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MODULI SPACES FOR LINEAR DIFFERENTIAL EQUATIONS AND THE PAINLEVÉ EQUATIONS

by Marius VAN DER PUT & Masa-Hiko SAITO (*)

Abstract. — A systematic construction of isomonodromic families of connections of rank two on the Riemann sphere is obtained by considering the analytic Riemann–Hilbert map $RH : \mathcal{M} \to \mathcal{R}$, where $\mathcal{M}$ is a moduli space of connections and $\mathcal{R}$, *the monodromy space*, is a moduli space for analytic data (*i.e.*, ordinary monodromy, Stokes matrices and links). The assumption that the fibres of RH (*i.e.*, the isomonodromic families) have dimension one, leads to ten moduli spaces $\mathcal{M}$. The induced Painlevé equations are computed explicitly. Except for the Painlevé VI case, these families have irregular singularities. The analytic classification of irregular singularities yields explicit spaces $\mathcal{R}$, which are families of affine cubic surfaces, related to Okamoto–Painlevé pairs. A weak and a strong form of the Riemann–Hilbert problem is treated. Our paper extends the fundamental work of Jimbo–Miwa–Ueno and is related to recent work on Painlevé equations.

Résumé. — Une construction systématique des familles isomonodromiques de connections de rang 2 sur la sphère de Riemann est obtenue de l'application analytique de Riemann–Hilbert $RH : \mathcal{M} \to \mathcal{R}$, où $\mathcal{M}$ est un espace de modules de connections et $\mathcal{R}$ est un espace de modules pour les données analytiques (*i.e.*, la monodromie usuelle, les matrices de Stokes et les "links"). La condition que les fibres de RH (*i.e.*, les familles isomonodromiques) sont de dimension un mène à dix espaces de modules $\mathcal{M}$. L'équation induite de Painlevé est calculée explicitement. À l'exception du cas Painlevé VI, les familles ont des singularités irrégulières. Utilisant la classification des singularités irrégulières, on obtient les espaces $\mathcal{R}$ comme familles explicites de surfaces affines cubiques liées aux pairs de Okamoto–Painlevé. Une forme faible et une forme forte du problème de Riemann–Hilbert sont démontrées. Notre article est une extension du travail fondamental de Jimbo-Miwa-Ueno et est en relation avec des travaux récents sur les équations de Painlevé.

Keywords: Moduli space for linear connections, irregular singularities, Stokes matrices, monodromy spaces, isomonodromic deformations, Painlevé equations.
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Introduction

The theme of this paper is a systematic construction of the ten isomonodromic families of connections of rank two on $\mathbb{P}^1$ inducing Painlevé equations. They are obtained by considering the complex analytic Riemann–Hilbert morphism $RH : \mathcal{M} \to \mathcal{R}$ from a moduli space $\mathcal{M}$ of connections to a categorical moduli space of analytic data (*i.e.,* ordinary monodromy, Stokes matrices and links) $\mathcal{R}$, here called the *monodromy space*. The fibres of RH are the isomonodromic families. One requires that an isomonodromic family has dimension 1, since it is then (locally) parametrized by one variable t and some combination $q(t)$ of the entries of the connection is a potential solution of some second order Painlevé equation. This condition leads to the ten families. Our method extends the work of Jimbo, Miwa and Ueno [17, 16], since we allow all possible irregular singularities including ramification and resonance.

There is a natural morphism $\mathcal{R} \to \mathcal{P}$, where $\mathcal{P}$ is a parameter space build from traces of matrices. For each of the ten families, the morphism $\mathcal{R} \to \mathcal{P}$ turns out to be a family of affine cubic surfaces with three lines at infinity. We will give explicit equations of $\mathcal{R}$ for these ten families in § 2 and § 3. The equation for Painlevé VI is classical [5, 14], and the equations for the other nine families seem to be new.

Since many aspects of the well known family with four regular singularities leading to Painlevé VI, has been studied in great detail ([2, 12, 13, 11, 14]), our emphasis will be on families with irregular singularities. Of the nine families with irregular singularities, six are again classical [16, 4]. The three remaining ones were also recently discovered in [21, 20]. The corresponding Painlevé equations appear already in [27] from the viewpoint of the Okamoto–Painlevé pairs.

The moduli spaces of connections $\mathcal{M}$ are strongly related to the Okamoto–Painlevé pairs (S, Y) of non fibre type [27, 25]. The latter determine uniquely each type of Painlevé equation [25]. We will give a brief description of this relation.

The surface S is the blow up of nine points (allowing for infinitely near points) in $\mathbb{P}^2$ (or equivalently eight points in the Hirzebruch surface Σ_2) which lie on an effective anti-canonical divisor of $\mathbb{P}^2$ or Σ_2. Let Y be the unique effective anti-canonical divisor of S. The Okamoto–Painlevé condition on Y implies that Y has the same configuration as a degenerate elliptic curve in the classification by Kodaira–Néron [22, 27, 25].

The configuration of the irreducible components of Y for the Okamoto–Painlevé pairs are given by the eight extended Dynkin diagrams

$$\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8,\ \tilde{E}_6,\ \tilde{E}_7,\ \tilde{E}_8.$$

Each Dynkin diagram gives rise to a (uni)versal global family provided with a unique vector field which induces a Painlevé equation [25].

One conjectures that a relative compactification of each of the ten families of connections $\pi : \mathcal{M} \to T \times \Lambda$ with parameter space $T \times \Lambda$, is isomorphic to one of the above global (uni)versal families. As a consequence of this conjecture, the fibres of π are the complement $S \setminus Y$ for a certain Okamoto–Painlevé pair (S, Y) of the given type. The conjecture has been proven for Okamoto–Painlevé pairs of type $\tilde{D}_4$, which corresponds to Painlevé VI. (For the construction of the moduli spaces of linear connections with only regular singularities and the Riemann-Hilbert correspondence for these, see [12, 13, 10]).

There is an explicit analytic morphism $\Lambda \to \mathcal{P}$, given by exponentials, which is compatible with the Riemann–Hilbert morphism $RH : \mathcal{M} \to \mathcal{R}$. The monodromy space $\mathcal{R} \to \mathcal{P}$ can have, as fibre, a singular (affine) cubic surface $\mathcal{R}_p$. As is conjectured and proved for the PVI case, the Riemann–Hilbert morphism yields an analytic resolution $(S \setminus Y) \to \mathcal{R}_p$. The singular points of type A_1, A_2, A_3, D_4 on the cubic surface yield 1, 2, 3 and 4 exceptional curves on $S \setminus Y$ which are called Riccati curves. The latter are related to Riccati solutions of the corresponding Painlevé equation. Since the Riccati curves on the Okamoto–Painlevé pairs are known ([26]), one can now link each of the ten monodromy spaces $\mathcal{R}$ to an Okamoto–Painlevé pair and an extended Dynkin diagram (see Table 2.1). We remark, as done in [21], that for the case $\tilde{D}_6$ there are two types of isomonodromic families corresponding to $\mathrm{PV}_{\mathrm{deg}}$ and $\mathrm{PIII}(D_6)$. The same holds for $\tilde{E}_7$.

In Section 4, a Zariski open set of the moduli space $\mathcal{M}$ of connections is described for each of the ten families. The corresponding isomonodromic equation produces an explicit Painlevé equation, confirming the statements of Table 2.1.

The contents of this paper is the following. The first section deals with the formal and analytic data attached to a differential module M over $\mathbb{C}(z)$. The connections on $\mathbb{P}^1$ inducing given formal and analytic data are studied. A weak and a strong form of the Riemann-Hilbert problem is treated. This result is also obtained by [3] in a slightly different setting.

In Section 2, "good" families of connections on $\mathbb{P}^1$ are described and studied. The monodromy space $\mathcal{R}$ is defined as a categorical quotient of the analytic data.

Then the ten families where the fibres of $RH : \mathcal{M} \to \mathcal{R}$ have dimension 1 are computed. The third Section contains the computation of the ten monodromy spaces $\mathcal{R} \to \mathcal{P}$ and the singularities of the fibres.

A theory of apparent singularities q is developed in Section 4. This is essential for the computation of the second order equation $q'' = R(q, q', t)$ (where R is a rational function of q', q, t) of the Painlevé type and of a corresponding symplectic structure with canonical coordinates p, q and a Hamiltonian equation. We obtain explicit Hamiltonian systems and explicit Painlevé equations for the nine families (see Subsections 4.3–4.11) which are natural from the view point of the Okamoto–Painlevé pairs. The explicit forms of equations depend on the choice of a cyclic vector, the choice of the parameter t and choices for the constants in the monodromy space. Though we will not tune up these data such that our explicit forms coincide with the classical Painlevé equations as in [7, 23, 16], one can transform one to the other by some birational transformation of coordinates. Most of these computations in Section 4 were made using *Mathematica*.

1. Singularities of a differential module

1.1. Summary

Let M be a differential module over $K = \mathbb{C}(z)$. The formal data (generalized local exponents, formal monodromy), and the analytic data (monodromy, Stokes matrices, links) of M are described. The weak form of the Riemann-Hilbert problem for arbitrary singularities has the positive answer:

Theorem 1.1. — *For given formal and analytic data, there exists a unique (up to isomorphism) differential module M inducing these data.*

A strong form of the Riemann–Hilbert problem is:

Theorem 1.2. — *Suppose that M is irreducible and has at least one (regular or irregular) singular point which is unramified. Then there is a connection $(\mathcal{V}, \nabla)$ on $\mathbb{P}^1$ representing M, such that $\mathcal{V}$ is free (i.e., a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}$) and the poles of the connection ∇ have the minimal order derived from the Katz invariant.*

Results concerning invariant lattices are developed for the proof of Theorem 1.2. That the strong Riemann–Hilbert problem has a negative answer if all the singularities of M are ramified, is shown by families of examples related to Painlevé equations.

Bolibruch's work [1] on the strong form of the Riemann–Hilbert problem is extended in the paper [3] to the case of irregular singularities. Our Theorems 1.1 and 1.2 clarify and supplement [3], by introducing links.

For the convenience of the reader, the useful compact way to describe the formal and the analytic singularities of differential modules (see [24] for more details) is explained in the next subsections. Explicit examples are given which will be used in the calculations for the monodromy spaces and the Painlevé equations.

1.2. The formal classification

This is the classification of differential modules $M = (M, \delta)$ over the differential field of the formal Laurent series $\mathbb{C}((t))$ (here t is the local parameter) , due to M. Hukuhara [9] and H. Turrittin [30]. For notational convenience we will use the derivation $t\frac{d}{dt}$ on $\mathbb{C}((t))$. The $\mathbb{C}$-linear map $\delta : M \to M$ has, by definition, the property $\delta(fm) = t\frac{df}{dt} \cdot m + f \cdot \delta(m)$ for $f \in \mathbb{C}((t)),\ m \in M$.

The module M is called *regular singular* (this includes regular) if there is an invariant lattice $\Lambda \subset M$, *i.e.,* $\Lambda \subset M$ is a free $\mathbb{C}[[t]]$-submodule containing a basis of M such that $\delta(\Lambda) \subset \Lambda$. A regular singular M has a basis $e_1, \dots, e_d$ such that the vector space $W := \oplus_{i=1}^{d} \mathbb{C}e_i$ is invariant under δ and such that the distinct eigenvalues $\lambda_1, \dots, \lambda_s$ (with $1 \leqslant s \leqslant d$) of δ acting on W satisfy $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. Using this basis the operator δ on M obtains the form $t\frac{d}{dt} + A$, where A is the matrix of δ operating on W. The λ_i are called the *local exponents*. These are only unique up to integers. The *(formal) monodromy* matrix is (up to conjugation) $e^{2\pi i A}$.

Clearly $\Lambda := \oplus_{i=1}^{d} \mathbb{C}[[t]]e_i$ is an invariant lattice. The *non resonant case* is defined by $s = d$, *i.e.,* the matrix A is diagonalizable and its eigenvalues $\lambda_1, \dots, \lambda_d$ satisfy $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. In the non resonant case the collection of all invariant lattices is $\{\oplus_{i=1}^{d} \mathbb{C}[[t]]t^{n_i}e_i \mid n_1, \dots, n_d \in \mathbb{Z}\}$ and the formal monodromy has d distinct eigenvalues.

The *solutions* of a regular singular module M, say, represented in matrix form $t\frac{d}{dt} + A$, are vectors v with $(t\frac{d}{dt} + A)v = 0$. One needs the following differential ring extension $\mathrm{Univ}_{rs} := \mathbb{C}((t))[\{t^a\}_{a \in \mathbb{C}}, \ell]$ of $\mathbb{C}((t))$ to obtain the vector space V of all solutions. The *symbols* t^a, ℓ are defined by the rules $t^a \cdot t^b = t^{a+b}$, t^1 coincides with $t \in \mathbb{C}((t))$. Further $t\frac{d}{dt}t^a = at^a,\ t\frac{d}{dt}\ell = 1$. The intuitive meaning of these symbols is rather clear: t^a stands for $e^{a \log t}$

and ℓ for $\log t$. Because these functions are multivalued, they are replaced by symbols.

Then V consists of the vectors v with coordinates in Univ_{rs} satisfying $(t\frac{d}{dt}+A)v=0$. In other words, $V=\{v\in \mathrm{Univ}_{rs}\otimes_{\mathbb{C}((t))}M \mid \delta(v)=0\}$. The ring Univ_{rs} has a $\mathbb{C}((t))$-linear differential automorphism γ, defined by $\gamma t^a=e^{2\pi i a}t^a,\ \gamma\ell=\ell+2\pi i$. Now γ induces on automorphism $\gamma\otimes id$ on $\mathrm{Univ}_{rs}\otimes M$, commuting with δ. Then V is invariant under γ and the restriction of γ to V, again written as γ or γ_V, is the formal monodromy. From the pair (V,γ_V) one recovers the differential module (M,δ_M) as the $\mathbb{C}((t))$-vector space of the γ-invariant elements of $\mathrm{Univ}_{rs}\otimes_{\mathbb{C}}V$. On the last space the operator δ is defined by $\delta(u\otimes v)=\delta(u)\otimes v$ for $u\in\mathrm{Univ}_{rs},\ v\in V$. The restriction of this δ to M is the δ_M. The above describes an equivalence between the category of the regular singular differential modules and the category of the pairs (V,γ) consisting of a finite dimensional vector space V and an $\gamma\in\mathrm{GL}(V)$. This equivalence respects all constructions of linear algebra, in particular tensor products.

This maybe somewhat abstract way to deal with regular singular differential modules extends to the case of *irregular singular* differential modules. It greatly simplifies the various classical classification results.

A typical example of an irregular singular module is the one-dimensional module $M=\mathbb{C}((z))e$ with $\delta e=(a+q)e$ with $q\in t^{-1}\mathbb{C}[t^{-1}],\ q\neq 0,\ a\in\mathbb{C}$. We call $a+q$ the *(generalized) local exponent* and q the *eigenvalue*. One observes that q is unique and a is unique up to a shift over an integer.

A more complicated example is the following. For any integer $n\geqslant 1$ we consider the field extension $\mathbb{C}((t^{1/n}))$ of degree n and an element $a+q\in\mathbb{C}+(t^{-1/n}\mathbb{C}[t^{-1/n}])$. Then we define the differential module $\mathbb{C}((t^{1/n}))e$ of rank one over $\mathbb{C}((t^{1/n}))$ by $\delta(e)=(a+q)e$. Now M is equal to this object, seen as a differential module over the field $\mathbb{C}((t))$. This module has dimension n. From these examples and the regular singular differential modules one can build, by constructions of linear algebra, all differential modules. In order to have solutions for all differential modules over $\mathbb{C}((t))$ we have to introduce new symbols $e(q)$ for $q\in\mathcal{Q}:=\bigcup_{n\geqslant 1}t^{-1/n}\mathbb{C}[t^{-1/n}]$. The rules are $t\frac{d}{dt}e(q)=q\cdot e(q)$ and $e(q_1)e(q_2)=e(q_1+q_2)$. One obtains the differential ring extension $\mathrm{Univ}:=\oplus_{q\in\mathcal{Q}}\mathrm{Univ}_{rs}\cdot e(q)$, equipped with the differential automorphism γ, extending the γ on Univ_{rs} by $\gamma e(q)=e(\gamma q)$. The meaning of $\gamma(q)$ is already defined since $\gamma(t^a)=e^{2\pi i a}t^a$ for any $a\in\mathbb{C}$. The intuitive meaning of $e(q)$ is rather evident, namely $e^{\int q\frac{dt}{t}}$. Since the latter is a multivalued function we avoid its use and use the symbol $e(q)$ instead.

The solution space V of a differential module M, say, represented by the matrix equation $t\frac{d}{dt}+A$ where A is a $d\times d$-matrix with coefficients in $\mathbb{C}((t))$, is defined as $V=\{v\in(\mathrm{Univ})^d \mid (t\frac{d}{dt}+A)(v)=0\}$. In other words $V=\{v\in \mathrm{Univ}\otimes_{\mathbb{C}((t))}M \mid \delta(v)=0\}$. As before, there is an action of γ on V. Moreover V has a direct sum decomposition $V=\oplus_{q\in\mathcal{Q}}V_q$ where $V_q:=\{v\in \mathrm{Univ}_{rs}\cdot e(q)\otimes_{\mathbb{C}((t))}M \mid \delta(v)=0\}$. As the dimension of V is finite (equal to $\dim_{\mathbb{C}((t))}M$), almost all V_q are 0. Clearly $\gamma(V_q)=V_{\gamma(q)}$. Thus we have attached to M a tuple $(V,\{V_q\},\gamma)$ consisting of a finite dimensional vector space V over $\mathbb{C}$ and subspaces V_q with $V=\oplus_{q\in\mathcal{Q}}V_q$ and an element $\gamma\in\mathrm{GL}(V)$ such that $\gamma(V_q)=V_{\gamma(q)}$ for all q. From this tuple one can recover (M,δ_M) as the $\mathbb{C}((t))$-vector space of the γ-invariant elements of $\oplus_{q\in\mathcal{Q}}\mathrm{Univ}_{rs}\cdot e(-q)\otimes_{\mathbb{C}}V_q$. By definition δ acts as zero on V and thus induces δ_M. In fact, $M\mapsto(V,\{V_q\},\gamma)$ defines an equivalence of categories commuting with all operations of linear algebra, and in particular with tensor product. Our formal classification is that of the tuples $(V,\{V_q\},\gamma)$.

The elements q with $V_q\neq 0$ are called the *eigenvalues* and γ, acting on V, is called the *formal monodromy*. The *Katz invariant* $r(M)$ of M is the maximum of the degrees in t^{-1} of the eigenvalues q.

Example 1.3. — We *illustrate* the above by classifying all differential modules M of dimension 2 such that Λ^2M is isomorphic to the trivial module $\mathbf{1}:=\mathbb{C}((t))e$ with $\delta e=0$. The possibilities for the tuple $(V,\{V_q\},\gamma)$ are:

(i) $V=V_0$ and $\gamma\in\mathrm{SL}(V)$. This is the *regular singular* case. By taking a logarithm $2\pi iA$ of γ one obtains the matrix equation $t\frac{d}{dt}+A$.

(ii) $V=V_q\oplus V_{-q}$ with $q=a_1t^{-r}+\cdots+a_rt^{-1}$ and $a_1\neq 0$.This is the *unramified irregular* case with eigenvalues $\pm q$ and Katz invariant r. Give the spaces V_q,V_{-q} a basis e_1 and e_2. Then the matrix of γ has the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$. A corresponding matrix differential equation can be written as $t\frac{d}{dt}+\begin{pmatrix} q+a & 0\\ 0 & -q-a\end{pmatrix}$ with $e^{2\pi ia}=\alpha$.

(iii) For the *ramified irregular* case $V=V_q\oplus V_{-q}$ with Katz invariant r, one must have $r=\frac{1}{2}+m,\ m\in\mathbb{Z},\ m\geqslant 0$ and $q=t^{1/2}(a_1t^{-r-1/2}+\cdots+a_*t^{-1}),\ a_1\neq 0$. This follows from $\gamma(q)=-q$. Consider a basis b_1 and b_2 for V_q and V_{-q} such that $\gamma(b_1)=b_2$. Then $\gamma(b_2)=-b_1$ since $\gamma\in\mathrm{SL}(V)$. For the computation of the corresponding differential module, it is easier to compute first the invariants under γ^2. This yields a differential module $N=\oplus_{i=1}^2\mathbb{C}((t^{1/2}))e_i$ over $\mathbb{C}((t^{1/2}))$ with $\delta(e_1)=qe_1,\ \delta(e_2)=-qe_2$. The element γ acts on N

by $\gamma e_1 = e_2,\ \gamma e_2 = e_1$ and $\gamma t^{1/2} = -t^{1/2}$. The module M of the invariants under γ has the basis $f_1 = e_1+e_2,\ f_2 = t^{1/2}(e_1-e_2)$. Write $q = t^{1/2}h$. Then δ on the basis f_1, f_2 yields the matrix differential equation $t\frac{d}{dt} + \begin{pmatrix} 0 & th \\ h & \frac{1}{2} \end{pmatrix}$.

Definition and examples 1.4. — *Invariant lattices.*

Let the differential module $M = (M, \delta)$ have Katz invariant r and let r^+ denote the smallest integer $\geqslant r$. A free submodule $\Lambda \subset M$ over $\mathbb{C}[[t]]$, containing a basis of M is usually called a *lattice*. We say that Λ is an *invariant lattice* if, moreover $t^{r^+}\delta\Lambda \subset \Lambda$. There exists an invariant lattice, in fact the *standard lattice* defined in [24], p. 307 and p. 311, is an invariant lattice. For later use we will compute all invariant lattices for the items of Example 1.3.

