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Giovanna CARNOVALE

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SPHERICAL CONJUGACY CLASSES AND THE BRUHAT DECOMPOSITION

by Giovanna CARNOVALE (*)

Abstract. — Let G be a connected, reductive algebraic group over an algebraically closed field of zero or good and odd characteristic. We characterize spherical conjugacy classes in G as those intersecting only Bruhat cells in G corresponding to involutions in the Weyl group of G.

Résumé. — Soit G un groupe algébrique réductif connexe, sur un corps algébriquement clos de caractéristique zéro ou bonne et impaire. Nous caractérisons les classes de conjugaison sphériques de G comme celles ayant une intersection seulement avec des cellules de Bruhat de G correspondantes à des involutions dans le groupe de Weyl de G.

Introduction

The Bruhat decomposition of a connected reductive algebraic group G over an algebraically closed field states that the two-sided cosets of G with respect to a Borel subgroup B (Bruhat cells) are naturally parametrized by the elements in the Weyl group of G and have a well-understood geometrical behaviour. It is a fundamental tool in the theory of algebraic groups, as it is relevant for the comprehension of the geometry of the flag variety G/B, for instance, in the computation of its cohomology. Besides, intersection of Bruhat cells corresponding to opposite Borel subgroups (double Bruhat cells) play a significant role in the description of the symplectic leaves of a natural Poisson structure on B ([10]). New interest has been raised by Bruhat cells and double Bruhat cells for their applications to total positivity

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([13]) and to the theory of cluster algebras. For instance, as it has been very recently shown, double Bruhat cells serve as a geometric model for cluster algebras of finite type, since every cluster algebra of finite type with principal coefficients at an arbitrary acyclic initial cluster can be realized as the coordinate ring of a certain reduced double Bruhat cell ([26]).

The interplay between conjugacy classes in an algebraic group and the Bruhat decomposition has been successfully exploited in the past. Probably the first results in this sense are in [24] where the Bruhat decomposition of a semisimple algebraic group G is used for the construction of a cross-section for the collection of regular conjugacy classes of G.

More recently, [11] and [12] have provided an analysis of the intersection of conjugacy classes in a Chevalley group with Bruhat cells corresponding to generalized Coxeter elements and their conjugates.

If we consider spherical conjugacy classes, that is, those conjugacy classes of a group G on which B acts with finitely many orbits, it is natural to inquire about their intersection with Bruhat cells. A characterization of spherical conjugacy classes has been given in terms of a formula involving the dimension of the class $\mathcal{O}$ and the maximal element w in the Weyl group W of G for which $\mathcal{O} \cap BwB$ is non-empty. This is obtained in [5] over the complex numbers and in [6] over an arbitrary algebraically closed field of zero or odd good characteristic. The motivation in [5] was the proof - in the spherical case - of a conjecture due to De Concini, Kac and Procesi on the dimension of irreducible representations of quantum groups at the roots of unity ([9]). The proof relied on the classifications of spherical nilpotent orbits ([19]) and of reductive spherical pairs ([4]) and on geometric properties of spherical homogeneous spaces in the complex setting ([4],[18]). In [6] a different approach was developed and a crucial step in the argument was that every spherical conjugacy class intersects only Bruhat cells BwB for w an involution in W. The aim of the present paper is to show that this property fully characterizes spherical conjugacy classes.

Theorem. — *Let G be a connected reductive algebraic group over an algebraically closed field of zero or good, odd characteristic. A conjugacy class $\mathcal{O}$ in G is spherical if and only if $\mathcal{O}$ intersects only Bruhat cells corresponding to involutions in the Weyl group of G.*

The paper is structured as follows: after fixing notation and recalling basic facts about spherical homogeneous spaces and conjugacy classes in §1, we analyse the case of G simple of type G_2 in full detail in §2. The reason for doing so is twofold. On the one hand we would like to give an idea of the techniques involved through an example, and on the other hand it would not be more efficient to treat the case of G_2 together with the others because separate descriptions for behaviour of roots with different length ratios are needed.

In §3 we restrict our attention to those conjugacy classes intersecting only Bruhat cells corresponding to involutions. For such a class $\mathcal{O}$ we consider the maximal element $w \in W$ for which $\mathcal{O} \cap BwB$ is non-empty and the set of B-orbits in $\mathcal{O}$ that are contained in BwB, the so-called maximal B-orbits. The properties of a special class of representatives x of maximal B-orbits are analyzed, allowing a description of the centralizer B_x in B. This is achieved by using the same strategy as in [6]. The proofs therein are rather laborious and need a case-by-case analysis but they apply also to the present situation so we use them referring to [6]. The hypothesis on the class $\mathcal{O}$ imposes restrictions on the representatives x in maximal B-orbits: for instance, if $x = \dot{w}\mathrm{v} \in N(T)U$ then v lies in the subgroup generated by the root subgroups X_α for which $w\alpha = -\alpha$. This condition is powerful for a general w but it is empty when w is the longest element w_0 in W and it acts as -1 in the geometric representation. For this reason we deal with this situation separately and an unpleasant case-by-case analysis is needed in the doubly-laced case. This is done in §4, where the theorem in this case is proved by showing the sufficient condition that the maximal B-orbits are finitely-many.

The rest of the paper is devoted to an estimate of the centralizer G_x in G of a representative x in a maximal B-orbit. Indeed, since $\mathcal{O}$ is parted into finitely many B-orbits if and only if it has a dense B-orbit ([3, 15, 17, 25]), we may conclude that $\mathcal{O}$ is spherical once we prove that the dimension of a maximal B-orbit equals the dimension of $\mathcal{O}$. In §5 we consider the general case and we construct some families of elements contained in $G_x \cap X_\alpha s_\alpha B$ for different roots α. We need different strategies according to the behaviour of α with respect to w. In particular, when $w\alpha = -\alpha$ we apply the results in §4. Once we have constructed enough elements in G_x we show using the intersections $G_x \cap B\sigma B$ and induction on the length of σ that the image of G_x through the projection of G on G/B is dense in the flag variety obtaining the sought equality of dimensions.

1. Preliminaries

Unless otherwise stated G will denote a connected, reductive algebraic group over an algebraically closed field k of characteristic 0 or odd and good ([23, §I.4]). When we write an integer as an element in k we shall mean the image of that integer in the prime field of k.

Let B be a Borel subgroup of G, let T be a maximal torus contained in B and let B^- be the Borel subgroup opposite to B. Let U (respectively U^-) be the unipotent radical of B (respectively B^-).

We shall denote by Φ the root system relative to (B,T); by $\Delta=\{\alpha_1,\dots,\alpha_n\}$ the corresponding set of simple roots and by Φ^+ the corresponding set of positive roots. We shall use the numbering of the simple roots in [2, Planches I-IX].

We shall denote by W the Weyl group associated with G and by s_α the reflection corresponding to the root α. By $\ell(w)$ we shall denote the length of the element $w\in W$ and by $\mathrm{rk}(1-w)$ we shall mean the rank of $1-w$ in the geometric representation of the Weyl group. By w_0 we shall denote the longest element in W and ϑ will be the automorphism of Φ given by $-w_0$. By Π we shall always denote a subset of Δ and $\Phi(\Pi)$ will indicate the corresponding root subsystem of Φ. We shall denote by W_Π the parabolic subgroup of W generated by the s_α for α in Π. Given an element $w\in W$ we shall denote by $\dot{w}$ a representative of w in the normalizer $N(T)$ of T. For any root α in Φ we shall write $x_\alpha(t)$ for the elements in the corresponding root subgroup X_α of G. Moreover, we choose $x_\alpha(1)$ and $x_\alpha(-1)$ so that $x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)=n_\alpha\in s_\alpha T$ so that the properties in [22, Lemma 8.1.4] hold.

If $\Pi\subset\Delta$ we shall indicate by P_Π the standard parabolic subgroup of G whose Levi component contains the root subgroups corresponding to roots in $\Phi(\Pi)$ and by P_Π^u its unipotent radical. If $\Pi=\{\alpha\}$ we shall simply write P_α and P_α^u.

For $w\in W$, we will put

$$\Phi_w:=\{\alpha\in\Phi^+\mid w^{-1}\alpha\in-\Phi^+\}\tag{1.1}$$

$$U^w=\langle X_\alpha\mid\alpha\in\Phi_w\rangle,\quad U_w=\langle X_\alpha\mid\alpha\in\Phi^+\setminus\Phi_w\rangle\tag{1.2}$$

so that $BwB=U^w\dot{w}B$ for any choice of $\dot{w}\in N(T)$. We shall denote by T^w the subgroup of T that is centralized by any representative $\dot{w}$ of w.

We shall make extensive use of Chevalley's commutator formula ([7, Theorem 5.2.2]): for α and β linearly independent roots and a, $b\in k$ there are structure constants $c_{\alpha\beta}^{ij}$ in the prime field of k such that

$$x_\alpha(a)x_\beta(b)=x_\beta(b)x_\alpha(a)\prod_{i,\,j>0}x_{i\alpha+j\beta}(c_{\alpha,\beta}^{ij}a^ib^j)\tag{1.3}$$

where the product is taken over all (i,j) such that $i\alpha + j\beta \in \Phi$ and in any order for which $i+j$ is increasing. Moreover, $c^{ij}_{\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ and 3 occurs only if Φ has a component of type G_2, so $c^{ij}_{\alpha\beta} \neq 0$.

Given an element $x \in G$ we shall denote by $\mathcal{O}_x$ the conjugacy class of x in G and by G_x (resp. B_x, resp. T_x) the centralizer of x in G (resp. B, resp. T). For a conjugacy class $\mathcal{O} = \mathcal{O}_x$ we shall denote by $\mathcal{V}$ the set of B-orbits into which $\mathcal{O}$ is parted.

Definition 1.1. — *Let K be a connected algebraic group. A homogeneous K-space is called spherical if it has a dense orbit for some Borel subgroup of K.*

It is well-known ([3], [25] in characteristic 0, [15], [17] in positive characteristic) that X is a spherical homogeneous G-space if and only if the set of B-orbits in X is finite.

2. B-orbits and Bruhat decomposition

Let $\mathcal{V}$ be the set of B-orbits in a conjugacy class $\mathcal{O}$ in $G = \bigcup_{w\in W} BwB$. There is a natural map $\phi\colon \mathcal{V} \to W$ associating to $v \in \mathcal{V}$ the element w in the Weyl group of G for which $v \subset BwB$.

It is shown in [6] for G simple that if $\mathcal{O}$ is spherical as a homogeneous space then the image of ϕ consists of involutions. The same proof holds for G reductive. This motivates the following definition.

Definition 2.1. — *Let G be a connected reductive algebraic group. A conjugacy class $\mathcal{O}$ in G is called quasi-spherical if the image of ϕ consists of involutions.*

Remark 2.2. — Regular conjugacy classes in simple algebraic groups of rank greater than 1 cannot be quasi-spherical. Indeed, by [24, Theorem 8.1] regular classes meet Bruhat cells corresponding to Coxeter elements.

2.1. The case of G_2

We aim at showing that every quasi-spherical conjugacy class is spherical. In order to illustrate this result explicitly, we analyze quasi-spherical conjugacy classes for G simple of type G_2 by inspection, making use of the classification of unipotent conjugacy classes (see, for instance, [16, Section 7.12]) and of the commutator formula (1.3).

THEOREM 2.3. — *Let G be simple of type G_2. Then every quasi-spherical conjugacy class of G is spherical.*

Proof. — Let α and β denote the short and long simple roots, respectively, and let $\mathcal{O}$ be a conjugacy class in G. We first assume that $\mathcal{O}$ is unipotent so it is either of type A_1, $\tilde{A}_1$, subregular or regular ([16, Section 7.18]). If $\mathcal{O}$ is of type A_1 or $\tilde{A}_1$ then $\mathcal{O}$ is spherical, hence quasi-spherical by [5, Proposition 6, Proposition 11] which hold for arbitrary k. Alternatively, one may use [19, Theorem 3.2] and [14, Theorem 4.14]. If $\mathcal{O}$ is regular it cannot be quasi-spherical by Remark 2.2. The element $u = x_\beta(1)x_{3\alpha+\beta}(1) \in G$ does not lie in the regular unipotent conjugacy class by [24, Lemma 3.2(c)]. Its conjugate $\dot{w}_0 u \dot{w}_0^{-1} = x_{-\beta}(a)x_{-3\alpha-\beta}(b)$ with $ab \neq 0$ lies in $Bs_\beta Bs_{3\alpha+\beta}B = Bs_\beta s_{3\alpha+\beta}B$ by [22, Lemma 8.1.4(i), Lemma 8.3.7] and $s_\beta s_{3\alpha+\beta}$ is not an involution. Then its class is not quasi-spherical and by exclusion it is the subregular unipotent conjugacy class, so the statement holds for unipotent conjugacy classes.

