ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, nº 7 (2009), p. 2733-2771.

<http://aif.cedram.org/item?id=AIF_2009__59_7_2733_0>

Ann. Inst. Fourier, Grenoble
59, 7 (2009) 2733-2771

ON A GENERAL DIFFERENCE GALOIS THEORY II

by Shuji MORIKAWA & Hiroshi UMEMURA

Abstract. — We apply the General Galois Theory of difference equations introduced in the first part to concrete examples. The General Galois Theory allows us to define a discrete dynamical system being infinitesimally solvable, which is a finer notion than being integrable. We determine all the infinitesimally solvable discrete dynamical systems on the compact Riemann surfaces.

Résumé. — Nous appliquons à des exemples concrets la théorie de Galois générale, pour les équations aux différences introduite, dans la première partie. La théorie de Galois générale nous permet de définir la notion, plus fine que l'intégrabilité, de résolubilité infinitésimale d'un système dynamique discret. Nous présentons la liste complète des systèmes dynamiques discrets, infinitésimalement résolubles sur les surfaces de Riemann compactes.

1. Introduction

Our Galois theory for difference equations is algebraic and is a realization of the idea of the second author sketched in section 7, [11]. We briefly explain how, not only in the differential case but also in the discrete case, our theory is related to Malgrange's idea [7], when nicely applied to the discrete case by Casale (*cf.* [1], [2], [14], [13]). We delightfully celebrate Professor Malgrange's 80th birthday showing how wonderful his idea is.

In this second part, we apply our general difference Galois theory to the question of integrability of discrete dynamical systems on algebraic varieties.

In theory of dynamical systems, we believe in general what is integrable, in any sense, is abelian. So far according to Poincaré, we observed integrability of dynamical system through its linearization along a carefully

Keywords: General difference Galois theory, dynamical system, integrable dynamical system, Galois groupoid.
Math. classification: 12Hxx, 37Fxx, 58Hxx, 14Hxx.

chosen particular solution, which brought us fruitful results that certain Hamiltonian systems are not integrable. General difference Galois theory offers us a more authentic invariant, Galois group, that allows us to measure, in a canonical way, the degree of integrability of dynamical systems. So the more abelian the Galois group is, the more integrable the dynamical system is.

We study in detail discrete dynamical systems (X, φ) of iteration of a rational map $\varphi : X \to X$ on an algebraic curve X defined over a field C of characteristic 0 and so in particular on a compact Riemann surface X if $C = \mathbb{C}$. We determine, under the assumption that the base field is $\mathbb{C}$, all the dynamical sysytems (X, φ) over an algebraic curve X such that the Lie algebra of their Galois group is finite dimensional (Theorems 5.16 and 6.5). The Theorem 6.5 is due to Casale [2]. We improved his original proof in several points.

For these dynamical systems, the Lie algebra is not only finite dimensional but also solvable. So we propose to call them infinitesimally solvable.

Thus we may regard that the classification yields us also the *list of infinitesimally solvable discrete dynamical systems on a Riemann surface.* We owe for this result much the precedent works [2], [9], [15]. We work in our algebraic framework trying to make the results and proofs accessible for wider public.

Several natural question arise.

(1) Prove Theorem 6.5 over an arbitrary field C of characteristic 0.
(2) Explore a similar result for a relative case or for a family of algebraic curves.
(3) Study an analogue of Theorem 6.5 in the positive characteristic case.

The first seems manageable. The third requires a differential or difference Galois theory in characteristic $p > 0$ that we suggested in [11], [12]. We hope, however, that it is also realizable without great difficulties.

Throughout the second part C denotes a field of characteristic 0. We study difference field extension $(L, \phi)/(C, \mathrm{Id}_C)$ or $C \subset C_L$ such that the field L is finitely generated over the field C.

Theorem with asterisk, Theorem*, is not proved. So it is not a Theorem in a strict sense of words. We can, however, sketch a proof. We do not use it and we want to prove it in a general and natural setting in future.

2. Groupoid

In the first part, our setting is algebraic. Our theory has, however, also geometric back ground. It is more naturally understood using groupoid.

Let $(L, \phi)/C)$ be a difference field extension such that the field L is finitely generated over the field C and such that the restriction $\phi|C$ is the identity map Id_C on C. According to our Definition 2.18 in the first part [8], our Galois group is a group functor Inf-gal on the category of $L^\natural$-algebras.

Let us *assume* that there exists a model V of the function field $C(V)$ such that the difference operator ϕ arises from a *regular endomorphism* $\varphi : V \to V$ of the model V. Then we can argue as we explained in [14] and [13] for the differential case, and we get a D-groupoid on $V \times V$ as Galois group of the extension L/C. The functor Inf-gal is a formal group over $L^\natural$. It describes the Lie algebra of the Galois groupoid over $V \times V$. We are going to determine the Lie algebra of the group functor Inf-gal.

Let us briefly recall the definition. See [14], Section 3. From now on, in this section, we assume $C = \mathbb{C}$ but as we see below, it works over any field C of characteristic 0.

Definition 2.1. — *A groupoid is a small category $\mathcal{C}$ in which all morphisms are isomorphisms. An object of $\mathcal{C}$ is called a vertex and a morphism in $\mathcal{C}$ is called an element of G.*

One of the most important example of groupoids is a group operation on a set.

Example 2.2. — A group operation (G, X) of a group G on a set X yields a groupoid in the following manner. The set $ob\,\mathcal{C}$ of the groupoid $\mathcal{C}$ is the set X. For $x, y \in X = ob\,\mathcal{C}$, we set $\mathrm{Hom}(x, y) = \{g \in G | gx = y\}$. If $g \in \mathrm{Hom}(x, y)$ and $h \in \mathrm{Hom}(y, z)$, then $gx = y$ and $hy = z$ by definition so that $z = hy = h(gx) = (hg)x$ and consequently $hg \in \mathrm{Hom}(x, z)$. So we can compose two morphisms. If $gx = y$, then $hy = x$, h being g^{-1} so that every morphism is an isomorphism.

Now let $\mathcal{C}$ be a groupoid . We set

$$Y := \{\text{morphisms in the category } \mathcal{C}\}$$

and

$$X := ob\,\mathcal{C}.$$

Let $\varphi \in Y$ so that $\varphi \in \mathrm{Hom}(A, B)$ for some $A, B \in ob\,\mathcal{C}$. Let us denote the source A of φ by $s(\varphi)$ and the target B of φ by $t(\varphi)$. So we get two maps $s : Y \to X$ and $t : Y \to X$. Let $(Y, t) \times (Y, s)$ be the fiber product of $t : Y \to X$ and $s : Y \to X$ so that

$$(Y, t) \times (Y, s) = \{(\varphi, \psi) \in Y \times Y | s(\varphi) = t(\psi)\}.$$

The composition of morphisms defines a map

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

The associativity of the composition is described by a commutative diagram that we do not make precise. See [4]. The existence of the identity map Id_A for every $A \in ob\,\mathcal{C}$ as well as the property called symmetry that every morphism is an isomorphism is also characterized in terms of maps and commutative diagrams.

Here is a summary of the above observation. Groupoid is described by two sets Y and X, two maps $s : Y \to X$ and $t : Y \to X$ and the composition maps

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

that satisfy certain commutative diagrams.

This allows us to generalize the notion of groupoid in a category in which the fiber product of arbitrary two objects exists. This is exactly by the same way as we define an algebraic group G requiring that, of all, G is an algebraic variety, the composition law $G \times G \to G$ is a morphism of algebraic varieties and so on.

Remark 2.3. — We have shown in example 2.2 that a groupoid is born from a group operation. We might as well think that groupoid generalizes group operation. This is false, however, in the following sense.

Let G_1 be the non-cyclic abelian group of order 4 and let G_2 be the cyclic group of order 4. Both G_1 and G_2 operate on the set X consisting of just 4 elements x_1, x_2, x_3 and x_4 in such a way that (G_1, X) and $G_2, X)$ are principal homogeneous spaces. It suffices to consider the left operation of each group on itself. The operaions (G_1, X) and (G_2, X) are not isomorphic but the groupoid $\mathrm{Gpd}(G_1, X)$ and $\mathrm{Gpd}(G_2, X)$ are the same category $\mathcal{C}$ of which the set $ob\,\mathcal{C}$ of objects is $\{x_1, x_2, x_3, x_4\}$ and $\mathrm{Hom}(x_i, x_j)$ consists of just one element for every $1 \leqslant i, j \leqslant 4$. If we pass from group operation to the groupoid it defines some informations are lost. In oter words the functor $(G, X) \to \mathrm{Gpd}(G, X)$ is not faithful.

Definition 2.4. — *Let $\mathcal{C}$ be a category in which fiber product exists. A groupoid in the category $\mathcal{C}$ consists of two objects $Y, X \in ob\,\mathcal{C}$, two morphisms $s : Y \to X$ and $t : Y \to X$ and a morphism*

$$\Phi : (Y, t) \times (Y, s) \to Y$$

etc, satisfying the above conditions (cf. Grothendieck [4]*).*

Example 2.5. — Let $\mathcal{C}$ be the category of algebraic varieties defined over a field k and let (G, V) be an operation of an algebraic group on an algebraic variety V defined over k. We have two morphisms p, h from $G \times V$ to V, namely the second projection p and the group operation $h(g, v) = gv$. Then $Y = G \times X$, $X = V$, $s = p$ and $t = h$ is a groupoid in the category $\mathcal{C}$ of algebraic varieties defined over k. Compare to Example 2.2.

For a complex manifold V, we can attach the space of its invertible jets $J^*(V \times V)$ that is a groupoid over $V \ \times V$ in the category of analytic spaces. We recall the definition for $V = \mathbb{C}$. The jet space $J(\mathbb{C} \times \mathbb{C})$ is an infinite dimensional analytic space $\mathbb{C} \times \mathbb{C}^{\mathbb{N}}$ with coordinate system $(x, , y_0, y_1, y_2, \cdots)$. We have two morphisms $s : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ and $t : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ given by

$$s((x, y_0, y_1, y_2, \cdots)) = x \qquad \text{and} \qquad t((x, , y_0, y_1, y_2, \cdots)) = y_0.$$

So we have a morphism $(s, t) : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ that makes $J(\mathbb{C} \times \mathbb{C})$ an infinite dimensional affine space over $\mathbb{C} \times \mathbb{C}$. The invertible jet space $J^*(\mathbb{C} \times \mathbb{C})$ is, by definition, the Zarisiki open set of $J(\mathbb{C} \times \mathbb{C})$. Namely,

$$J^*(\mathbb{C} \times \mathbb{C}) := \{(x, y_0, y_1, y_2, \cdots) \in J(\mathbb{C} \times \mathbb{C}) | y_1 \neq 0\}.$$

We simply denote $J^*(\mathbb{C} \times \mathbb{C})$ by J^* and we write the restrictions of the morphisms s, t to the Zariski open set J^* by the same letters. Now we explain J^* with two morphisms $s : J^* \to \mathbb{C}$ and $t : J^* \to \mathbb{C}$ is a groupoid. To this end we must define the composite morphism $\Phi : (J^*, t) \times (J^*, s) \to J^*$. Let

$$\varphi = (x, y_0, y_1, \cdots), \qquad \psi = (u, v_0, v_1, \cdots),$$

be points of J^* such that $y_0 = t(\varphi) = s(\psi) = u$, *i.e.*, (φ, ψ) is a point of $(J^*, t) \times (J^*, s)$. Then we set

$$\Phi(\psi, \varphi) := (x, v_0, y_1 v_1, y_2 v_1 + y_1^2 v_2, \cdots). \tag{2.1}$$

The n-th component of $\Phi(\psi, \varphi)$ is given by the following rule. Imagine formally that φ were a function of x taking the value y_0 at x, or $\varphi(x) = y_0$, with $\varphi'(x) = y_1, \varphi''(x) = y_2 \ldots$.

