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Alexey GLUTSYUK & Christophe SABOT

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STOKES MATRICES OF HYPERGEOMETRIC INTEGRALS

by Alexey GLUTSYUK & Christophe SABOT

Abstract. — In this work we compute the Stokes matrices of the ordinary differential equation satisfied by the hypergeometric integrals associated to an arrangement of hyperplanes in generic position. This generalizes the computation done by J.-P. Ramis for confluent hypergeometric functions, which correspond to the arrangement of two points on the line. The proof is based on an explicit description of a base of canonical solutions as integrals on the cones of the arrangement, and combinatorial relations between integrals on cones and on domains.

Résumé. — Dans cet article, nous calculons les matrices de Stokes de l'équation différentielle ordinaire satisfaites par les intégrales hypergéométriques, associées à un arrangement d'hyperplans en position générique. Cela généralise le calcul fait par J.-P. Ramis pour les fonctions hypergéométriques confluentes, qui correspondent à l'arrangement de deux points sur une droite. La démonstration est basée sur une description explicite d'une base de solutions canoniques comme intégrales sur les cônes de l'arrangement et les relations combinatoires entre les intégrales sur cônes et sur domaines.

1. Introduction and main result

The computation of the Stokes matrix of an ordinary differential equation with an irregular singular point is in general a difficult problem. In [10] and [3], J.-P.Ramis and A.Duval considered the case of confluent hypergeometric functions, and computed the associated Stokes matrices. In this paper we study a natural generalization: we consider an arrangement of hyperplanes in generic position and the hypergeometric integrals with an exponential term of the form $e^{-\lambda f_0}$ where f_0 is an extra linear form. Differentiating in λ leads to a differential equation satisfied by these integrals, with a regular singular point at 0 and an irregular singular point at

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infinity. The case of [10] is the case of the arrangement of two points on the line. The purpose of this paper is to compute explicitly the Stokes matrices of this equation. A differential equation of this type appears in the analysis of a probabilistic model of random environments ([11]), which was one of the motivation of this work.

Let $f_1, \cdots, f_N$ be N affine forms on $\mathbb{R}^k$, $N \geqslant k$, and set

$$H_j = \ker f_j.$$

We assume that the hyperplanes $H_1, \ldots, H_N$ are in generic position (all of them are distinct, any k planes intersect at a single point and the intersection of any $k+1$ planes is empty). We denote by

$$l_j(z) = f_j(z) - f_j(0),$$

the linear form directing f_j. We associate a positive weight α_j to each hyperplane H_j, and for any subset $U \subset \{1, \ldots, N\}$ we set

$$\alpha_U = \sum_{j \in U} \alpha_j. \tag{1.1}$$

The couple $(\mathbb{R}^k, (H_j)_{j=1,\cdots,N})$ defines an arrangement of hyperplanes. To any collection of k hyperplanes $H_{j_1}, \cdots, H_{j_k}$, $j_l \neq j_r$ for $l \neq r$, we associate the unique vertex of the arrangement

$$X = H_{j_1} \cap \cdots \cap H_{j_k}. \tag{1.2}$$

Depending on the context we will consider a vertex as a subset of $\{1, \ldots, N\}$ with k elements (i.e. in (1.2), $X = \{j_1, \ldots, j_k\}$) or a point of $\mathbb{R}^k$ (as in formula (1.2)). We denote by $\mathcal{X}$ the set of vertices of the arrangement. To any vertex $X = \{j_1, \ldots j_k\}$ we associate the differential form of maximal degree

$$\omega_X = \frac{df_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{df_{j_k}}{f_{j_k}},$$

where the elements of X are ordered so that the form $df_{j_1} \wedge \cdots \wedge df_{j_k}$ is positively oriented (for an arbitrary fixed orientation of the vector space $\mathbb{R}^k$).

A connected component Δ of $\mathbb{R}^k \setminus \cup_{j=1}^N H_j$ is called an *arrangement domain*. We denote by $\mathcal{D}$ the set of the arrangement domains. Let f_0 be a linear form on $\mathbb{R}^k$ in general position with respect to $(f_1, \ldots, f_N)$ (i.e., f_0 takes distinct values on the vertices of the arrangement and is nonconstant on each intersection line of $k-1$- ple of hyperplanes). We denote by $\mathcal{D}^\pm$ the set of the arrangement domains on which the form $\pm f_0$ is bounded from below. Since the arrangement is generic, it follows that the domains of $\mathcal{D}^+$ are the bounded domains or the unbounded domains Δ for which there exist some constants $A \in \mathbb{R}$ and $B > 0$ such that $f_0(x) \geqslant A + B\|x\|$ on Δ. To any domain Δ of $\mathcal{D}^+$ and any vertex X, we associate the integral

$$I_{\Delta,X}(\lambda) = \int_\Delta e^{-\lambda f_0}\Omega_X,\ \Omega_X = \left(\prod_{j=1}^N |f_j|^{\alpha_j}\right)\omega_X, \tag{1.3}$$

for $\mathrm{Re}(\lambda) > 0$.

Now we need to describe the edges of dimension 1 of the arrangement: to any subset $U = \{j_1, \dots, j_{k-1}\} \subset X$ we associate the edge of the arrangement

$$L_U = \cap_{j\in U} H_j,$$

which is a line in $\mathbb{R}^k$. Let e_U be the unique vector directing L_U, i.e. such that $L_U = X + \mathbb{R}e_U$, and normalized so that

$$f_0(e_U) = 1. \tag{1.4}$$

The general theory of hypergeometric integrals tells that these integrals are solutions of a differential equation. In our case, we can show (for the convenience of the reader, we give a proof of this result at the end of the paper) that for any domain Δ in $\mathcal{D}^+$, the vector

$$I_\Delta(\lambda) = (I_{\Delta,X}(\lambda))_{X\in\mathcal{X}}$$

satisfies the following ordinary differential equation

$$I' = -(\mathcal{A} + \frac{1}{\lambda}\mathcal{B})I, \tag{1.5}$$

where $\mathcal{A}$ is the diagonal matrix with diagonal terms

$$\mathcal{A}_{X,X} = f_0(X).$$

The matrix $\mathcal{B}$ is given by

$$\mathcal{B}_{X,X} = \alpha_X$$

on the diagonal and

$$\mathcal{B}_{X,Y} = 0,$$

if the vertices X, Y are distinct and do not lie in one and the same edge (or equivalently, $|X \cap Y| < k - 1$).

Finally, if $|X \cap Y| = k - 1$, we set $\{j\} = X \setminus Y$, $\{r\} = Y \setminus X$, $U = X \cap Y$,

$$\mathcal{B}_{X,Y} = \epsilon(j, r, U)\alpha_r,$$

where $\epsilon(j, r, U)$ depends on the relative orientation of f_j and f_r on the edge L_U:

$$\epsilon(j, r, U) = \mathrm{sgn}(l_j(e_U)l_r(e_U)).$$

There is a natural bijection between the set of vertices $\mathcal{X}$ and the domain set $\mathcal{D}^+$:

Definition 1.1. — *To each domain* $\Delta \in \mathcal{D}^+$ *we associate the unique vertex* $X(\Delta) \in \partial\Delta$ *that minimizes* f_0 *on* $\overline{\Delta}$*; the inverse of this application associates to any vertex* X *the unique domain* Δ_X *containing* X *in its boundary and on which* $f_0 - f_0(X) > 0$*, see Fig. 1.2.*

The Wronskian of the solutions $(I_\Delta(\lambda))_{\Delta\in\mathcal{D}^+}$ has been explicitely computed in the works by A.N. Varchenko [12, 13], in his joint work with Y. Markov and V. Tarasov [7], and in the joint work by A. Douai and H. Terao [2]. This Wronskian is nonzero. Hence, the functions $(I_\Delta(\lambda))_{\Delta\in\mathcal{D}^+}$ form a basis of solutions of the differential system (1.5) on the set $\{\mathrm{Re}(\lambda) > 0\}$. The differential equation (1.5) has a regular singular point at $\lambda = 0$ and an irregular singular point at $\lambda = \infty$. The question we address in this paper is the explicit computation of the Stokes matrices of this differential equation. J.-P. Ramis [10] computed the Stokes matrices of some confluent hypergeometric integrals, which corresponds to a particular case of our differential equations (cf. Example 1.9). A. Duval and C. Mitschi computed Stokes matrices and Galois groups for another class of differential equations: generalized confluent hypergeometric equations [3, 4, 8], which are equivalent to equations of type (1.5) with a matrix $\mathcal{A}$ having multiple zero eigenvalue. Earlier a similar result was obtained by M. Kohno and S. Ohkohchi [6] for some of the equations studied in [3, 4, 8]. It is not known whether there is a relation between the generalized confluent hypergeometric equations and the equations (1.5) coming from the hyperplane arrangements. K. Okubo considered generic equation of type (1.5) and provided a method of calculating its Stokes matrices as implicit solutions of equations in power series ([9], corollary (4.17)).

