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Fatou-Naïm-Doob limit theorems in the axiomatic system of Brelot


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1. Introduction.

Let $\Omega$ be a locally compact Hausdorff space which is connected and has a countable base. Let $\mathcal{H}$ be a class of real valued continuous functions, called harmonic functions, on open subsets of $\Omega$ such that for each open set $W \subset \Omega$, the set $\mathcal{H}_W$, consisting of all functions in $\mathcal{H}$ defined on $W$, is a real vector space. Let this class $\mathcal{H}$ satisfy the axioms 1, 2 and 3 of M. Brelot [1]. Let, moreover, there exist a potential $> 0$ on $\Omega$.

The classical Fatou-Naim-Doob limit theorems were extended to the axiomatic system of M. Brelot in [2]. But, besides the above mentioned axioms, we had assumed the validity of axioms D and $\mathcal{R}_a$ [2]. The object of this paper is to show that the Fatou-Naim-Doob limit theorems (cf. Theorem 8) hold good in the axiomatic set up without these supplementary axioms (viz. D and $\mathcal{R}_a$). The method consists in proving first, the limit theorems for a special class of superharmonic functions (cf. Theorem 4), and using it systematically to prove the general result. A novel feature in our proof is the consideration of a modified Dirichlet problem. We shall mostly follow the notation of [1, 2, 3].

Notation.

$S^+$ : The set of all non-negative superharmonic functions on $\Omega$.

$H^+$ : The set of all non-negative harmonic functions on $\Omega$.

$\Lambda$ : A compact base for $S^+$ (compact in the $T$-topology [3]).

$\Delta_1$ : The set of minimal harmonic functions contained in $\Lambda$. 

For any $E \subset \Omega$ and $\nu \in S^+$,

$$R^E_\nu = \inf \{ \omega : \omega \in S^+ \text{ and } \omega \geq \nu \text{ on } E \}. $$

For any $h \in \Delta_1$, $\mathcal{I}_h = \{ E \subset \Omega : R^E_h \equiv h \}$ [2]. A set $E$ is thin at $h \in \Delta_1$ if $R^E_h \equiv h$ (i.e. if $\mathcal{I}_h$ leaves no trace on $E$).

The limits of any function $f$ following $\mathcal{I}_h$, for any $h \in \Delta_1$, are called the fine limits of $f(x)$, as $x$ tends to $h$. To every harmonic function $\omega \in H^+$ corresponds a unique measure $\mu_\omega$ on $\Delta_1$, called the canonical measure corresponding to $\omega$, such that $\omega = \int h \mu_\omega(dh)$. For any regular domain $\delta \subset \Omega$, and $x \in \delta$, $d\mu_\omega$ is the measure on $\delta$ which associates to a finite continuous function $f$ on $\delta$ the integral $H_f(x)$. For the considerations below, let us fix a $u \in H^+$ with $u > 0$. Let $\mu_u$ be the canonical measure on $\Delta_1$, corresponding to $u$. A function $\nu$ on $\Omega$ is said to be super-$u$-harmonic (respectively-$u$-harmonic) if $u\nu$ is superharmonic (resp. harmonic) on $\Omega$.

### 2. Fine limits of bounded $u$-harmonic functions.

**Lemma 1.** — Let $V \subset \Omega$ be an open set. Then, for every $x \in \Omega$, the mapping $h \mapsto R^V_h(x)$ of $H^+ \to R^+$ is lower semi-continuous.

**Proof.** — Let $h_n \in H^+$ converge to $h \in H^+$. Let $\nu_n = R^V_{h_n}$. Then, $\nu_n$ is a non-negative superharmonic function on $\Omega$ and $\nu_n = h_n$ on $V$. Let $\nu = \liminf \nu_n$. Let $\omega$ be a regular domain of $\Omega$. Then,

$$\nu_n(y) \geq \int \nu_n(z)\rho^\omega_y(dz) \quad \text{for all } y \in \omega. $$

Hence,

$$\nu(y) = \liminf \nu_n(y) \geq \liminf \int \nu_n(z)\rho^\omega_y(dz) \geq \int \nu(z)\rho^\omega_y(dz) \quad \text{(Fatou's Lemma).}$$

(Note here that $\nu$ is a $\rho^\omega_y$-measurable function.) Since $\nu$ is also non-negative, it follows that $\nu$ is an $S_\mathcal{B}$-function, where $\mathcal{B}$ is the class of all regular domains of $\Omega$ [1]. Hence, $\nu$, the lower semi-continuous regularisation of $\nu$, is a superharmonic function. But $\nu(y) = h(y)$, for all $y \in V$, and hence $\nu = h$ on $V$.  

