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<http://aif.cedram.org/item?id=AIF_2009__59_6_2143_0>
GELFAND TRANSFORMS OF $SO(3)$-INVARIANT SCHWARTZ FUNCTIONS ON THE FREE GROUP $N_{3,2}$

by Véronique FISCHER & Fulvio RICCI

Abstract. — The spectrum of a Gelfand pair $(K \ltimes N, K)$, where $N$ is a nilpotent group, can be embedded in a Euclidean space. We prove that in general, the Schwartz functions on the spectrum are the Gelfand transforms of Schwartz $K$-invariant functions on $N$. We also show the converse in the case of the Gelfand pair $(SO(3) \ltimes N_{3,2}, SO(3))$, where $N_{3,2}$ is the free two-step nilpotent Lie group with three generators. This extends recent results for the Heisenberg group.

1. Introduction

Let $N$ be a connected, simply-connected, two-step nilpotent Lie group. Let $K$ be a compact group acting by automorphism on $N$. We assume that $(K \ltimes N, K)$ is a Gelfand pair. The Gelfand spectrum can be homeomorphically embedded in a Euclidean space as follows.

Let $\mathbb{D}(N)^K$ be the algebra of left-invariant and $K$-invariant differential operators on $N$ and $\{D_1, \ldots, D_q\}$ a finite set of essentially self-adjoint generators of $\mathbb{D}(N)^K$. We call $\mathcal{D}$ the ordered family $(D_1, \ldots, D_q)$. To each bounded $K$-spherical function $\phi$ on $N$ we assign the $q$-tuples of eigenvalues $\mu(\phi) = (\mu_1(\phi), \ldots, \mu_q(\phi))$, i.e. such that $D_j \phi = \mu_j(\phi) \phi$. The set $\Sigma_D$ of such $q$-tuples is in 1-1 correspondence with the Gelfand spectrum and the topology induced on it from $\mathbb{R}^q$ coincides with the Gelfand topology [5].
We define the Gelfand transform $\mathcal{G} : L^1(N)^K \to C_o(\Sigma_D)$ by:

$$\mathcal{G} F(\mu(\phi)) = \int_N F\check{\phi}.$$ 

We are interested in the following conjecture:

$\mathcal{G}$ establishes an isomorphism between $S(N)^K$ and $S(\Sigma_D)$ (as Fréchet spaces)

The validity of this statement is independent of the choice of $D$ (see Section 3); therefore once proved for one particular choice of $D$, it is true for any choice of $D$.

Proposition 3.3 of this paper shows the continuous inclusion $S(\Sigma_D) \hookrightarrow \mathcal{G}(S(N)^K)$. This property is already known for the case of the Heisenberg group [2, Theorem 5.5]. The proof relies on a generalisation [2, Theorem 5.2] of Hulanicki’s Schwartz kernel Theorem [14].

The converse inclusion has been recently shown for any Heisenberg Gelfand pair [2] and we prove it here for $(SO(3) \ltimes N_{3,2}, SO(3))$ where $N_{3,2}$ is the free two-step nilpotent Lie group with three generators: we realise $N_{3,2}$ as $\mathbb{R}^2 \times \mathbb{R}_y^3$, $\{0\} \times \mathbb{R}_y^3$ being the centre. It is known that $(SO(3) \ltimes N_{3,2}, SO(3))$ is a Gelfand pair [3, Theorem 5.12]. We will give explicit formulae for a family of three essentially self-adjoint operators $\mathcal{D}$ that generate $\mathbb{D}(N_{3,2})^{SO(3)}$, the bounded spherical functions and their corresponding eigenvalues for $\mathcal{D}$.

Historically, the first description of the image of the Schwartz space on the 2n+1-dimensional Heisenberg group $H_n$ under the group Fourier transform has been described by D. Geller [8]. In the same spirit, for a Heisenberg Gelfand pair $(K \ltimes H_n, K)$, a characterisation of the Gelfand transform of the radial Schwartz functions was given in [4] for closed subgroups $K$ of the unitary group $U(n)$. For more details, we refer the reader to the introductions of [1, 2].

Our goal here is to prove that for any Schwartz $SO(3)$-invariant function $F \in S(N_{3,2})^{SO(3)}$, there exists a Schwartz extension of its Gelfand transform:

$$i.e. \exists f \in S(\mathbb{R}^3) \quad f|_{\Sigma_D} = \mathcal{G} F.$$
Whenever it makes sense, we denote by $\mathcal{R}F$ the function on $N'$ given by integration of a function $F$ of $N_{3,2}$ on the central subgroup $\mathbb{R}^2_{(y_1, y_2)}$. The operator $\mathcal{R}$ maps $SO(3)$-invariant functions on $N_{3,2}$ to $K'$-invariant functions on $N'$, and Schwartz functions on $N_{3,2}$ to Schwartz functions on $N'$. $\mathcal{R}$ is 1-1, but does not send $\mathcal{S}(N_{3,2})^{SO(3)}$ onto $\mathcal{S}(N')^{K'}$ (see Proposition 4.3). The definition of $\mathcal{R}$ can be extended to left-invariant differential operators $D$ and any smooth compactly supported functions $F$ on $N$. We will see that the image of the operators in $\mathcal{D}$ by $\mathcal{R}$ completed with $-\partial^2_{x_3}$ gives a family $\mathcal{D}'$ of essentially self-adjoint generators for $\mathcal{D}(N')^{K'}$. Again we will give explicit formulae for $\mathcal{D}'$, the bounded spherical functions, and their corresponding eigenvalues for $\mathcal{D}'$.

The spectrum of $(K' \ltimes N', K')$ can be projected onto the spectrum of $(SO(3) \ltimes N_{3,2}, SO(3))$ in the following sense: composing an homomorphism of $L^1(N')^{K'}$ with $\mathcal{R}$ provides a mapping $\Pi : \Sigma_{\mathcal{D}'} \rightarrow \Sigma_{\mathcal{D}}$ between the two spectra, that is completely explicit here. In fact $\Pi$ maps continuously $\Sigma_{\mathcal{D}'}$ onto $\Sigma_{\mathcal{D}}$, but is 1-1 only on the regular part of the spectrum (see Section 2).

For any Schwartz $SO(3)$-invariant function $F \in \mathcal{S}(N_{3,2})^{SO(3)}$, we have:

$$\mathcal{G}'(\mathcal{R}F) = \mathcal{G}F \circ \Pi.$$ 

The existence of a Schwartz extension to $\mathbb{R}^4$ for $\mathcal{G}'(\mathcal{R}F)$, can be deduced easily from the Heisenberg case [1, 2]; it does not imply directly the existence of a Schwartz extension for $\mathcal{G}F$ but is constantly used all along the proof.

This article is organised as follows. In Section 2, we introduce the notations and the basic facts concerning the Gelfand spectra of $(SO(3) \ltimes N_{3,2}, SO(3))$ and $(K' \ltimes N', K')$. In Section 3, we give some general settings and the precise statements of our results. In Section 4 we describe $\mathcal{R}$ and the restriction mappings. In Section 5 we give the proof of Theorem 5 using an extension of a mean value formula due to Geller in the case of the Heisenberg group [8]. In the appendix, we give, for completeness, detailed proof of some results appearing in this paper and concerning differential operators and functional calculus on them.

2. The Gelfand spectra of $(SO(3) \ltimes N_{3,2}, SO(3))$

and $(K' \ltimes N', K')$

We realise $N_{3,2}$ as $\mathbb{R}_x^3 \times \mathbb{R}_y^3$ endowed with the law:

$$(x, y)(x', y') = (x + x', y + y' + \frac{1}{2} x \wedge x'),$$
where \( \wedge \) indicates the usual wedge product in \( \mathbb{R}^3 \). \( N_{3,2} \) denotes its Lie algebra. For \( j = 1, 2, 3 \), let \( X_j \) be the left-invariant vector field on \( N \) that equals \( \partial_{x_j} \) at 0, and \( Y_j \) the left-invariant vector field on \( N \) that equals \( \partial_{y_j} \). \((X_j)_{j=1,2,3}\) and \((Y_j)_{j=1,2,3}\) form the canonical basis of \( N_{3,2} \), and satisfy:

\[
[X_1, X_2] = Y_3, \quad [X_3, X_1] = Y_2, \quad [X_2, X_3] = Y_1.
\]

The group \( SO(3) \) acts on \( \mathbb{R}^3 \) and thus on \( N_{3,2} \) by acting simultaneously on each copy of \( \mathbb{R}^3 \). One checks easily that this action is by automorphisms on \( N_{3,2} \). \((SO(3) \ltimes N_{3,2}, SO(3))\) is a Gelfand pair \([3, \text{Theorem 5.12}]\).

Let us define the sub-Laplacian \( L \), the central Laplacian \( \Delta \) and a third operator \( D \) by:

\[
L = -\sum_{j=1}^{3} X_j^2, \quad \Delta = -\sum_{j=1}^{3} Y_j^2, \quad D = -\sum_{j=1}^{3} X_j Y_j.
\]

In Section 3, we will show that these operators form a family \( \mathcal{D} = (L, \Delta, D) \) of essentially self-adjoint operators that generate \( \mathbb{D}(N_{3,2})^{SO(3)} \).