The general theory (see [1, 5]) says that there is a unique formal linear invertible change of space variables at infinity that transforms (1.5) to its formal normal form:

$$Y' = -(\mathcal{A} + \frac{1}{\lambda} diag(\mathcal{B}))Y, \tag{1.6}$$

where diag($\mathcal{B}$) is the diagonal matrix formed by the diagonal terms of $\mathcal{B}$ (i.e. $\mathcal{B}_{X,X} = \alpha_X$). The previous formal change is given by a formal Laurent nonpositive power series in λ (with matrix coefficients; the free term is unit) that does not converge in general. On the other hand, on each sector $S_\pm \subset \mathbb{C}$ defined below there exists a unique holomorphic variable change (called *sectorial normalization*) transforming (1.5) to (1.6) for which the previous normalizing series is its asymptotic Laurent series at infinity. The latter statement holds true for the following sectors, see Fig. 1.1a:

$$(1.7)\quad S_{\pm}=\{\varepsilon-\frac{\pi}{2}<\pm\arg\lambda<\frac{3\pi}{2}-\varepsilon\};$$

with arbitrarily fixed $\varepsilon,\ 0<\varepsilon<\frac{\pi}{2}$.

Definition 1.2. — *The canonical solution base of (1.6) is the base of its solutions given by a diagonal fundamental matrix. The canonical sectorial solution base of (1.5) in $S_{\pm}$ is its pullback under the corresponding sectorial normalization.*

The canonical solution bases are uniquely defined up to multiplication of the base solutions by constants. We normalize them as follows. Let

$$V\to\mathbb{C}^*=\mathbb{C}\setminus 0$$

be the universal cover over $\mathbb{C}^*$. We lift both equations (1.5) and (1.6) and the sectorial normalizations to V. Take a holomorphic branch on V of the diagonal fundamental solution matrix W of the formal normal form (1.6). Fix connected components $S_0, S_1, S_2\subset V$ of the covering projection preimages of S_+, S_- and S_+ respectively that are ordered clockwise so that

$$(1.8)\qquad S_{01}=S_0\cap S_1\neq\emptyset,\ S_{12}=S_1\cap S_2\neq\emptyset,\ \text{see Fig. 1.1b.}$$

Definition 1.3. — *The normalized tuple of canonical sectorial solution bases of equation (1.5) in S_j, $j=0,1,2$, consists of the pullbacks of the previous holomorphic fundamental matrix W under the corresponding sectorial normalizations of (1.5). Then for any $j=0,1$ the pair of the previous solution bases in S_j and S_{j+1} is called a normalized base pair.*

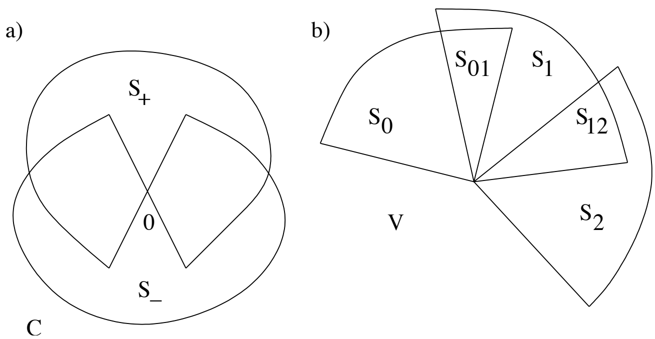

Figure 1.1. The sectors $S_{\pm}$, S_0, S_1, S_2

Remark 1.4. — A normalized base tuple (pair) is uniquely defined up to multiplication of the base functions by constants (independent on the sector).

Denote the previous normalized sectorial solution bases in S_j (more precisely, their fundamental matrices) by $Z_j(\lambda)$, $j = 0, 1, 2$. The transitions between them in the intersections S_{01}, S_{12} of their definition domains are given by constant matrices C_0, C_1 called *Stokes matrices*:

$$Z_1(\lambda) = Z_0(\lambda)C_0 \text{ in } S_{01},\ Z_2(\lambda) = Z_1(\lambda)C_1 \text{ in } S_{12}. \tag{1.9}$$

Remark 1.5. — The Stokes matrices are uniquely defined up to simultaneous conjugation by one and the same diagonal matrix.

In the present paper we find explicitly the above canonical sectorial solution bases (Proposition 2.4 in the next Section) and calculate the corresponding Stokes matrices (the next theorem).

We order all the vertices X of the hyperplane arrangement by the corresponding values $f_0(X)$ of the linear function f_0 (which are distinct by definition). The sectorial solution bases given by Proposition 2.4 in the sectors $S_\pm$ are numerated by the vertices X. Their X'- components are given by the integrals $I^\pm_{X,X'}$ over appropriate cones based at X of the (appropriately extended) forms $e^{-\lambda f_0}\Omega_{X'}$.

To describe the Stokes matrices, we need to introduce some notations. Let X be a vertex, we denote by $\mathcal{C}_X^+$ the unique (open) cone defined by the hyperplanes $(H_j)_{j\in X}$ on which $f_0 - f_0(X)$ is positive. Similarly, the cone $\mathcal{C}_X^-$ is the unique cone defined by the hyperplanes $(H_j)_{j\in X}$ on which $f_0 - f_0(X)$ is negative.

Definition 1.6. — *A pair (X, X') of distinct vertices $X, X' \in \mathcal{X}$ is said to be positive exceptional, if either $X' \notin \overline{\mathcal{C}}_X^+$, or $X' \in \mathcal{C}_X^+$ and there exists an arrangement hyperplane through X' that does not separate the domains $\Delta_{X'}$ and Δ_X (see Fig. 1.2). The latter hyperplane is then also called exceptional. A pair (X, X') is said to be negative exceptional, if it is positive exceptional with respect to the arrangement equipped with the new linear function $\widetilde{f_0} = -f_0$.*

Theorem 1.7. — *Consider the normalized tuple of canonical sectorial solution bases in S_0, S_1, S_2 (numerated by the vertices $X \in \mathcal{X}$) given by Proposition 2.4. The corresponding Stokes matrices*

$$C_j = (C_j(X', X))_{X',X\in\mathcal{X}},\ j = 0, 1,$$

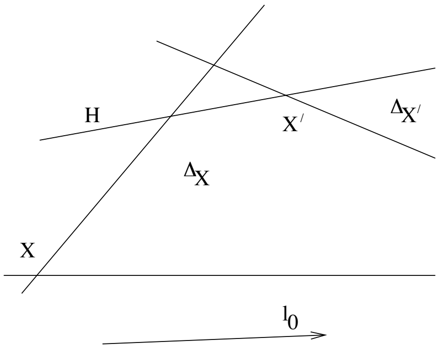

Figure 1.2. A positive exceptional pair (X, X'): the line H is exceptional

see (1.9), are given by the following formulas:

$$C_0(X, X) = C_1(X, X) = 1,$$

(1.10) $C_0(X', X) =$

$$\begin{cases} 0, \text{ if the pair } (X, X') \text{ is positive exceptional; otherwise} \\ (-1)^{|B|+|X'\setminus X|} e^{\pi i(\alpha_B - \alpha_A)} \prod_{j\in X'\setminus X} (2i \sin \pi\alpha_j); \end{cases}$$

(1.11) $C_1(X', X) =$

$$\begin{cases} 0 \text{ if the pair } (X, X') \text{ is negative exceptional; otherwise} \\ (-1)^{|B|+|X'\setminus X|} e^{\pi i(\alpha_X - \alpha_{X'} + \alpha_B - \alpha_A)} \prod_{j\in X'\setminus X} (2i \sin \pi\alpha_j), \end{cases}$$

where

(1.12) $A = \{j \mid H_j \text{ separates (strictly) } X \text{ from } X'\},$

$$B = \{j \mid H_j \text{ contains } X, X' \text{ and separates the cone } \mathcal{C}_X^+ \text{ from } \mathcal{C}_{X'}^+\}.$$

Remark 1.8. — The above set B coincides with the set defined in a similar way but with the upper index "+" of the cones replaced by "−". Indeed, any given hyperplane H through X and X' that separates the cones $\mathcal{C}_X^+$ and $\mathcal{C}_{X'}^+$ also separates $\mathcal{C}_X^-$ from $\mathcal{C}_{X'}^-$ and vice versa. This follows from the fact that the central symmetry with respect to X (X') sends $\mathcal{C}_X^+$ to $\mathcal{C}_X^-$ (respectively, $\mathcal{C}_{X'}^+$ to $\mathcal{C}_{X'}^-$) and changes the side of the cone under consideration with respect to H.

Example 1.9. — Let $k=1$, $N \geqslant 2$ and $X_1 < \cdots < X_N$ be N points on the real line, and set

$$f_i(z) = z - X_j, \quad j = 1, \dots, N, \ \ z \in \mathbb{R},$$

$$f_0(z) = z.$$

The matrix $\mathcal{A}$ is the diagonal matrix

$$\mathcal{A} = \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_N \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{pmatrix}.$$

The Stokes matrices are

$$C_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -2i \sin \pi\alpha_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -2ie^{-\pi i \sum_{s=2}^{N-1} \alpha_s} \sin \pi\alpha_N & -2ie^{-\pi i \sum_{s=3}^{N-1} \alpha_s} \sin \pi\alpha_N & \cdots & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & -2ie^{\pi i(\alpha_2-\alpha_1)} \sin \pi\alpha_1 & \cdots & -2ie^{\pi i(\alpha_N - \sum_{j=1}^{N-1} \alpha_j)} \sin \pi\alpha_1 \\ 0 & 1 & \cdots & -2ie^{\pi i(\alpha_N - \sum_{j=2}^{N-1} \alpha_j)} \sin \pi\alpha_2 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The case where $N=2$ and $X_1 = 0$, $X_2 = 1$ corresponds to the usual confluent hypergeometric case, which has been considered in [10].

