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GELFAND TRANSFORMS OF $SO(3)$-INVARIANT SCHWARTZ FUNCTIONS ON THE FREE GROUP $N_{3,2}$

by Véronique FISCHER & Fulvio RICCI

ABSTRACT. — The spectrum of a Gelfand pair $(K \ltimes N, K)$, where N is a nilpotent group, can be embedded in a Euclidean space. We prove that in general, the Schwartz functions on the spectrum are the Gelfand transforms of Schwartz K-invariant functions on N. We also show the converse in the case of the Gelfand pair $(SO(3) \ltimes N_{3,2}, SO(3))$, where $N_{3,2}$ is the free two-step nilpotent Lie group with three generators. This extends recent results for the Heisenberg group.

RÉSUMÉ. — Il est toujours possible d'injecter dans un espace euclidien le spectre d'une paire de Gelfand du type $(K \ltimes N, K)$, où N est un groupe de Lie nilpotent. Nous démontrons que de manière générale, les fonctions de la classe de Schwartz sur le spectre sont les transformées des fonctions de la classe de Schwartz sur N qui sont invariantes par K. Nous prouvons également l'inclusion inverse dans le cas où $N = N_{3,2}$ est le groupe de Lie nilpotent libre à trois générateurs et $K = SO(3)$. Ceci étend des résultats récents sur le groupe de Heisenberg.

1. Introduction

Let N be a connected, simply-connected, two-step nilpotent Lie group. Let K be a compact group acting by automorphism on N. We assume that $(K \ltimes N, K)$ is a Gelfand pair. The Gelfand spectrum can be homeomorphically embedded in a Euclidean space as follows.

Let $\mathbb{D}(N)^K$ be the algebra of left-invariant and K-invariant differential operators on N and $\{D_1, \ldots, D_q\}$ a finite set of essentially self-adjoint generators of $\mathbb{D}(N)^K$. We call $\mathcal{D}$ the ordered family $(D_1, \ldots, D_q)$. To each bounded K-spherical function ϕ on N we assign the q-tuples of eigenvalues $\mu(\phi) = (\mu_1(\phi), \ldots, \mu_q(\phi))$, i.e. such that $D_j \phi = \mu_j(\phi)\phi$. The set $\Sigma_{\mathcal{D}}$ of such q-tuples is in 1-1 correspondence with the Gelfand spectrum and the topology induced on it from $\mathbb{R}^q$ coincides with the Gelfand topology [5].

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We define the Gelfand transform $\mathcal{G} : L^1(N)^K \to C_o(\Sigma_{\mathcal{D}})$ by:

$$\mathcal{G}F(\mu(\phi)) = \int_N F\bar{\phi}.$$

We are interested in the following conjecture:

$\mathcal{G}$ establishes an isomorphism between $\mathcal{S}(N)^K$ and $\mathcal{S}(\Sigma_{\mathcal{D}})$
(as Fréchet spaces)

The validity of this statement is independent of the choice of $\mathcal{D}$ (see Section 3); therefore once proved for one particular choice of $\mathcal{D}$, it is true for any choice of $\mathcal{D}$.

Proposition 3.3 of this paper shows the continuous inclusion $\mathcal{S}(\Sigma_{\mathcal{D}}) \hookrightarrow \mathcal{G}(\mathcal{S}(N)^K)$. This property is already known for the case of the Heisenberg group [2, Theorem 5.5]. The proof relies on a generalisation [2, Theorem 5.2] of Hulanicki's Schwartz kernel Theorem [14].

The converse inclusion has been recently shown for any Heisenberg Gelfand pair [2] and we prove it here for $(SO(3) \ltimes N_{3,2}, SO(3))$ where $N_{3,2}$ is the free two-step nilpotent Lie group with three generators; we realise $N_{3,2}$ as $\mathbb{R}^3_x \times \mathbb{R}^3_y$, $\{0\} \times \mathbb{R}^3_y$ being the centre. It is known that $(SO(3) \ltimes N_{3,2}, SO(3))$ is a Gelfand pair [3, Theorem 5.12]. We will give explicit formulae for a family of three essentially self-adjoint operators $\mathcal{D}$ that generate $\mathbb{D}(N_{3,2})^{SO(3)}$, the bounded spherical functions and their corresponding eigenvalues for $\mathcal{D}$.

Historically, the first description of the image of the Schwartz space on the $2n+1$-dimensional Heisenberg group H_n under the group Fourier transform has been described by D. Geller [8]. In the same spirit, for a Heisenberg Gelfand pair $(K \ltimes H_n, K)$, a characterisation of the Gelfand transform of the radial Schwartz functions was given in [4] for closed subgroups K of the unitary group $U(n)$. For more details, we refer the reader to the introductions of [1, 2].

Our goal here is to prove that for any Schwartz $SO(3)$-invariant function $F \in \mathcal{S}(N_{3,2})^{SO(3)}$, there exists a Schwartz extension of its Gelfand transform:

$$\text{i.e.} \quad \exists f \in \mathcal{S}(\mathbb{R}^3) \qquad f_{|\Sigma_{\mathcal{D}}} = \mathcal{G}F.$$

In the proof, we will use the known result for the three-dimensional Heisenberg group H_1. For this, let us consider N', the quotient group of $N_{3,2}$ by the central subgroup $\mathbb{R}^2_{(y_1,y_2)}$, and K', the stabiliser of $\mathbb{R}^2_{(y_1,y_2)}$ in $SO(3)$. We will see that N' is isomorphic to $H_1 \times \mathbb{R}_{x_3}$, and K' is isomorphic to $U_1 \rtimes \mathbb{Z}_2$ (see Section 2). The Gelfand transform for the pair $(K' \ltimes N', K')$ will be denoted by $\mathcal{G}'$.

Whenever it makes sense, we denote by $\mathcal{R}F$ the function on N' given by integration of a function F of $N_{3,2}$ on the central subgroup $\mathbb{R}^2_{(y_1,y_2)}$. The operator $\mathcal{R}$ maps $SO(3)$-invariant functions on $N_{3,2}$ to K'-invariant functions on N', and Schwartz functions on $N_{3,2}$ to Schwartz functions on N'. $\mathcal{R}$ is 1-1, but does not send $\mathcal{S}(N_{3,2})^{SO(3)}$ onto $\mathcal{S}(N')^{K'}$ (see Proposition 4.3). The definition of $\mathcal{R}$ can be extended to left-invariant differential operators in such a way that $\mathcal{R}(DF) = (\mathcal{R}D)(\mathcal{R}F)$ for any left-invariant differential operators D and any smooth compactly supported functions F on N. We will see that the image of the operators in $\mathcal{D}$ by $\mathcal{R}$ completed with $-\partial^2_{x_3}$ gives a family $\mathcal{D}'$ of essentially self-adjoint generators for $\mathbb{D}(N')^{K'}$. Again we will give explicit formulae for $\mathcal{D}'$, the bounded spherical functions, and their corresponding eigenvalues for $\mathcal{D}'$.

The spectrum of $(K' \ltimes N', K')$ can be projected onto the spectrum of $(SO(3) \ltimes N_{3,2}, SO(3))$ in the following sense: composing an homomorphism of $L^1(N')^{K'}$ with $\mathcal{R}$ provides a mapping $\Pi : \Sigma_{\mathcal{D}'} \to \Sigma_{\mathcal{D}}$ between the two spectra, that is completely explicit here. In fact Π maps continuously $\Sigma_{\mathcal{D}'}$ onto $\Sigma_{\mathcal{D}}$, but is 1-1 only on the regular part of the spectrum (see Section 2).

For any Schwartz $SO(3)$-invariant function $F \in \mathcal{S}(N_{3,2})^{SO(3)}$, we have:

$$\mathcal{G}'(\mathcal{R}F) = \mathcal{G}F \circ \Pi.$$

The existence of a Schwartz extension to $\mathbb{R}^4$ for $\mathcal{G}'(\mathcal{R}F)$, can be deduced easily from the Heisenberg case [1, 2]; it does not imply directly the existence of a Schwartz extension for $\mathcal{G}F$ but is constantly used all along the proof.

This article is organised as follows. In Section 2, we introduce the notations and the basic facts concerning the Gelfand spectra of $(SO(3) \ltimes N_{3,2}, SO(3))$ and $(K' \ltimes N', K')$. In Section 3, we give some general settings and the precise statements of our results. In Section 4 we describe $\mathcal{R}$ and the restriction mappings. In Section 5 we give the proof of Theorem 5 using an extension of a mean value formula due to Geller in the case of the Heisenberg group [8]. In the appendix, we give, for completeness, detailed proof of some results appearing in this paper and concerning differential operators and functional calculus on them.

2. The Gelfand spectra of $(SO(3) \ltimes N_{3,2}, SO(3))$ and $(K' \ltimes N', K')$

We realise $N_{3,2}$ as $\mathbb{R}^3_x \times \mathbb{R}^3_y$ endowed with the law:

$$(x, y).(x', y') = (x + x', y + y' + \frac{1}{2} x \wedge x'),$$

where $\wedge$ indicates the usual wedge product in $\mathbb{R}^3$. $\mathcal{N}_{3,2}$ denotes its Lie algebra. For $j = 1, 2, 3$, let X_j be the left-invariant vector field on N that equals ∂_{x_j} at 0, and Y_j the left-invariant vector field on N that equals ∂_{y_j}. $(X_j)_{j=1,2,3}$ and $(Y_j)_{j=1,2,3}$ form the canonical basis of $\mathcal{N}_{3,2}$, and satisfy:

$$[X_1, X_2] = Y_3 \quad , \quad [X_3, X_1] = Y_2 \quad , \quad [X_2, X_3] = Y_1.$$

The group $SO(3)$ acts on $\mathbb{R}^3$ and thus on $N_{3,2}$ by acting simultanously on each copy of $\mathbb{R}^3$. One checks easily that this action is by automorphisms on $N_{3,2}$. $(SO(3) \ltimes N_{3,2}, SO(3))$ is a Gelfand pair [3, Theorem 5.12].

Let us define the sub-Laplacian L, the central Laplacian Δ and a third operator D by:

$$L = -\sum_{j=1}^{3} X_j^2 \quad , \quad \Delta = -\sum_{j=1}^{3} Y_j^2 \quad , \quad D = -\sum_{j=1}^{3} X_j Y_j.$$

In Section 3, we will show that these operators form a family $\mathcal{D} = (L, \Delta, D)$ of essentially self-adjoint operators that generate $\mathbb{D}(N_{3,2})^{SO(3)}$.