Similarly consider as if ψ were a function of u with $\psi(u) = v_0, \psi'(u) = v_1, \psi''(u) = v_2, \ldots$. Then $\Phi(\psi, \varphi)$ is the composite function $\psi \circ \varphi$, which is a function of x, so that its n-th component is the value of $d^n \psi \circ \varphi / dx^n$ at x. For example,

$$d(\psi \circ \varphi)/dx = \psi_u \varphi_x = y_1 v_1, \ d^2(\psi \circ \varphi)/dx^2 = \varphi_{xx} \psi_u + \varphi_x^2 \psi_{uu} = y_2 v_1 + y_1^2 u_2, \cdots$$

One can check this composition law is associative and the inverse of

$$\varphi = (x, y_0, y_1, \cdots)$$

is given by the inverse function $x(y_0)$ and its derivatives $d^n x(y_0)/dy_0^n$ for $n \in \mathbb{N}$, namely by

$$(y_0, x, 1/y_1, -y_2/y_1^3, \cdots).$$

We can very naturally extend this construction over a complex manifold of any dimension. Namely for an analytic manifold V, we construct pieces of jet spaces locally and glue them together. Locally for polydiscs U, W in $\mathbb{C}^n$, the space $J(U \times W)$ of jets over $U \times W$ is the infinite dimensional affine space over $U \times W$ with coordinate system

$$u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n$$

and formal higher derivatives

$$\frac{\partial^{|\nu|} w_j}{\partial u_1^{\nu_1} \partial u_2^{\nu_2}, \ldots \partial u_n^{\nu_n}},$$

where $1 \leqslant j \leqslant n$ and

$$0 \neq \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$$

with usual notation

$$|\nu| = \sum_{i=1}^{n} \nu_i.$$

The subspace of invertible jets $J^*(U \times W)$ is the Zariski open subset of $J(U \times W)$ on which the Jacobian

$$\frac{D(w_1, w_2, \ldots, w_n)}{D(u_1, u_2, \ldots, u_n)} = \det \left(\frac{\partial w_j}{\partial u_i} \right)_{1 \leqslant i,j \leqslant n} \neq 0.$$

We define in $J^*(V \times V)$ the composition law of groupoid structure by the law of calculating derivatives of composite functions, as well as inverse, just as in the 1-variable case.

The above construction of Lie groupoids, in the category of analytic spaces, also works in the category of algebraic varieties, or to be more correct in the category of schemes over a field C. The most important ingredient in the algebraic construction is the universal extension of derivations [13]. We do not go into the detail because it is technical and will be published elsewhere. So for a non-singular algebraic variety V defined over the field C of characteristic 0, we can define its invertible jet space $J^*(V \times V)$ that is an algebraic variety of infinite dimension, *i.e.*, an affine scheme over $V \times V$.

DEFINITION 2.6. — *Let V be an algebraic variety defined over a field of characteristic* 0. *An algebraic D-groupoid is a sub-groupoid of $J^*(V \times V)$ defined by a differential ideal.*

Let (G, V) be an operation of an algebraic group on an algebraic variety V defined over a field C. So the operation (G, V) defines a groupoid $\mathrm{Gpd}(G, V)$ in the category of schemes. Then there exists a natural morphism

$$\mathrm{Gpd}(G, V) \to J^*(V \times V) \tag{2.2}$$

of groupoids in the category of schemes.

THEOREM* 2.7. — *The image of the morphism 2.2 is an algebraic D-groupoid.*

When G is a finite group, we can prove the assertion of Theorem* 2.7 in a trivial way. So we may use it for a finite group.

3. Geometric Principle of Malgrange

We explain, geometrically, general difference Galois theory depends on a very simple principle. The geometric principle was discovered by Malgrange [7] in the differential case and Casale applied it successfully to the difference case ([1], [2]).

3.1. Let us assume that the dynamical system on an algebraic variety arises from a operation of algebraic group.

Let us consider an effective operation (G, V) of an algebraic group G on an algebraic variety V defined over a field C of characteristic 0. Namely, for an element $g \in G$ if $gx = x$ for every point $x \in V$, then $g = 1$. An element $g \in G$, to be more precise a C-valued point $g \in G(C)$, defines an automorphism

$$\varphi_g : V \to V \qquad x \mapsto gx$$

of the algebraic variery V over C and hence dually a C-automorphism

$$\varphi_g^* : C(V) \to C(V)$$

of the field $C(V)$ of rational functions on V. The algebraic counterpart of the dynamical system (V, φ_g) on the algebraic variety V of iteration of the automorppHism $\varphi_g : V \to V$ is the difference field $(C(V), \varphi_g^*)$.

Geometric Principle 1. — *The Malgrange Galois groupoid*

$$\text{MGal-gpd}(V, \varphi_g)$$

of the dynamical system (V, φ_g) *or equivalently the Galois groupoid*

$$\text{MGal-gpd}(C(V), \varphi_g^*)/(C, \text{Id}_C))$$

of the difference field extension $(C(V), \varphi_g^*/(C, \text{Id}_C)$ *should be the algebraic D-groupoid of the algebraic operation* $(\overline{< g >}, V)$*, where* $\overline{< g >}$ *denotes the Zariski closure of the sub-group* $< g >$ *of* G *generated by* g *or the smallest closed subgroup of* G *containing the element* g*. In particular the Galois groupoid is abelian.*

We can show in this case that our Galois group $\text{Inf-gal}(C(V)/C)$ of the difference field extension $(C(V), \varphi_g^*)/(C, \text{Id}_C))$ defined in the first part, which is a formal group over the field $L^\natural$ or an $L^\natural$-Lie algebra, is isomorphic to the Lie algebra

$$L^\natural \otimes_C \text{ (Lie algebra of the vector fields of the operation of the algebraic group } \overline{< g >} \text{on} V)$$

that is isomorphic to

$$L^\natural \otimes_C \text{Lie}\, \overline{< g >}.$$

Let us understand Geometric Principle 1 by examples.

Example 3.1. — Let us apply the above observation to the simplest case $(\text{PGL}_{2C}, \mathbb{P}^1_C)$. If we assume for simplicity that the field C is algebraically closed, then by Jordan's reduction theorem an element g of $\text{PGL}_2(C)$ is conjugate to either

$$(1) \qquad \begin{pmatrix} a & 0 \\ o & d \end{pmatrix}, \qquad \text{or} \qquad (2) \qquad \begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix}$$

according as (1) g is semi-simple or (2) unipotent. So we have

$$\overline{< g >} \text{ is } \begin{cases} \text{a finite group} & \text{if } g \text{ is of finite order,} \\ \text{isomorphic to } \mathbb{G}_m & \text{if } g \text{ is semi-simple and not of finite order,} \\ \text{isomorphic to } \mathbb{G}_a & \text{if } g \text{ is unipotent and } g \neq I_2. \end{cases}$$

Therefore we could conclude that the Malgrange Galois groupoid

$$\text{MGal-gpd}(\mathbb{P}^1_C, \varphi_g).$$

is an algebraic D-groupoid equivalent to the operation on $\mathbb{P}^1_C$ of

$$\begin{cases} \text{a finite group} & \text{if } g \text{ is of finite order,} \\ \mathbb{G}_m & \text{if } g \text{ is semi-simple and not of finite order,} \\ \mathbb{G}_a & \text{if } g \text{ is unipotent and } g \neq I_2. \end{cases}$$

Since Inf-gal is the Lie algebra of MGal-gpd, we would get

$$\text{Inf-gal}(\mathbb{P}^1_C, \varphi_g) \cong \begin{cases} 0 & \text{if } g \text{ is of finite order,} \\ \text{1-dimensional Lie algebra} & \text{if } g \text{ is not of finite order.} \end{cases}$$

3.2. *What do we do if the dynamical system on the algebraic variety does not come from an algebraic group operation?* Let $(L, \phi)/(C, \mathrm{Id}_C)$ be a difference field extension such that the field L is finite algebraic over $\phi(L)$. Let (V, φ) be a model of a differentia field extension L/C so that $C(V) \simeq L$ and $\varphi : V \cdots \to V$ is a rational map. It follows from the assumption that L is finite algebraic over $\phi(L)$, the rational map $\varphi : V \cdots \to V$ is generically surjective. Therefore we can find a non-empty Zariski open susets U_1 and U_2 of V such that the rational map φ restricted on U_1, the rational map $\varphi | U_1 : U_1 \to U_2$ is regular and étale.

Geometric Principle 2. — *Replace group action by the groupoid $J^*(V \times V)$ of invertible jets that is canonically attached to the variety V. The Galois groupoid*

$$\text{MGal-gpd}(V, \varphi)$$

of the dynamical system (V, φ_g) or equivalently the Galois groupoid

$$\text{MGal-gpd}(C(V), \varphi^*)/(C, \mathrm{Id}_C))$$

of the difference field extension $(C(V), \varphi^*)/(C, \mathrm{Id}_C)$ is the smallest algebraic D-groupoid $\mathcal{G}$ on V such that the differential ideal defining the algebraic D-groupoid $\mathcal{G}$ kills φ on $U_1 \times U_2$.

Definition 3.2. — *We adopt, according as Malgrange and Casale,* Geometric Principle 2 *as the definition of the Malgrange Galois groupoid*

$$\text{MGal-gpd}(V, \varphi)$$

of the dynamical system (V, φ) or equivalently the Galois groupoid

$$\mathrm{MGal} - \mathrm{Gpd}\,(\mathrm{L/C})$$

of the difference field extension L/C (cf. [2]*).*

The definition needs precision of the statement and existence of the smallest one.

(1) of all, we identify a D-groupoid on a Zariski open set of V with its restriction to a smaller Zariski open set. In other words, we work at the generic point of the variety V or over $\operatorname{Spec} C(V)$. We look for the smallest algebraic D-groupoid $\mathcal{G}_{C(V)}$ over $\operatorname{Spec} C(V) \times \operatorname{Spec} C(V))$, the defining differential ideal of which kills the rational map φ.

(2) The assumption of being minimum implies that the algebraic D-groupoid $\mathcal{G}_{C(V)}$ is reduced. Hence the finite basis theorem of Ritt, for example in [5], implies that the defining differential ideal of the smallest groupoid $\mathcal{G}_{C(V)}$ over $\operatorname{Spec} C(V) \times \operatorname{Spec} C(V)$ is differentially finitely generated.

(3) Since the number of coefficients of the differential generators of the defining ideal is finite, the algebraic D-groupoid $\mathcal{G}_{C(V)}$ extends to an algebraic D-groupoid over a Zariski open set of V.

As in the differential case, the advantage of our method in the first part is that it also gives us an explicit construction of MGal-gpd(L/C)*.* See Section 4 below.

Remarks 3.3. — We have to notice that Geometric Principles 1 and 2 are compatible. There are two points to check.

(i) For an effective algebraic group operation (G, V), the groupoid $\operatorname{Gpd}(G, V)$ is an algebraic D-groupoid on $V \times V$.
(ii) For a algebraic D-sub-groupoid G' of the D-groupoid $\operatorname{Gpd}(G, V)$, there exists a closed subgroup H of G such that $G' = \operatorname{Gpd}(H, V)$.

3.3. We do not prove the assertions (i) and (ii) in this note and so we are not allowed to use Geometric Principle 1. For, the proof that we can sketch, involves subtle points. In general, a fundamental fact such as (i) and (ii) should be stated and proved in a transparent way depending on few simple principles.