Example 1.10. — Let $k=2$ and for $z=(x,y)$

$$f_1(z) = x, \quad f_2(z) = y, \quad f_3(z) = x + y - 1,$$
$$f_0(z) = ax + by,$$

with $a > 0$, $b > 0$, $a > b$. The vertices of the arrangement are

$$X_1 := (0,0), \quad X_2 := (0,1), \quad X_3 := (1,0).$$

We have

$$f_0(X_1) = 0 < f_0(X_2) = b < f_0(X_3) = a,$$
$$X_1 = \{1,2\},\ X_2 = \{1,3\},\ X_3 = \{2,3\},$$
$$A = \emptyset \text{ for each pair } (X', X),$$
$$B = \{H_2\} \text{ if } \{X', X\} = \{X_1, X_3\},\ B = \emptyset \text{ otherwise},$$
$$C_0 = \begin{pmatrix} 1 & 0 & 0 \\ -2i\sin\pi\alpha_3 & 1 & 0 \\ 2ie^{\pi i\alpha_2}\sin\pi\alpha_3 & -2i\sin\pi\alpha_2 & 1 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 1 & -2ie^{i\pi(\alpha_3-\alpha_2)}\sin\pi\alpha_2 & 2ie^{\pi i(\alpha_2+\alpha_3-\alpha_1)}\sin\pi\alpha_1 \\ 0 & 1 & -2ie^{i\pi(\alpha_2-\alpha_1)}\sin\pi\alpha_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Canonical solutions at infinity. The plan of the proof of Theorem 1.7

2.1. Canonical solutions

Let X be a vertex and $\rho \in \mathbb{C}$, $|\rho| = 1$. We denote by $\mathcal{C}_X^\rho \subset \mathbb{C}^k$ the cone based at X and defined by

$$\mathcal{C}_X^\rho = \Big\{ z = X + \rho\Big(\sum_{j\in X} a_j e_{X\setminus\{j\}}\Big),\ \ a_j \in \mathbb{R}_+ \Big\}, \tag{2.1}$$

where $e_{X\setminus\{j\}}$ has been defined in (1.4).

Remark 2.1. — When $\rho = \pm 1$, one has $\mathcal{C}_X^\rho = \mathcal{C}_X^\pm$ (the cones $\mathcal{C}_X^\pm$ were defined in the Introduction, just before Theorem 1.7). For any vertex X one has $\Delta_X \subset \mathcal{C}_X^+ = \mathcal{C}_X^1$.

For any affine subspace $H \subset \mathbb{R}^k$ denote $\mathbb{C}H \subset \mathbb{C}^k$ its complexification.

Proposition 2.2. — *For any $j \in X$, the intersection $\mathbb{C}H_j \cap \overline{\mathcal{C}_X^\rho}$ is a face of the cone $\mathcal{C}_X^\rho$. For any $\rho \notin \mathbb{R}$ and $l \notin X$ one has*

$$\overline{\mathcal{C}_X^\rho} \cap \mathbb{C}H_l = \emptyset.$$

Proof. — Without loss of generality we prove the proposition assuming that $X = 0$ (translating the coordinates). The first statement of the proposition is obvious. Let us prove the second one. Suppose the contrary: there exist a $\rho \notin \mathbb{R}$ and a $l \notin X$ such that there exists a point $x_0 \in \overline{\mathcal{C}_0^\rho} \cap \mathbb{C}H_l$. By definition, $0 = X \notin \mathbb{C}H_l$, since $0 \notin H_l$ ($l \notin X$ by assumption). In particular, $x_0 \neq 0$. Set $x_1 = \operatorname{Re} x_0 \in \operatorname{Re}(\mathbb{C}H_l), x_2 = \operatorname{Im} x_0 \in \operatorname{Im}(\mathbb{C}H_l)$. One has

$$x_0 = \rho v, \ v \in \overline{\mathcal{C}_0^+} \setminus 0, \ x_1 = (\operatorname{Re}\rho) v \in \operatorname{Re}(\mathbb{C}H_l) = H_l, \tag{2.2}$$

$$x_2 = (\operatorname{Im}\rho) v \in \operatorname{Im}(\mathbb{C}H_l) = \operatorname{Im}(\mathbb{C}H_l') = H_l', \tag{2.3}$$

where H_l' is the real hyperplane through 0 parallel to H_l. One has $x_2 \neq 0$, since $x_0 \neq 0$ and $\operatorname{Im}\rho \neq 0$ ($\rho \notin \mathbb{R}$ by assumption). The vector x_1 lies in H_l', since it is proportional to $x_2 \in H_l' \setminus 0$. Therefore, x_1 lies simultaneously in two disjoint hyperplanes H_l and H_l', - a contradiction. □

For any $\rho \notin \mathbb{R}$, we consider the integral

$$I^\rho_{X,X'}(\lambda) = \int_{\mathcal{C}_X^\rho} e^{-\lambda f_0} \Omega_{X'}, \ \Omega_{X'} = \left(\prod_{j=1}^N |f_j|^{\alpha_j}\right) \omega_{X'}, \tag{2.4}$$

where the determination of the 1- form $\Omega_{X'}$ is chosen as follows. Take a simply connected domain $\widetilde{\mathcal{D}} \subset \mathbb{C}^k \setminus \cup_j \mathbb{C}H_j$ containing Δ_X and the union of the cones $\mathcal{C}_X^\rho$, $\operatorname{Im}\rho < 0$. The latter cones are simply connected, as is their union, and disjoint from the complex hyperplanes $\mathbb{C}H_j$ (see the previous proposition). The domain Δ_X, which is a convex (and hence, simply connected) polytope in $\mathcal{C}_X^1 \setminus \cup_j H_j$, is one-side adjacent to the union of the latter cones. Hence, the previous domain $\widetilde{\mathcal{D}}$ exists. Take the standard real branch of $\Omega_{X'}$ on the domain Δ_X and its immediate analytic extension to $\widetilde{\mathcal{D}}$.

Definition 2.3. — *Consider the integral $I^\rho_{X,X'}(\lambda)$, see (2.4). It is well-defined whenever λ is such that $\operatorname{Re}(\lambda\rho) > 0$. Moreover, for any $\lambda \notin i\mathbb{R}_-$, the integral does not depend on ρ such that $\operatorname{Im}\rho < 0$ and $\operatorname{Re}(\rho\lambda) > 0$ (when $\lambda \in i\mathbb{R}_-$, there is no such ρ). We denote by*

$$I^+_{X,X'}(\lambda)$$

the common value of $I^\rho_{X,X'}(\lambda)$ for $\operatorname{Im}(\rho) < 0$ and $\operatorname{Re}(\rho\lambda) > 0$. The function $I^+_{X,X'}(\lambda)$ is analytic on $\mathbb{C} \setminus i\mathbb{R}_-$. Similarly, we denote by $I^-_{X,X'}(\lambda)$ the common value of $I^\rho_{X,X'}(\lambda)$ for $\operatorname{Im}(\rho) > 0$ and $\operatorname{Re}(\rho\lambda) > 0$. The function $I^-_{X,X'}(\lambda)$ is well-defined and analytic on $\mathbb{C} \setminus i\mathbb{R}_+$.

We denote by $I^\pm_X(\lambda)$ the vector

$$I^\pm_X(\lambda) = (I^\pm_{X,X'}(\lambda))_{X' \in \mathcal{X}}, \ \Omega = (\Omega_{X'})_{X' \in \mathcal{X}}.$$

Proposition 2.4. — *Let $S_\pm$, V, $S_0, S_1, S_2 \subset V$ be as in (1.8). The vector functions $I^\pm_X(\lambda)$ (corresponding to all the vertices X) are solutions* *of (1.5) and form a canonical sectorial solution basis in the corresponding sector $S_\pm$ (see (1.7)). The liftings to S_0, S_1, S_2 of the solution bases*

$$I_X^+|_{S_+}, I_X^-|_{S_-}, e^{2\pi i\alpha_X} I_X^+|_{S_+}$$

respectively form a normalized tuple of sectorial solution bases (see Definition 1.3).

The Proposition is proved in 2.3.

At the end of the paper we also prove the following more precise asymptotic statement on the solutions $I_X^\pm$. We will not use it in the paper.

Proposition 2.5. — *For any vertex X, the function $I_X^\pm(\lambda)$ is a solution of (1.5), with the asymptotic behavior (uniform in the sector $S_\pm$)*

$$I_X^\pm(\lambda) \sim_{|\lambda|\to\infty} D_{X,X} e^{-\lambda f_0(X)} \lambda^{-\alpha_X} v_X,$$

where $(v_X)_{X\in\mathcal{X}}$ is the standard base of $\mathbb{R}^{|\mathcal{X}|}$,

$$D_{X,X} = \left(\prod_{j\in X} \Gamma(\alpha_j)\right) \prod_{j\in X} |l_j(e_{X\setminus\{j\}})|^{\alpha_j} \prod_{j\notin X} |f_j(X)|^{\alpha_j}.$$

2.2. The plan of the computation of Stokes operators

For the proof of Theorem 1.7 we have to calculate the transition matrices C_0, C_1 between the sectorial solution bases from Proposition 2.4. One has

$$(I_X^-)(\lambda) = (I_X^+)(\lambda) C_0 \text{ for } \lambda \in \mathbb{R}_+. \tag{2.5}$$

This follows from definition and the last statement of Proposition 2.4.

To calculate C_0, the strategy is to pass through the integrals $I_\Delta(\lambda)$, $\Delta \in \mathcal{D}^+$, which are well-defined on the axis $\lambda \in \mathbb{R}_+$.