Let us now consider a representative $x \in \mathcal{O} \cap B$ with Jordan decomposition $x = su \in TU$ with $s \neq 1$. Then G_s is connected and reductive ([16, Theorem 2.2, Theorem 2.11]). We shall analyze the different cases according to the semisimple rank srk of G_s.

If $\operatorname{srk} G_s = 0$ then $\mathcal{O}$ is regular, hence it is not quasi-spherical by Remark 2.2.

If $\operatorname{srk} G_s = 1$ and $u \neq 1$ then $\mathcal{O}$ is regular, hence it is not quasi-spherical. Let us assume $u = 1$. Up to conjugation by an element in $N(T)$ we may assume that G_s is either $H_1 = \langle T, X_{\pm\beta}\rangle$ or $H_2 = \langle T, X_{\pm\alpha}\rangle$.

If $G_s = H_1$ conjugation of s by $x_{-\alpha}(1)x_{-\alpha-\beta}(1)$ yields

$$s_1 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-2\alpha-\beta}(c)x_{-3\alpha-\beta}(d)x_{-3\alpha-2\beta}(e) \in \mathcal{O}$$

for $a, b, c, d, e \in k$ with $ab \neq 0$. Conjugation by a suitable element in $X_{-2\alpha-\beta}$ gives

$$s_2 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-3\alpha-\beta}(d')x_{-3\alpha-2\beta}(e') \in \mathcal{O}$$

for $d', e' \in k$. Conjugation by a suitable element in $X_{-3\alpha-\beta}$ gives

$$s_3 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-3\alpha-2\beta}(e') \in \mathcal{O}$$

and conjugation by a suitable element in $X_{-3\alpha-2\beta}$ gives

$$s_4 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b) \in \mathcal{O} \cap Bs_\alpha s_{\alpha+\beta}B$$

so $\mathcal{O}$ is not quasi-spherical.

Let $G_s = H_2$. Conjugation of s by $x_{-3\alpha-\beta}(1)x_{-\beta}(1)$ yields

$$s_1 = sx_{-\beta}(a)x_{-3\alpha-\beta}(b)x_{-3\alpha-2\beta}(c) \in \mathcal{O}$$

for some $a, b, c \in k$ with $ab \neq 0$. Conjugation by a suitable element in $X_{-3\alpha-2\beta}$ gives

$$s_2 = sx_{-\beta}(a)x_{-3\alpha-\beta}(b) \in \mathcal{O} \cap Bs_\beta s_{3\alpha+\beta}B$$

so $\mathcal{O}$ is not quasi-spherical, concluding the analysis if $\operatorname{srk} G_s = 1$.

Let $\operatorname{srk} G_s = 2$ with $s \neq 1$. Up to conjugation by an element in $N(T)$ we may assume that G_s is either

$$H_3 = \langle T, X_{\pm\beta}, X_{\pm(3\alpha+\beta)}, X_{\pm(3\alpha+2\beta)} \rangle \text{ or } H_4 = \langle T, X_{\pm\beta}, X_{\pm(2\alpha+\beta)} \rangle.$$

If $G_s = H_3$ of type A_2 and $u = 1$ then $\mathcal{O}$ is spherical by [5, Proposition 6, Theorem 16] whose proofs hold in arbitrary good odd characteristic. Let us assume that $u \neq 1$. If u is regular in H_3 then $\mathcal{O}$ is regular by [24, Corollary 3.7], hence it is not quasi-spherical. It remains to analyze the class of $x = sx_{-\beta}(1)$. Conjugating by $x_{-\alpha}(1)$ and reordering the terms gives

$$x_1 = sx_{-\beta}(1)x_{-\alpha-\beta}(b)x_{-2\alpha-\beta}(c)x_{-3\alpha-\beta}(d)x_{-3\alpha-2\beta}(e)x_{-\alpha}(f) \in \mathcal{O}$$

for some b, c, d, e, $f \in k$ with $f \neq 0$. We can get rid of the term in $X_{-\alpha-\beta}$ conjugating by a suitable element in $X_{-\alpha-\beta}$. Then we can get rid of the term in $X_{-2\alpha-\beta}$ conjugating by a suitable element in $X_{-2\alpha-\beta}$ and, finally, we can get rid of the term in $X_{-3\alpha-2\beta}$ by conjugating by a suitable element in $X_{-3\alpha-\beta}$ obtaining

$$x_2 = sx_{-\beta}(1)x_{-3\alpha-\beta}(b_1)x_{-\alpha}(f) \in \mathcal{O}$$

for some $b_1 \in k$. If $b_1 = 0$ then $\mathcal{O} \cap Bs_\beta s_\alpha B \neq \emptyset$ so $\mathcal{O}$ is not quasi-spherical. If $b_1 \neq 0$ we have, for some $h \in T$ and some nonzero $a_i \in k$:

$$\begin{aligned}
x_2 &= shx_\beta(a_1)n_\beta x_\beta(a_2)x_{3\alpha+\beta}(a_3)n_{3\alpha+\beta}x_{3\alpha+\beta}(a_4)x_{-\alpha}(f) \\
&= shx_\beta(a_1)n_\beta x_{3\alpha+\beta}(a_3)x_\beta(a_2)x_{3\alpha+2\beta}(a_5)n_{3\alpha+\beta}x_{3\alpha+\beta}(a_4)x_{-\alpha}(f) \\
&\in TX_\beta X_{3\alpha+2\beta}n_\beta n_{3\alpha+\beta}X_{3\alpha+2\beta}X_\beta x_{-\alpha}(f) \\
&\subset Bn_\beta n_{3\alpha+\beta}P_\alpha^u x_{-\alpha}(f) \subset Bn_\beta n_{3\alpha+\beta}x_{-\alpha}(f)U \subset BX_{2\alpha+\beta}n_\beta n_{3\alpha+\beta}U
\end{aligned}$$

so $\mathcal{O} \cap Bs_\beta s_{3\alpha+\beta}B \neq \emptyset$ and $\mathcal{O}$ is not quasi-spherical.

Let $G_s = H_4$ be of type $A_1 \times \tilde{A}_1$. If $u = 1$ then $\mathcal{O}$ is spherical by the argument in [5, Theorem 16]. If u has nontrivial components both in A_1 and in $\tilde{A}_1$ then $\mathcal{O}$ is regular, hence it is not quasi-spherical. We are left with the analysis of the classes of $y = sx_{-\beta}(1)$ and $z = sx_{-2\alpha-\beta}(1)$.
Conjugating y by $x_{-3\alpha-\beta}(1)$ we get $y_1 = sx_{-3\alpha-\beta}(a)x_{-\beta}(1)x_{-3\alpha-2\beta}(b)$ for some a, $b \in k$ with $a \neq 0$. Conjugation by a suitable element in $X_{-3\alpha-2\beta}$ yields

$$y_2 = sx_{-3\alpha-\beta}(a)x_{-\beta}(1) \in \mathcal{O} \cap Bs_{3\alpha+\beta}s_\beta B$$

hence $\mathcal{O}_y$ is not quasi-spherical.

Conjugating z by $x_{-\alpha}(1)$ we get $z_1 = sx_{-\alpha}(a)x_{-2\alpha-\beta}(1)x_{-3\alpha-\beta}(c)$ for some a, $c \in k$ with $a \neq 0$. Conjugating z_1 by a suitable element in $X_{-\alpha-\beta}$ we obtain the element $z_2 = sx_{-\alpha}(a)x_{-\alpha-\beta}(d)x_{-3\alpha-\beta}(c_1)x_{-3\alpha-2\beta}(c_2)$ for some $c_1, c_2, d \in k$ with $d \neq 0$. We can get rid of the term in $X_{-3\alpha-\beta}$ conjugating by a suitable element in $X_{-2\alpha-\beta}$ and then we can get rid of the term in $X_{-3\alpha-2\beta}$ conjugating by a suitable element in $X_{-3\alpha-2\beta}$.

Thus $z_3 = sx_{-\alpha}(1)x_{-\alpha-\beta}(d) \in \mathcal{O} \cap Bs_\alpha s_{\alpha+\beta}B$ hence $\mathcal{O}_z$ is not quasi-spherical. This exhausts the list of conjugacy classes for G of type G_2 and we have verified that all quasi-spherical conjugacy classes are spherical. □

3. Maximal B-orbits

Let $\mathcal{O}$ be a conjugacy class of G. Since $\mathcal{O}$ is an irreducible variety there exists a unique element in W for which $\mathcal{O} \cap BwB$ is dense in $\mathcal{O}$. We shall denote this element by $z_\mathcal{O}$. Denoting by $\overline{X}^Y$ the Zarisky closure of X in Y we have

$$\mathcal{O} \subset \overline{\mathcal{O}}^G = \overline{\mathcal{O} \cap Bz_\mathcal{O}B}^G \subset \overline{Bz_\mathcal{O}B}^G = \bigcup_{\sigma \leqslant z_\mathcal{O}} B\sigma B$$

so the element $z_\mathcal{O}$ is maximal in the image of ϕ (*cf.* [5, Section 1]). We will call *maximal orbits* the elements v in $\mathcal{V}$ for which $\phi(v) = z_\mathcal{O}$ and we shall denote by $\mathcal{V}_{\max}$ the set of maximal B-orbits in $\mathcal{O}$.

LEMMA 3.1. — *The following are equivalent for a conjugacy class $\mathcal{O}$ in G.*

(1) *$\mathcal{O}$ is spherical.*
(2) *$\mathcal{V}_{\max}$ contains only one element.*
(3) *$\mathcal{V}_{\max}$ is a finite set.*

Proof. — It follows from [5, Corollary 26], [6, Corollary 4.11] that if $\mathcal{O}$ is spherical then $\mathcal{V}_{\max}$ contains only one element, namely the dense B-orbit so 1 implies 2 and 2 trivially implies 3. Let us show that 3 implies 1. Since $\cup_{v\in\mathcal{V}_{\max}} v = \mathcal{O} \cap Bz_\mathcal{O}B$ is dense in $\mathcal{O}$ we have $\mathcal{O} \subset \cup_{v\in\mathcal{V}_{\max}} \overline{v}^\mathcal{O}$ with $\mathcal{O}$ irreducible ([16, Proposition 1.5]) and $\mathcal{V}_{\max}$ a finite set. Then there necessarily exists $v_0 \in \mathcal{V}_{\max}$ which is dense in $\mathcal{O}$. □

Let us analyze the maximal B-orbits in quasi-spherical conjugacy classes.

LEMMA 3.2. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class with $w = z_\mathcal{O}$. Let $v \in \mathcal{V}_{\max}$ and let $x = u\dot{w}\mathrm{v} \in v$ with $u \in U^w$, $\dot{w} \in N(T)$ and $\mathrm{v} \in U$. Then for every $\alpha \in \Delta$ such that $ws_\alpha > w$ in the Bruhat order we have:*

(1) $s_\alpha w = w s_\alpha$ *so* $w\alpha = \alpha$;
(2) $\mathrm{v} \in P_\alpha^u$, *the unipotent radical of* P_α;
(3) $X_{\pm\alpha}$ *commutes with* $\dot{w}$.

Proof. — This is proved as [6, Lemma 3.4], since the proof therein uses only maximality of w and that $\mathcal{O}$ is quasi-spherical. □

LEMMA 3.3. — *Let* $\mathcal{O}$ *be a quasi-spherical conjugacy class with* $w = z_{\mathcal{O}}$, *let* $\Pi = \{\alpha \in \Delta \mid w(\alpha) = \alpha\}$ *and let* w_Π *be the longest element in* W_Π. *Then* $w = w_\Pi w_0$.

