Let M, N be smooth manifolds, and $\mathfrak{Z}_M(M), \mathfrak{Z}_N(N)$ distributions of complex vector fields on M and N, respectively. Let $N \xrightarrow{\phi} M$ be a submersion onto its image. We say that ϕ is a Z-morphism if*

$$(3.8) \qquad\qquad \mathfrak{Z}_N(N) \subset [\phi^*\mathfrak{Z}_M](N).$$

Remark 3.6. — We keep the notation of Definition 3.5, and let ϖ, ι be the maps in (3.6). Set $\mathfrak{Z}_S(S) = [\iota^*\mathfrak{Z}_M](S)$. Then $N \xrightarrow{\phi} M$ is a Z-morphism if and only if $N \xrightarrow{\varpi} S$ is a Z-morphism.

LEMMA 3.7. — *Let M, N be two smooth real manifolds, and $N \xrightarrow{\varpi} M$ be a smooth submersion. Then*

$$(3.9) \qquad\qquad \mathbb{E}_{\varpi^*\mathfrak{Z}_M}(N) = [\varpi^*\mathbb{E}_{\mathfrak{Z}_M}](N)$$

$$(3.10) \qquad\qquad \mathbb{K}_{\varpi^*\mathfrak{Z}_M}(N) = [\varpi^*\mathbb{K}_{\mathfrak{Z}_M}](N)$$

$$(3.11) \qquad\qquad \Theta_{\varpi^*\mathfrak{Z}_M}(N) = [\varpi^*\Theta_{\mathfrak{Z}_M}](N).$$

Proof. — Again the statement becomes trivial after, by localization, we reduce to the case where $N = M \times \Omega$ with Ω open in $\mathbb{R}^k$ and ϖ being the projection onto the first factor. $\qquad\square$

PROPOSITION 3.8. — *Let M and N be smooth real manifolds, with assigned complex valued distributions of smooth complex vector fields $\mathfrak{Z}_M(M)$ and $\mathfrak{Z}_N(N)$, respectively. Let $N \xrightarrow{\phi} M$ be a smooth submersion onto the image and a Z-morphism. Let $q_0 \in N$ and $p_0 = \phi(q_0)$ be a regular point of $\mathfrak{Z}_M(M)$, according to Definition 2.15. Assume that*

   (1) *$\mathfrak{Z}_N(N)$ satisfies the higher Levi form concavity condition (2.32) at the point q_0;*
   (2) *the pullback $\phi_{q_0}^* : H_{p_0}^0 M \to H_{q_0}^0 N$ is injective.*

Then conditions (2.32) and (1.21) at p_0 are valid for $\mathfrak{Z}_M(M)$.

Proof. — Using Remark 3.6 we shall split the proof by separately considering the case in which ϕ is a submersion and the case where ϕ is an immersion.

First we assume that $N \xrightarrow{\phi} M$ is a smooth submersion. In this case $\mathbb{E}_{\mathfrak{Z}_N}(N) \subset \mathbb{E}_{\phi^*\mathfrak{Z}_M}(N)$. Moreover, (2) is automatically satisfied because $\mathfrak{Z}_N(N) \subset [\phi^*\mathfrak{Z}_M](N)$. The statement is trivially true, as it is easily checked by reducing it to the case where $N = M \times \Omega$, with Ω open in $\mathbb{R}^k$ and ϖ the projection onto the first coordinate.

To complete the proof of the general case, it suffices, by localization about q_0, to consider the case in which $N = S$ is a smooth submanifold of M, and $\mathfrak{Z}_N(N)$ are the restrictions to N of elements of $\mathfrak{Z}_M(M)$ with real and imaginary parts tangent to N at all points of N. By (2), (2.32) for $\mathfrak{Z}_N(N)$ at $q_0 = p_0$, implies that (2.32) is satisfied at p_0 for $\mathfrak{Z}_M(M)$. By the regularity assumption, this implies (1.21) at p_0 for $\mathfrak{Z}_M(M)$. □

4. Hypoellipticity for some differential operators of the first and of the second order

We keep the notation of §1, §2. Theorems 4.1, Corollary 4.2, and Theorem 4.3 below, which concern systems of first order partial differential operators, and second order partial differential operators closely related to sums of squares of vector fields, directly follow from the assumption that $\mathfrak{Z}(M)$ be subelliptic. The other results of this section, namely Theorems 4.4 and 4.8, refer to generalized parabolic second order operators, and are proved under conditions (4.26) and (4.43), respectively, that more directly involve the Lie structure of $\mathfrak{Z}(M)$ with respect to some *generalized time vector field*.

Let $E \xrightarrow{\pi} M$ be a complex vector bundle of rank r on M, endowed with a $\mathbb{C}$-linear connection

$$(4.1) \qquad \nabla : \mathfrak{X}(M) \times \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, E).$$

In a local trivialization $E|_U \simeq U \times \mathbb{C}^r$, the connection ∇ is described by the datum of a $\mathfrak{gl}(r, \mathbb{C})$-valued smooth one form $\gamma = (\gamma^\alpha_\beta) \in \mathcal{C}^\infty(U, \mathfrak{gl}(r, \mathbb{C}) \otimes T^*M)$. Using upper Greek letters for the components of the sections in $\mathcal{C}^\infty(U, E) \simeq \mathcal{C}^\infty(U, \mathbb{C}^r)$, we have

$$(4.2) \qquad (\nabla_X(\sigma))^\alpha = X\sigma^\alpha + \sum_{\beta=1}^r \gamma^\alpha_\beta(X)\sigma^\beta, \quad \text{for } \alpha = 1, \dots, r.$$

By $\mathbb{C}$-linearity, we can define, for each complex valued vector field $Z \in \mathfrak{X}^{\mathbb{C}}$, a linear partial differential operator

$$(4.3) \qquad \nabla_Z : \mathcal{C}^\infty(M, E) \ni \sigma \to \nabla_{\operatorname{Re} Z}(\sigma) + i\nabla_{\operatorname{Im} Z}(\sigma) \in \mathcal{C}^\infty(M, E).$$

Let us fix a smooth Riemannian metric g on M and a smooth Hermitian metric h on the fibers of $E \xrightarrow{\pi} M$. Then we can define the formal adjoint $\nabla_Z^* : \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, E)$ by

$$(4.4) \qquad \int_M h(\nabla_Z^* u, v) d\lambda_g = \int_M h(u, \nabla_Z v) d\lambda_g$$
$$\forall u, v \in \mathcal{C}^\infty(M, E), \text{ with } \operatorname{supp}(u) \cap \operatorname{supp}(v) \Subset M,$$

where $d\lambda_g$ is the Lebesgue density on M with respect to the Riemannian metric g.

We obtain an analogue of [12, Theorem 4.1]:

THEOREM 4.1. — *Let $E \xrightarrow{\pi} M$ be a smooth complex vector bundle of rank r on M and ∇ a $\mathbb{C}$-linear connection on $E \xrightarrow{\pi} M$. If $\mathfrak{Z}(M)$ is subelliptic at a point $p_0 \in M$, then any weak solution $u \in L^2_{\mathrm{loc}}(M, E)$ of*

$$(4.5) \qquad \nabla_Z u \in \mathcal{C}^\infty(M, E) \qquad \forall Z \in \mathfrak{Z}(M)$$

is equal, a.e. in an open neighborhood U of p_0 in M, to a smooth section of $E \xrightarrow{\pi} M$.

Formula (4.5) means that for every $Z \in \mathfrak{Z}(M)$ there is a smooth section $f_Z \in \mathcal{C}^\infty(M, E)$ such that

$$(4.6) \qquad \int_M h(u, \nabla_Z^* v) d\lambda_g = \int_M h(f_Z, v) d\lambda_g,$$
$$\forall v \in \mathcal{C}^\infty(M, E), \text{ with } \operatorname{supp}(v) \Subset M.$$

Proof. — The statement is local. Therefore by substituting for M a relatively compact open neighborhood of p_0, we can assume that $\mathfrak{Z}(M)$ is generated by a finite set of vector fields $L_1, \ldots, L_n$, and that, for some $\epsilon > 0$ and $C > 0$, we have the estimate

$$(4.7) \qquad \|v\|_\epsilon^2 \leqslant C \left(\sum_{j=1}^n \|L_j(v)\|_0^2 + \|v\|_0^2 \right), \quad \forall v \in \mathcal{C}_0^\infty(M).$$

We can also assume that E is the trivial bundle $M \times \mathbb{C}^r$ on M, so that (4.5) is equivalent to the system
(4.8)

$$L_j u^\alpha + \sum_{\beta=1}^r a_{j\,\beta}^\alpha(p) u^\beta = f_j^\alpha \in \mathcal{C}^\infty(M), \quad \text{for } j = 1, \ldots, n, \ \alpha = 1, \ldots, r,$$

with $a_{j\,\beta}^{\alpha} \in \mathcal{C}^{\infty}(M)$. Defining

(4.9)
$$
\begin{cases}
\mathfrak{d} : \mathcal{C}^{\infty}(M, \mathbb{C}^r) \to [\mathcal{C}^{\infty}(M, \mathbb{C}^r)]^n \quad \text{by} \\[2mm]
\mathfrak{d}((u^{\alpha})_{\alpha=1,\dots,r}) = \left(L_j u^{\alpha} + \displaystyle\sum_{\beta=1}^{r} a_{j\,\beta}^{\alpha}(p) u^{\beta} \right)_{\substack{1 \leqslant \alpha \leqslant r \\ 1 \leqslant j \leqslant n}}
\end{cases}
$$

we obtain from (4.7) that, with some new constant $C > 0$ and the same $\epsilon > 0$:

(4.10) $\|v\|_{\epsilon}^2 \leqslant C \left(\|\mathfrak{d}(v)\|_0^2 + \|v\|_0^2 \right), \quad \forall v \in \mathcal{C}_0^{\infty}(M, \mathbb{C}^r).$

We have, for $\chi \in \mathcal{C}_0^{\infty}(M)$,

(4.11) $(\mathfrak{d}(\chi\,u))_j^{\alpha} = \chi\,(\mathfrak{d}(u))_j^{\alpha} + (L_j(\chi)\,u^{\alpha}).$

In particular, if $u \in L_{\text{loc}}^2(M, \mathbb{C}^r)$ is a weak solution of (4.8), we have that $\mathfrak{d}(\chi\,u) \in \left[L^2(M, \mathbb{C}^r) \right]^n$. By applying Friedrichs' theorem on the identity of the weak and strong extensions of a first order partial differential operator, we can find a sequence $\{v_{\nu}\} \subset \mathcal{C}_0^{\infty}(M, \mathbb{C}^r)$ such that

(4.12) $v_{\nu} \to \chi\,u$ in $L^2(M, \mathbb{C}^r)$ and $\mathfrak{d}(v_{\nu}) \to \mathfrak{d}(\chi\,u)$ in $\left[L^2(M, \mathbb{C}^r) \right]^n$.

The subelliptic estimate (4.10) yields a uniform bound for the Sobolev ϵ-norm $\|v_{\nu}\|_{\epsilon}$. This implies that $\chi\,u \in W^{\epsilon}(M, \mathbb{C}^r)$, where $W^{\epsilon}(M, \mathbb{C}^r)$ denotes the Sobolev space of L^2-vector valued functions that have L^2-derivatives of the positive real order ϵ. Hence $u \in W_{\text{loc}}^{\epsilon}(M, \mathbb{C}^r)$.