For any $\Delta, \Delta' \in \mathcal{D}^+$ denote

$$\begin{aligned} \mathcal{H}(\Delta,\Delta') = \{j = 1,\dots,N \mid \\ \text{the hyperplane } H_j \text{ separates } \Delta \text{ from } \Delta'\}, \\ |\mathcal{H}(\Delta,\Delta')| = \text{ the cardinality of } \mathcal{H}(\Delta,\Delta'). \end{aligned} \tag{2.6}$$

Remark 2.6. — Each arrangement domain contained in a cone $\mathcal{C}_X^+$ belongs to $\mathcal{D}^+$. This follows from definition.

Lemma 2.7. — *For $\mathrm{Re}(\lambda) > 0$, we have*

$$I_X^+(\lambda) = \sum_{\Delta\subset\mathcal{C}_X^+} \eta(X,\Delta) I_\Delta(\lambda), \text{ and} \tag{2.7}$$

$$I_X^-(\lambda) = \sum_{\Delta\subset\mathcal{C}_X^+} \overline{\eta}(X,\Delta) I_\Delta(\lambda), \tag{2.8}$$

where

$$\eta(X,\Delta)=1,\ \text{if } \Delta=\Delta_X,\ \text{otherwise, } \eta(X,\Delta)=e^{\pi i\alpha_{\mathcal{H}(\Delta,\Delta_X)}}.$$

The lemma is proved below.

To calculate C_0, we have to express (I_X^-) via (I_X^+). The previous lemma expresses $I_X^\pm$ via the integrals I_Δ. Lemma 3.1 formulated in Section 3 provides the inverse expression of the integrals I_Δ via I_X^+. Afterwards C_0 is calculated by substituting the latter inverse expression to (2.8).

Recall that for any subset $B\subset\mathbb{R}^n$ $\chi_B:\mathbb{R}^n\to\mathbb{R}$ denotes the characteristic function of B:

$$\chi_B(x)\equiv 1 \text{ on } B, \chi_B(x)\equiv 0 \text{ on } \mathbb{R}^n\setminus B.$$

The proof of Lemma 3.1 is based on the next purely combinatorial identity, which holds true for arbitrary generic arrangement of hyperplanes and a linear function.

Lemma 2.8. — *Consider arbitrary generic hyperplane arrangement and a linear function, as at the beginning of the paper. Let $\mathcal{C}_X^+$ be the cones defined in the Introduction. For any arrangement domain $\Delta\in\mathcal{D}^+$ one has*

$$\chi_\Delta = \sum_{X\in\partial\Delta} \nu(\Delta,X)\chi_{C_X^+} \text{ on } \mathbb{R}^k\setminus\cup_{j=1}^N H_j, \tag{2.9}$$

$$\text{where } \nu(\Delta,X)=(-1)^{|\mathcal{H}(\Delta,\Delta_X)|},$$

$\mathcal{H}(\Delta,\Delta_X)$ was defined in (2.6).

This lemma is proved in Section 3.

A version of Lemma 2.8 was stated and proved by A.N.Varchenko and I.M.Gelfand in [14]. Namely they had shown that the characteristic function of a domain Δ_X can be uniquely presented as a linear combination (with coefficients ±1) of characteristic functions of some cones (of maybe different dimensions). They provided some implicit description of the coefficients of this linear combination without an explicit formula. Lemma 2.8 provides an explicit formula. Its proof uses a method different from that of [14].

Proof of Lemma 2.7. — Let us prove formula (2.7) of the lemma. Formula (2.8) then follows from (2.7), the equality

$$I_X^-(\lambda)=\overline{I_X^+(\lambda)} \text{ for any } \lambda\in\mathbb{R}_+ \tag{2.10}$$

(which follows from definition) and the complex conjugatedness of the right-hand sides of (2.7) and (2.8) (the integrals $I_\Delta(\lambda)$ are real for $\lambda\in\mathbb{R}_+$).

It suffices to show that the analytic extension of the integrals I_X^+ to the semiaxis $\lambda > 0$ is defined by formula (2.7). Indeed, the mapping

$$F_\rho : x \mapsto X + \rho(x - X)$$

is a real-linear isomorphism $\mathcal{C}_X^+ \to \mathcal{C}_X^\rho$ that tends to the identity, as $\rho \to 1$. The closure of the cone $\mathcal{C}_X^\rho$ is the union of the closures of the domains

$$\Delta(\rho) = F_\rho(\Delta), \Delta \subset \mathcal{C}_X^+.$$

The integral I_X^+ is the sum of the integrals of $e^{-\lambda f_0}\Omega|_{\mathcal{C}_X^\rho}$ over the domains $\Delta(\rho)$. Each latter integral tends (as $\rho \to 1$) to the integral over Δ of the form $e^{-\lambda f_0}\Omega$, where the branch $\Omega|_\Delta$ is the immediate analytic extension of $\Omega|_{\cup_{\operatorname{Im}\rho<0}\Delta(\rho)}$ to $\Delta = \Delta(1)$. We claim that thus extended branch $\Omega|_\Delta$ is the $\eta(X, \Delta)$- th multiple of the standard real branch of Ω on Δ (see (1.3)). Indeed, fix a $x \in \Delta$ and denote $L \subset \mathbb{C}^N$ the complex line containing the segment $[X, x]$. The latter segment intersects Δ_X by definition and since $x \in \mathcal{C}_X^+$. Fix a point of their intersection and denote it y. Fix a $\rho \in \mathbb{C}$, $|\rho| = 1$, with $\operatorname{Im}\rho < 0$. Denote

$$x_\rho = F_\rho(x) \in \Delta(\rho),\ \delta_\rho = \{F_{e^{i\theta}}(x) \mid \arg\rho \leqslant \theta \leqslant 0\}. \text{ Consider the path}$$

$$\gamma : [0, 1] \to \mathbb{C}^N \text{ from } y \text{ to } x :\ \gamma = [y, x_\rho] \circ \delta_\rho,$$

which goes first from y to x_ρ along the segment $[y, x_\rho]$ and then from x_ρ to x along the path δ_ρ. By construction, the previously constructed branch $\Omega|_\Delta$ is obtained by the analytic extension of the standard real branch of Ω on Δ_X along the path γ (all the points of γ except for its ends y and x lie in $\cup_{\operatorname{Im}\rho<0}\mathcal{C}_X^\rho$). For any hyperplane H_j intersecting the segment $[y, x]$ denote z_j the intersection point. We consider that the point x is chosen generic so that the points x_j are distinct. The path γ is isotopic in $L \setminus \cup_j H_j$ to the segment $[y, x]$ where small intervals (a_j, b_j) containing z_j are replaced by half-circles in L (with the same ends a_j and b_j) oriented counterclockwise (the notion "counterclockwise" is independent on the choice of affine complex coordinate on L). Extending the form Ω along a previous half-circle yields extra multiplier $e^{\pi i \alpha_j}$. This implies that the extended branch $\Omega|_\Delta$ is the standard real branch times $\eta(X, \Delta)$. This together with the previous discussion proves the lemma. □

2.3. The integrals $I_X^\pm$. Proof of Proposition 2.4.

The vector functions $I_X^\pm$ are linear combinations of integrals over domains Δ (Lemma 2.7). Therefore, they are solutions of (1.5), as are the latter integrals (see the Introduction). Now we have to show that they form canonical sectorial solution bases.

Given a ray in $\mathbb{C}$, we say that a vector function f is *asymptotically bigger* than another one g along the ray, if $g(z) = o(f(z))$, as $z \to \infty$ along the ray. A collection of functions is *asymptotically ordered* along a ray, if for any two distinct functions one is asymptotically bigger than the other one. We use the following characterization of canonical solution bases, which follows from the general theory of linear equations with irregular singularities.

Proposition 2.9. — *Consider arbitrary canonical solution base of (1.5) in the sector S_+ (or in S_-). The basic solutions in the given sector are asymptotically ordered along both semiaxes $\pm\lambda > 0$; their orderings along these semiaxes are opposite to each other. Vice versa, given arbitrary collection of solutions F_X (numerated by all the vertices X) of (1.5) in the sector $S_\pm$ under consideration. Let F_X be asymptotically ordered along the previous semiaxes and the corresponding orderings be opposite to each other. Then F_X is a canonical sectorial solution base.*

Addendum. — *Let $S_\pm$, S_0, S_1, S_2 be as in (1.7) and (1.8). Let $F^\pm = (F_X^\pm)_{X\in\mathcal{X}}$ be a pair of canonical sectorial solution bases in $S_\pm$ such that for any $X \in \mathcal{X}$ one has*

$$F_X^-(\lambda) = F_X^+(\lambda) + o(F_X^+(\lambda)), \text{ as } \lambda \in \mathbb{R}_+,\ \lambda \to +\infty. \tag{2.11}$$

Then the liftings to S_0, S_1, S_2 of the bases

$$F^+|_{S_+}, F^-|_{S_-}, (e^{2\pi i \alpha_X} F_X^+)|_{S_+}$$

form a normalized tuple of canonical sectorial bases.

Proof. — The statements of the proposition and the addendum are obvious for the formal normal form (1.6). Let us prove the statements of the addendum for (1.6) in more detail. Each solution base of (1.6) under consideration is defined by a diagonal fundamental matrix. Any two (locally defined) diagonal fundamental matrices are obtained one from the other by multiplication of the diagonal elements by appropriate constants. The latter constants comparing the fundamental matrices of $F^+|_{S_0}$ and $F^-|_{S_1}$ on S_{01} ($F^-|_{S_1}$ and $(e^{2\pi i\alpha_X} F_X^+)|_{S_2}$ on S_{12}) are unit, i.e., the three latter solution bases are holomorphic extensions of each other. This follows from (2.11) (for the former base pair) and the fact that the solution base $(e^{2\pi i\alpha_X} F_X^+)|_{S_+}$ is the image of $F^+|_{S_+}$ under the clockwise monodromy around 0. Hence, the lifted bases from the addendum form a normalized tuple (see Definition 1.3).