Proof. — By Lemma 3.2 if $\alpha \in \Delta$ and $w\alpha \in \Phi^+$ then $w\alpha = \alpha$. The statement follows from [21, Proposition 3.5]. □

The Lemmas above show that maximal B-orbits in quasi-spherical conjugacy classes behave similarly to the dense B-orbit v_0 in a spherical conjugacy class. The analysis of $z_{\mathcal{O}}$ given in [6] applies.

PROPOSITION 3.4. — *The following properties hold for a quasi-spherical conjugacy class* $\mathcal{O}$ *with* $w = z_{\mathcal{O}} = w_0 w_\Pi$.

(1) Π *is invariant with respect to* $\vartheta = -w_0$;
(2) *The restriction of* w_0 *to* $\Phi(\Pi)$ *coincides with* w_Π;
(3) $\Phi_w = \Phi^+ \setminus \Phi(\Pi)$, *notation as in* (1.1);
(4) $U_w = \langle X_\gamma \mid \gamma \in \Phi(\Pi) \cap \Phi^+ \rangle$ *and it normalizes* U^w, *notation as in* (1.2);
(5) U_w *commutes with* $\dot{w}$ *if* $x = u\dot{w}\mathrm{v} \in \mathcal{O} \cap U^w N(T) U$.

Proof. — The proof is as in [6, Section 3]. □

In [6] an analysis of the possible Π for which $\phi(v_0) = w_0 w_\Pi = z_{\mathcal{O}}$ for the dense B-orbit v_0 of a spherical conjugacy class in a simple algebraic group was given. The proof of [6, Lemma 4.1] can be adapted to the case of maximal B-orbits in quasi-spherical conjugacy classes, yielding the following statement.

LEMMA 3.5. — *Let* $\mathcal{O}$ *be a quasi-spherical conjugacy class and let* $w = w_0 w_\Pi = z_{\mathcal{O}}$. *Let* α *and* β *be simple roots with the following properties:* $(\beta, \beta) = (\alpha, \alpha)$; $w_0(\beta) = -\beta$; $\beta \not\perp \alpha$; $\beta \perp \alpha'$ *for every* $\alpha' \in \Pi \setminus \{\alpha\}$.

Then $\{\alpha\}$ *cannot be a connected component of* Π. *In particular, the list of the possible subsets* Π *for which* $z_{\mathcal{O}} = w_0 w_\Pi$ *for* G *simple coincides with the list given in* [6, Corollary 4.2].

Proof. — The proof follows as in [6, Lemma 4.1] since it only uses maximality of w and that $\mathcal{O}$ is quasi-spherical. There, the proof is given for G simple but it holds for G reductive, too. □

Let $\mathcal{O}$ be quasi-spherical with $w = z_{\mathcal{O}} = w_0 w_\Pi$ and let $\Phi_1 = \Phi \cap \mathrm{Ker}(1 + w)$. Then Φ_1 is a root subsystem of Φ and we put $\Phi_1^+ = \Phi^+ \cap \Phi_1$. If we write $w = \prod_j s_{\gamma_j}$ as a product of reflections with respect to mutually orthogonal roots then each γ_j lies in Φ_1. We shall denote by $W(\Phi_1)$ the subgroup of W generated by reflections with respect to roots in Φ_1, so $w \in W(\Phi_1)$.

Lemma 3.6. — *Let notation be as above and let $\beta \in \Phi$. Then $\beta \in \Phi_1$ if and only if $\beta \perp \Pi$ and $\vartheta\beta = \beta$.*

Proof. — If $\beta \perp \Pi$ and $\vartheta\beta = \beta$ then $w_\Pi \beta = \beta$ and $w_0\beta = -\beta$ thus $\beta \in \Phi_1$.
Conversely, if $w\beta = -\beta$ then for every $\alpha \in \Pi$ we have $\beta \perp \alpha$ because α and β lie in distinct eigenspaces of the orthogonal transformation w. Let now $\alpha \in \Phi$ and $w\beta = -\beta$. We have

$$(\vartheta\beta, \alpha) = -(w_0\beta, \alpha) = -(w w_\Pi \beta, \alpha) = -(w\beta, \alpha) = (\beta, \alpha)$$

and since this holds for every α, we have the statement. □

Let us denote by $G(\Phi_1)$ the subgroup of G generated by T and the root subgroups $X_{\pm\beta}$ with $\beta \in \Phi_1$. Let $U_{\Phi_1} = \langle X_\beta,\, \beta \in \Phi_1^+ \rangle$.

The following Lemma is an analogue of [6, Lemma 4.8, Remark 4.9] for quasi-spherical conjugacy classes.

Lemma 3.7. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class and let $z_{\mathcal{O}} = w_0 w_\Pi$. Let $\dot{w} \in N(T)$ be a representative of w. Then for every $x = \dot{w}t\mathrm{v} \in \dot{w}B \cap \mathcal{O}$ we have $\mathrm{v} \in U_{\Phi_1}$, $w \in W(\Phi_1)$ and x commutes with $(T^w)^\circ U_w$.*

Proof. — The proof when G is simple follows exactly as in [6, Lemmas 4.5, 4.6, 4.7, 4.8, 4.9]. Indeed, for their proofs we only need w to be maximal, $\mathcal{O}$ to be quasi-spherical and the list in [6, Corollary 4.2]. The general case is a consequence of the case of G simple. □

Lemma 3.8. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class and let $w = w_0 w_\Pi = z_{\mathcal{O}}$. Then $\langle X_{-\alpha}, \alpha \in \Pi \rangle$ commutes with every $x = \dot{w}t\mathrm{v} \in \dot{w}B \cap \mathcal{O}$.*

Proof. — It is not restrictive to assume G to be simple. By Lemmas 3.2 and 3.7 it is enough to show that $X_{-\alpha}$ commutes with v for every $\alpha \in \Pi$. If this were not the case, by (1.3) there would occur in the expression of v at least one root subgroup X_γ with nontrivial coefficient and with $\gamma - \alpha \in \Phi$. We consider such a γ of minimal height. By Lemma 3.7 and [2, Chapitre 6, §1.3] this could happen only if Φ is doubly-laced and α is a short root. Then we would also have $\alpha + \gamma \in \Phi$, which is impossible because X_α commutes with v by Lemma 3.7. □

A consequence of Lemma 3.7 is the following result.

Proposition 3.9. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class, let $w = z_{\mathcal{O}} = w_0 w_\Pi$ and let $v \in \mathcal{V}_{\max}$. Then* $\dim(v) = \ell(w) + \mathrm{rk}(1-w)$.

Proof. — Let n be the rank of G and let $x = \dot{w}\mathrm{v} \in v$. By Lemma 3.7 the centralizer B_x of x in B contains $(T^w)^\circ U_w$. On the other hand, if $b = u^w u_w t \in U^w U_w T$ commutes with x we have

$$\dot{w}\mathrm{v}u^w u_w t = u^w u_w t\dot{w}\mathrm{v} = u^w u_w \dot{w}(\dot{w}^{-1}t\dot{w})\mathrm{v} = u^w \dot{w} u_w(\dot{w}^{-1}t\dot{w})\mathrm{v}$$

where for the last equality we used Lemma 3.2. By uniqueness in the Bruhat decomposition we have $u^w = 1$ so $B_x \subset T_x U_w$ because $U_w \subset B_x$. Moreover, if $t \in T_x$ we have

$$\dot{w}(\dot{w}^{-1}t\dot{w})\mathrm{v} = t\dot{w}\mathrm{v} = \dot{w}\mathrm{v}t \in \dot{w}TU$$

and uniqueness of the decomposition in TU gives $t \in T^w$. Therefore $(T^w)^\circ U_w \subset B_x \subset T^w U_w$ and $\dim v = |\Phi^+| + n - (|\Phi^+| - \ell(w)) - (n - \mathrm{rk}(1-w)) = \ell(w) + \mathrm{rk}(1-w)$. □

4. The case $z_{\mathcal{O}} = w_0 = -1$

In this section Φ is such that w_0 acts as -1 in the geometric representation of W. If $\mathcal{O}$ is a quasi-spherical conjugacy class intersecting the big Bruhat cell Bw_0B then $\Phi_1 = \Phi$ and $\Pi = \emptyset$ so Lemma 3.7 gives no restriction to a representative $x = \dot{w}\mathrm{v} \in \mathcal{O} \cap \dot{w}U$. For this reason we use a different approach for such classes.

By Lemma 3.1 if a conjugacy class has finitely-many maximal B-orbits then it is spherical. The aim of this Section is to show that every quasi-spherical conjugacy class $\mathcal{O}$ intersecting Bw_0B has only finitely-many maximal B-orbits. This will be achieved by counting the possible representatives of a maximal B-orbit lying in $\dot{w}_0U$ for a fixed $\dot{w}_0 \in N(T)$. Next Lemma shows that every maximal B-orbit meets $\dot{w}_0U$.

Lemma 4.1. — *Let G be simple and let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0 = -1$. For any $v \in \mathcal{V}_{\max}$ and any representative $\dot{w}_0$ of w_0 in $N(T)$ we have $v \cap \dot{w}_0U \neq \emptyset$.*

Proof. — Let $x = u\dot{w}_0 t\mathrm{v} \in v \cap U\dot{w}_0B$. Then for every $s \in T$ we have $x_s = s^{-1}u^{-1}u\dot{w}_0t\mathrm{v}us = \dot{w}_0 s^2 t u' \in v \cap \dot{w}_0TU$ and since the map $s \mapsto s^2 \in T$ is onto ([1, III.8.9]) we may choose s so that $x_s \in v \cap \dot{w}_0U$. □

Lemma 4.2. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0 = -1$. Let $\dot{w}_0$ be a representative of w_0 and let $x = \dot{w}_0\mathrm{v} \in \mathcal{O} \cap \dot{w}_0U$,* *with* $\mathrm{v} = \prod_{\gamma\in\Phi^+} x_\gamma(c_\gamma)$ *in a fixed ordering. Let* α *and* β *be adjacent simple roots of the same length. Then the number of possibilities for* c_α *and* c_β *is finite and* $c_{\alpha+\beta}$ *is completely determined by the ordering,* c_α *and* c_β*.*

Proof. — Let $P = P_{\{\alpha,\beta\}}$ with unipotent radical P^u. Let us assume that α precedes β in the ordering. We have:

$$x = \dot{w}_0\mathrm{v} \in \dot{w}_0 x_\alpha(c_\alpha)x_\beta(c_\beta)x_{\alpha+\beta}(c_{\alpha+\beta})P^u.$$

For $h \in k$ we put $y(h) := n_\alpha x_\alpha(h) x x_\alpha(-h) n_\alpha^{-1}$. Then, for some nonzero structure constants $\theta_1, \theta_2, \theta_3, c^{11}_{\alpha\beta}$ and some $t_1 \in T$ we have

$$\begin{aligned} y(h) &\in n_\alpha \dot{w}_0 x_{-\alpha}(\theta_1 h) x_\alpha(c_\alpha - h) x_\alpha(h) x_\beta(c_\beta) x_{\alpha+\beta}(c_{\alpha+\beta}) x_\alpha(-h) n_\alpha^{-1} P^u \\ &= \dot{w}_0 t_1 x_\alpha(\theta_1\theta_2 h) x_{-\alpha}(\theta_3(c_\alpha - h)) n_\alpha x_\beta(c_\beta) x_{\alpha+\beta}(c_{\alpha+\beta} + h c_\beta c^{11}_{\alpha\beta}) n_\alpha^{-1} P^u. \end{aligned}$$