To show that $u \in \mathcal{C}^{\infty}(M, \mathbb{C}^r)$, we use the Sobolev embedding theorem: it suffices to show that $u \in W_{\text{loc}}^s(M, \mathbb{C}^r)$ for all $s > 0$. Assume that we already know that this is true for some $s_0 > 0$. If $\chi \in \mathcal{C}_0^{\infty}(M)$, then

$$\mathfrak{d}(\chi\,u) = [\mathfrak{d}, \chi](u) + \chi\,\mathfrak{d}(u) \in W^{s_0}(M, \mathbb{C}^r)$$

and has compact support. Fix any scalar pseudodifferential operator $\Lambda_q \in \Psi^q(M)$, with $s_0 - \epsilon < q \leqslant s_0$. Then, by the continuity properties of classical pseudodifferential operators, we obtain

(4.13) $\mathfrak{d}(\Lambda_q(\chi u)) = [\mathfrak{d}, \Lambda_q](\chi u) + \Lambda_q(\mathfrak{d}(\chi u)) \in L_{\text{loc}}^2(M, \mathbb{C}^r),$

because Λ_q and the commutator $[\mathfrak{d}, \Lambda_q](u)$ have orders $\leqslant s_0$ and $\chi\,u, \mathfrak{d}(\chi\,u) \in W^{s_0}(M)$ and have compact support in M. Thus, by the argument above, $\Lambda_q(\chi\,u) \in W^{\epsilon}(M, \mathbb{C}^r)$. This implies that $\chi\,u \in W^{q+\epsilon}(M, \mathbb{C}^r)$, with $q + \epsilon > s$. Since χ was an arbitrary smooth function with compact support in M, this yields $u \in W_{\text{loc}}^{q+\epsilon}(M, \mathbb{C}^r)$. By recurrence we obtain that $u \in W_{\text{loc}}^s(M, \mathbb{C}^r)$ for all $s > 0$, and hence is equal a.e. to a smooth section. The proof is complete. $\square$

COROLLARY 4.2. — *Assume that M is compact and that $\mathfrak{Z}(M)$ is subelliptic at all points $p \in M$. Then*

(1) *the space*

$$(4.14) \qquad \mathcal{O}_{\mathfrak{Z}}(M) = \{\sigma \in \mathcal{C}^{\infty}(M, E) \mid \nabla_Z(\sigma) = 0, \ \forall Z \in \mathfrak{Z}(M)\}$$

   is finite dimensional.

(2) *The map*

$$(4.15) \quad \mathcal{C}^{\infty}(M, E) \times \mathfrak{Z}(M) \ni (\sigma, Z) \to (\nabla_Z(\sigma), Z) \in \mathcal{C}^{\infty}(M, E) \times \mathfrak{Z}(M)$$

   has a closed range. By this we mean that, if $\{\sigma_{\nu}\}$ is a sequence of sections in $\mathcal{C}^{\infty}(M, E)$ and for each $Z \in \mathfrak{Z}(M)$ there is $f_Z \in \mathcal{C}^{\infty}(M, E)$ such that $\nabla_Z(\sigma_{\nu})$ converges uniformly to f_Z in M, then there exists a section $\sigma \in \mathcal{C}^{\infty}(M, E)$ with $\nabla_Z(\sigma) = f_Z$ for all $Z \in \mathfrak{Z}(M)$.

Proof. — (1) follows from the Sobolev embedding theorem, because $\mathcal{O}_{\mathfrak{Z}}(M)$ is an L^2-closed subspace of $\mathcal{C}^{\infty}(M, E)$ on which the L^2 and the ϵ-Sobolev norm, for some $\epsilon > 0$, are equivalent. Finally, (2) is a consequence of the fact that, since M is compact, for a finite set $L_1, \ldots, L_n \in \mathfrak{Z}(M)$, some $\epsilon > 0$ and some $\mathrm{const} > 0$, on the L^2-orthogonal complement of $\mathcal{O}_{\mathfrak{Z}}(M)$, we have the coercive estimate

$$(4.16) \ \|u\|_{\epsilon}^2 \leqslant \mathrm{const} \left(\sum_{j=1}^{n} \|L_j(u)\|_0^2 \right), \quad \forall u \in \mathcal{C}^{\infty}(M, E) \cap [\mathcal{O}_{\mathfrak{Z}}(M)]^{\perp}. \qquad \square$$

THEOREM 4.3. — *Let $\mathfrak{Z}(M)$ be a distribution of complex vector fields.*

(1) *We can find a locally finite family $\{L_j\} \subset \mathfrak{Z}(M)$, such that, for any choice of $a \in \mathcal{C}^{\infty}(M)$, the second order operator*

$$(4.17) \qquad\qquad P(u) = a\,u + \sum_j \bar{L}_j L_j(u)$$

   is hypoelliptic at all points of M at which $\mathfrak{Z}(M)$ is subelliptic.

(2) *If $\mathfrak{Z}(M)$ is finitely generated then, for any set of generators $L_1, \ldots, L_n$ of $\mathfrak{Z}(M)$, and for any choice of $Z_0 \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$ and $a \in \mathcal{C}^{\infty}(M)$, the operator*

$$(4.18) \qquad\qquad P(u) = \sum_{j=1}^{n} \bar{L}_j L_j(u) + Z_0(u) + a\,u$$

   is hypoelliptic at all points p of M where $\mathfrak{Z}(M)$ is subelliptic.

Let M' be the open subset of M of points p where $\mathfrak{Z}(M)$ is subelliptic. Then the operators P of (1) and (2) satisfy the following:

(4.19)
$$\begin{cases} \forall U^{\mathrm{open}} \Subset M', \quad \exists \epsilon > 0,\ C > 0 \ \text{such that} \\ \|u\|_{\epsilon}^2 \leqslant C\left(|(P(u)|u)_0| + \|u\|_0^2\right), \quad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

Proof. — Let $\{U_\nu\}$ be an open covering of M' by relatively compact open subsets, and, for each of them, let $L_1^{(\nu)}, \ldots, L_{n_\nu}^{(\nu)} \in \mathfrak{Z}(M)$ be chosen in such a way that, for suitable $\epsilon_\nu > 0$, $C_\nu > 0$, we have the estimate

(4.20)
$$\|u\|_{\epsilon_\nu}^2 \leqslant C_\nu \left(\sum_{h=1}^{n_\nu} \|L_h^{(\nu)}(u)\|_0^2 + \|u\|_0^2\right), \quad \forall u \in \mathcal{C}_0^\infty(U_\nu).$$

Take smooth functions $\chi_\nu \in \mathcal{C}_0^\infty(U_\nu)$ such that $\mathrm{supp}(\chi_\nu) \Subset U_\nu$, the family $\{\mathrm{supp}(\chi_\nu)\}$ is locally finite, and $\sum_\nu |\chi_\nu(p)|^2 > 0$ for all $p \in M'$. Then (1) holds with $\{L_j\} = \{\chi_\nu L_h^{(\nu)}\}$.

By [19], the hypoellipticity of (4.17) and of (4.18) is a consequence of (4.20). Since it suffices to prove that for each $p \in M'$, there is a small open neighborhood $U \Subset M'$ of p for which (4.20) holds true, we can reduce the proof to the case where $\mathfrak{Z}(M)$ is the $\mathcal{C}^\infty(M)$-module generated by $L_1, \ldots, L_n \in \mathfrak{Z}(M)$, and the operator P is of the form (4.18).

By integration by parts we obtain

(4.21)
$$-(Pu|u)_0 = \sum_{j=1}^n \|L_j(u)\|_0^2 - (L_0(u)|u)_0 + (u|L_{n+1}(u))_0 + (a'u|u)_0$$
$$\forall u \in \mathcal{C}_0^\infty(M).$$

Since $L_1, \ldots, L_n$ generate $\mathfrak{Z}(M)$, we obtain that (0.1) is valid for every relatively compact open subset $U \Subset M'$, and this in turn, together with (4.21), implies (4.20). The proof is complete. $\qquad\square$

THEOREM 4.4. — *We keep the notation of Definition 1.11. Assume that $\mathfrak{Z}(M)$ is generated by a finite set $L_1, \ldots, L_n$ of complex vector fields. Let $X_0 \in \mathfrak{X}(M)$ be a real vector field. Let*

(4.22)
$$\mathbb{T}_0'(M) = \mathcal{C}^\infty(M, \mathbb{R}) \cdot X_0 + \mathfrak{T}_{\mathfrak{Z}}(M)$$

(4.23)
$$\mathbb{T}_h'(M) = [\mathfrak{A}_{\mathfrak{Z}}(M), \mathbb{T}_{h-1}'(M)], \quad \text{for } h \geqslant 1,$$

(4.24)
$$\mathfrak{T}'(M) = \sum_{h=0}^\infty \mathbb{T}_h'(M).$$

In particular, $\mathfrak{T}'(M)$ is the $\mathfrak{A}_{\mathfrak{Z}}(M)$-Lie-submodule of $\mathfrak{X}(M)$ generated by X_0 and $\mathfrak{T}_{\mathfrak{Z}}(M)$. Then, for any choice of $Y_0 \in \mathbb{T}_{\mathfrak{Z}}^{(0)}(M)$ and $a \in \mathcal{C}^\infty(M)$, the

second order partial differential operator

$$(4.25) \qquad P(u) = a \cdot u + X_0(u) + iY_0(u) + \sum_{j=1}^{n} \bar{L}_j L_j(u)$$

is hypoelliptic at all points $p \in M$ where

$$(4.26) \qquad \{X(p) \mid X \in \mathfrak{T}'(M)\} = T_p M.$$

We divide the proof of Theorem 4.4 into several steps. First we prove

LEMMA 4.5. — Let $U \Subset M$ be an open set and assume that there are $\epsilon > 0$, $C > 0$ and $A_1, \ldots, A_r \in \Psi^0(U)$ such that

$$(4.27) \qquad \begin{cases} \|u\|_\epsilon^2 + \displaystyle\sum_{j=1}^{n} \|L_j(u)\|_0^2 \leqslant C \left(\displaystyle\sum_{h=1}^{r} |(P(u)|A_h(u))_0| + \|u\|_0^2 \right), \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

Then for every real $s \geqslant 0$, and every open subset U' with $U' \Subset U$, there is a constant $C' = C(s, U')$ such that

$$(4.28) \quad \|u\|_{s+\epsilon}^2 + \sum_{j=1}^{n} \|L_j(u)\|_s^2 \leqslant C' \left(\|P(u)\|_s^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}^\infty(U').$$

Proof. — Let $\Lambda_s \in \Psi^s(U)$ be elliptic. Then we have, with real constants $C_1 > 0$, $C_2 \geqslant 0$, uniformly for $u \in \mathcal{C}_0^\infty(U')$,

$$\|u\|_{s+\epsilon}^2 + \sum_{j=1}^{n} \|L_j(u)\|_s^2 \leqslant C_1 \left(\|\Lambda_s(u)\|_\epsilon^2 + \sum_{j=1}^{n} \|\Lambda_s(L_j(u))\|_0^2 \right)$$

$$\leqslant C_1 \left(\|\Lambda_s(u)\|_\epsilon^2 + \sum_{j=1}^{n} \|L_j(\Lambda_s(u))\|_0^2 \right) + C_2 \|u\|_s^2$$