Now given arbitrary differential equation (1.5). Consider the variable transformations inverse to its sectorial normalizations. These transformations send (1.6) to (1.5) and thus, the canonical sectorial solution bases of (1.6) to those of (1.5), and preserve the asymptotic orderings and relations (2.11). This together with the statements of the proposition and the addendum for (1.6) proves them for (1.5). □

One has

$$I_X^{\pm} = I_{\Delta_X} + o(I_{\Delta_X}), \text{ as } \lambda \in \mathbb{R}_+, \ \lambda \to +\infty, \tag{2.12}$$

This follows from Lemma 2.7, the inclusion $\Delta_X \subset \mathcal{C}_X^+$ and the inequality $f_0|_{\overline{\Delta}_{X'}} > f_0(X)$ valid for any vertex $X' \in \overline{\mathcal{C}}_X^+ \setminus X$ (which holds by definition). The integrals $I_X^{\pm}(I_{\Delta_X})$ are asymptotically ordered along the semiaxis $\lambda > 0$: $I_X^{\pm}(I_{\Delta_X})$ is asymptotically greater than $I_{X'}^{\pm}(I_{\Delta_{X'}})$, if and only if $f_0(X) < f_0(X')$. This follows from (2.12) and the previous inequality. The same integrals $I_X^{\pm}(I_{\Delta_X})$ are also asymptotically ordered along the opposite semiaxis $\lambda < 0$, and their latter order is opposite to the previous one. Indeed, let us prove the latter statement for I_X^+. Then for I_X^- the same statement follows from the one for I_X^+ and the relation $I_X^-(\lambda) = \overline{I_X^+(\bar{\lambda})}$ (which follows from (2.10)). The cone $\mathcal{C}_X^{-1} = \mathcal{C}_X^-$ is adjacent to the union $\cup_{\operatorname{Im}\rho<0}\mathcal{C}_X^{\rho}$ and its closure is a union of closures of some domains from $\mathcal{D}^-$ (denote $\Delta_X^- \subset \mathcal{C}_X^-$ the domain with vertex at X). The integral $I_X^+(\lambda)$ restricted to $\lambda \in \mathbb{R}_-$ can be expressed as a linear combination of the integrals over the previous domains, as in Lemma 2.7 and its proof. The integral $I_{\Delta_X^-}$ appears there with the coefficient $e^{-\pi i \alpha_X}$. One has

$$I_X^+(\lambda) = e^{-\pi i \alpha_X} I_{\Delta_X^-}(\lambda) + o(I_{\Delta_X^-}(\lambda)), \text{ as } \lambda \in \mathbb{R}_-, \ \lambda \to -\infty,$$

as in (2.12). This together with the arguments following (2.12) prove the previous asymptotic order statement.

The two asymptotic order statements proved above together with the previous proposition imply that the integrals $I_X^{\pm}$ form canonical solution bases in $S_{\pm}$. This proves the first part of Proposition 2.4.

Let us prove the second part of Proposition 2.4 (about the normalized base tuple). By the addendum, to do this, it suffices to prove equality (2.11) for the bases $F_X^{\pm} = I_X^{\pm}$. This equality follows immediately from (2.12). Proposition 2.4 is proved.

3. The relations between $I_X^{\pm}$ and I_Δ. Proof of Theorem 1.7

As it is shown (at the end of the section), Theorem 1.7 is implied by Lemma 2.7 and the following lemma. The proof of the latter is based on Lemma 2.8; both lemmas are proved below.

Lemma 3.1. — *For any $\Delta \in \mathcal{D}^+$ the following equalities hold for all $\lambda \in \mathbb{R}_+$:*

$$I_\Delta(\lambda) = \sum_{X\in\partial\Delta} \psi(\Delta, X) I_X^+(\lambda), \tag{3.1}$$

$$I_\Delta(\lambda) = \sum_{X\in\partial\Delta} \overline{\psi(\Delta, X)} I_X^-(\lambda), \quad \text{where} \tag{3.2}$$

$\psi(\Delta, X) = 1$ if $\Delta = \Delta_X$, otherwise, $\psi(\Delta, X) = (-1)^{|\mathcal{H}(\Delta,\Delta_X)|} e^{i\pi\alpha_{\mathcal{H}(\Delta,\Delta_X)}}$, the set $\mathcal{H}(\Delta, \Delta_X)$ was defined in (2.6).

Proof of Lemma 2.8. — Recall that all the vertices are ordered so that the function $X \mapsto f_0(X)$ is increasing. For any vertex X one has

$$\chi_{\mathcal{C}_X^+} = \sum_{\Delta\subset\mathcal{C}_X^+} \chi_\Delta = \sum_{\Delta\in\mathcal{D}^+} \theta(X, \Delta)\chi_\Delta \text{ on } \mathbb{R}^k \setminus \cup_{j=1}^N H_j, \text{ where} \tag{3.3}$$

$$\theta(X, \Delta) = 1 \text{ whenever } \Delta \subset \mathcal{C}_X^+;\ \theta(X, \Delta) = 0 \text{ otherwise.}$$

In other terms, the vector of the functions $\chi_{\mathcal{C}_X^+}$ is obtained from the vector of the functions χ_Δ by multiplication by the matrix $\theta(X, \Delta)$ with indices $X \in \mathcal{X}$ and $\Delta \in \mathcal{D}^+$.

For the proof of (2.9) we extend the values $\nu(\Delta, X)$ (which were defined in (2.9) for $X \in \partial\Delta$) up to a matrix (with the previous indices) by putting

$$\nu(\Delta, X) = 0 \text{ whenever } X \notin \partial\Delta.$$

We show that the matrices $\nu(\Delta, X)$ and $\theta(X, \Delta)$ are inverse to each other, i.e., for any two vertices X and X' one has

$$\sum_{\Delta\in\mathcal{D}^+} \theta(X, \Delta)\nu(\Delta, X') \text{ equals 0 if } X \neq X' \text{ and equals 1 if } X = X'. \tag{3.4}$$

This will prove the lemma.

The only nonzero terms of the sum in (3.4) correspond exactly to $\Delta \in D(X, X')$, where

$$D(X, X') = \{\Delta \subset \mathcal{C}_X^+ \mid X' \in \partial\Delta\}; \tag{3.5}$$

$$\text{one has } X \leqslant X',\ \text{if } D(X, X') \neq \emptyset.$$

Case $X = X'$. — Then

$$D(X, X') = \{\Delta_X\} \quad \text{and} \quad \theta(X, \Delta_X) = \nu(\Delta_X, X) = 1$$

by definition. This proves the second statement of (3.4).

Case $X > X'$. — Then all the terms of the sum in (3.4) vanish, see (3.5).

Case $X < X'$. — Let us introduce affine coordinates $x_1, \dots, x_k$ on $\mathbb{R}^k$ so that X' is the origin and the arrangement hyperplanes through X' are the coordinate hyperplanes. Fix a hyperplane $H = \{x_i = 0\}$ (which contains X') that does not contain X (it exists by definition).

If $X' \in \mathcal{C}_X^+$, then the domains $\Delta \in D(X, X')$ intersect a small neighborhood of X' by the coordinate quadrants (whose number equals 2^k). If $X' \in \partial\mathcal{C}_{X'}^+$, then locally near X' the cone $\mathcal{C}_X^+$ is the coordinate cone defined by the inequalities $\pm x_j > 0$ (for a certain collection of distinct indices $j \neq i$); the domains $\Delta \in D(X, X')$ are locally the coordinate quadrants in the latter cone. In both cases the domain collection $D(X, X')$ is split into pairs. The domains in each pair are *adjacent across* H: by definition, this means that they are adjacent to a common face in H (of the same dimension, as H), and thus, are separated from each other by H. For any two domains Δ_1 and Δ_2 adjacent across H one has

$$\nu(\Delta_1, X') + \nu(\Delta_2, X') = 0$$

(hence, the corresponding terms of the sum in (3.4) cancel out and the latter sum vanishes). Indeed, let H separate Δ_1 from Δ_2 and $\Delta_{X'}$ (otherwise we interchange Δ_1 and Δ_2). Then

$$\mathcal{H}(\Delta_1, \Delta_{X'}) = \mathcal{H}(\Delta_2, \Delta_{X'}) \cup H$$

by definition. This together with the definition of $\nu(\Delta_j, X')$, see (2.9), proves the previous cancellation statement, (3.4) and Lemma 2.8. $\square$

Proof of Lemma 3.1. — Let us prove (3.1) (then (3.2) follows by complex conjugation argument, see (2.10)). Let us substitute the expression (2.7) for I_X^+ via the integrals over domains to the right-hand side of (3.1). We show that for any $\Delta' \in \mathcal{D}^+$ the corresponding coefficients at $I_{\Delta'}$ obtained by this substitution cancel out, except for the unit coefficient corresponding to $\Delta' = \Delta$. This will prove the lemma. After the previous substitution the right-hand side of (3.1) takes the form

$$\sum_{X \in \partial\Delta} \sum_{\Delta' \subset \mathcal{C}_X^+} \eta(X, \Delta')\psi(\Delta, X) I_{\Delta'}, \ \eta(X, \Delta') \text{ are the same, as in (2.7).}$$