For example, the following assertion is not trivial. Let G_1 and G_2 be algebraic subgroups in the group of birational automorphism group $\operatorname{Bir}(V)$ of an algebraic variety V. Then the intersection $G_1 \cap G_2$ is an algebraic subgroup of $\operatorname{Bir}(V)$. Not only the proof but also the statement itself of this assertion requires a scheme theoretic definitions (*cf.* [3]).

4. Our Theory and Geometric Principle

We defined, in the part 1, the Galois group Inf-gal(L/K) that is a formal group or a Lie algebra over the field $L^{\natural}$, for a general difference field extension L/K. We explained in the differential case that how our theory [10] is related with Malgrange Theory. The method of [14] and [13] allows us to illustrate us that algebraic D-groupoid is also involved in our theory. Let $((L, \varphi)/C)$ be a difference field extension such that the field L is finitely generated over C, the field L is algebraic over $\varphi(L)$ and every element of C is fixed.

Proposition 4.1. — *Let $\mathcal{L}$ be the difference differential subring generated by $\iota(L)$ and $L^\sharp$ of $F(\mathbb{N}, L^\sharp)$. So the ring $L^\sharp[\iota(L)]$ is a subring of $\mathcal{L}$. The inclusion defines a morphism*

$$\Phi : \operatorname{Spec} \mathcal{L} \to \operatorname{Spec} L \times_C \operatorname{Spec} L.$$

$\operatorname{Spec} \mathcal{L}$ *has a structure of algebraic D-groupoid over* $\operatorname{Spec} L \times_C \operatorname{Spec} L$.

The ring $\mathcal{L}$ is a subring of the ring $F(\mathbb{N}, L)$ of functions that is reduced, and hence $\mathcal{L}$ is reduced. So that there exists a model V of the field L/C such that the algebraic D-groupoid $\mathcal{L}$ is defined over $V \times V$ (*cf.* (2) after Definition 3.2). Namely there exists an algebraic D-groupoid Gal-gpd(L/C) defined over $V \times V$ such that We have an isomorphism

$$\mathcal{L} \simeq \text{Gal-gpd}(L/C) \times_{V \times V} L \otimes L$$

of algebraic D-groupoids defined over $L \otimes_C L$. The algebraic D-groupoid Gal-gpd(L/C) is determined up to birational equivalence.

Definition 4.2. — *The algebraic D-groupoid* Gal-gpd(L/C) *is the Galois groupoid of the difference field extension L/C.*

Now we can speak of the Lie algebra Lie(Gal-gpd(L/C)) of the algebraic D-groupoid Gal-gpd(L/C), which is a C-Lie algebra of regular vector fields over the variety V.

Theorem 4.3. — *For a difference field extension L/C, the $L^\natural$-Lie algebra* Inf-gal(L/C) *is canonically isomorphic to the $L^\natural$-Lie algebra*

$$L^\natural \otimes_C \text{Lie}(\text{Gal-gpd}(L/C)).$$

We can argue as in the differential case ([14], [13]).

5. Examples and a general conclusion that follows from their observation

In this section, we apply our construction of the first part to various examples. On the way of calculation, we examine Geometric Principle 1.

We use the notation of the first part [8]. Namely, we start from a difference field extension L/K. We embed the difference field L into the difference ring $F(\mathbb{N}, L)$ of functions on the set $\mathbb{N}$ taking values in the field L by the universal Euler morphism

$$\iota : L \to F(\mathbb{N}, L),$$

which is a morphism of difference algebras. Here the difference operator of the ring $F(\mathbb{N}, L)$ is the shift operator Φ. Then we take a mutually commutative basis $\{D1, D_2, \dots D_d\}$ of the L-vector space $\mathrm{Der}(L/K)$ of derivations. The differential field consisting of the field L with the commutative derivations $\{D_1, D_2, \dots D_d\}$ is denoted by $L^\sharp$. So the ring $F(\mathbb{N}, L)$ of functions on $\mathbb{N}$ is a difference differential ring with respect to the shift operator Φ and the differential operators $D_1, D_2, \dots D_d$. We introduce a difference differential sub-algrbra $\mathcal{L}$ of $F(\mathbb{N}, L^\sharp)$ that is generated by $\iota(L)$ and the field $L^\sharp$ of constant functions on $\mathbb{N}$. Similarly we define difference differential sub-algebra $\mathcal{K}$ of $F(\mathbb{N}, L)$ generated by $\iota(K)$ and $L^\sharp$ so that we have an inclusion

$$\mathcal{K} \subset \mathcal{L} \subset F(\mathbb{N}, L^\sharp)$$

of difference differential algebras. Our Galois group is a group functor Inf-gal of infinitesimal automorphisms of difference differential algebra extension $\mathcal{L}/\mathcal{K}$. To see the infinitesimal deformations, we embed the difference differential algebra $F(\mathbb{N}, L^\sharp)$ in $F(\mathbb{N}, L^\natural[[W]])$ by the formal Taylor expansion

$$i : L^\sharp \to L^\natural[[W]], \qquad a \mapsto \sum_{\nu \in \mathbb{N}^d} \frac{1}{\nu!} D^\nu(a) W^\nu \tag{5.1}$$

called the universal Taylor expansion, where

$$\nu! = \nu_1!\nu_2! \cdots \nu_d!, \quad D^n u = D_1^{\nu_1} D_2^{\nu_2} \cdots D_d^{\nu} \quad W^\nu = W_1^{\nu_1} W_2^{\nu_2} \cdots W_d^{\nu_d}$$

for $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}^d$ and the W_i's are variables for $1 \leqslant i \leqslant d$. The universal Taylor expansion morphism is injective and is a differential mophism or it is compatible with derivations D_i and $\partial/\partial W_i$ for $1 \leqslant i \leqslant d$. So we have inclusions

$$\mathcal{K} \subset \mathcal{L} \subset F(\mathbb{N}, L^\sharp) \subset F(\mathbb{N}, L^\natural[[W]]) \tag{5.2}$$

of difference differential algebra with the shift operator Φ and the partial derivations $\partial/\partial W_i$ for $1 \leqslant i \leqslant d$.

According to our definition in the first part, our Galois group is a group functor Inf-gal(L/K) on the category of $L^\natural$-algebras. Let us now assume $K = C \subset C_L = \{a \in L |\ \phi(a) = a\}$. We argue carefully choosing a model V of the difference field L/C. Then as we explained for the differential case in

[14] and [13], we get a D-groupoid on $V \times V$ as Galois group of the extension L/C. The functor Inf-gal is a formal group over $L^\natural$. It describes the Lie algebra of the Galois groupoid over $V \times V$. We are going to determine the Lie algebra of the group functor Inf-gal.

Let us understand this procedure by Examples. Let us recall a notation. We denote an element $f \in F(\mathbb{N}, A)$ that is a function on $\mathbb{N}$ with values in the ring A by

$$f = \begin{bmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ f(0) & f(1) & f(2) & \cdots & f(n) & \cdots \end{bmatrix}.$$

When we use this notation, we denote the constant function $f \in F(\mathbb{N}, A)$ taking the value a by a. Namely

$$a = \begin{bmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ a & a & a & \cdots & a & \cdots \end{bmatrix}.$$

Example 5.1. — The simplest and intriguing example is Example 5.2 of the first part, diagonalizable autoumorphisms of $\mathbb{P}^1_C$. Namely, let $K = C$ and $L = C(x)$, x being a variable over C. Let $a \neq 0 \in C$ and we considered a C-automorphism

$$\phi : L \to L, \qquad x \mapsto ax$$

of the rational function field $L = C(x)$. So (L, ϕ) is a difference field and L/C is a difference field extension. Geometrically, we are interested in the discrete dynamical system on the affine line $\mathbb{A}^1_C$ or on the projective line $\mathbb{P}^1_C$ of iteration of the rational map

$$\mathbb{A}^1_C \to \mathbb{A}^1_C, \qquad z \mapsto az.$$

We showed the Lie algebra of Inf-gal(L/K) is 1-dimensional, or the formal group Inf-gal(L/K) is isomorphic to the formal completion $\widehat{\mathbb{G}}_{m,L^\natural}$ if a is not a root of unity. We can also conclude this by general Picard-Vessiot theory (*cf.* section 5 of the first part [8]). If there exists a non-zero integer l, such that $a^l = 1$ then Inf-gal$(L/K) = 0$. We observe here that the conclusion is compatible with the GEOMETRIC PRINCIPLE 1 in § 3.

Example 5.2. — Translations on $\mathbb{P}^1_C$. Let K and L be as in the previous example and let $0 \neq u \in C$. We consider a C-automorphism of the rational function field $C(x)$

$$C(x) \to C(x), \qquad x \mapsto x + u.$$

So geometrically we study the dynamical system on the affine line $\mathbb{A}_{1C}$ sending a point z of the affine line to $z + u$. We can apply Picard-Vessiot theory of linear difference equations to study this dynamical system. We calculate Galois group according to our definition. It follows from the definition

$$\begin{aligned} \iota(x) &= \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x & x+u & x+2u & \cdots \end{bmatrix} \\ &= x + \mathbb{N}u, \end{aligned} \tag{5.3}$$

where we set

$$\mathbb{N}u = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 0 & u & 2u & \cdots \end{bmatrix}.$$

We take derivation d/dx as a basis of the L-vector space $\mathrm{Der}(L/K)$. So the difference differential sub-algebra $\mathcal{L}$ of $F(\mathbb{N}, L^\sharp)$ that is generated by $L^\sharp$ and $\iota(x)$ invariant under the derivation d/dx and the shift operator Φ coincides with $L^\sharp[\mathbb{N}u]$. The element $\mathbb{N}u$ satisfies the following the difference and differential equations

$$\Phi(\mathbb{N}u) = u + \mathbb{N}u, \qquad \frac{d\mathbb{N}u}{dx} = 0. \tag{5.4}$$

Now let us calculate the Lie algebra of the functor $\text{Inf-gal}(L/K)$ over $L^\natural$. Let us take $L^\natural$-algebra $A := L^\natural[\epsilon]$ with $\epsilon^2 = 0$ and consider an infinitesimal deformation $\sigma \in F(L/K)$ of the inclusion (5.2) that we denote by i so that

$$i : \mathcal{L} \to F(\mathbb{N}, L^\natural[[W]]) \subset F(\mathbb{N}, A[[W]]).$$

and

$$\sigma : \mathcal{L} \to F(\mathbb{N}, A[[W]])$$

is a difference differential morphism satisfying

$$\sigma = i \bmod \epsilon \quad \text{ and } \quad \sigma|\mathcal{K} = i|\mathcal{K}.$$

Since $u \neq 0$, the element $u\mathbb{N} \in F(\mathbb{N}, L^\sharp) \subset F(\mathbb{N}, L^\natural)$ is transcendental over $\mathcal{K}$. It follows from (5.4)

$$\sigma(\mathbb{N}u) = \mathbb{N}u + a\epsilon \tag{5.5}$$

with $a \in L^\natural$. Conversely we can take any element $a \in L^\natural$ and define a difference differential morphism σ. So the Lie algebra of the group functor $\text{Inf-gal}(L/K)$ is one dimensional and the formal group $\text{Inf-gal}(L/K)$ is isomorphic to the formal completion $\widehat{\mathbb{G}}_a$. This example is also treated by Picard-Vessiot theory and compatible with the Geometric Principle of § 3.