(i) If the regular singular module M is *non resonant*, then M has a basis e_1, e_2 with $\delta e_1 = \frac{\theta}{2}e_1,\ \delta e_2 = -\frac{\theta}{2}e_2$ and $\theta \notin \mathbb{Z}$. The notation $\frac{\theta}{2}$ is chosen for historical reasons. The invariant lattices are $\mathbb{C}[[t]]t^{n_1}e_1 + \mathbb{C}[[t]]t^{n_2}e_2$ for any $n_1, n_2 \in \mathbb{Z}$.

A typical *resonant* case is $M = \mathbb{C}((t))e_1 + \mathbb{C}((t))e_2$ with $\delta e_1 = e_2,\ \delta e_2 = 0$. The invariant lattices are $\mathbb{C}[[t]]t^{n_1}e_1 + \mathbb{C}[[t]]t^{n_2}e_2$ with $n_1, n_2 \in \mathbb{Z}$ and $n_1 \geqslant n_2$.

(ii) M has a basis e_1, e_2 with $\delta e_1 = (q+a)e_1,\ \delta e_2 = -(q+a)e_2$. In this case $r^+ = r$ and the invariant lattices are $\mathbb{C}[[t]]t^{n_1}e_1 + \mathbb{C}[[t]]t^{n_2}e_2$ for any $n_1, n_2 \in \mathbb{Z}$.

(iii) M has basis f_1, f_2 with $\delta f_1 = hf_2,\ \delta f_2 = thf_1 + \frac{1}{2}f_2$. Now $r^+ = r + \frac{1}{2}$. The invariant lattices are only the lattices $t^n \cdot (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)$ and $t^n \cdot (\mathbb{C}[[t]]tf_1 + \mathbb{C}[[t]]f_2)$ where $n \in \mathbb{Z}$.

We omit the easy proofs for (i) and (ii). *The proof of case* (iii):

Consider the operator $\Delta = h^{-1}\delta$ on M. Thus $\Delta f_1 = f_2,\ \Delta f_2 = tf_1 + \frac{1}{2h}f_2$ and $\Delta(fm) = (h^{-1}t\frac{df}{dt}) \cdot m + f \cdot \Delta(m)$.

A lattice Λ is invariant if and only if $\Delta\Lambda \subset \Lambda$. If Λ is an invariant lattice then also $t^n \cdot \Lambda$ for any $n \in \mathbb{Z}$. The lattices generated by f_1, f_2 and by tf_1, f_2 are clearly invariant. Let Λ be any invariant lattice. After multiplication by some power of t we may suppose that $\Lambda \subset (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)$ and not contained in $t \cdot (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)$. If $\Lambda = \mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2$, then we are finished. If not we consider the invariant lattice $\Lambda + t \cdot (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)$. Since Δ induces on $(\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)/t \cdot (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2)$ a nilpotent map with only one proper invariant subspace, namely generated by the image of f_2, we have that $\Lambda + t \cdot (\mathbb{C}[[t]]f_1 + \mathbb{C}[[t]]f_2) = (\mathbb{C}[[t]]tf_1 + \mathbb{C}[[t]]f_2)$. It follows that Λ contains an element of the form $af_1 + f_2$ for some $a \in t\mathbb{C}[[t]]$. Now

$$\Delta(af_1 + f_2) - \Big(a + \frac{1}{2h}\Big)(af_1 + f_2) = \Big(t + h^{-1}t\frac{da}{dt} - \Big(a + \frac{1}{2h}\Big)a\Big)f_1 \in \Lambda.$$

Thus $tf_1 \in \Lambda$ and also $f_2 \in \Lambda$. Then $\Lambda = \mathbb{C}[[t]]tf_1 + \mathbb{C}[[t]]f_2$.

Comment. Two lattices Λ_1, Λ_2 in $\mathbb{C}((t))^2$ are called *equivalent* if there exists an integer n with $\Lambda_1 = t^n \cdot \Lambda_2$. Two classes of lattices $[\Lambda_1],\ [\Lambda_2]$ form an edge if the representatives Λ_1, Λ_2 can be chosen such that there are proper inclusions $t \cdot \Lambda_1 \subset \Lambda_2 \subset \Lambda_1$. One obtains a tree with vertices the classes of lattices and edges as above. If one replaces $\mathbb{C}$ by a finite field then this object is the well known Bruhat-Tits tree.

The classes of the invariant lattices form a subset of this tree. This subset is a line for the first case of (i) and a half line for the second case of (i). In case (ii), it is again a line and in case (iii) this subset consists of two vertices which form an edge.

1.3. The analytic classification

This is the classification of the differential modules M over the field of the convergent Laurent series $\mathbb{C}(\{t\})$. Again we use the derivation $t\frac{d}{dt}$. One associates to M the formal differential module $\widehat{M} = \mathbb{C}((t)) \otimes M$. If $\widehat{M}$ is *regular singular*, then one calls M also regular singular. In that case there exists also a basis $e_1, \ldots, e_d$ of M such that $W := \oplus_{i=1}^{d} \mathbb{C}e_i$ is invariant under δ and the distinct eigenvalues of the matrix A of δ on W do not differ by an integer. Then M is isomorphic to the module corresponding to the matrix differential operator $t\frac{d}{dt} + A$. The usual *topological monodromy* around $t = 0$ coincides with the formal monodromy and that ends the classification.

If $\widehat{M}$ is *irregular singular*, then it induces a tuple $(V, \{V_q\}, \gamma)$. The *singular directions* $d \in \mathbb{R}$ of M depend only on $\widehat{M}$ and are defined as follows. Let $q_1, \ldots, q_s$ denote the eigenvalues of $\widehat{M}$. A direction $d \in \mathbb{R}$ is singular for $q_i - q_j$ (with $i \neq j$) if the function $\exp(\int (q_i - q_j)\frac{dt}{t})$ has "maximal descent" for $r \to 0$ on the half line $t = re^{\mathrm{id}}$. More explicitly, if $q_i - q_j = \alpha_k t^{-k} + \cdots + \alpha_1 t^{-1},\ \alpha_k \neq 0$, then d is a singular direction if and only if $\alpha_k r e^{-idk}$ is a positive real number. The collection of the singular directions is the union over $i \neq j$ of the singular directions of $q_i - q_j$.

If a direction d is non singular, then there is a functorial map mults_d, the *multisummation in the direction* d, which maps the (symbolic) solution space V to the space of the actual solutions $V(S)$ in a certain sector S

at $t = 0$ around d. For every $v \in V$ the element $\mathrm{mults}_d(v)$ has v as its asymptotic expansion.

For each singular direction d, there is an analytic object, namely the Stokes map $St_d \in \mathrm{GL}(V)$. The Stokes map St_d is defined by comparing the multisummation at directions $d^- < d < d^+$ close to d. More precisely $\mathrm{mults}_{d^+} \circ St_d = \mathrm{mults}_{d^-}$. The map St_d is unipotent and has the form $\mathrm{Id} + \sum_{i,j} L_{i,j}$ where the sum is taking over all pairs i, j such that d is singular for $q_i - q_j$ and where L_{ij} is a linear map from V_{q_i} to V_{q_j}. The isomorphy class of M induces a tuple $(V, \{V_q\}, \gamma, \{St_d\})$, where the St_d are described above and where moreover the relation $\gamma^{-1} St_d \gamma = St_{d+2\pi}$ holds.

The **main result** of the asymptotic analysis of irregular singularities is: *The category of the differential modules over $\mathbb{C}(\{t\})$ is equivalent to the category of the tuples $(V, \{V_q\}, \gamma, \{St_d\})$, satisfying the above properties. This equivalence respects all constructions of linear algebra, in particular the tensor product.*

An **important property** that we will use is:

Let $0 \leqslant d_1 < \cdots < d_s < 2\pi$ denote the singular directions in $[0, 2\pi)$. Then the topological monodromy around the singular point is conjugated to $\gamma \circ St_{d_s} \circ \cdots \circ St_{d_1}$.

We note that this conjugation depends on the way the solution space at a point close to the singular point $t = 0$ is identified with the (formal) solution space V. Now we illustrate the above by continuing Examples 1.3.

Example 1.5. — Let M be an irregular differential module of dimension 2 over $\mathbb{C}(\{t\})$ such that $\Lambda^2 M = \{\mathbf{1}\}$.

(ii) If $\widehat{M}$ is unramified with Katz invariant r, then $V = V_q \oplus V_{-q}$, $q \in t^{-1}\mathbb{C}[t^{-1}]$ has degree r in t^{-1}. We recall that γ has the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ on any basis e_1, e_2 of V such that $V_q = \mathbb{C}e_1$, $V_{-q} = \mathbb{C}e_2$. For $q - (-q)$ there are r singular directions (in $[0, 2\pi)$) and the same holds for $(-q) - q$. The two pairs of singular directions intertwine. For the first ones the Stokes matrices (w.r.t. the basis e_1, e_2) have the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, and for the second ones the form is $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Thus the Stokes matrices are given by $2r$ constants c_i and the topological monodromy around $t = 0$ is up to conjugation (and we may choose the order) equal to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ c_{2r-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_{2r} \\ 0 & 1 \end{pmatrix}.$$

The basis e_1, e_2 is not unique, whereas the 1-dimensional spaces V_q and V_{-q} are. If we want Stokes data, independent of the choice of e_1, e_2, then we have to divide the space $\mathbb{A}^{2r}$ of the tuples $(c_1, \ldots, c_{2r})$ by the action of the group $\mathbb{G}_m$. For this action the c_{2i} can be given weight $+1$ and the c_{2i-1} weight -1.

(iii) If $\widehat{M}$ is ramified, then there are again $2r$ singular directions in $[0, 2\pi)$ and Stokes matrices of the form $\left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0 \\ * & 1\end{smallmatrix}\right)$. The singular directions intertwine. We choose now a basis e_1, e_2 of V with $V_q = \mathbb{C}e_1,\ V_{-q} = \mathbb{C}e_2$ and $\gamma e_1 = e_2, \gamma e_2 = -e_1$. The topological monodromy around $t = 0$ is conjugated to the product

$$\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} \cdot \begin{pmatrix}1 & 0 \\ c_1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & c_2 \\ 0 & 1\end{pmatrix} \cdots \begin{pmatrix}1 & 0 \\ c_{2r} & 0\end{pmatrix}.$$

In this case one may change the basis e_1, e_2 only into $\lambda e_1, \lambda e_2$ with $\lambda \in \mathbb{C}^*$. This does not have an effect on the Stokes data $(c_1, \ldots, c_{2r})$ and no division by $\mathbb{G}_m$ is needed.

1.4. The data for global differential modules

By a global differential module we mean a differential module M over the field $K = \mathbb{C}(z)$. We investigate the data that will describe M.

The first case that we consider is classical, namely:

The position of the singular points $\{p_1, \ldots, p_r\}$ of M is fixed and all the singular points are supposed to be regular singular.

One introduces the monodromy for M in the usual way. That is, one chooses a base point $b \in \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$ and loops $\alpha_1, \ldots, \alpha_r$ around the singular points, generating the fundamental group $\pi_1 := \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}, b)$. There is only one relation, namely $\alpha_1 \cdots \alpha_r = 1$. Then M induces a monodromy homomorphism

$$\mathrm{mon}_M : \pi_1 \to \mathrm{GL}(V(b)),$$

where $V(b)$ denotes the solution space at b. We note that $\mathrm{mon}_M(\alpha_i)$ is conjugated to the local monodromy at p_i (formal or topological). A weak solution of the Riemann–Hilbert problem reads ([24], Thm 6.15):

Proposition 1.6. — *The functor $M \mapsto \mathrm{mon}_M$ from the category of the differential modules with regular singularities at $\{p_1, \ldots, p_r\}$ to the category of the finite dimensional complex representations of π, is an equivalence of categories. This equivalence respects all constructions of linear algebra, in particular tensor products.*

Notation. — For any point $p \in \mathbb{P}^1$ we introduce the local parameter t_p, which is $z - p$ if $p \in \mathbb{C}$ and z^{-1} for $p = \infty$. The field K_p is the field of the meromorphic functions at p, *i.e.*, $\mathbb{C}(\{t_p\})$ and $\widehat{K}_p$ is the completion of K_p, *i.e.*, $\mathbb{C}((t_p))$. Further $O_p \subset K_p$ and $\widehat{O}_p \subset \widehat{K}_p$ are the valuation rings, *i.e.*, $O_p = \mathbb{C}\{t_p\}$ and $\widehat{O}_p = \mathbb{C}[[t_p]]$.

One associates to a global differential module M with fixed singularities $\{p_1, \ldots, p_r\}$, the data: the isomorphy classes of the $\{K_{p_i} \otimes_K M\}$ and the monodromy representation mon_M as above. Now we give an example showing that *this is not sufficient for the reconstruction of M.*

Example 1.7. — Two singular points 0 and ∞, both irregular.

At both points we prescribe local analytic data for the differential module M. In other words, we prescribe the two analytic differential modules $M_0 = K_0 \otimes M$ and $M_\infty = K_\infty \otimes M$. As we will see in Observations 1.10, this leads to a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$, where $\mathcal{M}$ is a vector bundle and $\nabla : \mathcal{M} \to \Omega(k[0] + k[\infty]) \otimes \mathcal{M}$ (for some $k > 0$). The restrictions T_0 and T_1 of this connection to the two open sets $\mathbb{P}^1 \setminus \{0\}$ and $\mathbb{P}^1 \setminus \{\infty\}$ are known from the given data M_0 and M_∞. We suppose now that the topological monodromies of M_0 and M_∞ are trivial. Thus the restrictions $T_{0,1}$ and $T_{1,0}$ of T_0 and T_1 to the open set $\mathbb{P}^1 \setminus \{0, \infty\}$ are trivial. The glueing is given by an isomorphism $T_{0,1} \to T_{1,0}$. Let m denote the dimension of M. Then an isomorphism is given by a linear bijection $L : \mathbb{C}^m \to \mathbb{C}^m$. Further L is unique up to multiplication (on the left, respectively on the right) by an automorphism of M_0 and M_∞. One can easily produce M_0 and M_∞ which have only $\mathbb{C}^*$ as group of automorphisms. Therefore the possible connections $(\mathcal{M}, \nabla)$ and also the possible differential modules M are in bijection with $\mathrm{PGL}(m, \mathbb{C})$.

Definition 1.8. — *Links and the formal and analytic data.*

What is missing is a "link" between the solution space $V(b)$ at the base point b with the (symbolic) solution spaces $V(p_i)$ at the singular points. This idea goes back to the work of Jimbo–Miwa–Ueno [17]. *We make the following construction to remedy this.*

As before, $\alpha_1, \ldots, \alpha_r$ are loops starting at b around the singular points. For each p_i we choose a point p_i^* on the loop close to p_i and a line segment $[p_i^*, p_i]$ which is a non–singular direction at p_i. The "*link*" $L_i : V(b) \to V(p_i)$ is defined by analytic continuation from $V(b)$ to $V(p_i^*)$, followed by the inverse of the multisummation map $\mathrm{mult}_S : V(p_i) \to V(p_i^*)$ in the direction $[p_i^*, p_i]$ (seen as an element in $\mathbb{R}$). The role of the monodromy map mon_M is taken over by links, *i.e.*, the linear bijections $L_1, \ldots, L_r$. The multisummation $\mathrm{mult}_S : V(p_i) \to V(p_i^*)$ (in the direction $[p_i, p_i^*]$) is used to identify the two vector spaces. Then the local topological monodromy top_i, along a circle starting in p_i^*, is expressed as a product of the Stokes maps and the formal monodromy at p_i. The relation $\alpha_1 \cdots \alpha_r = 1$ translates into

$$L_r^{-1} \circ \mathrm{top}_r \circ L_r \dots L_2^{-1} \circ \mathrm{top}_2 \circ L_2 \circ L_1^{-1} \circ \mathrm{top}_1 \circ L_1 = 1.$$

The "*formal and the analytic data*" for M are defined as:

(1) The position of the singular points $p_1, \dots, p_r$;
(2) for each i, the formal structure $(V(p_i), \{V(p_i)_q\}, \gamma_i)$ at p_i;
(3) for each i, the Stokes maps at p_i;
(4) the links $L_i : W \to V(p_i)$.
(5) These data are supposed to satisfy the relation

$$L_r^{-1} \circ \mathrm{top}_r \circ L_r \dots L_2^{-1} \circ \mathrm{top}_2 \circ L_2 \circ L_1^{-1} \circ \mathrm{top}_1 \circ L_1 = 1.$$

Here W stands for the space $V(b)$. The *formal part* of the data is (1) (the position of the singular points) and the eigenvalues q at each singular point. The *analytic part* of the data is the direct sum decompositions $\oplus_q V(p_i)_q$ of the spaces $V(p_i)$, including the permutation of the $V(p_i)_q$ induced by γ; further (3) and (4), since this combines the links and the Stokes maps. We observe that these "formal and analytic data" are considered up to the automorphisms of W and of the $V(p_i)$.

One might use L_1 to identify W with $V(p_1)$ and then one is only left with links $L_i : V(p_1) \to V(p_i)$ for $i = 2, \dots, r$. Another way to reduce the number of links by one, is to choose as base point b the singular point p_1 and define links $L_i : V(p_1) \to V(p_i)$ for $i = 2 \dots, r$.

THEOREM 1.9. — *For given "formal and analytic data", as above, there exists a differential module M over $K = \mathbb{C}(z)$ inducing the data. Moreover M is unique up to isomorphism.*

OBSERVATIONS 1.10. — *Global differential modules and connections.*

Before giving the proof of Theorem 1.9, we have to make the relation between differential modules M over $K = \mathbb{C}(z)$ and connections $(\mathcal{M}, \nabla)$ (with singularities) on $\mathbb{P}^1$ explicit.

Let a connection $(\mathcal{M}, \nabla)$ (with singularities) be given. We note that we may regard this connection either algebraically or analytically, because of the GAGA theorem. On proper Zariski-open subsets of $\mathbb{P}^1$ we sometimes see $\mathcal{M}$ as an analytic vector bundle. The generic fibre M of $\mathcal{M}$ is a vector space of finite dimension over K, equipped with a $\nabla : M \to \Omega_{K/\mathbb{C}} \otimes M$. After identifying $\Omega_{K/\mathbb{C}}$ with Kdz, this gives M the structure of a differential module.

On the other hand, let a differential module M be given. This is written as a (generic) connection $\nabla : M \to \Omega_{K/\mathbb{C}} \otimes M$. Consider a set $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ of points including the singular points of M. For each i one chooses an

$\widehat{O}_{p_i}$-lattice Λ_i in $\widehat{K}_{p_i} \otimes M$ and let k_i satisfy $\nabla(\Lambda_i) \subset t_i^{-k_i} dt_i \otimes \Lambda_i$ (where t_i is the local parameter at p_i). For $p \notin \{p_1, \dots, p_r\}$, the module $\widehat{K}_p \otimes M$ is non singular and there is a unique $\widehat{O}_p$-lattice Λ_p with $\nabla(\Lambda_p) \subset dt_p \otimes \Lambda_p$, where t_p denotes the local parameter at p. Then there exists a unique connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$ having the following properties (see [24], Lemma 6.16):

(1) $\mathcal{M}(V) \subset M$ for all, non empty, Zariski–open $V \subset \mathbb{P}^1$.
(2) There is a basis $e_1, \dots, e_m$ of M and a non empty Zariski–open subset $U \subset \mathbb{P}^1$ such that the restriction of $\mathcal{M}$ to U is the free algebraic vector bundle $O_U e_1 \oplus \cdots \oplus O_U e_m$.
(3) For each p_i one has $\widehat{\mathcal{M}}_{p_i} = \Lambda_i$.
(4) For $p \notin \{p_1, \dots, p_r\}$ one has $\widehat{\mathcal{M}}_p = \Lambda_p$.
(5) $\nabla : \mathcal{M} \to \Omega(\sum k_i[p_i]) \otimes \mathcal{M}$.

We still need another ingredient for the proof of Theorem 1.9. Let a differential module N over $K_p = \mathbb{C}(\{t_p\})$ be given and be written in the form $\nabla : N \to K_p dt_p \otimes N$. Choose any $\mathbb{C}\{t_p\}$-lattice $\Lambda \subset N$ and let $k \geqslant 0$ be such that $\nabla(\Lambda) \subset t_p^{-k} dt_p \otimes \Lambda$. Then the latter map extends to a connection $(\mathcal{N}, \nabla)$, defined on a suitable small disk around p and has the property $\nabla : \mathcal{N} \to \Omega(k[p]) \otimes \mathcal{N}$. We note that this extension depends on the choice of the lattice Λ or more precisely on the unique lattice Λ' in $\mathbb{C}((t_p)) \otimes N$ with $\Lambda' \cap N = \Lambda$.

Proof of Theorem 1.9. — We use the notation of Definition 1.8. For $r = 0$, the data set is empty. The only module M corresponding to this is the trivial differential module (of the required dimension, say m).

For $r = 1$, the data at p_1 determines a differential module M_1 over K_{p_1}. We choose a lattice $\Lambda \subset M_1$ (say the standard lattice) and then the connection $\nabla_1 : \Lambda \to \Omega(k \cdot [p_1]) \otimes \Lambda$ (some $k \geqslant 0$) extends to a connection $(\mathcal{M}_1, \nabla_1)$ on a small open disk D around p_1. The topological monodromy around p_1 of this connection is trivial. We consider the trivial connection $(\mathcal{M}_0, \nabla_0)$ (of the required rank m) on $\mathbb{P}^1 \setminus \{p_1\}$. The two connections can be glued over $D \setminus \{p_1\}$, because of the triviality of top_1, and there results a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$. Its generic fibre M is a differential module over K, inducing the given complete data.