Thus, using the inequality

$$\begin{cases} \qquad\qquad \forall \delta > 0, \quad \exists C_\delta > 0 \quad \text{s.t} \\ \|u\|_s^2 \leqslant \delta \|u\|_{s+\epsilon}^2 + C_\delta \|u\|_0^2, \quad \forall u \in \mathcal{C}_0^\infty(U), \end{cases}$$

we obtain that the left hand side of (4.28) is bounded by a constant times

$$\sum_{h=1}^{r} |(P(\Lambda_s(u))|A_h(\Lambda_s(u)))_0| + \|u\|_0^2.$$

The operator $Q = P + \sum_{j=1}^{n} L_j^* L_j$ is an operator of the first order, with principal part $X_0 + iY_0$. Therefore if $A \in \Psi^0(U)$, we obtain

$$(P(\Lambda_s(u))|A(\Lambda_s(u)))_0 = (\Lambda_s(Q(u))|A(\Lambda_s(u)))_0 + O(\|u\|_s^2)$$

$$- \sum_{j=1}^{n} (L_j(\Lambda_s(u))|L_j(\Lambda_s(A(u))))_0$$

$$= (\Lambda_s(Q(u))|A(\Lambda_s(u)))_0 + O(\|u\|_s^2)$$

$$+ \sum_{j=1}^{n} (\bar{L}_j \Lambda_s L_j(u)|A(\Lambda_s(u)))_0 + O(\|L_j(u)\|_s \|u\|_s)$$

$$+ \sum_{j=1}^{n} ([\Lambda_s, L_j](u)|[L_j, A \circ \Lambda_s](u))_0$$

$$+ \sum_{j=1}^{n} ([\Lambda_s, L_j](u)|A \circ \Lambda_s(L_j(u)))_0$$

$$= (\Lambda_s(P(u))|A \circ \Lambda_s(u))_0 + O(\|u\|_s^2 + \sum_{j=1}^{n} \|L_j(u)\|_s^2),$$

$$\forall u \in \mathcal{C}^\infty(U'),$$

where we use $O(N(u))$ to indicate some quantity whose modulus is bounded by a constant times $N(u)$. This computation yields

$$(4.29) \quad \begin{cases} \forall U' \Subset U, \ \forall s \in \mathbb{R}_+, \ \exists A_1^{(2s)}, \ldots, A_r^{(2s)} \in \Psi^{(2s)}(U), \ \exists C_s' > 0 \text{ s.t} \\ \|u\|_{s+\epsilon}^2 + \sum_{j=1}^{n} \|L_j(u)\|_s^2 \leqslant C_s' \left(\sum_{h=1}^{r} \left| (P(u)|A_h^{(2s)}(u))_0 \right| + \|u\|_0^2 \right), \\ \hfill \forall u \in \mathcal{C}_0^\infty(U'). \end{cases}$$

Clearly (4.29) implies (4.28). $\square$

It is known (see e.g. [14, 18]) that

LEMMA 4.6. — If (4.28) is valid for all $s \in \mathbb{R}_+$ and all open subset $U' \Subset U$, then P is $\mathcal{C}^\infty$-hypoelliptic in U.

End of the proof of Theorem 4.4. — By the previous Lemmas, we only need to prove (4.27). First we note that, for all $u \in \mathcal{C}_0^\infty(U)$,

$$|\text{Re}\,(X_0(u)|u)_0| = \frac{1}{2} \left| \int_U X_0(u \bar{u}) d\lambda_g \right|$$

$$= \frac{1}{2} \left| \int_U |u|^2 X_0^*(1) d\lambda_g \right| \leqslant \frac{1}{2} \left(\sup_{p \in U} |X_0^*(1)| \right) \|u\|_0^2$$

Since, for some positive constant C_0 depending on U,

$$|(iY_0(u) + a\,u|u)_0| \leqslant C_0\|u\|_0 \left(\|u\|_0 + \sum_{j=1}^n \|L_j(u)\|_0\right), \quad \forall u \in \mathcal{C}_0^\infty(U),$$

we obtain, upon integrating by parts, that with a constant $C_1 > 0$,

$$(4.30) \qquad \sum_{j=1}^n \|L_j(u)\|_0^2 \leqslant -2(P(u)|u)_0 + C_1\|u\|_0^2, \quad \forall u \in \mathcal{C}_0^\infty(U).$$

Next we note that, taking for $A_0 \in \Psi^0(U)$ the composition with X_0 of an elliptic pseudodifferential operator $\Lambda_{-1} \in \Psi^{-1}(U)$, we have, with some constant $C_2 > 0$,

$$\|X_0(u)\|_{-\frac{1}{2}}^2 \leqslant C_2\left(|(X_0(u)|A_0(u))_0| + \|u\|_0^2\right), \quad \forall u \in \mathcal{C}_0^\infty(U).$$

We obtain, with constants $C_3, C_4 > 0$,

$$|(X_0(u)|A_0(u))| \leqslant |(P(u)|A_0(u))_0| + \left|\sum_{j=1}^n (L_j(u)|L_j(A_0(u)))_0\right|$$

$$+ C_3\|u\|\left(\|u\| + \sum_{j=1}^n \|L_j(u)\|_0\right)$$

$$\leqslant |(P(u)|A_0(u))_0| + \sum_{j=1}^n \|L_j(u)\|_0^2 + C_4\|u\|_0^2,$$

$$\forall u \in \mathcal{C}_0^\infty(U).$$

To complete the proof, it suffices to show that, if $Y \in \mathfrak{X}(M)$ satisfies, for some $\delta > 0$, for some $A_1, \ldots, A_r \in \Psi^0(U)$, and a constant $C' > 0$, the estimate

$$(4.31) \quad \|Y(u)\|_{\delta-1}^2 \leqslant C'\left(\sum_{h=1}^r |(P(u)|A_h(u))_0| + \|u\|_0^2\right), \quad \forall u \in \mathcal{C}_0^\infty(U),$$

and $X \in \mathbb{A}_3(M)$, then we have, with some $A_1', \ldots, A_{r'}' \in \Psi^0(U)$, and constants $\delta' > 0$, $C'' > 0$,

$$(4.32) \quad \begin{cases} \|[X,Y](u)\|_{\delta'-1}^2 \leqslant C''\left(\sum_{h=1}^{r'} |(P(u)|A_h'(u))_0| + \|u\|_0^2\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

Recall that the estimate (1.13) holds for $Z = X$, and hence

$$(4.33) \qquad \|X(u)\|_0^2 \leqslant c_X\left(|(P(u)|u)_0| + \|u\|_0^2\right) \quad \forall u \in \mathcal{C}_0^\infty(U),$$

with a constant $c_X > 0$. Then, with positive constants $c_i > 0$ and a $T \in \Psi^{\delta-1}(U)$, we get

$$\|[X,Y](u)\|_{\frac{\delta}{2}-1}^2 \leqslant c_0 \left(|([X,Y](u)|T(u))_0| + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U).$$

Assuming, as we can, that $\delta \leqslant \frac{1}{2}$, we obtain

$$|(XY(u)|T(u))_0| \leqslant |(T^*(Y(u))|X(u))_0| + |([T,X]^*(Y(u))|u)_0| + c_1\|u\|_0^2$$
$$\leqslant c_2\|Y(u)\|_{\delta-1}(\|X(u)\|_0 + \|u\|_0) + c_1\|u\|_0^2, \quad \forall u \in \mathcal{C}^\infty(U).$$

The last term, in view of (4.33), can be estimated by the right hand side of (4.32). Likewise

$$|(YX(u)|T(u))_0| \leqslant |(X(u)|T(Y(u)))_0| + |(X(u)|[T,Y](u))_0| + c_3\|u\|_0^2$$
$$\leqslant c_4\|X(u)\|_0 \left(\|Y(u)\|_{\delta-1} + \|u\|_0\right) + c_3\|u\|_0^2,$$
$$\forall u \in \mathcal{C}_0^\infty(U),$$

and in view of (4.33), also this last term can be estimated by the right hand side of (4.32). The proof is complete. $\qquad\square$

Remark 4.7. — Theorem 4.4 is weaker than the analogous statement in [14] in the case where the L_j's are real. Indeed, for $L_j \in \mathfrak{X}(M)$, setting $X_0 = L_0$, our assumption requires that all commutators $[L_{j_1}, \ldots, L_{j_{r-1}}, L_{j_r}]$ with $r \in \mathbb{Z}_+$, and $0 \leqslant j_h \leqslant n$, and $j_h > 0$ for $h < r$, span the tangent space T_pM. The statement in [14], also proved in [16] and [8], allows $j_h = 0$ also for $1 \leqslant h < r$. This motivates us to consider separately the special case where $\mathbb{E}_{\mathfrak{Z}}(M) = \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$: Theorem 4.8 below generalizes the case where all the L_j's are real.

THEOREM 4.8. — We keep the notation of Definition 1.11. Assume that

$$(4.34) \qquad L_1, \ldots, L_n \in \mathfrak{X}^{\mathbb{C}}(M) \text{ generate } \mathfrak{Z}(M),$$

$$(4.35) \qquad \mathbb{E}_{\mathfrak{Z}}(M) \supset \overline{\mathfrak{Z}(M)},$$

$$(4.36) \qquad L_0 \text{ is a real vector field},$$

$$(4.37) \qquad L_{n+1} \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)},$$

$$(4.38) \qquad a \in \mathcal{C}^\infty(M).$$

Let us define:

$$(4.39) \qquad \mathbb{A}'(M) = \mathbb{A}'_{(0)}(M) = \mathbb{A}_{\mathfrak{Z}}(M) + \mathcal{C}^\infty(M,\mathbb{R})\,L_0$$

$$(4.40) \qquad \mathbb{A}'_{(h)}(M) = [\mathbb{A}'(M), \mathbb{A}'_{(h-1)}(M)] \quad \text{for } h \geqslant 1,$$

$$(4.41) \qquad \mathfrak{T}''(M) = \sum_{h=0}^{\infty} \mathbb{A}'_{(h)}$$

Then the second order differential operator

$$(4.42) \qquad P(u) = L_0(u) + \sum_{j=1}^{n} \bar{L}_j L_j(u) + L_{n+1}(u) + a\,u$$

is hypoelliptic at all points $p \in M$ where

$$(4.43) \qquad \{X(p) \mid X \in \mathfrak{T}''(M)\} = T_p M.$$

Proof. — We shall prove that, for every $X \in \mathfrak{T}''(M)$,

$$(4.44) \qquad \begin{cases} \forall U^{\text{open}} \Subset M, \ \exists \epsilon > 0, \ \exists C > 0, \ \text{s.t.} \\ \|X(u)\|_{\epsilon-1}^2 \leqslant C \left(\|(P(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

We observe that the proof of Theorem 4.4 shows that, if $Y \in \mathfrak{X}(M)$ satisfies (4.44), and $X \in \mathbb{A}_3(M)$, then $[X,Y]$ also satisfies (4.44) (with $\epsilon/2$ substituting ϵ). Thus to show that all $X \in \mathfrak{T}''(M)$ satisfy (4.44), it suffices to prove the following ☐

LEMMA 4.9. — *If $Y \in \mathfrak{X}(M)$ satisfies (4.44), then $[L_0, Y]$ also satisfies (4.44).*

Proof. — We closely follow the argument in [16, p.66-68]. Condition (4.35) means that, for every $U \Subset M$ there is a constant $C_0 > 0$ such that

$$(4.45) \qquad \sum_{j=1}^{n} \|\bar{L}_j(u)\|_0^2 \leqslant C_0 \left(\sum_{j=1}^{n} \|L_j(u)\|_0^2 + \|u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U).$$

Thus, by (4.30), we obtain, with a constant C_1 that only depends on $U \Subset M$,

$$(4.46) \qquad \begin{cases} \displaystyle\sum_{j=1}^{n} \left(\|L_j(u)\|_0^2 + \|\bar{L}_j(u)\|_0^2 \right) \leqslant C_1 \left(|(P(u)|u)_0| + \|u\|_0^2 \right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

Let $Y \in \mathfrak{X}(M)$ satisfy (4.44). We have, with $T_{2\delta-1} \in \Psi^{2\delta-1}(U)$, and for all $u \in \mathcal{C}_0^\infty(U)$,

$$\|[L_0, Y](u)\|_{\delta-1}^2 \leqslant C_2 \left(|([L_0, Y](u)|T_{2\delta-1}(u))_0| + \|u\|_0^2 \right)$$
$$\leqslant C_2 \left(|(L_0 Y(u)|T_{2\delta-1}(u))_0| + |(Y L_0(u)|T_{2\delta-1}(u))_0| + \|u\|_0^2 \right).$$

We shall estimate separately each summand inside the parentheses in the last term. In the following U will be a relatively compact open subset of M and all functions u will be smooth and have compact support in some fixed relatively compact open subset of U.