It follows that \( \varphi > \varphi \geq R^v_h \) on \( \Omega \). This gives the required lower semi-continuity.

**Corollary.** — For any regular domain \( \delta \) of \( \Omega \) and all \( x \in \delta \), the function \( h \rightarrow \int R^v_h(x) \varphi^v(\delta^v) \) is lower semi-continuous on \( H^+ \).

The corollary follows from the lemma by the use of Fatou’s lemma.

**Lemma 2.** — The set \( \varepsilon_v \) of points of \( \Delta_1 \), where an open set \( V \subset \Omega \) is thin, is a borel subset of \( \Delta_1 \).

**Proof.** — Let \( \{ \varepsilon_n \} \) be a countable covering of \( \Omega \) by regular domains. Let, for each \( n \), \( x_n \in \varepsilon_n \). Define,

\[
F_n = \{ h \in \Delta \cap H^+: \int R^v_h(y) \varphi^v(\delta^v) < h(x_n) \}.
\]

In view of the above lemma and its corollary, \( F_n \) is a borel subset of \( \Delta \) (in fact, a \( K^\sigma \) — set). Hence, \( F_n = F_n \cap \Delta_1 \) is a borel subset of \( \Delta_1 \). It can be proved as in [2], that \( \bigcup F_n \) is precisely the set \( \delta_v \). The lemma is proved.

**Theorem 1.** — Let \( V \subset \Omega \) be any open set. Then \( R^v \equiv u \) if and only if \( \mu_v(\varepsilon_v) = 0 \).

**Proof.** — Let \( \mu_v(\varepsilon_v) = 0 \). For any \( x \in \Omega \), we have,

\[
R^v(x) = \int R^v_h(x) \mu_v(dh) \quad (\text{Th. 22.3, [3]}).
\]

Since \( R^v_h(x) = h(x) \), for all \( h \in \Delta_1 - \varepsilon_v \), and \( \mu_v(\varepsilon_v) = 0 \), we get,

\[
R^v(x) = \int h(x) \mu_v(dh) = u(x).
\]

This is true whenever \( x \in \Omega \).

Conversely, suppose that \( R^v \equiv u \). Let \( \{ \varepsilon_n \} \) be a sequence covering \( \Omega \), each \( \varepsilon_n \) being a regular domain, and consider the sets \( F_n \subset \Delta_1 \), as defined in the above lemma.

Let \( \nu_k \) be the swept-out measure corresponding to the measure \( d\varphi^v_k \) relative to the sweeping out on \( V \) (Th. 10. 1, [3]). (Note that \( d\varphi^v_k \) is with the compact support \( \varepsilon_k \)). The measure \( \nu_k \) is such that, for any \( \nu \in S^+ \),

\[
\int \nu(y) \varphi_k(dy) = \int R^v(x) \varphi^v_k(dy).
\]
We have,
\[
\int R^u_y(y) \varphi^\delta_{\mathcal{A}}(dy) = \int u(z) \nu_u(dz) = \int \nu_k(dz) \int h(z) \mu_u(dh) \\
= \int \mu_u(dh) \int h(z) \nu_k(dz) = \int \mu_u(dh) \int R^u_h(y) \varphi^\delta_{\mathcal{A}}(dy) 
\]
(Lebesgue-Fubini Theorem).

Now,
\[
\int h(x_k) \mu_u(dh) = u(x_k) = \int R^u_y(y) \varphi^\delta_{\mathcal{A}}(dy) \quad \text{(hypothesis)} \\
= \int \mu_u(dh) \int R^u_h(y) \varphi^\delta_{\mathcal{A}}(dy) \quad \text{(from (1)).}
\]

It follows that,
\[
\int \left[ h(x_k) - \int R^u_h(y) \varphi^\delta_{\mathcal{A}}(dy) \right] \mu_u(dh) = 0 \quad \text{(2)}.
\]

Since the integrand in the above equation is always \( \geq 0 \), we get, \( h(x_k) = \int R^u_h(y) \varphi^\delta_{\mathcal{A}}(dy) \), for all \( h \in \Delta_1 \), except for a set of \( \mu_u \)-measure zero. But the exceptional set where the inequality does not hold good is precisely \( F^u \). Hence,
\[
\mu_u(F^u) = 0.
\]

It follows, from the above lemma, that \( \mu_u(\mathcal{A}_v) = 0 \). The theorem is proved.