The bounded spherical functions and their corresponding eigenvalues for $\mathcal{D}$ are known explicitly. Let us define some notation first: for any vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we will write $\tilde{x}$ or $x\tilde{}$ for (x_1, x_2) and, occasionally, $[x]_3$ for x_3. $\mathcal{L}_l(u) = (1/l!) e^{u/2} (d/du)^l u^l e^{-u}$ denotes the l-Laguerre function of order 0 on $\mathbb{R}$. Then the bounded spherical functions on $N_{3,2}$ are:

$$\phi_{\lambda,l,r}(x, y) = \int_{k \in SO(3)} e^{-i\lambda [k.y]_3} \mathcal{L}_l \Big(\frac{\lambda}{2} |[k.x]\tilde{}|^2\Big) e^{-ir[k.x]_3} dk \ ,$$

$$\lambda > 0,\ l \in \mathbb{N},\ r \in \mathbb{R}.$$

and

$$\phi_{0,R}(x, y) = \int_{k \in SO(3)} e^{-iR[k.x]_3}\, dk = \frac{\sin(R|x|)}{R|x|} dk \ , \quad R \geqslant 0;$$

their eigenvalues for $\mathcal{D}$ are given by:

$$\begin{aligned} \mu_{\phi_{\lambda,l,r}} &= \big(\lambda(2l+1) + r^2, \lambda^2, \lambda r\big)\,, \\ \mu_{\phi_{0,R}} &= \big(R^2, 0, 0\big)\,. \end{aligned}$$

The Gelfand spectrum $\Sigma_{\mathcal{D}}$ of $(SO(3) \ltimes N_{3,2}, SO(3))$ is then realised as the union of the collection of the $\mu_{\phi_{\lambda,l,r}}$, $\lambda > 0$, $l \in \mathbb{N}$, $r \in \mathbb{R}$ (the regular part of the spectrum), with the collection of the μ_{ϕ_R}, $R \in \mathbb{R}$ (the singular part of the spectrum). Calling (η_1, η_2, η_3) the coordinates corresponding to L, Δ, D respectively, $\Sigma_{\mathcal{D}}$ is then the union, for $l \geqslant 0$, of the surfaces Γ_l defined by the equation $\eta_3^2 = \eta_2\big(\eta_1 - (2l+1)\sqrt{\eta_2}\big)$. They all meet together on the positive η_2-axis, the singular part of $\Sigma_{\mathcal{D}}$.

N' is the quotient group of $N_{3,2}$ by the central subgroup $\mathbb{R}^2_{(y_1,y_2)}$; we realise N' as $\mathbb{R}^3_x \times \mathbb{R}_t$ endowed with the law:

$$(x,t).(x',t') = (x+x', t+t' + \frac{1}{2}[x \wedge x']_3).$$

$\mathcal{N}'$ denotes its Lie algebra; this is a quotient of $\mathcal{N}_{3,2}$ by $\mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ and $q = q_{\mathcal{N}_{3,2}} : \mathcal{N}_{3,2} \to \mathcal{N}'$ denotes the quotient mapping. $q(X_j) = X'_j$ is the left-invariant vector field X'_j on N' that equals ∂_{x_j} at 0, $j = 1,2,3$; $q(Y_1) = q(Y_2) = 0$, and $q(Y_3) = T$ is the left-invariant vector field on N that equals ∂_t. In particular $X'_3 = \partial_{x_3}$ lies in the centre of $\mathcal{N}'$. $(X'_j)_{j=1,2,3}$ and T form the canonical basis of $\mathcal{N}'$. It is easy to see that N' is isomorphic to $H_1 \times \mathbb{R}$. Let K' be the stabiliser of $\mathbb{R}^2_{(y_1,y_2)} \subset \mathbb{R}^3_y$ in $SO(3)$. The group K' is $S\big(O(2) \times O(1)\big) \cong O(2) \cong U_1 \rtimes \mathbb{Z}_2$.

$(K' \ltimes N', K')$ is a Gelfand pair and its bounded spherical functions are explicitly known:

$$\phi'_{\lambda,l,r}(x,t) = \cos(\lambda t + r x_{x_3}) \mathcal{L}_l(\frac{\lambda}{2}|[k.x]\check{}|^2) dk \quad , \quad \lambda > 0, l \in \mathbb{N}, r \in \mathbb{R},$$

and

$$\phi'_{\zeta,r}(x,y) = J_o(\zeta|\tilde{x}|)\cos(r x_3) \quad , \quad \zeta, r \in \mathbb{R},$$

J_o being the Bessel function of order 0.

We define the following operators:

$$L' = -\sum_{j=1}^{3} {X'_j}^2 \quad , \quad \Delta' = -T^2 \quad , \quad D' = -X'_3 T.$$

The operators L', Δ', D' and $-{X'_3}^2$ are K'-invariant, essentially self-adjoint and generate $\mathbb{D}(N')^{K'}$ (see Proposition 3.1 and Subsection 5.1). We set the family $\mathcal{D}' = (L', \Delta', D', -{X'_3}^2)$.

The eigenvalues of the bounded spherical functions for $\mathcal{D}'$ are given by:

$$\begin{aligned} \mu_{\phi'_{\lambda,l,r}} &= \big(\lambda(2l+1) + r^2, \lambda^2, \lambda r, r^2\big), \\ \mu_{\phi'_{\zeta,r}} &= \big(\zeta^2 + r^2, 0, 0, r^2\big). \end{aligned}$$

As in the case of $(SO(3) \ltimes N_{3,2}, SO(3))$, the Gelfand spectrum $\Sigma'_{\mathcal{D}'}$ of $(K' \ltimes N', K')$ is then realised as the union of a regular and a singular part: the regular part is the collection of the $\mu_{\phi'_{\lambda,l,r}}$, $\lambda > 0$, $l \in \mathbb{N}$, $r \in \mathbb{R}$, and the singular part is the collection of the $\mu_{\phi'_{\zeta,r}}$, $\zeta, r \in \mathbb{R}$.

With coordinates $(\eta_1, \eta_2, \eta_3, \eta_4)$ corresponding to $L', \Delta', D', -{X'_3}^2$ respectively, $\Sigma'_{\mathcal{D}'}$ is the union of the set $\{(\eta_1, 0, 0, \eta_4) : 0 \leqslant \eta_4 \leqslant \eta_1\}$ (the singular set) and the two-dimensional surfaces Γ'_l, $l \geqslant 0$, defined by the system of equations

$$\begin{cases} \eta_3^2 = \eta_2\big(\eta_1 - (2l+1)\sqrt{\eta_2}\big) \\ \eta_3^2 = \eta_2\eta_4 \ . \end{cases}$$

Notice that the projection onto the hyperplane $\eta_4 = 0$ parallel to the η_4-axis maps $\Sigma'_{\mathcal{D}'}$ onto $\Sigma_{\mathcal{D}}$, and is bijective between the two regular sets. This fact, alluded to already in the introduction, will be relevant in view of the mapping $\mathcal{R}$ defined in Subsection 4.1.

Let us give an equivalent and intrinsic point of view of this fact. As explained in the introduction, the spectrum of $(K' \ltimes N', K')$ can be projected in the following sense onto the spectrum of $(SO(3) \ltimes N_{3,2}, SO(3))$: the composition of an homomorphism of $L^1(N')^{K'}$ with $\mathcal{R}$ (see also Subsection 4.1) provides a mapping between the two Gelfand spectra. Realising the Gelfand spectra as explained in the introduction (see also Section 3), this mapping $\Pi : \Sigma_{\mathcal{D}'} \to \Sigma_{\mathcal{D}}$ is then given by:

$$\begin{aligned} \Pi\left(\lambda(2l+1) + r^2, \lambda^2, \lambda r, r^2\right) &= \left(\lambda(2l+1) + r^2, \lambda^2, \lambda r\right), \\ \Pi\left(\zeta^2 + r^2, 0, 0, r^2\right) &= \left(\zeta^2 + r^2, 0, 0\right). \end{aligned}$$

Π maps continuously $\Sigma_{\mathcal{D}'}$ onto $\Sigma_{\mathcal{D}}$. Moreover Π maps homeomorphically the regular part of $\Sigma_{\mathcal{D}'}$ onto the regular part of $\Sigma_{\mathcal{D}}$; Π maps the irregular part of $\Sigma_{\mathcal{D}'}$ onto the irregular part of $\Sigma_{\mathcal{D}}$, but this correspondence is not 1-1.

3. Results

In this section, we describe the general settings of our work and explain the conjecture $\mathcal{G}(\mathcal{S}(N)^K) = \mathcal{S}(\Sigma_{\mathcal{D}})$. We will give the precise statement of our main result in Theorem 3.5.

Let N be a connected, simply-connected Lie group, $\mathcal{N}$ its Lie algebra, $\exp : \mathcal{N} \to N$ the exponential mapping and $(E_i)_{i=1}^p$ a basis of $\mathcal{N}$. The canonical basis $(E_i)_{i=1}^p$ of $\mathcal{N}$ being chosen, this induces a Lebesgue measure dX on $\mathcal{N}$ and, via the exponential map, a Haar measure dn on N; the spaces $L^p(N)$ are defined with respect to this Haar measure. When N is a graded Lie group, following [7, ch1.D], we fix a homogeneous gauge $|.|$ on N and we keep the same notation for the basis (E_j) of $\mathcal{N}$ and the associated left-invariant vector fields on N; we set the following family of semi-norms parametrised by $a \in \mathbb{N}$ on the Schwartz space $\mathcal{S}(N)$ which induces the usual Fréchet space structure on $\mathcal{S}(N)$:

$$\|F\|_{a,N} = \sup_{n \in N, d(I) \leqslant a} (1 + |n|)^a |E^I F(n)|.$$

$\mathcal{P}(\mathcal{N})$ denotes the algebra of polynomials on $\mathcal{N}$ with real coefficients where $\mathcal{N}$ is then identified with the Euclidean vector space $\mathbb{R}^p$. $\mathbb{D}(N)$ denotes the algebra of real left-invariant differential operators on N, as operators defined on $C_c^\infty(N)$, the space of smooth, compactly-supported functions on N.

To $P \in \mathcal{P}(\mathcal{N})$, we associate $D_P \in \mathbb{D}(N)$ by:

$$D_P F(n) = \left[P(i^{-1}\partial_u) F(n \exp(\sum_{j=1}^{p} u_j E_j)) \right]_{|u=0}.$$

We obtain the symmetrisation mapping $P \mapsto D_P$, that is a linear isomorphism between the algebras $\mathbb{D}(N)$ and $\mathcal{P}(\mathcal{N})$ [12, Ch.II Theorems 4.3 and 4.9]

In the appendix we show:

Proposition 3.1. — *If $(K \ltimes N, K)$ is a Gelfand pair, each operator of $\mathbb{D}(N)^K$ is essentially self-adjoint, that is, it admits a unique self-adjoint extension to an unbounded operator of $L^2(N)$.*

Furthermore the operators of $\mathbb{D}(N)^K$ commute strongly, in the sense that the spectral resolutions of their self-adjoint extensions commute.

We will use the same notation for an operator of $\mathbb{D}(N)^K$ and its self-adjoint extension.