We have

$$P^*(u) = -L_0(u) + \sum_{j=1}^{n} L_j^* L_j(u) + L_{n+1}'(u) + a'u,$$

with $L_{n+1}' \in \mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$ and $a' \in \mathcal{C}^\infty(M)$. Hence using (4.46),

$$(*) \quad \begin{aligned} |(L_0 Y(u)|T_{2\delta-1}(u))_0| &\leqslant |(P^* Y(u)|T_{2\delta-1}(u))_0| \\ &+ \sum_{j=1}^{n} |(L_j^* L_j Y(u)|T_{2\delta-1}(u))_0| + C_3(\|u\|_0^2 + \|Y(u)\|_{2\delta-1}^2 + \|P(u)\|_0^2). \end{aligned}$$

The term in parenthesis can be estimated by a constant times $(\|u\|_0^2 + \|P(u)\|_0^2)$, provided we choose a δ so small that Y satisfies (4.44) in U with some $\epsilon \geqslant 2\delta$. For the first summand on the right hand side of $(*)$, we have

$$(P^* Y(u)|T_{2\delta-1}(u))_0 = (T_{2\delta-1}^*(Y(u))|P(u))_0 + (Y(u)|[P, T_{2\delta-1}](u))_0.$$

We obtain

$$|(T_{2\delta-1}^*(Y(u))|P(u))_0| \leqslant C_4 \left(\|Y(u)\|_{2\delta-1}^2 + \|P(u)\|_0^2 \right).$$

The right hand side is bounded by a constant times $(\|u\|_0^2 + \|P(u)\|_0^2)$, provided again that (4.44) holds for Y with $\epsilon \geqslant 2\delta$.

For the commutator $[P, T_{2\delta-1}]$, we have

$$[P, T_{2\delta-1}] = \sum_{j=1}^{n} T_{2\delta-1}' L_j^*(u) + T_{2\delta-1}'' L_j(u) + T_{2\delta-1}''',$$

with $T_{2\delta-1}', T_{2\delta-1}'', T_{2\delta-1}''' \in \Psi^{2\delta-1}(U)$.

Thus, because of (4.46), also the term $|(Y(u)|[P, T_{2\delta-1}](u))_0|$ is bounded by a constant times $(\|u\|_0^2 + \|P(u)\|_0^2)$, provided again that (4.44) holds for Y with $\epsilon \geqslant 2\delta$.

We have

$$T_{2\delta-1}^* L_j^* L_j = L_j^* L_j T_{2\delta-1}^* + L_j B_{2\delta-1}' + L_j^* B_{2\delta-1}'' + B_{2\delta-1}'''$$

with $B_{2\delta-1}', B_{2\delta-1}'', B_{2\delta-1}''' \in \Psi^{2\delta-1}(U)$. Thus with a constant $C_4 > 0$,

$$\begin{aligned} \left| (L_j^* L_j Y(u)|T_{2\delta-1}(u))_0 \right| &\leqslant \left| (L_j T_{2\delta-1}^*(u)|L_j(u))_0 \right| + \left| (B_{2\delta-1}' Y(u)|L_j^*(u))_0 \right| \\ &+ \left| (B_{2\delta-1}'' Y(u)|L_j(u))_0 \right| + \left| (B_{2\delta-1}''' Y(u)|u)_0 \right| \\ &\leqslant C_4 \left(\|L_j T_{2\delta-1}^* Y(u)\|_0^2 + \|L_j(u)\|_0^2 + \|\bar{L}_j(u)\|_0^2 \right. \\ &\left. + \|Y(u)\|_{2\delta-1}^2 + \|u\|_0^2 \right). \end{aligned}$$

Therefore, provided again that (4.44) holds for Y with some $\epsilon \geqslant 2\delta$, all terms but those of the form $\|L_j T_{2\delta-1}^*(u)\|_0^2$ are bounded by a constant times

$(\|u\|_0^2 + \|P(u)\|_0^2)$. Thus we only need to bound the terms $\|L_j T_{2\delta-1}^*(u)\|_0^2$. We have, by (4.30), with some constant $C_5 > 0$,

$$\sum_{j=1}^n \|L_j T_{2\delta-1}^* Y(u)\|_0^2 \leqslant C_5 \left(|(P(T_{2\delta-1}^* Y(u))|T_{2\delta-1}^* Y(u))_0| + \|T_{2\delta-1}^* Y(u)\|_0^2 \right).$$

The last summand inside the parentheses on the right hand side is bounded by a constant times $(\|u\|_0^2 + \|P(u)\|_0^2)$, provided again that (4.44) holds for Y with some $\epsilon \geqslant 2\delta$. Let us consider the first one. Note that the composition $T_{2\delta} = T_{2\delta-1}^* \circ Y$ is a pseudodifferential operator in $\Psi^{2\delta}(U)$. We have the commutation formula

$$[P, T_{2\delta}] = \sum_{j=1}^n (F'_{j2\delta} L_j + F''_{j2\delta} \bar{L}_j) + F'''_{2\delta}$$

with $F'_{j2\delta}, F''_{j2\delta}, F'''_{2\delta} \in \Psi^{2\delta}(U)$.

Thus

$$(P(T_{2\delta-1}^* Y(u))|T_{2\delta-1}^* Y(u))_0 = (P(T_{2\delta}(u))|T_{2\delta-1}^* Y(u))_0$$

$$= (P(u)|T_{2\delta}^* T_{2\delta-1}^* Y(u))_0 + \sum_{j=1}^n (L_j(u)|[F'_{j2\delta}]^* T_{2\delta-1}^* Y(u))_0$$

$$+ \sum_{j=1}^n (\bar{L}_j(u)|[F''_{j2\delta}]^* T_{2\delta-1}^* Y(u))_0 + (u|[F'''_{2\delta}]^* T_{2\delta-1}^* Y(u))_0.$$

Hence we obtain, with some constant $C_6 > 0$,

$$\|L_j T_{2\delta-1}^* Y(u)\|_0^2 \leqslant C_6 \Big(\|P(u)\|_0 \|Y(u)\|_{4\delta-1} + \sum_{j=1}^n \left(\|L_j(u)\|_0^2 + \|\bar{L}_j(u)\|_0^2 \right)$$
$$+ \|Y(u)\|_{4\delta-1}^2 + \|u\|_0^2 \Big),$$

which can be bounded by the right hand side of (4.44), provided Y satisfies (4.44) with $\epsilon \geqslant 4\delta$.

Finally we note that

$$(4.47) \qquad |(YL_0(u)|T_{2\delta-1}(u))_0| \leqslant C_7 (|(T_{2\delta-1}^*(u)|L_0 Y(u))_0| + \|L_0(u)\|_{2\delta-1}^2$$
$$+ \|Y(u)\|_{2\delta-1}^2 + \|u\|_0^2).$$

Thus, by repeating the discussion above with $T_{2\delta-1}^*$ replacing $T_{2\delta-1}$, we find that the left hand side of (4.47), provided Y satisfies (4.44) with $\epsilon \geqslant 4\delta$, is bounded by a constant times $(\|u\|_0^2 + \|P(u)\|_0^2)$. This concludes the proof of the Lemma. $\qquad \square$

End of the Proof of Theorem 4.8. — By the discussion at the beginning of the proof, and by Lemma 4.9, we obtain that for every relatively compact open subset U of M, which is contained in the open subset M'

of M, consisting of the points p where (4.43) is satisfied, there are positive constants $\epsilon > 0$ and $c_0 > 0$ such that

$$(4.48) \quad \begin{cases} \|u\|_\epsilon^2 + \sum_{j=1}^n \left(\|L_j(u)\|_0^2 + \|\bar{L}_j(u)\|_0^2 \right) \leqslant c_0 \left(\|u\|_0^2 + \|P(u)\|_0^2 \right), \\ \\ \hfill \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

One easily shows by recurrence that, if U is a relatively compact open subset of M, then there is a positive constant $\epsilon > 0$, and, for every real $s \geqslant 0$ another constant $c_s \geqslant 0$, such that

$$(4.49) \quad \begin{cases} \|u\|_{\epsilon+s}^2 + \sum_{j=1}^n \left(\|L_j(u)\|_s^2 + \|\bar{L}_j(u)\|_s^2 \right) \leqslant c_0 \left(\|u\|_0^2 + \|P(u)\|_s^2 \right), \\ \\ \hfill \forall u \in \mathcal{C}_0^\infty(U). \end{cases}$$

The hypoellipticity of P in U, with a gain of ϵ derivatives, follows in a standard way from (4.49) (see e.g. [14]). $\qquad\square$

5. Applications to almost CR manifolds

In this section we shall consider the case where M is an almost CR manifold of CR dimension n and CR codimension k, and $\mathfrak{Z}(M)$ is the distribution of vector fields of type $(0,1)$ on M. This means that conditions (i), (ii) and (iii) below are satisfied:

(i) $\qquad\qquad\qquad$ M has real dimension $2n + k$,

(ii) $\qquad\qquad\qquad$ $\mathfrak{Z}(M)$ has constant rank n,

(iii) $\qquad\qquad\qquad$ $\mathfrak{Z}(M) \cap \overline{\mathfrak{Z}(M)} = \{0\}$,

(iv) $\qquad\qquad\qquad$ $[\mathfrak{Z}(M), \mathfrak{Z}(M)] \subset \mathfrak{Z}(M)$.

When the formal integrability condition (iv) is also satisfied, we say that M is a CR manifold.

When M is an almost CR manifold, it is customary to write $T^{0,1}M$ for the complex bundle with fibers $T_p^{0,1}M = Z_pM$. The Z-morphisms of §3 are then the CR maps, i.e. the smooth maps $\phi : N \to M$ with $d\phi^\mathbb{C}(T_q^{0,1}N) \subset T_{\phi(q)}^{0,1}M$ for all $q \in N$.

LEMMA 5.1. — *If an almost CR manifold M satisfies the higher order Levi form concavity condition at a point p, then M is of finite type at p.*

The next statements clarify in what sense condition (2.32) is a pseudo-concavity condition.