Example 5.3. — Monomials. Let K, L and u be as in example 5.1. Let l be a non-zero integer different form ± 1. We consider the field morphism

$$\phi : C(x) \to C(x) \qquad x \mapsto ux^l.$$

We calculate Inf-gal$((L,\phi)/K)$. Geometrically speaking, we are interested in the Galois group of the dynamical system

$$\mathbb{A}^1_C \to \mathbb{A}^1_C \qquad z \mapsto uz^l.$$

As we can argue similarly, we may assume $u = 1$ and $l = 2$. (If the field C is algebraically closed, the dynamical system for an arbitrary u is equivalent to the case $u = 1$.) It follows from Definition 2.6 in the first part [8],

$$\iota(x) = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x & x^2 & x^4 & \cdots \end{bmatrix} \tag{5.6}$$

and so that we have

$$\Phi(\iota(x)) = \iota(X)^2 \tag{5.7}$$

and

$$x\frac{d}{dx}\iota(x) = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x & 2x^2 & 3x^3 & \cdots \end{bmatrix}.$$

Therefore

$$\begin{aligned} \iota(x^{-1})x\frac{d}{dx}\iota(x) &= \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x^{-1} & x^{-2} & x^{-3} & \cdots \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x & 2x^2 & 4x^4 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 1 & 2 & 4 & \cdots \end{bmatrix} \\ &= 2^{\mathbb{N}}. \end{aligned} \tag{5.8}$$

So

$$\frac{d}{dx}\left(\iota(x^{-1})x\frac{d}{dx}\iota(x)\right) = 0 \tag{5.9}$$

It follows from (5.8)

$$\mathcal{L} = L^\sharp[\iota(x), 2^{\mathbb{N}}] \qquad \text{and} \qquad \mathcal{K} = L^\sharp. \tag{5.10}$$

Let us determine the Lie algebra of the group functor Inf-gal(L/K). To this end let us take $L^\natural$-algebra $A := L^\natural[\epsilon]$ with $\epsilon^2 = 0$. Let $\sigma \in F(L/K)(A)$ so that σ is an infinitesimal deformation of the inclusion that we denote by i so that

$$i : \mathcal{L} \to F(\mathbb{N}, L^\natural[[W]]) \subset F(\mathbb{N}, A[[W]])$$

and so the infinitesimal deformation

$$\sigma : \mathcal{L} \to F(\mathbb{N}, A[[W]])$$

is a difference differential morphism satisfying

$$\sigma = i \bmod \epsilon \quad \text{and} \quad \sigma|\mathcal{K} = i|\mathcal{K}.$$

Therefore the difference differential morphism σ is determined by the image $\sigma(\iota(x)) \in F(\mathbb{N}, L^\natural[[W]])$. We writing $\iota(x)$ by $\tilde{X}$, $\sigma(\tilde{X})$ should satisfy

$$\Phi(\sigma(\tilde{X})) = \Phi(\tilde{X})^2 \quad \text{and} \quad \frac{d}{dW}\left(\sigma(\tilde{X})^{-1}(x+W)\frac{d}{dW}\sigma(\tilde{X})\right) = 0 \tag{5.11}$$

by (5.7) and (5.9). Conversely if an element $\sigma(\tilde{X}) \in F(\mathbb{N}, , L^\natural[[W]])$ satisfies two equations (5.11), since $2^{\mathbb{N}}$ is transcendental over $\mathcal{K} = L^\sharp$, we there exists a unique difference differential infinitesimal deformation morphism $\mathcal{L} \to F(\mathbb{N}, A[[W]])$ that sends $\tilde{X}$ to $\sigma(\tilde{X})$. Consequently there exists an element $a \in A[[W]]$ such that

$$\sigma(\tilde{X}) = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x+W+a\epsilon & (x+W+a\epsilon)^2 & (x+W+a\epsilon)^4 & \cdots \end{bmatrix} \tag{5.12}$$

satisfying

$$\frac{d}{dW}\left(\sigma(\tilde{X})^{-1}(x+W)\frac{d}{dW}\sigma(\tilde{X})\right) = 0. \tag{5.13}$$

The latter is equivalent to the linear differential equation

$$\left((x+W)^2\frac{d^2}{dW^2} - (x+W)\frac{d}{dW} + 1\right) a = 0. \tag{5.14}$$

Elements

$$x+W, \qquad (x+W)\log\left(1+\frac{W}{x}\right) = \frac{W}{x} - \frac{1}{2}\left(\frac{W}{x}\right)^2 + \frac{1}{3}\left(\frac{W}{x}\right)^3 - \cdots$$

of $L^\natural[[W]]$ are two linearly independent solutions of linear defferential equation (5.14). So there exist $c_1, c_2 \in L^\natural$ such that

$$a = c_1(x+W) + c_2(x+W)\log\left(1+\frac{W}{x}\right).$$

The image $\sigma_a(\tilde{X})$ is explicitly given by

$$\begin{aligned}
\sigma_a(\tilde{X}) &= \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x+W+a\epsilon & (x+W+a\epsilon)^2 & (x+W+a\epsilon)^4 & \cdots \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 2 & \cdots \\ x+W & (x+W)^2 & (x+W)^4 & \cdots \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 1 & 2 & \cdots \\ a\epsilon & 2(x+W)a\epsilon & 4(x+W)^3a\epsilon & \cdots \end{bmatrix} \\
&= \tilde{X}\left(1+\frac{2^{\mathbb{N}}a\epsilon}{x+W}\right). \qquad (5.15)
\end{aligned}$$

Since the infinitesimal deformation σ depends only on $a\epsilon \in A[[W]] = C[\epsilon][[W]]$, let us denote σ by σ_a so that

$$\sigma_a \in F(L/K)(A).$$

The corresponding difference differential infinitesimal automorphism of

$$\mathcal{L}\widehat{\otimes}_{L^\sharp} A[[W]]/\mathcal{K}\widehat{\otimes}_{L^\sharp} A[[W]]$$

is evidently given by

$$\mathcal{L}\widehat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{L}\widehat{\otimes}_{L^\sharp} A[[W]], \qquad \tilde{X} \mapsto \tilde{X}\left(1 + \frac{2^{\mathbb{N}} a\epsilon}{x+W}\right).$$

Since $\log\left(1+\frac{W}{x}\right)$ is an element of $L^\natural[[W]]$ but is not in $\mathcal{L}$, it is inevitable to introduce the completion $\mathcal{L}\widehat{\otimes}_{L^\sharp} L^\natural[[W]]$. In other words, *this shows theoretical necessity of considering the completion.*

Hence in particular the dimension of the $L^\natural$-Lie algebra of the formal group Inf-gal(L/K) is 2. What is the group structure of the group of infinitesimal automorphisms? To this end we have to know the image $\sigma_{a\epsilon}(2^{\mathbb{N}})$ of $2^{\mathbb{N}}$ under $\sigma_{a\epsilon}$.

Lemma 5.4.

$$\sigma_{a\epsilon}(2^{\mathbb{N}}) = 2^{\mathbb{N}}\left(1 + \frac{1}{x+W}\left(\frac{da}{dW} - a\right)\epsilon\right). \tag{5.16}$$

We taking the logarithmic derivative of (5.15) with respect to d/dW, the Lemma follows from (5.8) and (5.15).

Lemma 5.5. — *Let us consider the $L^\natural$-algebra $B := L^\natural[\epsilon, \epsilon']$ such that $\epsilon^2 = \epsilon'^2 = 0$. Let $\sigma_{a\epsilon}, \sigma_{b\epsilon'}$ be infinitesimal difference differential $\mathcal{K}\widehat{\otimes}_{L^\sharp} B[[W]]$-automorphisms of $\mathcal{L}\widehat{\otimes}_{L^\sharp} B[[W]]$ corresponding to the solutions $a = a(W)$, $b = b(W) \in L^\natural[[W]]$ of linear differential equation (5.14). Namely*

$$\sigma_1(\tilde{X}) = \tilde{X}\left(1 + \frac{2^{\mathbb{N}} a\epsilon}{x+W}\right), \qquad \sigma_2(\tilde{X}) = \tilde{X}\left(1 + \frac{2^{\mathbb{N}} b\epsilon'}{x+W}\right). \tag{5.17}$$

The commutation relation of the automorphisms $\sigma_{a\epsilon}$ and $\sigma_{b\epsilon'}$ is

$$\sigma_{a\epsilon}^{-1}\sigma_{b\epsilon'}^{-1}\sigma_{a\epsilon}\sigma_{b\epsilon'} = \sigma_{c\epsilon\epsilon'}, \tag{5.18}$$

where

$$c = \left(a\frac{db}{dW} - \frac{da}{dW}b\right)\epsilon\epsilon'.$$

It is sufficient to check that the images of the generator $\tilde{X}$ by the both automorphisms in the left and right of (5.18) coincide. This follows from lemma 5.4 and (5.17).

Definition 5.6. — *We denote by* $\mathrm{AF}_1(R)$ *the group of affine transformations of dimension* 1 *with coefficients in a ring* R. *So*

$$\mathrm{AF}_1(R) = \{(a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathrm{GL}_2(R) | a_{21} = 0, a_{22} = 1\}.$$

The Lie algebra of the R*-algebraic group* AF_{1R} *is denoted by* af_{1R}.

The Lie algebra $af_1(R)$ is a 2-dimensional Lie algebra of $gl_2(R)$ of 2×2 matrices with entries in R spanned by

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in gl_2(R)$$

satisfying

$$[e, f] = f. \tag{5.19}$$

Corollary 5.7. — *The Lie algebra of the formal group* $\text{Inf-gal}(L/K)$ *is isomorphic to the Lie algebra* $af_{1L^\natural}$. *The group functor* $\text{Inf-gal}(L/K)$ *is isomorphic to the formal group* $\widehat{\mathrm{AF}}_{1L^\natural}$.

It is sufficient to show the assertion. The $L^\natural$-vector space of solutions in $L^\natural[[W]]$ of the linear differential equation (5.14) is 2-dimensional and spanned by

$$x + W, \qquad (x + W)\log\left(1 + \frac{W}{X}\right).$$

Corollary now follows from (5.19) if we notice

$$\left[(x + W)\,\frac{d}{dW}, (x + W)\log\left(1 + \frac{W}{X}\right)\right] = (x + W)\frac{d}{dW}.$$

Remarks 5.8. — Examples 5.1, 5.2 and 5.3 are all of the same type. They are dynamical systems on an algebraic group G defined over C and arises in the following way. Let b be a C-valued point of the algebraic group G and let $h : G \to G$ be a C-endomorphism of the algebraic group G. We introduce the endomorphism

$$\varphi : G \to G, \qquad z \mapsto h(z)b \tag{5.20}$$

of algebraic variety G so that we can consider the dynamical system (G, φ). The group $G = \mathbb{G}_{aC}$ for Examples 5.1, 5.2, and $G = \mathbb{G}_{mC}$ for Example 5.3. We treat the elliptic curve case in Example 5.11 below (See also [9]).