**Corollary.** — The greatest harmonic minorant of \( R^u_Y \) is the function \( \int h \mu^u_Y(dh) \) where \( \mu^u_Y \) is the restriction of \( \mu_u \) to \( \Delta_1 - \mathcal{A}_v \). Hence, \( R^u_Y \) is a potential if \( Y \) is thin \( \mu_u \)-almost everywhere on \( \Delta_1 \).

The proof of the corollary is exactly as in (Cor. Th. II. 2, [2]).

**Theorem 2.** — Let \( \omega > 0 \) be a potential on \( \Omega \). Then \( \frac{\omega}{u} \) has the fine limit zero, at \( \mu_u \)-almost every element of \( \Delta_1 \).

**Proof.** — It is enough to show that, for every rational number \( r > 0 \), the set \( \mathcal{V}_r = \left\{ x \in \Omega : \frac{\omega(x)}{u(x)} > r \right\} \) is thin \( \mu_u \)-almost everywhere. But, since \( \frac{\omega}{u} \) is a lower semi-continuous function, \( \mathcal{V}_r \) is an open subset of \( \Omega \). Further, \( R^u_{\mathcal{V}_r} \leq \frac{\omega}{r} \) Hence \( R^u_{\mathcal{V}_r} \) is a potential and it follows (Cor. to Theorem 1) that \( \mathcal{V}_r \) is
thin at $\mu_\ast$-almost every element of $\Delta_1$. This is true for every $r > 0$. The proof is completed easily.

The following result is an important corollary of the above theorem.

**Theorem 3.** — Let $\nu$ and $\omega$ be two non-negative harmonic functions on $\Omega$ such that their canonical measures $\mu_\nu$ and $\mu_\omega$ on $\Delta_1$ are singular relative to each other. If, $\nu > 0$ on $\Omega$, then, $\frac{\omega}{\nu}$ has the fine limit zero, at $\mu_\nu$-almost every element of $\Delta_1$.

*Proof.* — Let $\nu' = \text{Inf} (\nu, \omega)$. Then it is clear that $\nu'$ is a potential on $\Omega$. By the above theorem, we can find a set $E \subset \Delta_1$ of $\mu_\ast$-measure zero such that, for every $h \in \Delta_1 - E$, 

\[
\mathrm{fine \; lim} \frac{\nu'(x)}{\nu(x)} = 0.
\]

From this we easily deduce that, the fine limit

\[
\mathrm{fine \; lim} \frac{\omega(x)}{\nu(x)} = 0,
\]

for every $h \in \Delta_1 - E$. This completes the proof.

**Theorem 4.** — Let $w$ be a bounded $u$-harmonic function on $\Omega$. Then, $w$ has a fine limit at $\mu_\ast$-almost every element of $\Delta_1$.

*Proof.* — Define, for a $\mu_\ast$-summable function $f$ on $\Delta_1$,

\[
\sigma_f = \int f(h) \frac{h}{u} \mu_\ast (dh).
\]

For the characteristic function $\chi_E$ of a $\mu_\ast$-measurable set $E \subset \Delta_1$, let us denote by $\sigma_E$ the function $\sigma_{\chi_E}$ and $\sigma''_E$ the function $\sigma_{\chi''_E}$. $\sigma_f$ is a $u$-harmonic function, for every such $f$. Now, for a $\mu_\ast$-measurable set $E \subset \Delta_1$, since $\sigma_E \leq 1$ on $\Omega$,

\[
\mathrm{fine \; lim \; sup} \sigma_E(x) \leq 1 \quad \text{for all } h \in \Delta_1.
\]

If either $\mu_\ast (E)$ or $\mu_\ast (\bigcup E)$ is zero, then $\sigma_E' = 1$ (or respectively $\sigma_E = 1$), and the fine limits of $\sigma_E$ and $\sigma''_E$, exist at all points of $\Delta_1$. On the other hand, suppose $\mu_\ast (E) \neq 0$ and also $\mu_\ast (\bigcup E) \neq 0$. 