Proposition 5.2. — *Let M be a CR manifold of hypersurface type, i.e. of CR codimension $k = 1$. If $\mathfrak{Z}(M)$ satisfies the higher Levi form concavity condition (2.32) at a point p, which is regular for $\mathfrak{Z}(M)$, then M is strictly pseudoconcave at p.*

Proof. — Let $0 \neq \xi \in H_p^0 M$. We want to prove that $\mathcal{L}_\xi$ is indefinite. Assume by contradiction that this is not the case. Replacing, if needed, ξ by $(-\xi)$, we can assume that $\mathcal{L}_\xi \geqslant 0$. By assumption (2.32) we can choose $Z_0 \in \mathfrak{Z}(M)$ and $Z_1, \ldots, Z_r \in \mathbb{K}_{\mathfrak{Z}}(M) \cup \overline{\mathbb{K}_{\mathfrak{Z}}(M)}$ satisfying

$$(*) \qquad \qquad \xi([Z_1, \ldots, Z_r, \bar{Z}_0]) \neq 0.$$

We can take r minimal with this property. In particular, we have

$$(**) \qquad \qquad [Z_2, \ldots, Z_r, \bar{Z}_0](p) = Z(p) + \overline{W}(p),$$

with $Z, W \in \mathfrak{Z}(M)$. Assume that $Z_1 \in \mathbb{K}_{\mathfrak{Z}}(M)$. Then $\xi([Z_1, Z]) = 0$, because of the integrability condition (iv), and then, from $(*)$, we have $\mathcal{L}_\xi(Z_1, \overline{W}) \neq 0$ and $\mathcal{L}_\xi(Z_1, \overline{Z}_1) = 0$, yielding a contradiction. Likewise, if $Z_1 \in \overline{\mathbb{K}_{\mathfrak{Z}}(M)}$, we have $\xi([Z_1, \bar{W}]) = 0$ by (iv), and hence, from $(*)$, $\mathcal{L}_\xi(Z, Z_1) \neq 0$, while $\mathcal{L}_\xi(\bar{Z}_1, Z_1) = 0$, contradicting the assumption that $\mathcal{L}_\xi \geqslant 0$. The proof is complete. $\qquad \square$

Corollary 5.3. — *Let N be a generic CR submanifold of a CR manifold M (this means that $T_p^{0,1} N = T_p^{0,1} M \cap T_p^{\mathbb{C}} N$ and the restriction map $H_p^0 M \to H_p^0 N$ is injective for all $p \in N \subset M$). If M is of hypersurface type and $\mathfrak{Z}(N)$ satisfies the higher Levi form concavity condition (2.32) at a point p_0, regular for $\mathfrak{Z}(M)$, then M is strictly pseudoconcave at p_0.*

Proof. — The statement follows from Propositions 3.8 and 5.2. $\qquad \square$

Lemma 5.4. — *Let M, N be almost CR manifolds, and $\varpi \colon N \to M$ a CR map and a smooth submersion. We denote by $\mathfrak{Z}(M)$ and $\mathfrak{Z}(N)$ the distributions of $(0, 1)$ vector fields on M, N, respectively. If M is strictly pseudoconvex at a point p_0, i.e. if there is $\xi_0 \in H_{p_0}^0 M$ with $\mathcal{L}_{\xi_0} > 0$, and moreover p_0 is regular for $\mathfrak{Z}(M)$, then the higher Levi form concavity condition (2.32) for $\mathfrak{Z}(N)$ is not satisfied at any point $q_0 \in \varpi^{-1}(p_0)$.*

Proof. — Replacing M by an open neighborhood of p_0 in M, we can assume that M is strictly pseudoconvex at all points. Then $\mathbb{K}_{\mathfrak{Z}}(M) = 0$, and hence $\mathbb{K}_{\mathfrak{Z}}(N)$ is contained in the complexification of the vertical distribution $\mathfrak{V}^\varpi(N)$. Indeed, if $\xi \in H_p^\oplus M$ and $q \in \varpi^{-1}(p)$, the pullback $\varpi^*(q)(\xi)$ belongs to $H_q^\oplus N$.

Thus if $\eta_0 = \varpi^*(q_0)(\xi_0)$ for $q_0 \in \varpi^{-1}(p_0)$, then η_0 does not satisfy (2.32). Indeed, η_0 vanishes on the pullback of $\mathfrak{Z}(M) + \overline{\mathfrak{Z}(M)}$, and this distribution is a $\mathfrak{V}^\varpi(N)$-module. $\qquad\square$

Theorem 4.1 and Corollary 4.2 yield

THEOREM 5.5. — *Let M be an almost CR manifold and assume that condition (1.21) is satisfied by the distribution $\mathfrak{Z}(M)$ of its $(0,1)$ vector fields at all points of M. Then*

(1) *If $E \xrightarrow{\pi} M$ a Hermitian vector bundle on M, endowed with a $\mathbb{C}$-linear connection ∇, then all weak solutions $u \in L^2_{\mathrm{loc}}(M,E)$ of*

$$\nabla_Z(u) \in \mathcal{C}^\infty(M,E), \quad \forall Z \in \mathfrak{Z}(M)$$

are smooth sections of $E \xrightarrow{\pi} M$.

In particular, all CR sections of E (i.e. weak L^2_{loc} solutions of $\nabla_Z(u) = 0$, for all $Z \in \mathfrak{Z}(M)$) are smooth.

(2) *In case M is compact, the space of CR sections of E is a finite dimensional $\mathbb{C}$-linear space.*

(3) *If M is a compact CR manifold, then the cohomology groups $H^{p,1}_{\bar{\partial}_M}(M)$, for $p = 1, \ldots, n+k$, of the tangential Cauchy-Riemann complexes, are Hausdorff.*

6. Subellipticity conditions for homogeneous CR manifolds

Let M be a CR manifold, homogeneous for the CR action of a Lie group $\mathbf{G}$. Fix a base point $\mathbf{o} \in M$ and denote by

$$(6.1) \qquad\qquad \varpi : \mathbf{G} \ni g \to g \cdot \mathbf{o} \in M$$

the associated principal bundle. In [20] we associated to M and the base point $\mathbf{o}$ the CR algebra $(\mathfrak{g}, \mathfrak{q})$, that is the pair consisting of

- the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$,
- the complex Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}^\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ given by

$$(6.2) \qquad\qquad \mathfrak{q} = [d\varpi_e]^{-1}_\mathbb{C}(T^{0,1}_\mathbf{o} M),$$

where $[d\varpi_e]_\mathbb{C} : \mathfrak{g}^\mathbb{C} \to T^\mathbb{C}_\mathbf{o} M$ is the complexification of the differential $d\varpi_e : \mathfrak{g} \to T_\mathbf{o} M$ of ϖ at the identity.

For $X \in \mathfrak{g}$ we denote by X^* the left-invariant vector field in $\mathbf{G}$ with $X^*(e) = X$. Let

$$(6.3) \qquad \mathfrak{q}^* = \{X^* + iY^* \mid X,Y \in \mathfrak{g}, \ X + iY \in \mathfrak{q}\},$$

$$(6.4) \qquad \mathfrak{Z}(\mathbf{G}) = \mathcal{E}(M) \otimes \mathfrak{q}^* = \text{the vector distribution spanned by } \mathfrak{q}^*.$$

The following statement is straightforward.

PROPOSITION 6.1. — Let $M = \mathbf{G}/\mathbf{G_o}$ be a $\mathbf{G}$-homogeneous CR manifold, $\mathfrak{Z}(M)$ the distribution of $(0,1)$-vector fields on M, and $\mathfrak{Z}(\mathbf{G})$ the distribution on $\mathbf{G}$ defined by (6.4). Then

(1) The distributions $\mathfrak{Z}(M)$ and $\mathfrak{Z}(\mathbf{G})$ are regular at all points;
(2) the principal bundle fibration $\mathbf{G} \xrightarrow{\varpi} M$ is a Z-morphism,

$$(6.5) \qquad\qquad \mathfrak{Z}(\mathbf{G}) = [\varpi^* \mathfrak{Z}](\mathbf{G})$$

and conditions (2), (3), (4) of Proposition 3.8 are satisfied.

Proposition 6.1 can be used to reduce the question of the subellipticity of the distribution of $(0,1)$-vector fields of a homogeneous CR manifold to Lie algebra computations.

Let $(\mathfrak{g}, \mathfrak{q})$ be the CR algebra associated with the $\mathbf{G}$-homogeneous CR manifold M and its base point $\mathbf{o}$. Let

$$(6.6) \qquad \mathfrak{t}^0 = H_e^0 \mathbf{G} = \{\xi \in \mathfrak{g}^* \mid \xi(\mathrm{Re}(Z)) = 0, \ \forall Z \in \mathfrak{q}\}.$$

To each $\xi \in \mathfrak{t}^0$ we associate the Levi form

$$(6.7) \qquad\qquad \mathcal{L}_\xi(Z, \bar{W}) = i\xi([Z, \bar{W}]), \ \text{for } Z, W \in \mathfrak{q}.$$

We also set

$$(6.8) \qquad\qquad \mathfrak{t}^\oplus = \{\xi \in \mathfrak{t}^0 \mid \mathcal{L}_\xi \geqslant 0\},$$

$$(6.9) \qquad\qquad \mathfrak{k}_\mathfrak{q} = \{Z \in \mathfrak{q} \mid \mathcal{L}_\xi(Z, \bar{Z}) = 0, \ \forall \xi \in \mathfrak{t}^\oplus\}.$$

Since all points of M and of $\mathbf{G}$ are regular for $\mathfrak{Z}(M)$ and $\mathfrak{Z}(\mathbf{G})$, respectively, we obtain from Proposition 2.13:

LEMMA 6.2. — Let $(\mathfrak{g}, \mathfrak{q})$ be a CR algebra. The set $\mathfrak{k}_\mathfrak{q}$ is a linear subspace of $\mathfrak{q}$ and is equal to the set

$$\left\{ Z_0 \in \mathfrak{q} \,\middle|\, \exists Z_1, \ldots, Z_r \in \mathfrak{q}, \ s.t. \ \sum_{j=0}^r [Z_j, \bar{Z}_j] \in \mathfrak{q} + \bar{\mathfrak{q}} \right\}.$$

The elements

$$Z^* = (\mathrm{Re}\, Z)^* + i(\mathrm{Im}\, Z)^*, \quad \text{for } Z \in \mathfrak{k}_\mathfrak{q},$$

generate the distribution $\mathbb{K}_\mathfrak{Z}(\mathbf{G})$, that is equal to $\Theta_\mathfrak{Z}(\mathbf{G})$.

As a consequence of Propositions 2.16, 6.1 and Lemma 6.2 we have

PROPOSITION 6.3. — *Let $M = \mathbf{G}/\mathbf{G_o}$ be a homogeneous CR manifold and let $(\mathfrak{g}, \mathfrak{q})$ be the CR algebra associated with M and the base point $\mathbf{o}$. We denote by $\mathfrak{Z}(M)$ the distribution of $(0,1)$-vector fields on M. Then the following are equivalent.*

(1) $\mathfrak{Z}(M)$ *satisfies condition* (2.32).
(2) $\mathfrak{Z}(M)$ *satisfies condition* (1.21).
(3) $\mathfrak{Z}(\mathbf{G})$ *satisfies condition* (2.32).
(4) $\mathfrak{Z}(\mathbf{G})$ *satisfies condition* (1.21).
(5) $(\mathfrak{g}, \mathfrak{q})$ *satisfies the condition:*

$$(6.10) \qquad \begin{cases} \forall \xi \in \mathfrak{t}^{\oplus} \setminus \{0\}, \quad \exists Z_0 \in \mathfrak{q} \text{ and } Z_1, \dots, Z_r \in \mathfrak{k}_{\mathfrak{q}} \cup \bar{\mathfrak{k}}_{\mathfrak{q}} \\ \qquad s.t. \quad i\xi([Z_1, \dots, Z_r, \bar{Z}_0]) \neq 0. \end{cases}$$

7. Orbits of a real form in a complex flag manifold

In this section we investigate the subellipticity of the distribution of the $(0,1)$-vector fields of the homogeneous CR manifolds which are real orbits of real forms in complex flag manifolds. The study of their CR geometry has been already started in [1, 2], to which we refer for the complete explanation of many details.

7.1. Complex flag manifolds and orbits of a real form

We recall that a *complex flag manifold* is a closed complex projective variety, that is a coset space of a connected semisimple complex Lie group $\mathbf{G}^{\mathbb{C}}$ with respect to a complex *parabolic* subgroup.