By Hilbert's Basis Theorem, if a group K acts orthogonally on some Euclidean space $\mathbb{R}^p$, the algebra $\mathcal{P}(\mathbb{R}^p)^K$ of K-invariant polynomials on $\mathbb{R}^p$ is finitely generated [12, Ch.II Corollary 4.10]. If $\rho_1, \ldots, \rho_q$ is a set of generators, we call $\{\rho_1, \ldots, \rho_q\}$ a Hilbert basis for $(\mathbb{R}^p, K)$ and $\rho = (\rho_1, \ldots, \rho_q)$ the corresponding Hilbert mapping. Furthermore, if $\rho = (\rho_1, \ldots, \rho_q)$ and $\rho' = (\rho'_1, \ldots, \rho'_{q'})$ are two Hilbert mappings for $(\mathbb{R}^p, K)$, then there exists $Q = (Q_1, \ldots, Q_q)$, $Q_j \in \mathcal{P}(\mathbb{R}^{q'})$, such that $\rho = Q \circ \rho'$ (and viceversa). We will make extensive use of G. Schwarz's Theorem [16]: every K-invariant smooth function on $\mathbb{R}^p$ can be expressed as a smooth function of a Hilbert basis ρ of $(\mathbb{R}^p, K)$. In other words, the Hilbert map, ρ, induces an application given by $\rho^*(h) = h \circ \rho$. Moreover ρ^* is a linear continuous mapping from $C^\infty(\mathbb{R}^q)$ onto $C^\infty(\mathbb{R}^p)^K$, and also from $\mathcal{S}(\mathbb{R}^q)$ onto $\mathcal{S}(\mathbb{R}^p)^K$ [2, Theorem 6.1].

Assume that $(K \ltimes N, K)$ is a Gelfand pair. Any family of generators of $\mathbb{D}(N)^K$ is obtained as the symmetrisation of a Hilbert basis, and conversely, if $\{\rho_1, \ldots, \rho_q\}$ denotes a Hilbert basis for $(\mathcal{N}, K)$, then $\{D_{\rho_1}, \ldots, D_{\rho_q}\}$ is a set of generators of $\mathbb{D}(N)^K$.

Let us fix $(\rho_1, \dots, \rho_q)$ an ordered Hilbert basis for $(\mathcal{N}, K)$, to which we associate the ordered family of operators $\mathcal{D}_\rho = (D_{\rho_1}, \dots, D_{\rho_q})$. We denote by $\Sigma_{\mathcal{D}_\rho}$, the set of the q-tuples of eigenvalues $\mu(\phi) = (\mu_1(\phi), \dots, \mu_q(\phi))$ of $\mathcal{D}_\rho$ for the bounded K-spherical functions ϕ on N. As mentioned in the introduction and proved in [5], $\Sigma_{\mathcal{D}_\rho}$ is the realisation of the Gelfand spectrum associated to $\mathcal{D}_\rho$, in the sense that the set $\Sigma_{\mathcal{D}_\rho}$ of such q-tuples is in 1-1 correspondence with the Gelfand spectrum and the topology induced on it from $\mathbb{R}^q$ coincides with the Gelfand topology. In the appendix, we will show that $\Sigma_{\mathcal{D}_\rho}$ is also the joint spectrum of $\mathcal{D}_\rho$:

Proposition 3.2. — *Let $(\rho_1, \dots, \rho_q)$ be an ordered Hilbert basis for $(\mathcal{N}, K)$. The joint spectrum of the family of strongly commuting, self-adjoint operators $\mathcal{D}_\rho = (D_{\rho_1}, \dots, D_{\rho_q})$ is $\Sigma_{\mathcal{D}_\rho}$.*

For a closed subset E of $\mathbb{R}^q$, $\mathcal{S}(E)$ denotes the space of restrictions to E of Schwartz functions, endowed with the quotient topology of $\mathcal{S}(\mathbb{R}^q)/\{f : f_{|E} = 0\}$; we will often define a class in this quotient as being given as the restriction of a Schwartz function on $\mathbb{R}^q$. The spectrum $\Sigma_{\mathcal{D}}$ is a closed subset of $\mathbb{R}^q$. We are interested in the conjecture $\mathcal{S}(N)^K \overset{\mathcal{G}}{\sim} \mathcal{S}(\Sigma_{\mathcal{D}})$. The existence of a polynomial mapping between two Hilbert mappings implies that the validity of this conjecture is independent of the choice of $\mathcal{D}$ (see [2, Section 3]). The continuous inclusion $\mathcal{S}(\Sigma_{\mathcal{D}}) \hookrightarrow \mathcal{G}(\mathcal{S}(N)^K)$ relies on the following statement, which is a generalisation of Hulanicki's Schwartz Kernel Theorem proved in the appendix:

Proposition 3.3. — *Let $(\rho_1, \dots, \rho_q)$ be an ordered Hilbert basis for $(\mathcal{N}, K)$, and $\mathcal{D}_\rho = (D_{\rho_1}, \dots, D_{\rho_q})$ the associated family of operators.*

Let m be in $\mathcal{S}(\mathbb{R}^q)$. The operator $m(\mathcal{D}_\rho)$ is a convolution operator with a K-invariant Schwartz kernel $M = M_{m,\mathcal{D}_\rho} \in \mathcal{S}(N)^K$:

$$\forall F \in L^2(N) \qquad m(\mathcal{D}_\rho)F = F * M.$$

The Gelfand transform of M is:

$$\mathcal{G}M = m_{|\Sigma_{\mathcal{D}_\rho}}.$$

Furthermore the mapping $m \in \mathcal{S}(\mathbb{R}^q) \mapsto M_{m,\mathcal{D}_\rho} \in \mathcal{S}(N)^K$ is continuous.

For the Gelfand spectra of Heisenberg groups or the free two-step nilpotent Lie groups, the inclusion of the spectrum in the image of the Hilbert mapping:

$$\Sigma_{\mathcal{D}_\rho} \subset \operatorname{im} \rho,$$

is true,independently of the choice of the Hilbert mapping ρ (but we do not know if it is true in general). Here we will use this property only in the case of $N' = H_1 \times \mathbb{R}$, where it is known, the spectrum and the Hilbert mapping being explicit.

Lemma 3.4. — *The polynomials $|x|^2$, $|y|^2$ and $x \cdot y$ generate the algebra of polynomials on $\mathbb{R}^3_x \times \mathbb{R}^3_y$ that are invariant under the simultaneous action of $SO(3)$ on each copy of $\mathbb{R}^3$.*

Proof. — If $P(x,y)$ is an $SO(3)$-invariant polynomial on $\mathbb{R}^3_x \times \mathbb{R}^3_y$, then for each independent vectors $x, y \in \mathbb{R}^3$, we have $P(x,y) = P(-x,-y)$ because the linear transformation that equals $-\mathrm{Id}$ on the vector space spanned by x and y, and 1 on the orthogonal complement line, is in $SO(3)$; this shows that P is invariant under $-\mathrm{Id}_{\mathbb{R}^3}$, and thus also under the simultaneous action of $O(3)$ on each copy of $\mathbb{R}^3$. This implies:

$$\mathcal{P}(\mathbb{R}^3_x \times \mathbb{R}^3_y)^{SO(3)} = \mathcal{P}(\mathbb{R}^3_x \times \mathbb{R}^3_y)^{O(3)}.$$

By [10, Theorem 4.2.2.(1)], $\mathcal{P}(\mathbb{R}^3_x \times \mathbb{R}^3_y)^{O(3)}$ is spanned by $|x|^2$, $|y|^2$ and $x \cdot y$. $\square$

Thus $\rho(x,y) = (|x|^2, |y|^2, x \cdot y)$ gives a Hilbert mapping for $(\mathcal{N}_{3,2}, SO(3))$. We compute easily that the associated family of operators by symmetrisation is $\mathcal{D} = (L, \Delta, D)$ defined in Section 2, where we give also an explicit description of the associated realisation of the Gelfand spectrum.

Theorem 3.5. — *The Gelfand transform of any Schwartz $SO(3)$-invariant function on $N_{3,2}$ admits a Schwartz extension to $\mathbb{R}^3$:*

$$\forall F \in \mathcal{S}(N_{3,2})^{SO(3)} \qquad \mathcal{G}F \in \mathcal{S}(\Sigma_{\mathcal{D}}).$$

Moreover the mapping $F \in \mathcal{S}(N_{3,2})^{SO(3)} \mapsto \mathcal{G}F \in \mathcal{S}(\Sigma_{\mathcal{D}})$ is an isomorphism of Fréchet spaces.

The group $N_{3,2}$ admits a slightly bigger group of automorphisms than $SO(3)$, namely $O(3)$ acting by:

$$k(x,y) = \big(kx, (\det k)ky\big) \quad , \quad k \in O(3),$$

It is easily verified that $\{|x|^2, |y|^2, (x \cdot y)^2\}$ gives a Hilbert basis for $(\mathcal{N}_{3,2}, O(3))$ and the associated family of operators by symmetrisation is $\tilde{\mathcal{D}} = (L, \Delta, D^2)$. Following the same lines as in [2, Section 8], we have the following.

Corollary 3.6. — *The Gelfand transform is an isomorphism between $\mathcal{S}(N_{3,2})^{O(3)}$ and $\mathcal{S}(\Sigma_{\tilde{\mathcal{D}}})$ as Fréchet spaces.*

From now on, N will stand for $N_{3,2}$ and K for $SO(3)$.

4. $\mathcal{R}$ and restriction mappings

4.1. The mapping $\mathcal{R}$

In the introduction, we denoted by $\mathcal{R}F$ the function on N' given by integration of a function F of N on the central subgroup $\mathbb{R}^2_{(y_1,y_2)}$ whenever it makes sense, for example on $L^1(N)$. It is sometimes convenient to consider $\mathcal{R}$ as acting between functions defined on the Lie algebras, rather than on the groups. We will do so without any further mention. The operator $\mathcal{R}$ maps K-invariant functions on N to K'-invariant functions on N', integrable functions on N to integrable functions on N' continuously, Schwartz functions on N to Schwartz functions on N' continuously. It respects convolution on the groups and abelian convolution on the Lie algebras.

We extend the definition of $\mathcal{R}$ to the algebra $\mathbb{D}(N)$ of left-invariant differential operators on N in the following way: if $D \in \mathbb{D}(N)$, then we define $D' = \mathcal{R}D \in \mathbb{D}(N')$ by

$$(D'G) \circ q = D(G \circ q), \quad G \in C_c^\infty(N'),$$

where $q = q_N : N \to N'$ is the quotient mapping. Note that if $D = D_P \in \mathbb{D}(N)$, $P \in \mathcal{P}(\mathcal{N})$, then easy changes of variables, see e.g. (A.1) below, leads to:

$$\forall F \in C_c^\infty(N) \quad , \quad \forall G \in C_c^\infty(N') \qquad \langle \mathcal{R}(DF), G\rangle = \langle \mathcal{R}F, D_{P_{|\mathcal{N}'}} G\rangle.$$

This shows that $\mathcal{R}D_P = D_Q$ where $Q = P_{|\mathcal{N}'}$ is the restriction of P to $\mathcal{N}'$.