Then, \( u\sigma_E \) and \( u\sigma'_E \) are two harmonic functions \( > 0 \) on \( \Omega \) and their canonical measures on \( \Delta_1 \) (viz. \( \mu_u \) restricted to \( E \) and \( \int E \)) are singular relative to each other. Hence, by the Theorem 3, \( \sigma_E/\sigma'_E \) has the fine limit zero at \( \mu_u \)-almost every element of \( \Delta_1 - E \). It follows then that,

\[
\text{fine lim sup}_{x \to h} \sigma_E(x) \leq \text{fine lim}_{x \to h} \frac{\sigma_E(x)}{\sigma'_E(x)} = 0
\]

for \( \mu_u \)-almost every element of \( \Delta_1 - E \), as \( \sigma'_E(x) \leq 1 \). Hence,

\[
\text{fine lim sup}_{x \to h} \sigma_E(x) \leq \chi_E(h)
\]

for \( \mu_u \)-almost every \( h \in \Delta_1 \) ... (3).

In particular, the inequality (3) is valid for the complement of \( E \) and we deduce that,

\[
\text{fine lim inf}_{x \to h} \sigma_E(x) \geq \chi_E(h) \quad \text{for} \quad \mu_u \text{-almost every } h \in \Delta_1.
\]

In any case we get, for the characteristic function \( \chi_E \) of a \( \mu_u \)-measurable set \( E \) contained in \( \Delta_1 \),

\[
\text{fine lim sup} \sigma_E(x) = \chi_E(h) \quad \text{for} \quad \mu_u \text{-almost every } h \in \Delta_1 \ldots (4).
\]

Suppose, now, \( f \geq 0 \) is a \( \mu_u \)-measurable function on \( \Delta_1 \). Then, there exists an increasing sequence of non-negative simple functions \( s_n \) such that \( \lim_{n \to \infty} s_n = f \). We deduce easily from (4) that

\[
\text{fine lim sup} \sigma_n(x) = s_n(h) \quad \text{for} \quad \mu_u \text{-almost every } h \in \Delta_1.
\]

Hence, \( \sigma_f \) satisfies,

\[
\text{fine lim inf} \sigma_f(x) \geq s_n(h) \quad \text{for} \quad \mu_u \text{-almost every } h \in \Delta_1.
\]

Now, it is easily seen that,

\[
\text{fine lim inf} \sigma_f(x) \geq f(h) \quad \text{for} \quad \mu_u \text{-almost every } h \in \Delta_1 \ldots (5).
\]

Let us now consider a bounded \( \mu_u \)-measurable function \( g \) on \( \Delta_1 \) (say \( |g| \leq M \)). Then, applying the inequality (5) to the
two functions $\sigma_{M \pm g}$, and noting that, $\sigma_{M \pm g} = M \pm \sigma_g$, we get that

$$\text{fine lim}_{x \to h} \sigma_g(x) = g(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1.$$ 

Now, the proof of the theorem is completed by noting that any bounded $u$-harmonic function $w$ is equal to $w \sigma_g$, for some bounded $\mu_u$-measurable function $g$ on $\Delta_1$; this $g$ is unique (depending on $w$) up to a set of $\mu_u$-measure zero.

**Remark 1.** — In the course of the proof of the theorem, we have shown that, for any $f \geq 0$, which is $\mu_u$-measurable,

$$\text{fine lim inf}_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) \geq f(h_0),$$

for $\mu_u$-almost every $h_0 \in \Delta_1$ (viz. the inequality (5)).

**Remark 2.** — For any bounded $u$-harmonic function $w$ on $\Omega$, if $g(h) = \text{fine lim}_{x \to h} w(x)$, (the function $g$ is defined up to a set of $\mu_u$-measure zero), then $g$ is $\mu_u$-measurable and

$$w(x) = \int g(h) \frac{h(x)}{u(x)} \mu_u(dh).$$

In particular, if the fine limit is $\geq 0$ for $\mu_u$-almost every element of $\Delta_1$, then $w$ is non-negative.

**Remark 3.** — For any bounded super-$u$-harmonic function $\nu$ on $\Omega$, the fine lim $\nu(x)$ exists for $\mu_u$-almost every $h \in \Delta_1$.