For any X, Δ, Δ' such that $X \in \partial\Delta$, $\Delta' \subset \mathcal{C}_X^+$ one has

$$\eta(X, \Delta')\psi(\Delta, X) = (-1)^{|\mathcal{H}(\Delta_X, \Delta)|} e^{i\pi\alpha_{\mathcal{H}(\Delta,\Delta')}}. \tag{3.6}$$

Indeed, recall that by definition,

$$\eta(X, \Delta') = e^{\pi i\alpha_{\mathcal{H}(\Delta',\Delta_X)}}, \ \psi(\Delta, X) = (-1)^{|\mathcal{H}(\Delta_X,\Delta)|} e^{i\pi\alpha_{\mathcal{H}(\Delta_X,\Delta)}}. \tag{3.7}$$

Formula (3.6) follows from (3.7) and the fact that for any $\Delta \in \mathcal{D}^+, X \in \partial\Delta$ and $\Delta' \subset \mathcal{C}_X^+$ one has

$$\mathcal{H}(\Delta, \Delta_X) \cap \mathcal{H}(\Delta_X, \Delta') = \emptyset, \ \mathcal{H}(\Delta, \Delta_X) \cup \mathcal{H}(\Delta_X, \Delta') = \mathcal{H}(\Delta, \Delta'). \tag{3.8}$$

Indeed, each hyperplane $H \in \mathcal{H}(\Delta_X, \Delta')$, which separates Δ_X from Δ', by definition, also separates Δ from Δ'. Otherwise H separates Δ from Δ_X (hence, $X \in H$). Therefore, H does not cut the cone $\mathcal{C}_X^+$ and thus, cannot separate its subdomains Δ_X and Δ', - a contradiction. Each $H \in \mathcal{H}(\Delta, \Delta_X)$ separates Δ from Δ', since it separates Δ from the cone $\mathcal{C}_X^+ \supset \Delta', \Delta_X$ (which follows from definition). Thus,

$$\mathcal{H}(\Delta, \Delta_X) \cup \mathcal{H}(\Delta_X, \Delta') \subset \mathcal{H}(\Delta, \Delta').$$

Vice versa, each hyperplane $H \in \mathcal{H}(\Delta, \Delta')$ separates Δ from Δ' (by definition), and Δ_X is either on the Δ'- s or on the Δ- s side. These two (incompatible) cases take place, when $H \in \mathcal{H}(\Delta, \Delta_X)$ (respectively, $H \in \mathcal{H}(\Delta_X, \Delta')$). This proves (3.8) and (3.6).

Now by (3.6), the right-hand side of (3.1) equals the linear combination of the integrals $I_{\Delta'}$ with the coefficients

$$e^{i\pi\alpha_{\mathcal{H}(\Delta,\Delta')}} \sum_{X \in \partial\Delta, \ \Delta' \subset \mathcal{C}_X^+} (-1)^{|\mathcal{H}(\Delta_X,\Delta)|}.$$

The latter sum over vertices X equals the value on Δ' of the characteristic function combination (2.9) by definition. Hence, it vanishes, if $\Delta' \neq \Delta$, and equals 1 if $\Delta' = \Delta$ (Lemma 2.8). This proves Lemma 3.1. □

Proof of Theorem 1.7. — Let $C_0 = (C_0(X', X))_{X',X \in \mathcal{X}}$ be the Stokes matrix (1.9) corresponding to the normalized base tuple in S_0, S_1, S_2 from Proposition 2.4. One has

$$I_X^-(\lambda) = \sum_{X' \in \mathcal{X}} C_0(X', X) I_{X'}^+(\lambda) \text{ for all } \lambda \in \mathbb{R}_+, \tag{3.9}$$

by definition. Let us calculate the coefficients $C_0(X', X)$. Lemma 2.7 gives formula (2.8) for I_X^- as a linear combination of the integrals I_Δ with constant coefficients. Replacing each I_Δ in (2.8) by its expression (3.1) via the integrals $I^+_{X'}$ yields (3.9) with

$$C_0(X',X)=\sum_{\Delta\in D(X,X')}\gamma(X,X',\Delta), \tag{3.10}$$

$D(X,X')$ is the same, as in (3.5),

$$\gamma(X,X',\Delta)=\overline{\eta}(X,\Delta)\psi(\Delta,X')= (-1)^{|\mathcal{H}(\Delta,\Delta_{X'})|}e^{\pi i(\alpha_{\mathcal{H}(\Delta,\Delta_{X'})}-\alpha_{\mathcal{H}(\Delta,\Delta_X)})}. \tag{3.11}$$

In the case, when $X'=X$, obviously $C_0(X',X)=1$. If $X'\notin\overline{\mathcal{C}}^+_X$, then $C_0(X',X)=0$, since the previous sum contains no terms.

Thus, everywhere below in the calculation of C_0 we consider that $X'\in\overline{\mathcal{C}}^+_X\setminus X$. Let us calculate the sum (3.10). To do this, we extend (literally) the definition of $\mathcal{H}(\Delta_1,\Delta_2)$ to the case, when each Δ_j is an arbitrary union of domains in $\mathcal{D}^+$, by putting $\mathcal{H}(\Delta_1,\Delta_2)$ to be the number of the arrangement hyperplanes separating Δ_1 from Δ_2. Then we extend analogously the definition of the values $\gamma(X,X',\Delta)$ (for Δ being a union of domains) by writing formula (3.11) with thus generalized $\mathcal{H}(\Delta,\Delta_{X'})$, $\mathcal{H}(\Delta,\Delta_X)$.

Fix an arbitrary arrangement hyperplane H_j through X' that does not contain X. Recall that the domains from $D(X,X')$ are split into pairs of domains adjacent across H_j (see the above proof of Lemma 2.8). To calculate (3.10), we first fix a pair of domains $\Delta_1,\Delta_2\in D(X,X')$ adjacent across H_j and compute their contribution to (3.10). Afterwards we deduce formula (1.10) by extending the latter computation for $\Delta_{1,2}$ being appropriate unions of domains.

Case 1. — The pair (X,X') is positive exceptional and the hyperplane H_j is exceptional (see Definition 1.6; then $X'\in\mathcal{C}^+_X$ and no arrangement hyperplane through X' contains X; thus, H_j can be chosen arbitrary, e.g., exceptional). We claim that

$$\gamma(X,X',\Delta_1)+\gamma(X,X',\Delta_2)=0. \tag{3.12}$$

Indeed, by definition, the domains Δ_X and $\Delta_{X'}$ lie on the same side from H_j. Let Δ_1 also lie on the same side; then Δ_2 lies on the other side (otherwise, we interchange Δ_1 and Δ_2). One has

$$\mathcal{H}(\Delta_2,\Delta_X)=\mathcal{H}(\Delta_1,\Delta_X)\cup H_j,\ \mathcal{H}(\Delta_2,\Delta_{X'})=\mathcal{H}(\Delta_1,\Delta_{X'})\cup H_j,$$

since H_j is the only arrangement hyperplane separating Δ_1 and Δ_2. This together with (3.11) implies (3.12).

Case 2. — The pair (X,X') is not positive exceptional. (This includes the case, when $X'\in\partial\mathcal{C}^+_X$, since then any hyperplane through X' that does not contain X (thus, H_j) separates $\Delta_{X'}$ from Δ_X. This follows from definition and the increasing of the function f_0 along the segment $[X, X']$ oriented from X to X'.) We claim that

$$\gamma(X, X', \Delta_1) + \gamma(X, X', \Delta_2) = -(2i \sin \pi\alpha_j)\gamma(X, X', \Delta_1 \cup \Delta_2), \tag{3.13}$$

and this equality remains valid in the case, when Δ_1 and Δ_2 are *adjacent across H_j unions of domains* from $D(X, X')$: each domain in Δ_1 is adjacent across H_j to a domain in Δ_2 and vice versa.

Indeed, without loss of generality we consider that Δ_1, $\Delta_{X'}$ are separated by H_j from Δ_2 and Δ_X (interchanging Δ_1 and Δ_2 if necessary). By definition, one has

$$\mathcal{H}(\Delta_2, \Delta_{X'}) = \mathcal{H}(\Delta_1, \Delta_{X'}) \cup H_j,\ \mathcal{H}(\Delta_1, \Delta_{X'}) = \mathcal{H}(\Delta_1 \cup \Delta_2, \Delta_{X'}),$$
$$\mathcal{H}(\Delta_1, \Delta_X) = \mathcal{H}(\Delta_2, \Delta_X) \cup H_j,\ \mathcal{H}(\Delta_2, \Delta_X) = \mathcal{H}(\Delta_1 \cup \Delta_2, \Delta_X).$$

Hence, by (3.11),

$$\gamma(X, X', \Delta_1) = e^{-\pi i\alpha_j}\gamma(X, X', \Delta_1 \cup \Delta_2),$$
$$\gamma(X, X', \Delta_2) = -e^{\pi i\alpha_j}\gamma(X, X', \Delta_1 \cup \Delta_2).$$

The two latter formulas imply (3.13).

If the pair (X, X') is positive exceptional, then $C_0(X', X) = 0$. Indeed, fix an exceptional hyperplane H_j. The domain collection $D(X, X')$ is split into pairs of adjacent domains across H_j. The terms in the sum (3.10) corresponding to two adjacent domains cancel out by (3.12), hence the sum vanishes.