The mapping $\mathcal{R}$ on functions is dual to the restriction mapping from $\mathcal{N}$ to $\mathcal{N}'$ in the following sense. Let us denote $\mathcal{F}_y$ and $\mathcal{F}_t$ the Fourier transform with respect to the variables $y \in \mathbb{R}^3$ and $t \in \mathbb{R}$ respectively given by:

$$\begin{aligned} \mathcal{F}_y F(x,\hat{y}) &= \int_{\mathbb{R}^3} F(x,y) e^{-iy.\hat{y}} dy, \\ \mathcal{F}_t G(x,\hat{t}) &= \int_{\mathbb{R}} G(x,t) e^{-it.\hat{t}} dt; \end{aligned}$$

whenever it makes sense for a function G on N' and a function F on N, identified with functions on $\mathcal{N}'$ and $\mathcal{N}$ respectively, we have:

$$G = \mathcal{R}F \iff \mathcal{F}_t G = \mathcal{F}_y F_{|\mathcal{N}'} \tag{4.1}$$

In the following subsection, we describe the restriction mapping.

4.2. Restriction and radialisation mappings

For a function f on $\mathcal{N}$, we denote by Rest $f = f_{|\mathcal{N}'}$ its restriction to $\mathcal{N}'$. We set:

$$\mathcal{N}_o = \mathbb{R}^3_x \times (\mathbb{R}^3_y \backslash \{0\}) \quad \text{and} \quad \mathcal{N}'_o = \mathbb{R}^3_x \times (\mathbb{R}_t \backslash \{0\}).$$

In the next lemma, we define the radialisation mapping Radial :

LEMMA 4.1. — *For a function $h \in C^\infty(\mathcal{N}'_o)^{K'}$ and $(x, y) \in \mathcal{N}_o$, the following:*

$$\text{Radial}\,(h)(x,y) = h(k^{-1}x, t) \quad , \quad \textit{where} \quad y = k(0,0,t) \textit{ for some } k \in K.$$

defines a K-invariant function Radial (h)*, that is smooth on $\mathcal{N}_o$.*

Radial *is an isomorphism between the topological vector spaces $C^\infty(\mathcal{N}_o)^K$ and $C^\infty(\mathcal{N}'_o)^{K'}$, whose inverse is* Rest .

Proof. — For a function $h \in C^\infty(\mathcal{N}'_o)^{K'}$ and $(x, y) \in \mathcal{N}_o$, it is easy to see that Radial $(h)(x, y)$ is well defined and K-invariant.

Let us show that Radial $(h) \in C^\infty(\mathcal{N}_o)^K$. We choose a basis $(A_j)_{j=1,2,3}$ for the Lie algebra of K. At a point $(x_0, y_0) \in \mathcal{N}_0$ (with $y_0 = k_0(t_0, 0, 0)$, $t_0 = |y_0| \neq 0$) we choose a local coordinate system $(x, y) = \big(x, k(t, 0, 0)\big) = \sigma(x, k, t)$, where $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and k varies in a small two-dimensional surface in K containing k_0 and transversal to k_0K'. This change of variables does not affect the derivatives in x, whereas

$$\partial_{y_j} = c_{j,0}(k,t)\partial_t + \sum_{j'=1,2,3} c_{j,j'}(k,t) A_{j'}.$$

By homogeneity, $c_{j,0} \in C^\infty(K_k \times \mathbb{R}^*_t)$ is homogeneous of degree 0 in t, and the $c_{j,j'} \in C^\infty(K_k \times \mathbb{R}^*_t)$ homogeneous of degree (-1) in t. More generally, we can write the derivative

$$\partial^I_y = \partial^{i_1}_{y_1} \partial^{i_2}_{y_2} \partial^{i_3}_{y_3} \quad , \quad I = (i_1, i_2, i_3) \in \mathbb{N}^3,$$

as:

$$\partial^I_y = \sum c_{I,J}(k,t) \partial^{j_0}_t A^{j_1}_1 A^{j_2}_2 A^{j_3}_3, \tag{4.2}$$

where the sum is over $J = (j_0, j_1, j_2, j_3) \in \mathbb{N}^4$, with $|J| = |I|$, and the $c_{I,J} \in C^\infty(K_k \times \mathbb{R}^*_t)$ are homogeneous of degree $(j_0 - |I|)$ in t. As the function $(x; k, t) \mapsto h(k^{-1}x, t)$ is smooth on $\mathbb{R}^3 \times K \times \mathbb{R}$, (4.2) implies that Radial h is smooth on $\mathcal{N}_o$. Furthermore $h \in C^\infty(\mathcal{N}'_o) \mapsto$ Radial $h \in C^\infty(\mathcal{N}_o)$ is continuous. $\square$

Lemma 4.1 implies that the mapping Rest is 1-1 on $C^\infty(\mathcal{N})^K$ and on $\mathcal{S}(\mathcal{N})^K$. Let us determine Rest $(C^\infty(N)^K)$. We will need the following notation:

- For $f \in C^\infty(\mathcal{N})^K$, we denote by $P_M^{(f)}(x,t)$ the homogeneous polynomial of degree M in the Taylor expansion of $f(x,\cdot)$ at $y=0$:
$$P_M^{(f)}(x,y) = \sum_{|j|=M} \frac{1}{j!}\partial_y^j f(x,0) y^j .$$
- For $g \in C^\infty(\mathcal{N}')^{K'}$, we denote by $Q_M^{(g)}(x,t)$ the homogeneous polynomial of degree M in the Taylor expansion of $g(x,\cdot)$ at $t=0$:
$$Q_M^{(g)}(x,t) = \frac{1}{M!}\partial_t^M g(x,0) t^M .$$

We see:
$$Q_M^{(\mathrm{Rest}\, f)} = \mathrm{Rest}\, P_M^{(f)} \quad \text{and thus} \quad \mathrm{Radial}\, Q_M^{(\mathrm{Rest}\, f)} = P_M^{(f)} .$$

Thus a function $g \in C^\infty(\mathcal{N}')^{K'}$ that is the restriction of some function $f \in C^\infty(\mathcal{N})^K$, necessarily has the following property:

Property (R). *For any $M \in \mathbb{N}$,* Radial $(Q_M^{(g)})$ *extends to a smooth function on $\mathcal{N}$ which is a homogeneous polynomial in y of degree M, with smooth coefficients in x.*

It turns out that this condition is also sufficient:

Proposition 4.2. — *Let $g \in C^\infty(\mathcal{N}')^{K'}$.*

The function g is in the image of Rest *if and only if it satisfies Property (R).*

In this case, Radial (g) *extends to a K-invariant smooth function f on $\mathcal{N}$, whose restriction to $\mathcal{N}'$ is g and we have $Q_M^{(g)} = \mathrm{Rest}\,[P_M^{(f)}]$. Moreover if in addition $g \in \mathcal{S}(\mathcal{N}')^{K'}$, then* Radial $(g) \in \mathcal{S}(\mathcal{N})^K$

In the proof, we adapt the ideas of the Euclidean setting [13, Theorem 2.4].

Proof. — Let $g \in C^\infty(\mathcal{N}')^{K'}$ satisfying Property (R). For each M, we denote P_M the extension of Radial $(Q_M^{(g)})$ to a smooth function on $\mathcal{N}$ that is a homogeneous polynomial in y of degree M, with smooth coefficients in x.

Let $M_o \in \mathbb{N}$. The Taylor Formula gives:

$$g(x,t) - \sum_{j=0}^{M_o} Q_j^{(g)}(x,t) = \frac{t^{M_o+1}}{M_o!}\int_0^1 (1-w)^{M_o}\left(\partial_t^{M_o+1} g\right)(x,wt)dw. \tag{4.3}$$

Let $I_o \in \mathbb{N}^3$ with $|I_o| = M_o + 1$. We have on $\mathcal{N}_o$:

$$\partial_y^{I_o}\text{Radial}\,(g) = \partial_y^{I_o}\left(\text{Radial}\,(g) - \sum_{j=0}^{M_o} P_j\right) = \partial_y^{I_o}\left[\text{Radial}\left(g - \sum_{j=0}^{M_o} Q_j^{(g)}\right)\right].$$

Now for any $(x,y) \in \mathcal{N}_o$, $y \neq 0$ can be written $y = k(0,0,t)$, $t \in \mathbb{R}^*$, $k \in K$, and by (4.2) and (4.3), $\left(\partial_y^{I_o}\text{Radial}\,(g)\right)(x,y)$ can be written as the sum over $J \in \mathbb{N}^4$, $|J| = M_o + 1$, of:

$$\frac{c_{I_o,J}(k,t)}{M_o!}\int_0^1 (1-w)^{M_o}\partial_t^{j_0}A_1^{j_1}A_2^{j_2}A_3^{j_3}\left[t^{M_o+1}\left(\partial_t^{M_o+1}g\right)(k^{-1}x, wt)\right]dw.$$

This last term remains bounded if $0 < |y| = |t| \leqslant 1$ because $c_{I_o,J}$ is homogeneous of degree $j_o - (M_o+1)$. This implies that $\partial_y^{I_o}\text{Radial}\,(g)$ is bounded on a compact neighborhood of $(x,0)$ for any x, and any I_o and M_o. It is easy to see that for any $I \in \mathbb{N}^3$, $\partial_x^I\partial_y^{I_o}\text{Radial}\,(g)$ satisfies the same conditions. Local boundedness of all derivatives is sufficient to imply that Radial$\,(g)$ has a smooth extension to $\mathcal{N}$. □

We deduce easily from (4.1) the following characterisation of $\mathcal{R}(\mathcal{S}(\mathcal{N})^K)$:

Proposition 4.3. — *Let $G \in \mathcal{S}(N')^{K'}$. The function G is in $\mathcal{R}(\mathcal{S}(N)^K)$ if and only if either of the following equivalent conditions is satisfied:*

(i) *$\mathcal{F}_t G$ (identified with a function on $\mathcal{N}'$) satisfies Property (R);*

(ii) *denoting by $\mathcal{F}_{x,t}$ the Fourier transform with respect to the variables $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$ given by:*

$$\mathcal{F}_{x,t}G(\hat{x},\hat{t}) = \int_{\mathbb{R}^3\times\mathbb{R}} G(x,t)e^{-ix.\hat{x}}e^{-it.\hat{t}}dxdt\ ,$$

$\mathcal{F}_{x,t}G$ (identified with a function on $\mathcal{N}'$) satisfies Property (R);

From G. Schwarz's Theorem, it follows (compare with [13, Theorem 2.4]):

Corollary 4.4. — *Let $G \in \mathcal{S}(N')^{K'}$. The function G is in $\mathcal{R}(\mathcal{S}(N)^K)$ if and only if either of the following equivalent conditions is satisfied:*

(i) *for each $j \in \mathbb{N}$ there exist Schwartz functions $a_{j,i} \in \mathcal{S}(\mathbb{R})$, $i = 0,\dots,j$ satisfying:*

$$\forall x \in \mathbb{R}^3 \qquad \int_{\mathbb{R}} G(x,t)t^j dt = \sum_{i=0}^{j} a_{i,j}(|x|^2)x_3^i\ ;$$

(ii) *for each $j \in \mathbb{N}$ there exist Schwartz functions $b_{j,i} \in \mathcal{S}(\mathbb{R})$, $i = 0,\dots,j$ satisfying:*

$$\forall \zeta \in \mathbb{R}^3 \qquad \int_{\mathbb{R}^3}\int_{\mathbb{R}} G(x,t)t^j e^{-ix.\zeta}dtdx = \sum_{i=0}^{j} b_{i,j}(|\zeta|^2)\zeta_3^i.$$

5. Proof of Theorem 3.5

Here we give the proof of Theorem 3.5. It is based on the properties of mappings explained in Section 4 and on results already shown on the Heisenberg group [1, 2]. These two key ingredients are used in the proofs of a "Geller-type" Lemma (Subsection 5.2) and of Theorem 3.5 (Subsection 5.3).