**Theorem 5.** — *(The Minimum Principle).* Let $\nu$ be a lower bounded super-$u$-harmonic function on $\Omega$. Suppose that, for every $h \in \Delta_1 - E$, fine lim sup $\nu(x) \geq 0$, where $E$ is a set with $\mu_u^*(E) = 0$. Then, $\nu$ is $\geq 0$ on $\Omega$.

**Proof.** — Let $\alpha > 0$ be such that $\nu \geq - \alpha$. Consider

$$\nu' = \text{Inf}(\nu, 1).$$

Then $\nu'$ is a super-$u$-harmonic function such that $\nu' \geq - \alpha$. The theorem would be proved if we show that $\nu' \geq 0$ on $\Omega$. 
Now, it is easily seen that fine lim sup $\nu'(x) \geq 0$, for all $h \in \Delta_1 - E$. But, we know, (by the Remark 3 following the Theorem 4) that, the limit of $\nu'$ exists, following $\mathcal{F}_h$, for $\mu_a$-almost every $h \in \Delta_1$; and this fine limit is precisely the fine limit of $u_1$, where $u_1$ is the greatest $u$-harmonic minorant of $\nu'$. Hence, we have that the fine limit of $u'$ is $\geq 0$ at $\mu_a$-almost every element of $\Delta_1$. It follows that $u_1 \geq 0$ (from the Remark 2, Theorem 4). A fortiori, $\nu' \geq 0$. This completes the proof of the theorem.

3. A Dirichlet problem.

Let $\Sigma$ be the set of all lower bounded super-$u$-harmonic functions on $\Omega$. Corresponding to any extended real valued function $f$ on $\Delta_1$, define,

$$
\Sigma_f = \{ \nu \in \Sigma : \exists \text{ a set } E_\nu \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for all } h \in \Delta_1 - E_\nu, \text{ fine lim inf } \nu(x) \geq f(h) \}
$$

$$
\bar{\Sigma}_f = \{ \nu \in \Sigma : \exists \text{ a set } F_\nu \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for all } h \in \Delta_1 - F_\nu, \text{ fine lim sup } \nu(x) \geq f(h) \}
$$

**Definition.** — Corresponding to any extended real valued function $f$ on $\Delta_1$, define, for all $x \in \Omega$,

$$
\mathcal{H}_{f,u}(x) = \text{Inf}\{ \nu(x) : \nu \in \Sigma_f \}
$$

$$
\mathcal{K}_{f,u}(x) = - \mathcal{H}_{-f,u}(x)
$$

and

$$
\mathcal{D}_{f,u}(x) = \text{Inf}\{ \nu(x) : \nu \in \bar{\Sigma}_f \}.
$$

It is easy to see that $\Sigma_f$ is a saturated family of super-$u$-harmonic functions [1]. Hence $\mathcal{H}_{f,u}$ is either identically $\pm \infty$ or it is a $u$-harmonic function. Moreover, from the minimum principle, we deduce that $\mathcal{H}_{f,u} \geq \mathcal{K}_{f,u}$ on $\Omega$.

Also $\mathcal{H}_{f,u} \geq \mathcal{D}_{f,u}$.

**Definition 2.** — Let $u(\mathcal{R})$ be the class of extended real valued functions $f$ on $\Delta_1$ such that, $\mathcal{H}_{f,u} = \mathcal{K}_{f,u}$ and this function $u$-harmonic on $\Omega$. For functions $f \in u(\mathcal{R})$, we denote $\mathcal{H}_{f,u} = \mathcal{K}_{f,u} = \mathcal{D}_{f,u}$.
**Lemma 3.** — *Every bounded* \( \mu_u \)-measurable function \( f \) on \( \Delta_1 \) belongs to \( u(3\mathbb{R}) \) and moreover

\[
\mathcal{H}_{f,u} = \int f(h) \frac{h}{u} \mu_u(dh).
\]

**Proof.** — The \( u \)-harmonic function \( \sigma_f = \int f(h) \frac{h}{u} \mu_u(dh) \) satisfies,

\[
\text{fine lim}_{x \to h} \sigma_f(x) = f(h) \quad \text{for } \mu_u \text{-almost every } h \in \Delta_1
\]

(Theorem 4). Hence, \( \mathcal{H}_{f,u} \leq \sigma_f \leq \mathcal{H}_{f,u} \). This completes the proof.