4.2. Here is the geometric interpretation of the Example 5.3. We work on the projective line $\mathbb{P}^1_C$, or rather over the algebraic group $\mathbb{G}_{mC}$. We consider

$$\phi : \mathbb{G}_{mC} \to \mathbb{G}_{mC}, \qquad x \mapsto x^2.$$

It is natural to consider the invariant 1-form dx/x on the algebraic group $\mathbb{G}_{mC}$. The endomorphism ϕ transforms

$$\phi^*\left(\frac{dx}{x}\right) = 2\frac{dx}{x}.$$

It follows from (5.9)

$$\frac{d\tilde{X}}{\tilde{X}} = 2^G \frac{dx}{x}. \tag{5.21}$$

The function $\iota(x) = \tilde{X} \in F(\mathbb{N}, L)$ on $\mathbb{N}$ is a solution to the difference equation

$$\Phi(\tilde{X}) = \tilde{X}^2$$

with initial condition $\tilde{X}(0) = x$. Namely, we are interested in the dynamical system

$$x \mapsto \tilde{X}.$$

Moreover $\tilde{X}$ satisfies the differential equation (5.21). Let us observe that the vector fields

$$x\frac{d}{dx} \qquad \text{and} \qquad x\log x\frac{d}{dx} \tag{5.22}$$

are a basis of C-vector space of vector fields that leave the differential form dx/x semi-invariant. In fact, let $\theta_1 = xd/dx$. For ϵ with $\epsilon^2 = 0$, we have

$$\frac{d(x+\epsilon x)}{x+\epsilon x} = \frac{1+\epsilon}{1+\epsilon}\frac{dx}{x} = \frac{dx}{x}.$$

For $\theta_2 = (x\log x)d/dx$,

$$\begin{aligned}\frac{d(x+\epsilon x\log x)}{x+\epsilon x\log x} = \frac{(1+\epsilon\log x+1)dx}{x(1+\epsilon\log x)} &= (1+(\log x+1))(1-\epsilon\log x)\frac{dx}{x}\\ &= (1+\epsilon)\frac{dx}{x}.\end{aligned}$$

4.3. If the base field $C = \mathbb{C}$, we have an analytic covering morphism

$$\pi : \mathbb{C} \to \mathbb{C}^*, \qquad u \mapsto x = \exp u \tag{5.23}$$

so that $u = \log x$. We can use the covering morphsim π to understand vector fields (5.22) better. The invariant vector field dx/x on the Lie group $\mathbb{C}^*$ is transformed to the invariant 1-form dx on the additive group $\mathbb{C}$. $\pi^*(dx/x) = du$. More generally we have the following table. The corresponding vector fields in a law are compatible through the morphism (5.23).

	On the group $\mathbb{C}^*$	On the group $\mathbb{C}$
Invariant 1-form ω	dx/x	$\pi^* d/dx = du$
Vector field leaving ω invariant	xd/dx	d/du
Vector field leaving ω semi-invariant	$(x \log x)d/dx$	ud/du

Example 5.9. — Chebyshev polynomials. Let us recall the definitio of the Chebyshev polynomials. Let us consider the automorphism

$$\phi_+ : \mathbb{G}_{mC} \to \mathbb{G}_{mC}, \qquad z \mapsto z^{-1}$$

of the multiplicative group $\mathbb{G}_{mC}$. So

$$\Gamma = \{\mathrm{Id}_{\mathbb{G}_{mC}}, \phi_+\}$$

is a subgroup of the automorphism group of $\mathbb{G}_{m,C}$. For a positive integer d,

$$\tilde{\phi_d} : \mathbb{G}_{mC} \to \mathbb{G}_{mC}, \qquad z \mapsto z^d$$

is an endomorphism of $\mathbb{G}_{mC}$ commuting with the elements of the group Γ. Therefore the morphism $\tilde{\phi_d}$ induces an endomorphism

$$\phi_d : \mathbb{G}_{mC}/\Gamma \to \mathbb{G}_{mC}/\Gamma$$

of the quotient space $\mathbb{G}_{mC}/\Gamma$ making the following diagram commutative.

$$\begin{array}{ccc} \mathbb{G}_{mC} & \xrightarrow{\tilde{\phi_d}} & \mathbb{G}_{mC} \\ \downarrow & & \downarrow \\ \mathbb{G}_{mC}/\Gamma & \xrightarrow[\phi_d]{} & \mathbb{G}_{mC}/\Gamma, \end{array} \tag{5.24}$$

where the vertical morphism is the quotient morpism.

The morpism ϕ_d is the geometric interpretation of the Chebyshev polynomial of degree d. To get the usual concrete form of Chebyshef polynomials, we choose the coordinate system z on the multiplicative group $\mathbb{G}_{mC}$ such that $\mathbb{G}_{mC} = \operatorname{Spec} C[z, z^{-1}]$. So $\mathbb{G}_{mC}/\Gamma = \operatorname{Spec}[z + z^{-1}] \simeq \mathbb{A}^1_C$. We writing $y := z + z^{-1}$, the polynomial $f_d(y) := \phi_d^*(y) \in C[y]$ is the Chebyshev polynomial of degree d, where $\phi_d^* : C[y] \to C[y]$ is the C-algebra morphism dual to the scheme morphism $\phi_d : \operatorname{Spec} C[y] \to \operatorname{Spec} C[y]$. For example $d = 2$, we get

$$f_2(y) = \phi_2^*(y + y^{-1}) = \tilde{\phi_2}^*(z + z^{-1}) = z^2 + z^{-2} = y^2 - 2$$

so that the Chebyshev polynomial of the second degree $f_2(y) = y^2 - 2$. Similary

$$f_3(y) = y^3 - 3y, \qquad f_4(y) = y^4 - 4y^2 + 2, \dots .$$

Conclusion. *The Lie algebra* Inf-gal$((C(y), f_d(y))/C)$ *is isomorphic to* $al_1(L)$ *for every integer* $d \geqslant 2$.

This follows from the argument of Example 5.3 and the commutative diagram (5.24). Let us have a closer look for $d = 2$. We keep in mind the commutative diagram (5.24). We set $L := C(y)$ and $\tilde{L} := C(z)$ so that $\tilde{L}/L$ is an algebraic extension of degree 2. We study the dynamical system or the difference field (L, ϕ_2^*) by using the covering $(\tilde{L}, \tilde{\phi}_2^*)$. We consider the morphism $\iota : L \to F(\mathbb{N}, \tilde{L}^\natural)$ as well as the morphism $\iota : \tilde{L} \to F(\mathbb{N}, \tilde{L}^\natural)$. So the latter is the extension of of the former. Let us denote $\iota(z)$ by $\tilde{Z}$ and $\iota(y)$ by $\tilde{Y}$. As we have seen in Example 5.3,

$$\frac{d\tilde{Z}}{dz} = 2^{\mathbb{N}} \frac{\tilde{Z}}{z}. \tag{5.25}$$

Since ι is an algebra morphsim, it follows from

$$y = z + z^{-1} \tag{5.26}$$

that

$$\tilde{Y} = \tilde{Z} + \tilde{Z}^{-1}. \tag{5.27}$$

Now the equations (5.25), (5.21) and (2.1) give

$$\left(\frac{d\tilde{W}}{dw}\right)^2 = \left(2^N \frac{\tilde{W}^2 - 4}{w^2 - 4}\right)^2 \tag{5.28}$$

so that

$$\frac{d\tilde{W}}{dw} = \pm 2^N \frac{\tilde{W}^2 - 4}{w^2 - 4}. \tag{5.29}$$

Now we arrive at the conclusion by the equality (5.29) and the argument of Example 5.3.

Example 5.10 (Curves of genus 1). — Let E be a non-singular projective algebraic curve of genus 1 defined over a field C of characteristic 0.

We know that the curve E is isomorphic to a cubic curve in $\mathbb{P}^2$ defined by

$$y_0 y_2^2 = 4y_1^3 - g_2 y_0^2 y_1 - g_3 y_0^3, \qquad g_2, g_3 \in C, \tag{5.30}$$

with $4g_2 3 + 27g_3^2 \neq 0$. So the rational function field of the curve $C(E) = C(y, z)$ with $y = y_1/y_0$ and $z = y_2/y_0$. So we have

$$z^2 = 4y^3 - g_2 y - g_3. \tag{5.31}$$

See [9], 6.3. We call (5.31) Weierstrass form of the elliptic curve E so that we have $C(E) \simeq C(y, z)$, y, z satisfying (5.31).

Let now $\varphi : E \to E$ be an endomorphism of a non-singular projective curve E of genus 1 defined over C. We assume that φ is dominant so that we can speak of the discrete dynamical system of iteration of φ on the curve E or the difference field $(C(E), \varphi^*)$, where

$$\varphi^* : C(E) \to C(E)$$

is an endomorphism of the rational function field $C(E)$ sending a rational function $f \in C(E)$ to $f \circ \varphi$.

We know that the curve E has C-rational points and if we take a C-rational point e of E, then the curve E has a unique commutative algebraic group structure such that e is the 0 of the group. When we consider not only the curve C but also the group structure on E we denote the algebraic group by (E, e).

We also know that, given an endomorphism φ of the genus 1 curve E, then φ is an endomorphism of the algebraic group (E, e) if and only if $\varphi(e) = e$. Furthermore when an endomorphism φ of the curve E is given, we have either

(1) φ is an endomorphism of the algebraic group (E, e), or
(2) there exist a C-rational point $a \in E$ and the endomorpshism $\psi : (E, e) \to (E, e)$ of the algebraic group such that

$$\varphi(q) = \psi(q) + a$$

for every point $q \in E$.

(5.10.1) If $\varphi : E \to E$ is an automorphism of finite order, then the Lie algebra of Inf-gal$(L/C) = 0$.

(5.10.2) If $\varphi : E \to E$ is an automorphism of the curve E and if it is not of finite order, then the Lie algebra of Inf-gal(L/C) is the 1-dimensional $L^\natural$-Lie algebra.

(5.10.3) If $\varphi : E \to E$ is an endomorphism of the elliptic curve E and if deg$\varphi \geqslant 2$, then the Lie algebra Inf-gal(L/C) is isomorphic to $af_1(L)$.

As we noticed in § 3, (5.10.1) is a particular instance of GEOMETRIC PRINCIPLE 1 that we can prove directly in this case. (5.10.2) is examined as Examples 5.1 and 5.2. Let us prove (5.10.3). In fact, the endomorphisme ring R of the elliptic curve (E, e) is well-known (*cf.* [9], Proposition 6.25). The important fact is the endomorphism ring R is commutative algebra integral over $\mathbb{Z}$. As other endomorphisms are treated in a similar way. Let us study a particular endomorphism $\varphi_l(q) = lq$ for every point q of E, l being an integer not equal to ± 1.

We use the Weierstrass form (5.31) so that we identify $C(E) = C(y, z)$, which we denote by L. Let us consider $\iota : L \to F(\mathbb{N}, L^\natural)$. We denote $\iota(y)$ by $\tilde{Y}$ and ιz by $\tilde{Z}$. So we have

$$\tilde{Z}^2 = 4\tilde{Y}^3 - g_2\tilde{Y} - g_3.$$

The derivation d/dy of the field $C(y)$ of rational functions of 1-variable is extended to a unique derivation of the field $C(E) = C(y, z)$ which we denote also by d/dy. We know that zd/dy is translation invariant regular vector field on the elliptic curve E and its dual 1-form $(1/z)dy$ is also translation invariant. We take as a basis of the $L^\natural$ vector space $\mathrm{Der}(L/C)$, the derivation $zd/dy \in \mathrm{Der}(L/C)$.

We know that $\varphi_l^*(1/z)dy = l(1/z)dy$. So for every $f \in C(E)$, we have

$$z\frac{d\varphi_l^*(f)}{dy} = lz\frac{df}{dy}. \tag{5.32}$$

Now the argument of Example 4.3 gives us the result.

Example 5.11. — Lattès discovered, in 1918, discrete dynamical systems of iteration of a rational map on $\mathbb{P}^1_{\mathbb{C}}$ of which the Fatou set is empty. So they are considered to be quite chaotic. They are, however, not so wild. In fact their Galois group Inf-gal is isomorphic to the solvable Lie algebra af_1. These dynamical systems are related with the elliptic curves. They come from an ellipic curve E over the field C of characteristic 0 in the following manner. We choose once for all a point e of E. So the group structure (E, e) on E is fixed.

Let $\Gamma \neq 1$ be a group of automorphisms of the elliptic curve (E, e). We know the group Γ is necessarily finite. We explained above the three types of endomorphism ring R in the proof of (5.10.3). So the group Γ is a subgroup of the finite group of units of the above listed rings R. We can show that if Γ is not trivial, the quotient E/Γ is isomorphic to the projective line $\mathbb{P}^1_C$.