5.1. The Gelfand pair $(K' \ltimes N', K')$

We easily check that

$$\rho'(x,t) = (|x|^2, t^2, x_3 t, x_3^2),$$

defines a Hilbert mapping of $(\mathcal{N}', K')$, which satisfies:

$$\rho'(x,t) = \left(\rho_{|\mathcal{N}'}(x,(0,0,t)), x_3^2\right) \quad \text{and} \quad \mathcal{D}' = \mathcal{D}_{\rho'}.$$

From the Heisenberg case [1, 2], we deduce:

Lemma 5.1. — *For any $G \in \mathcal{S}(N')^{K'}$, there exists $\tilde{g} \in \mathcal{S}(\mathbb{R}^4)$ with $\mathcal{G}'G = \tilde{g}_{|\Sigma_{\mathcal{D}'}}$.*

Furthermore the mapping $G \in \mathcal{S}(N')^{K'} \mapsto \mathcal{G}'G \in \mathcal{S}(\Sigma_{\mathcal{D}'})$ is continuous.

Precisely, continuity of the last mapping means that

$$\begin{aligned} &\forall a \in \mathbb{N} \quad \exists C = C(a) > 0 \quad \exists a' \in \mathbb{N} \quad \forall G \in \mathcal{S}(N')^{K'} \\ &\exists \tilde{g} \in \mathcal{S}(\mathbb{R}^4) \quad \tilde{g}_{|\Sigma_{\mathcal{D}'}} = \mathcal{G}'G \quad \|\tilde{g}\|_{a,\mathbb{R}^4} \leqslant C \|G\|_{a',N'}. \end{aligned} \tag{5.1}$$

Notice that the extension $\tilde{g}$ depends on the Schwartz semi-norm $\|.\|_{a,\mathbb{R}^4}$.

5.2. The Geller-type Lemma

In this subsection, we will state and prove a "Geller-type Lemma", extending [8, 2]. For this purpose we will need the following remark.

Let $F \in \mathcal{S}(N)^K$. The mapping

$$R \mapsto \mathcal{G}F(R^2, 0, 0) = \int F(x,y) e^{-iRx_1} dx\, dy,$$

is a Schwartz even function on $\mathbb{R}$; by Whitney's Theorem, there exists a Schwartz function $f_o \in \mathcal{S}(\mathbb{R})$ such that

$$\forall R \in \mathbb{R} \qquad f_o(R^2) = \mathcal{G}F(R^2, 0, 0);$$

by Hulanicki's Schwartz Kernel Theorem or Proposition 3.3, $f_o(L)$ is a convolution operator with a K-invariant Schwartz kernel which we denote by $\mathcal{G}F(L,0,0)$ (for brevity reasons, in this section we will often denote a convolution operator and its kernel by the same symbol).

PROPOSITION 5.2 (Geller-type Lemma). — *Let $F \in \mathcal{S}(N)^K$. There exist $F_1 \in \mathcal{S}(N)^K$ and $F_2 \in \mathcal{S}(N)^K$ satisfying:*

$$F = \mathcal{G}(F)(L,0,0) + \Delta F_1 + DF_2.$$

Proof. — Let F be in $\mathcal{S}(N)^K$ and $G = \mathcal{R}F \in \mathcal{S}(N')^{K'}$. By Lemma 5.1 there exists $\tilde{g} \in \mathcal{S}(\mathbb{R}^4)$ with $\tilde{g}_{|\Sigma_{\mathcal{D}'}} = \mathcal{G}'G$. By Proposition 3.3, the operator given by:

$$\int_{w=0}^{1} \partial_2 \tilde{g}\,(L, w\Delta, 0, 0)\, dw,$$

is a convolution operator with a K-invariant Schwartz kernel which we denote by $F_1 \in \mathcal{S}(N)^K$. By spectral calculus, we have:

$$\Delta F_1 = \tilde{g}\,(L,\Delta,0,0) - \tilde{g}\,(L,0,0,0)\,.$$

We will have finished the proof of Proposition 5.2 once we have shown:

$$\exists F_2 \in \mathcal{S}(N)^K \qquad F - \mathcal{G}F\,(L,0,0) - \Delta F_1 = DF_2 \tag{5.2}$$

We denote by $H \in \mathcal{S}(N)^K$ and $I \in \mathcal{S}(N')^{K'}$ the functions given by:

$$\begin{aligned} H &= F - \mathcal{G}F\,(L,0,0) - \Delta F_1 = F - \tilde{g}\,(L,\Delta,0,0)\,, \\ I &= \mathcal{R}H = G - \tilde{g}\,(L',\Delta',0,0)\,. \end{aligned}$$

The Gelfand transform of I is given by:

$$\mathcal{G}'I\,(\mu_{\phi'}) = \mathcal{G}'G\,(\mu_{\phi'}) - \tilde{g}\,(L'\phi'(0), \Delta'\phi'(0), 0, 0)\,. \tag{5.3}$$

On the singular part of the spectrum, (5.3) yields to:

$$\mathcal{G}'I\left(\mu_{\phi'_{\zeta,r}}\right) = \tilde{g}\left(\zeta^2 + r^2, 0, 0, r^2\right) - \tilde{g}\left(\zeta^2 + r^2, 0, 0, 0\right) = 0,$$

because $\tilde{g}\left(\zeta^2 + r^2, 0, 0, r^2\right) = \tilde{g}\left(\zeta^2 + r^2, 0, 0, 0\right)$ as $\mathcal{G}'G = \mathcal{G}F \circ \Pi$; this implies:

$$\forall x \in \mathbb{R}^3 \qquad \int_{\mathbb{R}} I(x,t)dt = 0. \tag{5.4}$$

On the regular part of the spectrum, (5.3) yields to:

$$\mathcal{G}'I\left(\mu_{\phi'_{\lambda,l,r}}\right) = \tilde{g}\left(\lambda(2l+1) + r^2, \lambda^2, \lambda r, r^2\right) - \tilde{g}\left(\lambda(2l+1) + r^2, \lambda^2, 0, 0\right),$$

and in particular for $r = 0$:

$$\mathcal{G}'I\left(\mu_{\phi'_{\lambda,l,0}}\right) = \tilde{g}\left(\lambda(2l+1), \lambda^2, 0, 0\right) - \tilde{g}\left(\lambda(2l+1), \lambda^2, 0, 0\right) = 0;$$

this implies for all $\lambda > 0$:

$$\forall l \in \mathbb{N} \qquad \int_{N'} I(x,t) e^{-i\lambda t} \mathcal{L}_l \left(\frac{\lambda}{2} |\tilde{x}|^2 \right) dx dt = 0,$$

and $\{\mathcal{L}_l\}_{l \in \mathbb{N}}$ being an orthogonal basis of $L^2(\mathbb{R}^+)$,

$$\forall \tilde{x} \in \mathbb{R}^2 \qquad \int_{\mathbb{R}^2} I(\tilde{x}, x_3; t) e^{-i\lambda t} dx_3 dt = 0.$$

Eventually, we get:

$$\forall \tilde{x} \in \mathbb{R}^2 \quad , \quad \forall t \in \mathbb{R} \qquad \int_{\mathbb{R}} I(\tilde{x}, x_3; t) dx_3 = 0. \tag{5.5}$$

Let us set:

$$G_2(x,t) = \int_{-\infty}^{x_3} \int_{-\infty}^{t} I\left(\tilde{x}, w; s\right) ds\, dw.$$

Because of (5.4) and (5.5), we see that $G_2 \in \mathcal{S}(N')^{K'}$. Let us show that $\mathcal{F}_{x,t} G_2$ (identified with a function on $\mathcal{N}'$) satisfies Property (R). We have for $\hat{t} \neq 0$ and $\hat{x}_3 \neq 0$:

$$\mathcal{F}_{x,t} G_2(\hat{x}, \hat{t}) = (\hat{x}_3 \hat{t})^{-1} \mathcal{F}_{x,t} I(\hat{x}; \hat{t})$$

and

$$Q_{M-1}^{(\mathcal{F}_{x,t} G_2)}(\hat{x}; \hat{t}) = (\hat{x}_3 \hat{t})^{-1} Q_M^{(\mathcal{F}_{x,t} I)}(\hat{x}; \hat{t}).$$

By Proposition 4.3(ii), as $I = \mathcal{R}H$, $\mathcal{F}_{x,t} I$ (identified with a function on $\mathcal{N}'$) satisfies Property (R), that is Radial $\left(Q_M^{(\mathcal{F}_{x,t} I)} \right)$ extends to a smooth K-invariant function on $\mathcal{N}$ which is a homogeneous polynomial in y of degree M, with Schwartz coefficients in x. By G. Schwarz's Theorem, there exists a function $\tilde{Q}_M \in C^\infty(\mathbb{R}^3)$ of the form:

$$\tilde{Q}_M(r_1, r_2, r_3) = \sum_{2j_1 + j_2 = M} c_j(r_1) r_2^{j_1} r_3^{j_2} \quad , \quad c_j \in \mathcal{S}(\mathbb{R}),$$

satisfying:

$$\text{Radial} \left(Q_M^{(\mathcal{F}_{x,t} I)} \right) = \tilde{Q}_M \circ \rho.$$

That is:

$$Q_M^{(\mathcal{F}_{x,t} I)}(\hat{x}, \hat{t}) = \tilde{Q}_M(|\hat{x}|^2, \hat{t}^2, \hat{x}_3 \hat{t}) = \sum_{2j_1 + j_2 = M} c_j(|\hat{x}|^2) \hat{t}^{2j_1} (\hat{x}_3 \hat{t})^{j_2}.$$