A *real form* $\mathbf{G}$ of $\mathbf{G}^{\mathbb{C}}$ is a *real* Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ whose Lie algebra $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$, i.e. such that $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$. We shall write $\bar{Z}$ for the conjugate of an element $Z \in \mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. The left action of $\mathbf{G}$ decomposes F into a finite set of $\mathbf{G}$-orbits (see [24]). All $\mathbf{G}$-orbits $M = \mathbf{G} \cdot \mathbf{o}$ are generically embedded CR submanifolds of F. It turns out that F and also M, if we take $\mathbf{G}$ connected, are completely determined by the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, by its real form $\mathfrak{g}$, and by the complex parabolic Lie subalgebra $\mathfrak{q} \subset \mathfrak{g}^{\mathbb{C}}$ of the isotropy subgroup $\mathbf{Q}$ at $\mathbf{o}$. Thus we shall write $M = M(\mathfrak{g}, \mathfrak{q})$, for the homogeneous CR manifold M. We denote by $\mathbf{G_o} = \mathbf{Q} \cap \mathbf{G}$ the isotropy subgroup at the base point $\mathbf{o}$ and by $\mathbf{G} \xrightarrow[\mathbf{G_o}]{\varpi} M$ the principal $\mathbf{G_o}$-bundle $\varpi \colon \mathbf{G} \ni g \to g \cdot \mathbf{o} \in M$.

The Lie algebra $\mathfrak{g_o}$ of $\mathbf{G_o}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (see e.g. [24, 1, 2]). Its complexification $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} + i\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathcal{R}$ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$, and $\mathfrak{g}_\alpha^{\mathbb{C}} = \{Z \in \mathfrak{g}^{\mathbb{C}} \mid [H, Z] = \alpha(H)Z, \ \forall H \in \mathfrak{h}^{\mathbb{C}}\}$ the eigenspace corresponding to the root $\alpha \in \mathcal{R}$. Since $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{q}$, the subalgebra $\mathfrak{q}$ is $\mathrm{ad}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}})$-invariant. Hence we have

$$(7.1) \qquad \mathfrak{q} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{Q}} \mathfrak{g}_\alpha^{\mathbb{C}}, \quad \text{for}$$

$$(7.2) \qquad \mathcal{Q} = \{\alpha \in \mathcal{R} \mid \mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{q}\}.$$

To say that $\mathfrak{q}$ is parabolic means that $\mathcal{Q}$ contains a positive system of roots $\mathcal{R}^+$. Let $\prec$ be a corresponding partial order on the linear span $\mathfrak{h}_{\mathbb{R}}^*$ of $\mathcal{R}$, and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of simple positive roots in $\mathcal{R}^+$. Every root $\alpha \in \mathcal{R}$ can be written in a unique way as a linear combination with integral coefficients of the elements of $\mathcal{B}$:

$$(7.3) \qquad \alpha = \sum_{j=1}^{\ell} k_j \alpha_j,$$

where all k_j's are either $\geqslant 0$, or $\leqslant 0$, according to whether α is positive or negative with respect to $\prec$. We define the *support* of α to be the set

$$(7.4) \qquad \mathrm{supp}(\alpha) = \{\alpha_j \in \mathcal{B} \mid k_j \neq 0\}.$$

Let

$$(7.5) \qquad \Phi = \{\alpha \in \mathcal{B} \mid -\alpha \notin \mathcal{Q}\}$$

be the set of simple roots α whose opposite $(-\alpha)$ does not belong to $\mathcal{Q}$. Then

$$(7.6) \qquad \mathcal{Q} = \mathcal{R}^+ \cup \{\alpha \prec 0 \mid \mathrm{supp}(\alpha) \cap \Phi = \emptyset\}.$$

Since $\mathcal{Q}$ is completely determined by Φ, we shall write $\mathcal{Q}_\Phi$, $\mathfrak{q}_\Phi$, $\mathbf{Q}_\Phi$ for the parabolic set of roots, the complex parabolic subalgebra and the complex parabolic subgroup, respectively, that are attached to any special choice of the subset Φ of $\mathcal{B}$. We shall also introduce the notation

$$(7.7) \qquad \begin{aligned} \mathcal{Q}_\Phi^n &= \{\alpha \succ 0 \mid \mathrm{supp}(\alpha) \cap \Phi \neq \emptyset\} \\ \mathcal{Q}_\Phi^r &= \{\alpha \mid \alpha, -\alpha \in \mathcal{Q}\} = \{\alpha \in \mathcal{R} \mid \mathrm{supp}(\alpha) \cap \Phi = \emptyset\}. \end{aligned}$$

The conjugation in $\mathfrak{g}^{\mathbb{C}}$ induced by the real form $\mathfrak{g}$ defines, by duality, a conjugation $\alpha \to \bar{\alpha}$ in $\mathcal{R}$. We partition $\mathcal{R}$ into three subsets:

$$(7.8) \qquad \mathcal{R}_{\mathrm{re}} = \{\alpha \in \mathcal{R} \mid \bar{\alpha} = \alpha\} \qquad \text{real roots,}$$

$$(7.9) \qquad \mathcal{R}_{\mathrm{im}} = \{\alpha \in \mathcal{R} \mid \bar{\alpha} = -\alpha\} \qquad \text{imaginary roots,}$$

$$(7.10) \qquad \mathcal{R}_{\mathrm{cx}} = \{\alpha \in \mathcal{R} \mid \bar{\alpha} \neq \pm\alpha\} \qquad \text{complex roots.}$$

The Cartan subalgebra $\mathfrak{h}$ is invariant under a Cartan involution ϑ of $\mathfrak{g}$, whose set of fixed points $\mathfrak{k}$ is a maximal compact Lie subalgebra of $\mathfrak{g}$. With $\mathfrak{p} = \{X \mid \vartheta(X) = -X\}$, we have that for imaginary α, the eigenspace $\mathfrak{g}_\alpha^\mathbb{C}$ is contained either in $\mathfrak{k}^\mathbb{C} = \mathfrak{k} + i\mathfrak{k}$, or in $\mathfrak{p}^\mathbb{C} = \mathfrak{p} + i\mathfrak{p}$. In the first case we say that α is a *compact* root. Compact roots form a root subsystem $\mathcal{R}_\bullet$ of $\mathcal{R}$.

The **G**-homogeneous CR structure of M is defined, as in §6, by assigning the subspace

$$(7.11) \qquad T_\mathbf{o}^{0,1} = d\varpi(e)(\mathfrak{q}).$$

Here e is the identity of **G** and we denote by the same symbol $d\varpi(e)$ the complexification of the differential $d\varpi(e) : \mathfrak{g} = T_e\mathbf{G} \to T_\mathbf{o}M$.

For each $Z \in \mathfrak{g}^\mathbb{C}$ we can consider the *fundamental vector field* $Z^* \in \mathfrak{X}^\mathbb{C}(M)$. Its real and imaginary parts are the infinitesimal generators of the flows associated to the left translations by $\exp(t\mathrm{Re}(Z))$ and $\exp(t\mathrm{Im}(Z))$, respectively.

Fix a Chevalley basis[1] $\{H_\alpha \mid \alpha \in \mathcal{B}\} \cup \{Z_\alpha \mid \alpha \in \mathcal{R}\}$ of $\mathfrak{g}^\mathbb{C}$. The restrictions to **G** of the vector fields H_α, for $\alpha \in \mathcal{B}$, and Z_α, for $\alpha \in \mathcal{Q}_\Phi$, form a basis for the pullback $\mathfrak{z}(\mathbf{G})$ of $\mathfrak{z}(M)$.

Since $\mathfrak{g}^\mathbb{C}$ is semisimple, the Killing form $\kappa_{\mathfrak{g}^\mathbb{C}}$ is nondegenerate. Thus, as in [1, §13], we can use the Killing form to identify the complexification $\mathfrak{t}^{0\,\mathbb{C}}$ of the space $\mathfrak{t}^0$ of (6.6) with the linear span of

$$(7.12) \qquad Z_\alpha \quad \text{for} \quad \alpha \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n.$$

This is obtained by associating to Z_α the linear form

$$(7.13) \quad f_\alpha : \mathfrak{g}^\mathbb{C} \ni Z \to \kappa_{\mathfrak{g}^\mathbb{C}}(Z_{-\alpha}, Z) = \mathrm{trace}\left(\mathrm{ad}_{\mathfrak{g}^\mathbb{C}}(Z_{-\alpha}) \circ \mathrm{ad}_{\mathfrak{g}^\mathbb{C}}(Z)\right) \in \mathbb{C}.$$

Correspondingly we obtain the complexified Levi forms

$$(7.14) \qquad \mathbf{L}_\alpha(Z, \bar{W}) = if_\alpha([Z, \bar{W}]), \text{ for } Z, W \in \mathfrak{q}_\Phi.$$

When $\alpha \in \mathcal{R}_{\mathrm{re}} \cap \mathcal{Q}$, it actually corresponds to a Levi form (2.7) at $\mathbf{o}$. The intersection $\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$ does not contain imaginary roots. To a pair of complex roots $\alpha, \bar{\alpha} \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$, correspond the two Hermitian symmetric

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[1] This means that $Z_\alpha \in \mathfrak{g}_\alpha^\mathbb{C}$ for all $\alpha \in \mathcal{R}$ and that, for $\alpha \in \mathcal{B}$, we also have $[H_\alpha, Z_\alpha] = 2Z_\alpha, [H_\alpha, Z_{-\alpha}] = -2Z_{-\alpha}, [Z_\alpha, Z_{-\alpha}] = -H_\alpha$; and moreover that the linear map defined by $H_\alpha \to -H_\alpha$ and $Z_\alpha \to Z_{-\alpha}$ is an involutive automorphism of the Lie algebra $\mathfrak{g}^\mathbb{C}$. Moreover, $Z_\alpha \in \mathfrak{g}$ when $\alpha \in \mathcal{R}_{\mathrm{re}}$ and, if $\alpha \in \mathcal{R}_{\mathrm{im}}$, we have $\bar{Z}_\alpha = Z_{-\alpha}$ when α is compact and $\bar{Z}_\alpha = -Z_{-\alpha}$ when α is not compact. In general, $Z_{\bar{\alpha}} = t_\alpha \bar{Z}_\alpha$, with $t_\alpha = \pm 1$, for all roots $\alpha \in \mathcal{R}$, and we can take $t_\alpha = 1$ when $\alpha \in \mathcal{R}_{\mathrm{re}}$ (see e.g. [5]).

forms obtained by polarization from the Hermitian quadratic forms

$$[\operatorname{Re} \mathbf{L}_\alpha] (Z, \bar{Z}) = \frac{i}{2} \left(f_\alpha([Z, \bar{Z}]) - \overline{f_\alpha([Z, \bar{Z}])} \right),$$

$$[\operatorname{Im} \mathbf{L}_\alpha] (Z, \bar{Z}) = \frac{1}{2} \left(f_\alpha([Z, \bar{Z}]) + \overline{f_\alpha([Z, \bar{Z}])} \right).$$

Note that $L_\alpha(Z, \bar{W}) = \pm \overline{L_{\bar{\alpha}}(W, \bar{Z})}$, when $Z_{\bar{\alpha}}$ equals $\pm \bar{Z}_\alpha$, respectively. Thus each Levi form can be written as a linear combination of the $\mathbf{L}_\beta$'s for $\beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$, the coefficients of $\mathbf{L}_\beta$ and $\mathbf{L}_{\bar{\beta}}$, for $\beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \cap \mathcal{R}_{\mathrm{cx}}$ being either conjugate or anticonjugate according to the sign in the equality $Z_{\bar{\beta}} = \pm \bar{Z}_\beta$.