Let h_1 and h_2 be the solutions of

$$X^2(\theta_1\theta_2\theta_3) - c_\alpha\theta_1\theta_2\theta_3 X - 1 = 0$$

so that $-(\theta_1\theta_2 h_i)^{-1} = (c_\alpha - h_i)\theta_3$. By [22, Lemma 8.1.4 (i)] we have

$$\begin{aligned} y(h_i) &\in \dot{w}_0 t_1 n_\alpha t_2 x_{\beta+\alpha}(c'_\beta) x_\beta(\theta_4(c_{\alpha+\beta} + h_i c_\beta c^{11}_{\alpha\beta})) P^u \\ &\subset \dot{w}_0 n_\alpha t_3 x_\beta(\theta_4(c_{\alpha+\beta} + h_i c_\beta c^{11}_{\alpha\beta})) P^u_\beta \subset \mathcal{O} \cap B w_0 s_\alpha B \end{aligned}$$

for some $t_2, t_3 \in T$, some $c'_\beta \in k$ and some nonzero structure constant θ_4. Since $w_0 s_{\alpha+\beta}\beta = \alpha \in \Phi^+$, conjugation of $y(h_i)$ by n_β would yield an element in $\mathcal{O} \cap B w_0 s_{\alpha+\beta} s_\beta B$ unless

$$c_{\alpha+\beta} + h_i c_\beta c^{11}_{\alpha\beta} = 0. \tag{4.1}$$

As $s_{\alpha+\beta}s_\beta$ is not an involution, (4.1) must hold for both $i = 1, 2$ thus we have either $h_1 = h_2$ so that

$$\Delta_\alpha = \theta_1^2\theta_2^2\theta_3^2 c_\alpha^2 + 4\theta_1\theta_2\theta_3 = 0, \quad \text{or} \tag{4.2}$$

$$c_\beta = c_{\alpha+\beta} = 0. \tag{4.3}$$

Let us now consider, for $l \in k$, the element

$$\begin{aligned} z(l) &= n_\beta x_\beta(l) x x_\beta(-l) n_\beta^{-1} \\ &\in n_\beta x_\beta(l) \dot{w}_0 x_\beta(c_\beta) x_\alpha(c_\alpha) x_{\alpha+\beta}(c_{\alpha+\beta} + c_\alpha c_\beta c^{11}_{\alpha\beta}) x_\beta(-l) n_\beta^{-1} P^u. \end{aligned}$$

Repeating the same argument for β we see that there are nonzero structure constants $\eta_1, \eta_2, \eta_3, \eta_4$ so that if l_j is a solution of

$$\eta_1\eta_2\eta_3 X^2 - c_\beta\eta_1\eta_2\eta_3 X - 1 = 0$$

then

$$z(l_j) \in \mathcal{O} \cap \dot{w}_0 n_\beta T x_\alpha(\eta_4(c_{\alpha+\beta} + c_\alpha c_\beta c^{11}_{\alpha\beta} - l_j c_\alpha c^{11}_{\alpha\beta})) P^u_\alpha$$

so conjugation by n_α would yield an element in $\mathcal{O}\cap Bw_0s_{\alpha+\beta}s_\alpha B$ unless

$$c_{\alpha+\beta}+c_\alpha c_\beta c^{11}_{\alpha\beta}-l_jc_\alpha c^{11}_{\alpha\beta}=0 \tag{4.4}$$

for both $j=1,2$. This forces either $l_1=l_2$ and therefore

$$\Delta_\beta=\eta_1^2\eta_2^2\eta_3^2c_\beta^2+4\eta_1\theta_2\theta_3=0,\quad \text{or} \tag{4.5}$$

$$c_\alpha=c_{\alpha+\beta}=0. \tag{4.6}$$

If (4.2) holds then $c_\alpha\neq 0$ so (4.5) holds. Then the possibilities for c_α and c_β are finite. Besides, by (4.4) the coefficient $c_{\alpha+\beta}$ is completely determined by c_α, c_β, and the structure constants.

If (4.3) holds then (4.5) cannot hold so $c_\alpha=c_\beta=c_{\alpha+\beta}=0$, whence the statement. □

Lemma 4.3. — *Let $\mathcal{O}$, $z_{\mathcal{O}}$, x, v be as in Lemma 4.2. Let α and β be adjacent simple roots with $2(\alpha,\alpha)=(\beta,\beta)$. Then $c_{2\alpha+\beta}$ and $c_{\alpha+\beta}$ are completely determined by the ordering, c_α and c_β.*

Proof. — It is not restrictive to assume that α precedes β and $\alpha+\beta$ in the ordering. Let $P=P_{\{\alpha,\beta\}}$ and P^u be its unipotent radical. Then

$$x\in \dot{w}_0x_\alpha(c_\alpha)x_\beta(c_\beta)x_{\alpha+\beta}(c_{\alpha+\beta})x_{2\alpha+\beta}(c_{2\alpha+\beta})P^u.$$

Conjugation by $n_\alpha x_\alpha(h)$ for $h\in k$ yields an element $y(h)$ lying in

$$\dot{w}_0t_1x_\alpha(\eta_1h)x_{-\alpha}(\eta_2(c_\alpha-h))x_\beta(\eta_3(c_{2\alpha+\beta}+hc_{\alpha+\beta}c^{11}_{\alpha,\alpha+\beta}+c_\beta h^2c^{21}_{\alpha,\beta}))P^u_\beta$$

for some $t_1\in T$ and some nonzero structure constants η_1,η_2,η_3. If h_1, h_2 are the solutions of

$$\eta_1\eta_2X^2-c_\alpha\eta_1\eta_2X-1=0$$

then $y(h_1)$, $y(h_2)$ lie in $\mathcal{O}\cap Bw_0s_\alpha B$ and $n_\beta y(h_i)n_\beta^{-1}\in\mathcal{O}\cap Bw_0s_{\alpha+\beta}s_\beta B$ unless

$$c_{2\alpha+\beta}+h_ic_{\alpha+\beta}c^{11}_{\alpha,\alpha+\beta}+c_\beta h_i^2c^{21}_{\alpha,\beta}=0 \tag{4.7}$$

for both $i=1,2$. Besides, $h_1+h_2=c_\alpha$ and $h_1h_2=-(\eta_1\eta_2)^{-1}$. Thus we have either

$$\Delta_\alpha=(\eta_1\eta_2c_\alpha)^2+4\eta_1\eta_2=0\ \text{ and }\ c_{2\alpha+\beta}=ac_\alpha c_{\alpha+\beta}+bc_\beta c_\alpha^2 \tag{4.8}$$

$$\text{or}\ \ c_{2\alpha+\beta}=cc_\beta \text{ and } c_{\alpha+\beta}=dc_\beta c_\alpha \tag{4.9}$$

for a, b, c, $d\in k$. On the other hand, reordering terms we have:

$$x\in\dot{w}_0x_\beta(c_\beta)x_\alpha(c_\alpha)x_{\alpha+\beta}(c_{\alpha+\beta}+c_\alpha c_\beta c^{11}_{\alpha\beta})x_{2\alpha+\beta}(c_{2\alpha+\beta}+c_\alpha^2c_\beta c^{21}_{\alpha\beta})P^u.$$

Conjugation by $n_\beta x_\beta(l)$ for $l\in k$ gives an element

$$z(l)\in\dot{w}_0t_2x_\beta(\theta_1l)x_{-\beta}(\theta_2(c_\beta-l))x_\alpha(\theta_3(c_{\alpha+\beta}+c_\alpha c_\beta c^{11}_{\alpha\beta}-lc_\alpha c^{11}_{\alpha\beta}))P^u_\alpha$$

for some $t_2 \in T$ and some nonzero structure constants $\theta_1, \theta_2, \theta_3, c^{11}_{\alpha\beta}$. For the solutions l_1, l_2 of

$$\theta_1\theta_2 X^2 - c_\beta\theta_1\theta_2 X - 1 = 0$$

the corresponding elements $z(l_1)$, $z(l_2)$ lie in $\mathcal{O}\cap Bw_0s_\beta B$ and $n_\alpha z(l_i)n_\alpha^{-1} \in Bw_0s_{2\alpha+\beta}s_\alpha B$ unless

$$c_{\alpha+\beta} + c_\alpha c_\beta c^{11}_{\alpha\beta} - l_i c_\alpha c^{11}_{\alpha\beta} = 0 \tag{4.10}$$

for $i = 1, 2$. It follows that we have either

$$\Delta_\beta = (\theta_1\theta_2 c_\beta)^2 + 4\theta_1\theta_2 = 0 \quad \text{and} \quad c_{\alpha+\beta} = dc_\alpha c_\beta \tag{4.11}$$

$$\text{or } c_{\alpha+\beta} = c_\alpha = 0. \tag{4.12}$$

Arguing as in Lemma 4.2 we see that $c_{\alpha+\beta}$ and $c_{2\alpha+\beta}$ are completely determined by the ordering, c_α and c_β. □

Lemma 4.4. — *Let Φ be a simply- or doubly-laced root system for which $w_0 = -1$. Let $\mathcal{O}$, $z_{\mathcal{O}}$, x, v be as in Lemma 4.2. Then, for every $\gamma = \sum_{j\in J} n_j\alpha_j \in \Phi^+$ with $J \subset \{1,\ldots,n\}$ there is a polynomial $p_\gamma(X) \in k[x_j,\, j \in J]$ depending only on the fixed ordering of the positive roots and the structure constants of G such that the coefficient c_γ in the expression of v is the evaluation of $p_\gamma(X)$ at $x_j = c_{\alpha_j}$ for every $j \in J$.*

Proof. — We shall proceed by induction on the height ht of the root γ. Let us assume that the claim holds for all γ with $\mathrm{ht}(\gamma) \leqslant m-1$. By Lemmas 4.2 and 4.3 the statement holds for $m = 1, 2$ so we assume $m \geqslant 3$.

Let $\nu \in \Phi^+$ with $\mathrm{ht}(\nu) = m$. Then there exists $\beta \in \Delta$ for which $\mathrm{ht}(s_\beta\nu) \leqslant m-1$. We put

$$y = n_\beta x n_\beta^{-1} = \dot{w}_0 t \prod_{\gamma\in\Phi^+} x_{s_\beta\gamma}(\theta_\gamma c_\gamma) \tag{4.13}$$

for some nonzero structure constants θ_γ. Here the products have to be intended in the fixed ordering of the γ's. We have:

$$y = \dot{w}_0 t (\prod_{\gamma<_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma)) x_{-\beta}(\theta_\beta c_\beta) (\prod_{\gamma>_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma))$$

where $<_o$ indicates that a root precedes another in the fixed ordering and the expression makes sense also if $c_\beta = 0$. Then

$$\begin{aligned} y &= \dot{w}_0 t x_{-\beta}(\theta_\beta c_\beta) x_{-\beta}(-\theta_\beta c_\beta)(\prod_{\gamma<_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma)) x_{-\beta}(\theta_\beta c_\beta) \prod_{\gamma>_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma) \\ &= x_\beta(\eta c_\beta)\dot{w}_0 t (x_{-\beta}(-\theta_\beta c_\beta)(\prod_{\gamma<_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma)) x_{-\beta}(\theta_\beta c_\beta)) \prod_{\gamma>_o\beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma) \end{aligned}$$

for some nonzero structure constant η. Conjugation by $x_\beta(-\eta c_\beta)$ yields

$$z = \dot{w}_0 t(x_{-\beta}(-\theta_\beta c_\beta)(\prod_{\gamma <_o \beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma))x_{-\beta}(\theta_\beta c_\beta))(\prod_{\gamma >_o \beta} x_{s_\beta\gamma}(\theta_\gamma c_\gamma))x_\beta(\eta c_\beta)$$
$$= \dot{w}_0 t \prod_{\gamma\in\Phi^+} x_\gamma(d_\gamma) \in \dot{w}_0 tU \cap \mathcal{O}$$

where the last equality indicates reordering of root subgroups. By the induction hypothesis applied to z and $s_\beta\nu$, the coefficient $d_{s_\beta\nu}$ is evaluation at the d_α for α in the support of $s_\beta\nu$ of a polynomial $q_{s_\beta\nu}(X)$. Besides, each d_μ differs from $\theta_{s_\beta\mu}c_{s_\beta\mu}$ by a (possibly trivial) sum of monomials in the $\theta_{\mu'}c_{\mu'}$, c_β and the structure constants coming from application of (1.3) when reordering root subgroups. More precisely, we have

$$d_\mu = \theta_{s_\beta\mu}c_{s_\beta\mu} + \sum *(\prod_{l=1}^{p} c_{\nu_l}^{i_l})c_\beta^j \tag{4.14}$$

where $*$ denotes a coefficient depending on the structure constants and the sum is taken over the possible decompositions $\mu = \sum_{l=1}^p i_l s_\beta\nu_l + j\beta$ for $i_l > 0$ and $j \geqslant 0$. In particular, if μ is simple there is no such decomposition: in this case $d_\mu = \theta_{s_\beta\mu}c_{s_\beta\mu}$ and by Lemmas 4.2 and 4.3 the coefficient $c_{s_\beta\mu}$ is evaluation of a polynomial at the c_α for α in the support of $s_\beta\mu$, and such support is contained in $\{\beta, \mu\}$. Thus, by the induction hypothesis $d_{s_\beta\nu}$ is evaluation of a polynomial $\overline{q}(X)$ at the c_α for α in the support of ν. We wish to prove that the same holds for c_ν. Contribution to $d_{s_\beta\nu}$ as in (4.14) may occur when

$$s_\beta\nu = \sum_{l=1}^{p} i_l s_\beta\nu_l + j\beta \tag{4.15}$$

for $i_l > 0$ and $j \geqslant 0$. Then $\mathrm{ht}(s_\beta\nu_l) < \mathrm{ht}(s_\beta\nu) \leqslant m-1$. We wish to show that $\mathrm{ht}(\nu_l) \leqslant m-1$ for every l so we may apply the induction hypothesis to c_{ν_l}. Suppose that there is a decomposition (4.15) and an l for which $\mathrm{ht}(\nu_l) \geqslant m$ for some l. Since

$$m \leqslant \mathrm{ht}(\nu_l) \leqslant \mathrm{ht}(s_\beta\nu_l) + 2 \leqslant \mathrm{ht}(s_\beta\nu) - 1 + 2 \leqslant m$$

we would necessarily have $\mathrm{ht}(\nu) = \mathrm{ht}(\nu_l) = m$; $\mathrm{ht}(s_\beta\nu_l) = m-2$; $\mathrm{ht}(s_\beta\nu) = m-1$ thus $s_\beta\nu = s_\beta\nu_l + \alpha$ for some $\alpha \in \Delta$. Applying s_β to this equality we would have $\nu = \nu_l + s_\beta\alpha$ contradicting $\mathrm{ht}(\nu) = \mathrm{ht}(\nu_l)$.