The bounded spherical functions and their corresponding eigenvalues for \( \mathcal{D} \) are known explicitly. Let us define some notation first: for any vector \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), we will write \( \tilde{x} \) or \( x^r \) for \( (x_1, x_2) \) and, occasionally, \([x]_3\) for \( x_3 \). \( L_l(u) = (1/!)e^{u/2}(d/du)^l u^le^{-u} \) denotes the \( l \)-Laguerre function of order 0 on \( \mathbb{R} \). Then the bounded spherical functions on \( N_{3,2} \) are:

\[
\phi_{\lambda, l, r}(x, y) = \int_{k \in SO(3)} e^{-ik[k.y]_3} L_l \left( \frac{\lambda}{2} |[k.x]_3|^2 \right) e^{-lr|[k.x]_3} dk,
\]

\[ \lambda > 0, \ l \in \mathbb{N}, \ r \in \mathbb{R}. \]

and

\[
\phi_{0, R}(x, y) = \int_{k \in SO(3)} e^{-ik[k.x]_3} dk = \frac{\sin(R|x|)}{R|x|} dk, \quad R > 0;
\]

their eigenvalues for \( \mathcal{D} \) are given by:

\[
\mu_{\phi_{\lambda, l, r}} = (\lambda(2l + 1) + r^2, \lambda^2, \lambda r), \quad \mu_{\phi_{0, R}} = (R^2, 0, 0).
\]

The Gelfand spectrum \( \Sigma_{\mathcal{D}} \) of \((SO(3) \ltimes N_{3,2}, SO(3))\) is then realised as the union of the collection of the \( \mu_{\phi_{\lambda, l, r}}, \lambda > 0, l \in \mathbb{N}, r \in \mathbb{R} \) (the regular part of the spectrum), with the collection of the \( \mu_{\phi_{R}}, R \in \mathbb{R} \) (the singular part of the spectrum). Calling \((\eta_1, \eta_2, \eta_3)\) the coordinates corresponding to \( L, \Delta, D \) respectively, \( \Sigma_{\mathcal{D}} \) is then the union, for \( l \geq 0 \), of the surfaces \( \Gamma_l \) defined by the equation \( \eta_3^2 = \eta_2 (\eta_1 - (2l + 1)\sqrt{\eta_2}) \). They all meet together on the positive \( \eta_2 \)-axis, the singular part of \( \Sigma_{\mathcal{D}} \).
$N'$ is the quotient group of $N_{3,2}$ by the central subgroup $\mathbb{R}^2_{(y_1,y_2)}$; we realise $N'$ as $\mathbb{R}^3_+ \times \mathbb{R}_t$ endowed with the law:

$$(x,t).(x',t') = (x + x', t + t' + \frac{1}{2}(x \wedge x')_3).$$

$N'$ denotes its Lie algebra; this is a quotient of $\mathcal{N}_{3,2}$ by $\mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ and $q = q_{N_{3,2}} : \mathcal{N}_{3,2} \to N'$ denotes the quotient mapping. $q(X_j) = X'_j$ is the left-invariant vector field $X'_j$ on $N'$ that equals $\partial_{x_j}$ at $0$, $j = 1, 2, 3$; $q(Y_1) = q(Y_2) = 0$, and $q(Y_3) = T$ is the left-invariant vector field on $N$ that equals $\partial_t$. In particular $X'_3 = \partial_{x_3}$ lies in the centre of $N'$. $(X'_j)_{j=1,2,3}$ and $T$ form the canonical basis of $N'$. It is easy to see that $N'$ is isomorphic to $H_1 \times \mathbb{R}$. Let $K'$ be the stabiliser of $\mathbb{R}^2_{(y_1,y_2)} \subset \mathbb{R}^3_+$ in $SO(3)$. The group $K'$ is $SO(2) \times O(1) \cong O(2) \cong U_1 \times \mathbb{Z}_2$.

$(K' \ltimes N', K')$ is a Gelfand pair and its bounded spherical functions are explicitly known:

$$\phi_{\lambda,l,r}'(x,t) = \cos(\lambda t + rx_{x_3}) \mathcal{L}_l(\frac{\lambda}{2} ||k.x||^2) dk , \quad \lambda > 0, l \in \mathbb{N}, r \in \mathbb{R},$$

and

$$\phi'_{\zeta,r}(x,y) = J_o(\zeta |x|) \cos(rx_{x_3}) , \quad \zeta, r \in \mathbb{R},$$

$J_o$ being the Bessel function of order 0.

We define the following operators:

$$L' = -\sum_{j=1}^{3} X'_j^2 , \quad \Delta' = -T^2 , \quad D' = -X'_3 T.$$ 

The operators $L'$, $\Delta'$, $D'$ and $-X'_3^2$ are $K'$-invariant, essentially self-adjoint and generate $\mathbb{D}(N')^{K'}$ (see Proposition 3.1 and Subsection 5.1). We set the family $D' = (L', \Delta', D', -X'_3^2)$.

The eigenvalues of the bounded spherical functions for $D'$ are given by:

$$\mu_{\phi_{\lambda,l,r}'} = (\lambda(2l + 1) + r^2, \lambda^2, \lambda r, r^2),$$

$$\mu_{\phi'_{\zeta,r}} = (\zeta^2 + r^2, 0, 0, r^2).$$

As in the case of $(SO(3) \ltimes N_{3,2}, SO(3))$, the Gelfand spectrum $\Sigma_{D'}$ of $(K' \ltimes N', K')$ is then realised as the union of a regular and a singular part: the regular part is the collection of the $\mu_{\phi_{\lambda,l,r}'}$, $\lambda > 0$, $l \in \mathbb{N}$, $r \in \mathbb{R}$, and the singular part is the collection of the $\mu_{\phi'_{\zeta,r}}$, $\zeta, r \in \mathbb{R}$.

With coordinates $(\eta_1, \eta_2, \eta_3, \eta_4)$ corresponding to $L'$, $\Delta'$, $D'$, $-X'_3^2$ respectively, $\Sigma_{D'}$ is the union of the set $\{(\eta_1, 0, 0, \eta_4) : 0 \leq \eta_4 \leq \eta_1\}$ (the
singular set) and the two-dimensional surfaces $\Gamma'_l$, $l \geq 0$, defined by the system of equations
\[
\begin{aligned}
\eta_3^2 &= \eta_2 (\eta_1 - (2l + 1) \sqrt{\eta_2}) \\
\eta_3^2 &= \eta_2 \eta_4.
\end{aligned}
\]

Notice that the projection onto the hyperplane $\eta_4 = 0$ parallel to the $\eta_4$-axis maps $\Sigma'_{D'}$ onto $\Sigma_D$, and is bijective between the two regular sets. This fact, alluded to already in the introduction, will be relevant in view of the mapping $R$ defined in Subsection 4.1.

Let us give an equivalent and intrinsic point of view of this fact. As explained in the introduction, the spectrum of $(K' \ltimes N', K')$ can be projected in the following sense onto the spectrum of $(SO(3) \ltimes N_{3,2}, SO(3))$: the composition of an homomorphism of $L^1(N')^{K'}$ with $R$ (see also Subsection 4.1) provides a mapping between the two Gelfand spectra. Realising the Gelfand spectra as explained in the introduction (see also Section 3), this mapping $\Pi : \Sigma'_{D'} \to \Sigma_D$ is then given by:
\[
\begin{aligned}
\Pi \left( \lambda(2l + 1) + r^2, \lambda^2, \lambda r, r^2 \right) &= \left( \lambda(2l + 1) + r^2, \lambda^2, \lambda r \right), \\
\Pi \left( \zeta^2 + r^2, 0, 0, r^2 \right) &= \left( \zeta^2 + r^2, 0, 0 \right).
\end{aligned}
\]

$\Pi$ maps continuously $\Sigma'_{D'}$ onto $\Sigma_D$. Moreover $\Pi$ maps homeomorphically the regular part of $\Sigma'_{D'}$ onto the regular part of $\Sigma_D$; $\Pi$ maps the irregular part of $\Sigma'_{D'}$ onto the irregular part of $\Sigma_D$, but this correspondence is not 1-1.

\section{3. Results}

In this section, we describe the general settings of our work and explain the conjecture $G(S(N)^K) = S(\Sigma_D)$. We will give the precise statement of our main result in Theorem 3.5.

Let $N$ be a connected, simply-connected Lie group, $\mathcal{N}$ its Lie algebra, $\exp : \mathcal{N} \to N$ the exponential mapping and $(E_i)_{i=1}^p$ a basis of $\mathcal{N}$. The canonical basis $(E_i)_{i=1}^p$ of $\mathcal{N}$ being chosen, this induces a Lebesgue measure $dX$ on $\mathcal{N}$ and, via the exponential map, a Haar measure $dn$ on $N$; the spaces $L^p(N)$ are defined with respect to this Haar measure. When $N$ is a graded Lie group, following [7, ch1.D], we fix a homogeneous gauge $|.|$ on $N$ and we keep the same notation for the basis $(E_j)$ of $\mathcal{N}$ and the associated left-invariant vector fields on $N$; we set the following family of semi-norms parametrised by $a \in \mathbb{N}$ on the Schwartz space $S(N)$ which induces the usual Fréchet space structure on $S(N)$:
\[
\|F\|_{a,N} = \sup_{n \in \mathcal{N}, d(I) \leq a} (1 + |n|)^a |E^l F(n)|.
\]
\( \mathcal{P}(N) \) denotes the algebra of polynomials on \( N \) with real coefficients where \( N \) is then identified with the Euclidean vector space \( \mathbb{R}^p \). \( \mathcal{D}(N) \) denotes the algebra of real left-invariant differential operators on \( N \), as operators defined on \( C_c^{\infty}(N) \), the space of smooth, compactly-supported functions on \( N \).

To \( P \in \mathcal{P}(N) \), we associate \( D_P \in \mathcal{D}(N) \) by:

\[
D_P F(n) = \left. P(i^{-1} \partial_u) F(n \exp(\sum_{j=1}^{p} u_j E_j)) \right|_{u=0}.
\]

We obtain the symmetrisation mapping \( P \mapsto D_P \), that is a linear isomorphism between the algebras \( \mathcal{D}(N) \) and \( \mathcal{P}(N) \) [12, Ch.II Theorems 4.3 and 4.9]

In the appendix we show:

**Proposition 3.1.** — If \((K \ltimes N, K)\) is a Gelfand pair, each operator of \( \mathcal{D}(N)^K \) is essentially self-adjoint, that is, it admits a unique self-adjoint extension to an unbounded operator of \( L^2(N) \).

Furthermore the operators of \( \mathcal{D}(N)^K \) commute strongly, in the sense that the spectral resolutions of their self-adjoint extensions commute.

We will use the same notation for an operator of \( \mathcal{D}(N)^K \) and its self-adjoint extension.

By Hilbert’s Basis Theorem, if a group \( K \) acts orthogonally on some Euclidean space \( \mathbb{R}^p \), the algebra \( \mathcal{P}(\mathbb{R}^p)^K \) of \( K \)-invariant polynomials on \( \mathbb{R}^p \) is finitely generated [12, Ch.II Corollary 4.10]. If \( \rho_1, \ldots, \rho_q \) is a set of generators, we call \( \{\rho_1, \ldots, \rho_q\} \) a Hilbert basis for \( (\mathbb{R}^p, K) \) and \( \rho = (\rho_1, \ldots, \rho_q) \) the corresponding Hilbert mapping. Furthermore, if \( \rho = (\rho_1, \ldots, \rho_q) \) and \( \rho' = (\rho'_1, \ldots, \rho'_q) \) are two Hilbert mappings for \( (\mathbb{R}^p, K) \), then there exists \( Q = (Q_1, \ldots, Q_q), Q_j \in \mathcal{P}(\mathbb{R}^{q'}) \), such that \( \rho = Q \circ \rho' \) (and viceversa). We will make extensive use of G. Schwarz’s Theorem [16]: every \( K \)-invariant smooth function on \( \mathbb{R}^p \) can be expressed as a smooth function of a Hilbert basis \( \rho \) of \( (\mathbb{R}^p, K) \). In other words, the Hilbert map, \( \rho \), induces an application given by \( \rho^*(h) = h \circ \rho \). Moreover \( \rho^* \) is a linear continuous mapping from \( C^{\infty}(\mathbb{R}^q) \) onto \( C^{\infty}(\mathbb{R}^p)^K \), and also from \( \mathcal{S}(\mathbb{R}^q) \) onto \( \mathcal{S}(\mathbb{R}^p)^K \) [2, Theorem 6.1].