Because of (5.5), we have:

$$\forall \tilde{\hat{x}} \in \mathbb{R}^2 \quad , \quad \forall \hat{t} \in \mathbb{R} \qquad \mathcal{F}_{x,t} I(\tilde{\hat{x}}, 0; \hat{t}) = 0,$$

thus the term $c_j(|\hat{x}|^2)$ with $j = (j_1, 0)$ is zero: we can factor out one $(\hat{x}_3 \hat{t})$. This implies that for $M > 0$, Radial$\,(Q_{M-1}^{(\mathcal{F}_{x,t} G_2)}(\hat{x}; \hat{t}))$ extends to a smooth function on $\mathcal{N}$ which is a homogeneous polynomial in y of degree $(M-1)$, with smooth coefficients in x. Thus $\mathcal{F}_{x,t}G_2$ satisfies Property (R). By Proposition 4.3(ii), there exists $F_2 \in \mathcal{S}(N)^K$ such that $\mathcal{R}F_2 = G_2$. As $D'G_2 = I = \mathcal{R}H$ and $\mathcal{R}$ being 1-1 on K-invariant functions, we obtain $DF_2 = H$. This proves (5.2). $\square$

Applying recursively Proposition 5.2, we obtain F as a sum of functions of the form $\left(\mathcal{G}\left[\Delta^{j_1} D^{j_2} F_j\right]\right)(L,0,0)$ with a rest. As the degrees of homogeneity of the operators D and Δ with respect to the variable y are three and four respectively, we will be interested in a sum over $2j_1 + j_2 \leqslant M$:

Corollary 5.3. — *Let $F \in \mathcal{S}(N)^K$. There exists a family $(F_j)_{j\in\mathbb{N}^2}$ of Schwartz functions $F_j \in \mathcal{S}(N)^K$ satisfying for any $M \in \mathbb{N}$:*

$$F - \sum_{2j_1+j_2\leqslant M} \left(\mathcal{G}\left[\Delta^{j_1} D^{j_2} F_j\right]\right)(L,0,0) = \sum_{2j_1+j_2=M+1} \Delta^{j_1} D^{j_2} F_j.$$

5.3. End of the proof

Here we complete the proof of Theorem 3.5.

Existence of the extension. Let $F \in \mathcal{S}(N)^K$, $G = \mathcal{R}F$, $f = \mathcal{G}F$, $g = \mathcal{G}'G$, F_j, the associated functions in Corollary 5.3, and $f_j = \mathcal{G}F_j$, $j \in \mathbb{N}^2$. By Lemma 5.1 we choose $\tilde{g}$, $\tilde{g}_j \in \mathcal{S}(\mathbb{R}^4)$ Schwartz extensions of g and $\mathcal{G}'(\mathcal{R}F_j)$, $j \in \mathbb{N}^2$, respectively. We set $\tilde{g}_{\rho'} = \tilde{g} \circ \rho'$.

Let us fix M. For $\xi = (\tilde{\xi}, \xi_3) \in \mathbb{R}^3$, setting $r = \xi_3$ and $\lambda_l = |\tilde{\xi}|^2/(2l+1)$, we have $\rho'(\xi, \lambda_l) \in \Sigma_{\mathcal{D}'}$ and:

$$\begin{aligned}
\tilde{g}_{\rho'}(\xi,\lambda_l) &= \tilde{g}\circ\rho'(\xi,\lambda_l) = g(\lambda_l(2l+1)+r^2, \lambda_l^2, \lambda_l r, r^2)\\
&= f(\lambda_l(2l+1)+r^2, \lambda_l^2, \lambda_l r)\\
&= \sum_{2j_1+j_2\leqslant M} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l+1)+r^2, 0, 0)\\
&\qquad + \sum_{2j_1+j_2=M+1} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l+1)+r^2, \lambda_l^2, \lambda_l r).
\end{aligned}$$

Thus:

$$\begin{aligned}
&|\tilde{g}_{\rho'}(\xi,\lambda_l) - \sum_{2j_1+j_2\leqslant M} \lambda_l^{2j_1}(\lambda_l r)^{j_2} f_j(\lambda_l(2l+1)+r^2, 0, 0)|\\
&\qquad \leqslant \left(\sum_{2j_1+j_2=M+1} \|\tilde{g}_j\|_{M+1,\mathbb{R}^4}\right) |\lambda_l|^{M+1}.
\end{aligned}$$

This characterises the Taylor expansion of $\tilde{g}_{\rho'}(\xi,.)$: for $\xi=(\tilde{\xi},\xi_3)$ first with $\tilde{\xi}\neq 0$, and then for all $\tilde{\xi}$, we have:

$$\begin{aligned} Q_M^{(\tilde{g}_{\rho'})}(\xi,t) &= \sum_{2j_1+j_2=M} t^{2j_1}(t\xi_3)^{j_2} f_j(|\xi|^2,0,0) \\ &= \sum_{2j_1+j_2=M} (\rho_2'(\xi,t))^{j_1}(\rho_3'(\xi,t))^{j_2} f_j(\rho_1'(\xi,t),0,0). \end{aligned}$$

This shows that $\tilde{g}_{\rho'}$ satisfies Property (R). By Proposition 4.2, there exists $f_1\in\mathcal{S}(\mathcal{N})^K$ such that $\operatorname{Rest} f_1=\tilde{g}_{\rho'}$ and $f_1=\operatorname{Radial}\tilde{g}_{\rho'}$. By G. Schwarz's Theorem (see also [2, Theorem 6.1]), there exists $\tilde{f}\in\mathcal{S}(\mathbb{R}^3)$ such that $f_1=\tilde{f}\circ\rho$. We have:

$$\operatorname{Rest} f_1=\tilde{g}_{\rho'}=\tilde{f}\circ\rho_{|\mathcal{N}'}=g\circ\rho'.$$

For any point $s=(\lambda(2l+1)+r^2,\lambda^2,\lambda r)\in\Sigma_\rho$, the point $s'=(s,r^2)\in\Sigma_{\rho'}$ is in $\operatorname{im}\rho'$; it follows that s is in $\operatorname{im}\rho$ and $\tilde{f}(s)=\tilde{g}(s')=g(s')=f(s)$. Thus $\tilde{f}$ is an extension of f.

Continuity. Now that we have shown that the Gelfand transform of a function $F\in\mathcal{S}(N)^K$ admits a Schwartz extension, we still have to prove the continuity of $F\in\mathcal{S}(N)^K\mapsto\mathcal{G}F\in\mathcal{S}(\Sigma_\rho)$. We will use the following two lemmas. The first one states the improvement due to Mather [15] of G. Schwarz's Theorem as well as some straightforward consequences:

Lemma 5.4. — *Let $(\rho_1,\dots,\rho_q)$ be a minimal and homogeneous Hilbert basis for $(\mathbb{R}^p,K)$, and ρ the corresponding Hilbert mapping.*

The induced application $\rho^*:\tilde{h}\mapsto\tilde{h}\circ\rho$ on $\mathcal{S}(\mathbb{R}^q)$ is split-surjective, i.e. it admits a linear continuous right inverse $\sigma:\mathcal{S}(\mathbb{R}^p)^K\to\mathcal{S}(\mathbb{R}^q)$ for ρ^*, that is $\rho^*\circ\sigma$ is the identity mapping of $\mathcal{S}(\mathbb{R}^p)^K$.

We fix such σ. For any $h\in\mathcal{S}(\operatorname{im}\rho)$, the function $h\circ\rho$ is well defined and in $\mathcal{S}(\mathbb{R}^p)^K$, the function $\tilde{h}=\sigma(h\circ\rho)\in\mathcal{S}(\mathbb{R}^q)$ defines a Schwartz extension which we will call the Mather extension of $h\in\mathcal{S}(\operatorname{im}\rho)$. We have: $\tilde{h}\circ\rho=h\circ\rho$. The linear mapping $\tilde{h}\longmapsto\tilde{h}$ of $\mathcal{S}(\mathbb{R}^q)$ is continuous.

It is easy to check that the Hilbert mapping, ρ, of $(\mathcal{N},K)$ is minimal and homogeneous.

The second lemma follows from Rest being a 1-1 continuous mapping, from the Closed Graph Theorem and Lemma 5.4:

Lemma 5.5. — *To any $\tilde{g}\in\mathcal{S}(\mathbb{R}^4)$ such that there exists $\tilde{f}\in\mathcal{S}(\mathbb{R}^3)$ satisfying $\tilde{f}\circ\rho_{|\mathcal{N}'}=\tilde{g}\circ\rho'$, we associate the Mather extension $\tilde{f}_1$ of $\tilde{f}$. The mapping $\tilde{g}\mapsto\tilde{f}_1$ is well-defined, continuous and linear:*

$$\forall a\in\mathbb{N}\quad\exists C>0\quad\exists a'\in\mathbb{N}\quad \left\|\tilde{f}_1\right\|_{a,\mathbb{R}^3}\leqslant C\left\|\tilde{g}\right\|_{a',\mathbb{R}^4} \tag{5.6}$$

Let $a_o \in \mathbb{N}$. Let a_1 corresponding to a' in (5.6) for $a = a_o$.

Let $F \in \mathcal{S}(N)^K$, $G = \mathcal{R}F$, $f = \mathcal{G}F$, $g = \mathcal{G}'G$. By (5.1), there exists $a_2 \in \mathbb{N}$ such that we have indepently of G:

$$\|\tilde{g}\|_{a_1,\mathbb{R}^4} \leqslant C\,\|G\|_{a_2,N}\,.$$

As $\mathcal{R}$ is continuous, there exists $a_3 \in \mathbb{N}$ such that we have indepently of F:

$$\|\mathcal{R}F\|_{a_2,N} \leqslant C\,\|F\|_{a_3,N}\,.$$

Thus we have:

$$\left\|\tilde{f}_1\right\|_{a_o,\mathbb{R}^3} \leqslant C_1\,\|\tilde{g}_o\|_{a_1,\mathbb{R}^4} \leqslant C_2\,\|G\|_{a_2,N'} \leqslant C_3\,\|F\|_{a_3,N}\,.$$

Notice that a_3 and C_3 depend only on a_o, and that $\tilde{f}_1$ depends on F and also on a_o because $\tilde{g}_o$ depends on a_1.

Appendix A.

We adopt again the notation of Section 3 and assume that $(K \ltimes N, K)$ is a Gelfand pair. Here we give the proofs of Propositions 3.1, 3.2 and 3.3.

A.1. Proof of Proposition 3.1

This proof is an easy generalisation of [2, Lemma 5.3] which is a similar result given in the case of the Heisenberg group, using [3].