Let now $\varphi_1 : E \to E$ be an endomorphism of the elliptic curve E. So we can find an endomorphism $\varphi_1 : (E, e) \to (E, e)$ of the algebraic group and a point $t \in E$ such that

$$\varphi_0(q) = \varphi_1(q) + t.$$

Now we assume that the map $\varphi_0 : E \to E$ commutes with every element of the group $\Gamma \subset \mathrm{Aut}(E, e)$. The condition is satisfied if and only if $2t = 0$, if we recall the fact that the ring $R = \mathrm{End}(E, e)$ is a commutative ring. So we get a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi_0} & E \\ \downarrow & & \downarrow \\ E/\Gamma \simeq \mathbb{P}^1_C & \xrightarrow[\varphi]{} & E/\Gamma \simeq \mathbb{P}^1_C, \end{array} \tag{5.33}$$

where the vertical morphisms are the quotient morpism.

Definition 5.12. — *A Lattès map associated with the elliptic curve E and the morphism $\varphi_0 : E \to E$ is the induced map $\varphi : \mathbb{P}^1_C \to \mathbb{P}^1_C$.*

Therefore, we get a dynamical system

$$(\mathbb{P}^1_C, \varphi)$$

or equivalently the difference field

$$(C(\mathbb{P}^1_C), \varphi^*).$$

The argument of studying the dynamical system of the Chebyshev polynomials allows us to conclude the following

Proposition 5.13. — *If the endomorphism φ_0 is not an isomorphism, the Lie algebra of Galois group* Inf-gal*$(C(\mathbb{P}^1_C), \varphi^*)$ of a dynamical system of Lattès is isomorphic to af_1.*

We analyzed concrete examples. We summarize the arguments in a form of Theorem. To this end, we need a definition.

Definition 5.14. — *Let G be an algebraic group defined over C and let Γ be a finite group of automorphisms of the algebraic group G. Let $\varphi_1 : G \to G$ be an endomorphism of the algebraic group G and t be a C-rational point of G. We call the map $\varphi_0 : G \to G$ such that $\varphi_0(z) = \varphi_1(z).t$ for every point $z \in G$, an End-translation map.*

Let φ_0 be an End-translation map commuting with every element of the automorphism group Γ. So φ_0 induces a map $\varphi : G/\Gamma \to G/\Gamma$, which we call a generalized Chebyshef-Lattès map. More generally, it is convenient to call a map birationally equivalent to a generalized Chebyshef-Lattès map, of generalized Chebyshef-Lattès type.

Remarks 5.15. — (1) For a logical simplicity, we do not exclude the case $\Gamma = 1$. Hence in all the examples we have so far examined, the morphism $\varphi : V \to V$ of the dynamical system is either automorphism of finite order or of generalized Chebyshev-Lattès type.

(2) For the multiplcative group $\mathbb{G}_m$, the automorphism $\Gamma = \{\pm 1\}$ of $\mathbb{G}_m$ and an integer $|l| \geqslant 2$, the map

$$\varphi_0 : \mathbb{G}_m \to \mathbb{G}_m, \qquad x \mapsto -x^l$$

engenders a polynomial generalizing the Chebyshev polynomials but it is equivalent to the Chebyshev polynomial if we can solve the algebraic equation $\xi^{l-1} = -1$ in C.

Theorem 5.16. — *Let $\varphi : V \to V$ be a dominant endomorphism of an algebraic curve defined over a field C of characteristic 0. Then the Lie algebra of* Inf-gal$((C(V)\varphi)/(C, \mathrm{Id}_C))$ *is at most af_1, more precisely either of dimension $\leqslant 1$ or isomorphic to af_1 for the following 2 types of dynamical systems. In particular, the dynamical system is finite dimensional and infinitesimally solvable according to Definition 6.2 below.*

(1) *$\varphi : V \to V$ is an automorphism of finite order.*
(2) *The dynamical systems of generalized Chebyshev-Lattès type map.*

Remark 5.17. — We can make statements of the Theorem more precise. Let (V, φ) be a dynamical system in the list.

(1) If φ is finite order, then Inf-gal$(C(V)/C)$ is 0.
(2) If φ is an automorphism of infinite order, then Inf-gal$(C(V)/C)$ is 1-dimensional.
(3) If the degree of the map φ is $\geqslant 2$, then the Lie algebra of Inf-gal $(C(V)/C)$ is isomorphic to af_1.

6. Solvable dynamical systems on an algebraic curve defined over $\mathbb{C}$

There are extensive works on the integrability of discrete dynamical systems (Theorem 6.1 below, [15]). One of the definitions of integrability is the existence of independent commutative rational maps.

Theorem 6.1 (Julia, Fatou, Ritt and Erëmenko). — *Let $\varphi, \psi \in \mathbb{C}(z)$ be rational maps of degree $\geqslant 2$ with coefficients in $\mathbb{C}$ such that*

$$\varphi \circ \psi = \psi \circ \varphi.$$

Then either (1) there exist integers m, n such that $\varphi^{\circ m} = \psi^{\circ n}$, or (2) the rational maps φ and ψ are birationally equivalent to monomials, Chebyshev polynomials or Lattès maps.

See [9], 6.8 Theorem 6.79.

Theorem 6.1 says if we adopt as a definition of integrabillity the existence of independent commuting rational maps, the integrable discrete dynamical system of iteration of a rational map $\varphi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ with $\deg \varphi \geqslant 2$ are exhausted by monomials, Chebyshev polynomials or Lattès maps.

General difference Galois theory also allows us to measure integrability. We propose the following

Definition 6.2. — *Let V be an algebraic variety defined over a field C of characteristec 0 and $\varphi : V \to V$ be a dominant rational map also defined over C. We say that the dynamical system (V, φ) is infinitesimally abelian, respectively solvable or semi-simple if the Lie algebra of he Galois group* $\text{Inf-gal}(C(V), \varphi)$ *is abelian, respectively solvable or semi-simple.*

We recall several results that we use in the proof of Theorem 6.5 below.

From now on, we work over the field $\mathbb{C}$ of complex numbers and we prove the converse of Theorem 5.16.

We have to consider algebraic D-groupoid defined on an open algebraic curve. Let us fix notations. Let us work over the affine line $\mathbb{A}^1_{\mathbb{C}}$. The coordinate ring of the space of invertible jets is, by definition, $\mathbb{C}[x][y, y_1, y_2, \cdots, y_1^{-1}]$ with derivation

$$\frac{\partial}{\partial x} + \sum_{i=0}^{\infty} y_{i+1} \frac{\partial}{\partial y_i},$$

where the y_i's are variables for $i \in \mathbb{N}$ (y_0 being y). When we consider a D-groupoid over an open set Spec (R) of the affine line, we replace the coordinate ring $\mathbb{C}[x]$ of $\mathbb{A}^1_{\mathbb{C}}$ by R. So if we are interested in a local definition of D-groupoid on a rational curve, we replace the coordinate ring $\mathbb{C}[x]$ by the field $\mathbb{C}(x)$.

Proposition 6.3. — *An algebraic D-groupoid of finite dimension over a rational curve that is a Zariski open set of Spec $\mathbb{C}[x]$ is a sub-groupoid of the groupoid defined by a differential ideal of $\mathbb{C}(x)[y, y_1, y_2, \cdots y_1^{-1}]$ generated by*

$$G_3(\nu) := \nu(y) y_1^2 + 2\frac{y_3}{y_1} - 3\left(\frac{y_2}{y_1}\right)^2 - \nu(x),$$

where $\nu(x) \in \mathbb{C}(x)$.

See Casale [2], Proposition 9.

We can not avoid transcendental method. So we have to know the behavior of the D-groupoid in Definition 6.3 under the inverse image. Let us now assume that we are in analytic situation so that the function $\nu(x)$ is a holomorphic function defined on a open set U of $\mathbb{C}$. So $G_3(\nu)$ defines an analytic D-groupoid over $U \times U$ in the category of analytic spaces and $\varphi : W \to U$ a holomorphic map of an open set W of $\mathbb{C}$. Then we have

Proposition 6.4.

$$\varphi^* G_3(\nu) = G_3(\nu \circ \varphi {\varphi'}^2 + S(\varphi)),$$

where $S(\varphi)$ is the Schwarzian derivative of the map φ with respect to the coordinate system on W.

See Casale [2], Proposition 8.

We determine infinitesimally solvable dynamical systems over curves. In other words, we prove the converse of Theorem 5.16, when the base field is $\mathbb{C}$.

Theorem 6.5 (Casale). — *Let (V, φ) be a discrete dynamical system over an algebraic curve V defined over the complex number field $\mathbb{C}$ so that*

$$\varphi : V \to V$$

is a dominant $\mathbb{C}$-rational map. Then the following three conditions 1, 2 and 3 for the dynamical system (V, φ) are equivalent.

(1) *The dimension of the Galois group*

$$\text{Inf-gal}((\mathbb{C}(V), \varphi^*)/(\mathbb{C}, \mathrm{Id}_{\mathbb{C}}))$$

is finite.

(2) *The dynamical system (V, φ) is birationally equivalent to one of the following dynamical systems.*
   (a) *The endomorphism $\varphi : V \to V$ is an automorphism of finite order.*
   (b) *The dynamical systems of generalized Chebyshev-Lattès type.*

(3) *The dynamical system (V, φ) is infinitesimally solvable.*

Remark 6.6. — Here is the list of concrete dynamical systems of 2 in the Theorem.

(1) The morhism φ is an automorphism of finite order.
(2) The curve V is the projective line $\mathbb{P}^1_{\mathbb{C}}$ and the morphism φ is an automorphism.

(3) The curve V is the projective line $\mathbb{P}^1_{\mathbb{C}}$ and there exists an integer $n \neq 0$ such that the morphism $\varphi(x) = x^n$, where x is an inhomogeneous coordinate system on the projective line.
(4) The curve V is the projective line $\mathbb{P}^1_{\mathbb{C}}$ and the map φ is a Chebyshev polynomial.
(5) V is an elliptic curve and φ is an endomorphism of the elliptic curve V.
(6) Examples of Lattès in Example 5.11.

We have shown that the condition 2 implies the condition 3 in Theorem 5.16.

The condition 1 is an evident consequence of the condition 3 if we recall the well-known fact that a Lie algebra $\mathfrak{g}$ of holomorphic vector fields on an open disc on $\mathbb{C}$ is not solvable if $\dim \mathfrak{g} \geqslant 3$.

So we have to show that the condition 1 implies the condition 2. As we noticed in § 4, the Galois group is 0 if φ is a birational map of finite order. In particular, we may assume the genus g of the curve V is $\leqslant 1$. In fact, if $g \geqslant 2$, then the endomorphsm φ is birational by the Hurwitz's formula. Further more by a theorem of Hurwitz, the group of birational automophisms of V is finite. We replacing V by its non-singular model if necessary, φ is an automorphism of finite order of V.

If $g = 1$, then, taking the non-singular model, we may assume that the φ is either the translation by a point of the elliptic curve V or an endomorphism of the elliptic curve. In both cases, we examined that the dimension of the Galois group is of at most 2. Namely if either φ is an automorphism or V is a curve of genus $\geqslant 1$, the dynamical system (V, φ) satisfies all the conditions of the Theorem.