Assume that \((K \ltimes N, K)\) is a Gelfand pair. Any family of generators of \( \mathcal{D}(N)^K \) is obtained as the symmetrisation of a Hilbert basis, and conversely, if \( \{\rho_1, \ldots, \rho_q\} \) denotes a Hilbert basis for \( (N, K) \), then \( \{D_{\rho_1}, \ldots, D_{\rho_q}\} \) is a set of generators of \( \mathcal{D}(N)^K \).
Let us fix \((\rho_1, \ldots, \rho_q)\) an ordered Hilbert basis for \((\mathcal{N}, K)\), to which we associate the ordered family of operators \(D_\rho = (D_{\rho_1}, \ldots, D_{\rho_q})\). We denote by \(\Sigma_{D_\rho}\), the set of the \(q\)-tuples of eigenvalues \(\mu(\phi) = (\mu_1(\phi), \ldots, \mu_q(\phi))\) of \(D_\rho\) for the bounded \(K\)-spherical functions \(\phi\) on \(N\). As mentioned in the introduction and proved in [5], \(\Sigma_{D_\rho}\) is the realisation of the Gelfand spectrum associated to \(D_\rho\), in the sense that the set \(\Sigma_{D_\rho}\) of such \(q\)-tuples is in 1-1 correspondence with the Gelfand spectrum and the topology induced on it from \(\mathbb{R}^q\) coincides with the Gelfand topology. In the appendix, we will show that \(\Sigma_{D_\rho}\) is also the joint spectrum of \(D_\rho\):

**Proposition 3.2.** — Let \((\rho_1, \ldots, \rho_q)\) be an ordered Hilbert basis for \((\mathcal{N}, K)\). The joint spectrum of the family of strongly commuting, self-adjoint operators \(D_\rho = (D_{\rho_1}, \ldots, D_{\rho_q})\) is \(\Sigma_{D_\rho}\).

For a closed subset \(E\) of \(\mathbb{R}^q\), \(S(E)\) denotes the space of restrictions to \(E\) of Schwartz functions, endowed with the quotient topology of \(S(\mathbb{R}^q)/\{f : f|_E = 0\}\); we will often define a class in this quotient as being given as the restriction of a Schwartz function on \(\mathbb{R}^q\). The spectrum \(\Sigma_D\) is a closed subset of \(\mathbb{R}^q\). We are interested in the conjecture \(S(N)^K \overset{?}{\cong} S(\Sigma_D)\). The existence of a polynomial mapping between two Hilbert mappings implies that the validity of this conjecture is independent of the choice of \(D\) (see [2, Section 3]). The continuous inclusion \(S(\Sigma_D) \hookrightarrow G(S(N)^K)\) relies on the following statement, which is a generalisation of Hulanicki’s Schwartz Kernel Theorem proved in the appendix:

**Proposition 3.3.** — Let \((\rho_1, \ldots, \rho_q)\) be an ordered Hilbert basis for \((\mathcal{N}, K)\), and \(D_\rho = (D_{\rho_1}, \ldots, D_{\rho_q})\) the associated family of operators.

Let \(m\) be in \(S(\mathbb{R}^q)\). The operator \(m(D_\rho)\) is a convolution operator with a \(K\)-invariant Schwartz kernel \(M = M_{m,D_\rho} \in S(N)^K\):

\[
\forall F \in L^2(N) \quad m(D_\rho)F = F \ast M.
\]

The Gelfand transform of \(M\) is:

\[
\mathcal{G}M = m|_{\Sigma_{D_\rho}}.
\]

Furthermore the mapping \(m \in S(\mathbb{R}^q) \mapsto M_{m,D_\rho} \in S(N)^K\) is continuous.

For the Gelfand spectra of Heisenberg groups or the free two-step nilpotent Lie groups, the inclusion of the spectrum in the image of the Hilbert mapping:

\[
\Sigma_{D_\rho} \subset \text{im } \rho,
\]

is true, independently of the choice of the Hilbert mapping \(\rho\) (but we do not know if it is true in general). Here we will use this property only in
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the case of $N' = H_1 \times \mathbb{R}$, where it is known, the spectrum and the Hilbert mapping being explicit.

**Lemma 3.4.** — The polynomials $|x|^2$, $|y|^2$ and $x \cdot y$ generate the algebra of polynomials on $\mathbb{R}_x^3 \times \mathbb{R}_y^3$ that are invariant under the simultaneous action of $SO(3)$ on each copy of $\mathbb{R}^3$.

**Proof.** — If $P(x, y)$ is an $SO(3)$-invariant polynomial on $\mathbb{R}_x^3 \times \mathbb{R}_y^3$, then for each independent vectors $x, y \in \mathbb{R}^3$, we have $P(x, y) = P(-x, -y)$ because the linear transformation that equals $-\text{Id}$ on the vector space spanned by $x$ and $y$, and $1$ on the orthogonal complement line, is in $SO(3)$; this shows that $P$ is invariant under $-\text{Id}_{\mathbb{R}^3}$, and thus also under the simultaneous action of $O(3)$ on each copy of $\mathbb{R}^3$. This implies:

$$\mathcal{P}(\mathbb{R}_x^3 \times \mathbb{R}_y^3)^{SO(3)} = \mathcal{P}(\mathbb{R}_x^3 \times \mathbb{R}_y^3)^{O(3)}.$$  

By [10, Theorem 4.2.2.(1)], $\mathcal{P}(\mathbb{R}_x^3 \times \mathbb{R}_y^3)^{O(3)}$ is spanned by $|x|^2$, $|y|^2$ and $x \cdot y$. □

Thus $\rho(x, y) = (|x|^2, |y|^2, x \cdot y)$ gives a Hilbert mapping for $(N_{3,2}, SO(3))$. We compute easily that the associated family of operators by symmetrisation is $\tilde{D} = (L, \Delta, D)$ defined in Section 2, where we give also an explicit description of the associated realisation of the Gelfand spectrum.

**Theorem 3.5.** — The Gelfand transform of any Schwartz $SO(3)$-invariant function on $N_{3,2}$ admits a Schwartz extension to $\mathbb{R}^3$:

$$\forall F \in \mathcal{S}(N_{3,2})^{SO(3)} \quad \mathcal{G} F \in \mathcal{S}(\Sigma_D).$$

Moreover the mapping $F \in \mathcal{S}(N_{3,2})^{SO(3)} \mapsto \mathcal{G} F \in \mathcal{S}(\Sigma_D)$ is an isomorphism of Fréchet spaces.

The group $N_{3,2}$ admits a slightly bigger group of automorphisms than $SO(3)$, namely $O(3)$ acting by:

$$k(x, y) = (kx, (\det k)ky) \quad , \quad k \in O(3),$$

It is easily verified that $\{|x|^2, |y|^2, (x \cdot y)^2\}$ gives a Hilbert basis for $(N_{3,2}, O(3))$ and the associated family of operators by symmetrisation is $\tilde{D} = (L, \Delta, D^2)$. Following the same lines as in [2, Section 8], we have the following.

**Corollary 3.6.** — The Gelfand transform is an isomorphism between $\mathcal{S}(N_{3,2})^{O(3)}$ and $\mathcal{S}(\Sigma_{\tilde{D}})$ as Fréchet spaces.

From now on, $N$ will stand for $N_{3,2}$ and $K$ for $SO(3)$.
4. \( \mathcal{R} \) and restriction mappings

4.1. The mapping \( \mathcal{R} \)

In the introduction, we denoted by \( \mathcal{R}F \) the function on \( N' \) given by integration of a function \( F \) of \( N \) on the central subgroup \( \mathbb{R}^2(y_1, y_2) \) whenever it makes sense, for example on \( L^1(N) \). It is sometimes convenient to consider \( \mathcal{R} \) as acting between functions defined on the Lie algebras, rather than on the groups. We will do so without any further mention. The operator \( \mathcal{R} \) maps \( K \)-invariant functions on \( N \) to \( K' \)-invariant functions on \( N' \), integrable functions on \( N \) to integrable functions on \( N' \) continuously, Schwartz functions on \( N \) to Schwartz functions on \( N' \) continuously. It respects convolution on the groups and abelian convolution on the Lie algebras.

We extend the definition of \( \mathcal{R} \) to the algebra \( \mathbb{D}(N) \) of left-invariant differential operators on \( N \) in the following way: if \( D \in \mathbb{D}(N) \), then we define \( D' = \mathcal{R}D \in \mathbb{D}(N') \) by

\[
(D'G) \circ q = D(G \circ q), \quad G \in C_c^\infty(N'),
\]

where \( q = q_N : N \to N' \) is the quotient mapping. Note that if \( D = D_P \in \mathbb{D}(N), P \in \mathcal{P}(N) \), then easy changes of variables, see e.g. (A.1) below, leads to:

\[
\forall F \in C^\infty_c(N) \quad , \quad \forall G \in C^\infty_c(N') \quad \langle \mathcal{R}(DF), G \rangle = \langle RF, D_P|_{N'} \rangle G.
\]

This shows that \( \mathcal{R}D_P = D_Q \) where \( Q = P|_{N'} \) is the restriction of \( P \) to \( N' \).