Let us check that the operators of $\mathbb{D}(N)$ are symmetric. In fact, $C_c^\infty(N)$ is equipped with the Hilbert inner product $\langle F_1, F_2\rangle = \int_N F_1(n)\bar{F}_2(n)dn$. For any $D = D_P \in \mathbb{D}(N)$, $P \in \mathcal{P}(\mathcal{N})$, we have $\langle DF_1, F_2\rangle = \langle F_1, DF_2\rangle$ because:

$$\begin{aligned}\langle DF_1, F_2\rangle &= \left[P(i^{-1}\partial_u)\int_N F_1(n\exp(\sum_{j=1}^p u_jE_j))\bar{F}_2(n)dn\right]_{|u=0}\\ &= \left[P(i^{-1}\partial_u)\int_N F_1(n_1))\bar{F}_2(n_1\exp(-\sum_{j=1}^p u_jE_j)dn\right]_{|u=0} \qquad \text{(A.1)}\end{aligned}$$

after the change of variable $n_1 = n\exp(\sum_{j=1}^p u_jE_j)$.

Let us recall some facts about Gelfand pairs of the form $(K \ltimes N, K)$ [3]. Let $\hat{N}$ be the set of (the classes of) unitary representations on N. For each $\pi \in \hat{N}$, let K_π be the stabilizer of π in K. There exists a decomposition of the Hilbert space $\mathcal{H}_\pi$ into finite-dimensional irreducible subspaces

$\mathcal{H}_{\pi,\alpha}$ under the projective action of K_π on $\mathcal{H}_\pi$. Each bounded K-spherical function ϕ on N is in 1-1 correspondence with π and α, in the sense that $\phi = \phi_{\pi,\alpha}$ can be written as:

$$\phi(n) = \int_K \langle \pi(kn)u, u\rangle dk,$$

where u is any unit vector in $\mathcal{H}_{\pi,\alpha}$ (and dk the Haar probability measure of K). Let $D \in \mathbb{D}(N)^K$. For each $\pi \in \hat{N}$, each subspace $\mathcal{H}_{\pi,\alpha}$ is an eigenspace for the operator $d\pi(D)$ and its eigenvalue is $\mu_{\pi,\alpha,D}$ satisfies: $D\phi_{\pi,\alpha} = \mu_{\pi,\alpha,D}\phi_{\pi,\alpha}$.

Note that the trace $\mathrm{tr}_{\mathcal{H}_\pi}$ of operators on $\mathcal{H}_\pi$ can be computed as the sum over α of traces $\mathrm{tr}_{\mathcal{H}_{\pi,\alpha}}$ of operators on $\mathcal{H}_{\pi,\alpha}$.

We denote by β the Plancherel measure on $\hat{N}$:

$$\|F\|_2^2 = \int_{\hat{N}} \mathrm{tr}_{\mathcal{H}_\pi}\left[\pi(F)\pi(F)^*\right] d\beta(\pi) \quad , \quad F \in C_c^\infty(N).$$

Now let us prove Proposition 3.1.

Let $D \in \mathbb{D}(N)^K$. It is easy to see that there exists a unique self-adjoint extension of D, whose domain is the space of function $F \in L^2(N)$ satisfying:

$$\int_{\hat{N}} \sum_\alpha |\mu_{\pi,\alpha,D}|^2 \mathrm{tr}_{\mathcal{H}_{\pi,\alpha}}\left[\pi(F)\pi(F)^*\right] d\beta(\pi) < \infty.$$

Let us also denote by D the self-adjoint extension. Following [2, Lemma 5.3], we construct a realisation $E = E_D$ of the spectral resolution of D in the following way. Given ω a Borel subset of $\mathbb{R}$, we define the operator $E(\omega)$ on $L^2(N)$ by:

$$\pi(E(\omega)F) = \sum_\alpha \chi_\omega(\mu_{\pi,\alpha,D})\pi(F)\Pi_{\pi,\alpha},$$

where χ_ω is the characteristic function of ω and $\Pi_{\pi,\alpha}$ the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_{\pi,\alpha}$. Then $E = \{E(\omega)\}$ defines a resolution of the identity, and for $F \in \mathcal{S}(N)$,

$$\int_{\mathbb{R}} \xi dE(\xi)F = DF.$$

Therefore $E = E_D$ is the spectral resolution of D.

One readily checks that if $D_1, D_2 \in \mathbb{D}(N)^K$, then for any Borel sets ω_1, ω_2, the operators $E_{D_1}(\omega_1)$ and $E_{D_2}(\omega_2)$ commute.

A.2. Proof of Proposition 3.2

Let us recall the definition of the joint spectrum of a given strongly commuting family of self-adjoint operators $T_1, \dots, T_q$ (densily defined) on a Hilbert space $\mathcal{H}$: it is the set $S_{T_1,\dots,T_q}$ of the q-tuples $\mu=(\mu_1,\dots,\mu_q)\in\mathbb{R}^q$ for which there do not exist bounded operators $U_1,\dots,U_q$ on $\mathcal{H}$ satisfying:

$$\sum_{j=1}^{q}(\mu_j-T_j)U_j=\sum_{j=1}^{q}U_j(\mu_j-T_j)=\mathrm{Id}_{\mathcal{H}}.$$

Let $(\rho_1,\dots,\rho_q)$ be an ordered Hilbert basis for $(\mathcal{N},K)$ and $\mathcal{D}_\rho$ the associated family of strongly commuting self-adjoint operators on $L^2(N)$.

For each $\pi\in\hat{N}$, we decompose its Hilbert space $\mathcal{H}_\pi=\oplus_\alpha^\perp\mathcal{H}_{\pi,\alpha}$ as in the proof of Proposition 3.1 in Section A.1 and we have for $j=1,\dots,q$:

$$d\pi(D_{\rho_j})_{|\mathcal{H}_{\pi,\alpha}}=\mu_j(\phi_{\pi,\alpha})\mathrm{Id}_{\mathcal{H}_{\pi,\alpha}}.$$

This implies the inclusion $\Sigma_{\mathcal{D}_\rho}\subset S_{\mathcal{D}_\rho}$.

For the converse inclusion, we will need the following Lemma, an easy consequence of the Plancherel formula:

Lemma A.1. — *If a function m is continuous and compactly-supported on the Gelfand spectrum, then there exists a K-invariant function $M\in L^2(N)^K$ whose Gelfand transform is m. Furthermore, the convolution operator with kernel M defined on $C_c^\infty(N)$ extends to a bounded operator on $L^2(N)$ with operator norm* $\sup|m|$.

We will also use a dyadic decomposition on $\mathbb{R}^+$: there exists a smooth, non-negative function ψ, supported in the interval $[\frac{1}{2},2]$ and satisfying:

$$\forall x>0\qquad\sum_{a\in\mathbb{Z}}\psi(2^{-a}x)=1.$$

We set $\psi_a(x)=\psi(2^{-a}x)$ if $a\geqslant 1$, and $\psi_o(x)=\sum_{a<1}\psi(2^{-a}x)$.

Let $\mu^o=(\mu_1^o,\dots,\mu_q^o)\in\mathbb{R}^q\backslash\Sigma_{\mathcal{D}_\rho}$. We define the fonctions $m_{a,j}$, $a\geqslant 0$, $j=1,\dots,q$ by:

$$m_{a,j}(\mu)=\frac{\mu_j^o-\mu_j}{\sum_{j'=1}^q{\mu_{j'}^o}^2-\mu_{j'}^2}\psi_a(|\mu|)\quad,\quad\mu\in\Sigma_{\mathcal{D}_\rho}.$$

Each function $m_{a,j}$ is continuous and compactly supported in $\Sigma_{\mathcal{D}_\rho}$ and because μ^o is not in the closed set $\Sigma_{\mathcal{D}\rho}$, there exists a constant $C=C(\mu^o)>0$, independent of a and j, such that:

$$\sup_{\mu\in\Sigma_{\mathcal{D}_\rho}}|m_{a,j}(\mu)|\leqslant C2^{-a}.$$

We denote by $U_{a,j}$ the convolution operator whose kernel admits $m_{a,j}$ as Gelfand transform; by Lemma A.1, this operator is bounded on $L^2(N)$ with norm less than $C2^{-a}$. The operator $\sum_{a\geqslant 0}U_{a,j}$ is thus also a bounded operator on $L^2(N)$, which we denote by U_j. We check that for any representation $\pi \in \hat{N}$, we have on each subspace $\mathcal{H}_{\pi,\alpha}$:

$$\pi(U_j)_{|\mathcal{H}_{\pi,\alpha}} = \frac{\mu_j^o - \mu_j(\phi_{\pi,\alpha})}{\sum_{j'=1}^q {\mu_{j'}^o}^2 - \mu_{j'}(\phi_{\pi,\alpha})^2},$$

from which we deduce:

$$\sum_{j=1}^q \pi(U_j)\left(\mu_j^o - \pi(D_j)\right) = \sum_{j=1}^q \left(\mu_j^o - \pi(D_j)\right)\pi(U_j) = \mathrm{Id}_{\mathcal{H}}.$$

This implies:

$$\sum_{j=1}^q U_j\left(\mu_j^o - D_j\right) = \sum_{j=1}^q \left(\mu_j^o - D_j\right)U_j = \mathrm{Id}_{L^2(N)},$$

that is, μ^o is not in the joint spectrum S_{D_ρ}.

This shows the inclusion $\Sigma_{\mathcal{D}_\rho} \supset S_{\mathcal{D}_\rho}$ and concludes the proof of Proposition 3.2.

A.3. Proof of Proposition 3.3

With the Plancherel formula (see proof of Proposition 3.1 in Subsection A.1), it is easy to see that if $m \in \mathcal{S}(\mathbb{R}^d)$ and if $m(\mathcal{D}_\rho)$ is a convolution operator whose kernel is $M \in \mathcal{S}(N)^K$, then the Gelfand transform of M coincides with m on $\Sigma_{\mathcal{D}_\rho}$.

The proof of the rest of Proposition 3.3 relies mainly on the generalisation [2, Theorem 5.2] of Hulanicki's Schwartz Kernel Theorem.

We will also use the following Lemma which is well-known to specialists (see [11], where the estimate given below in (A.2) is established for general Rockland operators):

Lemma A.2. — *Let N be a graded Lie group, $\mathcal{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \dots \mathcal{V}_l$ its graded Lie algebra, (X_i) a basis of $\mathcal{V}_1$, $L = -\sum X_i^2$ the associated sub-Laplacian.*

For any homogeneous left-invariant differential operator D on N of degree $2d$, there exists a constant $C = C(D) > 0$ such that we have:

$$\forall F \in C_c^\infty(N) \quad \|DF\|_2 \leqslant C\left\|L^d F\right\|_2 . \tag{A.2}$$

Furthermore $\tilde{D} = 2CL^d - D$ is a positive Rockland operator on N.

For the sake of completeness, we give a proof of this Lemma.