Let now the pair (X, X') be not positive exceptional. Let us numerate all the hyperplanes $H_{j_1}, \dots, H_{j_q}$ through X' that do not contain X (one has $q \leqslant k$). If $X' \in \mathcal{C}_X^+$, then $q = k$ and these are all the arrangement hyperplanes through X'. Otherwise, if $X' \in \partial\mathcal{C}_X^+$, then $q < k$ and these are all the arrangement hyperplanes through X' that do not contain faces of the cone $\mathcal{C}_X^+$. In both cases one has $\{j_1, \dots, j_q\} = X' \setminus X$. The terms in the sum (3.10) correspond to the domains $\Delta_1, \dots \Delta_{2^q}$, which we numerate as follows. Fix an arbitrary domain $\Delta_1 \in D(X, X')$. Let Δ_2 be the domain adjacent across H_{j_1} to Δ_1, Δ_3 (Δ_4) be the domain adjacent across H_{j_2} to Δ_1 (respectively, Δ_2), etc., for any $s = 1, \dots, q-1$ the domains $\Delta_{2^s+1}, \dots \Delta_{2^{s+1}}$ be adjacent across $H_{j_{s+1}}$ to $\Delta_1, \dots, \Delta_{2^s}$. We claim that for any $s = 1, \dots, q$

$$\sum_{l=1}^{2^s} \gamma(X, X', \Delta_l) = \gamma(X, X', \cup_{l=1}^{2^s} \Delta_l) \prod_{r=1}^{s} (-2i \sin \pi\alpha_{j_r}), \tag{3.14}$$

$$(3.15)\qquad \sum_{l=2^s+1}^{2^{s+1}} \gamma(X,X',\Delta_l) = \gamma(X,X',\cup_{l=2^s+1}^{2^{s+1}}\Delta_l)\prod_{r=1}^{s}(-2i\sin\pi\alpha_{j_r}),\ \text{whenever } s<q.$$

We prove both statements (3.14), (3.15) by induction in s.

The induction base for $s=1$ follows from (3.13) and the fact that Δ_3, Δ_4 are adjacent across H_{j_1} (by definition).

Induction step. Let (3.14), (3.15) be proved for a given $s<q$. Let us prove (3.14) for s replaced by $s+1$. The domain unions from (3.14) and (3.15) are adjacent across $H_{j_{s+1}}$ to each other by definition. Adding equalities (3.14) and (3.15) and applying (3.13) to the γ's in the right-hand side yields (3.14) for s replaced by $s+1$. Equality (3.15) for $s+1\leqslant q$ is proved analogously. The induction step is over and statements (3.14), (3.15) are proved.

Formula (3.14) with $s=q$ says that the sum (3.10) equals

$$\gamma(X,X',\widetilde{\Delta})\prod_{s=1}^{q}(-2i\sin\pi\alpha_{j_s}),\ \text{where } \widetilde{\Delta}=D(X,X').$$

The latter expression coincides with the right-hand side in (1.10), by (3.11) (applied to $\widetilde{\Delta}$) and since $A=\mathcal{H}(\widetilde{\Delta},\Delta_X)$, $B=\mathcal{H}(\widetilde{\Delta},\Delta_{X'})$, $q=|X'\setminus X|$ (by definition). This proves (1.10).

Now let us prove (1.11). The Stokes matrix C_1 is the transition matrix between the canonical solution bases $I^-_X(\lambda)$ and $e^{2\pi i\alpha_X}I^+_X(\lambda)$, $\lambda\in\mathbb{R}_-$, by definition and Proposition 2.4. To calculate it, we consider the variable change $\lambda\mapsto-\lambda$, which transforms the equation (1.5) = (1.5)(f_0) to the new one (denoted (1.5)($-f_0$)). The latter equation corresponds to the same hyperplane arrangement equipped with the new linear function

$$\widetilde{f}_0=-f_0.$$

We express the Stokes matrix C_1 via the (already calculated) Stokes matrix $\widetilde{C}_0$ of the new equation (1.5)($-f_0$).

Denote $J^\pm_X(\lambda)$ the canonical basic solutions of (1.5)($-f_0$) in the sector $S_\pm$: the solutions given by Proposition 2.4 (denoted there by $I^\pm_X(\lambda)$). The variable change $\lambda\mapsto-\lambda$ transforms the canonical sectorial basic solutions of (1.5)(f_0) in $S_\pm$ to those of (1.5)($-f_0$) in $S_\mp$. The Stokes matrix $\widetilde{C}_0$ compares the bases $J^\pm_X$ over the ray $\lambda>0$, as in (2.5). To express C_1 via $\widetilde{C}_0$, we show that

$$(3.16)\qquad I^-_X(\lambda)=e^{\pi i\alpha_X}J^+_X(-\lambda)\ \text{for all } \lambda\in S_-.$$

Then one has

$$e^{2\pi i\alpha_X} I^+_X(\lambda) = e^{\pi i\alpha_X} J^-_X(-\lambda) \text{ for any } \lambda \in S_+. \tag{3.17}$$

Indeed, $I^-_X|_{S_1}$, $e^{2\pi i\alpha_X} I^+|_{S_2}$ form a normalized base pair, as do $J^+_X|_{S_0}$, $J^-_X|_{S_1}$ (Proposition 2.4 applied to (1.5)($\pm f_0$)). This together with (3.16) and Remark 1.4 implies (3.17). One has

$$e^{\pi i\alpha_X} J^-_X(\lambda) = \sum_{X'} C_1(X', X) e^{\pi i\alpha_{X'}} J^+_{X'}(\lambda),\ \lambda \in \mathbb{R}_+, \tag{3.18}$$

by definition and (3.16), (3.17). Formula (3.18) together with the (already proved) formula (1.10) for the Stokes matrix $\widetilde{C}_0$ yields (1.11). Here we replace "positive exceptional" by "negative exceptional", since the sign of the function f_0 (which defines the cone $\mathcal{C}^+_X$) is changed.

Let us prove (3.16). Let $\mathcal{C}^\rho_X$, $\rho \in \mathbb{C}$, $|\rho| = 1$, be the cones defined in (2.1). By definition,

$$J^+_X = (J^+_{X,X'})_{X' \in \mathcal{X}},\ J^+_{X,X'} = J^\rho_{X,X'}(\lambda) = \int_{\mathcal{C}^\rho_X} e^{\lambda f_0(x)} \Omega_{X'}, \tag{3.19}$$

$$I^-_{X,X'}(-\lambda) = I^\rho_{X,X'}(-\lambda) = \int_{\mathcal{C}^\rho_X} e^{\lambda f_0(x)} \Omega_{X'};\ \operatorname{Im}\rho > 0,\ \operatorname{Re}(\rho\lambda) < 0. \tag{3.20}$$

In formulas (3.19) (respectively, (3.20)) the analytic branch of $\Omega_{X'}$ (denoted $\Omega^+_{X'}$ (respectively, $\Omega^-_{X'}$)) in the union $\hat{C} = \cup_{\operatorname{Im}\rho>0} \mathcal{C}^\rho_X$ is defined as a result of immediate analytic extension of its standard real branch in a neighborhood of X in $\mathcal{C}^-_X = \mathcal{C}^{-1}_X$ (respectively, $\mathcal{C}^+_X = \mathcal{C}^1_X$) to the latter union. One has

$$\Omega^-_{X'} = e^{\pi i\alpha_X} \Omega^+_{X'}. \tag{3.21}$$

(This together with (3.19) and (3.20) implies (3.16).) Indeed, consider a point $x_0 \in \Delta_X \subset \mathcal{C}^+_X$ and a path

$$\Gamma : [0,1] \to \hat{C} \cup \Delta_X,\ \Gamma(t) = X + e^{i\pi t}(x_0 - X);$$

x_0 being close enough to X in order that $\Gamma(1) \in \mathcal{C}^-_X$ be not separated from X by arrangement hyperplanes. The restriction $\Omega^-_{X'}|_{\Delta_X}$ is the standard real branch of $\Omega_{X'}$. The result of its analytic extention from x_0 along Γ is $e^{i\pi\alpha_X}$ times the real branch of $\Omega_{X'}$ defined near $\Gamma(1)$. The latter branch equals $\Omega^+_{X'}$ by definition. This proves (3.21) and hence (3.16). The proof of Theorem 1.7 is complete. □

4. Appendix: the differential equation (1.5)

Proof of (1.5). — The proof of (1.5) is based on two types of relations. The first one comes from the fact that $f_0 - f_0(X)$ is a linear combination of f_j with $j \in X$ for any vertex X: the latter are linearly independent and vanish at X, as does $f_0 - f_0(X)$. Thus, there exist constants $(c_{0,j})_{j\in X}$ such that

$$f_0(z) = f_0(X) + \sum_{j\in X} c_{0,j} f_j(z) \quad \forall z \in \mathbb{R}^k.$$

The second relation is of a cohomological type. Let $U = \{j_1, \dots, j_{k-1}\}$ and

$$\omega_U = df_{j_1}/f_{j_1} \wedge \cdots \wedge df_{j_{k-1}}/f_{j_{k-1}},$$

where the indices of U are ordered so that the form

$$df_0 \wedge df_{j_1} \wedge \cdots \wedge df_{j_{k-1}}$$

be positively oriented. We have

$$\begin{aligned}(4.1)\quad & d\left(e^{-\lambda f_0}\left(\prod_j |f_j|^{\alpha_j}\right)\omega_U\right) \\ &= \left(e^{-\lambda f_0}\prod_j |f_j|^{\alpha_j}\right)\left(-\lambda df_0 \wedge \omega_U + \sum_{j\notin U} \alpha_j \frac{df_j}{f_j} \wedge \omega_U\right)\end{aligned}$$