Thus induction applies and

$$c_\nu = *\overline{q}(c_\alpha)_{\alpha\in\mathrm{supp}(\nu)} + \sum * \prod_{l=1}^{p} (p_{\nu_l}(c_\alpha)_{\alpha\in\mathrm{supp}(\nu_l)})^{i_l} c_\beta^j$$

is evaluation of a polynomial depending only on the structure constants. □

Remark 4.5. — The proof of Lemma 4.4 can be adapted to show that if Φ is simply-laced and $\mathrm{v} \in P^u_\alpha$ for some $\alpha \in \Delta$ then $\mathrm{v} = 1$ so $x = \dot{w}_0$ and $\mathcal{O}$ is a symmetric space.

Proposition 4.6. — *Let Φ be irreducible, simply-laced, with $w_0 = -1$ and let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0$. Then $\mathcal{O}$ is spherical.*

Proof. — If $\Phi = \{\pm\alpha\}$ is of type A_1 the statement follows from even dimensionality of conjugacy classes ([20, Prop. 4.3]). Indeed, $\dim \mathcal{O} \leqslant 2$ and for a representative x of a maximal B-orbit in $\mathcal{O}$ we have $\dim B.x = \ell(s_\alpha) + \mathrm{rk}(1 - s_\alpha) = 2$ by Proposition 3.9. Thus, $B.x$ is dense in $\mathcal{O}$.

We assume now that the rank of Φ is at least two. By Lemma 4.1 every $v \in \mathcal{V}_{\max}$ meets $\dot{w}_0 U$ for every choice of $\dot{w}_0$ and by Lemmas 4.2 and 4.4 there is only a finite number of elements in $\mathcal{O} \cap \dot{w}_0 U$. We conclude using Lemma 3.1. □

Proposition 4.7. — *Let G be simple of type F_4 and let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0$. Then $\mathcal{O}$ is spherical.*

Proof. — By Lemma 3.1 we need to show that there are only finitely many maximal B-orbits. By Lemma 4.1 it is enough to show that there are only finitely many elements in $\dot{w}_0 U \cap \mathcal{O}$ for a fixed $\dot{w}_0 \in N(T)$. By Lemma 4.4 it is enough to show that there are only finitely many possibilities for c_α for $\alpha \in \Delta$. Applying Lemma 4.2 to the pair α_1, α_2 we see that for $x = \dot{w}_0 \mathrm{v} \in \mathcal{O} \cap \dot{w}_0 U$ there is a finite number of possibilities for the coefficients c_{α_1} and c_{α_2} in v. Applying Lemma 4.2 to the pair α_3, α_4 we see that there is a finite number of possibilities for c_{α_3} and c_{α_4}, concluding the proof. □

Proposition 4.8. — *Let Φ be irreducible of type C_n and let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0$. Then $\mathcal{O}$ is spherical.*

Proof. — If $n = 2$ by Proposition 3.9 we have $6 = \dim B = \dim v$ for every maximal B-orbit v. On the other hand $\dim \mathcal{O} < 2|\Phi^+| = 8$ because $\mathcal{O}$ cannot be regular (see Remark 2.2). It follows from even dimensionality of conjugacy classes ([20, Prop. 4.3]) that $\dim \mathcal{O} = \dim v$ so v is dense and $\mathcal{O}$ is spherical.

Let us now assume that $n \geqslant 3$. Let $\dot{w}_0 \in N(T)$ be fixed and let $x = \dot{w}_0 \mathrm{v} = \dot{w}_0 \prod_{\gamma\in\Phi^+} x_\gamma(c_\gamma)$ be as in Lemma 4.2. By Lemmas 3.1, 4.1 and 4.4 it is enough to prove that there is a finite number of possibilities for c_α

for $\alpha \in \Delta$. By Lemma 4.2 we have either $\Delta_{\alpha_i} = 0$ for $i = 1, \ldots, n-1$ or $c_{\alpha_i} = 0$ for $i = 1, \ldots, n-1$. In the first case, Lemma 4.3 with $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n$ gives $\Delta_{\alpha_n} = 0$ so there are finitely many possibilities for all c_γ. We shall thus focus on the case $c_{\alpha_i} = 0$ for $i \leqslant n-1$. Then Lemma 4.3 with $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n$ gives $c_{\alpha+\beta} = 0$. We claim that $c_\gamma = 0$ for every short root. We proceed by induction as in Lemma 4.4 and we look at the possible contribution as in (4.14) to c_ν with $\nu = s_\beta \mu$ and $\mathrm{ht}(\mu) < \mathrm{ht}(\nu)$. This would correspond to a decomposition of the short root $s_\beta \nu = \sum i_j s_\beta \nu_j + i\beta$ with $i_j > 0$ and $i \geqslant 0$. If $i > 0$ we have nontrivial contribution only if β is a long root, for $c_\beta = 0$ if β is short. Thus, both for $i = 0$ and $i > 0$ there is at least one ν_j which is short and then $\mathrm{ht}(s_\beta \nu_j) \leqslant \mathrm{ht}(s_\beta \nu) - 1 = m - 2$ so $\mathrm{ht}(\nu_j) \leqslant m - 2 + 1$. By the induction hypothesis $c_{\nu_j} = 0$ and there is no contribution coming from this decomposition, so the claim is proved.

In other words, putting $\gamma_n = \alpha_n$ and $\gamma_i = s_i s_{i+1} \cdots s_{n-1} \alpha_n$ for $i = 1, \ldots, n-1$ we have $x = \dot{w}_0 \prod_{i=1}^{n} x_{\gamma_i}(a_i)$ for some $a_i \in k$. We claim that there can be only finitely many elements of this type in a fixed class $\mathcal{O}$. It is not restrictive to assume that $G = Sp_{2n}(k)$. Then, G is the subgroup of $GL_{2n}(k)$ of matrices preserving the bilinear form associated with the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to the canonical basis of k^{2n}. We choose B as the subgroup of G of matrices of the form $\begin{pmatrix} X & XA \\ 0 & {}^t X^{-1} \end{pmatrix}$ where X is an invertible upper triangular matrix, ${}^t X^{-1}$ is its inverse transpose and A is a symmetric matrix. Then the computations above translate into:

$$x = x(D, A) = \dot{w}_0 \mathrm{v} = \begin{pmatrix} 0 & D \\ -D^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D^{-1} & -D^{-1}A \end{pmatrix}$$

for some diagonal matrices D and A with D fixed and invertible. It is immediate to verify that for two diagonal matrices A and A', the characteristic polynomials of $x(D, A)$ and $x(D, A')$ are the same only if $D^{-1}A$ and $D^{-1}A'$ coincide up to a permutation of the diagonal entries. Therefore there are only finitely many matrices of the form $x(A, D)$ in a single conjugacy class $\mathcal{O}$. Since by Lemma 4.1 each maximal B-orbit in $\mathcal{O}$ contains some $x(A, D)$ or some of the finitely many representatives with $\Delta_{\alpha_j} = 0$ for every j, there are only finitely many maximal B-orbits in $\mathcal{O}$ and we may conclude by using Lemma 3.1. □

Proposition 4.9. — *Let Φ be irreducible of type B_n and let $\mathcal{O}$ be a quasi-spherical conjugacy class with $z_{\mathcal{O}} = w_0$. Then $\mathcal{O}$ is spherical.*

Proof. — The case $n = 2$ is dealt with in Proposition 4.8 so we may assume $n \geqslant 3$. Let $\dot{w}_0 \in N(T)$ be fixed and let $x = \dot{w}_0\mathrm{v}$ be as in Lemma 4.2. We shall show that there is a finite number of possibilities for c_α for $\alpha \in \Delta$. It follows from Lemma 4.2 that we have either $c_\alpha = 0$ for every long simple root α or $\Delta_\alpha = 0$ for every long simple root α. In the first case, Lemma 4.3 with $\alpha = \alpha_n$ and $\beta = \alpha_{n-1}$ shows that (4.11) cannot be satisfied so $c_{\alpha_n} = 0$ as well. Hence, there is no freedom for the c_α in this case and we shall focus on the second case. Let $\Delta' = \{\alpha_1, \dots, \alpha_{n-1}\}$ and $P = P_{\Delta'}$. Then $x = \dot{w}_0\mathrm{v}_1\mathrm{v}_2$ with $\mathrm{v}_1 \in \langle X_\alpha, \alpha \in \Delta'\rangle$ and $\mathrm{v}_2 \in P^u$ and we might assume that the fixed ordering of the roots is compatible with this decomposition. By Lemma 4.4 the factor v_1 is completely determined by the c_α for $\alpha \in \Delta'$ so there are finitely many possibilities for it. If there were infinitely many elements in $\mathcal{O}\cap\dot{w}_0\mathrm{v}$, there would be infinitely many elements in $\mathcal{O} \cap \dot{w}_0\mathrm{v}_1P^u$ for some v_1. We shall show that this cannot be the case. It is not restrictive to assume that $G = SO_{2n+1}(k)$. We describe G as the subgroup of $SL_{2n+1}(k)$ of matrices preserving the bilinear form associated with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ with respect to the canonical basis of k^{2n+1}. We may choose B to be the subgroup of matrices of the form $\begin{pmatrix} 1 & 0 & {}^t\gamma \\ -X\gamma & X & XA \\ 0 & 0 & {}^tX^{-1} \end{pmatrix}$ where X is an invertible $n \times n$ upper triangular matrix, ${}^tX^{-1}$ is its inverse transpose, γ is a column in k^n, ${}^t\gamma$ is its transpose and the symmetric part of A is $-2^{-1}\gamma{}^t\gamma$. The above discussion and Lemma 4.1 translate into the assumption that there would be infinitely many conjugate matrices of the form

$$x(V,\lambda) = \dot{w}_0\mathrm{v}_1\mathrm{v}_2 = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & {}^tV^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & {}^t\gamma \\ -\gamma & I & A \\ 0 & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} (-1)^n & 0 & {}^t\gamma \\ 0 & 0 & {}^tV^{-1} \\ -V\gamma & V & VA \end{pmatrix}$$

where: γ is a vector in k^n; V is a fixed upper triangular unipotent matrix; A is a matrix whose symmetric part is $-2^{-1}\gamma{}^t\gamma$ and by Lemma 4.4 the coefficients of A and γ depend polynomially on $\lambda = \gamma_n$ and the coefficients of V. The characteristic polynomial $q_\lambda(T)$ of $x(V,\lambda)$ depends polynomially on λ thus $q_\lambda(T) = q_\mu(T)$ for at most finitely many μ in k unless $q_\lambda(T)$ is independent of λ. We claim that this is not the case. In order to prove this,

we need a more explicit description of V.
Using Lemma 4.2 one can show that, up to conjugation in $SO_{2n+1}(k)$ by diagonal matrices of type $\mathrm{diag}(1, -I_j, I_{n-j}, -I_j, I_{n-j})$ the matrix V is an upper triangular unipotent matrix with all 2's in the first off-diagonal. Inductively as in Lemma 4.4, using Lemma 4.2 one sees that V is the upper triangular unipotent matrix with only 2's above the diagonal. Thus it is enough to exhibit two matrices x_1 and x_2 of shape $\begin{pmatrix} (-1)^n & 0 & {}^t\gamma \\ 0 & 0 & {}^tV^{-1} \\ -V\gamma & V & M \end{pmatrix}$ with V as above, lying in quasi-spherical conjugacy classes and with distinct characteristic polynomials.