**Proposition 1.** — Let \( \{f_n\} \) be an increasing sequence of extended real functions such that \( \mathcal{H}_{f_n,u} \geq -\infty \). Then,

\[
\lim_{n \to \infty} \mathcal{H}_{f_n,u} = \mathcal{H}_{f,u}.
\]

**Proof.** — Since \( \mathcal{H}_{f_n,u} \leq \mathcal{H}_{f,u} \), for every \( n \), it is enough to show that \( \mathcal{H}_{f,n} \leq \lim_{n \to \infty} \mathcal{H}_{f_n,u} \) when the limit is not \( +\infty \).

Let \( x_0 \in \Omega \). Given \( \varepsilon > 0 \), choose for every \( n \), an element \( \nu_n \in \Sigma_{f_n} \) such that

\[
\mathcal{H}_{f_n,u}(x_0) \geq \nu_n(x_0) - \frac{\varepsilon}{2^n}.
\]

Consider \( \omega = \lim_{n \to \infty} \mathcal{H}_{f_n,u} + \sum_{n=1}^{\infty} (\nu_n - \mathcal{H}_{f_n,u}) \). It is easily seen that \( \omega \) is a super-\( u \)-harmonic function. Moreover \( \omega \geq \nu_n \) for every \( n \). Hence \( \omega \) is lower bounded on \( \Omega \). Also, if \( E_{v_n} \) is the set contained in \( \Delta_1 \) such that \( \mu_u(E_{v_n}) = 0 \) and for all \( h \in \Delta_1 - E_{v_n} \), fine \( \text{lim inf}_{x \to h} \nu_n(x) \geq f_n(h) \), then,

\[
\text{fine lim inf}_{x \to h} \omega(x) \geq f(h),
\]

for all \( h \in \Delta_1 - \bigcup_{n=1}^{\infty} E_{v_n} \). It follows that \( \omega \in \Sigma_{\mathcal{H}_{f,u}} \). Hence \( \omega \geq \mathcal{H}_{f,u} \).

But,

\[
\mathcal{H}_{f,u}(x_0) \leq \omega(x_0) \leq \lim \mathcal{H}_{f_n,u}(x_0) + \varepsilon.
\]

The proof is now completed easily.

The following proposition is proved easily.
PROPOSITION 2. — \( u(\mathbb{R}) \) is a real vector space. Moreover, for \( f, g \in u(\mathbb{R}) \), \( \mathcal{H}_{f+g} = \mathcal{H}_f + \mathcal{H}_g \).

LEMMA 4. — For any non-negative extended real valued function \( f \) on \( \Delta_1 \), \( \mathcal{H}_{f} = 0 \) is equivalent to the fact that \( f = 0 \) \( \mu_{\ast} \)-almost everywhere.

Proof. — Suppose \( f = 0 \) except on a set of \( \mu_{\ast} \)-measure zero.
Let \( \nu \in \Sigma_f \). Then clearly \( \frac{1}{n} \nu \in \Sigma_f \), for all positive integers \( n \).
Hence \( \mathcal{H}_{f} = 0 \).
Conversely, suppose \( \mathcal{H}_{f} = 0 \). Let \( A_n = \left\{ h : f(h) > \frac{1}{n} \right\} \).
Then the characteristic function \( \gamma_{A_n} \) of \( A_n \in \Delta_1 \) has the property that \( \mathcal{H}_{\gamma_{A_n}} = 0 \). The lemma would be proved if we show that for any set \( A \subset \Delta_1 \), \( \mathcal{H}_{\gamma_{A}} = 0 \) implies that \( \mu_{\ast}(A) = 0 \).
Let \( \nu \in \Sigma_{\gamma_{A}} \). That is, there exists a set \( \mathcal{E}_{\nu} \) of \( \mu_{\ast} \)-measure zero such that \( \liminf_{x \to h} \nu(x) \geq \gamma_{A}(h) \), for all \( h \in \Delta_1 - \mathcal{E}_{\nu} \). Given \( \varepsilon > 0 \), let \( V_{\varepsilon} = \{ x \in \Omega : \nu(x) > 1 - \varepsilon \} \). Then, \( V_{\varepsilon} \) is an open set and \( V_{\varepsilon} \) is not thin at any point of \( h \in A - \mathcal{E}_{\nu} \). Now,
\[
\frac{u_{\nu}}{1 - \varepsilon} \geq R^{\Sigma}_{\nu} \geq \int h\chi_{A - \mathcal{E}_{\nu}}(h) \, \mu_{\ast}(dh) = \int h\chi_{A}(h) \, \mu_{\ast}(dh).
\]
This inequality is true for all \( \varepsilon > 0 \). Hence
\[
\nu \geq \int \frac{h}{u} \chi_{A}(h) \, \mu_{\ast}(dh).
\]
In turn, this inequality is true for all \( \nu \in \Sigma_{\chi_{A}} \), and we deduce,
\[
\mathcal{H}_{\gamma_{A}} \geq \frac{1}{u} \int h \chi_{A}(h) \, \mu_{\ast}(dh).
\]
Hence, if \( \mathcal{H}_{\gamma_{A}} = 0 \), then \( \int h\chi_{A}(h) \, \mu_{\ast}(dh) = 0 \). Now, we deduce easily that \( \mu_{\ast}(A) = 0 \). This completes the proof.