The mapping \( \mathcal{R} \) on functions is dual to the restriction mapping from \( N \) to \( N' \) in the following sense. Let us denote \( \mathcal{F}_y \) and \( \mathcal{F}_t \) the Fourier transform with respect to the variables \( y \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \) respectively given by:

\[
\mathcal{F}_y F(x, \hat{y}) = \int_{\mathbb{R}^3} F(x, y) e^{-i y \cdot \hat{y}} dy,
\]

\[
\mathcal{F}_t G(x, \hat{t}) = \int_{\mathbb{R}} G(x, t) e^{-i t \cdot \hat{t}} dt;
\]

whenever it makes sense for a function \( G \) on \( N' \) and a function \( F \) on \( N \), identified with functions on \( N' \) and \( N \) respectively, we have:

\[
(4.1) \quad G = \mathcal{R}F \iff \mathcal{F}_t G = \mathcal{F}_y F|_{N'},
\]

In the following subsection, we describe the restriction mapping.
4.2. Restriction and radialisation mappings

For a function $f$ on $\mathcal{N}$, we denote by $\text{Rest} f = f|_{\mathcal{N}'}$ its restriction to $\mathcal{N}'$. We set:

$$\mathcal{N}_o = \mathbb{R}_x^3 \times (\mathbb{R}_y^3 \setminus \{0\}) \quad \text{and} \quad \mathcal{N}_o' = \mathbb{R}_x^3 \times (\mathbb{R}_t \setminus \{0\}).$$

In the next lemma, we define the radialisation mapping $\text{Radial}$:

**Lemma 4.1.** — For a function $h \in C^\infty(\mathcal{N}_o')^{K'}$ and $(x, y) \in \mathcal{N}_o$, the following:

$$\text{Radial}(h)(x, y) = h(k^{-1}x, t), \quad \text{where} \quad y = k(0, 0, t) \text{ for some } k \in K.$$

defines a $K$-invariant function $\text{Radial}(h)$, that is smooth on $\mathcal{N}_o$.

$\text{Radial}$ is an isomorphism between the topological vector spaces $C^\infty(\mathcal{N}_o)^K$ and $C^\infty(\mathcal{N}_o')^{K'}$, whose inverse is $\text{Rest}$.

**Proof.** — For a function $h \in C^\infty(\mathcal{N}_o')^{K'}$ and $(x, y) \in \mathcal{N}_o$, it is easy to see that $\text{Radial}(h)(x, y)$ is well defined and $K$-invariant.

Let us show that $\text{Radial}(h) \in C^\infty(\mathcal{N}_o)^K$. We choose a basis $(A_j)_{j=1,2,3}$ for the Lie algebra of $K$. At a point $(x_0, y_0) \in \mathcal{N}_o$ (with $y_0 = k_0(t_0, 0, 0)$, $t_0 = |y_0| \neq 0$) we choose a local coordinate system $(x, y) = (x, k(t, 0, 0)) = \sigma(x, k, t)$, where $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $k$ varies in a small two-dimensional surface in $K$ containing $k_0$ and transversal to $k_0K'$. This change of variables does not affect the derivatives in $x$, whereas

$$\partial_{y_j} = c_{j,0}(k, t)\partial_t + \sum_{j'=1,2,3} c_{j,j'}(k, t)A_{j'}.$$

By homogeneity, $c_{j,0} \in C^\infty(K_k \times \mathbb{R}_t^*)$ is homogeneous of degree 0 in $t$, and the $c_{j,j'} \in C^\infty(K_k \times \mathbb{R}_t^*)$ homogeneous of degree $(-1)$ in $t$. More generally, we can write the derivative

$$\partial^I_y = \partial^i_{y_1}\partial^j_{y_2}\partial^k_{y_3}, \quad I = (i_1, i_2, i_3) \in \mathbb{N}^3,$$

as:

$$\partial^I_y = \sum c_{I,J}(k, t)\partial^j_\xi A^j_1 A^j_2 A^j_3,$$

where the sum is over $J = (j_0, j_1, j_2, j_3) \in \mathbb{N}^4$, with $|J| = |I|$, and the $c_{I,J} \in C^\infty(K_k \times \mathbb{R}_t^*)$ are homogeneous of degree $(j_0 - |I|)$ in $t$. As the function $(x; k, t) \rightarrow h(k^{-1}x, t)$ is smooth on $\mathbb{R}^3 \times K \times \mathbb{R}$, (4.2) implies that $\text{Radial} h$ is smooth on $\mathcal{N}_o$. Furthermore $h \in C^\infty(\mathcal{N}_o') \rightarrow \text{Radial} h \in C^\infty(\mathcal{N}_o)$ is continuous. □
Lemma 4.1 implies that the mapping \( \text{Rest} \) is 1-1 on \( C^\infty(N)^K \) and on \( \mathcal{S}(N)^K \). Let us determine \( \text{Rest} (C^\infty(N)^K) \). We will need the following notation:

- For \( f \in C^\infty(N)^K \), we denote by \( P^{(f)}_M(x, t) \) the homogeneous polynomial of degree \( M \) in the Taylor expansion of \( f(x, \cdot) \) at \( y = 0 \):
  \[
P^{(f)}_M(x, y) = \sum_{|j| = M} \frac{1}{j!} \frac{\partial^j_y f(x, 0)}{j!} y^j.
  \]

- For \( g \in C^\infty(N)^{K'} \), we denote by \( Q^{(g)}_M(x, t) \) the homogeneous polynomial of degree \( M \) in the Taylor expansion of \( g(x, \cdot) \) at \( t = 0 \):
  \[
  Q^{(g)}_M(x, t) = \frac{1}{M!} \frac{\partial^M_t g(x, 0)}{M!} t^M.
  \]

We see:
  \[
  Q^{(\text{Rest} f)}_M = \text{Rest} P^{(f)}_M \quad \text{and thus} \quad \text{Radial} Q^{(\text{Rest} f)}_M = P^{(f)}_M.
  \]

Thus a function \( g \in C^\infty(N)^{K'} \) that is the restriction of some function \( f \in C^\infty(N)^K \), necessarily has the following property:

**Property (R).** For any \( M \in \mathbb{N} \), \( \text{Radial} (Q^{(g)}_M) \) extends to a smooth function on \( N \) which is a homogeneous polynomial in \( y \) of degree \( M \), with smooth coefficients in \( x \).

It turns out that this condition is also sufficient:

**Proposition 4.2.** — Let \( g \in C^\infty(N)^{K'} \).

The function \( g \) is in the image of \( \text{Rest} \) if and only if it satisfies Property (R).

In this case, \( \text{Radial} (g) \) extends to a \( K \)-invariant smooth function \( f \) on \( N \), whose restriction to \( N' \) is \( g \) and we have \( Q^{(g)}_M = \text{Rest} [P^{(f)}_M] \). Moreover if in addition \( g \in \mathcal{S}(N)^{K'} \), then \( \text{Radial} (g) \in \mathcal{S}(N)^K \).

In the proof, we adopt the ideas of the Euclidean setting [13, Theorem 2.4].

**Proof.** — Let \( g \in C^\infty(N)^{K'} \) satisfying Property (R). For each \( M \), we denote \( P_M \) the extension of \( \text{Radial} (Q^{(g)}_M) \) to a smooth function on \( N \) that is a homogeneous polynomial in \( y \) of degree \( M \), with smooth coefficients in \( x \).

Let \( M_0 \in \mathbb{N} \). The Taylor Formula gives:

\[
(4.3) \quad g(x, t) - \sum_{j=0}^{M_0} Q^{(g)}_j(x, t) = \frac{t^{M_0+1}}{M_0!} \int_0^1 (1 - w)^{M_0} \left( \partial^M_t g \right)(x, wt) dw.
\]
Let $I_o \in \mathbb{N}^3$ with $|I_o| = M_o + 1$. We have on $N_o$:

$$
\partial_y^{I_o} \text{Radial} (g) = \partial_y^{I_o} \left( \text{Radial} (g) - \sum_{j=0}^{M_o} P_j \right) = \partial_y^{I_o} \left[ \text{Radial} \left( g - \sum_{j=0}^{M_o} Q_j \right) \right].
$$

Now for any $(x, y) \in N_o$, $y \neq 0$ can be written $y = k(0, 0, t)$, $t \in \mathbb{R}^*$, $k \in K$, and by (4.2) and (4.3), $(\partial_y^{I_o} \text{Radial} (g)) (x, y)$ can be written as the sum over $J \in \mathbb{N}^4$, $|J| = M_o + 1$, of:

$$
\frac{c_{I_o,J}(k, t)}{M_o!} \int_0^1 (1 - w)^{M_o} \partial_t^{j_0} A_1^{j_1} A_2^{j_2} A_3^{j_3} \left[ t^{M_o+1} \left( \partial_t^{M_o+1} g \right) \right] (k^{-1} x, wt) \, dw.
$$

This last term remains bounded if $0 < |y| = |t| \leq 1$ because $c_{I_o,J}$ is homogeneous of degree $j_0 - (M_o + 1)$. This implies that $\partial_y^{I_o} \text{Radial} (g)$ is bounded on a compact neighborhood of $(x, 0)$ for any $x$, and any $I_o$ and $M_o$. It is easy to see that for any $I \in \mathbb{N}^3$, $\partial_x^{I} \partial_y^{I_o} \text{Radial} (g)$ satisfies the same conditions. Local boundedness of all derivatives is sufficient to imply that Radial $(g)$ has a smooth extension to $N$.

We deduce easily from (4.1) the following characterisation of $\mathcal{R}(\mathcal{S}(N)^K)$:

**Proposition 4.3.** — Let $G \in \mathcal{S}(N')^{K'}$. The function $G$ is in $\mathcal{R}(\mathcal{S}(N)^K)$ if and only if either of the following equivalent conditions is satisfied:

(i) $\mathcal{F}_I G$ (identified with a function on $N'$) satisfies Property (R);

(ii) denoting by $\mathcal{F}_{x,t}$ the Fourier transform with respect to the variables $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$ given by:

$$
\mathcal{F}_{x,t} G(\hat{x}, \hat{t}) = \int_{\mathbb{R}^3 \times \mathbb{R}} G(x, t) e^{-ix \cdot \hat{x}} e^{-it \cdot \hat{t}} \, dx \, dt,
$$

$\mathcal{F}_{x,t} G$ (identified with a function on $N'$) satisfies Property (R);

From G. Schwarz’s Theorem, it follows (compare with [13, Theorem 2.4]):

**Corollary 4.4.** — Let $G \in \mathcal{S}(N')^{K'}$. The function $G$ is in $\mathcal{R}(\mathcal{S}(N)^K)$ if and only if either of the following equivalent conditions is satisfied:

(i) for each $j \in \mathbb{N}$ there exist Schwartz functions $a_{j,i} \in \mathcal{S}(\mathbb{R})$, $i = 0, \ldots, j$ satisfying:

$$
\forall x \in \mathbb{R}^3 \int_{\mathbb{R}} G(x, t) t^j \, dt = \sum_{i=0}^{j} a_{i,j} (|x|^2)x_i^i;
$$

(ii) for each $j \in \mathbb{N}$ there exist Schwartz functions $b_{j,i} \in \mathcal{S}(\mathbb{R})$, $i = 0, \ldots, j$ satisfying:

$$
\forall \zeta \in \mathbb{R}^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}} G(x, t) t^j e^{-ix \cdot \zeta} \, dt \, dx = \sum_{i=0}^{j} b_{i,j} (|\zeta|^2) \zeta_i^i.
$$
5. Proof of Theorem 3.5

Here we give the proof of Theorem 3.5. It is based on the properties of mappings explained in Section 4 and on results already shown on the Heisenberg group [1, 2]. These two key ingredients are used in the proofs of a “Geller-type” Lemma (Subsection 5.2) and of Theorem 3.5 (Subsection 5.3).