Let n be even. For ζ a square root of 2 in k we take

$$\gamma_1 = 2\zeta(1, -1, 1, \ldots, -1) \ \text{ and } \ M_1 = \begin{pmatrix} 0 & -2 & 2 & -2 & \cdots \\ 2 & -4 & 2 & -2 & \ddots \\ 2 & -2 & 0 & \ddots & \ddots \\ \ddots & -2 & \ddots & \ddots & \ddots \\ 2 & \ddots & \ddots & \ddots & -4 \end{pmatrix}.$$

Then the matrix x_1 with $\gamma = \gamma_1$ and $M = M_1$ is conjugate to $a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -A_1 & 0 \\ 0 & 0 & -{}^tA_1 \end{pmatrix}$ where A_1 is the upper triangular unipotent matrix with $(1, 0, 1, \ldots, 0, 1)$ on the first upper off-diagonal and zero elsewhere. Using the Jordan decomposition of a_1 and the formulas in [8, §13.1] which hold in good characteristic by [16, Theorem 7.8] we see that the dimension of the conjugacy class $\mathcal{O}_{a_1}$ of a_1 is $n^2 + n = \ell(w_0) + \mathrm{rk}(1 - w_0)$. Since x_1 lies in $\dot{w}_0 U$ we deduce from [5, Theorem 5], [6, Corollary 4.10] that $\mathcal{O}_{a_1}$ is spherical, hence quasi-spherical. Therefore $x = x_1(V, 2\zeta)$. Let us now consider $M_2 = \begin{pmatrix} 4 & -2 & 2 & -2 & \cdots \\ 2 & 0 & 2 & -2 & \ddots \\ 2 & -2 & 4 & \ddots & \ddots \\ \ddots & -2 & \ddots & \ddots & \ddots \\ 2 & \ddots & \ddots & \ddots & 0 \end{pmatrix}$. Then the matrix x_2 with $\gamma = 0$ and $M = M_2$ is unipotent and lies in the conjugacy class corresponding to the Young diagram $(3, 2^{n-2}, 1^2)$ whose dimension is again $n^2 + n$. The class $\mathcal{O}_{x_2}$ is thus spherical (see also [19, Theorem 3.2] and [14, Theorem 4.14]) hence quasi-spherical. It follows that $x_2 = x(V, 0)$ and the characteristic polynomials of x_1 and x_2 are different.

Let now n be odd and let ξ be a square root of -2 in k. We may consider $\gamma_3 = -2\xi(1, -1, 1, \dots, -1, 1)$ and M_3, constructed as M_2, and the corresponding matrix x_3. One verifies that x_3 is unipotent and lies in the conjugacy class associated with the Young diagram $(3, 2^{n-1})$, whose dimension is $n^2 + n$. As above, this class is spherical, hence quasi-spherical so $x_3 = x(V, -2\xi)$. On the other hand, taking $\gamma = 0$ and M_4 constructed as M_1 we get a matrix x_4 which is conjugate to $a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -A_2 & 0 \\ 0 & 0 & -{}^tA_2 \end{pmatrix}$ where A_2 is the upper triangular unipotent matrix with $(1, 0, \dots, 1, 0)$ on the first upper off-diagonal and zero elsewhere. As for n even we see that the dimension of $\mathcal{O}_{a_2}$ is $n^2 + n$ and since x_4 lies in $\dot{w}_0 U$ we deduce as above that $\mathcal{O}_{a_2}$ is spherical, hence quasi-spherical. Thus, $x_4 = x(V, 0)$ and the characteristic polynomials of x_3 and x_4 are distinct. It follows that in a fixed class $\mathcal{O}$ there can only be finitely many elements of type $x(V, \lambda)$. By Lemma 4.1 each maximal B-orbit contains an element of type $x(V, \lambda)$ or a representative with all $c_\alpha = 0$, so there are only finitely many of them. We may conclude using Lemma 3.1. □

Remark 4.10. — If H is a connected reductive algebraic group the radical $R(H)$ of H is a central torus ([22, Proposition 7.3.1]) contained in all Borel subgroups of H. Thus, a conjugacy class $\mathcal{O}$ in H is spherical (resp. quasi-spherical) if and only if its projection into the semisimple group $K = H/R(H)$ is spherical (resp. quasi-spherical). Moreover, a conjugacy class in K is spherical (resp. quasi-spherical) if and only if its projection into each simple factor of K is spherical (resp. quasi-spherical). Thus, the results we obtained so far apply also to the case of reductive groups. In particular, if G is connected, reductive with $w_0 = -1$ and $\mathcal{O}$ is quasi-spherical in G with $z_{\mathcal{O}} = w_0$ then $\mathcal{O}$ is spherical.

5. Curves in the centralizer

In this section we aim at the understanding of G_x for $x \in \dot{w}U \cap \mathcal{O}$ with $\mathcal{O}$ quasi-spherical and $w = z_{\mathcal{O}}$. We shall focus on searching suitable families of elements in G_x by making use of the particular form of the chosen representative x guaranteed by Lemma 3.7. By Lemma 3.8 if $w\alpha = \alpha$ then $X_{-\alpha} \subset X_\alpha s_\alpha B \cap G_x$. Now we aim at finding elements in $X_\gamma s_\gamma B \cap G_x$ for the remaining roots $\gamma \in \Phi^+$, namely those such that $w\gamma \in -\Phi^+$. We shall first analyze those γ for which $w\gamma = -\gamma$, that is, $\gamma \in \Phi_1$, by looking at $G_x \cap G(\Phi_1) = G(\Phi_1)_x$. By Lemma 3.7, $x \in G(\Phi_1)$ so since w is the longest element in $W(\Phi_1)$ we may use the results obtained in §4.

LEMMA 5.1. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class in G, let $w = z_{\mathcal{O}}$ and let $x \in \dot{w}U \cap \mathcal{O}$. Let $\alpha \in \Phi_1 \cap \Delta$. Then for all but finitely many $c \in k$ there exists $b_c \in B$ such that $x_\alpha(c)n_\alpha b_c \in G_x \cap G(\Phi_1)$.*

Proof. — We have $x \in G(\Phi_1)$ by Lemma 3.7. The element w is the longest element in $W(\Phi_1)$ and its restriction to $W(\Phi_1)$ is -1. Let us consider the conjugacy class $\mathcal{O}'$ of x in $G(\Phi_1)$. It is quasi-spherical because $B_1 = B \cap G(\Phi_1)$ is a Borel subgroup of $G(\Phi_1)$ containing T. Therefore $\mathcal{O}'$ is spherical in $G(\Phi_1)$ by Remark 4.10.

Let P be the minimal parabolic subgroup of $G(\Phi_1)$ associated with α and let P^u be its unipotent radical. Let $x = \dot{w}x_\alpha(a)\mathrm{v}$ with $\mathrm{v} \in P^u$ and, for any nonzero $c \in k$, let $y_c = n_\alpha^{-1}x_\alpha(-c)xx_\alpha(c)n_\alpha \in \mathcal{O}'$. We have, for some nonzero structure constants θ_1, θ_2, θ_3 and for t_1, $t_2 \in T$:

$$\begin{aligned} y_c &\in n_\alpha^{-1}(\dot{w}x_{-\alpha}(-\theta_1 c)x_\alpha(a+c)P^u)n_\alpha^{-1} \\ &= \dot{w}t_1x_\alpha(-\theta_2\theta_1 c)x_{-\alpha}(\theta_3(a+c))P^u \\ &= \dot{w}t_1x_\alpha(-\theta_2\theta_1 c)x_{-\alpha}((\theta_2\theta_1 c)^{-1})x_{-\alpha}(-(\theta_2\theta_1 c)^{-1} + \theta_3(a+c))P^u \\ &= \dot{w}n_\alpha t_2 x_\alpha(\theta_2\theta_1 c)x_{-\alpha}(-(\theta_2\theta_1 c)^{-1} + \theta_3(a+c))P^u \end{aligned}$$

where for the last equality we have used [22, Lemma 8.1.4 (i)]. Then, if

$$-(\theta_2\theta_1 c)^{-1} + \theta_3(a+c) \neq 0$$

we have $y_c \in B_1ws_\alpha B_1s_\alpha B_1 = B_1wB_1$ because $ws_\alpha < w$ in the Bruhat ordering. Thus, for all but finitely many $c \in k$ the element y_c lies in $B_1wB_1 \cap \mathcal{O}'$. By Lemma 3.1 applied to $\mathcal{O}'$ this intersection is the dense B_1-orbit $B_1.x$ so for all but finitely many $c \in k$ there is $b_c \in B_1$ such that $b_c^{-1}y_cb_c = x$, that is, $x_\alpha(c)n_\alpha b_c \in G_x \cap G(\Phi_1)$. □

Next we shall consider the roots γ for which $w\gamma \neq \pm\gamma$. The set of all such roots is $\Phi_2 = \Phi \setminus (\Phi_1 \cup \Phi(\Pi))$. Let $\Phi_2^+ = \Phi_2 \cap \Phi^+$. Every $\alpha \in \Phi_2^+$ determines the following subsets of Φ:

$$\Phi^+(\alpha) = \bigcup_{j>0}(j\alpha + \mathrm{Ker}(1+w)) \cap \Phi^+ \subset \Phi^+(\Pi) \cup \Phi_2^+;$$

$$\Phi^-(\alpha) = \bigcup_{j>0}(j\alpha + \mathrm{Ker}(1+w)) \cap (-\Phi^+) \subset (\Phi(\Pi) \cup \Phi_2) \cap (-\Phi^+)$$

and $\Phi(\alpha) = \Phi^+(\alpha) \cup \Phi^-(\alpha)$. Fixing an ordering of the roots, we define:

$$U_\alpha = \prod_{\beta\in\Phi^+(\alpha)} X_\beta \subset U; \qquad U_\alpha^- = \prod_{\beta\in\Phi^-(\alpha)} X_\beta \subset U^-. \tag{5.1}$$

Lemma 5.2. — *Let Φ be a simply or doubly-laced irreducible root system, let $w = w_0 w_\Pi$ and let $\alpha \in \Phi_2^+$. We have:*

1. *The subsets U_α and U_α^- are subgroups of U and U^-, respectively.*
2. *The subgroups U_α and U_α^- are independent of the chosen ordering of the roots.*
3. $U_\alpha \cap U_{\Phi_1} = 1$.
4. *If $X_{-\beta} \in U_\alpha^-$ then $X_\beta \notin \langle U_{\Phi_1}, U_\alpha\rangle = U_{\Phi_1} U_\alpha$.*
5. $U_\alpha U_\alpha^- = U_\alpha^- U_\alpha$.
6. $w(\Phi(\alpha)) \subset \Phi(\alpha)$
7. *If Φ is simply-laced and $w\alpha + \alpha \notin -\Phi^+$ then $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha$.*
8. *If Φ is doubly-laced and $w\alpha+\alpha \notin -(\Phi^+ \cup 2\Phi^+)$ then $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha$.*
9. *If Φ is doubly-laced and $w\alpha + \alpha = 2\beta \in -2\Phi^+$ then $X_\beta \subset U_\alpha^-$ and $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha X_\beta$. Besides, X_β commutes with U_α.*
10. *If Φ is simply- or doubly-laced and $w\alpha + \alpha = \beta \in -\Phi^+$ then $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha X_\beta$. Besides, X_β commutes with U_α.*