So we may assume that the curve V is the projective line $\mathbb{P}^1_{\mathbb{C}}$ and we have to show condition 1 implies 2. Since every automorphism of $\mathbb{P}^1_{\mathbb{C}}$ satisfies condition 2, we may further assume that $\deg \varphi \geqslant 2$. The condition 1 says that the transcendence degree of the algebra $\mathcal{L}$ over $L^\natural$ is finite. Equivalently, the algebraic D-groupoid $\operatorname{Spec} \mathcal{L}$ over $\operatorname{Spec} L^\natural$ is finite dimensional. In other words, there exists an algebraic D-groupoid G of finite dimension, which is a model of $\operatorname{Spec} \mathcal{L}$, defined over a Zariski open set of $\mathbb{P}^1_{\mathbb{C}}$ such that $\varphi \circ n$ is a solution to G or solution to the defining differential ideal of G. Now, it follows from Proposition 6.3 that there exists a rational function $\nu(x)$ such that $y = \varphi^{\circ n}$ satisfies the differential equation $G_3(\nu) = 0$ for every $n \in \mathbb{N}$, x being an appropriate inhomogeneous coordinate system on $\mathbb{P}^1_{\mathbb{C}}$. We prove for sufficiently big n, the iteration $\varphi^{\circ n}$ is one of the maps of the condition 2. Since we assume $\deg \varphi \geqslant 2$, it follows from Theorems 1.29 and 1.35 of [9] that there are many repelling periodic points of φ. In fact, we can find an integer n such that there exists a point $p \in \mathbb{P}^1_{\mathbb{C}}$ such that p is a repelling fixed point of $\varphi^{\circ n}$ and such that the rational function $\nu(x)$ defining $G_3(\nu)$ is regular at p. Therefore, we have $|\lambda| > 1$, λ being $(\varphi^{\circ n})'(p) = d\varphi^{\circ n}(p)/dx$.

It follows from a Theorem of Kœnigs that we can linearize the dynamical system locally around the point p. In other words, we can find a holomorphic map $F : W \to \mathbb{P}^1_{\mathbb{C}}$ form an open neighborhood W of the origin $0 \in \mathbb{C}$ such that $F'(0) = 1$ and such that F is locally equivariant, namely

$$F(\lambda w) = \varphi^{\circ n}(F(w)) \text{ for every point } w \text{ in a neighborhood of } 0 \in \mathbb{C}. \tag{6.1}$$

Thanks to equation (6.1), since $|\lambda| > 1$, we can extend the holomorphic map F over the whole complex plain $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$. We denote this extension of F also by F so that we have a holomorphic map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ equivariant with respect to the multiplication by λ and the morphism $\varphi^{\circ n}$.

$$F(\lambda w) = \varphi^{\circ n}(F(w)) \quad \text{for every point } w \in \mathbb{C}\ . \tag{6.2}$$

So the multiplication by λ on $\mathbb{C}$ sending $w \mapsto \lambda w$ is a solution to the inverse image $G_3(\overline{\nu})$ of $G_3(\nu)$. Therefore

$$\overline{\nu}(\lambda w)\lambda^2 = \overline{\nu}(w). \tag{6.3}$$

Since ν is holomorphic at $w = 0$ and hence expanded into a power series in w, consequently by equation (6.3) we conclude $\overline{\nu} = 0$. It is convenient to introduce a few notations.

Notation. — Let t be a point of $\mathbb{C}$ such that the holomorphic map F is unramified at the point t. We denote by F_t the restriction of F in a neighborhood of t. So F_t is a local isomorphism and hence we can speak of its local inverse F_t^{-1}.

Let r, s be two points of $\mathbb{C}$ such that the holomorphic map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is unramified at r and s. We set

$$F_{sr} := F_s^{-1} \circ F_r$$

that is a local holomorhic isomorphism of a neighborhood of r to a neighborhood of s.

So it follows from the definition of F_{sr}that we have

$$F = F \circ F_{sr} \tag{6.4}$$

in a neighborhood of the point r. Consequently equation (6.4) implies

$$G_3(0) = F^* G_3(\nu) = F^*_{sr} \circ F^* G_3(\nu) = F^*_{sr} G_3(0). \tag{6.5}$$

This shows that $S(F_{sr}) = 0$ by Proposition 6.4. In other words, F_{sr} is a Möbius transformation of $\mathbb{C}$ and in particular it can be extended to a rational map $F_{sr} : \mathbb{C} \to \mathbb{C}$. Now it follows from (6.4) that we have globally

$$F = F \circ F_{sr}. \tag{6.6}$$

This is an identity between meromorphic functions on $\mathbb{C}$.

Case I. — *There exist points $r, s \in \mathbb{C}$ such that F is unramified at r, s and such that the Möbius transformation F_{sr} has a pole on $\mathbb{C}$.*

There exists a point $t \in \mathbb{C}$ such that the transformation F_{sr} maps the point $t \in \mathbb{C}$ to the infinity. Now it follows from (6.6) that F can be holomorphically extended over $\mathbb{P}^1_{\mathbb{C}}$ so that we have

$$F : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}. \tag{6.7}$$

Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$ be the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ generated by the subset

$$\{F_{sr} \in \mathrm{PGL}_2(\mathbb{C}) |\ F \text{ is unramified at } r, s \in \mathbb{C} \text{ with } F(r) = F(s)\}.$$

Lemma 6.7. — *Γ is a finite group.*

Proof of Lemma. — If Γ were a infinite group, we could find infinitely many distinct points $p_0, p_1, p_2, \ldots$ on $\mathbb{P}^1_{\mathbb{C}}$ such that for every $i \in \mathbb{N}$, the points p_i are on the Γ-orbit Γp_0 of the point p_0 and such that the limit p of the points p_i, for the usual topology, exists on the projective line $\mathbb{P}^1_{\mathbb{C}}$. So by (6.6), (6.7), $F(p_0) = F(p_1) = F(p_2) = \cdots = F(p)$ and hence F is a constant function, which is a contradiction. □

Lemma 6.8. — *The map $F : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ factors through the quotient $\pi; \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}/\Gamma$ and induces an isomorphism $\overline{F} : \mathbb{P}^1_{\mathbb{C}}/\Gamma \tilde{\to} \mathbb{P}^1_{\mathbb{C}}$ so that we have*

$$\overline{F} \circ \pi = F.$$

In other words, $F : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is the quotient morphism of $\mathbb{P}^1_{\mathbb{C}}$ with respect to the action of the group Γ.

Proof of Lemma. — In fact, the map F factors through the quotient map π by (6.6) inducing the map $\overline{F} : \mathbb{P}^1_{\mathbb{C}}/\Gamma \to \mathbb{P}^1_{\mathbb{C}}$. We have to show that F is an isomorphism. This follows from the following observation. Let us take a small disk D on $\mathbb{P}^1_{\mathbb{C}}$ such that the map F is unramified on $F^{-1}(D)$. So the inverse image $F^{-1}(D)$ is a disjoint union of connected open sets mapped isomorphically to D by F. If we take two points $r, s \in F^{-1}(D)$ such that $F(r) = F(s)$, then F_{sr} interchanges the connected components of $F^{-1}(D)$ sending the point r to s. So elements of the group Γ interchanges transitively the connective components of $F^{-1}(D)$. Therefore the degree of the map $\overline{F}$ is 1. □

Now we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\Lambda} & \mathbb{P}^1_{\mathbb{C}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{P}^1_{\mathbb{C}}/\Gamma = \mathbb{P}^1_{\mathbb{C}} & \xrightarrow[\varphi^{\circ n}]{} & \mathbb{P}^1_{\mathbb{C}}/\Gamma = \mathbb{P}^1_{\mathbb{C}}. \end{array} \tag{6.8}$$

Lemma 6.9. — *Denoting by $\Lambda : \mathbb{C} \to \mathbb{C}$ the affine transformation of the multiplication by λ, we have in the group* $\mathrm{PGL}_2(\mathbb{C})$

$$\Lambda\Gamma\Lambda^{-1} \subset \Gamma.$$

This follows from Lemma 6.8 and diagram (6.8)

Corollary 6.10. — $\Lambda\Gamma\Lambda^{-1} = \Gamma$.

This follows from the Lemmas 6.7 and 6.9.

We conclude now from commutative diagram 6.8 and Corolary 6.10 that $\varphi^{\circ n} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$. So φ itself is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$.

Now we pass to

Case II. — *For every $r, s \in \mathbb{P}^1_{\mathbb{C}}$ such that F is unramified at r and s with $F(r) = F(s)$, the Möbius transformation F_{sr} is regular on $\mathbb{C}$ so that F_{sr} is an affine transformation of $\mathbb{C}$.*

Let us define the subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$ as in Case I. Then Γ is a subgroup of $\mathrm{AF}_1(\mathbb{C})$ of affine transformations. So the group Γ operates on $\mathbb{C}$ and the function $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is Γ-invariant. It is convenient to use the matrix representation of the affine transformation group $\mathrm{AF}_1(\mathbb{C})$. Namely to an affine transformation

$$w \mapsto aw + b, \qquad 0 \neq a, b \in \mathbb{C}$$

corresponds the matrix

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

In other words, we identify the affine transformation group $\mathrm{AF}_1^*(\mathbb{C})$ with the group of matrices of the above form. Let U be the unipotent radical of $\mathrm{AF}_1(\mathbb{C})$ so that

$$U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} | b \in \mathbb{C} \right\}.$$

We have an exact sequence

$$1 \to U \to \mathrm{AF}_1 \to \mathbb{C}^* \to 1.$$

which induces an exact sequence

$$1 \to U \cap \Gamma \to \Gamma \to \Gamma/(U \cap \Gamma) \to 1. \tag{6.9}$$

Lemma 6.11. — *The rank of the torsion free abelian group* $U_0 := U \cap \Gamma$ *is at most* 2.

Since Γ-invariant meromorphic function F is U_0-invariant, the assertion is well-known in theory of elliptic functions and easy to prove.

Now we study case by case quickly.

Sub-case II.1. — $U_0 = 0$.

In this case, the group Γ is isomorphic to Γ/U_0 that is a subgroup of $\mathbb{C}^*$. So the group Γ is abelian and consists of only semi-simple elements. So we can simultaneously diagonalize all the elements of the subgroup Γ. Namely, choosing an appropriate affine coordinate of the line $\mathbb{C}$, we may assume that

$$\Gamma \subset \mathbb{C}^* = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} | 0 \neq a \in \mathbb{C} \right\} \subset \mathrm{AF}_1(\mathbb{C}).$$

The argument of Lemma 6.7 allows us to show that Γ is a finite group and then we can appy the method of the proof of Lemmas 6.8 and 6.9 to this case to conclude that the rational map $\varphi^{\circ n} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defining the dynamical system is an automorphism of the projective line $\mathbb{P}^1_{\mathbb{C}}$. So φ itself is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$.

The next case to examine is

Case II.2. — *Rank* $U_0 = 1$.

We may assume that the group $U_0 \subset \mathrm{AF}_1(\mathbb{C})$ is the subgroup of all the translations by integers. *i.e.*,

$$U_0 = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} | n \in \mathbb{N} \right\}.$$

So $\mathbb{C}/\Gamma_0$ is isomorphic to the group $\mathbb{C}^*$ and the map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ factors through the quotient $\pi : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ giving a map $\overline{F} : \mathbb{C}^* \to \mathbb{P}^1_{\mathbb{C}}$ so that $F = \overline{F} \circ \pi$.