Let N be another differential module over K inducing the given (formal and analytic) data. Then $K_{p_1} \otimes N$ is isomorphic to $K_{p_1} \otimes M$ and we choose in $K_{p_1} \otimes N$ the lattice which maps to the lattice $\Lambda \subset K_{p_1} \otimes M$. This yields a connection $(\mathcal{N}, \nabla_N)$ with only p_1 as singularity. Outside p_1 the two connections are isomorphic and the same holds above a small enough disk D around p_1. The two isomorphisms above $D \setminus \{p_1\}$ will differ by an element in $\mathrm{GL}_m(\mathbb{C})$ (where $m = \dim M$). The isomorphism between the connections above $\mathbb{P}^1 \setminus \{p_1\}$ can be changed by any element in $\mathrm{GL}_m(\mathbb{C})$. Then, after this change, the two connections are isomorphic and then N is isomorphic to M.

Now we suppose that $r \geqslant 2$. The monodromy determines a connection $(\mathcal{M}_0, \nabla_0)$ on $\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$. The analytic data at p_i determine a differential module over K_{p_i}. For this differential module we choose the standard lattice as before. This extends to a connection $(\mathcal{M}_i, \nabla_i)$ on a small disk D_i around the point p_i.

Since top_i is conjugated to the monodromy of the loop λ_i, we have that the restrictions of $(\mathcal{M}_0, \nabla_0)$ and $(\mathcal{M}_i, \nabla_i)$ to $D_i \setminus \{p_i\}$ are isomorphic. A priori, many isomorphisms are possible. However, the link L_i determines the isomorphism. Namely, one takes the isomorphism such that the map $V(b) \xrightarrow{\alpha} V(p_i^*) \xrightarrow{\beta} V(p_i)$ is equal to the given L_i, where α is the analytic continuation for the connection $(\mathcal{M}_0, \nabla_0)$ and β is the inverse for the multisummation $V(p_i) \to V(p_i^*)$ for the connection $(\mathcal{M}_i, \nabla_i)$. Glueing yields a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$ and its generic fibre has the required properties.

Consider another differential module N which produces the same (formal and analytic) data. Then N yields a connection $(\mathcal{N}, \nabla_N)$. This connection is chosen such that the local connections at the points p_i are standard, as above. This connection is, above $\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$ and above each of the small enough disks D_i, isomorphic to the same items for $(\mathcal{M}, \nabla)$. The links L_i imply that these isomorphisms glue to a global isomorphism between $(\mathcal{N}, \nabla_N)$ and $(\mathcal{M}, \nabla)$. Thus N is isomorphic to M. □

OBSERVATIONS 1.11. — (1) In the construction in the proof of Theorem 1.9 from the given formal and analytic data to a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$, one can change the lattices Λ_i at the points p_i. We note that this corresponds to an elementary transformation in [12], Section 3. This will change the connection on $\mathbb{P}^1$. However the corresponding differential module does not change.

(2) By Proposition 1.6, the links are superfluous in case all the singularities are regular singular. Another way to see this is to take the standard lattice at each point p_i. Then the glueing of the connection $(\mathcal{M}_0, \nabla_0)$ to the connection $(\mathcal{M}_i, \nabla_i)$ on a small disk D_i around p_i is *unique*, since the connection $(\mathcal{M}_i, \nabla_i)$ on D_i and its restriction to $D_i \setminus \{p_i\}$ have the same group of automorphisms, namely the elements in $\mathrm{GL}(m, \mathbb{C})$ commuting with the topological monodromy.

(3) Theorem 1.9 is the *weak solution* of the Riemann–Hilbert problem for differential modules with any type of singularities.

Definition and examples 1.12. — *The strong Riemann-Hilbert problem.*

This problem can be formulated as follows:

Let M be the weak solution for the given formal and analytic data.

Does there exists a connection $(\mathcal{M},\nabla)$ with generic fibre M and free *vector bundle $\mathcal{M}$ such that $\nabla:\mathcal{M}\to\Omega(\sum_p(r^+(p)+1)[p])\otimes\mathcal{M}$?*

In the above, the sum $\sum_p$ is taken over the singular points p of M, $r(p)$ is the Katz invariant of $\widehat{K}_p\otimes M$ and $r^+(p)$ is the smallest integer $\geqslant r(p)$.

One observes that in case that all the singularities are regular singular (this means that $r(p)=0$ for every singular point p) the above is the classical strong form of the Riemann–Hilbert problem.

In the proof of Theorem 1.9 one can choose at each singular point an invariant lattice which exists according to Definition and examples 1.4. One arrives at a connection $(\mathcal{M},\nabla)$ which satisfies all conditions with the exception that the vector bundle $\mathcal{M}$ is possibly not free. Under the assumptions of Theorem 1.2, the invariant lattices can be changed to obtain a free vector bundle. Now we give families of examples, closely related to the Painlevé equations, where the strong Riemann–Hilbert problem has a *negative* answer.

(1) *The differential module M is given by the matrix differential equation $\frac{d}{dz}+\left(\begin{smallmatrix}0&f\\1&0\end{smallmatrix}\right)$, where $f\in\mathbb{C}[z]$ has degree 3. For M there is no solution for the strong Riemann–Hilbert problem.*

Proof. — The only singular point ∞ of M has Katz invariant $r=5/2$ and $r^+=3$. *Suppose that* M can be represented by a connection $(\mathcal{V},\nabla)$ with $\mathcal{V}$ free and $\nabla:\mathcal{V}\to\Omega(4[\infty])\otimes\mathcal{V}$. Write $V=H^0(\mathcal{V})$. Then $\nabla:V\to H^0(\Omega(4[\infty]))\otimes V$ and $\partial:=\nabla_{\frac{d}{dz}}$ has, with respect to a basis of V, the form $\frac{d}{dz}+B$ where B is a polynomial matrix of degree $\leqslant 2$.

For computational convenience we may suppose that $f=f_3z^3+f_1z+f_0$ with $f_3\neq 0$. There exists $A\in\mathrm{GL}_2(\mathbb{C}(z))$ with $A^{-1}(\frac{d}{dz}+\left(\begin{smallmatrix}0&f\\1&0\end{smallmatrix}\right))A=\frac{d}{dz}+B$. One easily verifies that $A\in\mathrm{GL}_2(\mathbb{C}[z])$ and we may assume that A has determinant 1. We use the notation $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^t=\left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)$. Write $A=A_0+A_1z+\cdots+A_sz^s$ with constant matrices A_i and $A_s\neq 0$. Then $A^{-1}=A_0^t+\cdots+A_s^tz^s$ and $A^{-1}A'+A^{-1}\left(\begin{smallmatrix}0&f\\1&0\end{smallmatrix}\right)A=B$ has degree $\leqslant 2$.

Suppose first that $s\geqslant 2$. We compute the coefficients of z-powers in the expression $A^{-1}A'+A^{-1}\left(\begin{smallmatrix}0&f\\1&0\end{smallmatrix}\right)A$. The coefficient of z^{2s+3} is $A_s^t\left(\begin{smallmatrix}0&f_3\\0&0\end{smallmatrix}\right)A_s$ is zero and thus A_s has the form $\left(\begin{smallmatrix}*&*\\0&0\end{smallmatrix}\right)$. The coefficient of z^{2s+2} is then also zero. The coefficient $A_{s-1}^t\left(\begin{smallmatrix}0&f_3\\0&0\end{smallmatrix}\right)A_{s-1}$ of z^{2s+1} is zero and then A_{s-1} has the form $\left(\begin{smallmatrix}*&*\\0&0\end{smallmatrix}\right)$. The coefficient of z^{2s} yields that $A_s^t\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)A_s=0$. This implies $A_s=0$ in contradiction with the assumption.

In the case $s=1$ one finds again $A_1^t\left(\begin{smallmatrix}0 & f_3\\ 0 & 0\end{smallmatrix}\right)A_1=0$ and observes then that the term $(f_3z^3+f_1z)A^{-1}\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)A$ has degree $\leqslant 2$. This implies that the term is 0, contradicting that A is invertible. □

Comments.

(a) The scalar equation $(\frac{d}{dz})^2-f$, obtained from M by using the first basis vector as cyclic vector, has only ∞ as singularity, *i.e.*, there is no *apparent singularity* (see 4.2).

(b) In the above negative answer for the strong Riemann–Hilbert problem one can replace f by any polynomial of odd degree $\geqslant 3$.

(c) Consider a differential module M of dimension two which has only ∞ as singular point and with $r(\infty)=5/2$ and $\Lambda^2M\cong\mathbf{1}$. Suppose that M can be represented by a connection $(\mathcal{V},\nabla)$ with $\mathcal{V}\cong O\oplus O(-2)$. Write more explicitly $\mathcal{V}=Oe_1\oplus O(-2[\infty])e_2$. Then $\partial:=\nabla_{\frac{d}{dz}}$ has the form $\frac{d}{dz}+\left(\begin{smallmatrix}0 & f\\ 1 & 0\end{smallmatrix}\right)$. One computes that the invariant lattices at ∞ have generators $\{z^ne_1,z^{n-1}e_2\}$ or $\{z^ne_1,z^{n-2}e_2\}$ over $\mathbb{C}[[z^{-1}]]$ (with any $n\in\mathbb{Z}$). The connections $(\mathcal{V},\nabla)$ with $\nabla:\mathcal{V}\to\Omega(4[\infty])\otimes\mathcal{V}$ representing M are of two types, namely $\mathcal{V}\cong O(n)\oplus O(n-1)$ and $\mathcal{V}\cong O(n)\oplus O(n-2)$.

(d) Consider again of differential module M of rank two, $\Lambda^2M\cong\mathbf{1}$, only ∞ as singular point and $r(\infty)=5/2$. Suppose now that the strong Riemann-Hilbert problem has a *positive answer* for M. Then M can be represented by a matrix differential equation of the form $\frac{d}{dz}+A_0+A_1z+\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)z^2$. Using the invariant lattices at ∞ one finds that the vector bundles of the connections $\nabla:\mathcal{V}\to\Omega(4[\infty])\otimes\mathcal{V}$, representing M, are of two types namely isomorphic to $O(n)\oplus O(n)$ or $O(n)\oplus O(n-1)$ (for any $n\in\mathbb{Z}$. Further M has a cyclic vector (essentially unique) which produces a scalar differential equation with precisely one apparent singularity (see 4.2).

(2) *Let M be a 2-dimensional differential module over $\mathbb{C}(z)$ with $\Lambda^2M=\mathbf{1}$, $r(0)=r(\infty)=1/2$ and no singularities $\neq 0,\infty$. Suppose that there exists a connection $\nabla:\mathcal{V}\to\Omega(2[0]+2[\infty])\otimes\mathcal{V}$ with generic fibre M and $\mathcal{V}\cong O\oplus O(-2)$. Then M can be represented by a matrix differential equation of the form*

$$z\frac{d}{dz}+\begin{pmatrix}0 & c_{-1}z^{-1}+c_0+c_1z\\ 1 & m\end{pmatrix}\ \text{with } c_{-1}\neq 0\neq c_1 \text{ and } m\in\mathbb{Z}.$$

In particular, the strong Riemann–Hilbert problem has a positive solution for M. The special phenomenon *is that the scalar equation, associated to this matrix differential equation w.r.t. the first basis vector reads $\delta(\delta-m)-(c_{-1}z^{-1}+c_0+c_1z)$, with $\delta:=z\frac{d}{dz}$, and therefore has* no apparent singularities!

Proof. — Identify $\mathcal{V}$ with $Oe_1 \oplus O(-2[\infty])e_2$. The operator $\delta := \nabla_{z\frac{d}{dz}}$ satisfies:

$\delta e_1 \in (\mathbb{C}z^{-1} + \mathbb{C} + \mathbb{C}z)e_1 + \mathbb{C}z^{-1}e_2$ and

$\delta e_2 \in (\mathbb{C}z^{-1} + \mathbb{C} + \mathbb{C}z + \mathbb{C}z^2 + \mathbb{C}z^3)e_1 + (\mathbb{C}z^{-1} + \mathbb{C} + \mathbb{C}z)e_2$.

Since the module is irreducible, we can change e_2 into $\lambda e_2 + (* + *z + *z^2)e_1$ with suitable $\lambda \in \mathbb{C}^*$, $* \in \mathbb{C}$ and obtain $\delta e_1 = z^{-1}e_2$. The condition $\Lambda^2 M = \mathbf{1}$ implies that $\delta e_2 = (a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3)e_1 + me_2$ with $m \in \mathbb{Z}$. Conjugation of the corresponding matrix differential equation with $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ yields the matrix differential equation

$$z\frac{d}{dz} + \begin{pmatrix} 0 & a_{-1}z^{-2} + a_0 z^{-1} + a_1 + a_2 z + a_3 z^2 \\ 1 & m-1 \end{pmatrix}.$$

The assumptions $r(0) = r(\infty) = 1/2$ imply $a_{-1} = a_3 = 0$ and $a_0 \neq 0 \neq a_2$. This is the required form. The formula for the scalar equation is obvious. □

Theorem 1.13. — *Suppose that the differential module M over $K = \mathbb{C}(z)$ is irreducible and has a (regular or irregular) singularity which is unramified. Then the strong Riemann-Hilbert problem has a solution for M.*

Proof. — We will adapt the proof of Theorem 6.22, [24] to the present more general situation. As shown in the proof of Theorem 1.9, there exists a connection $(\mathcal{M}, \nabla)$ such that $\nabla : \mathcal{M} \to \Omega(\sum(k_i+1)[p_i]) \otimes \mathcal{M}$, where the p_i are the singular points of M and k_i is the smallest integer $\geqslant$ the Katz invariant at p_i. The irreducibility of M implies that the defect of any $\mathcal{M}$ is bounded by a number only depending on M, see Proposition 6.21, [24].

Let $p = p_1$ be an unramified singular point. Then we want to prove the equivalent of Lemma 6.20, [24], namely for any integer $N > 1$, there exists a lattice Λ for $\widehat{K}_p \otimes M$ such that Λ has a basis $e_1, \ldots, e_m$ with the property $t_p \cdot \nabla e_i = dt_p \otimes ((c_i + a_i)e_i + \sum_{i \neq j} a_{i,j}e_j)$, with $c_i \in \mathbb{C}$; the $a_i \in t_p^{-1}\mathbb{C}[t_p^{-1}]$ belong to the set of the eigenvalues of M at p and $a_{i,j} \in t_p^N \mathbb{C}[[t_p]]$.

It suffices to show the above for an indecomposable direct summand of $\widehat{K}_p \otimes M$. This direct summand has only one eigenvalue and the formal monodromy γ has only one Jordan block. Then the proof of Lemma 6.20, [24], yields the required lattice for this indecomposable direct summand.

Finally, as in the proof of Theorem 6.22, [24], one can change the lattice Λ step by step to obtain a connection $(\mathcal{M}, \nabla)$ where $\mathcal{M}$ has defect 0. Taking the tensor product with $O(k[p_1])$ for a suitable k makes $\mathcal{M}$ free. □

2. Families of differential modules

2.1. Good families and the monodromy space $\mathcal{R}$

The aim is to study the formal and analytic data of a *family* of differential modules $M(\underline{u})$ depending on some parameters $\underline{u}$. Of course, this notion has to be made explicit. A rough approximation would be a matrix differential equation $\frac{d}{dz} + A(z, \underline{u})$ where each entry of the $m \times m$-matrix $A(z, \underline{u})$ is a rational function in z with coefficients depending analytically on the parameters $\underline{u}$. We recall that for a point p, the local parameter is t_p (equal to $z - p$ or z^{-1}). Further we use the notation of Subsection 1.2: $V = V(p)$ for the formal solution space at p, the eigenvalues are q_* and correspond to subspaces V_{q_*} of V. The singular directions at p are defined in Subsection 1.3.

In order to have meaningful analytic data (as functions of $\underline{u}$) one has to make some assumptions. A *good family* is defined by the properties:

(1) The number r of the singular points is fixed. The position of these points $\{p_1, \ldots, p_r\}$ may vary, but only slightly.
(2) For every singular point p, the degrees in t_p^{-1} of the eigenvalues q_i and the degrees in t_p^{-1} of the differences $q_i - q_j,\ i \neq j$ is fixed.
(3) For every singular point p and every eigenvalue q_i, the dimension of the space V_{q_i} is fixed.
(4) The top coefficients $c_{i,j}$ of the $q_i - q_j$ may vary in a certain restricted way. Namely, we *impose* that any singular direction for the singular point p is a singular direction for a unique difference $q_i - q_j,\ i \neq j$ of the eigenvalues at p and that these singular directions vary slightly. As a consequence, the order of the directions at p does not change in the family.

COMMENTS 2.1. — (a) From (1) it follows that one can take a base point b and loops $\alpha_1, \ldots, \alpha_r$ around the singular points, valid for all $M(\underline{u})$. From a point p_i^* on α_i and close to p_i, the direction of the line segment $[p_i^*, p_i]$ is non singular and lies between the "same" singular directions for the family $M(\underline{u})$. This follows from (2), (3) and (4). Suppose that the family $M(\underline{u})$ satisfies (1)–(3), and that for $M(\underline{0})$ every singular direction of each singular point belongs to a unique difference $q_i - q_j,\ i \neq j$ of eigenvalues. Then condition (4) is valid for the restriction of this family to a suitable neighbourhood of $\underline{u} = \underline{0}$.

(b) Let a differential module M over $K = \mathbb{C}(z)$ be given and assume that every singular direction of each singular point belongs to a unique difference of eigenvalues. *We sketch the proof of the statement that there exists a local analytic family $M(\underline{u})$, satisfying (1)–(4), such that $M(\underline{0}) = M$.*

Write M as a matrix differential equation $\frac{d}{dz} + \sum_{i=1}^{r}(\sum_{n=1}^{k_i} \frac{A(i,n)}{(z-p_i)^n})$, with constant matrices $A(i,n)$. Here, the singular points $p_1, \ldots, p_r$ are for notational convenience different from ∞ (and thus $\sum_{i=1}^{r} A(i,1) = 0$). We do not impose a condition on k_i in relation with the Katz invariant at the point p_i. One considers the family

$$\frac{d}{dz} + A(z,\underline{v}) := \frac{d}{dz} + \sum_{i=1}^{r}\left(\sum_{n=1}^{k_i} \frac{A(i,n)+V(i,n)}{(z-p_i-v_i)^n}\right),$$

where the $V(i,n)$ are matrices of indeterminates and the v_i are also indeterminates. Let $\underline{v}$ denote the collection of all indeterminates. We consider this family in a small enough polydisk D around $\underline{0} \in \mathbb{C}^N$. The conditions (1)–(3) on $\frac{d}{dz} + A(z,\underline{v})$ define a Zariski closed subset $Z \subset \mathbb{C}^N$. The subfamily $\frac{d}{dz} + A(z,\underline{u})$ with $\underline{u}$ belonging to the locally closed set $U = D \cap Z \neq \emptyset$ satisfies conditions (1)–(3). Further condition (4) is satisfied for this subfamily since D small enough.

A priori, $\underline{0}$ is a singular point of U. If needed, one can, by resolution of singularities, return to the case that U is a small open polydisk around the point $\underline{0}$.

(c) The statement in (b) justifies the naive way of writing a family, satisfying (1)–(4), locally as $\frac{d}{dz} + A(z,\underline{u})$. As mentioned in the introduction, the theory of Okamoto–Painlevé pairs has the aim to improve on this. In this paper however, we will deal with the naive local situation.

Let $M(\underline{u})$ be a good family of dimension m for $\underline{u}$, close to $\underline{0}$. According to Definition 1.8, the *formal data* of the family are the position of the singular points $\{p_j\}_{j=1}^{r}$ and the eigenvalues q at the singular point p_j. Now we describe the items which do not vary in the family:

(a) $V(j)$, the formal solution space at p_j.
(b) The direct sum decomposition $V(j) = \oplus_{i \in I_j} V(j,i)$, given by the eigenvalues.
(c) The dimension of the spaces $V(j,i)$.
(d) The permutation τ_j of the $V(j,i)$, satisfying $\dim V(j,\tau_j i) = \dim V(j,i)$, induced by the action of γ on the eigenvalues.
(e) The order of the singular directions for any p_j. This yields a sequence of Stokes maps $\{St_k(j)\}_{k=1}^{n_j}$.
(f) Each $St_k(j)$ has the form $1 + R_k$, with $R_k := i_t \circ M(j,k) \circ pr_s$ with prescribed $s,t \in I_j,\ s \neq t$ (depending on k) and pr_s is the projection $V(j) \to V(j,s)$ (with kernel $\oplus_{h \neq s} V(j,h)$) and $i_t : V(j,t) \to V(j)$

is the canonical injection. Moreover, $M(j,k) : V(j,s) \to V(j,t)$ is a linear map which is not constant in the family.

(g) A vector space W of dimension m, representing the solution space $V(b)$ for a given base point b.

The *analytic data* of $M(\underline{u})$ are tuples $(\{\gamma_j\}, \{L_j\}, \{St_k(j)\})$ satisfying:

(1) For each j, a map $\gamma_j \in \mathrm{GL}(V(j))$ with $\gamma_j(V(j,i)) = V(j, \tau_j i)$ for all i.
(2) $St_k(j) \subset \mathrm{GL}(V(j))$ of the form described in (f), determined by the linear map $M(j,k) : V(j,s) \to V(j,t)$.
(3) Linear bijections $L_j : W \to V(j)$ for $j = 1, \ldots, r$.
(4) Define $\mathrm{top}_j := \gamma_j \circ St_{n_j}(j) \circ \cdots \circ St_1(j)$. The data should satisfy the relation $L_r^{-1} \circ \mathrm{top}_r \circ L_r \cdots \circ L_1^{-1} \circ \mathrm{top}_1 \circ L_1 = 1$.