5.1. The Gelfand pair \((K', N', K')\)

We easily check that

\[
\rho'(x, t) = (|x|^2, t^2, x_3, x_3^2),
\]
defines a Hilbert mapping of \((N', K')\), which satisfies:

\[
\rho'(x, t) = (\rho|_{N'}(x, (0, 0, t)), x_3^2) \quad \text{and} \quad D' = D_{\rho'}.
\]

From the Heisenberg case [1, 2], we deduce:

**Lemma 5.1.** — For any \(G \in S(N')^K\), there exists \(\tilde{g} \in S(R^4)\) with \(\tilde{G}'G = \tilde{g}|_{\Sigma_{D'}}\).

Furthermore the mapping \(G \in S(N')^K' \mapsto \tilde{G}'G \in S(\Sigma_{D'})\) is continuous.

Precisely, continuity of the last mapping means that

\[
\forall a \in \mathbb{N} \quad \exists C = C(a) > 0 \quad \exists a' \in \mathbb{N} \quad \forall G \in S(N')^K' \quad \exists \tilde{g} \in S(R^4) \quad \tilde{g}|_{\Sigma_{D'}} = \tilde{G}'G \quad \|\tilde{g}\|_{a, R^4} \leq C \|G\|_{a', N'}. \tag{5.1}
\]

Notice that the extension \(\tilde{g}\) depends on the Schwartz semi-norm \(\|\|_{a, R^4}\).

5.2. The Geller-type Lemma

In this subsection, we will state and prove a “Geller-type Lemma”, extending [8, 2]. For this purpose we will need the following remark.

Let \(F \in S(N)^K\). The mapping

\[
R \mapsto \mathcal{G}F(R^2, 0, 0) = \int F(x, y)e^{-iRx_1}dx\,dy,
\]
is a Schwartz even function on \(R\); by Whitney’s Theorem, there exists a Schwartz function \(f_o \in S(R)\) such that

\[
\forall R \in \mathbb{R} \quad f_o(R^2) = \mathcal{G}F(R^2, 0, 0);
\]
by Hulanicki’s Schwartz Kernel Theorem or Proposition 3.3, \( f_o(L) \) is a convolution operator with a \( K \)-invariant Schwartz kernel which we denote by \( \mathcal{G}F(L,0,0) \) (for brevity reasons, in this section we will often denote a convolution operator and its kernel by the same symbol).

**Proposition 5.2** (Geller-type Lemma). — Let \( F \in \mathcal{S}(N)^K \). There exist \( F_1 \in \mathcal{S}(N)^K \) and \( F_2 \in \mathcal{S}(N)^K \) satisfying:

\[
F = \mathcal{G}(F)(L,0,0) + \Delta F_1 + DF_2.
\]

**Proof.** — Let \( F \) be in \( \mathcal{S}(N)^K \) and \( G = RF \in \mathcal{S}(N')^{K'} \). By Lemma 5.1 there exists \( \tilde{g} \in \mathcal{S}^\prime(\mathbb{R}^4) \) with \( \tilde{g}_{|\Sigma_{\vartheta'}} = \mathcal{G}'G \). By Proposition 3.3, the operator given by:

\[
\int_{w=0}^1 \partial_2 \tilde{g}(L,w\Delta,0,0)\,dw,
\]

is a convolution operator with a \( K \)-invariant Schwartz kernel which we denote by \( F_1 \in \mathcal{S}(N)^K \). By spectral calculus, we have:

\[
\Delta F_1 = \tilde{g}(L,\Delta,0,0) - \tilde{g}(L,0,0,0).
\]

We will have finished the proof of Proposition 5.2 once we have shown:

\[
\exists F_2 \in \mathcal{S}(N)^K \quad F - \mathcal{G}F(L,0,0) - \Delta F_1 = DF_2
\]

We denote by \( H \in \mathcal{S}(N)^K \) and \( I \in \mathcal{S}(N')^{K'} \) the functions given by:

\[
H = F - \mathcal{G}F(L,0,0) - \Delta F_1 = F - \tilde{g}(L,\Delta,0,0),
\]

\[
I = RF = G - \tilde{g}(L',\Delta',0,0).
\]

The Gelfand transform of \( I \) is given by:

\[
\mathcal{G}'I(\mu_{\varphi'}) = \mathcal{G}'G(\mu_{\varphi'}) - \tilde{g}(L'\varphi'(0),\Delta'\varphi'(0),0,0).
\]

On the singular part of the spectrum, (5.3) yields to:

\[
\mathcal{G}'I\left(\mu_{\varphi',\varphi},r\right) = \tilde{g}(\zeta^2 + r^2,0,0,r^2) - \tilde{g}(\zeta^2 + r^2,0,0,0) = 0,
\]

because \( \tilde{g}(\zeta^2 + r^2,0,0,r^2) = \tilde{g}(\zeta^2 + r^2,0,0,0) \) as \( \mathcal{G}'G = \mathcal{G}F \circ \Pi \); this implies:

\[
\forall x \in \mathbb{R}^3 \int_{\mathbb{R}} I(x,t)\,dt = 0.
\]

On the regular part of the spectrum, (5.3) yields to:

\[
\mathcal{G}'I\left(\mu_{\varphi',\lambda,t},r\right) = \tilde{g}(\lambda(2l+1) + r^2,\lambda^2,\lambda r,r^2) - \tilde{g}(\lambda(2l+1) + r^2,\lambda^2,0,0),
\]

and in particular for \( r = 0 \):

\[
\mathcal{G}'I\left(\mu_{\varphi',\lambda,t},0\right) = \tilde{g}(\lambda(2l+1),\lambda^2,0,0) - \tilde{g}(\lambda(2l+1),\lambda^2,0,0) = 0;
\]

\[
\mathcal{G}'I\left(\mu_{\varphi',\lambda,t},0\right) = \tilde{g}(\lambda(2l+1),\lambda^2,0,0) - \tilde{g}(\lambda(2l+1),\lambda^2,0,0) = 0;
\]
this implies for all $\lambda > 0$:
\[
\forall l \in \mathbb{N} \quad \int_{\mathbb{N}^l} I(x, t) e^{-i\lambda t} \mathcal{L}_l \left( \frac{\lambda}{2} |\tilde{x}|^2 \right) \, dx dt = 0,
\]
and $\{\mathcal{L}_l\}_{l \in \mathbb{N}}$ being an orthogonal basis of $L^2(\mathbb{R}^+)$,
\[
\forall \tilde{x} \in \mathbb{R}^2 \quad \int_{\mathbb{R}^2} I(\tilde{x}, x_3; t) e^{-i\lambda t} \, dx_3 dt = 0.
\]
Eventually, we get:
\[(5.5) \quad \forall \tilde{x} \in \mathbb{R}^2, \quad \forall t \in \mathbb{R} \quad \int_{\mathbb{R}} I(\tilde{x}, x_3; t) \, dx_3 = 0.
\]
Let us set:
\[
G_2(x, t) = \int_{-\infty}^{x_3} \int_{-\infty}^{t} I(\tilde{x}, w; s) \, ds dw.
\]
Because of $(5.4)$ and $(5.5)$, we see that $G_2 \in \mathcal{S}(\mathbb{N}^{'K})$. Let us show that $F_{x,t} G_2$ (identified with a function on $\mathcal{N'}$) satisfies Property (R). We have for $\hat{t} \neq 0$ and $\hat{x}_3 \neq 0$:
\[
F_{x,t} G_2(\hat{x}, \hat{t}) = (\hat{x}_3 \hat{t})^{-1} F_{x,t} I(\hat{x}; \hat{t})
\]
and
\[
Q_{M-1}^{(F_{x,t} G_2)}(\hat{x}; \hat{t}) = (\hat{x}_3 \hat{t})^{-1} Q_{M}^{(F_{x,t} I)}(\hat{x}; \hat{t}).
\]
By Proposition 4.3(ii), as $I = RH$, $F_{x,t} I$ (identified with a function on $\mathcal{N'}$) satisfies Property (R), that is Radial $\left( Q_{M}^{(F_{x,t} I)} \right)$ extends to a smooth $K$-invariant function on $\mathcal{N}$ which is a homogeneous polynomial in $y$ of degree $M$, with Schwartz coefficients in $x$. By G. Schwarz’s Theorem, there exists a function $\tilde{Q}_M \in C^\infty(\mathbb{R}^3)$ of the form:
\[
\tilde{Q}_M(r_1, r_2, r_3) = \sum_{2j_1 + j_2 = M} \tilde{c}_j(r_1) r_2^{j_1} r_3^{j_2}, \quad \tilde{c}_j \in \mathcal{S}(\mathbb{R}),
\]
satisfying:
\[
\text{Radial} \left( Q_{M}^{(F_{x,t} I)} \right) = \tilde{Q}_M \circ \rho.
\]
That is:
\[
Q_{M}^{(F_{x,t} I)}(\hat{x}, \hat{t}) = \tilde{Q}_M(|\hat{x}|^2, \hat{t}^2, \hat{x}_3 \hat{t}) = \sum_{2j_1 + j_2 = M} \tilde{c}_j(|\hat{x}|^2) \hat{t}^{2j_1} (\hat{x}_3 \hat{t})^{j_2}.
\]
Because of $(5.5)$, we have:
\[
\forall \tilde{x} \in \mathbb{R}^2, \forall \hat{t} \in \mathbb{R} \quad F_{x,t} I(\hat{x}, 0; \hat{t}) = 0,
\]
thus the term $\tilde{c}_j(|\tilde{x}|^2)$ with $j = (j_1, 0)$ is zero; we can factor out one $(\hat{x}_3 \hat{t})$. This implies that for $M > 0$, Radial $\left( Q_{M-1}^{(F_{x,t} G_2)}(\hat{x}; \hat{t}) \right)$ extends to a
smooth function on \( \mathcal{N} \) which is a homogeneous polynomial in \( y \) of degree \( (M - 1) \), with smooth coefficients in \( x \). Thus \( F_{x,t}G_2 \) satisfies Property (R). By Proposition 4.3(ii), there exists \( F_2 \in \mathcal{S}(N)^K \) such that \( \mathcal{R}F_2 = G_2 \). As \( D'G_2 = I = \mathcal{R}H \) and \( \mathcal{R} \) being 1-1 on \( K \)-invariant functions, we obtain \( DF_2 = H \). This proves (5.2).