Lemma 6.12. — *Replacing* Γ *by an inner automorphism of* $AF_1(\mathbb{C})$, *we may assume that we have either*

$$\Gamma = U_0, \text{ or } \Gamma = < U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} > .$$

Proof of Lemma. — Let us assume $U_0 \neq \Gamma$. We take an element $R \in \Gamma \setminus U_0$ so that

$$R = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{ with } 0 \neq a, b \in \mathbb{C}.$$

As R is not an element of U_0, we have $a \neq 1$. Since U_0 is a normal subgroup of the group Γ, the element $R \in \Gamma$ normalizes the group U_0. So for any integer n

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & an \\ 0 & 1 \end{bmatrix} \in U_0.$$

Hence $an \in \mathbb{Z}$ for every integer n and consequently $a = \pm 1$. As we assume $R \notin U_0$, $a = -1$. Namely

$$R = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}.$$

Now let us denote by B the unipotent matrix

$$\begin{bmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{bmatrix}.$$

By an easy calculation,

$$BRB^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So if we consider the inner automorphism

$$Ad(B) : \mathrm{AF}_1(\mathbb{C}) \to \mathrm{AF}_1(\mathbb{C}), \qquad X \mapsto BXB^{-1},$$

then the automorphism $Ad(B)$ of the group $\mathrm{AF}_1(\mathbb{C})$ leaves the subgroup U_0 invariant and transforms R to

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Namely, we have shown that

$$< U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} > \subset Ad(B)(\Gamma).$$

We prove that this inclusion is the identity. To this end let $S \in \Gamma$ and show that

$$Ad(B)(S) \in < U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} > .$$

We may assume $S \notin U_0$. By the argument above, we have

$$S = \begin{bmatrix} -1 & c \\ 0 & 1 \end{bmatrix} \qquad \text{with } c \in \mathbb{C}.$$

Since the product

$$SR = \begin{bmatrix} -1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & c-b \\ 0 & 1 \end{bmatrix}$$

of two elements of the group Γ is an element of Γ and unipotent so that it is in U_0. Hence $c - b = n \in \mathbb{Z}$. It suffices to notice

$$S = \begin{bmatrix} -1 & b+n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$$

which implies

$$AdB(S) = Ad(B)(R)Ad(B)\left(\begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$$
$$\in \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} U_0$$

showing

$$Ad(S) \in \; < U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} > .$$

We may assume by Lemma 6.12 that either

$$(1) \quad \Gamma = U_0, \qquad \text{or} \quad (2) \quad \Gamma = \; < U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} > .$$

In the case (1), the quotient $\mathbb{C}/\Gamma$ exists and isomorphic to $\mathbb{C}/\mathbb{Z} = \mathbb{C}^*$. The argument of Lemma 6.8 shows the map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is he quotient morphism $\pi : \mathbb{C} \to \mathbb{C}^*$. Letting

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{AF}_1(\mathbb{C}).$$

We show by the argument of Lemma 6.9

$$\Lambda U_0 \Lambda^{-1} \subset U_0,$$

which implies $0 \neq \lambda \in \mathbb{N}$ and the map $\varphi^{\circ n} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is equivalent to the map

$$\mathbb{C}^* \to \mathbb{C}^*, \qquad x \mapsto x^n$$

for the integer n. We are in the case of (3) of the Theorem. $\square$

Remark 6.13. — We do not necessarily have $\Lambda U_0 \Lambda^{-1} = U_0$.

In the second case (2)

$$\Gamma = \; < U_0, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} >,$$

the quotient $\mathbb{C}/\Gamma$ exists and the map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is the quotient morphism. So $\Lambda U_0 \Lambda^{-1} \subset U_0$. therefore $\lambda = d \in \mathbb{Z}$ and the map $\varphi^{\circ n} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defining the dynamical system is the Chebyshev polynomial of degree $|d|$. We notice here that d and $-d$ give the same Chebyshev polynomial.

Subcase II.3. — *Rank* $U_0 = 2$.

Since the meromorphic function $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is Γ-invariant and hence $U_0 = U \cap \Gamma$- invariant. So the quotient $\mathbb{C}/U_0$ is an elliptic curve. The quotient group Γ/U_0 operates on the elliptic curve as automorphisms of the algebraic group if we choose appropriately the element e of the elliptic curve. Since e is a fixed point of the Γ/U_0, the point e is not the image of the origin 0 of the complex plain $\mathbb{C}$ by the quotient map $\mathbb{C} \to \mathbb{C}/U_0$. In other words, the multiplication by λ on $\mathbb{C}$ defines a isomorphism $E \to E$ but this isomorphism is not an isomorphism of the algebraic group (E, e). We know that Γ/U_0 is a finite group and they are classified. Anyhow the quotient $\mathbb{C}/\Gamma$ exists and we argue as in Subcase II. 2 to conclude that $\varphi^{\circ n}$ is one of the examples of Lattès.

We have so far shown that $\varphi^{\circ n}$ is one of the maps of the condition 2. We have to prove that φ itself is so. We may assume that φ is not of finite order automorphism. Othe cases are treated in a similar way, we prove that if $\varphi^{\circ n}$ is a Lattès, then φ is itself a Lattès map. We may assume that we are in the situation of Subcase II.3, We keep the notation there. There exists a fixed point $q \in \mathbb{P}^1_{\mathbb{C}}$ of φ. We need a simple

Lemma 6.14. — *Let U be a small open disc centered at the origin* 0 *of the complex plane $\mathbb{C}$ and let $g : U \to U$ be a non constant holomorphic map such that $g(0) = 0$. Let W be another open disc centered at a point q in $\mathbb{C}$ and $h : W \to V$ be a non constant holomorphic map such that $h(q) = 0$. Then there exists a holomorphic map $\tilde{g} : W' \to W$ defined in a neighborhood W' of the point q on W such that $\tilde{h}(g) = q$ and such that $g \circ h = h \circ \tilde{g}$. In other words, we can lift $g : U \to U$ over W locally around the point q.*

We may assume that W is also centered at 0 and there exists an integer $l \geqslant 1$ such that $h(s) = s^l$, s being the coordinate on W. Let t be the coordinate on U so that we can write $g(t) = t^n g_1(t)$ for every $t \in U$, where n is a positive integer and $g_1(t)$ is holomorhic in t with $g_1(0) \neq 0$. So we can find locally at the origin a holomorphic function $g_2(t)$ such that $g_2(t)^l = g_1(t)$. It is sufficient to set $\tilde{g}(s) = s^n g_2(s^l)$.

Let us come back to the proof. Since φ is of degree $\geqslant 2$, there exists a fixed point $q \in \mathbb{P}^1_{\mathbb{C}}$ of φ so that $\varphi(q) = q$. Since $\varphi^{\circ n}$ is a Lattès map, the quotient map $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is surjective. So we can find a point $q' \in \mathbb{C}$ such that $F(q') = q$. We apply Lemma 6.14 to conclude that locally around the point q', there exists a holomorphic function $\psi : W \to \mathbb{C}$ defined in a neighborhood of $q' \in \mathbb{C}$ such that $\varphi \circ F = F \circ \psi$ on the neighborhood W of q'. Since φ satisfies the differential equation $G_3(\nu)$, ψ satisfies $F^*G_3(\nu) = G_3(0)$ so that the Schwarzian $S(\psi) = 0$. Namely we can extend ψ globally as a Möbius transformation of $\mathbb{C}$. We denote the extension of ψ by Λ_1.

Lemma 6.15. — *For an element $\gamma \in \Gamma$, there exists an element $\gamma' \in \Gamma$ such that*

$$\gamma' \circ \Lambda_1 = \Lambda_1 \circ \gamma. \tag{6.10}$$

Let w be a general point of $\mathbb{C}$. We have $F(w) = F(\gamma w)$ so that

$$F(\Lambda(w)) = \varphi F(w) = \varphi F(\gamma w) = F(\Lambda\gamma w).$$

Since $F : \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ is the quotient map, this implies there exists an element $\gamma' \in \Gamma$ such that $\gamma \circ \Lambda_1 = \gamma' \circ \Lambda_1$.

Using the matrix representation, let

$$\Lambda_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C}).$$

We show that $c = 0$ and hence Λ_1 is an affine transformation. In fact, let us suppose in Lemma 6.15 $\gamma \neq I_2$ and consequently $\gamma' \neq I_2$, and write them in the matrix form

$$\gamma = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \qquad \gamma' = \begin{bmatrix} 1 & u' \\ 0 & 1 \end{bmatrix}$$

so that $u, u' \neq 0$. Then the condition (6.10) implies $c = 0$. Now we can follow all the arguments of Subcase II.3 replacing Λ by Λ_1 to conclude that φ is a Lattès map. This completes the proof of the Theorem.

Corollary 6.16. — *Let $\varphi(c, x) = x^2 - c$ with $c \in \mathbb{C}$ be a family of polynomials of degree 2. The following conditions on the complex number c are equivalent.*

(1) *The dynamical system $(\mathbb{P}^1_{\mathbb{C}}, \varphi(c, x))$ is solvable.*
(2) *The complex number c is equal to 0 or 2.*

Proof. — In fact, it is sufficient to notice that a dynamical system $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by a polynomial $f(x)$ of degree 2 is birationally equivalent to one of the dynamical systems in 2, (b) of Theorem if and only if the polynomial $f(x)$ is birationally equivalent to either the monomial x^2 or the Chebyshev polynomial $2x^2 - 1$ of degree 2. □

Remark 6.17. — For all the other values of c, the formal group Inf-gal $(\mathbb{P}^1_{\mathbb{C}}, \varphi(c,x))$ is isomorphic to $\Gamma_{1L^\natural}$ introduced in section 1 of the first part, or isomorphic to the formal group of all the coordinate transformations of 1-variable. So the Lie algebra of the formal group Inf-gal$(\mathbb{P}^1_{\mathbb{C}}, \varphi(c,x))$ is infinite dimensional and simple $L^\natural$-Lie algebra. In fact, it is well-known since Lie [6] that a Lie-Ritt sub-functor F of $\Gamma_{1L^\natural}$ coincides with the whole functor $\Gamma_{1L^\natural}$ if $\dim_{L^\natural} F \geqslant 4$.

Corollary 6.18 (Corollary to proof). — *The following condition on a rational map $\varphi(x)$ is equivalent.*

(1) *There exists an integer $n \geqslant 1$ such that $\varphi^{\circ n}$ is birationally equivalent to a monomial (respectively a Chebyshef polynomial, or a Lattès map).*
(2) *The rational map φ is birationally equivalent to a monomial (respectively a Chebyshev polynomial or a Lattès map).*

We proved this equivalence in the last part of the Proof of Theorem. The monomial case is trivial and the Chebyshev case is known (*cf.* [9], Theorem 6.9 (b)). The Lattès case seems new.

7. Concluding remarks

The proof of Theorem 6.5 is transcendental. So it is natural to expect an algebraic proof.

Question 7.1. — *Let (V, φ) be a discrete dynamical system over an algebraic curve V defined over a field C of characteristic* 0 *so that*

$$\varphi : V \to V$$

is a dominant C-rational map. Then, are the following conditions 1, 2 and 3 for the dynamical system (V, φ) equivalent?

(1) *The dimension of the Galois group*

$$\text{Inf-gal}((C(V), \varphi^*)/(C, \mathrm{Id}_C))$$

is finite.

(2) *The dynamical system (V, φ) is birationally equivalent to one of the following dynamical systems.*
   (a) *The endomorphism $\varphi : V \to V$ is an automorphism of finite order.*
   (b) *The dynamical systems of generalized Chebyshev-Lattès type.*

(3) *The dynamical system* (V, φ) *is infinitesimally solvable.*

The question seems plausible under an additional assumption that the field C is algebraically closed.

PROBLEM. — *It is challenging to explore a characteristic* $p > 0$ *version of Theorem 6.5 according to the idea that we suggested in* [11]. *We have to introduce the algebraic D-groupoid in characteristic* $p > 0$ *using iterative higher derivations of Hasse-Schmidt. Then it seems that we need algebraic D-semigroupoids more than D-groupoids. For, we can not invert a rational map even locally for the étale topology due to inseparability of the rational map.*

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Manuscrit reçu le 23 avril 2009,
accepté le 20 septembre 2009.

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