Let *AnalyticData* denote the set of all tuples. This has a natural structure of an affine variety over $\mathbb{C}$. Two tuples $(\{\gamma_j\}, \{L_j\}, \{St_k(j)\})$ and $(\{\gamma_j'\}, \{L_j'\}, \{St_k(j)'\})$ are called *equivalent*, if there exists $\sigma_j \in \mathrm{GL}(V(j))$, $j = 1, \ldots, r$, $\sigma \in \mathrm{GL}(W)$ such that each σ_j preserves the direct sum decomposition $\oplus V(j,i)$ and further $\sigma_j \circ L_j = L_j' \circ \sigma,\ j = 1, \ldots, r$ and $\sigma_j \circ \gamma_j = \gamma_j' \circ \sigma_j,\ j = 1, \ldots, r$. In other words, the equivalence relation on *AnalyticData* is given by the action of the reductive linear algebraic group $G := \mathrm{GL}(W) \times \prod_{j,i} \mathrm{GL}(V(j,i))$.

The *monodromy space* $\mathcal{R}$ is by definition *AnalyticData*$//G$, the categorical quotient. This is again an affine variety. In general this quotient is not a geometric one. In particular, a closed point of $\mathcal{R}$ can correspond to many equivalence classes. One may use L_1 to identify each space $V(1)$ with W to reduce the space *AnalyticData* and the group G acting on it.

The map, which associates to $\underline{u}$ in D (a small polydisk around $\underline{0}$) the tuple $(\{\gamma_j\}, \{L_j\}, \{St_k(j)\})$, is analytic. Indeed, it is rather clear that analytic continuation depends in an analytic way on $\underline{u}$. That the same is valid for multisummation follows from [24], Proposition 12.20, p. 314. Thus $D \to$ *AnalyticData* is analytic and hence $D \to \mathcal{R} :=$ *AnalyticData*$//G$ is analytic. The next example illustrates the above for a relatively simple case.

Example 2.2. — The monodromy space $\mathcal{R}$ for the local family $M(\underline{u})$ with $M(\underline{0})$ given by the matrix equation

$$z\frac{d}{dz} + \begin{pmatrix} z^{-1} + a_1 & 0 & 0 \\ 0 & \omega z^{-1} + a_2 & 0 \\ 0 & 0 & \omega^2 z^{-1} + a_3 \end{pmatrix}, \text{ where } \omega = e^{2\pi i/3}.$$

A good choice (compare [24], 12.3) for the family $M(\underline{u})$ is

$$z\frac{d}{dz}+\begin{pmatrix}(1+u_1)z^{-1}+a_1+u_2 & u_3 & u_4\\ u_5 & (1+u_6)\omega z^{-1}+a_2+u_7 & u_8\\ u_9 & u_{10} & (1+u_{11})\omega^2 z^{-1}+a_3+u_{12}\end{pmatrix}.$$

The singular points are $z=0,\ \infty$ and $r(0)=1,\ r(\infty)=0$. The space *AnalyticData* consist of the formal monodromy and six Stokes matrices at 0, the link between 0 and ∞ and the formal (=topological) monodromy at ∞. This link and the topological monodromy at ∞ are determined by the data at 0 up to an automorphism of the solution space at ∞.

The eigenvalues at $z=0,\underline{u}=\underline{0}$ are $q_1=z^{-1},\ q_2=\omega z^{-1},\ q_3=\omega^2 z^{-1}$.

The order of the six singular directions in $\mathbb{R}/2\pi\mathbb{Z}$ is given by the differences $q_1-q_2, q_1-q_3, q_2-q_3, q_2-q_1, q_3-q_1, q_3-q_2$. The topological monodromy top_0 at $z=0$ is then the following product of matrices

$$\begin{pmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&c_6\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&c_5\\0&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&c_4&0\\0&1&0\\0&0&1\end{pmatrix}$$
$$\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&c_3&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\c_2&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\c_1&1&0\\0&0&1\end{pmatrix}.$$

The entries $(\alpha_1,\alpha_2,\alpha_3)$ of the first matrix, the formal monodromy, are

$$(\alpha_1,\alpha_2,\alpha_3)=(e^{2\pi i(a_1+u_2)},e^{2\pi i(a_2+u_7)},e^{2\pi i(a_3+u_{12})}).$$

The $c_1,\dots,c_6$ are analytic functions of $\underline{u}$, produced by multisummation (in this case just Borel summation). The topological monodromy top_∞ at ∞ is conjugated to top_0. Under the condition that $a_i-a_j\notin\mathbb{Z}$ for $i\neq j$, one has that top_∞ is equal to $e^{2\pi i A}$ with

$$A=\begin{pmatrix}a_1+u_2&u_3&u_4\\u_5&a_2+u_7&u_8\\u_9&u_{10}&a_3+u_{12}\end{pmatrix}.$$

The group

$$G:=\left\{\begin{pmatrix}t_1&0&0\\0&t_2&0\\0&0&t_3\end{pmatrix}\mid t_1t_2t_3=1\right\}\subset \mathrm{GL}(V(0))$$

acts, by conjugation, on the data $(\alpha_1,\alpha_2,\alpha_3,c_1,\dots,c_6)$. Thus $\mathcal{R}$ is the affine space with coordinate ring $\mathbb{C}[\alpha_1,\alpha_1^{-1},\dots,\alpha_3,\alpha_3^{-1},c_1,\dots,c_6]^G$. A computation shows that this ring is $\mathbb{C}[\alpha_1,\alpha_1^{-1},\dots,\alpha_3,\alpha_3^{-1},x_{14},x_{25},x_{36},x_{135},x_{246}]$ where the only relation is $x_{135}x_{246}-x_{14}x_{25}x_{36}=0$. Here $x_{14}=c_1c_4,\ x_{25}=c_2c_5,\ x_{135}=c_1c_3c_5$ et cetera. It is natural to see the eigenvalues $(\alpha_1,\alpha_2,\alpha_3)$ of the formal monodromy as a parameter space $\mathcal{P}$. The fibers of $\mathcal{R}\to\mathcal{P}$ are isomorphic to the 4-dimensional affine space with coordinate ring

$\mathbb{C}[x_{14}, x_{25}, x_{36}, x_{135}, x_{246}]$ with relation $x_{135}x_{246} - x_{14}x_{25}x_{36} = 0$. The singular locus of this space has three components, they are the image of the locus where the differential equation is reducible $\{\underline{u}|\ M(\underline{u})$ is reducible$\}$.

The group G also acts on the local family $M(\underline{u})$ and we obtain a local Riemann–Hilbert morphism $RH : \{\underline{u} \in \mathbb{C}^{12}|\ \|\underline{u}\| < \epsilon\}//G \to \mathcal{R}$. This map does not depend on the coefficients $(1{+}u_1), (1{+}u_6)\omega, (1{+}u_{11})\omega^2$ of q_1, q_2, q_3. The fibres of RH are, by definition, the isomonodromic families. They are parametrized by the three variables $t_1 =: u_1,\ t_2 := u_6,\ t_3 := u_{11}$. Using $z \mapsto \lambda z$, one may normalize to $(1 + u_{11}) = 1$ and thus the isomonodromic family is parametrized by t_1, t_2. One expects that a suitable expression in the other u_i satisfies a Painlevé type of partial differential equations w.r.t. the variables t_1, t_2. In fact it is possible to convert the situation into a one variable case for PVI (*cf.* [2]).

Remarks 2.3. — The papers of M. Jimbo, T. Miwa and K. Ueno.

The above introduction of families of differential modules and their formal and analytic data can be seen as an extension of the papers [17], [16], which we will describe now, using our terminology.

In [17], [16] the base point b is taken to be ∞ and this point is supposed to be (irregular) singular. Further the irregular singularities p are of a simple kind, namely all the eigenvalues (generalized exponents) q_i are in $t_p^{-1}\mathbb{C}[t_p^{-1}]$, and all q_i and $q_i - q_j$ for $i \neq j$ have the same degree in t_p^{-1} (there is one exception, related to the Painlevé I equation). We note that Example 2.2 is of the type considered in [17]. In particular, Borel summation or, better, k-summation is sufficient for the asymptotic analysis of the singularity. A theorem of Y. Sibuya [28] gives the required input from asymptotics. The “links”, that we defined, are present in their work and the family of linear differential equations is presented as a matrix differential equation $\frac{d}{dz} + A(z, \underline{u})$. The origin of the examples, in the appendix of [16], of families related to Painlevé I–VI, is probably classical (discovered by R. Fuchs [6], P. Painlevé [23], R. Garnier [8] et al.). Another source for similar examples are the ones discovered by H. Flaschka and A.C. Newell [4]. Later work of B. Malgrange [18, 19], clarifies and extends the papers [17], [16].

The new tool “multisummation” and the precise construction of the Stokes matrices, enables to generalize the work of Jimbo, Miwa and Ueno. Especially, as we will show in the next subsection, a “complete” list of the equations related to Painlevé I–VI can be derived. Further, the monodromy spaces $\mathcal{R}$ for the analytic data can now be computed and studied in detail.

2.2. Finding the list. Tables for connections and $\mathcal{R}$

We consider a local family $M(\underline{u})$ of differential modules, represented by a matrix equation $\frac{d}{dz}+A(z,\underline{u})$ where $A(z,\underline{u})$ is a 2×2-matrix with trace 0 and $\underline{u}$ lies in a small polydisk D around $\underline{0}$. The possibilities of the formal structure at the singular points is given in Example 1.3. The local Riemann–Hilbert map $RH : D \to \mathcal{R}$ forgets the formal data, namely the position of the set of singular points S and the coefficients of the eigenvalues at the irregular singular points.

The position of the points S contributes $\max(-3+\#S,0)$ to the dimension of the fibre, because of the automorphisms of $\mathbb{P}^1$. A singular point p with Katz invariant $r(p)$ contributes to the fibre the dimension $r(p)$ if $r(p)\in\mathbb{Z}_{\geqslant 0}$ and $r^+(p)=\frac{1}{2}+r(p)$ if $r(p)\in\frac{1}{2}+\mathbb{Z}_{\geqslant 0}$. Further, in the space D one divides by the action of a subgroup of the automorphisms of $\mathbb{P}^1$.

The requirement that the fibres of RH have dimension 1 produces the list:

$\#S>4$ is excluded.

$\#S=4$, then $S=\{0,1,\infty,t\}$, only regular singular points, *i.e.*, all $r(p)=0$.

$\#S=3$, then $S=\{0,1,\infty\}$ and only one irregular point with $r(p)\in\{1,\frac{1}{2}\}$. $\#S=2$, then $S=\{0,\infty\}$, the contribution of the singular points to the dimension of the fibre is 2, since we divide by the group $z\mapsto az$.

$\#S=1$, then $S=\{\infty\}$, the contribution of the singular point to the dimension of the fibre is 3, since we divide by the group $z\mapsto az+b$.

Columns 3–6 of the next table present the ten resulting cases. In the second column one finds the classification of the related Painlevé equation (some classes are divided into subclasses). The first column gives the extended Dynkin diagram of the corresponding Okamoto–Painlevé pair (see the Introduction). The space $\mathcal{R}$ is mapped to a space of parameters $\mathcal{P}$ (related to the parameters spaces of the Painlevé equations) consisting of traces or eigenvalues of the matrices involved in the construction of $\mathcal{R}$.

We will not number these ten families, but indicate them by their Katz invariants, *e.g.*, $(0,0,0,0)$, $(0,0,1),\dots$, $(0,-,3/2)$, $(-,-,3)$, $(-,-,5/2)$. For a differential module M corresponding to one of the types, the strong Hilbert-Riemann problem has a positive answer, except possibly for $(1/2,-,1/2)$ and $(-,-,5/2)$ (see Definitions and examples 1.12). For these two types we only consider the modules $M(\underline{u})$ for which the strong Riemann–Hilbert problem does have a positive answer. The strong Riemann-Hilbert problem for a family $M(\underline{u})$ is more subtle. It seems that connections on the vector bundle $O\oplus O(-1)$ is better adapted to families. For the Painlevé VI

Dynkin	Painlevé equation	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{P}$
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	$\mathrm{PV}_{\mathrm{deg}}$= PIII(D6)	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

Table 2.1. Classification of Families

case this is type of vector bundle is considered in [12, 13]. Here however we will represent a family $M(\underline{u})$ by connections on $O \oplus O$. This defines, in general, an affine Zariski open subset of the space of all connections. However, the monodromy space $\mathcal{R}$ classifies the analytic data (modulo some equivalence) for the complete space of all connections. For each type there are many possibilities. We will make choices that are helpful for the computation of the Painlevé equations and are moreover close to classical formulas.

It turns out that $\mathcal{R} \to \mathcal{P}$ is a *family of affine cubic surfaces*. There are two sources for the singularities of the fibres. The first one is *reducibility* of systems and is connected with the singularities of $\mathcal{R}$ itself. The other source is *resonance*, *i.e.*, at least one of the matrices involved in the construction of $\mathcal{R}$ has a difference of eigenvalues belonging to $\mathbb{Z}\setminus\{0\}$. Section 3 provides the computations of the families $\mathcal{R} \to \mathcal{P}$. We will also describe the corresponding one-dimensional families of differential modules $M(t)$. This subsection ends with a list indicating the families of connections and presenting the families $\mathcal{R} \to \mathcal{P}$ of affine cubic surfaces by an equation. The monodromy space $\mathcal{R}$ for $(0,0,0,0)$ is classical(*cf.* [5, 15]), the others seem to be new.

(0,0,0,0). PVI. $\frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$, all $\mathrm{tr}(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - s_1x_1 - s_2x_2 - s_3x_3 + s_4 = 0$, with
$s_i = a_ia_4 + a_ja_k, \quad (i,j,k) =$ a cyclic permutation of $(1,2,3)$,
$s_4 = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4$ with $a_1, a_2, a_3, a_4 \in \mathbb{C}$.

(0,0,1). PV. $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+t/2\cdot\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$, all $\mathrm{tr}(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2-(s_1+s_2s_3)x_1-(s_2+s_1s_3)x_2-s_3x_3+s_3^2+s_1s_2s_3+1=0$ with $s_1,s_2\in\mathbb{C},\ s_3\in\mathbb{C}^*$.

(0,0,1/2). $\mathrm{PV}_{\mathrm{deg}}$. $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+\left(\begin{smallmatrix}0&t^2\\0&0\end{smallmatrix}\right)$, all $\mathrm{tr}(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+s_0x_1+s_1x_2+1=0$ with $s_0,s_1\in\mathbb{C}$.

(1,-,1). PIII(D6). $z\frac{d}{dz}+A_0z^{-1}+A_1+\left(\begin{smallmatrix}\frac{t}{2}&0\\0&-\frac{t}{2}\end{smallmatrix}\right)z$, all $\mathrm{tr}(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+(1+\alpha\beta)x_1+(\alpha+\beta)x_2+\alpha\beta=0$ with $\alpha,\beta\in\mathbb{C}^*$.

(1/2,-,1). PIII(D7). $z\frac{d}{dz}+A_0z^{-1}+A_1+\left(\begin{smallmatrix}\frac{t}{2}&0\\0&-\frac{t}{2}\end{smallmatrix}\right)z$, all $\mathrm{tr}(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+\alpha x_1+x_2=0$ with $\alpha\in\mathbb{C}^*$.

(1/2,-,1/2). PIII(D8). $z\frac{d}{dz}+\left(\begin{smallmatrix}0&0\\-q&0\end{smallmatrix}\right)z^{-1}+\left(\begin{smallmatrix}\frac{p}{q}&-\frac{t}{q}\\1&-\frac{p}{q}\end{smallmatrix}\right)+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)z$.

$x_1x_2x_3+x_1^2-x_2^2-1=0$.

(0,-,2). PIV. $z\frac{d}{dz}+A_0+A_1z+\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)z^2$.

$x_1x_2x_3+x_1^2-(s_2^2+s_1s_2)x_1-s_2^2x_2-s_2^2x_3+s_2^2+s_1s_2^3=0$ with $s_1\in\mathbb{C},\ s_2\in\mathbb{C}^*$.

(0,-,3/2). PIIFN. $z\frac{d}{dz}+A_0+\left(\begin{smallmatrix}0&t+q\\1&0\end{smallmatrix}\right)z+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)z^2$

$x_1x_2x_3+x_1-x_2+x_3+s=0$, with $s\in\mathbb{C}$.

(-,-,3). PII. $\frac{d}{dz}+A_0+A_1z+\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)z^2$, all $\mathrm{tr}(A_*)=0$.

$x_1x_2x_3-x_1-\alpha x_2-x_3+\alpha+1=0$ with $\alpha\in\mathbb{C}^*$.

(-,-,5/2). PI. $\frac{d}{dz}+\left(\begin{smallmatrix}p&t+q^2\\-q&-p\end{smallmatrix}\right)+\left(\begin{smallmatrix}0&q\\1&0\end{smallmatrix}\right)z+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)z^2$.

$x_1x_2x_3+x_1+x_2+1=0$.

Table of the equations of the monodromy spaces for the 10 families.

3. Computation of the monodromy spaces

3.1. Family $(0,0,0,0)$ and Painlevé PVI

For completeness we describe this classical family. The family of differential modules is represented by $\frac{d}{dz}+A(z,t):=\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}$ with constant 2×2 matrices having trace 0. Dividing by the action, by conjugation, of PSL_2 one finds a moduli space $\mathcal{M}$ (say the categorical quotient) of differential modules with dimension 7.

The monodromy data are given by the tuples $(M_1,M_2,M_3,M_4)\in \mathrm{SL}_2^4$ satisfying $M_1\cdots M_4=1$. This defines an affine space of dimension 9. The categorical quotient $\mathcal{R}$ of this space under the action, by conjugation with PSL_2, has dimension 6. The fibres of $RH:\mathcal{M}\to\mathcal{R}$ are parametrized by $t\in\mathbb{P}^1\setminus\{0,1,\infty\}$.

The coordinate ring of $\mathcal{R}$ is generated over $\mathbb{C}$ by $x_1,x_2,x_3,a_1,a_2,a_3,a_4$ with $a_i=\mathrm{tr}(B_i)$ and $x_1=\mathrm{tr}(B_2B_3)$, $x_2=\mathrm{tr}(B_1B_3)$, $x_3=\mathrm{tr}(B_1B_2)$. There is only one relation ([5, 14]), namely (as in the list)

$$x_1x_2x_3+x_1^2+x_2^2+x_3^2-s_1x_1-s_2x_2-s_3x_3+s_4=0.$$

The morphism $\mathcal{R}\to\mathcal{P}:=\mathbb{C}^4$, given by $(x_1,\dots,a_4)\mapsto(s_1,\dots,s_4)$, is a family of affine cubic surfaces with "three lines at infinity". For information concerning the singularities of $\mathcal{R}$ and of the fibres we refer to [[15], [14], [11]].

3.2. Family $(0,0,1)$ and Painlevé PV

For a differential module of type $(0,0,1)$, the strong Riemann-Hilbert problem has a positive answer. Indeed, the lattices at 0 and 1 can be chosen arbitrary. By tradition one supposes that the corresponding local exponents are $\pm\theta_0/2$ and $\pm\theta_1/2$. From Definition and examples 1.4 one concludes that there exists a unique lattice at ∞ leading to a free vector bundle. By tradition, the generalized local exponents at ∞ are $\pm(tz+\theta_\infty)/2$. The module is then represented by the matrix differential equation $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+A_\infty$, for certain constant matrices A_0,A_1,A_∞ with trace 0. A_∞ is normalized by $A_\infty=t/2\cdot\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ with $t\in\mathbb{C}^*$. Further $-\theta_i^2/4$ is the determinant of A_i for $i=0,1$ and, θ_∞ is the $(1,1)$ entry of A_0+A_1.

3.2.1. The moduli space $\mathcal{R}$ for the analytic data

The symbolic solution space V at ∞ is written as $V_q \oplus V_{-q}$. Let e_1, e_2 be basis vectors for V_q and V_{-q}. Starting at ∞ one makes loops around 0 and 1, producing monodromy matrices M_1, M_2, with respect to the basis $\{e_1, e_2\}$. Let M_∞ be the topological monodromy at ∞, then we have the relation $M_1M_2M_\infty = 1$. Further $M_\infty = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ f_1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & f_2\\ 0 & 1\end{smallmatrix}\right)$, where the first matrix is the formal monodromy and the others are the two Stokes matrices. In the sequel, we will eliminate the choice of the basis vectors e_1, e_2 of V_q and V_{-q} for the matrix equation. One considers the isomorphism $\mathbb{C}^* \times \mathbb{C} \times \mathbb{C} \to \{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2 \mid a \neq 0\}$, given by

$$(\alpha, f_1, f_2) \mapsto \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ f_1 & 1\end{pmatrix}\begin{pmatrix}1 & f_2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & \alpha f_2\\ \alpha^{-1}f_1 & \alpha^{-1}(1+f_1f_2)\end{pmatrix}.$$

One concludes that the matrices $M_j = \left(\begin{smallmatrix}a_j & b_j\\ c_j & d_j\end{smallmatrix}\right) \in \mathrm{SL}_2$ for $j = 1, 2$ determine M_∞ and that $c_1b_2 + d_1d_2 = \alpha$ is non zero. Therefore, the pairs M_1, M_2 that occur define an affine variety with coordinate ring

$$R = \mathbb{C}\Big[a_1, \ldots, d_2, \frac{1}{c_1b_2 + d_1d_2}\Big]/(a_1d_1 - b_1c_1 - 1, a_2d_2 - b_2c_2 - 1).$$

The group $\mathbb{G}_m = \{\left(\begin{smallmatrix}c & 0\\ 0 & 1\end{smallmatrix}\right) \mid c \in \mathbb{C}^*\}$ acts on V and thus on the matrices M_1, M_2. For this action the weights are: $+1$ for b_1, b_2; -1 for c_1, c_2 and 0 for a_1, d_1, a_2, d_2. The subring R_0 of R, consisting of the invariants under the action of $\mathbb{G}_m$ is the subring consisting of the elements of weight 0. The moduli space $\mathcal{R}$ for the analytic data is $\mathrm{Spec}(R_0)$.