Applying recursively Proposition 5.2, we obtain \( F \) as a sum of functions of the form \( (G \left[ \Delta^{j_1} D^{j_2} F_j \right]) (L, 0, 0) \) with a rest. As the degrees of homogeneity of the operators \( D \) and \( \Delta \) with respect to the variable \( y \) are three and four respectively, we will be interested in a sum over \( 2j_1 + j_2 \leq M \):

**Corollary 5.3.** — Let \( F \in \mathcal{S}(N)^K \). There exists a family \( (F_j)_{j \in \mathbb{N}^2} \) of Schwartz functions \( F_j \in \mathcal{S}(N)^K \) satisfying for any \( M \in \mathbb{N} \):

\[
F - \sum_{2j_1 + j_2 \leq M} (G \left[ \Delta^{j_1} D^{j_2} F_j \right]) (L, 0, 0) = \sum_{2j_1 + j_2 = M+1} \Delta^{j_1} D^{j_2} F_j.
\]

**5.3. End of the proof**

Here we complete the proof of Theorem 3.5.

**Existence of the extension.** Let \( F \in \mathcal{S}(N)^K, G = \mathcal{R}F, f = GF, g = G'G, F_j, \) the associated functions in Corollary 5.3, and \( f_j = GF_j, j \in \mathbb{N}^2 \). By Lemma 5.1 we choose \( \tilde{g}, \tilde{g}_j \in \mathcal{S}(\mathbb{R}^4) \) Schwartz extensions of \( g \) and \( G'(\mathcal{R}F_j), j \in \mathbb{N}^2 \), respectively. We set \( \tilde{g}_{\rho'} = \tilde{g} \circ \rho' \).

Let us fix \( M \). For \( \xi = (\xi_1, \xi_3) \in \mathbb{R}^3 \), setting \( r = \xi_3 \) and \( \lambda_l = |\xi|^2/(2l + 1) \), we have \( \rho'(\xi, \lambda_l) \in \Sigma_{D'} \) and:

\[
\tilde{g}_{\rho'}(\xi, \lambda_l) = \tilde{g} \circ \rho'(\xi, \lambda_l) = g(\lambda_l(2l + 1) + r^2, \lambda_l^2, \lambda_l r, r^2) = f(\lambda_l(2l + 1) + r^2, \lambda_l^2, \lambda_l r)
\]

\[
= \sum_{2j_1 + j_2 \leq M} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l + 1) + r^2, 0, 0) + \sum_{2j_1 + j_2 = M+1} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l + 1) + r^2, \lambda_l^2, \lambda_l r).
\]

Thus:

\[
|\tilde{g}_{\rho'}(\xi, \lambda_l) - \sum_{2j_1 + j_2 \leq M} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l + 1) + r^2, 0, 0)|
\]

\[
\leq \left( \sum_{2j_1 + j_2 = M+1} \|\tilde{g}_j\|_{M+1, \mathbb{R}^4} \right) \lambda_l^{|M+1|}. 
\]
This characterises the Taylor expansion of \( \tilde{g}_{\rho'}(\xi,.) \): for \( \xi = (\tilde{\xi}, \xi_3) \) first with \( \tilde{\xi} \neq 0 \), and then for all \( \xi_i \), we have:

\[
Q^M_\tilde{g}_{\rho'}(\xi, t) = \sum_{2j_1 + j_2 = M} t^{2j_1} (t \xi_3)^{j_2} f_j(|\xi|^2, 0, 0)
\]

This shows that \( \tilde{g}_{\rho'} \) satisfies Property (R). By Proposition 4.2, there exists \( f_1 \in \mathcal{S}(\mathcal{N})^K \) such that \( \text{Rest} f_1 = \tilde{g}_{\rho'} \) and \( f_1 = \text{Radial} \tilde{g}_{\rho'} \). By G. Schwarz’s Theorem (see also [2, Theorem 6.1]), there exists \( \tilde{f} \in \mathcal{S}(\mathbb{R}^3) \) such that \( f_1 = \tilde{f} \circ \rho \). We have:

\[
\text{Rest} f_1 = \tilde{g}_{\rho'} = \tilde{f} \circ \rho_{\mathcal{N}'} = g \circ \rho'.
\]

For any point \( s = (\lambda(2l + 1) + r^2, \lambda^2, \lambda r) \in \Sigma_\rho \), the point \( s' = (s, r^2) \in \Sigma_{\rho'} \) is in \( \text{im} \rho' \); it follows that \( s \) is in \( \text{im} \rho \) and \( \tilde{f}(s) = \tilde{g}(s') = g(s') = f(s) \). Thus \( \tilde{f} \) is an extension of \( f \).

**Continuity.** Now that we have shown that the Gelfand transform of a function \( F \in \mathcal{S}(\mathcal{N})^K \) admits a Schwartz extension, we still have to prove the continuity of \( F \in \mathcal{S}(\mathcal{N})^K \mapsto GF \in \mathcal{S}(\Sigma_\rho) \). We will use the following two lemmas. The first one states the improvement due to Mather [15] of G. Schwarz’s Theorem as well as some straightforward consequences:

**Lemma 5.4.** Let \( (\rho_1, \ldots, \rho_q) \) be a minimal and homogeneous Hilbert basis for \( (\mathbb{R}^p, K) \), and \( \rho \) the corresponding Hilbert mapping.

The induced application \( \rho^* : \hat{h} \mapsto \hat{h} \circ \rho \) on \( \mathcal{S}(\mathbb{R}^q) \) is split-surjective, i.e. it admits a linear continuous right inverse \( \sigma : \mathcal{S}(\mathbb{R}^p)^K \rightarrow \mathcal{S}(\mathbb{R}^q) \) for \( \rho^* \), that is \( \rho^* \circ \sigma \) is the identity mapping of \( \mathcal{S}(\mathbb{R}^p)^K \).

We fix such \( \sigma \). For any \( h \in \mathcal{S}(\text{im} \rho) \), the function \( h \circ \rho \) is well defined and in \( \mathcal{S}(\mathbb{R}^p)^K \), the function \( \hat{h} = \sigma(h \circ \rho) \in \mathcal{S}(\mathbb{R}^q) \) defines a Schwartz extension which we will call the Mather extension of \( h \in \mathcal{S}(\text{im} \rho) \). We have:

\[
\hat{h} \circ \rho = h \circ \rho.
\]

The linear mapping \( \hat{h} \mapsto \hat{h} \) of \( \mathcal{S}(\mathbb{R}^q) \) is continuous.

It is easy to check that the Hilbert mapping, \( \rho \), of \( (\mathcal{N}, K) \) is minimal and homogeneous.

The second lemma follows from Rest being a 1-1 continuous mapping, from the Closed Graph Theorem and Lemma 5.4:

**Lemma 5.5.** To any \( \tilde{g} \in \mathcal{S}(\mathbb{R}^4) \) such that there exists \( \tilde{f} \in \mathcal{S}(\mathbb{R}^3) \) satisfying \( \tilde{f} \circ \rho_{\mathcal{N}'} = \tilde{g} \circ \rho' \), we associate the Mather extension \( \tilde{f}_1 \) of \( \tilde{f} \). The mapping \( \tilde{g} \mapsto \tilde{f}_1 \) is well-defined, continuous and linear:

\[
\forall a \in \mathbb{N} \quad \exists C > 0 \quad \exists a' \in \mathbb{N} \quad \| \tilde{f}_1 \|_{a, \mathbb{R}^3} \leq C \| \tilde{g} \|_{a', \mathbb{R}^4}
\]
Let $a_0 \in \mathbb{N}$. Let $a_1$ corresponding to $a'$ in (5.6) for $a = a_o$.

Let $F \in \mathcal{S}(N)^K, G = \mathcal{R}F, f = \mathcal{G}F, g = \mathcal{G}'G$. By (5.1), there exists $a_2 \in \mathbb{N}$ such that we have indepently of $G$:

$$\|\tilde{g}\|_{a_1, R^4} \leq C \|G\|_{a_2, N}.$$

As $\mathcal{R}$ is continuous, there exists $a_3 \in \mathbb{N}$ such that we have indepently of $F$:

$$\|\mathcal{R}F\|_{a_2, N} \leq C \|F\|_{a_3, N}.$$

Thus we have:

$$\|\tilde{f}_1\|_{a_o, R^3} \leq C_1 \|	ilde{g}_o\|_{a_1, R^4} \leq C_2 \|G\|_{a_2, N'} \leq C_3 \|F\|_{a_3, N}.$$

Notice that $a_3$ and $C_3$ depend only on $a_o$, and that $\tilde{f}_1$ depends on $F$ and also on $a_o$ because $\tilde{g}_o$ depends on $a_1$.

### Appendix A.

We adopt again the notation of Section 3 and assume that $(K \ltimes N, K)$ is a Gelfand pair. Here we give the proofs of Propositions 3.1, 3.2 and 3.3.

#### A.1. Proof of Proposition 3.1

This proof is an easy generalisation of [2, Lemma 5.3] which is a similar result given in the case of the Heisenberg group, using [3].

Let us check that the operators of $\mathbb{D}(N)$ are symmetric. In fact, $C^\infty_c(N)$ is equipped with the Hilbert inner product $\langle F_1, F_2 \rangle = \int_N F_1(n) \bar{F}_2(n)dn$. For any $D = D_P \in \mathbb{D}(N), P \in \mathcal{P}(N)$, we have $\langle DF_1, F_2 \rangle = \langle F_1, DF_2 \rangle$ because:

$$\langle DF_1, F_2 \rangle = \left[ P(i^{-1} \partial_u) \int_N F_1(n \exp(\sum_{j=1}^p u_j E_j)) \bar{F}_2(n)dn \right]_{u=0}$$

$$\langle DF_1, F_2 \rangle = \left[ P(i^{-1} \partial_u) \int_N F_1(n_1) \bar{F}_2(n_1 \exp(-\sum_{j=1}^p u_j E_j))dn \right]_{u=0} \quad \text{(A.1)}$$

after the change of variable $n_1 = n \exp(\sum_{j=1}^p u_j E_j)$.