THEOREM 6. — Every \( \mu_{\ast} \)-summable function \( f \) on \( \Delta_1 \) belongs to \( u(\mathbb{R}) \) and moreover, \( \mathcal{H}_{f}(x) = \int f(h) \frac{h(x)}{u(x)} \mu_{\ast}(dh) \) on \( \Omega \).

Proof. — Suppose \( f \) is a non-negative \( \mu_{\ast} \)-summable function on \( \Delta_1 \). For each positive integer \( n \), if \( f_n = \inf \langle f, n \rangle \), then
\( f_n \in u(\mathbb{R}) \) and \( \mathcal{H}_{f_n,u} = \int f_n(h) \frac{h}{u} \lambda_u (dh) \). (Lemma 3). Hence, we have,

\[
\mathcal{H}_{f,u} = \lim_{n \to \infty} \mathcal{H}_{f_n,u} \quad \text{(Proposition 1)}
\]

\[
= \lim_{n \to \infty} \int f_n(h) \frac{h}{u} \mu_u(dh)
\]

\[
= \int f(h) \frac{h}{u} \mu_u(dh).
\]

Also,

\[
\int f(h) \frac{h}{u} \mu_u(dh) = \lim \mathcal{H}_{f_n,u} \leq \mathcal{H}_{f,u}.
\]

It follows that

\[ f \in u(\mathbb{R}) \quad \text{and} \quad \mathcal{H}_{f,u} = \int f(h) \frac{h}{u} \mu_u(dh). \]

Now the proof is completed easily.

**Remark.** — It can be proved that any function \( f \in u(\mathbb{R}) \) is necessarily equal \( \mu_u \)-almost everywhere to a \( \mu_u \)-summable function and that \( \mathcal{H}_{f,u} \) is precisely \( \int f(h) \frac{h}{u} \mu_u(dh) \).

### 4. The Main Result.

**Theorem 7.** — *Let \( f \geq 0 \) be an extended real valued function on \( \Delta_1 \). Then, \( \overline{D}_{f,u} = \mathcal{H}_{f,u} \).*

**Proof.** — It is enough to show that \( \overline{D}_{f,u} \geq \mathcal{H}_{f,u} \).

First of all consider a function \( f \geq 0 \) which is bounded, say \( f \leq M \). Consider \( \overline{\Sigma}_f^M = \{ \nu \in \overline{\Sigma}_f : \nu \leq M \} \). We assert that

\[
\overline{D}_{f,u} = \inf \{ \nu : \nu \in \overline{\Sigma}_f^M \}. \text{ For, suppose } \nu \in \overline{\Sigma}_f. \text{ Then } \nu_M = \inf (\nu, M) \text{ is a super-\( u \)-harmonic function and satisfies}
\]

\[
\lim\inf_{x \searrow h} \nu_M(x) \geq f(h),
\]

for \( \mu_u \)-almost every \( h \in \Delta_1 \). Hence, \( \nu \geq \nu_M \geq \inf \{ \nu : \nu \in \overline{\Sigma}_f^M \} \). Hence \( \overline{D}_{f,u} \geq \inf \{ \nu : \nu \in \overline{\Sigma}_f^M \} \). The opposite inequality is obvious.