Proof. — We refer to [7, ch.6.A] for the definition and the properties of kernels of type $\alpha \in [0, Q[$, where Q is the homogeneous dimension of the group. We will also use the fact that the sub-Laplacian, L, has a fundamental solution, - L being a homogeneous positive Rockland operator of order two - and that the same is true for L^d, $d = 1, 2, \ldots$. By [6], for $2d < Q$, there exists a fundamental solution G_d of L^d such that $G_d \in C^\infty(N \backslash \{0\})$ is homogeneous of degree $2d - Q$.

For any composition of left-invariant vector fields $X^I = X_{i_1} X_{i_2} \cdots X_{i_k}$ with $k < 2d$, it is easy to check that $X^I G_d \in C^\infty(N \backslash \{0\})$ is a homogeneous function H of degree $-Q + 1$, smooth away from the origin. One further differentiation gives a homogeneous distribution of degree $-Q$. Being a derivative, it automatically satisfies the cancellation condition (63) of [17, ch.XIII.5.3]. In fact, let ϕ be a function supported on the unit ball and normalized in the C^1-norm. For any $X \in \mathcal{V}_1$ and $r > 0$,

$$\langle XH, \phi(r\cdot) \rangle = -r \int_N H(x) X\phi(rx)\, dx = -\int_N H(x) X\phi(x)\, dx,$$

which is bounded independently of ϕ and r.

This implies that for every I of length $2d$, the kernel $X^I G_d$ satisfies the L^2-boundedness condition (6.3) of [7, ch.6.A], and thus is of type 0. The operator $X^I L^{-d}$ being $L^2(N)$-bounded, we have:

$$\forall F \in C_c^\infty(N) \quad \left\| X^I F \right\|_2 \leqslant C \left\| L^d F \right\|_2 . \tag{A.3}$$

If $2d \geqslant Q$, L^d does not have a homogeneous fundamental solution, but, according to [9], it has a fundamental solution G_d which is the sum of two terms, one homogeneous of degree $2d - Q$, and the other of the form $P(x) \log |x|$, where P is a polynomial, homogeneous of degree $2d - Q$, and $|x|$ is any smooth homogeneous norm on N. This implies that, if the length k of I satisfies $2d - Q < k < 2d$, then $X^I G_d$ is a homogeneous function of degree $-Q + 2d - k$. We can then repeat the previous argument to conclude that (A.3) holds for every d.

Let D be a homogeneous left-invariant differential operator on N of degree $2d$. As D can be written as a linear combination of monomials X^I, with I of degree $2d$, we see that the property (A.3) implies (A.2). Let $C = C(D)$ be the L^2-operator norm of DL^{-d}. In particular the $L^2(N)$-norm of the operator $D(CL^d)^{-1}$ is one and $I - \frac{1}{2} D(CL^d)^{-1}$ is an invertible operator on $L^2(N)$. The differential operator $\tilde{D} = 2CL^d - D$ is a $2d$-homogeneous, left-invariant, symmetric and positive on $C_c^\infty(N)$. To finish the proof, it remains to prove the defining property of Rockland operators, that is, for any non-trivial, irreducible, unitary representation π of N, $\pi(\tilde{D})$ is injective on smooth vectors; this is true because we can write:

$$\pi(\tilde{D}) = \pi(2CL^d - D) = 2C\pi\left(I - \frac{1}{2}D(CL^d)^{-1}\right)\pi(L)^d,$$

and $I - \frac{1}{2}D(CL^d)^{-1}$ is invertible and L a Rockland operator. □

Before proving Proposition 3.3, let us define some notation. We equip the two-step nilpotent Lie algebra $\mathcal{N}$ with an Euclidean product such that K acts orthogonally. K stabilises the centre $\mathcal{Z}$ of $\mathcal{N}$, and its orthogonal complement $\mathcal{V} = \mathcal{Z}^\perp$. The decomposition $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ endows N with a structure of graded Lie group. $Q = \dim \mathcal{V} + 2\dim \mathcal{Z}$ is the homogeneous dimension of the group. For the symmetrisation mapping, we assume that the basis $(E_i)_{i=1}^p$ is given as a basis $(E_i)_{i=1}^{p_1}$ of $\mathcal{V}$ completed with a basis $(E_i)_{i=p_1}^p$ of $\mathcal{Z}$. As the action of K on $\mathcal{P}(\mathcal{N})$ respects the degree-graduation in both the $\mathcal{Z}$ and $\mathcal{V}$-variables, there exist bi-homogeneous Hilbert basis $\{\rho_1, \dots, \rho_q\}$ in the sense that each polynomial ρ_j is homogeneous in the $\mathcal{Z}$-variables and in the $\mathcal{V}$-variables. For a bi-homogeneous Hilbert basis $\{\rho_1, \dots, \rho_q\}$, we denote by $d_j^{(1)}$ the degree of homogenity of ρ_j in the $\mathcal{V}$-variables, and by $d_j^{(2)}$ the degree of homogenity of ρ_j in the $\mathcal{Z}$-variables; $d_j = d_j^{(1)} + 2d_j^{(2)}$ is the degree of homogeneity of the operator D_{ρ_j} for the structure of graded Lie group of N.

Let us start the proof of Proposition 3.3. We notice that it suffices to show the result for one Hilbert mapping because of the existence of a polynomial mapping between two Hilbert mappings. We choose a bi-homogeneous ordered Hilbert basis $\rho = (\rho_1, \dots, \rho_q)$ with the two following properties. First $\rho_1(\sum_{j=1}^p u_j E_j) = \sum_{j=1}^{p_1} |u_j|^2$. Second, the polynomials $\rho_1, \dots, \rho_{q_1}$ are of even degree of homogeneity in the $\mathcal{V}$-variables and the polynomials $\rho_{q_1+1}, \dots, \rho_q$ are of odd degree of homogeneity in the $\mathcal{V}$-variables.

Let m be in $\mathcal{S}(\mathbb{R}^q)$. S denotes the set of all the sequences $\epsilon : \{q_1 + 1, \dots, q\} \to \{0, 1\}$. Using Whitney's Theorem or G. Schwarz's Theorem, there exists a family of Schwartz functions $(\tilde{m}_\epsilon)_{\epsilon \in S}$, $\tilde{m}_\epsilon \in \mathcal{S}(\mathbb{R}^q)$ satisfying for all $(r_1, \dots, r_q) \in \mathbb{R}^q$:

$$m(r_1, \dots, r_q) = \sum_{\epsilon \in S} r^\epsilon \tilde{m}_\epsilon(r_1, \dots, r_{q_1}, r_{q_1+1}^2, \dots, r_q^2),$$

where we use the notation $r^\epsilon = r_{q_1+1}^{\epsilon(q_1+1)} \dots r_q^{\epsilon(q)}$.

The operator $\tilde{D}_1 = D_{\rho_1}$ is the sub-Laplacian of N which is a positive Rockland operator. By Lemma A.2 there exist constants c_j, $j = 2, \dots, q$ such that

- for $j = 2, \dots, q_1$, the operator $\tilde{D}_j = -D_{\rho_j} + c_j D_{\rho_1}^{\frac{d_j}{2}}$ is a positive Rockland operator on N
- for $j = q_1 + 1, \dots, q$, the operator $\tilde{D}_j = -D_{\rho_j}^2 + c_j D_{\rho_1}^{d_j}$ is a positive Rockland operator on N

For $r = (r_1, \dots, r_q) \in \mathbb{R}^q$, we set $[A(r)]_1 = r_1$ and:

$$\begin{array}{rcll} [A(r)]_j & = & -r_j + c_j r_1^{d_j/2} & , \quad j = 2, \dots, q_1 \\ [A(r)]_j & = & -r_j + c_j r_1^{d_j} & , \quad j = q_1 + 1, \dots, q \end{array} .$$

This defines an application $A : \mathbb{R}^q \to \mathbb{R}^q$ which is a C^∞-diffeomorphism of $\mathbb{R}^q$ and whose Jacobian equals $(-1)^{q-1}$ at any point. Thus if h is in $\mathcal{S}(\mathbb{R}^q)$ then $h \circ A^{-1}$ is in $\mathcal{S}(\mathbb{R}^q)$.

We have:

$$m(D_{\rho_1}, \dots, D_{\rho_q}) = \sum_{\epsilon \in S} D_{\rho'}^{\epsilon} \tilde{m}_\epsilon(D_{\rho_1}, \dots, D_{\rho_{q_1}}, D_{\rho_{q_1+1}}^2, \dots, D_{\rho_q}^2) \tag{A.4}$$

(using the notation $D_{\rho'}^{\epsilon} = D_{\rho_{q_1+1}}^{\epsilon(q_1+1)} \dots D_{\rho_q}^{\epsilon(q)}$) and:

$$\tilde{m}_\epsilon(D_{\rho_1}, \dots, D_{\rho_{q_1}}, D_{\rho_{q_1+1}}^2, \dots, D_{\rho_q}^2) = \tilde{m}_\epsilon \circ A^{-1}(\tilde{D}_1, \dots, \tilde{D}_q) \tag{A.5}$$

Each operator given by (A.5) is a Schwartz multiplier $\tilde{m}_\epsilon \circ A^{-1} \in \mathcal{S}(\mathbb{R}^q)$ of a strongly commutative family of positive Rockland operators $\tilde{D}_j$, $j = 1, \dots q$. By [2, Theorem 5.2], it is a convolution operator with a Schwartz kernel $M_{\tilde{m}_\epsilon \circ A^{-1}, (\tilde{D}_j)}$. Because of the expression (A.4), we deduce that the operator $m(D_{\rho_1}, \dots, D_{\rho_q})$ is also a convolution operator with a Schwartz kernel $M_{m, \mathcal{D}_\rho} = \sum_{\epsilon \in S} D_{\rho'}^{\epsilon} M_{\tilde{m}_\epsilon \circ A^{-1}, (\tilde{D}_j)}$.

The continuity of $m \in \mathcal{S}(\mathbb{R}^q) \mapsto M_{m, \mathcal{D}_\rho} \in \mathcal{S}(N)^K$ is a direct consequence of the following facts:

- by Schwarz-Mather's Theorem, the mappings $m \in \mathcal{S}(\mathbb{R}^q) \mapsto m_\epsilon \in \mathcal{S}(\mathbb{R}^{q_1} \times [0, \infty[^{q-q_1})$, $\epsilon \in S$, are continuous
- the application A being a C^∞-diffeomorphism of $\mathbb{R}^q$ with $(-1)^{q-1}$ as jacobian, the mapping $A^{-1^*} : h \in \mathcal{S}(\mathbb{R}^q) \mapsto h \circ A^{-1} \in \mathcal{S}(\mathbb{R}^q)$ is continuous
- by [2, Theorem 5.2], the application that maps $m \in \mathcal{S}(\mathbb{R}^q)$ to the kernel $M_{m, (\tilde{D}_j)}$ of the operator $m(\tilde{D}_1, \dots, \tilde{D}_q)$ is continuous

The proof of Proposition 3.3 is thus complete.

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