For the calculation of R_0 we may at first forget the localization at the degree 0 element $c_1b_2 + d_1d_2$. Now, using the two relations, we find that R has a basis over $\mathbb{C}$, consisting of the monomials

$$a_1^* a_2^* d_1^* d_2^* b_1^{n_1} b_2^{n_2} c_1^{m_1} c_2^{m_2} \text{ with } n_1m_1 = 0,\ n_2m_2 = 0 \text{ and any integers } * \geqslant 0.$$

It follows that R_0 is equal to $\mathbb{C}[a_1, d_1, a_2, d_2, b_1c_2, b_2c_1, \frac{1}{b_2c_1+d_1d_2}]$, where the six generators have only one relation namely $b_1c_2 \cdot b_2c_1 = (-1+a_1d_1)(-1+a_2d_2)$. The *singular locus of* $\mathcal{R}$ is given by the additional equations $0 = b_1c_2 = b_2c_1 = a_1d_1 - 1 = a_2d_2 - 1$. One observes that this describes the *reducible analytic data*, given by $b_1 = b_2 = 0$ or $c_1 = c_2 = 0$ (or equivalently the corresponding reducible differential equations). The coordinate ring of the singular locus of $\mathcal{R}$ is $\mathbb{C}[d_1, d_1^{-1}, d_2, d_2^{-1}]$.

Introduce new variables $s_1 := a_1 + d_1,\ s_2 := a_2 + d_2,\ s_3 := b_2c_1 + d_1d_2$, *i.e.*, the traces of M_1, M_2 and the eigenvalue α of the formal monodromy at ∞ and the new variable $d_3 := b_1c_2 + d_1d_2 - s_2d_1 - s_1d_2 + s_1s_2 + s_3$. Exchange these variables against a_1, a_2, b_2c_1 and b_1c_2. Then the ring R_0 obtains the form $R_0 = \mathbb{C}[d_1, d_2, d_3, s_1, s_2, s_3, s_3^{-1}]/(R(s_1, s_2, s_3))$, where $R(s_1, s_2, s_3)$ is equal to

$$d_1d_2d_3 + d_1^2 + d_2^2 - (s_1 + s_2s_3)d_1 - (s_2 + s_1s_3)d_2 - s_3d_3 + s_3^2 + s_1s_2s_3 + 1.$$

In the sequel we will write $x_i = d_i$ for $i = 1, 2, 3$. The inclusion

$$\mathbb{C}[s_1, s_2, s_3, s_3^{-1}] \subset \mathbb{C}[x_1, x_2, x_3, s_1, s_2, s_3, s_3^{-1}]/(R(s_1, s_2, s_3))$$

induces a surjective morphism

$$\pi : \mathcal{R} \to \mathcal{P} = \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[s_1, s_2, s_3, s_3^{-1}]),$$

which maps a given tuple (M_1, M_2, M_∞) to (s_1, s_2, s_3). Thus $\pi : \mathcal{R} \to \mathcal{P}$ is a family of affine cubic surfaces with equation $F = 0$ with

$$F = x_1x_2x_3 + x_1^2 + x_2^2 - (s_1 + s_2s_3)x_1 - (s_2 + s_1s_3)x_2 - s_3x_3 + s_3^2 + s_1s_2s_3 + 1.$$

We note that this equation (or the cubic surface) has a symmetry, given by interchanging (x_1, s_1) and (x_2, s_2) (*i.e.*, interchanging M_1, M_2).

3.2.2. The singularities of $\mathcal{R}$ and the fibres of $\mathcal{R} \to \mathcal{P}$

The inclusion of the singular locus $\mathrm{Spec}(\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}])$ of $\mathcal{R}$ into $\mathcal{R}$ has the explicit form

$$(x_1, x_2) \in (\mathbb{C}^*)^2 \mapsto (x_1, x_2, x_1x_2 + x_1^{-1}x_2^{-1}, x_1 + x_1^{-1}, x_2 + x_2^{-1}, x_1x_2) \in \mathcal{R}(\mathbb{C}).$$

The image of the induced morphism $\mathrm{Spec}(\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]) \to \mathcal{P}$ lies in $\mathcal{P}_{\mathrm{red}} := \mathrm{Spec}(\mathbb{C}[s_1, s_2, s_3, s_3^{-1}]/(R_1))$, where R_1 is the irreducible element $R_1 = (s_3 + s_3^{-1})^2 - s_1s_2(s_3 + s_3^{-1}) + s_1^2 + s_2^2 - 4$. More precisely, since R_1 is irreducible, one has an inclusion $\mathbb{C}[s_1, s_2, s_3, s_3^{-1}]/(R_1) \to \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$, given by $s_1 \mapsto x_1 + x_1^{-1},\ s_2 \mapsto x_2 + x_2^{-1},\ s_3 \mapsto x_1x_2$. This identifies the first ring with the subring $\mathbb{C}[x_1 + x_1^{-1}, x_2 + x_2^{-1}, x_1x_2, x_1^{-1}x_2^{-1}]$ of the second one. It easily follows that $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ is the normalization of $\mathbb{C}[s_1, s_2, s_3, s_3^{-1}]/(R_1)$ in its field of fractions.

The singular locus of $\mathcal{P}_{\mathrm{red}}$ itself is easily computed to be the union of two disjoint components given by the ideals $(s_3 - 1, s_1 - s_2)$ and $(s_3 + 1, s_1 + s_2)$. The map $\tau : \mathrm{Spec}(\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]) \to \mathcal{P}_{\mathrm{red}}$ is an isomorphism outside the singular locus of $\mathcal{P}_{\mathrm{red}}$ and further satisfies:

$$\tau^{-1}(s_1, s_1, 1) = \left\{ \left(\frac{1}{2}\left(s_1 \pm \sqrt{s_1^2 - 4} \right), \frac{1}{2}\left(s_1 \mp \sqrt{s_1^2 - 4} \right) \right) \right\}$$

for $s_1 \neq \pm 2$ and $\tau^{-1}(\pm 2, \pm 2, 1) = \pm(1, 1)$;

$$\tau^{-1}(s_1,-s_1,-1)=\left\{\left(\frac{1}{2}\left(s_1\pm\sqrt{s_1^2-4}\right),-\frac{1}{2}\left(s_1\mp\sqrt{s_1^2-4}\right)\right)\right\}$$

for $s_1\neq\pm2$ and $\tau^{-1}(\pm2,\mp2,-1)=(\pm1,\mp1)$.

If for a fixed point $p\in\mathcal{P}$, the fibre $\pi^{-1}(p)$ has a singular point, then the ideal $(\frac{d}{dx_1}F(x_1,x_2,x_3,p),\frac{d}{dx_2}F(x_1,x_2,x_3,p),\frac{d}{dx_3}F(x_1,x_2,x_3,p))$ is not the unit ideal and it follows that the ideal $I:=(F,\frac{d}{dx_1}F,\frac{d}{dx_2}F,\frac{d}{dx_3}F)\cap\mathbb{C}[s_1,s_2,s_3,s_3^{-1}]$ lies in the maximal ideal of $\mathbb{C}[s_1,s_2,s_3,s_3^{-1}]$ defined by the point p. Using a Gröbner basis one verifies that I is generated by $(s_1^2-4)(s_2^2-4)R_1(s_1,s_2,s_3)$. Thus singular points in $\pi^{-1}(p)$ occur for p lying on one of the five divisors on $\mathcal{P}$ defined by $s_1=\pm2,\ s_2=\pm2,R_1=0$. The first four divisors correspond to resonance for the matrices M_1,M_2 and the last one to reducibility. A singular point in $\pi^{-1}(p)$, with p lying on only one of the divisors has type A_1. If p lies on more than one divisor, the singularity type can be different. The following table, of importance for the comparison with the Okamoto-Painlevé pairs, gives the rather complicated structure of the singularities of the fibres (see Table 3.1).

3.3. Family $(0,0,1/2)$ and Painlevé $\mathbf{PV_{deg}}$

A differential module of this type is irreducible and by Theorem 1.11 can be represented by a matrix differential equation of the form $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+A_\infty$ with $\mathrm{tr}(A_0)=\mathrm{tr}(A_1)=0$ and A_∞ nilpotent. The generalized eigenvalues at ∞ are $\pm t\cdot z^{1/2}$ and $t\in\mathbb{C}^*$. One may normalize by $A_\infty=\begin{pmatrix}0&t^2\\0&0\end{pmatrix}$.

For the computation of *monodromy space* $\mathcal{R}$ we give the solution space V at ∞ a basis e_1,e_2 such that $V_q=\mathbb{C}e_1,\ V_{-q}=\mathbb{C}e_2,\gamma e_1=e_2,\ \gamma e_2=-e_1$. Let M_0,M_1,M_∞ denote the topological monodromies at $0,1,\infty$ on the basis e_1,e_2. Then $M_\infty=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\e&1\end{pmatrix}$ and one finds $M_0M_1\begin{pmatrix}-e&-1\\1&0\end{pmatrix}=1$. Changing the basis at ∞ does not effect these data. Therefore $\mathcal{R}$ has dimension $3+3+1-3=4$ (for M_0,M_1,M_∞ and the 3 equations). One considers the map $\mathcal{R}\to\mathcal{P}:=\mathbb{C}\times\mathbb{C}$ which sends the tuple to $(s_0,s_1):=(\mathrm{tr}(M_0),\mathrm{tr}(M_1))$.

Write $M_1=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}$. The equation $M_0M_1M_\infty=1$ determines M_0 in terms of M_1,M_∞. In particular, $s_0=-c_1+b_1+a_1e$. Thus $\mathcal{R}$ is the space, given by the 5 variables a_1,b_1,c_1,d_1,e and the equation $a_1d_1-b_1c_1=1$. Use s_0 and $s_1=a_1+d_1$ to eliminate c_1 and d_1. Then the single equation between b_1,a_1,e,s_0,s_1 reads $a_1b_1e+a_1^2+b_1^2-a_1s_1-b_1s_0+1=0$. With the choice $x_1=-b_1,\ x_2=-a_1,\ x_3=e$ this equation is

$$x_1x_2x_3+x_1^2+x_2^2+s_0x_1+s_1x_2+1=0 \text{ and shows that } \mathcal{R}\to\mathcal{P}$$

s_1	s_2	R_1	Singular points (x_1, x_2, x_3)	Type of the singularities
2	$\neq \pm 2$	$\neq 0$	$(1, s_3, s_2)$	A_1
2	2	$\neq 0$	$(1, s_3, 2), (s_3, 1, 2)$	$A_1 + A_1$
2	2	0	$(1, 1, 2)$	A_3
2	-2	$\neq 0$	$(-s_3, -1, -2), (1, s_3, -2)$	$A_1 + A_1$
2	-2	0	$(1, -1, -2)$	A_3
2	$\neq \pm 2$	0	$(1, s_3, s_3 + s_3^{-1})$	A_2
-2	$\neq \pm 2$	$\neq 0$	$(-1, -s_3, -s_2)$	A_1
-2	2	$\neq 0$	$(-1, -s_3, -2), (s_3, 1, -2)$	$A_1 + A_1$
-2	2	0	$(-1, -1, -2)$	A_3
-2	-2	$\neq 0$	$(-1, -s_3, 2), (-s_3, -1, -2)$	$A_1 + A_1$
-2	-2	0	$(-1, -1, 2)$	A_3
-2	$\neq \pm 2$	0	$(-1, -s_3, s_3 + s_3^{-1})$	A_2
$\neq \pm 2$	2	$\neq 0$	$(s_3, 1, s_{)}$	A_1
$\neq \pm 2$	2	0	$(s_3, 1, s_3 + s_3^{-1})$	A_2
$\neq \pm 2$	-2	$\neq 0$	$(-s_3, -1, -s_1)$	A_1
$\neq \pm 2$	-2	0	$(-s_3, -1, s_3 + s_3^{-1})$	A_2
$\neq \pm 2$	$\neq \pm 2$	0	(a_1, a_2, a_3)	A_1
$\neq \pm 2$	s_1	0	$(\alpha, \beta, 2), (\beta, \alpha, 2)$	$A_1 + A_1$
$\neq \pm 2$	$-s_1$	0	$(\alpha, -\beta, 2), (-\beta, \alpha, 2)$	$A_1 + A_1$

This table of the singularities of the fibres uses the notation

$$(a_1, a_2, a_3) = (\frac{s_3^2 - 1}{s_2 s_3 - s_1}, \frac{s_3(s_2 s_3 - s_1)}{s_3^2 - 1}, s_3 + s_3^{-1}) \text{ and}$$

$$\alpha = \frac{1}{2}\left(s_1 + \sqrt{s_1^2 - 4}\right), \beta = \frac{1}{2}\left(s_1 - \sqrt{s_1^2 - 4}\right).$$

As usual, the symbol $A_n,\ n \geqslant 1$ stands for the surface singularity given by the local equation $x^2 + y^2 + z^{n+1} = 0$.

Table 3.1. Singularities for the monodomy spaces for PV.

is a family of affine cubic surfaces. We note that there are no reducible cases and that $\mathcal{R}$ is nonsingular. The singularities of the fibres occur only for the loci $s_0 = \pm 2$ and/or $s_1 = \pm 2$, corresponding to resonance. The fibres for $(s_0, s_1) = (\pm 2, \neq \pm 2)$ and $(s_0, s_1) = (\neq \pm 2, \pm 2)$ contain one singular point and the fibers for $(s_0, s_1) = (\pm 2, \pm 2)$ contain two singular points. All these surface singularities are of type A_1.

3.4. Family $(1, -, 1)$ and Painlevé PIII(D6)

Due to the ample choice of invariant lattices at 0 and at ∞, any differential module of this type can be represented by a matrix differential equation $z\frac{d}{dz}+A_0z^{-1}+A_1+A_2z$. By a transformation $z\mapsto \lambda z$ one arrives at eigenvalues $\pm\frac{t}{2}z^{-1}$ at 0 and $\pm\frac{t}{2}z$ at ∞ with $t\in\mathbb{C}^*$. Moreover one can normalize such that $A_2=\frac{t}{2}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. There are more normalizations possible.

The affine space *AnalyticData* is described as follows.

The formal solution space $V(0)$ at 0 is given a basis e_1,e_2 such that the formal monodromy, the Stokes matrices and the topological monodromy top_0 have the form

$$\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix},\ \begin{pmatrix}1&0\\a_1&1\end{pmatrix},\ \begin{pmatrix}1&a_2\\0&1\end{pmatrix},\ \begin{pmatrix}\alpha&\alpha a_2\\\alpha^{-1}a_1&\alpha^{-1}(1+a_1a_2)\end{pmatrix}.$$

The last matrix is written as $\left(\begin{smallmatrix}m_1&m_2\\m_3&m_4\end{smallmatrix}\right)$. It is characterized by $m_1\neq 0,\ m_1m_4-m_2m_3=1$ and it determines α,a_1,a_2. Moreover, $e_1\wedge e_2$ is a fixed global solution of the second exterior power.

The formal solution space $V(\infty)$ at ∞ is provided with a basis f_1,f_2, such that $f_1\wedge f_2$ is again this fixed global solution and the formal monodromy, the Stokes maps and the topological monodromy top_∞ have the matrices

$$\begin{pmatrix}\beta&0\\0&\beta^{-1}\end{pmatrix},\ \begin{pmatrix}1&0\\b_1&1\end{pmatrix},\ \begin{pmatrix}1&b_2\\0&1\end{pmatrix},\ \begin{pmatrix}\beta&\beta b_2\\\beta^{-1}b_1&\beta^{-1}(1+b_1b_2)\end{pmatrix}.$$

The link $L:V(0)\to V(\infty)$ satisfies:

(i) $\mathrm{top}_\infty\circ L=L\circ\mathrm{top}_0$, this follows from $M_1M_\infty=1$.

(ii) L maps $e_1\wedge e_2$ to $f_1\wedge f_2$. Thus the matrix $\left(\begin{smallmatrix}\ell_1&\ell_2\\\ell_3&\ell_4\end{smallmatrix}\right)$ of L w.r.t. the given bases has determinant 1.

One uses (i) to forget the data for ∞. The coordinate ring for *Analytic Data* is the localization of $\mathbb{C}[m_1,\dots,m_4,\ell_1,\dots,\ell_4]/(m_1m_4-m_2m_3-1,\ell_1\ell_4-\ell_2\ell_3-1)$, given by $0\neq\alpha=m_1$ and $0\neq\beta=\ell_1\ell_4m_1+\ell_2l_4m_3-\ell_1\ell_3m_2-\ell_2\ell_3m_4$.

The monodromy space $\mathcal{R}$ is obtained by dividing *AnalyticData* by the action of the elements $(\gamma,\delta)\in\mathbb{G}_m\times\mathbb{G}_m$, which is induced by the base change $e_1,e_2,f_1,f_2\mapsto\gamma e_1,\gamma^{-1}e_2,\delta f_1,\delta^{-1}f_2$.

The new matrices are $\begin{pmatrix}m_1&\gamma^2m_2\\\gamma^{-2}m_3&m_4\end{pmatrix}$ and $\begin{pmatrix}\gamma^{-1}\delta\ell_1&\gamma\delta\ell_2\\\gamma^{-1}\delta^{-1}\ell_3&\gamma\delta^{-1}\ell_4\end{pmatrix}$.

The ring of invariants for the action of $\mathbb{G}_m\times\mathbb{G}_m$ is computed to be (a localization of) $\mathbb{C}[m_1,m_4,\ell_1\ell_4,m_2\ell_1\ell_3,m_3\ell_2\ell_4]$. We note that m_2m_3 and $\ell_2\ell_3$ are omitted because of the determinant $=1$ relation. There is only one relation between these five generators namely (recall $\alpha=m_1$)

$$(m_2\ell_1\ell_3)\cdot(m_3\ell_2\ell_4)+(-\alpha m_4+1)\cdot(\ell_1\ell_4)\cdot(\ell_1\ell_4-1)=0.$$

Writing $y_1 := \ell_1\ell_4,\ y_2 := m_2\ell_1\ell_3,\ y_3 := m_3\ell_2\ell_4$ and using the formula for β one obtains the equation and the formula

$$y_2y_3 + (-\alpha m_4 + 1)y_1(y_1 - 1) = 0 \text{ and } \beta = \alpha y_1 + y_3 - y_2 - (y_1 - 1)m_4.$$

Using the formula for β one eliminates y_3 and obtains the equation

$$y_2(\beta - \alpha y_1 + y_2 + (y_1 - 1)m_4) + (-\alpha m_4 + 1)y_1(y_1 - 1) = 0.$$

For fixed α, β this is a cubic equation in y_1, y_2, m_4. After a series of simple transformations, one obtains the following equation for $\mathcal{R} \to \mathcal{P} = \mathbb{C}^* \times \mathbb{C}^*$

$$x_1x_2x_3 + x_1^2 + x_2^2 + (1 + \alpha\beta)x_1 + (\alpha + \beta)x_2 + \alpha\beta = 0.$$

The discriminant of $\mathcal{R} \to \mathcal{P}$ has the formula $(\alpha-\beta)^2(\alpha\beta-1)^2$ and therefore the fiber above (α, β) with $\alpha \neq \beta, \beta^{-1}$ is non singular. The singular locus of $\mathcal{R}$ consists of the two non intersecting lines

$$L_1: \quad \alpha = \beta,\ (x_1, x_2, x_3) = (0, -\alpha, \alpha + \alpha^{-1}) \text{ and}$$

$$L_2: \quad \alpha = \beta^{-1},\ (x_1, x_2, x_3) = (-1, 0, \alpha + \alpha^{-1}).$$

They correspond to the reducible connections (or equivalently reducible monodromy data). All the singularities of the fibres are obtained by intersecting with L_1 or L_2. The corresponding surface singularities are of type A_1. If $\alpha \neq \pm 1$ and $\beta = \alpha^{\pm 1}$, then there is only one singular point in the fiber. If $\alpha = \beta = \pm 1$, then the fiber has two singular points.

3.5. Family $(1/2, -, 1)$ and Painlevé PIII(D7)

By Theorem 1.2, any differential module of this type is represented by a matrix differential equation $z\frac{d}{dz} + A_0z^{-1} + A_1 + A_2z$. One may normalize $A_2 = \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} \end{pmatrix}$. After a transformation $z \mapsto \lambda z$ one may suppose that the eigenvalues at 0 are $\pm z^{-1/2}$ and $\pm\frac{t}{2} \cdot z$ at ∞. Assuming that A_0 and A_2 have no common eigenvector leads to the explicit family

$$z\frac{d}{dz} + A_0z^{-1} + A_1 + \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} \end{pmatrix} z.$$

For the description of the space *AnalyticData*, the formal solution space $V(0)$ at 0 is given the basis e_1, e_2 for which the formal monodromy, the Stokes matrix and topological monodromy top_0 have the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix},\ \begin{pmatrix} -e & -1 \\ 1 & 0 \end{pmatrix}.$$

The formal solution space $V(\infty)$ at ∞ is given a basis f_1, f_2 for which the formal monodromy, the Stokes maps and the topological monodromy top_∞ have the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & \alpha c_2 \\ \alpha^{-1}c_1 & \alpha^{-1}(1+c_1c_2) \end{pmatrix}.$$

One writes $\text{top}_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \neq 0$ and determinant 1. It is assumed that $e_1 \wedge e_2$ and $f_1 \wedge f_2$ are the same global solution of the second exterior power of the differential equation. The link $L : V(0) \to V(\infty)$ has therefore a matrix $\left(\begin{smallmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{smallmatrix}\right)$ with determinant 1.