7.2. Semidefinite Levi forms

LEMMA 7.1. — *Let $\mathcal{R}$ be a root system. There exist no triples $\alpha, \beta, \gamma \in \mathcal{R}$ with*

$$\begin{cases} \alpha + \bar{\alpha}, \; \beta + \bar{\beta}, \; \gamma + \bar{\gamma} \in \mathcal{R}, \\ \alpha + \bar{\alpha} \neq \beta + \bar{\beta}, \quad \alpha + \bar{\beta} = \gamma + \bar{\gamma}. \end{cases}$$

Proof. — Note that α, β, γ belong to the same irreducible component of $\mathcal{R}$. Thus we may as well assume that $\mathcal{R}$ is irreducible. We argue by contradiction.

The root subsystem $\mathcal{R}'$ generated by $\{\alpha + \bar{\alpha}, \beta + \bar{\beta}, \gamma + \bar{\gamma}\}$ is of type B_2. Indeed, it is not of type G_2 because it is a proper subsystem of a larger root system, since all roots in $\mathcal{R}'$ are real, while the fact that $\alpha + \bar{\alpha}$ is a root implies that $\mathcal{R}_{\mathrm{cx}}$ is not empty. Moreover, from $(\alpha + \bar{\alpha}) + (\beta + \bar{\beta}) = 2(\gamma + \bar{\gamma})$, we deduce that $\mathcal{R}'$ has rank 2 and contains roots of different lengths. Thus we can choose a basis $\{e_j\}$ in $\mathfrak{h}_\mathbb{R}^*$ such that

$$\alpha + \bar{\alpha} = 2e_1, \quad \beta + \bar{\beta} = 2e_2, \quad \gamma + \bar{\gamma} = e_1 + e_2,$$

and, furthermore, that

$$\alpha = e_1 + a e_3, \quad \beta = e_2 + a e_3, \quad \gamma = (1/2)e_1 + (1/2)e_2 + b e_3 + c e_4$$

for some $a, b, c \in \mathbb{R}$, and $a > 0$.

Let $\langle \alpha | \beta \rangle = 2(\alpha | \beta)/(\alpha | \alpha)$. Recall that, if $\alpha_1, \alpha_2 \in \mathcal{R}$ are not proportional, then $(\langle \alpha_1 | \alpha_2 \rangle \langle \alpha_2 | \alpha_1 \rangle) \in \{0, 1, 2, 3\}$. Since

$$\langle \alpha | \gamma + \bar{\gamma} \rangle \langle \gamma + \bar{\gamma} | \alpha \rangle = \frac{2}{1 + a^2},$$

we have $a = 1$, that is $\alpha = e_1 + e_3$ and $\beta = e_2 + e_3$. Then

$$\langle \gamma | \alpha + \bar{\alpha} \rangle \langle \alpha + \bar{\alpha} | \gamma \rangle = \frac{1}{(1/2) + b^2 + c^2}$$

implies that $b^2 + c^2 = 1/2$. Hence $\|\gamma\|^2 = 1$. This gives a contradiction, because $\|\alpha\|^2 = 2$ and $\|\alpha + \bar{\alpha}\|^2 = 4$, and at most two different root lengths are allowed in an irreducible root system. $\qquad\square$

As a consequence of Proposition 6.3 we obtain

PROPOSITION 7.2. — *Let $M(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a **G**-orbit in the complex flag manifold F, and $\mathfrak{Z}(M)$ the distribution of its $(0,1)$-vector fields. Set*

$$(7.15) \qquad \mathcal{K}_\Phi = \left\{ \alpha \in \mathcal{R} \mid \mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{k}_{\mathfrak{q}_\Phi} \right\}.$$

Then

$$(7.16) \qquad \mathcal{K}_\Phi = \left\{ \alpha \in \mathcal{Q} \,\middle|\, \begin{array}{l} \text{either } -(\alpha + \bar{\alpha}) \notin \mathcal{Q} \\ \text{or } \mathbf{L}_{\alpha + \bar{\alpha}} \text{ is indefinite} \end{array} \right\}$$

and we have

$$(7.17) \qquad \mathfrak{k}_{\mathfrak{q}_\Phi} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{K}_\Phi} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

A sufficient condition, in order that $\mathfrak{Z}(M)$ satisfy the higher Levi form concavity condition (2.32), is that

$$(7.18) \qquad \begin{cases} \text{for each } \gamma \in \mathcal{Q}_\Phi^n \text{ with } \gamma = \bar{\gamma} \text{ and } \mathbf{L}_\gamma \geqslant 0, \\ \text{there exist } \alpha_0 \in \bar{\mathcal{Q}}_\Phi, \text{ and } \alpha_1, \dots, \alpha_r \in \mathcal{K}_\Phi \cup \bar{\mathcal{K}}_\Phi \text{ s.t.} \\ \sum_{j=0}^h \alpha_j \in \mathcal{R} \text{ for } 1 \leqslant h \leqslant r, \quad -\gamma = \sum_{j=0}^r \alpha_j. \end{cases}$$

Proof. — Denote by $\mathcal{H}^\oplus$ the set of real roots $\gamma \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$ such that the corresponding Levi form $\mathbf{L}_\gamma$ is semidefinite.

Consider any Levi form $\mathbf{L} = \sum_{\beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n} c_\beta \mathbf{L}_\beta$, with $c_{\bar{\beta}} = \pm \bar{c}_\beta$. We claim that if $\mathbf{L}$ is semidefinite then

$$\bigcap_{\gamma \in \mathcal{H}^\oplus} \ker \mathbf{L}_\gamma \subset \ker \mathbf{L}.$$

This claim implies (7.16) and (7.17), from which the last statement follows.

With respect to the basis $\{Z_\alpha\}_{\alpha \in \mathcal{Q} \setminus \bar{\mathcal{Q}}}$, the complexified Levi forms $\mathbf{L}_\beta$ have mutually non overlapping nonzero entries, and each of them has at most one nonzero entry in each row (or column). Note that, for $\beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \cap \mathcal{R}_{\mathrm{cx}}$, the matrix of $\mathbf{L}_\beta$ has no diagonal entries. Partition $\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$

into the following subsets:

$$\mathcal{H}_0 = \left\{ \beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \cap \mathcal{R}_{\mathrm{re}} \mid \mathbf{L}_\beta \text{ is semidefinite (hence diagonal)} \right\},$$

$$\mathcal{H}_1 = \left\{ \beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \cap \mathcal{R}_{\mathrm{re}} \mid \mathbf{L}_\beta \text{ is indefinite and diagonal} \right\},$$

$$\mathcal{H}_2 = \left\{ \beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \mid \mathbf{L}_\beta \text{ has no nonzero entry on the diagonal} \right\},$$

$$\mathcal{H}_3 = \left\{ \beta \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n \cap \mathcal{R}_{\mathrm{re}} \,\middle|\, \mathbf{L}_\beta \begin{array}{l} \text{is indefinite and has nonzero} \\ \text{entries on and off the diagonal} \end{array} \right\}.$$

If a coefficient c_β in the decomposition of $\mathbf{L}$ is different from zero for some $\beta \in \mathcal{H}_1$, then $\mathbf{L}$ is indefinite. The same is true for $\beta \in \mathcal{H}_3$. Indeed, β is real and there exists γ such that $\beta = \gamma + \bar{\gamma}$ because $\mathbf{L}_\beta$ has a nonzero diagonal entry. Moreover, there are roots $\alpha, \alpha' \in \mathcal{Q}$ such that $\beta = \alpha + \bar{\alpha}'$ because $\mathbf{L}_\beta$ has a nonzero entry off of the main diagonal. By Lemma 7.1, the restriction of $\mathbf{L}_\beta$ to $\mathfrak{g}^\alpha + \mathfrak{g}^{\alpha'}$ is indefinite. Hence the only possible nonzero coefficients are those corresponding to roots in $\mathcal{H}_0 = \mathcal{H}^\oplus$ and $\mathcal{H}_2$. The statement then follows from the following statement about Hermitian matrices:

Let $A = (a_{i,j})_{1 \leqslant i,j \leqslant \ell}$ be a Hermitian symmetric matrix with $a_{i,i} = 0$ for $i = 1, \dots, \ell$, and $D = \mathrm{diag}(d_{1,1}, \dots, d_{\ell,\ell})$. If $D + A$ is semidefinite then $\ker D \subset \ker D + A$. $\square$

By using Proposition 3.8 we obtain

PROPOSITION 7.3. — *Let $\mathfrak{g}$ be a semisimple real Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{B}$ a system of simple roots of $\mathcal{R}^+$. Let $\Psi \subset \Phi \subset \mathcal{B}$.*

If $\mathfrak{z}(M(\mathfrak{g}, \mathfrak{q}_\Phi))$ is subelliptic, then $\mathfrak{z}(M(\mathfrak{g}, \mathfrak{q}_\Psi))$ is also subelliptic.

Proof. — The inclusions $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$ and $\mathbf{Q}_\Phi \subset \mathbf{Q}_\Psi$ induce a smooth submersion $M(\mathfrak{q}, \mathfrak{q}_\Phi) \to M(\mathfrak{q}, \mathfrak{q}_\Psi)$, that is also a CR map and satisfies the hypotheses of Proposition 3.8. $\square$

7.3. Minimal orbits

In [24] it is shown that there is a unique $\mathbf{G}$ orbit in F (the *minimal orbit*) that is compact. It is connected and has minimal dimension. Minimal orbits have been studied, from the point of view of CR geometry, in [1]. If $(\mathfrak{g}, \mathfrak{q})$ is the CR algebra of a minimal orbit M, then $\mathfrak{g}_\mathbf{o}$ contains a maximally vectorial Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This choice of $\mathfrak{h}$ is equivalent to the fact that $\mathcal{R}_{\mathrm{im}} = \mathcal{R}_\bullet$, i.e. that all imaginary roots are compact (see [3]).

The fact that $M = M(\mathfrak{g}, \mathfrak{q})$ is minimal is then equivalent to the possibility of choosing the positive root system $\mathcal{R}^+ \subset \mathcal{Q}$ in such a way that

$$(7.19) \qquad \text{for } \alpha \in \mathcal{R}_{\mathrm{cx}} \quad \alpha \succ 0 \Longleftrightarrow \bar{\alpha} \succ 0.$$

This leads to a complete classification of the minimal orbits in terms of *cross-marked Satake diagrams* (see e.g. [1]), i.e. by systems $\Phi \subset \mathcal{B}$, where $\mathcal{B}$ are the simple roots of a positive system $\mathcal{R}^+$ satisfying (7.19). Using these Φ's, we give below a complete classification of the minimal orbits for which the distribution of $(0, 1)$ vector fields satisfies the higher order Levi concavity condition (2.32).

When $\mathfrak{g}$ decomposes into a sum $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(\ell)}$ of simple ideals, the CR manifold $M(\mathfrak{g}, \mathfrak{q})$ is a Cartesian product $M(\mathfrak{g}^{(1)}, \mathfrak{q}^{(1)}) \times \cdots \times M(\mathfrak{g}^{(\ell)}, \mathfrak{q}^{(\ell)})$, (see [1, p.490]). Thus we can reduce the question of the validity of the higher order Levi form concavity condition, and also of the subellipticity and hypoellipticity of $\mathfrak{z}(M(\mathfrak{g}, \mathfrak{q}))$, to the case where $\mathfrak{g}$ is a simple real Lie algebra. Namely, this property will be valid for $M(\mathfrak{g}, \mathfrak{q})$ if, and only if, it is valid for each factor $M(\mathfrak{g}^{(h)}, \mathfrak{q}^{(h)})$, for $1 \leqslant h \leqslant \ell$.

Thus we state the following classification theorem for the case of a simple $\mathfrak{g}$.

