



ANNALES

DE

L'INSTITUT FOURIER

Laurent DUCROHET

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Tome 59, n° 4 (2009), p. 1641-1669.

http://aif.cedram.org/item?id=AIF_2009__59_4_1641_0

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THE FROBENIUS ACTION ON RANK 2 VECTOR BUNDLES OVER CURVES IN SMALL GENUS AND SMALL CHARACTERISTIC

by Laurent DUCROHET

ABSTRACT. — Let X be a general proper and smooth curve of genus 2 (resp. of genus 3) defined over an algebraically closed field of characteristic p . When $3 \leq p \leq 7$, the action of Frobenius on rank 2 semi-stable vector bundles with trivial determinant is completely determined by its restrictions to the 30 lines (resp. the 126 Kummer surfaces) that are invariant under the action of some order 2 line bundle over X . Those lines (resp. those Kummer surfaces) are closely related to the elliptic curves (resp. the abelian surfaces) that appear as the Prym varieties associated to double étale coverings of X . We are therefore able to compute the explicit equations defining Frobenius action in these cases. We perform some of these computations and draw some geometric consequences.

RÉSUMÉ. — Soit X une courbe générale, propre et lisse de genre 2 (resp. de genre 3) définie sur un corps algébriquement clos de caractéristique p . Lorsque $3 \leq p \leq 7$, l'action de Frobenius sur les fibrés vectoriels semi-stable de rang 2 et de déterminant trivial est entièrement déterminée par ses restrictions aux 30 droites (resp. aux 126 surfaces de Kummer) invariantes sous l'action d'un fibré en droites d'ordre 2 sur X . Ces lignes (resp. ces surfaces de Kummer) sont étroitement liées aux courbes elliptiques (resp. aux surfaces abéliennes) qui apparaissent comme variétés de Prym associées aux revêtements étales doubles de X . Nous sommes par conséquent en mesure de calculer les équations explicites définissant l'action de Frobenius dans ces cas. Nous faisons quelques-uns de ces calculs et nous en tirons quelques conséquences géométriques.

1. Introduction

Let k be a algebraically closed field of positive characteristic p and let X be an irreducible, proper and smooth curve of genus g over k . Let X_s ($s \in \mathbb{Z}$ since k is perfect) be the p^s -twist of X and let J (resp. J_s) denote its Jacobian variety (resp. the p^s -twist of its Jacobian variety). Also, let

Keywords: Vector bundles, Frobenius, Prym varieties.

Math. classification: 14H60.

Θ denote a symmetric principal polarization for J (associated to a theta characteristic κ_0). Denote by $S_X(r)$ the (coarse) moduli space of semi-stable rank r vector bundles with trivial determinant over X . The map $E \mapsto F_{\text{abs}}^* E$ defines a rational map $S_X(r) \dashrightarrow S_X(r)$ the k -linear part $V_r : S_{X_1}(r) \dashrightarrow S_X(r)$ of which is called the (generalized) Verschiebung.

Our interest in this situation stems from the fact (see [9]) that a stable rank r vector bundle E over X corresponds to an (irreducible) continuous representation of the algebraic fundamental group $\pi_1(X)$ in $\text{GL}_r(\bar{k})$ (endowed with the discrete topology) if and only if one can find an integer $n > 0$ such that $F_{\text{abs}}^{(n)*} E \cong E$. Thus, natural questions about the generalized Verschiebung $V_r : S_{X_1}(r) \dashrightarrow S_X(r)$ arise like, e.g., its surjectivity, its degree, the density of Frobenius-stable bundles (i.e., those vector bundles whose pull-back by Frobenius iterates are all semi-stable), the loci of Frobenius-destabilized bundles...

For general (g, r, p) , not much seems to be known (see the introductions of [10] and [11] for an overview of this subject).

We will focus on the rank $r = 2$ in low genus ($g = 2$ or 3) and we will let S_X stand for $S_X(2)$. When $k = \mathbb{C}$, Narasimhan and Ramanan have given explicit descriptions for S_X as a subvariety of $|2\Theta|$. Namely, S_X is isomorphic to $|2\Theta|$ in the genus 2 case (see [16]) and S_X identifies with the Coble quartic surface in $|2\Theta|$ in the genus 3 non-hyperelliptic case (see [17]). These descriptions also hold over any algebraically closed field of odd characteristic and V_2 lifts to a rational map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ (see Proposition 3.2 for the genus 3 case) given by a system of $J[2]$ -equivariant homogeneous polynomials of degree p .