The equation $\text{top}_0 \cdot L^{-1}\text{top}_\infty L = 1$ can be written as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = L\left(\begin{smallmatrix} 0 & 1 \\ -1 & -e \end{smallmatrix}\right)L^{-1}$. This eliminates $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (and thus α, c_1, c_2). The coordinate ring of *Analytic Data* is therefore $\mathbb{C}[\ell_1, \ldots, \ell_4, e]/(\ell_1\ell_4 - \ell_2\ell_3 - 1)$. The elements $\mu \in \mathbb{G}_m$ act on *AnalyticData* by the base change $f_1,\ f_2 \mapsto \mu f_1,\ \mu^{-1} f_2$. The elements $\ell_1, \ell_2, \ell_3, \ell_4$ have weights $-1, -1, 1, 1$ for this action.

The coordinate ring of $\mathcal{R}$ is generated by the variables $e, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}$ where $\ell_{ij} := \ell_i \ell_j$. There are two relations, namely $\ell_{14} - \ell_{23} = 1$ and $\ell_{14}\ell_{23} = \ell_{13}\ell_{24}$. $\mathcal{R}$ has dimension 3. The map $\mathcal{R} \to \mathcal{P} = \mathbb{C}^*$ is defined by: an element in $\mathcal{R}$ is mapped to $\alpha = -\ell_{24} - \ell_{13} - \ell_{23}e$, one of the eigenvalues of the formal monodromy at ∞. Eliminate $\ell_{14} = \ell_{23} + 1$. Then we have the equation $(\ell_{23} + 1)\ell_{23} + \ell_{13}(\alpha + \ell_{13} + \ell_{23}e) = 0$ (here ℓ_{24} is eliminated). We obtain a family $\mathcal{R} \to \mathcal{P} = \mathbb{C}^*$ of non singular affine cubic surfaces given by the equation $\ell_{13}\ell_{23}e + \ell_{13}^2 + \ell_{23}^2 + \alpha\ell_{13} + \ell_{23} = 0$.

3.6. Family $(1/2, -, 1/2)$ and Painlevé PIII(D8)

We consider differential modules of this type for which the strong Riemann–Hilbert problem has a solution (see Definition and examples 1.12, part (2)). Then a matrix differential equation $z\frac{d}{dz} + A_0 z^{-1} + A_1 + A_2 z$, with nilpotent A_0 and A_2, represents the module. Further assuming that the eigenvectors of A_0 and A_2 are distinct one can normalize to an equation of the form (see 4.7)

$$z\frac{d}{dz} + \begin{pmatrix} 0 & 0 \\ -q & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} \frac{p}{q} & -\frac{t}{q} \\ 1 & -\frac{p}{q} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z.$$

The space *AnalyticData* is build as follows. The formal solution space $V(0)$ at 0 is given a basis e_1, e_2, unique up to multiplication by the same constant, such that $V(0)_{z^{-1/2}} = \mathbb{C}e_1,\ V(0)_{-z^{-1/2}} = \mathbb{C}e_2$, such that the formal monodromy has matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. There is one Stokes matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. The topological monodromy at 0 is the product, *i.e.*, $\begin{pmatrix} -a & -1 \\ 1 & 0 \end{pmatrix}$.

At ∞, one has similarly a basis f_1, f_2 for $V(\infty)$ with topological monodromy $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} -b & -1 \\ 1 & 0 \end{pmatrix}$. The matrix of the link $L : V(0) \to V(\infty)$ with respect to these basis satisfies $L : e_1 \wedge e_2 \mapsto f_1 \wedge f_2$, because we assume, as we may, that $e_1 \wedge e_2$ and $f_1 \wedge f_2$ are the same global solution of the second exterior power. Thus $L =: \begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix}$ has determinant 1. The identity

$$\begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix}\begin{pmatrix} -a & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -b & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix}$$

describes the generators and relations of the coordinate ring of *Analytic Data*. The only admissible bases change is $e_1, e_2, f_1, f_2 \mapsto \lambda e_1, \lambda e_2, \lambda f_1, \lambda f_2$ with $\lambda \in \mathbb{C}^*$ acts trivially on *AnalyticData* and thus this space coincides with $\mathcal{R}$.

After elimination of b, ℓ_1, ℓ_3 one is left with the variables a, ℓ_2, ℓ_4 and one equation, namely $a\ell_2\ell_4 + \ell_4^2 - \ell_2^2 - 1 = 0$, or in other variables

$$x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0.$$

This defines a non singular affine cubic surface.

3.7. Family $(0, -, 2)$ and Painlevé PIV

The singularity at ∞ of a module of this type guarantees that the strong Riemann-Hilbert problem has a solution. There is a corresponding matrix differential equation which can be normalized to $z\frac{d}{dz} + A_0 + A_1 z + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2$. Using the transformation $z \mapsto \lambda z$ one may suppose that the eigenvalues at ∞ are $\pm(z^2 + \frac{t}{2} \cdot z)$. The ingredients for *AnalyticData* are the following.

The symbolic solution space at ∞ is written as $V_q \oplus V_{-q}$ and one takes a basis $\{e_1\}$ and $\{e_2\}$ for V_q and V_{-q}. With respect to the basis $\{e_1, e_2\}$ the topological monodromy top_∞ at ∞ has the form

$$\mathrm{top}_\infty = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix},$$

where the first matrix is the formal monodromy and the others are the 4 Stokes matrices. Let top_0 denote the monodromy at 0 written on the basis e_1, e_2. The condition $\mathrm{top}_0 \cdot \mathrm{top}_\infty = 1$ implies that top_∞ determines top_0. The coordinate ring of *AnalyticData* is $\mathbb{C}[\alpha, \alpha^{-1}, a_1, \ldots, a_4]$. An element $\lambda \in \mathbb{G}_m$ acts on *AnalyticData* by the base change $e_1, e_2 \mapsto \lambda e_1, \lambda^{-1} e_2$. The weights of $\alpha, a_1, a_2, a_3, a_4$ for this action are $0, +1, -1, +1, -1$. Therefore $\mathcal{R}$

has coordinate ring $\mathbb{C}[\alpha,\alpha^{-1},a_{12},a_{14},a_{23},a_{34}]$, where $a_{ij}:=a_ia_j$. There is only one relation namely $a_{12}a_{34}-a_{14}a_{23}=0$.

The *singular points* of $\mathcal{R}$ are given by the equations $a_{12}=a_{14}=a_{23}=a_{34}=0$. This coincides with the locus where the monodromy data (or equivalently the differential modules) are reducible (namely $a_1=a_3=0$ or $a_2=a_4=0$).

The morphism $\mathcal{R}\to\mathcal{P}:=\mathbb{C}\times\mathbb{C}^*$, where $\mathcal{P}$ is a space of parameters, is given by $(\alpha,a_1,\dots,a_4)\mapsto(\mathrm{tr}(\mathrm{top}_1),\alpha)$. Now $\mathrm{tr}(\mathrm{top}_1)=\mathrm{tr}(\mathrm{top}_\infty)$ and

$$\mathrm{tr}(\mathrm{top}_\infty)=\alpha(1+a_{23})+\alpha^{-1}(a_{14}+a_{34}+a_{12}+a_{12}a_{34}+1).$$

Write $s_2=\alpha$ and $s_1=\mathrm{tr}(\mathrm{top}_\infty)$ and exchange a_{23} with s_1 by the formula

$$a_{23}=s_2^{-1}s_1-s_2^{-2}(a_{14}+a_{34}+a_{12}+a_{12}a_{34}+1)-1.$$

Then the coordinate ring of $\mathcal{R}$ has the form $\mathbb{C}[s_1,s_2,s_2^{-1},a_{12},a_{14},a_{34}]$ and there is one relation, namely

$$a_{12}a_{14}a_{34}+(s_2^2a_{12}a_{34}+a_{14}^2+a_{14}a_{12}+a_{14}a_{34})+a_{14}(1+s_2^2-s_1s_2)=0.$$

One makes the following substitutions

$$a_{14}=x_1-s_2^2,\ a_{12}=x_2-1,\ a_{34}=x_3-1 \text{ and the relation } R \text{ reads}$$

$$x_1x_2x_3+x_1^2-(s_2^2+s_1s_2)x_1-s_2^2x_2-s_2^2x_3+s_2^2+s_1s_2^3=0,$$

and thus $\mathcal{R}\to\mathcal{P}=\mathbb{C}\times\mathbb{C}^*$ is a family of affine cubic surfaces.

3.7.1. Singular loci of $\mathcal{R}$ and the fibres of $\mathcal{R}\to\mathcal{P}$

We have already remarked that the singular points of $\mathcal{R}$ correspond to reducibility and are given by $x_1=s_2^2,\ x_2=1,\ x_3=1,\ s_1=s_2+s_2^{-1}$.

For a fixed $s=(s_1,s_2)\in\mathcal{P}$, the singular locus of the fibre is given by the (relative) Jacobian ideal, generated by $R,\partial R/\partial x_1,\partial R/\partial x_2,\partial R/\partial x_3$. A Gröbner basis for this ideal produces the following results.

The fiber has singular points if and only if s satisfies the equation

$$\Delta(s):=(s_1-2)(s_1+2)(s_2^2-s_1s_2+1)=0.$$

We define three divisors of $\mathcal{P}$ by $D_1^\pm=\{s_1=\pm2\},D_{\mathrm{red}}=\{s_2^2-s_1s_2+1=0\}$. We have seen that D_{red} corresponds to the locus of the *reducible differential equations*. Further $s_1=e^{\pi i\theta_0}+e^{-\pi i\theta_0}$, where $\pm\theta_0/2$ are the local exponents at $z=0$ of the differential equation. Thus $s_1=\pm2$ corresponds to the *resonant case* $\theta_0\in\mathbb{Z}$. The table gives the singularities and their type of the fibres.

	$s=(s_1,s_2)$	(x_1,x_2,x_3)	Type
D_1^+ not $D_{\rm red}$	$(2,s_2), s_2\neq 1$	(s_2,s_2,s_2)	A_1
D_1^- not $D_{\rm red}$	$(-2,s_2), s_2\neq -1$	$(-s_2,-s_2,-s_2)$	A_1
$D_{\rm red}$ not $D_1^{\pm}$	$(s_2+s_2^{-1},s_2), s_2\neq \pm 1$	$(s_2^2,1,1)$	A_1
$D_1^+\cap D_{\rm red}$	$(2,1)$	$(1,1,1)$	A_2
$D_1^-\cap D_{\rm red}$	$(-2,-1)$	$(1,1,1)$	A_2

3.8. Family $(0,-,3/2)$ and Painlevé PIIFN

A module of this type can be represented, by Theorem 1.2, by a matrix differential equation $z\frac{d}{dz}+A_0+A_1z+A_2A_2$ with A_2 nilpotent. One can use the transformation $z\mapsto\lambda z$ and choose a basis such that the explicit form is $z\frac{d}{dz}+\left(\begin{smallmatrix} a & c\\ -b & -a\end{smallmatrix}\right)+\left(\begin{smallmatrix} 0 & t+b\\ 1 & 0\end{smallmatrix}\right)z+\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)z^2$. The eigenvalues at ∞ are $\pm(z^{3/2}+\frac{t}{2}\cdot z^{1/2})$.

The space *AnalyticData* is formed as follows. The formal solution space V at ∞ is given a basis e_1,e_2 such that the formal monodromy and the three Stokes maps have the matrices

$$\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\ \begin{pmatrix} 1 & 0\\ a_1 & 0\end{pmatrix},\ \begin{pmatrix} 1 & a_2\\ 0 & 1\end{pmatrix},\ \begin{pmatrix} 1 & 0\\ a_3 & 1\end{pmatrix}.$$

The topological monodromy top_∞ is the product of these matrices and top_0 is the inverse of top_∞. Further, the base change $e_1,e_2\mapsto\lambda e_1,\lambda e_2$ does not effect the matrices. It follows that the coordinate ring of $\mathcal{R}$ is $\mathbb{C}[a_1,a_2,a_3]$. One computes that the trace s of the topological monodromy at 0 is $s=-a_1a_2a_3-a_1+a_2-a_3$. The map $\mathcal{R}\to\mathcal{P}$ is given by $(a_1,a_2,a_3)\mapsto s$. Thus $\mathcal{R}\to\mathcal{P}$ is a family of affine cubic surfaces given by the equation $a_1a_2a_3+a_1-a_2+a_3+s=0$.

The *singularities of the fibres* occur only for the resonant case $\theta_0\in\mathbb{Z}$, where $\pm\theta_0/2$ are the local exponents at $z=0$. Since $s=e^{\pi i\theta_0}+e^{-\pi i\theta_0}$, this corresponds to $s=\pm 2$. For $s=2$ one finds one singular point $(a_1,a_2,a_3)=(-1,1,-1)$ and for $s=-2$ one singular point $(a_1,a_2,a_3)=(1,-1,1)$. The type of the singularity is A_1 in both cases.

3.9. Family $(-,-,3)$ and Painlevé PII

The family of connections. A differential module of this type can be represented by a matrix differential equation $\frac{d}{dz}+A_0+A_1z+A_2z^2$, because of the singularity at ∞. Using a transformation $z\mapsto\lambda z+\mu$ and by choosing a suitable basis one arrives at the explicit form, having eigenvalues $\pm(z^3 + \frac{t}{2}\cdot z)$ at ∞, namely

$$\frac{d}{dz} + \begin{pmatrix} a_{10}+z^2 & a_{21}z+a_{20} \\ a_{31}z+a_{30} & -a_{10}-z^2 \end{pmatrix} \text{ and } t = a_{10} + a_{21}a_{31}/2.$$

The group $\mathbb{G}_m$ acts by conjugation, in fact by $a_{21}z + a_{20} \mapsto \lambda(a_{21}z + a_{20})$ and $a_{31}z + a_{30} \mapsto \lambda^{-1}(a_{31}z + a_{30})$. In general, this cannot be used to normalize the equation even further. (See Subsection 4.10).

The space *AnalyticData* consists of the formal monodromy and six Stokes matrices. The formal solution space V at ∞ is given a basis e_1, e_2 such that the formal monodromy and the six Stokes maps have the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & b_6 \\ 0 & 1 \end{pmatrix}.$$

The product of all of them is the topological monodromy at ∞ and hence is equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The coordinate ring of *AnalyticData* is therefore generated by $\alpha, \alpha^{-1}, b_1, \dots, b_6$ and the matrix identity defines the ideal of the relations $I \subset \mathbb{C}[\alpha, \alpha^{-1}, b_1, \dots, b_6]$. The basis e_1, e_2 is unique up to the action of the elements $\lambda \in \mathbb{G}_m$, given by $e_1, e_2 \mapsto \lambda e_1, \lambda^{-1}e_2$.

Call the six Stokes matrices $M_1, \dots, M_6$. Then $M_3M_4M_5M_6$ is equal to

$$\begin{pmatrix} \alpha^{-1}(1+b_1b_2) & -\alpha b_2 \\ -\alpha^{-1}b_1 & \alpha \end{pmatrix} \text{ which is the inverse of } \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} M_1M_2.$$

We note that the product of the three matrices determines α, b_1, b_2. Further one computes that $\alpha = b_3b_6 + (1+b_3b_4)(1+b_5b_6)$. Thus the coordinate ring of *AnalyticData* is $\mathbb{C}[b_3, b_4, b_5, b_6, \frac{1}{b_3b_6+(1+b_3b_4)(1+b_5b_6)}]$. For the group $\mathbb{G}_m$ the variables b_1, b_3, b_5 have weight -1, the variables b_2, b_4, b_6 have weight $+1$ and α has weight 0. Write $b_{ij} := b_ib_j$ for $i < j$. Then the coordinate ring of $\mathcal{R}$ is the ring of the $\mathbb{G}_m$-invariant elements and this is

$$\mathbb{C}\left[b_{34}, b_{36}, b_{45}, b_{56}, \frac{1}{b_{36} + (1+b_{34})(1+b_{56})}\right].$$

There is only one relation, namely $b_{34}b_{56} = b_{36}b_{45}$. We use the identity $\alpha = b_{36} + (1+b_{34})(1+b_{56})$ to exchange b_{36} with α. Then the coordinate ring of $\mathcal{R}$ has the form $\mathbb{C}[\alpha, \alpha^{-1}, b_{34}, b_{45}, b_{56}]$ and there is only one relation now. Define $x_1 = b_{34} + 1 = \mathrm{tr}(M_3M_4) - 1,\ x_2 = b_{45} + 1 = \mathrm{tr}(M_4M_5) - 1,\ x_3 = b_{56} + 1 = \mathrm{tr}(M_5M_6) - 1$. Then this relation reads $x_1x_2x_3 - x_1 - \alpha x_2 - x_3 + \alpha + 1 = 0$ and defines a family $\mathcal{R} \to \mathcal{P} = \mathbb{C}^*$ of cubic surfaces.

The locus in the affine space *AnalyticData* of reducible data has two components. The first one is given by $\alpha = 1,\ b_1 = b_3 = b_5 = 0$ and the second one by $\alpha = 1,\ b_2 = b_4 = b_6 = 0$. These loci are mapped to the unique singular point $\alpha = 1,\ x_1 = x_2 = x_3 = 1$ of $\mathcal{R}$.

For $\alpha \neq 1$, the affine cubic surface, given by the above equation, has no singularities. The infinite part of the cubic surface consists of three lines. The three intersection points of these lines are the infinite singularities. The cubic surface for $\alpha = 1$ has one extra singular point, namely $x_1 = x_2 = x_3 = 1$. (This cubic surface is the Cayley surface). The type of the surface singularities is A_1.

3.10. Family $(-,-,5/2)$ and Painlevé PI

According to Definition and examples 1.12, a differential module of this type need not have a solution for the strong Riemann-Hilbert problem. We deal here with the modules for which there is a solution, *i.e.*, are represented by a matrix differential equation $\frac{d}{dz} + A_0 + A_1 z + A_2 z^2$ with nilpotent A_2 which can be normalized into $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The map $z \mapsto \lambda z + \mu$ is used to normalize the eigenvalues at ∞ to $\pm(z^{5/2} + \frac{t}{2} \cdot z^{1/2})$. Conjugation with a constant matrix of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ leads to the normalization

$$\frac{d}{dz} + \begin{pmatrix} p & t+q^2 \\ -q & -p \end{pmatrix} + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2.$$

The space *AnalyticData* is given by the formal monodromy and 5 Stokes maps which are on a basis e_1, e_2 of the formal solution space at ∞ given by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_5 & 1 \end{pmatrix}.$$

Their product is the topological monodromy and thus equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The base change $e_1, e_2 \mapsto \lambda e_1, \lambda e_2$ does not effect these matrices. Hence the coordinate ring of $\mathcal{R}$ is generated by $a_1, \ldots, a_5$ and their relations are given by the above matrix identity.

After eliminating a_2 by $a_2 = 1 + a_4 a_5$ and a_1 by $a_1 = -1 - a_3 a_4$, one obtains for the remaining variables a_3, a_4, a_5 just one equation and $\mathcal{R}$ is a non singular affine cubic surface with three lines at infinity, given by $a_3 a_4 a_5 + a_3 + a_5 + 1 = 0$.

4. The Painlevé equations

4.1. Finding the Painlevé equations

For each of the ten families of Section 3, with the exception of $(0,0,0,0)$, which is the well known classical case leading to PVI, we want to derive a corresponding Painlevé equation $q'' = R(q, q', t)$.

We choose one of the other nine cases. A Zariski open part $\mathcal{M}^0$ of the corresponding moduli space is represented by a suitable matrix differential operator. Recall that there is a morphism $pr : \mathcal{M}^0 \to T \times \Lambda$, where T denotes the space of the "time variable" t and the parameter space Λ consists of the local exponents for the regular singular points and the constant term of the generalized local exponents at the irregular singular points.

Choose $\lambda \in \Lambda$, let $a \in \mathcal{P}$ be the image of λ in the parameter space of $\mathcal{R}$. Write $\mathcal{M}^0_\lambda = pr^{-1}(T \times \{\lambda\})$ and $\mathcal{R}_a$ for the fibre of $\mathcal{R} \to \mathcal{P}$ at a. Then the Riemann–Hilbert map restricts to $RH_\lambda : \mathcal{M}^0_\lambda \to \mathcal{R}_a$ and the fibres of RH_λ are parametrized by t. In particular, $\mathcal{M}^0_\lambda$ has dimension 3. This space is represented by an explicit family of differential operators $\frac{d}{dz} + A$, where the entries of A are rational functions in z with coefficients depending on three explicit variables, say f, g, t. Later on we will make a rather special choice for f, g.

An isomonodromic family $\frac{d}{dz} + A = \frac{d}{dz} + A(z,t)$ on $\mathbb{P}^1$, parametrized by t, is a fibre of some RH_λ. The earlier variables f, g are now functions of t. Let S denote the singular locus. On $\mathbb{P}^1 \setminus S$ there exists a multivalued fundamental matrix $Y(z,t)$, *i.e.*, $(\frac{d}{dz} + A(z,t))Y(z,t) = 0$, normalized by $\det Y(z,t) = 1$. By isomonodromy, $\frac{d}{dt}Y(z,t)$ and $Y(z,t)$ have the same behaviour for Stokes and monodromy and thus $B(z,t) := -\frac{d}{dt}Y(z,t) \cdot Y(z,t)^{-1}$ is univalued and extends in a meromorphic way at the set S. Moreover $B := B(z,t)$ has trace 0 since $\det Y(z,t) = 1$. Therefore the entries of B are rational functions in z and are analytic in t. It follows that the operators $\frac{d}{dz} + A(z,t)$ and $\frac{d}{dt} + B(z,t)$ commute. This is equivalent to the identity

$$\frac{d}{dt}A = \frac{d}{dz}B + [B, A] \text{ and } \operatorname{tr}(B) = 0.$$

This equality is seen as a differential equation for matrices B, rational in z and with trace 0. Assume (as we will in the examples) that $\frac{d}{dz} + A$ is irreducible, then B is unique. Indeed, the difference C of two solutions is rational in z and satisfies $\frac{d}{dz}C = [C, A]$. Thus $C(\frac{d}{dz} + A) = (\frac{d}{dz} + A)C$ and C is an endomorphism of $\frac{d}{dz} + A$. By irreduciblity, C is a multiple of the identity and $C = 0$ because $\operatorname{tr}(C) = 0$.