Let us recall some facts about Gelfand pairs of the form $(K \ltimes N, K)$ [3]. Let $\hat{N}$ be the set of (the classes of) unitary representations on $N$. For each $\pi \in \hat{N}$, let $K_\pi$ be the stabilizer of $\pi$ in $K$. There exists a decomposition of the Hilbert space $\mathcal{H}_\pi$ into finite-dimensional irreducible subspaces
$\mathcal{H}_{\pi,\alpha}$ under the projective action of $K\pi$ on $\mathcal{H}_\pi$. Each bounded $K$-spherical function $\phi$ on $N$ is in 1-1 correspondence with $\pi$ and $\alpha$, in the sense that $\phi = \phi_{\pi, \alpha}$ can be written as:

$$\phi(n) = \int_K \langle \pi(kn)u, u \rangle dk,$$

where $u$ is any unit vector in $\mathcal{H}_{\pi,\alpha}$ (and $dk$ the Haar probability measure of $K$). Let $D \in \mathbb{D}(N)^K$. For each $\pi \in \hat{N}$, each subspace $\mathcal{H}_{\pi,\alpha}$ is an eigenspace for the operator $d\pi(D)$ and its eigenvalue is $\mu_{\pi,\alpha,D}$ satisfies:

$$D\phi_{\pi,\alpha} = \mu_{\pi,\alpha,D}\phi_{\pi,\alpha}.$$

Note that the trace $tr\mathcal{H}_{\pi}$ of operators on $\mathcal{H}_\pi$ can be computed as the sum over $\alpha$ of traces $tr\mathcal{H}_{\pi,\alpha}$ of operators on $\mathcal{H}_{\pi,\alpha}$.

We denote by $\beta$ the Plancherel measure on $\hat{N}$:

$$\|F\|_2^2 = \int_{\hat{N}} tr\mathcal{H}_{\pi} \left[ \pi(F)\pi(F)^* \right] d\beta(\pi), \quad F \in C_c^\infty(N).$$

Now let us prove Proposition 3.1.

Let $D \in \mathbb{D}(N)^K$. It is easy to see that there exists a unique self-adjoint extension of $D$, whose domain is the space of function $F \in L^2(N)$ satisfying:

$$\int_{\hat{N}} \sum_{\alpha} |\mu_{\pi,\alpha,D}|^2 tr\mathcal{H}_{\pi,\alpha} \left[ \pi(F)\pi(F)^* \right] d\beta(\pi) < \infty.$$

Let us also denote by $\mathcal{D}$ the self-adjoint extension. Following [2, Lemma 5.3], we construct a realisation $E = E_D$ of the spectral resolution of $D$ in the following way. Given $\omega$ a Borel subset of $\mathbb{R}$, we define the operator $E(\omega)$ on $L^2(N)$ by:

$$\pi(E(\omega)F) = \sum_{\alpha} \chi_\omega(\mu_{\pi,\alpha,D})\pi(F)\Pi_{\pi,\alpha},$$

where $\chi_\omega$ is the characteristic function of $\omega$ and $\Pi_{\pi,\alpha}$ the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_{\pi,\alpha}$. Then $E = \{E(\omega)\}$ defines a resolution of the identity, and for $F \in \mathcal{S}(N)$,

$$\int_\mathbb{R} \xi dE(\xi)F = DF.$$

Therefore $E = E_D$ is the spectral resolution of $D$.

One readily checks that if $D_1, D_2 \in \mathbb{D}(N)^K$, then for any Borel sets $\omega_1$, $\omega_2$, the operators $E_{D_1}(\omega_1)$ and $E_{D_2}(\omega_2)$ commute.

### A.2. Proof of Proposition 3.2

Let us recall the definition of the joint spectrum of a given strongly commuting family of self-adjoint operators $T_1, \ldots, T_q$ (densely defined) on
a Hilbert space $\mathcal{H}$: it is the set $S_{T_1,\ldots,T_q}$ of the $q$-tuples $\mu = (\mu_1,\ldots,\mu_q) \in \mathbb{R}^q$ for which there do not exist bounded operators $U_1,\ldots,U_q$ on $\mathcal{H}$ satisfying:

$$\sum_{j=1}^{q}(\mu_j - T_j)U_j = \sum_{j=1}^{q}U_j(\mu_j - T_j) = \text{Id}_\mathcal{H}.$$ 

Let $(\rho_1,\ldots,\rho_q)$ be an ordered Hilbert basis for $(\mathcal{N},K)$ and $\mathcal{D}_\rho$ the associated family of strongly commuting self-adjoint operators on $L^2(N)$.

For each $\pi \in \hat{N}$, we decompose its Hilbert space $\mathcal{H}_\pi = \bigoplus_\alpha \mathcal{H}_\pi,\alpha$ as in the proof of Proposition 3.1 in Section A.1 and we have for $j = 1,\ldots,q$:

$$d\pi(D_{\rho_j})|_{\mathcal{H}_\pi,\alpha} = \mu_j(\phi_{\pi,\alpha})\text{Id}_{\mathcal{H}_\pi,\alpha}.$$ 

This implies the inclusion $\Sigma_{\mathcal{D}_\rho} \subset S_{\mathcal{D}_\rho}$.

For the converse inclusion, we will need the following Lemma, an easy consequence of the Plancherel formula:

**Lemma A.1.** — If a function $m$ is continuous and compactly-supported on the Gelfand spectrum, then there exists a $K$-invariant function $M \in L^2(N)^K$ whose Gelfand transform is $m$. Furthermore, the convolution operator with kernel $M$ defined on $C_c^\infty(N)$ extends to a bounded operator on $L^2(N)$ with operator norm $\sup |m|$.

We will also use a dyadic decomposition on $\mathbb{R}^+$: there exists a smooth, non-negative function $\psi$, supported in the interval $[\frac{1}{2},2]$ and satisfying:

$$\forall x > 0 \sum_{a \in \mathbb{Z}} \psi(2^{-a}x) = 1.$$ 

We set $\psi_a(x) = \psi(2^{-a}x)$ if $a \geq 1$, and $\psi_0(x) = \sum_{a<1} \psi(2^{-a}x)$.

Let $\mu^o = (\mu_1^o,\ldots,\mu_q^o) \in \mathbb{R}^q \setminus \Sigma_{\mathcal{D}_\rho}$. We define the functions $m_{a,j}$, $a \geq 0$, $j = 1,\ldots,q$ by:

$$m_{a,j}(\mu) = \frac{\mu_j^o - \mu_j}{\sum_{j'=1}^{q}\mu_{j'}^o - \mu_{j'}^2} \psi_a(|\mu|), \quad \mu \in \Sigma_{\mathcal{D}_\rho}.$$ 

Each function $m_{a,j}$ is continuous and compactly supported in $\Sigma_{\mathcal{D}_\rho}$ and because $\mu^o$ is not in the closed set $\Sigma_{\mathcal{D}_\rho}$, there exists a constant $C = C(\mu^o) > 0$, independent of $a$ and $j$, such that:

$$\sup_{\mu \in \Sigma_{\mathcal{D}_\rho}} |m_{a,j}(\mu)| \leq C2^{-a}.$$ 

We denote by $U_{a,j}$ the convolution operator whose kernel admits $m_{a,j}$ as Gelfand transform; by Lemma A.1, this operator is bounded on $L^2(N)$ with norm less than $C2^{-a}$. The operator $\sum_{a \geq 0} U_{a,j}$ is thus also a bounded
operator on $L^2(N)$, which we denote by $U_j$. We check that for any representation $\pi \in \hat{N}$, we have on each subspace $\mathcal{H}_{\pi, \alpha}$:

$$\pi(U_j)_{|\mathcal{H}_{\pi, \alpha}} = \frac{\mu^\alpha_j - \mu_j(\phi_{\pi, \alpha})}{\sum_{j'=1}^q \mu^\alpha_{j'} - \mu_{j'}(\phi_{\pi, \alpha})^2},$$

from which we deduce:

$$\sum_{j=1}^q \pi(U_j) (\mu^\alpha_j - \pi(D_j)) = \sum_{j=1}^q (\mu^\alpha_j - \pi(D_j)) \pi(U_j) = \text{Id}_{\mathcal{H}}.$$ 

This implies:

$$\sum_{j=1}^q U_j (\mu^\alpha_j - D_j) = \sum_{j=1}^q (\mu^\alpha_j - D_j) U_j = \text{Id}_{L^2(N)},$$

that is, $\mu^\alpha$ is not in the joint spectrum $S_{\mathcal{D}^\rho}$.

This shows the inclusion $\Sigma_{\mathcal{D}^\rho} \supset S_{\mathcal{D}^\rho}$ and concludes the proof of Proposition 3.2.

**A.3. Proof of Proposition 3.3**

With the Plancherel formula (see proof of Proposition 3.1 in Subsection A.1), it is easy to see that if $m \in \mathcal{S}(\mathbb{R}^d)$ and if $m(\mathcal{D}^\rho)$ is a convolution operator whose kernel is $M \in \mathcal{S}(N)^K$, then the Gelfand transform of $M$ coincides with $m$ on $\Sigma_{\mathcal{D}^\rho}$.

The proof of the rest of Proposition 3.3 relies mainly on the generalisation [2, Theorem 5.2] of Hulanicki’s Schwartz Kernel Theorem.

We will also use the following Lemma which is well-known to specialists (see [11], where the estimate given below in (A.2) is established for general Rockland operators):

**Lemma A.2.** — Let $N$ be a graded Lie group, $\mathcal{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \ldots \mathcal{V}_l$ its graded Lie algebra, $(X_i)$ a basis of $\mathcal{V}_1$, $L = -\sum X_i^2$ the associated sub-Laplacian.

For any homogeneous left-invariant differential operator $D$ on $N$ of degree $2d$, there exists a constant $C = C(D) > 0$ such that we have:

$$(A.2) \quad \forall F \in C_c^\infty(N) \quad \|DF\|_2 \leq C \|L^dF\|_2.$$ 

Furthermore $\tilde{D} = 2CL^d - D$ is a positive Rockland operator on $N$.

For the sake of completeness, we give a proof of this Lemma.
Proof. — We refer to [7, ch.6.A] for the definition and the properties of kernels of type \( \alpha \in [0, Q] \), where \( Q \) is the homogeneous dimension of the group. We will also use the fact that the sub-Laplacian, \( L \), has a fundamental solution, \( -L \) being a homogeneous positive Rockland operator of order two - and that the same is true for \( L^d, d = 1, 2, \ldots \). By [6], for \( 2d < Q \), there exists a fundamental solution \( G_d \) of \( L^d \) such that \( G_d \in C^\infty(N\setminus\{0\}) \) is homogeneous of degree \( 2d - Q \).