Proof. — The first two assertions follow from iterated application of (1.3). Statement 3 follows directly from the definition of $\Phi(\alpha)$. Statement 4 is easily seen by looking at the coefficient of α in the expression of β. Statement 5 follows from 4 and (1.3). The sixth statement follows once we write $w = \prod_\gamma s_\gamma$ for mutually orthogonal roots $\gamma \in \Phi_1$. Let us prove 7 and 8. If $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U$ then $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha$ because w is the product of reflections with respect to roots in Φ_1. Hence, it is enough to show that $w\mu \in \Phi^+$ for all $\mu \in \Phi^-(\alpha)$. If we had $w\mu \in -\Phi^+$ for $\mu = j\alpha + y$ with $j > 0$ and $y \in \mathrm{Ker}(1 + w)$ we would have $\mu \in \Phi(\Pi)$ so $w\mu = \mu$, that is

$$2\mu = \mu + w\mu = j\alpha + y + jw\alpha - y = j(\alpha + w\alpha) \in -2\Phi. \tag{5.2}$$

Thus $\alpha + w\alpha \notin \Phi$ for it could neither be a positive nor a negative root, so $(\alpha, w\alpha) \geqslant 0$. Taking $(2\mu, 2\mu)$ we would have

$$2(\mu, \mu) = j^2((\alpha, \alpha) + (\alpha, w\alpha)) \geqslant j^2(\alpha, \alpha). \tag{5.3}$$

If $(\alpha, \alpha) = (\mu, \mu)$ then $j = 1$ and $(\alpha, w\alpha) = (\alpha, \alpha)$ which is impossible proving statement 7. If $(\alpha, \alpha) = 2(\mu, \mu)$ we have again $j = 1$ and (5.2) gives $2\mu = \alpha + w\alpha$ contradicting our assumption in the doubly-laced case. If $2(\alpha, \alpha) = (\mu, \mu)$ we have $j^2 \leqslant 4$ so $j \leqslant 2$. Then either $j = 2$ and $\mu = \alpha + w\alpha \in -\Phi^+$ against our assumptions, or $j = 1$ and $3(\alpha, \alpha) = 4(\alpha, w\alpha)$. Since this can never happen, $\mu \notin \Phi(\Pi)$ and statement 8 holds.

Let us prove 9. Let $\mu = j\alpha + y \in \Phi^-(\alpha)$, with $y \in \mathrm{Ker}(1+w)$ and $j > 0$ and let us assume that $w\mu \in -\Phi^+$. It follows from the proof of 7 and 8 that we have $2\mu = j(\alpha + w\alpha) = 2j\beta$. Hence $j = 1$ and $\beta = \mu$ so X_β is the only root subgroup in U_α^- that is mapped onto a negative root subgroup under conjugation by $\dot{w}$, and it is mapped onto itself. Moreover, for every $\gamma = i\alpha + y' \in \Phi^+(\alpha)$ with $i > 0$ and $y' \in \mathrm{Ker}(1+w)$ we have $2(\beta, \gamma) = (\alpha + w\alpha, i\alpha + y') = i(\alpha, \alpha) + i(\alpha, w\alpha)$ because $\alpha + w\alpha$ is orthogonal to $\mathrm{Ker}(1+w)$. Since $(\alpha, \alpha) = (w\alpha, w\alpha)$ we have $s_\alpha(w\alpha) \in \{w\alpha - \alpha,\ w\alpha,\ w\alpha + \alpha,\}$ so $2\frac{(\alpha, w\alpha)}{(\alpha,\alpha)} \in \{0, \pm 1\}$. Thus $(\beta, \gamma) > 0$ and therefore $\beta + \gamma \notin \Phi$ so X_β commutes with with X_γ and $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha X_\beta U_\alpha = U_\alpha X_\beta$.

Let us prove the last assertion. Let us assume that $\beta = \alpha + w\alpha \in -\Phi$. If for some root $\nu = j\alpha + y \in \Phi^-(\alpha)$ we had $w\nu \in -\Phi^+$ we would have, as before, $w\nu = \nu$ and $2\nu = \nu + w\nu = j(\alpha + w\alpha) = j\beta \in 2\Phi$ so $j = 2$ and $\beta = \nu$. Thus $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha X_\beta U_\alpha$. As in the proof of 9 we verify that $\beta + \gamma \notin \Phi$ for every $\gamma \in \Phi^+(\alpha)$ whence X_β commutes with U_α and $\dot{w} U_\alpha^- \dot{w}^{-1} \subset U_\alpha X_\beta$. □

Lemma 5.3. — *Let G be a simple algebraic group, let $\mathcal{O}$ be a quasi-spherical conjugacy class with $w = z_{\mathcal{O}} = w_0 w_\Pi$ and let $x = \dot{w}\mathrm{v} \in \mathcal{O} \cap \dot{w}U$. Let $\alpha \in \Phi_2$ be such that $\alpha + w\alpha \notin -\Phi^+$. Let us also assume, if Φ is doubly-laced, that $\alpha + w\alpha \notin -2\Phi^+$. Then for every $c \in k$ there exists an element in $x_{w\alpha}(c)U^w \cap G_x$.*

Proof. — Since $w\alpha \neq \alpha$ we have $w\alpha \in -\Phi^+$. For every $c \in k$ we consider the elements

$$y(c) = x_\alpha(c) x x_\alpha(-c) = \dot{w} x_{w\alpha}(\theta c) \mathrm{v} x_\alpha(-c) \in \mathcal{O}$$

where θ is a nonzero structure constant, and the elements

$$\begin{aligned} z(c) &= x_{w\alpha}(\theta c) \dot{w} x_{w\alpha}(\theta c) \mathrm{v} x_\alpha(-c) x_{w\alpha}(-\theta c) \\ &= \dot{w} x_\alpha(\eta\theta c)(x_{w\alpha}(\theta c) \mathrm{v} x_{w\alpha}(-\theta c))(x_{w\alpha}(\theta c) x_\alpha(-c) x_{w\alpha}(-\theta c)) \in \mathcal{O} \end{aligned}$$

where η is a nonzero structure constant. By making use of (1.3) we shall show that for a suitable $u_c \in U_\alpha$, possibly trivial, we have

$$u_c^{-1} z(c) u_c = x \prod_{\gamma \in \Phi^+ \setminus \Phi_1} x_\gamma(c_\gamma) \in \dot{w}U \cap \mathcal{O}.$$

Lemma 3.7 will force $c_\gamma = 0$ for every $\gamma \in \Phi^+ \setminus \Phi_1$ so $u_c^{-1} z(c) u_c = x$ and we have $u_c^{-1} x_{w\alpha}(\theta c) x_\alpha(c) = u'_c x_\alpha(c) x_{w\alpha}(\theta c) \in U_\alpha x_{w\alpha}(\theta c) \in G_x$. Taking inverses will give the statement because c is arbitrary, $U_\alpha \subset U = U_w U^w$ and $U_w \subset G_x$ by Lemma 3.7.

By hypothesis $w\alpha+\alpha$ is either in Φ^+ or it is not a root. Therefore we have $x_{w\alpha}(\theta c)x_\alpha(-c)x_{w\alpha}(-\theta c) = \mathrm{v}' \in U_\alpha$. Besides, we have $\mathrm{v} = \prod_{\gamma_i\in\Phi_1^+} x_{\gamma_i}(c_i)$ so that

$$x_{w\alpha}(\theta c)\mathrm{v}x_{w\alpha}(-\theta c) = \prod_{i=1}^{r}(x_{\gamma_i}(c_i)\prod_{a_i,b_i>0} x_{a_i\gamma_i+b_iw\alpha}(d_{abi}))$$

where we intend $d_{abi} = 0$ if $a_i\gamma_i + b_iw\alpha \notin \Phi$. We proceed as follows: if $w\alpha+\gamma_1 \in -\Phi^+$ we apply (1.3) in order to move the term in $X_{w\alpha+\gamma_1}$ to the left of $x_{\gamma_1}(c_1)$ whereas if $w\alpha+\gamma_1 \in \Phi^+$ we apply (1.3) in order to move the term in $X_{w\alpha+\gamma_1}$ to the right of $x_{\gamma_r}(c_r)$. At each step we might get extra factors either in U_α or in U_α^- and we repeat the procedure. Formula (1.3) can always be applied because we need never to interchange factors in X_β with factors in $X_{-\beta}$ (*cf.* Lemma 5.2(4)). Therefore we have:

$$x_{w\alpha}(\theta c)\mathrm{v}x_{w\alpha}(-\theta c) = u^-\mathrm{v}u \in U_\alpha^-\mathrm{v}U_\alpha$$

because the coefficients of the terms in v are never modified. By Lemma 5.2 (5) we have $x_\alpha(\eta\theta c)u^- = u_-u_+ \in U_\alpha^-U_\alpha$ and thus

$$z(c) = \dot{w}u_-u_+\mathrm{v}u\mathrm{v}' = \dot{w}u_-\mathrm{v}u' = u_c\dot{w}\mathrm{v}u' \subset U_\alpha\dot{w}\mathrm{v}U_\alpha$$

where for the second equality and the inclusions we have used Lemma 5.2 (4,7,8). Conjugation by u_c^{-1} yields $u_c^{-1}z(c)u_c \in \mathcal{O}\cap\dot{w}\mathrm{v}U_\alpha$ hence the term in U_α vanishes by Lemma 3.7. Then $u_c^{-1}x_{w\alpha}(\theta c)x_\alpha(c) = u_c'x_\alpha(c)x_{w\alpha}(\theta c) \in G_x\cap U_\alpha x_{w\alpha}(\theta c)$ and we have the statement. □

Lemma 5.4. — *Let $\mathcal{O}$ be a quasi-spherical conjugacy class, let $w = z_{\mathcal{O}} = w_0w_\Pi$. If for some $\gamma \in \Phi^+\setminus\Phi(\Pi)$ and for every scalar c there is an element in $x_{-\gamma}(c)U^w$ centralizing $x \in \mathcal{O}\cap\dot{w}U$ then for every $\gamma' \in W_\Pi\gamma$ and for every $d \in k$ there is an element in $x_{-\gamma'}(d)U^w$ centralizing x.*

Proof. — By Lemmas 3.7 and 3.8, the centralizer of x contains $X_{\pm\alpha}$ hence n_α, for every $\alpha\in\Pi$. Conjugation by n_α preserves U^w and U_w and maps $X_{-\gamma}$ onto $X_{-s_\alpha(\gamma)}$, whence the statement. □

Lemma 5.5. — *Let G be a simple algebraic group with Φ doubly-laced. Let $\mathcal{O}$ be a quasi-spherical conjugacy class with notation as above. Let $\alpha\in\Phi_2^+$ be such that $w\alpha+\alpha = 2\beta \in -2\Phi^+$. Then for $x\in\dot{w}U_\alpha\cap\mathcal{O}$ and for every $c\in k$ we have $x_{w\alpha}(c)U^w\cap G_x \neq \emptyset$.*

Proof. — Let $z(c)$ be defined as in the proof of Lemma 5.3. We have again

$$x_{w\alpha}(\theta c)\mathrm{v}x_{w\alpha}(-\theta c) = u^-\mathrm{v}u \in U_\alpha^-\mathrm{v}U_\alpha.$$

Let us first assume that $\beta\in-\Pi$. Then $u^-=x_\beta(a)u_-$ with $u_-\in U_\alpha^-\cap\dot w^{-1}U_\alpha\dot w$ by Lemma 5.2 (2,9). We have

$$z(c)=\dot w x_\alpha(\theta\eta c)x_\beta(a)u_-\mathrm{v}ux_\alpha(-c)=\dot w x_\beta(a)x_\alpha(\theta\eta c)u_-\mathrm{v}ux_\alpha(-c)$$

by Lemma 5.2 (9). Applying repeatedly (1.3) and Lemma 5.2 (5) we have for some $u'\in U_\alpha$, u'_-, $\mathrm{v}_-\in U_\alpha^-$, and $a'\in k$

$$z(c)=\dot w x_\beta(a)u'_-\mathrm{v}u'=\dot w x_\beta(a+a')\mathrm{v}_-\mathrm{v}u'$$

with $u_c=\dot w\mathrm{v}_-\dot w^{-1}\in U_\alpha$. We claim that $a+a'\neq 0$. Otherwise, for some nonzero structure constant θ' we would have, by Lemma 5.2 (9),

$$z(c)=x_\beta((a+a')\theta')u_c\dot w\mathrm{v}u'\in Bs_\beta BwB=Bs_\beta wB$$

with $s_\beta w>w$ contradicting maximality of w. Thus, $a+a'=0$ and we may proceed as in the proof of Lemma 5.3. Moreover, $u_cX_\alpha\subset U^w$.