For the actual computation of B for the cases of Subsection 2.2, the following remarks are useful. If $z = c$ is a regular, or regular singular point (without resonance), then B has no pole at c. If the Katz invariant $r = r(c) > 0$ is an integer, and the top coefficient of the eigenvalues at c do not depend on t, then $\mathrm{ord}_c(B) \geqslant -r+1$. If however this top coefficient depends on t, then $\mathrm{ord}_c(B) \geqslant -r$. If the Katz invariant $r(c) = m + \frac{1}{2}$ with integer $m \geqslant 0$, then $\mathrm{ord}_c(B) \geqslant -m-1$. The above matrix equation yields explicit differential equations for f, g as functions of t, and an explicit B.

The symbols p, q denote a *preferred choice* for the variables f, g. To define and find them we consider a pair $(t, \lambda) \in T \times \Lambda$ and the 2-dimensional space $\mathcal{M}^0_{t,\lambda} := pr^{-1}(\{(t,\lambda)\})$. Let $\frac{d}{dz} + A$ be the corresponding matrix differential operator and let a cyclic vector e be given. The monic scalar differential operator $L := (\frac{d}{dz})^2 + a_1 \frac{d}{dz} + a_0$ defined by $Le = 0$ has, in general, a number of new singularities, called *apparent singular points*. In Subsection 4.2, we will find *good cyclic vectors* e, defined by the condition that there is only one apparent singular point. This singular point, varying in the family $\mathcal{M}^0_{(t,\lambda)}$, is the choice for q. To make this explicit, suppose that $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ and that the first basis vector is the cyclic vector e. Then

$$L = (\frac{d}{dz})^2 - \frac{c'}{c} \cdot \frac{d}{dz} - a' - a^2 - bc + a \cdot \frac{c'}{c}, \text{ where } a' = \frac{da}{dz} \text{ etc.}$$

Thus c has as rational function in z a simple zero at q and this yields a pole at q with residue 1 in the coefficient of $\frac{d}{dz}$ in L. Now p is defined as the residue at q of the "constant term" $-a' - a^2 - bc + a \cdot \frac{c'}{c}$ of L, multiplied by a factor $F \in \{1, q, q^2, q(q-1)\}$ depending on the family $\mathcal{M}$. This factor is introduced for geometrical reasons in connection with the Okamoto–Painlevé pairs [25, 29] (see the formulas of 4.3).

A Zariski open, dense part of the space $\mathcal{M}^0_{(t,\lambda)}$ is now parametrized by p, q. On this space we introduce the *symplectic structure* by the closed 2-form $\frac{dp \wedge dq}{F}$ (with $F \in \{1, q, q^2, q(q-1)\}$) and thus p, q are canonical coordinates. The Zariski open subset of the space $\mathcal{M}^0_\lambda$ is parametrized by p, q, t. This space has a foliation given by the isomonodromy families, *i.e.*, the fibres of RH_λ. There is an *Hamiltonian function* $H = H(p, q, t)$, rational in p, q and t, such that this foliation coincides with the foliation deduced from the closed 2-form $\Omega = \frac{dp \wedge dq}{F} - dH \wedge dt$ on $\mathcal{M}^0_\lambda$. More precisely, the vector field $v = \frac{\partial}{\partial t} + v_p \frac{\partial}{\partial p} + v_q \frac{\partial}{\partial q}$ describing ismonodromic families satisfies $v \cdot \Omega = 0$ (see [25], Section 6 and [29], Subsection (2.3)).

The important fact is that for an isomonodromic family, q as function of t satisfies the Painlevé equation $q'' = R(q, q', t)$ that we are looking for.

The functions p, q of t satisfy the Hamiltonian equations, modified with the factor $F \in \{1, q, q^2, q(q-1)\}$, thus $p' = F \cdot \frac{\partial H}{\partial q},\ q' = -F \cdot \frac{\partial H}{\partial p}$.

4.2. Apparent singularities

Let M denote a differential module over $\mathbb{C}(z)$ of rank 2 with $\det M \cong \mathbf{1}$, with singular points $0, 1, \infty$ and represented by a connection $(\mathcal{V}, \nabla)$ with $\mathcal{V}$ free and $\nabla : \mathcal{V} \to \Omega(n_0[1] + n_1[1] + n_\infty[\infty]) \otimes \mathcal{V}$ for integers $n_0, n_1, n_\infty \geqslant 1$. Put $V := H^0(\mathbb{P}^1, \mathcal{V})$. Then $M = \mathbb{C}(z) \otimes V$ and $\partial := \nabla_{\frac{d}{dz}} = \frac{d}{dz} + B_0 + B_1 + B_\infty$ with $B_0,\ B_1,\ B_\infty$ polynomials in $z^{-1},\ (z-1)^{-1},\ z$ of degrees $\leqslant n_0,\ n_1, -1 + n_\infty$ and with coefficients in $\mathrm{End}(V)$. The free module $N := \mathbb{C}[z, \frac{1}{z(z-1)}] \otimes V$ over $\mathbb{C}[z, \frac{1}{z(z-1)}]$ is invariant under ∂ and can be considered as a differential module over $\mathbb{C}[z, \frac{1}{z(z-1)}][\frac{d}{dz}]$.

Let $e \in M = \mathbb{C}(z) \otimes V$ be a cyclic vector, producing the scalar equation $(\partial^2 + a_1\partial + a_0)e = 0$. The poles of a_1, a_0, different from $0, 1, \infty$ are called the *apparent singularities*. Let $s \neq 0, 1, \infty$ have local parameter $u = z - s$. The elements $e \in \mathbb{C}((u)) \otimes V$ are written as formal Laurent series $\sum_{n \geqslant *} v_n u^n$ with all $v_n \in V$. Now $\mathrm{ord}_s(e)$, the order of $e \neq 0$ at s, is defined to be the minimal integer d with $v_d \neq 0$.

We will use the second exterior power $\Lambda^2 N = \mathbb{C}[z, \frac{1}{z(z-1)}] \otimes \Lambda^2 V$. Let $e \in N$ be a cyclic vector and $N^0 \subset N$ the submodule generated by e and ∂e. Then $\Lambda^2 N^0 = b \cdot \Lambda^2 N$ for some monic polynomial b with $b(0) \neq 0,\ b(1) \neq 0$. The following lemma is an explicit calculation corresponding to [13], Subsection 4.2.

Lemma 4.1. — *The zero's of b are the apparent singular points.*

Proof. — We fix a point $s \neq 0, 1, \infty$ and show that $\mathrm{ord}_s(b) > 0$ if and only if s is an apparent singularity. First we consider the case that $\mathrm{ord}_s(e) = 0$. Since s is non singular, $\mathbb{C}[[u]] \otimes V$ has a free basis w_1, w_2 over $\mathbb{C}[[u]]$ with $\partial w_1 = \partial w_2 = 0$. Write $e = c_1 w_1 + c_2 w_2$ with $\min(\mathrm{ord}(c_1), \mathrm{ord}(c_2)) = 0$. We may suppose that $\mathrm{ord}(c_1) = 0$ and $\mathrm{ord}(c_2) = m \geqslant 1$. The equation for the cyclic vector is $\partial^2 + a_1\partial + a_0$ with $a_1 = \frac{(-c_1'' c_2 + c_1 c_2'')}{(c_1 c_2' - c_1' c_2)},\ a_0 = \frac{(-c_1'' c_2' + c_1' c_2'')}{(c_1 c_2' - c_1' c_2)}$. If $m = 1$, then $\mathrm{ord}_s(c_1 c_2' - c_1' c_2) = 0$ and s is not an apparent singularity. If $m \geqslant 2$, then

$$\mathrm{ord}_s(c_1 c_2' - c_1' c_2) = m - 1,\ \mathrm{ord}_s(-c_1'' c_2 + c_1 c_2'') = m - 2,$$
$$\mathrm{ord}_s(-c_1'' c_2' + c_1' c_2'') \geqslant m - 2.$$

Thus $\mathrm{ord}_s(a_1) = -1, \mathrm{ord}_s(a_0) \geqslant -1$ and s is an apparent singularity.

Suppose now that $e = u^n f$, $n \geqslant 1, \mathrm{ord}_s(f) = 0$. The equation for e is obtained from the scalar equation $\partial^2 + a_1\partial + a_0$ for f by the substitution $\partial \mapsto \partial - nu^{-1}$ and reads $\partial^2 + (-2nu^{-1} + a_1)\partial + (n^2+n)u^{-2} - na_1u^{-1} + a_0$. This introduces a pole if there was no pole before and a pole of order 2 if there was already a pole.

For e with $\mathrm{ord}_s(e)$ one has $e \wedge \partial e = (c_1c_2' - c_1'c_2)w_1 \wedge w_2$ and $\mathrm{ord}_s(c_1c_2' - c_1'c_2) = m-1$ and for $e = u^n f$ one has $e \wedge \partial e = u^{2n} f \wedge \partial f$. From this the statement follows. □

By multiplying a given cyclic vector e with $\prod_{s\neq 0,1}(z-s)^{-\mathrm{ord}_s(e)}$, the number of zero's of b (counted with multiplicity) goes down. The cyclic vectors with minimal degree for b have the form $e \in \mathbb{C}[z, \frac{1}{z(z-1)}] \otimes V$ (or even, after multiplying with $z^*(z-1)^*$ one has $e \in \mathbb{C}[z] \otimes V$) and $\mathrm{ord}_s(e) = 0$ for all $s \neq 0,1$. We note that the condition $\mathrm{ord}_s(e) = 0$ is equivalent to b has at most simple zero's. We call a cyclic vector e *good* if the corresponding b has degree one (and thus there is only one apparent singularity).

Application of Lemma 4.1 for finding good cyclic vectors $v \in V$.

Family $(0,0,1)$, $\partial = \frac{d}{dz} + z^{-1}A_0 + (z-1)^{-1}A_1 + A_\infty$ and $A_0, A_1, A_\infty \in \mathrm{End}(V)$. We only consider good cyclic vectors $v \in V$. The operator ∂ is multiplied by $z(z-1)$. The condition that v produces only one apparent singularity is equivalent to $v \wedge (z(z-1)A_\infty(v) + (z-1)A_0(v) + zA_1(v))$ (as element of $\mathbb{C}[z] \otimes \Lambda^2 V$) has only one zero $\neq 0,1$. This is equivalent to v is an eigenvector of A_∞ or A_0 or A_1. Thus in total there are 6 good cyclic vectors, in general.

Family $(0,0,1/2)$, same formula for ∂ but with A_∞ nilpotent. The good cyclic vectors are the eigenvectors of the matrices A_0, A_1, A_∞. There are, in general, 5 good cyclic vectors.

Family $(1,-,1)$. Now we work with a module over $\mathbb{C}[z, z^{-1}]$. Here $z^2\partial = z^2\frac{d}{dz} + A_2 + A_1z + A_0z^2$. The condition on v is $v \wedge (A_2(v) + A_1(v)z + A_0(v)z^2)$ has only one zero $\neq 0$. Thus v is an eigenvector of A_2 or of A_0. In general, there are total 4 good cyclic vectors and they come in pairs.

Family $(1/2,-,1)$. As before, but now A_2 is nilpotent and, in general, there are in 3 good cyclic vectors.

Family $(1/2,-,1/2)$. As before, but both A_2 and A_0 are nilpotent. Thus, in general, 2 good cyclic vectors.

Family $(0,-,2)$. $z\partial = z\frac{d}{dz} + A_0 + zA_1 + z^2A_2$. Then a good cyclic vector is eigenvector of A_0 or of A_2. Thus, in general 4 good cyclic vectors.

Family $(0,-,3/2)$. As before, but now, in general, 3 good cyclic vectors because A_2 is nilpotent.

Family $(-,-,3)$. Now we work over the ring $\mathbb{C}[z]$ and $\partial = \frac{d}{dz} + A_0 + zA_1 + z^2A_2$. The possible v are eigenvector for A_2. Thus 2 good cyclic vectors.
Family $(-,-,5/2)$. As before, but only one good cyclic vector, since A_2 is nilpotent.

Remarks 4.2. — (1) In general there are more complicated good cyclic vectors, than those in V. However for $(-,-,5/2)$ there is only one good cyclic vector.

(2) If a cyclic vector $v \in V$ is eigenvector of two of the matrices, then the corresponding scalar equation has no apparent singularity.

4.3. Family $(0,0,1)$ and Painlevé V, PV($\tilde{D}_5$)

In terms of the parameters (p,q) of Subsection 4.1, the family reads

The singularities z	0	1	∞
Katz invariant	0	0	1
generalized local exponents	$\pm\frac{\theta_0}{2}$	$\pm\frac{\theta_1}{2}$	$\pm(\frac{t}{2}z + \frac{\theta_\infty}{2})$

$$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + A_\infty = \frac{d}{dz} + \frac{1}{z(z-1)}A \quad \text{with} \tag{4.1}$$

$$A_0 = \begin{pmatrix} -p - \frac{1}{2}q\,(qt - t + \theta_\infty) & \frac{(q-1)\left(\left(p+\frac{1}{2}q(qt-t+\theta_\infty)\right)^2 - \frac{\theta_0^2}{4}\right)}{q} \\ -\frac{q}{q-1} & p + \frac{1}{2}q\,(qt - t + \theta_\infty) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} p + \frac{(q-1)(qt+\theta_\infty)}{2} & \left(\frac{\theta_1}{2}\right)^2 - \left(p + \frac{(q-1)(qt+\theta_\infty)}{2}\right)^2 \\ 1 & -\left(p + \frac{(q-1)(qt+\theta_\infty)}{2}\right) \end{pmatrix}$$

$$A_\infty = \begin{pmatrix} -\frac{t}{2} & 0 \\ 0 & \frac{t}{2} \end{pmatrix}.$$

Write $A = z(z-1)(A_\infty + A_0/z + A_1/(z-1)) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}$ with

$$\begin{aligned}
a(z) &= p + \frac{1}{2}\left(-2qt + t - \theta_\infty\right)(z-q) - \frac{1}{2}t(z-q)^2, \\
b(z) &= -\frac{z\left(\left(p+\frac{1}{2}(q-1)q\right)^2 - \frac{\theta_1^2}{4} + (q-1)\left(\frac{\theta_0^2}{4} - \frac{\theta_1^2}{4} - \frac{q\theta_\infty^2}{4}\right)\right)}{q} \\
&\quad - \frac{(q-1)\left(p+\frac{1}{2}\left((q-1)qt - \theta_0 + q\theta_\infty\right)\right)\left(p+\frac{1}{2}\left((q-1)qt + \theta_0 + q\theta_\infty\right)\right)}{q}, \\
c(z) &= -\frac{(z-q)}{q-1}.
\end{aligned}$$

The first basis vector is chosen as cyclic vector and following Subsection 4.1, q is the unique zero of $c(z)$ and $p = a(q)$.

The parameter (p,q) gives canonical coordinates on an affine Zariski open set $U_0 \cong \mathbb{C} \times (\mathbb{C} \setminus \{0,1\})$ of the moduli space $\mathcal{M}_{t,\lambda}$ of the connections with fixed t and fixed generalized local exponents λ. The symplectic form on U_0, which is natural from the view point of Okamoto–Painlevé pair, is given by $\frac{dp\wedge dq}{q(q-1)}$.

The matrix different operator $\frac{d}{dt} + B$, commuting with $\nabla_{\frac{d}{dz}}$, has the form

$$B = zB_0 + \frac{1}{t}B_1 \text{ where } B_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and}$$

$$B_1 = \begin{pmatrix} -\frac{p}{q-1} - \frac{1}{2}(q-1)t - \frac{\theta_\infty}{2} & -\frac{\left(p+\frac{1}{2}(q-1)qt\right)^2 - \frac{\theta_1^2}{4} + (q-1)\left(\frac{\theta_0^2}{4} - \frac{\theta_1^2}{4} - \frac{q\theta_\infty^2}{4}\right)}{q} \\ -\frac{1}{q-1} & \frac{p}{q-1} + \frac{1}{2}(q-1)t + \frac{\theta_\infty}{2} \end{pmatrix}.$$

From $\left[\frac{d}{dt} + B, \nabla_{\frac{d}{dz}}\right] = 0$ one deduces the following.

Painlevé V, PV($\tilde{D}_5$)

(4.2)
$$\left\{\begin{aligned}
\frac{dq}{dt} &= \frac{2p}{t} \\
\frac{dp}{dt} &= \frac{(2q-1)p^2}{(q-1)qt} + \frac{\theta_0^2(q-1)^2 - \theta_1^2 q^2}{4q(q-1)t} + \frac{1}{4}(q-1)q\left(2qt - t + 2\theta_\infty - 2\right).
\end{aligned}\right.$$

The equation (4.2) is equivalent to the second order differential equation of q

(4.3)
$$q'' = \frac{(2q-1)\left(q'\right)^2}{2(q-1)q} - \frac{q'}{t} + \frac{(q-1)q\left(2qt - t + 2\theta_\infty - 2\right)}{2t} + \frac{\theta_0^2}{2qt^2} + \frac{\theta_1^2}{2(q-1)t^2}.$$

By a rational transformation of (p,q), this can be transformed into the classical Painlevé V in [16].

Now we compute the Hamiltonian function $H_V = H_V(p,q,t,\theta)$ for (4.2), which is a rational function of (p,q,t). It is defined by the property that the foliation given by the 2-form $\Omega = \frac{dp\wedge dq}{q(q-1)} - dH_V \wedge dt$ on $U_0 \times (\mathbb{C}\setminus\{0\})$ coincides with the foliation given by isomonodromy. The latter is given by the vector field v, equivalent to (4.2), satisfying $v\cdot\Omega = 0$ and of the form

$$v = \frac{\partial}{\partial t} + v_p\frac{\partial}{\partial p} + v_q\frac{\partial}{\partial q}, \text{ with } v_p = \frac{dp}{dt},\ v_q = \frac{dq}{dt}.$$

Now $0 = v\cdot\Omega = dH_V + v_p\dfrac{dq}{q(q-1)} - v_q\dfrac{dp}{q(q-1)}$, is equivalent to

$$\left\{\begin{array}{ccc} \dfrac{dp}{dt} & = & q(q-1)\frac{\partial H_V}{\partial q} \\ \dfrac{dq}{dt} & = & -q(q-1)\frac{\partial H_V}{\partial p}. \end{array}\right. \tag{4.4}$$

Comparing this with (4.2), one obtains the following expression for H_V

$$H_V(p,q,t) = -\frac{p^2}{(q-1)qt} - \frac{\theta_0^2}{4qt} + \frac{\theta_1^2}{4(q-1)t} + \frac{1}{4}q\left(qt - t + 2\theta_\infty - 2\right). \tag{4.5}$$

4.4. Family $(0,0,1/2)$ and degenerate Painlevé V, $\mathrm{PV}_{\mathrm{deg}}(\tilde{D}_6)$

$\mathrm{PV}_{\mathrm{deg}}$ stands for "degenerate PV" (*cf.* [21]) which turns out to be equivalent to Painlevé equation of type $\mathrm{PIII}(\tilde{D}_6)$. The first basis vector is chosen as cyclic vector, the (p,q) are as in Subsection 4.1 and the family reads

The singular points z	0	1	∞
Katz invariant	0	0	$\frac{1}{2}$
generalized local exponents	$\pm\frac{\theta_0}{2}$	$\pm\frac{\theta_1}{2}$	$\pm tz^{\frac{1}{2}}$

$$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + A_\infty = \frac{d}{dz} + \frac{1}{z(z-1)}A \text{ with} \tag{4.6}$$

$$A_0 = \begin{pmatrix} -p & \frac{\theta_0^2-4p^2}{4q} \\ q & p \end{pmatrix} \quad A_1 = \begin{pmatrix} p & \frac{4p^2-\theta_1^2}{4(q-1)} \\ 1-q & -p \end{pmatrix} \quad A_\infty = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} p & L \\ z-q & -p \end{pmatrix},\ L := \frac{(q+z-1)p^2}{(q-1)q} + \frac{(z-1)\theta_0^2}{4q} - \frac{z\theta_1^2}{4(q-1)} + t^2(z-1)z.$$

The operator $\frac{d}{dt}+B$ with $\left[\frac{d}{dt}+B,\nabla_{\frac{d}{dz}}\right]=0$ satisfies $B=zB_0+B_1$, where

$$B_0=\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}, B_1=\begin{pmatrix} 0 & \dfrac{2p^2}{(q-1)qt}+\dfrac{\theta_0^2}{2qt}-\dfrac{\theta_1^2}{2(q-1)t}+2(q-1)t \\ \dfrac{2}{t} & 0 \end{pmatrix}.$$

Solving $\left[\frac{d}{dt}+B,\nabla_{\frac{d}{dz}}\right]=0$ with *Mathematica* yields the following.