We see that the orientation of $df_j \wedge df_{j_1} \wedge \cdots \wedge df_{j_{k-1}}$ depends on the relative orientation of the linear forms df_j and df_0 on the edge L_U. More precisely, its orientation is equal to the sign of $l_j(e_U)$ (where e_U is defined in (1.4), and $l_j = f_j - f_j(0)$ is the linear form associated with f_j). Hence, we have

$$(4.2)\qquad df_j/f_j \wedge \omega_U = \epsilon(j, U)\omega_{U\cup\{j\}},$$

where

$$\epsilon(j, U) = \operatorname{sgn}(l_j(e_U)).$$

We prove that the integral of the differential in (4.1) over $\Delta \in \mathcal{D}^+$ vanishes by applying Stokes formula and showing that the boundary terms do not contribute. Since the integrand $e^{-\lambda f_0}\prod_j |f_j|^{\alpha_j}\omega_U$ may diverge on the boundary we first apply Stokes formula in the subdomain Δ^η defined as follows. Let ϵ_i^Δ be the sign of f_i on Δ, and $I^\Delta = \{i,\ \overline{\Delta}\cap H_i \neq \emptyset\}$ the subset of the hyperplanes tangent to the domain Δ. We set for $\eta > 0$

$$\Delta^\eta = \{z \in \Delta,\ \ f_i(z)\epsilon_i^\Delta \geqslant \eta\ \forall i \in I^\Delta\}.$$

Since the integrand is exponentially decreasing at infinity, we just have to evaluate the following integral

$$|\int_{\partial\Delta^\eta}\left(e^{-\lambda f_0}\prod|f_j|^{\alpha_j}\right)\omega_U| \leqslant \sum_{i\in I^\Delta}|\int_{\partial\Delta^\eta\cap\{f_i\epsilon_i^\Delta=\eta\}}\left(e^{-\lambda f_0}\prod|f_j|^{\alpha_j}\right)\omega_U|.$$

Now, if $i\in U$ then ω_U vanishes on the set $\{f_i=\eta\}$. On the other hand, if $i\notin U$, then

$$\int_{\partial\Delta^\eta\cap\{f_i\epsilon_i^\Delta=\eta\}}\left(e^{-\lambda f_0}\prod|f_j|^{\alpha_j}\right)\omega_U \ \sim \eta^{\alpha_i}\int_{\partial\Delta\cap H_i}\left(e^{-\lambda f_0}\prod_{j\neq i}|f_j|^{\alpha_j}\right)\omega_U,$$

as η tends to 0. The integral on $\partial\Delta\cap H_i$ is finite, since all α_r are positive. Therefore, applying Stokes formula and taking the limit as $\eta\to 0$, we get

$$\lambda\int_\Delta\left(e^{-\lambda f_0}\prod_j|f_j|^{\alpha_j}\right)df_0\wedge\omega_U=\sum_{j\notin U}\epsilon(j,U)\alpha_j I_{\Delta,U\cup\{j\}}. \tag{4.2$_{\rm bis}$}$$

We are now in a position to prove the result.

$$\begin{aligned} dI_{\Delta,X}/d\lambda &= -\int_\Delta\left(e^{-\lambda f_0}\prod_j|f_j|^{\alpha_j}\right)f_0\omega_X \\ &= -f_0(X)I_{\Delta,X}-\left(\sum_{j\in X}c_{0,j}\int_\Delta\left(e^{-\lambda f_0}\prod_r|f_r|^{\alpha_r}\right)f_j\omega_X\right). \end{aligned} \tag{4.3}$$

Since $df_0=\sum_{j\in X}c_{0,j}df_j$, we see that by (4.2),

$$df_0\wedge\omega_{X\setminus\{j\}}=c_{0,j}df_j\wedge\omega_{X\setminus\{j\}}=\epsilon(j,X\setminus\{j\})c_{0,j}f_j\omega_X.$$

Hence, the sum in (4.3) becomes

$$\sum_{j\in X}\epsilon(j,X\setminus\{j\})\int_\Delta\left(e^{-\lambda f_0}\prod_r|f_r|^{\alpha_r}\right)df_0\wedge\omega_{X\setminus\{j\}}.$$

Using the cohomological relation (4.2$_{\rm bis}$) we get

$$dI_{\Delta,X}/d\lambda=-f_0(X)I_{\Delta,X}$$
$$-1/\lambda\left(\sum_{j\in X}\sum_{r\notin X\setminus\{j\}}\epsilon(j,X\setminus\{j\})\epsilon(r,X\setminus\{j\})\alpha_r I_{\Delta,X\setminus\{j\}\cup\{r\}}\right)$$

□

Proof of Proposition 2.5. — A point z in $\mathcal{C}_X^\rho$ has the form

$$z = X + \rho \sum_{j\in X} a_j e_{X\setminus\{j\}}, \quad (a_j)_{j\in X} \in (\mathbb{R}_+)^X.$$

Thus, for $j \in X$ we have $f_j(X) = 0$ (by definition) and

$$f_j(z) = \rho a_j l_j(e_{X\setminus\{j\}}), \tag{4.4}$$

where l_j is the linear form associated with f_j. For $r \notin X$

$$f_r(z) = f_r(X) + \rho \sum_{j\in X} a_j l_r(e_{X\setminus\{j\}}), \tag{4.5}$$

and

$$f_0(z) = f_0(X) + \rho \sum_{j\in X} a_j, \tag{4.6}$$

since by convention $l_0(e_{X\setminus\{j\}}) = 1$, see (1.4). Let us write down the integral $I^+_{X,X'}$ over the cone C_X^ρ as an integral over the variables $u_j = \lambda\rho a_j$, $j \in X$, $a_j \in \mathbb{R}_+$. Denote

$$J_{X,X'} = |\det\left(l_r(e_{X\setminus\{j\}})_{\substack{j\in X,\\ r\in X'}}\right)|$$

For any vertex X' denote

$$\mathbf{1}_{j\in X'} = \begin{cases} 1, \text{ if } j \in X' \\ 0 \text{ otherwise} \end{cases}, \quad \mathbf{1}_{j\notin X'} = \begin{cases} 1, \text{ if } j \notin X' \\ 0 \text{ otherwise} \end{cases}.$$

For all ρ and λ such that $\mathrm{Im}(\rho) < 0$ and $\mathrm{Re}(\lambda\rho) > 0$, substituting (4.4)-(4.6) to the integral $I^+_{X,X'}$ we get

$$\begin{aligned} I^+_{X,X'}(\lambda) &= e^{-\lambda f_0(X)} J_{X,X'} \lambda^{-\sum_{j\in X}(\alpha_j - \mathbf{1}_{j\in X'}+1)} \\ &\times \left(\prod_{j\in X} |l_j(e_{X\setminus\{j\}})|^{\alpha_j - \mathbf{1}_{j\in X'}}\right)\left(\prod_{r\notin X} |f_r(X)|^{\alpha_r - \mathbf{1}_{r\in X'}}\right) \\ &\times \int_{(\lambda\rho\mathbb{R}_+^*)^X} \left(\prod_{j\in X} e^{-u_j} u_j^{\alpha_j - \mathbf{1}_{j\in X'}}\right)\left(\prod_{r\notin X} h_r^{\alpha_r - \mathbf{1}_{r\in X'}}\right) \prod_{j\in X} du_j, \end{aligned}$$

where

$$h_r = 1 + \lambda^{-1} \sum_{j\in X} u_j l_r(e_{X\setminus\{j\}})/f_r(X)$$

(In $h_r^{\alpha_r} = e^{\alpha_r \ln h_r}$ the determination of the logarithm is just obtained by analytic extension of the logarithm: $h_r = 1$, $\ln h_r = 0$ at $u = 0$; the analytic extension of $\ln h_r$ to $\mathcal{C}_X^\rho$ is well-defined, since $h_r(z) = f_r(X)^{-1} f_r(z) \neq 0$ there

(Proposition 2.2). Now, when λ tends to infinity, then h_r converges pointwise to 1. Using the dominated convergence theorem we see that $I_{X,X'}(\lambda)$ is equivalent to

$$D_{X,X'}e^{-\lambda f_0(X)}\lambda^{-\sum_{j\in X}(\alpha_j+\mathbf{1}_{j\notin X'})}$$

(and it can be made uniform in λ in the domain S_+) where $D_{X,X'}$ is the following constant:

$$J_{X,X'}\Bigg(\prod_{j\in X}\Gamma(\alpha_j+\mathbf{1}_{j\notin X'})|l_j(e_{X\setminus\{j\}})|^{\alpha_j-\mathbf{1}_{j\in X'}}\Bigg)\Bigg(\prod_{r\notin X}|f_r(X)|^{\alpha_r-\mathbf{1}_{r\in X'}}\Bigg).$$

Clearly, the term obtained for $X = X'$ is dominating and we get that

$$I_X(\lambda)\sim D_{X,X}\lambda^{-\alpha_X}e^{-\lambda f_0(X)}v_X,$$

where $D_{X,X}$, v_X are as in Proposition 2.5, since $J_{X,X}=\prod_{j\in X}|l_j(e_{X\setminus\{j\}})|$. □

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Alexey GLUTSYUK
École normale supérieure de Lyon
Unité de Mathématiques pures et appliquées
46 allée d'Italie
69364 Lyon 07 (France)
aglutsyu@umpa.ens-lyon.fr

Christophe SABOT
Université de Lyon 1
Institut Camille Jordan
43 bd du 11 nov. 1918
69622 Villeurbanne cedex (France)
sabot@math.univ-lyon1.fr