Now, let \( \nu \in \overline{\Sigma}_f^M \). Then, by Theorem 4, Remark 3, the
fine limit $\nu(x)$ exists for all $h \in \Delta_1 - E_\nu$, where $\mu_u(E_\nu) = 0$.
But, by the defining property of $\nu \in \tilde{\Sigma}_f$, fine lim sup $\nu(x) \geq f(h)$
for all $h \in \Delta_1 - F_\nu$, where $\mu_u(F_\nu) = 0$. It follows that,
\[
\text{fine lim inf } \nu(x) \geq f(h),
\]
for all $h \in \Delta_1 - (E_\nu \cup F_\nu)$. Hence, $\nu \geq \mathcal{H}_{f_u}$. This is true for
all $\nu \in \tilde{\Sigma}_f$ and we get that $\overline{D}_{f,u} \geq \mathcal{H}_{f_u}$.

Let us now consider any $f \geq 0$. Let, for every positive
integer $n$, $f_n = \inf (f, n)$. Then, we have,
\[
\overline{D}_{f,u} \geq \lim \overline{D}_{f_n,u} = \lim \mathcal{H}_{f_n,u} = \mathcal{H}_{f,u}.
\]
This completes the proof of the theorem.

**Theorem 7.** — For every $\mu_u$-summable function $f$ on $\Delta_1$,
\[
\text{fine lim sup } \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0),
\]
for $\mu_u$ almost every $h_0 \in \Delta_1$.

**Proof.** — It is enough to prove the theorem assuming that $f \geq 0$. Define, for every $h_0 \in \Delta_1$,
\[
\phi'(h_0) = \text{fine lim sup } \int f(h) \frac{h(x)}{u(x)} \mu_u(dh).
\]
Let $\phi = \sup (\phi', f)$ and $\nu \in \tilde{\Sigma}_f$. Then, $\nu \geq \int f(h) \frac{h}{u} \mu_u(dh)$
and we see easily that the fine lim sup $\nu(x) \geq \phi(h)$, for $\mu_u$-almost
every $h \in \Delta_1$. It follows that $\nu \in \tilde{\Sigma}_\phi$. This is true for all $\nu \in \tilde{\Sigma}_f$.
Hence, $\mathcal{H}_{f_1,u} \geq \mathcal{H}_{\phi,u}$. But $\overline{D}_{f,u} = \mathcal{H}_{f,u} \leq \mathcal{H}_{\phi,u}$. This implies
that $\phi \in u(\mathcal{R})$ and $\mathcal{H}_{\phi,u} = \mathcal{H}_{f,u}$. Again, $\phi - f \geq 0$ and $\mathcal{H}_{\phi-f,u} = 0$.
We get, from the Lemma 4, that, $\phi = f$, $\mu_u$-almost everywhere.
Hence,
\[
\text{fine lim sup } \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) \leq f(h_0)
\]
for $\mu_u$-almost every $h_0 \in \Delta_1$. But we have already proved that
the fine lim inf is $\geq f(h_0)$ for $\mu_u$-almost every $h_0 \in \Delta_1$. Hence, we get,

$$\liminf_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0)$$

for $\mu_u$-almost every $h_0 \in \Delta_1$, completing the proof of the theorem.

**Theorem 8.** (Fatou-Naïm-Doob). For any $\nu \in S^+$, $\frac{\nu}{u}$ has a finite limit at $\mu_u$-almost every element of $\Delta_1$.

**Proof.** — Let $\nu$ be the canonical measure on $\Delta_1$ corresponding to the greatest harmonic minorant of $\nu$. Let $\nu_1$ (respectively $\nu_2$) be the absolutely continuous (resp. singular) part of $\nu$ relative to $\mu_u$. Let $f$ be the Radon-Nikodym derivative of $\nu_1$ relative to $\mu_u$ ($f$ is defined upto a set of $\mu_u$ measure zero). Then

$$\nu = \nu_1 + \nu_2 + \nu_3$$

where $\nu_3$ is a potential, $\nu_2 = \int h \nu_2 (dh)$ and $\nu_1 = \int f(h) h \mu_u(dh)$. Now, $\frac{\nu_1}{u}$ has the fine limit $f$ (note that $f$ is finite $\mu_u$ almost everywhere), for $\mu_u$-almost every element of $\Delta_1$. Also, $\frac{\nu_2 + \nu_3}{u}$ has the fine limit zero at $\mu_u$-almost every element of $\Delta_1$. This completes the proof of the theorem.

**BIBLIOGRAPHY**


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