For the Satake diagrams characterizing the different simple real Lie algebras, their labels and those of the roots of a simple positive system, we refer to [10], or to the Appendix in [1].

THEOREM 7.4. — *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be the CR algebra of a minimal orbit M, with $\mathfrak{g}$ simple and assume that M is a CR manifold of finite type in the sense of [4]. Then $\mathfrak{z}(M)$ satisfies the higher order Levi form concavity condition (2.32) if and only if one of the following conditions is satisfied*

(1) *$\mathfrak{g}$ is either of the complex type, or compact, or of one of the real types AII, AIIIb, B, CIIb, DI, DII, DIIIa, EII, EIV, EVI, EVII, EIX;*

(2) *$\mathfrak{g} \simeq \mathfrak{su}(p, q)$ is of type AIIIa $-$ AIV and $\Phi \cap \{\alpha_p, \alpha_q\} = \emptyset$;*

(3) *$\mathfrak{g} \simeq \mathfrak{sp}(p, \ell - p)$, with $2p < \ell$, is of type CIIa and either Φ is all contained in $\{\alpha_{2j-1} \mid 1 \leqslant j < p\} \cup \{\alpha_j \mid 2p < j \leqslant \ell\}$, or Φ is all contained in $\{\alpha_{2j-1} \mid 1 \leqslant j \leqslant p\}$;*

(4) *$\mathfrak{g} \simeq \mathfrak{so}^*(2\ell)$, with $\ell \in 2\mathbb{Z}_+$, $\Phi \cap \{\alpha_{\ell-1}, \alpha_\ell\} = \emptyset$;*

(5) *$\mathfrak{g}$ is of type EIII and $\Phi \subset \{\alpha_3, \alpha_4, \alpha_5\} = \mathcal{R}_\bullet \cap \mathcal{B}$;*

(6) *$\mathfrak{g}$ is of type FII and $\Phi \subset \{\alpha_1, \alpha_2\}$.*

Note that the condition for M being of finite type in the sense of [4] is explicitly described in terms of Φ in [1, Theorem 9.1].

Proof. — We use the results of [1, §13, §14]. We stress the fact that we are assuming that M is of finite type, so that some choices of Φ are excluded from our consideration because of [1, Theorem 9.1].

First we prove that if (2.32) holds true, then $(\mathfrak{g}, \mathfrak{q}_\Phi)$ satisfies one of the conditions (1) . . . (6). If $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is not one of those listed in the statement, in

all cases, with the two exceptions of CIIa with $\alpha_{2p-1} \in \Phi$ and $\Phi \cap \{\alpha_h \mid h > 2p\} \neq \emptyset$, and FII with $\alpha_3 \in \Phi$, then M admits a **G**-equivariant fibration onto one of the manifolds described in [1, Examples 14.1, 14.2, and 14.3]. These are strictly pseudoconvex, hence M does not satisfy the higher order Levi form concavity condition by Lemma 5.4. The two remaining cases will be discussed below, while proving the opposite implications.

We know from [1, Theorem 13.4] that M is essentially pseudoconcave, and hence in particular (7.18) and (2.32) hold, in case one of the following is satisfied:

(1′) case (1),
(2′) case (2) with either $\Phi \subset \mathcal{R}_\bullet$, or $\Phi \cap \mathcal{R}_\bullet = \emptyset$,
(3′) case (3) with either $\Phi \subset \{\alpha_1, \ldots, \alpha_{2p-1}\}$, or $\Phi \subset \{\alpha_{2p+1}, \ldots, \alpha_\ell\}$,
(4′) case (4),
(5′) case (5) when either $\alpha_4 \in \Phi \subset \mathcal{R}_\bullet$, or $\Phi = \{\alpha_3, \alpha_5\}$;
(6′) case (6) when $\Phi \subset \{\alpha_1, \alpha_2\}$.

In all of these situations the subellipticity of $\mathfrak{Z}(M)$ was already proved in [12].

To complete the proof, we need only consider the cases in the list in which M is not essentially pseudoconcave. Next we proceed to a case by case discussion.

AIIIa

In this case $\mathfrak{g} \simeq \mathfrak{su}(p,q)$ with $2 \leqslant p < q$. Let $\Phi = \{\alpha_{j_1}, \ldots, \alpha_{j_k}\}$. We need to discuss the case where $\Phi \cap \{\alpha_p, \alpha_q\} = \emptyset$, but Φ intersects both $\mathcal{R}_\bullet$ and its complement $\mathcal{B} \setminus \mathcal{R}_\bullet$. By the condition that M is of finite type, we know from [1, Theorem 13.4] that Φ does not contain at the same time a simple root α_j, with $1 \leqslant j < p$, and its symmetrical α_{p+q-j}. Since Φ and $\Phi' = \{\alpha_{p+q-j} \mid \alpha_j \in \Phi\}$ define anti-isomorphic CR manifolds, we can assume in the proof that

$$1 \leqslant j_1 < \cdots < j_a < p < j_{a+1} < \cdots < j_b < q < j_{b+1} < \cdots < j_k < j_{k+1} = p + q,$$

$$p - j_a < j_{b+1} - q.$$

The real roots that correspond to non zero semidefinite Levi forms are

$$\gamma_s = \sum_{j=s}^{p+q-s} \alpha_j \quad \text{for} \quad p + q - j_{b+1} < s \leqslant j_a.$$

Indeed, those $\alpha \in \bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ for which $\mathfrak{g}_\alpha^\mathbb{C}$ is not contained in the kernel of the Levi form $\mathbf{L}_\gamma$ must satisfy $\mathrm{supp}(\alpha) \cap \Phi = \mathrm{supp}(\gamma) \cap \Phi$.

Thus all roots α in $\bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ with $\mathrm{supp}(\alpha) \cap \Phi \not\supset \{\alpha_{j_a}, \alpha_{j_b}\}$ belong to $\overline{\mathcal{K}}_\Phi$. This is the case in particular for all the simple roots not in $\mathcal{Q}$ and for the roots $-\sum_{j=p}^{q-1}\alpha_j$ and $-\sum_{j=p+1}^{q}\alpha_j$, whose conjugates are $-\alpha_q$ and $-\alpha_p$, respectively. This implies that all roots in $-(\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n)$ are sums of roots in $\mathcal{K}_\Phi \cup \overline{\mathcal{K}}_\Phi$, giving condition (7.18).

CIIa

We need to consider the cases where $\Phi = \{\alpha_{j_1}, \ldots, \alpha_{j_k}\}$ contains at the same time a root α_j with $j < 2p$ and a root α_j with $j > 2p$. This means that $k \geqslant 2$, and we can order the indices in such a way that

$$1 \leqslant \cdots < j_a < 2p < j_{a+1} < \cdots \leqslant \ell,$$

with j_r odd for $1 \leqslant r \leqslant a$. The positive real roots are

$$\gamma_r = \alpha_{2r-1} + \alpha_\ell + 2\sum_{j=2r}^{\ell-1}\alpha_j.$$

Since their supports contain $\alpha_{j_{a+1}}$, they all belong to $\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$. Let $j_a = 2q-1$. The Levi forms $\mathbf{L}_{\gamma_r}$ are indefinite if $r > q$ by [1, (ii), p.520]. Next we note that $\mathbf{L}_{\gamma_r}$ with $1 \leqslant r < q$ are identically zero, because there is no root α in $\bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ with $\mathrm{supp}(\alpha) \cap \Phi = \mathrm{supp}(\gamma_r) \cap \Phi$. Indeed, if α is a negative root whose support contains $\mathrm{supp}(\gamma_r) \cap \Phi$ for some $1 \leqslant r < q$, then α_{2q-2} and α_{2q} both belong to $\mathrm{supp}(\alpha)$ and hence also to $\mathrm{supp}(\bar{\alpha})$. This implies that also $\alpha_{2q-1} \in \Phi$ belongs to the support of both α and $\bar{\alpha}$, showing that $-\alpha \in \mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$.

It was shown in [1, p.520, (iii)] that $\mathbf{L}_{\gamma_q}$ is not zero and is positive semidefinite. The set of pairs $\beta, \beta' \in \bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ with $\beta + \bar{\beta}' = \gamma_q$ was shown in [1, p.520, (iii)] to consist of the pairs (β_s, β_s) with

$$\bar{\beta}_s = -\sum_{j=2q}^{s}\alpha_j \quad \text{for} \quad 2p \leqslant s < j_{a+1}.$$

We now distinguish two cases. If $q < p$, the root $-\bar{\alpha}_{2p}$ belongs to $\mathcal{K}_\Phi$. Then the root system generated by $\mathcal{K}_\Phi \cup \overline{\mathcal{K}}_\Phi$ contains $-(\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n)$ and hence the higher order Levi form concavity condition (7.18) is satisfied.

When $p = q$, then no element of $(\mathcal{K}_\Phi \cup \overline{\mathcal{K}}_\Phi) \setminus \mathcal{R}^+$ contains α_{2p} in its support. Thus condition (7.18) fails for γ_q because, in the expression for $(-\gamma_q)$ in (7.3), the coefficient of α_{2p} is (-2), while for all roots in $\bar{\mathcal{Q}}_\Phi$ it is $\geqslant (-1)$.

EIII

We only need to consider the cases where either $\Phi = \{\alpha_3\}$, or $\Phi = \{\alpha_5\}$. Due to the symmetry of the EIII diagram, we can restrict our attention to the case where $\Phi = \{\alpha_3\}$.

We note that $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the unique positive real root. It belongs to $\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$ and the corresponding Levi form $\mathbf{L}_\gamma$ is semidefinite, having rank 1. Indeed, $\alpha = -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$ is the unique root in $\bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ for which $\alpha + \bar{\alpha} = -\gamma$, and there is no other pair $\beta, \beta' \in \bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ for which $\beta + \bar{\beta}' = -\gamma$. Thus all simple roots belong to $\mathcal{K}_\Phi \cup \overline{\mathcal{K}}_\Phi$, and hence the higher Levi form concavity condition (7.18) is satisfied.

FII

We are left to discuss the case where $\{\alpha_3\} \subset \Phi \subset \{\alpha_1, \alpha_2, \alpha_3\}$. There is only one positive real root, namely $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$, that belongs to $\mathcal{Q}_\Phi^n \cap \bar{\mathcal{Q}}_\Phi^n$. The corresponding Levi form $\mathbf{L}_\gamma$ has rank one, and hence it is semidefinite. The set $\mathcal{K}_\Phi$ is the complement of $\{-\alpha_4\}$ in $\mathcal{Q}_\Phi$. Then no element of $(\mathcal{K}_\Phi \cup \overline{\mathcal{K}}_\Phi) \setminus \mathcal{R}^+$ contains α_4 in its support. Thus condition (7.18) fails for γ because, in the expression for $(-\gamma)$ in (7.3), the coefficient of α_4 is (-2), while for all roots in $\bar{\mathcal{Q}}_\Phi$ it is $\geqslant (-1)$. $\qquad\square$

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Andrea ALTOMANI
University of Luxembourg
Research Unity in Mathematics
162a, avenue de la Faïencerie
1511 Luxembourg (Luxembourg)
andrea.altomani@uni.lu

C. Denson HILL
Stony Brook University
Department of Mathematics

Stony Brook, NY 11794 (USA)
dhill@math.sunysb.edu

Mauro NACINOVICH
II Università di Roma "Tor Vergata"
Dipartimento di Matematica
Via della Ricerca Scientifica
00133 Roma (Italy)
nacinovi@mat.uniroma2.it

Egmont PORTEN
Sweden University
Department of Mathematics
85170 Sundsvall (Sweden)
Egmont.Porten@miun.se