For any composition of left-invariant vector fields \( X^I = X_{i_1}X_{i_2} \cdots X_{i_k} \) with \( k < 2d \), it is easy to check that \( X^I G_d \in C^\infty(N\setminus\{0\}) \) is a homogeneous function \( H \) of degree \( -Q + 1 \), smooth away from the origin. One further differentiation gives a homogeneous distribution of degree \( -Q \). Being a derivative, it automatically satisfies the cancellation condition (63) of [17, ch.XIII.5.3]. In fact, let \( \phi \) be a function supported on the unit ball and normalized in the \( C^1 \)-norm. For any \( X \in V_1 \) and \( r > 0 \),

\[
\langle XH, \phi(r \cdot) \rangle = -r \int_N H(x)X\phi(rx) \, dx = -\int_N H(x)X\phi(x) \, dx,
\]

which is bounded independently of \( \phi \) and \( r \).

This implies that for every \( I \) of length \( 2d \), the kernel \( X^I G_d \) satisfies the \( L^2 \)-boundedness condition (6.3) of [7, ch.6.A], and thus is of type 0. The operator \( X^I L^{-d} \) being \( L^2(N) \)-bounded, we have:

\[
(A.3) \quad \forall F \in C^\infty_c(N) \quad \|X^I F\|_2 \leq C \|L^d F\|_2.
\]

If \( 2d \geq Q \), \( L^d \) does not have a homogeneous fundamental solution, but, according to [9], it has a fundamental solution \( G_d \) which is the sum of two terms, one homogeneous of degree \( 2d - Q \), and the other of the form \( P(x) \log |x| \), where \( P \) is a polynomial, homogeneous of degree \( 2d - Q \), and \( |x| \) is any smooth homogeneous norm on \( N \). This implies that, if the length \( k \) of \( I \) satisfies \( 2d - Q < k < 2d \), then \( X^I G_d \) is a homogeneous function of degree \( -Q + 2d - k \). We can then repeat the previous argument to conclude that (A.3) holds for every \( d \).

Let \( D \) be a homogeneous left-invariant differential operator on \( N \) of degree \( 2d \). As \( D \) can be written as a linear combination of monomials \( X^I \), with \( I \) of degree \( 2d \), we see that the property (A.3) implies (A.2). Let \( C = C(D) \) be the \( L^2 \)-operator norm of \( DL^{-d} \). In particular the \( L^2(N) \)-norm of the operator \( D(\mathcal{C}L^d)^{-1} \) is one and \( I - \frac{1}{2}D(\mathcal{C}L^d)^{-1} \) is an invertible operator on \( L^2(N) \). The differential operator \( \tilde{D} = 2CL^d - D \) is a \( 2d \)-homogeneous,
left-invariant, symmetric and positive on $C_c^\infty(N)$. To finish the proof, it remains to prove the defining property of Rockland operators, that is, for any non-trivial, irreducible, unitary representation $\pi$ of $N$, $\pi(\tilde{D})$ is injective on smooth vectors; this is true because we can write:

$$\pi(\tilde{D}) = \pi(2CL^d - D) = 2C\pi\left(I - \frac{1}{2}D(CL^d)^{-1}\right)\pi(L)^d,$$

and $I - \frac{1}{2}D(CL^d)^{-1}$ is invertible and $L$ a Rockland operator. \hfill □

Before proving Proposition 3.3, let us define some notation. We equip the two-step nilpotent Lie algebra $N$ with an Euclidean product such that $K$ acts orthogonally. $K$ stabilises the centre $Z$ of $N$, and its orthogonal complement $V = Z^\perp$. The decomposition $N = V \oplus Z$ endows $N$ with a structure of graded Lie group. $Q = \dim V + 2\dim Z$ is the homogeneous dimension of the group. For the symmetrisation mapping, we assume that the basis $(E_i)_{1 \leq i \leq p}$ is given as a basis $(E_i)_{1 \leq i \leq p_1}$ of $Z$. As the action of $K$ on $P(N)$ respects the degree-graduation in both the $Z$ and $V$-variables, there exist bi-homogeneous Hilbert basis $\{\rho_1, \ldots, \rho_q\}$ in the sense that each polynomial $\rho_j$ is homogeneous in the $Z$-variables and in the $V$-variables. For a bi-homogeneous Hilbert basis $\{\rho_1, \ldots, \rho_q\}$, we denote by $d_j^{(1)}$ the degree of homogeneity of $\rho_j$ in the $V$-variables, and by $d_j^{(2)}$ the degree of homogeneity of $\rho_j$ in the $Z$-variables; $d_j = d_j^{(1)} + 2d_j^{(2)}$ is the degree of homogeneity of the operator $D_{\rho_j}$ for the structure of graded Lie group of $N$.

Let us start the proof of Proposition 3.3. We notice that it suffices to show the result for one Hilbert mapping because of the existence of a polynomial mapping between two Hilbert mappings. We choose a bi-homogeneous ordered Hilbert basis $\rho = (\rho_1, \ldots, \rho_q)$ with the two following properties. First $\rho_1(\sum_{j=1}^p u_j E_j) = \sum_{j=1}^{p_1} |u_j|^2$. Second, the polynomials $\rho_1, \ldots, \rho_{q_1}$ are of even degree of homogeneity in the $V$-variables and the polynomials $\rho_{q_1+1}, \ldots, \rho_q$ are of odd degree of homogeneity in the $V$-variables.

Let $m$ be in $S(\mathbb{R}^q)$. $S$ denotes the set of all the sequences $\epsilon : \{q_1 + 1, \ldots, q\} \to \{0, 1\}$. Using Whitney’s Theorem or G. Schwarz’s Theorem, there exists a family of Schwartz functions $(\tilde{m}_\epsilon)_{\epsilon \in S}$, $\tilde{m}_\epsilon \in S(\mathbb{R}^q)$ satisfying for all $(r_1, \ldots, r_q) \in \mathbb{R}^q$:

$$m(r_1, \ldots, r_q) = \sum_{\epsilon \in S} r^\epsilon \tilde{m}_\epsilon(r_1, \ldots, r_{q_1}, r_{q_1+1}^2, \ldots, r_q^2),$$

where we use the notation $r^\epsilon = r_1^{\epsilon(q_1+1)} \ldots r_q^{\epsilon(q)}$. 

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The operator $\tilde{D}_1 = D_{\rho_1}$ is the sub-Laplacian of $N$ which is a positive Rockland operator. By Lemma A.2 there exist constants $c_j$, $j = 2, \ldots, q$ such that

- for $j = 2, \ldots, q_1$, the operator $\tilde{D}_j = -D_{\rho_j} + c_j D_{\rho_1}^{d_j}$ is a positive Rockland operator on $N$,
- for $j = q_1 + 1, \ldots, q$, the operator $\tilde{D}_j = -D_{\rho_j}^2 + c_j D_{\rho_1}^{d_j}$ is a positive Rockland operator on $N$.

For $r = (r_1, \ldots, r_q) \in \mathbb{R}^q$, we set $[A(r)]_1 = r_1$ and:

\[
[A(r)]_j = -r_j + c_j r_1^{d_j/2}, \quad j = 2, \ldots, q_1 \\
[A(r)]_j = -r_j + c_j r_1^{d_j}, \quad j = q_1 + 1, \ldots, q.
\]

This defines an application $A : \mathbb{R}^q \to \mathbb{R}^q$ which is a $C^\infty$-diffeomorphism of $\mathbb{R}^q$ and whose Jacobian equals $(-1)^{q-1}$ at any point. Thus if $h$ is in $S(\mathbb{R}^q)$ then $h \circ A^{-1}$ is in $S(\mathbb{R}^q)$.

We have:

\[
(A.4) \quad m(D_{\rho_1}, \ldots, D_{\rho_q}) = \sum_{\epsilon \in S} D_{\rho'}^\epsilon \tilde{m}_\epsilon(D_{\rho_1}, \ldots, D_{\rho_{q_1}^1}, D_{\rho_{q_1}^2}, \ldots, D_{\rho_q})
\]

(using the notation $D_{\rho'}^\epsilon = D_{\rho_{q_1}^1}^{\epsilon(1)} \ldots D_{\rho_q}^{\epsilon(q)}$) and:

\[
(A.5) \quad \tilde{m}_\epsilon(D_{\rho_1}, \ldots, D_{\rho_{q_1}^1}, D_{\rho_{q_1}^2}, \ldots, D_{\rho_q}) = \tilde{m}_\epsilon \circ A^{-1}(\tilde{D}_1, \ldots, \tilde{D}_q)
\]

Each operator given by (A.5) is a Schwartz multiplier $\tilde{m}_\epsilon \circ A^{-1} \in S(\mathbb{R}^q)$ of a strongly commutative family of positive Rockland operators $\tilde{D}_j$, $j = 1, \ldots, q$. By [2, Theorem 5.2], it is a convolution operator with a Schwartz kernel $M_{\tilde{m}_\epsilon \circ A^{-1}}(\tilde{D}_j)$. Because of the expression (A.4), we deduce that the operator $m(D_{\rho_1}, \ldots, D_{\rho_q})$ is also a convolution operator with a Schwartz kernel $M_{m, D_{\rho}} = \sum_{\epsilon \in S} D_{\rho}^\epsilon M_{\tilde{m}_\epsilon \circ A^{-1}}(\tilde{D}_j)$.

The continuity of $m \in S(\mathbb{R}^q) \mapsto M_{m, D_{\rho}} \in S(N)^K$ is a direct consequence of the following facts:

- by Schwarz-Mather’s Theorem, the mappings $m \in S(\mathbb{R}^q) \mapsto m_\epsilon \in S(\mathbb{R}^{q_1} \times [0, \infty[^{q-q_1})$, $\epsilon \in S$, are continuous,
- the application $A$ being a $C^\infty$-diffeomorphism of $\mathbb{R}^q$ with $(-1)^{q-1}$ as jacobian, the mapping $A^{-1} : h \in S(\mathbb{R}^q) \mapsto h \circ A^{-1} \in S(\mathbb{R}^q)$ is continuous,
- by [2, Theorem 5.2], the application that maps $m \in S(\mathbb{R}^q)$ to the kernel $M_{m_\epsilon}(\tilde{D}_j)$ of the operator $m(\tilde{D}_1, \ldots, \tilde{D}_q)$ is continuous.

The proof of Proposition 3.3 is thus complete.
BIBLIOGRAPHY


Manuscrit reçu le 29 août 2008,
accepté le 12 décembre 2008.

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