Degenerate Painlevé V, $\mathrm{PV}_{\mathrm{deg}}(\tilde{D}_6)$

$$\text{(4.7)}\qquad \begin{cases} \dfrac{dq}{dt} &=& \dfrac{4p}{t} \\ \dfrac{dp}{dt} &=& \dfrac{2(2q-1)p^2}{(q-1)qt}+\dfrac{(q-1)\theta_0^2}{2qt}-\dfrac{q\theta_1^2}{2(q-1)t}+2q(q-1)t \end{cases}$$

$$\text{(4.8)}\qquad q''=\frac{(2q-1)\left(q'\right)^2}{2(q-1)q}-\frac{q'}{t}+\frac{2(q-1)\theta_0^2}{qt^2}-\frac{2q\theta_1^2}{(q-1)t^2}+8(q-1)q.$$

The 2-form on $\mathbb{C}\times(\mathbb{C}\setminus\{0,1\})\times(\mathbb{C}\setminus\{0\})$, natural for the Okamoto–Painlevé pair of type $\tilde{D}_6$, is given by

$$\Omega=\frac{dp\wedge dq}{q(q-1)}-dH_{dV}\wedge dt,\ \text{ where } H_{dV}=H_{dV}(p,q,t,\theta) \text{ is equal to}$$

$$\text{(4.9)}\qquad H_{dV}(p,q,t,\theta) = -\frac{2p^2}{(q-1)qt}-\frac{\theta_0^2}{2qt}+\frac{\theta_1^2}{2(q-1)t}+2qt,$$

$$\text{(4.10)}\qquad = \frac{2(p^2-\left(\frac{\theta_0}{2}\right)^2)}{tq}-\frac{2(p^2-\left(\frac{\theta_1}{2}\right)^2)}{t(q-1)}+2qt.$$

We note that equation (4.7) is equivalent to the Hamiltonian system

$$\text{(4.11)}\qquad \begin{cases} \dfrac{dq}{dt} &=& -q(q-1)\dfrac{\partial H_{dV}}{\partial p}, \\ \dfrac{dp}{dt} &=& q(q-1)\dfrac{\partial H_{dV}}{\partial q}. \end{cases}$$

4.5. Family $(1,-,1)$ and Painlevé III, PIII($\tilde{D}_6$)

As before, the first basis vector is chosen to be the cyclic vector, (p,q) are as introduced in Subsection 4.1 and the operator $\frac{d}{dt}+B$ commuting with $\nabla_{\frac{d}{dz}}$ has the form $B=zB_0+B_1+\frac{1}{z}B_2$. We present now the data.

$$\text{(4.12)}\qquad \nabla_{\frac{d}{dz}}=\frac{d}{dz}+\frac{1}{z^2}A_0+\frac{1}{z}A_1+A_2=\frac{d}{dz}+\frac{1}{z^2}A.$$

The singular points z	0	∞
Katz invariant	1	1
generalized local exponents	$\pm(\frac{t}{2}z^{-1}+\frac{\theta_0}{2})$	$\pm(\frac{t}{2}z+\frac{\theta_\infty}{2})$

$$A_0=\begin{pmatrix} \frac{1}{2}\left(-tq^2-\theta_\infty q+2p\right) & \frac{t^2q^4+2t\theta_\infty q^3+\theta_\infty^2q^2-4ptq^2-4p\theta_\infty q+4p^2-t^2}{4q} \\ -q & \frac{1}{2}\left(tq^2+\theta_\infty q-2p\right)\end{pmatrix},$$

$$A_1=\begin{pmatrix} \frac{\theta_\infty}{2} & \frac{t^2q^4-\theta_\infty^2q^2-4ptq^2-2t\theta_0q+4p^2-t^2}{4q^2} \\ 1 & -\frac{\theta_\infty}{2}\end{pmatrix},\quad A_2=\begin{pmatrix}\frac{t}{2} & 0\\ 0 & -\frac{t}{2}\end{pmatrix},$$

$$A=\begin{pmatrix} p+\frac{1}{2}(z-q)\left(qt+zt+\theta_\infty\right) & L \\ (z-q) & -p-\frac{1}{2}(z-q)\left(qt+zt+\theta_\infty\right)\end{pmatrix},$$

$$\begin{aligned} L &= \frac{q\left(tq^2+\theta_\infty q-2p+t\right)\left(tq^2+\theta_\infty q-2p-t\right)}{4q^2} \\ &\qquad\qquad +\frac{z\left(t^2q^4-\left(\theta_\infty^2+4pt\right)q^2-2t\theta_0q+4p^2-t^2\right)}{4q^2}. \end{aligned}$$

$$B_0=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}, B_1=\begin{pmatrix} q+\frac{\theta_\infty}{2t} & \frac{t^2q^4-\theta_\infty^2q^2-4ptq^2-2t\theta_0q+4p^2-t^2}{4q^2t} \\ \frac{1}{t} & -q-\frac{\theta_\infty}{2t}\end{pmatrix},$$

$$B_2=\begin{pmatrix} \frac{tq^2+\theta_\infty q-2p}{2t} & \frac{-4p^2+\left(1-q^4\right)t^2+2q^2t(2p-q\theta_\infty)+q\theta_\infty(4p-q\theta_\infty)}{4qt} \\ \frac{q}{t} & \frac{-tq^2-\theta_\infty q+2p}{2t}\end{pmatrix}.$$

Solving the equation $[\frac{d}{dt}+B,\nabla_{\frac{d}{dz}}]=0$ yields the following.

Painlevé III, PIII($\tilde{D}_6$).

$$\left\{\begin{aligned} \frac{dq}{dt} &= \frac{4p+q}{t} \\ \frac{dp}{dt} &= \frac{4p^2}{qt}+\frac{p}{t}+tq^3+q^2-\frac{t}{q}-\theta_0+q^2\theta_\infty\,. \end{aligned}\right. \tag{4.13}$$

The system (4.13) is equivalent to the following second order equation.

$$q''=\frac{(q')^2}{q}-\frac{q'}{t}-\frac{4\theta_0}{t}+\frac{4(\theta_\infty+1)q^2}{t}+4q^3-\frac{4}{q}. \tag{4.14}$$

The equations (4.13) or (4.14) are defined on $\mathbb{C}\times\mathbb{C}\setminus\{0\}\times\mathbb{C}\setminus\{0\}$ and the 2-form Ω on this affine open set is

$$\Omega=\frac{dp\wedge dq}{q^2}-dH_{III}\wedge dt$$

where $H_{III} = H_{III}(p,q,t,\theta)$ is a Hamiltonian function for PIII given by

$$H_{III}(p,q,t,\theta) = -\frac{2p^2}{q^2t} - \frac{p}{qt} + q + \frac{q^2t}{2} + \frac{t}{2q^2} + \frac{\theta_0}{q} + q\theta_\infty. \tag{4.15}$$

As before, the equation (4.13) is equivalent to the Hamiltonian system:

$$\begin{cases} \dfrac{dq}{dt} & = & -q^2\dfrac{\partial H_{III}}{\partial p}, \\ \dfrac{dp}{dt} & = & q^2\dfrac{\partial H_{III}}{\partial q}. \end{cases} \tag{4.16}$$

4.6. Family $(1/2,-,1)$: Painlevé IIID_7, PIII($\tilde{D}_7$)

This family can be written as

$$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + \frac{1}{z^2}A_0 + \frac{1}{z}A_1 + A_2 = \frac{d}{dz} + \frac{1}{z^2}A. \tag{4.17}$$

The items p,q,B are as before and the form of B is $zB_0 + B_1$. We give now the explicit data and the results on the Painlevé equation and the Hamiltonian.

The singular points z	0	∞
Katz invariant	$1/2$	1
generalized local exponents	$\pm z^{-1/2}$	$\pm(\frac{t}{2}z + \frac{\theta_\infty}{2})$

$$A_0 = \begin{pmatrix} \frac{1}{2}\left(-tq^2 - \theta_\infty q + 2p\right) & \frac{\left(tq^2+\theta_\infty q-2p\right)^2}{4q} \\ -q & \frac{1}{2}\left(tq^2 + \theta_\infty q - 2p\right) \end{pmatrix}$$

$$A_1 = \begin{pmatrix} \frac{\theta_\infty}{2} & \frac{t^2q^4-\theta_\infty^2q^2-4ptq^2-4q+4p^2}{4q^2} \\ 1 & -\frac{\theta_\infty}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} \end{pmatrix} \text{ and}$$

$$A = \begin{pmatrix} p + \frac{1}{2}(z-q)\left(qt+zt+\theta_\infty\right) & \frac{q(-tq^2-\theta_\infty q+2p)^2+z(t^2q^4-\theta_\infty^2q^2-4ptq^2-4q+4p^2)}{4q^2} \\ (z-q) & -p - \frac{1}{2}(z-q)\left(qt+zt+\theta_\infty\right) \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{q}{2} + \frac{\theta_\infty}{2t} & \frac{t^2q^4-\left(\theta_\infty^2+4pt\right)q^2-4q+4p^2}{4q^2t} \\ \frac{1}{t} & -\frac{q}{2} - \frac{\theta_\infty}{2t} \end{pmatrix}.$$

Painlevé IIID_7, PIII($\tilde{D}_7$)

$$\left\{\begin{array}{rcl} \dfrac{dq}{dt} & = & \dfrac{2p}{t} \\ \\ \dfrac{dp}{dt} & = & \dfrac{2p^2}{tq}+\dfrac{tq^3}{2}+\dfrac{1}{2}\left(\theta_\infty+1\right)q^2-\dfrac{1}{t}. \end{array}\right. \tag{4.18}$$

The system (4.18) is equivalent to the following second order equation.

$$q''=\frac{(q')^2}{q}-\frac{q'}{t}+\frac{(\theta_\infty+1)\,q^2}{t}+q^3-\frac{2}{t^2} \tag{4.19}$$

$$H_{IIID_7}(p,q,t,\theta) \quad = \quad -\frac{p^2}{q^2t}+\frac{q^2t}{4}+\frac{1}{2}q\left(\theta_\infty+1\right)+\frac{1}{qt}. \tag{4.20}$$

$$\left\{\begin{array}{rcl} \dfrac{dq}{dt} & = & -q^2\dfrac{\partial H_{IIID_7}}{\partial p}, \\ \dfrac{dp}{dt} & = & q^2\dfrac{\partial H_{IIID_7}}{\partial q}. \end{array}\right. \tag{4.21}$$

4.7. Family $(1/2,-,1/2)$: Painlevé IIID_8, PIII($\tilde{D}_8$)

We present the data and the results of the computation.

$$\nabla_{\frac{d}{dz}}=\frac{d}{dz}+\frac{1}{z^2}A_0+\frac{1}{z}A_1+A_2=\frac{d}{dz}+\frac{1}{z^2}A. \tag{4.22}$$

The singular points z	0	∞
Katz invariant	$1/2$	1
generalized local exponents	$\pm\sqrt{t}\cdot z^{-1/2}$	$\pm z^{1/2}$

$$A_0=\begin{pmatrix} 0 & 0 \\ -q & 0 \end{pmatrix},\quad A_1=\begin{pmatrix} \frac{p}{q} & -\frac{t}{q} \\ 1 & -\frac{p}{q} \end{pmatrix},\quad A_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A=\begin{pmatrix} \frac{pz}{q} & \frac{z(qz-t)}{q} \\ z-q & -\frac{pz}{q} \end{pmatrix},\ B=B_0+\frac{1}{z}B_1 \text{ where}$$

$$B_0=\begin{pmatrix} 0 & \frac{1}{q} \\ 0 & 0 \end{pmatrix},\quad B_1=\begin{pmatrix} 0 & 0 \\ \frac{q}{t} & 0 \end{pmatrix}.$$

Painlevé IIID_8, PIII(D_8).

$$\left\{\begin{array}{rcl} \dfrac{dq}{dt} &=& \dfrac{2p+q}{t} \\ \\ \dfrac{dp}{dt} &=& \dfrac{2p^2}{qt}+\dfrac{p}{t}+\dfrac{q^2}{t}-1=\dfrac{q^3+pq-tq+2p^2}{qt}. \end{array}\right. \tag{4.23}$$

The system (4.23) is equivalent to the following second order equation.

$$q''=\frac{(q')^2}{q}-\frac{q'}{t}+\frac{2q^2}{t^2}-\frac{2}{t} \tag{4.24}$$

$$\Omega=\frac{dp\wedge dq}{q^2}-dH_{IIID_8}\wedge dt,\ \ H_{IIID_8}=-\frac{p^2}{q^2t}-\frac{p}{qt}+\frac{1}{q}+\frac{q}{t}. \tag{4.25}$$

The equation (4.23) is equivalent to the following Hamiltonian system:

$$\left\{\begin{array}{rcl} \dfrac{dq}{dt} &=& -q^2\dfrac{\partial H_{IIID_8}}{\partial p}, \\ \dfrac{dp}{dt} &=& q^2\dfrac{\partial H_{IIID_8}}{\partial q}. \end{array}\right. \tag{4.26}$$

4.8. Family $(0,-,2)$ and Painlevé IV, PIV($\tilde{E}_6$)

The family of connection with this data can be written as

$$\nabla_{\frac{d}{dz}}=\frac{d}{dz}+\frac{1}{z}A_0+A_1+zA_2=\frac{d}{dz}+\frac{1}{z}A, \tag{4.27}$$

$$A_0=\begin{pmatrix} -q^2-\dfrac{tq}{2}+p & \dfrac{q^4+tq^3+\frac{t^2q^2}{4}-2pq^2-tpq+p^2-\frac{\theta_0^2}{4}}{q} \\ -q & q^2+\dfrac{tq}{2}-p \end{pmatrix},$$

$$A_1=\begin{pmatrix} \frac{t}{2} & 2q^2+tq-2p+\theta_\infty \\ 1 & -\frac{t}{2} \end{pmatrix},\quad A_2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The singular points z	0	∞
Katz invariant	0	2
generalized local exponents	$\pm\frac{\theta_0}{2}$	$\pm(z^2+\frac{t}{2}z+\frac{\theta_\infty}{2})$

$$B=zB_1+B_2, B_1=\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, B_2=\begin{pmatrix} \frac{q}{2}+\frac{t}{4} & q^2+\frac{tq}{2}-p+\frac{\theta_\infty}{2} \\ \frac{1}{2} & -\frac{q}{2}-\frac{t}{4} \end{pmatrix}.$$

Painlevé IV, PIV($\tilde{E}_6$)

$$\left\{\begin{array}{lcl} \dfrac{dq}{dt} & = & p \\[2ex] \dfrac{dp}{dt} & = & \dfrac{3q^3}{2}+tq^2+\dfrac{1}{8}\left(t^2+4\theta_\infty+4\right)q+\dfrac{4p^2-\theta_0^2}{8q} \\ & = & \dfrac{12q^4+8tq^3+t^2q^2+4\theta_\infty q^2+4q^2+4p^2-\theta_0^2}{8q}. \end{array}\right. \tag{4.28}$$

The system (4.28) is equivalent to the following second order equation.

$$q''=\frac{(q')^2}{2q}+\frac{3q^3}{2}+tq^2+\frac{1}{8}\left(t^2+4\theta_\infty+4\right)q-\frac{\theta_0^2}{8q}. \tag{4.29}$$

$$H_{IVE_6}(p,q,t,\theta)=-\frac{p^2}{2q}+\frac{q^3}{2}+\frac{tq^2}{2}+\frac{(t^2+4\theta_\infty+4)q}{8}+\frac{\theta_0^2}{8q}. \tag{4.30}$$

Equation (4.28) is equivalent to the following Hamiltonian system:

$$\left\{\begin{array}{lcl} \dfrac{dq}{dt} & = & -q\dfrac{\partial H_{IVE_6}}{\partial p}, \\ \dfrac{dp}{dt} & = & q\dfrac{\partial H_{IVE_6}}{\partial q}. \end{array}\right. \tag{4.31}$$

4.9. Family $(0,-,3/2)$ and Painlevé II, PIIFN($\tilde{E}_7$)

PIIFN($\tilde{E}_7$) stands for the Flaschka–Newell equation [4] which is equivalent to the Painlevé equation PII. A family of connection with this data is

(4.32)

$$\nabla_{\frac{d}{dz}}=\frac{d}{dz}+\frac{1}{z}A_0+A_1+zA_2=\frac{d}{dz}+\frac{1}{z}A,\ A=\begin{pmatrix} p & \frac{p^2+qz^2-\theta_0^2+q^2z-2qtz}{q} \\ z-q & -p \end{pmatrix}.$$

$$A_0=\begin{pmatrix} p & \frac{p^2-\frac{\theta_0^2}{4}}{q} \\ -q & -p \end{pmatrix},\quad A_1=\begin{pmatrix} 0 & q+t \\ 1 & 0 \end{pmatrix},\quad A_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$B:=B_0+zB_1,\ B_1=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\quad B_0=\begin{pmatrix} 0 & 2q+t \\ 1 & 0 \end{pmatrix}.$$

The singular points z	0	∞
Katz invariant	0	$3/2$
generalized local exponents	$\pm\frac{\theta_0}{2}$	$\pm(z^{3/2}+\frac{t}{2}z^{1/2})$

Painlevé II, PIIFN ($\tilde{E}_7$)

$$\left\{\begin{array}{lcl}\dfrac{dq}{dt} & = & 2p \\ \dfrac{dp}{dt} & = & \dfrac{\left(2q^3+tq^2+p^2-\frac{\theta_0^2}{4}\right)}{q} = 2q^2+tq+\dfrac{p^2-\frac{\theta_0^2}{4}}{q}.\end{array}\right. \tag{4.33}$$

The system (4.33) is equivalent to the following second order equation.

$$\begin{aligned} q'' &= \frac{(q')^2}{2q}+4q^2+2tq-\frac{\theta_0^2}{2q} \qquad (4.34)\\ \Omega &= \frac{dp\wedge dq}{q}-dH_{IIFNE_7}\wedge dt \quad \text{with}\end{aligned}$$

$$H_{IIFNTE_7} = -\frac{p^2-\frac{\theta_0^2}{4}}{q}+q^2+tq. \tag{4.35}$$

Equation (4.33) is equivalent to the following Hamiltonian system:

$$\left\{\begin{array}{lcl}\dfrac{dq}{dt} & = & -q\dfrac{\partial H_{IIFNE_7}}{\partial p}, \\ \dfrac{dp}{dt} & = & q\dfrac{\partial H_{IIFNE_7}}{\partial q}.\end{array}\right. \tag{4.36}$$

4.10. Family $(-,-,3)$ and Painlevé II, PII($\tilde{E}_7$)

We present the data and the results of the computation.

$$\nabla_{\frac{d}{dz}} = \frac{d}{dz}+A_0+zA_1+z^2A_2 = \frac{d}{dz}+A \text{ where} \tag{4.37}$$

$$A_0 = \begin{pmatrix} p-q^2 & 2q^3-2pq+tq+\theta_\infty \\ -q & q^2-p \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 2q^2-2p+t \\ 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A = \begin{pmatrix} p+z^2-q^2 & (q+1)t-2(p-q^2)(z+q)z+\theta_\infty \\ z-q & -p-z^2+q^2 \end{pmatrix}.$$

$$B := B_0+zB_1,\ B_0 = \begin{pmatrix} \frac{q}{2} & q^2-p+\frac{t}{2} \\ \frac{1}{2} & -\frac{q}{2} \end{pmatrix},\ B_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The singular points z	∞
Katz invariant	3
generalized local exponents	$\pm(z^3+\frac{t}{2}z+\frac{\theta_\infty}{2})$

Painlevé II, PII($\tilde{E}_7$)

$$\begin{cases} \dfrac{dq}{dt} &= p \\ \dfrac{dp}{dt} &= 2q^3+tq+\frac{\theta_\infty+1}{2}. \end{cases} \tag{4.38}$$

$$q''=2q^3+qt+\frac{\theta_\infty+1}{2} \tag{4.39}$$

$$\Omega = dp\wedge dq - dH_{IIE_7}\wedge dt, \text{ where}$$

$$H_{IIE_7}(p,q,t,\theta)=\frac{1}{2}(-p^2+q^4+tq^2+(\theta_\infty+1)q). \tag{4.40}$$

Equation (4.38) is equivalent to the following Hamiltonian system:

$$\begin{cases} \dfrac{dq}{dt} &= -\dfrac{\partial H_{IIE_7}}{\partial p}, \\ \dfrac{dp}{dt} &= \dfrac{\partial H_{IIE_7}}{\partial q}. \end{cases} \tag{4.41}$$

4.11. Family $(-,-,5/2)$ and Painlevé I, PI($\tilde{E}_8$)

The family of connection with the data can be written as

The singular points z	∞
Katz invariant	$\frac{5}{2}$
generalized local exponents	$\pm(z^{5/2}+\frac{t}{2}z^{1/2})$

$$\nabla_{\frac{d}{dz}}=\frac{d}{dz}+A_0+zA_1+z^2A_2=\frac{d}{dz}+A, \text{ where} \tag{4.42}$$

$$A_0=\begin{pmatrix} p & q^2+t \\ -q & -p \end{pmatrix}, \quad A_1=\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \quad A_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.43}$$

$$A = \begin{pmatrix} p & q^2 + zq + z^2 + t \\ z - q & -p \end{pmatrix}.$$

$$B := B_0 + zB_1, \quad B_0 = \begin{pmatrix} 0 & 2q \\ 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Painlevé I, PI($\tilde{E}_8$)

$$\left\{ \begin{array}{lcl} \dfrac{dq}{dt} & = & 2p \\ \\ \dfrac{dp}{dt} & = & 3q^2 + t \end{array} \right. \tag{4.44}$$

The system (4.44) is equivalent to the following second order equation.

$$q'' = 6q^2 + 2t \tag{4.45}$$

(4.46) $\Omega = dp \wedge dq - dH_{IE_8} \wedge dt, \ H_{IE_8}(p, q, t, \theta) \quad = \quad -p^2 + q^3 + tq$

Equation (4.44) is equivalent to the following Hamiltonian system:

$$\left\{ \begin{array}{lcl} \dfrac{dq}{dt} & = & -\dfrac{\partial H_{IE_8}}{\partial p}, \\ \dfrac{dp}{dt} & = & \dfrac{\partial H_{IE_8}}{\partial q}. \end{array} \right. \tag{4.47}$$

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