If $\beta\notin-\Pi$ then there is $\sigma\in W_\Pi$ such that $\sigma\beta\in-\Pi$ and $\sigma\alpha\in\Phi^+$ because the support of α contains at least one simple root outside Π. Since w is the identity on $\Phi(\Pi)$ it commutes with σ and we have $\sigma w\alpha\in-\Phi^+$ and $\sigma\alpha+w\sigma\alpha\in-2\Pi$. By the first part of the proof for every $c\in k$ there is an element in $x_{\sigma w\alpha}(c)U^w$ centralizing x and we may apply Lemma 5.4 to get the statement. □

LEMMA 5.6. — *Let G be a simple algebraic group with Φ simply or doubly-laced. Let $\mathcal{O}$ be a quasi-spherical conjugacy class with notation as above. Let $\alpha\in\Phi^+$ be such that $w\alpha+\alpha=\beta\in-\Phi$. Then for $x\in\dot wU\cap\mathcal{O}$ and for every $c\in k$ we have $x_{w\alpha}(c)U^w\cap G_x\neq\emptyset$.*

Proof. — Let us first assume that $\beta\in-\Pi$. Then $X_\beta\subset U_\alpha^-$ and $\dot wU_\alpha^-\dot w^{-1}\subset X_\beta U_\alpha$ by Lemma 5.2(10). We have $\beta+w\alpha=2w\alpha+\alpha=w(w\alpha+2\alpha)\notin\Phi$. This follows as in the proof of Lemma 5.2 (9,10).

Let $z(c)$ be as in the as in the proofs of Lemmas 5.3 and 5.5. As above we have:

$$x_{w\alpha}(\theta c)\mathrm{v}x_{w\alpha}(-\theta c)=u^-\mathrm{v}u\in U_\alpha^-\mathrm{v}U_\alpha.$$

We may apply (1.3) to move $x_\alpha(\eta\theta c)$ to the right of v in the expression of $z(c)$. Then we have

$$z(c)=\dot wu_-\mathrm{v}u^+x_{w\alpha}(\theta c)x_\alpha(-c)x_{w\alpha}(-\theta c)$$

with $u_-=x_\beta(h)u'_-\in X_\beta U_\alpha^-$, $u_c=\dot wu'_-\dot w^{-1}\in U_\alpha\cap U^w$ and $u^+\in U_\alpha$. Applying once more (1.3) to $x_{w\alpha}(\theta c)x_\alpha(-c)$ gives only a nontrivial extra term in X_β by Lemma 5.2 (10). Then, for some h_1, $h_2\in k$ and some $u'\in U_\alpha$ we have $z(c)=x_\beta(h_1)u_c\dot w\mathrm{v}u'x_\beta(h_2)$. Conjugation by $u_c^{-1}x_\beta(-h_1)$ yields an element $z'(c)$ in $xU_\alpha X_\beta U_\alpha\cap\mathcal{O}\subset BwBX_\beta B\subset Bws_\beta B\cup BwB$.

Maximality of w forces $h_1 + h_2 = 0$ so the X_β-factor in $z'(c)$ is trivial. Lemma 3.7 implies that $z'(c) = x$ so, using that $\beta+\alpha$, $\beta+w\alpha \notin \Phi$ we have

$$u_c^{-1}x_\beta(-h_1)x_{w\alpha}(\theta c)x_\alpha(c) = u_c^{-1}x_\alpha(c)x_{w\alpha}(\theta c)x_\beta(h) \in G_x$$

with $u_c \in U^w$. By Lemma 3.8 we have $x_\beta(h) \in G_x$ so $u_c^{-1}x_\alpha(c)x_{w\alpha}(\theta c) \in G_x$ and taking the inverse yields the statement for $\beta \in -\Pi$.
If $\beta \notin -\Pi$ we may apply Lemma 5.4 as we did in Lemma 5.5. □

We have constructed enough elements in G_x and we are ready to prove the main result of this paper.

Theorem 5.7. — *Let G be a connected, reductive algebraic group over an algebraically closed field k of characteristic zero or good and odd. Then every quasi-spherical conjugacy class $\mathcal{O}$ in G is spherical.*

Proof. — By Remark 4.10 it is enough to prove the statement for G simple. Type G_2 has already been discussed in Section 2.1 so we only need to consider Φ simply or doubly-laced. Moreover, when $z_{\mathcal{O}} = w_0 = -1$ the statement has been proved in Propositions 4.6, 4.7, 4.8, 4.9 so we shall prove the remaining cases. Let v be a maximal B-orbit in $\mathcal{O}$. We will prove the statement by showing that $\dim(\mathcal{O}) = \dim(v)$, so that v is dense in $\mathcal{O}$. To this end, we need to show that for some $x \in v$ we have $\dim G_x = \dim B_x + |\Phi^+|$. We will do so by using $x \in \dot{w}U \cap \mathcal{O}$ for $w = z_{\mathcal{O}}$.

Let us consider the restriction π_x to G_x of the natural projection π of G onto the flag variety G/B. Let gB be in the image of π_x. We may assume that $g \in G_x$ and then it is not hard to verify that $\pi_x^{-1}(gB) = gB_x$ so each non-empty fiber has dimension equal to $\dim B_x$. Since $\dim G/B = |\Phi^+|$ it is enough to prove that π_x is dominant and use [22, Theorem 5.1.6]. We shall prove that $\pi_x(G_x) \cap \pi(B\sigma B)$ is dense in $\pi(B\sigma B)$ for every $\sigma \in W$. In particular, this is true for $\sigma = w_0$ so $\pi_x(G_x) \cap \pi(Bw_0B)$ is dense in $\pi(Bw_0B)$ thus $\pi_x(G_x)$ is dense in G/B.

More precisely, if we identify $\pi(B\sigma B) = \pi(U^\sigma\sigma B)$ with the affine space $\mathbb{A}^{\ell(\sigma)}$ through the map $\pi(u\dot{\sigma}B) = \pi(\prod_{\gamma\in\Phi_\sigma} x_\gamma(c_\gamma)\dot{\sigma}B) \mapsto (c_\gamma)_{\gamma\in\Phi_\sigma}$, we will show by induction on $\ell(\sigma)$ that $\pi_x(G_x) \cap \mathbb{A}^{\ell(\sigma)}$ contains the complement in $\mathbb{A}^{\ell(\sigma)}$ of finitely many hyperplanes.

For $\sigma = 1$ there is nothing to say. Suppose that the statement holds for $\ell(\sigma) \leqslant s$ and let us consider $\tau \in W$ with $\ell(\tau) = s+1$. Then $\tau = \sigma s_\alpha$ for some $\sigma \in W$ with $\ell(\sigma) = s$ and some $\alpha \in \Delta$ with $\sigma\alpha \in \Phi^+$. Besides, $\Phi_\tau = \Phi_\sigma \cup \{\sigma\alpha\}$ so $U^\tau = U^\sigma X_{\sigma\alpha}$. By the induction hypothesis the set U' of elements u in U^σ for which $u\dot{\sigma}b$ lies in G_x for some $b \in B$ contains the complement of finitely many hyperplanes in $U^\sigma \cong \mathbb{A}^{\ell(\sigma)}$.

There are three possibilities: $\alpha \in \Pi$, $\alpha \in \Delta \cap \Phi_1$ and $\alpha \in \Delta \cap \Phi_2$. If $\alpha \in \Pi$ we have $X_\alpha n_\alpha \subset G_x$ by Lemma 3.7. Then for every $u \in U'$ and every $c \in k$ there is $b \in B$ for which $(u\dot{\sigma}b)(x_\alpha(c)n_\alpha) \in G_x$. Let $b = x_\alpha(r)\mathrm{v}$ for $r \in k$ and $\mathrm{v} \in P_\alpha^u$. Then for some $\mathrm{v}' \in P_\alpha^u$ and for some nonzero structure constant η we have

$$(u\dot{\sigma}b)(x_\alpha(c)n_\alpha) = u\dot{\sigma}x_\alpha(r+c)n_\alpha\mathrm{v}' = ux_{\sigma\alpha}(\eta(r+c))\dot{\sigma}n_\alpha\mathrm{v}' \in G_x.$$

Since c is arbitrary and $\eta \neq 0$, if $\alpha \in \Pi$ then $\pi_x(G_x) \cap \pi(B\tau B)$ contains $\pi(U'X_{\sigma\alpha}\tau B)$ so it contains the complement of finitely many hyperplanes in $\mathbb{A}^{\ell(\tau)}$.

Let now $\alpha \in \Delta \cap \Phi_1$. By Lemma 5.1 for all but finitely many $c \in k$ there is $b_c \in B$ such that $x_\alpha(c)n_\alpha b_c \in G_x$. Thus, for every $u \in U'$ and for those c there is $b \in B$ for which $(u\dot{\sigma}b)(x_\alpha(c)n_\alpha b_c) \in G_x$. Let $b = x_\alpha(r)\mathrm{v}$ for $r \in k$ and $\mathrm{v} \in P_\alpha^u$. Then for some $\mathrm{v}' \in P_\alpha^u$ and for some nonzero structure constant η we have

$$(u\dot{\sigma}b)(x_\alpha(c)n_\alpha b_c) = u\dot{\sigma}x_\alpha(r+c)n_\alpha\mathrm{v}'b_c = ux_{\sigma\alpha}(\eta(r+c))\dot{\sigma}n_\alpha\mathrm{v}'b_c \in G_x.$$

Since all but finitely many c were possible and $\eta \neq 0$, also in this case $\pi_x(G_x) \cap \pi(B\tau B)$ contains $\pi(U'x_{\sigma\alpha}(c)\tau B)$ for all but finitely many c, thus it contains the complement of finitely many hyperplanes in $\mathbb{A}^{\ell(\tau)}$.

Finally, let $\alpha \in \Phi_2 \cap \Delta$. Then by Lemmas 5.3, 5.5, and 5.6 for every $c \in k$ there exists $u_c \in U$ such that $x_{-\alpha}(c)u_c \in G_x$. For $c \neq 0$ this element is equal to $x_\alpha(c^{-1})t_c n_\alpha x_\alpha(c^{-1})u_c$ for some $t_c \in T$ by [22, Lemma 8.1.4 (i)]. Thus, for every $u \in U'$ and $c \neq 0$ there is $b \in B$ for which $(u\dot{\sigma}b)(x_\alpha(c^{-1})t_c n_\alpha x_\alpha(c^{-1})u_c) \in G_x$. Let $b = x_\alpha(r)\mathrm{v}$ for $r \in k$ and $\mathrm{v} \in P_\alpha^u$. Then for some $\mathrm{v}' \in P_\alpha^u$, $t_c' \in T$ and for some nonzero structure constant η we have

$$\begin{aligned}(u\dot{\sigma}b)(x_\alpha(c^{-1})t_c n_\alpha x_\alpha(c^{-1})u_c) &= u\dot{\sigma}x_\alpha(r+c^{-1})n_\alpha t_c'\mathrm{v}'x_\alpha(c^{-1})u_c \\ &= ux_{\sigma\alpha}(\eta(r+c^{-1}))\dot{\sigma}n_\alpha t_c'\mathrm{v}'x_\alpha(c^{-1})u_c \in G_x.\end{aligned}$$

Then again, $\pi_x(G_x) \cap \pi(B\tau B)$ contains the complement of finitely many hyperplanes in $\mathbb{A}^{\ell(\tau)}$ and we have the statement. □

As a consequence of Theorem 5.7 we get the sought characterization.

THEOREM 5.8. — *Let $\mathcal{O}$ be a conjugacy class in a connected reductive algebraic group G over a field of zero or good odd characteristic. Then $\mathcal{O}$ is spherical if and only if $\mathcal{O} \subset \bigcup_{w^2=1} BwB$.*

Proof. — This is obtained combining Theorem 5.7 with [6, Theorem 2.7], whose proof holds also for G connected and reductive. □

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Giovanna CARNOVALE
University of Padova
Dipartimento di Matematica Pura ed Applicata
via Trieste 63
Padova, 35121 (Italy)
carnoval@math.unipd.it