In genus 2 and characteristic $p = 3$, Laszlo and Pauly gave in [11] the cubic equations of \tilde{V} in showing that this rational map coincided with the polar map of a quartic surface, isomorphic to Kum_X , embedded in $|2\Theta_1|$. The proof uses the fact that the action of Frobenius is equivariant under the action of $J[2]$ in odd characteristic as well as a striking relationship (see [4]) between cubics and quartics on $|2\Theta_1|$ that are invariant under the action of $J[2]$. For a general odd p , the base locus of \tilde{V} coincides at least set-theoretically with the locus of Frobenius destabilized bundles (i.e., those stable vector bundle E such that F^*E is unstable) and it has been much studied (see [8] and [18]).

In this article, we shall suppose that the characteristic p of the base field is odd. Given a line bundle τ of order 2 over X and a τ -invariant (i.e., satisfying $E \otimes \tau \xrightarrow{\sim} E$) semi-stable degree 0 vector bundle E , one can give E a structure of invertible $\mathcal{O}_X \oplus \tau$ -module. In other words, if $\pi : \tilde{X} \rightarrow X$ is

the degree 2 étale cover corresponding to τ , there is a degree 0 line bundle L over \tilde{X} such that $E \cong \pi_*(L)$. Because π is étale, one has $F_{\text{abs}}^*(\pi_*(L)) \cong \pi_*(F_{\text{abs}}^*(L))$. Furthermore, requiring E to have trivial determinant forces L to be in some translate of the Prym variety P_τ associated to π (which has genus $g-1$). On the one hand, the associated morphism $P_\tau \rightarrow |2\Theta|^\tau$ factors through the Kummer morphism $P_\tau \rightarrow P_\tau/\{\pm\}$ and, on the other hand, as multiplication by p over an abelian variety commutes with the inversion, it induces an endomorphism of $P_\tau/\{\pm\}$. If $g = 2$, the Prym varieties are elliptic curves and there are formulae (see [23]) that allow us to compute explicitly the k -linear part $\tilde{V}_\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of this endomorphism. If $g = 3$, the Prym varieties are Jacobian varieties of genus 2 curves and one can use the genus 2 case results.

Through representations of Heisenberg groups, we prove the key result of this article

THEOREM 1.1. — *Let k be an algebraically closed field of characteristic $p = 3, 5$ or 7 . Let X be a smooth, proper, general curve of genus 2 or 3 over k . There is a rational map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ extending the generalized Verschiebung $V_2 : S_{X_1} \dashrightarrow S_X$ that is completely determined by its restrictions $\tilde{V}_\tau : |2\Theta_1|^{\tau_1} \dashrightarrow |2\Theta|$ to the τ_1 -invariant locus of $|2\Theta_1|$, τ_1 ranging in the non zero elements of $J_1[2]$.*

Therefore, one can explicitly compute the equations of $V_2 : S_{X_1} \dashrightarrow S_X$ and we perform these computations in genus 2 and characteristic 3, 5 and 7, as well as in genus 3 and characteristic 3.

In the genus 2 case, we give the following generalization of the results of [11] in characteristic 3 :

PROPOSITION 1.2. — *Let X be a general, proper and smooth curve of genus 2 over an algebraically closed field of characteristic p . For $p = 3, 5$ or 7 , there is a degree $2p - 2$ irreducible hypersurface H in $|2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$*

$$\tilde{V}^{-1}(\text{Kum}_X) = \text{Kum}_{X_1} + 2H$$

holds scheme-theoretically.

Randomly choosing curves, the following pattern arises, analogous to the characteristic 3 case : The base locus of V is strictly contained in the singular locus of H . The latter has dimension 0, is contained in the stable locus of S_{X_1} , as well as in the inverse image of the singular points of Kum_X which is 1-dimensional. Unfortunately, the Groebner basis computation

required to check this statement for the generic curve seems too heavy and we could not check this result globally.

In the genus 3 case, the results of [11] in characteristic 3 generalize as follows :

THEOREM 1.3. — *Assume that X is a general, smooth and projective curve of genus 3 over an algebraically closed field of characteristic 3.*

There is an embedding $\alpha : \text{Cob}_X \hookrightarrow |2\Theta_1|$ such that the cubic equations of the rational map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ lifting the generalized Verschiebung $V_2 : S_{X_1} \dashrightarrow S_X$ are given by the 8 partial derivatives of the quartic equation of $\alpha(\text{Cob}_X) \subseteq |2\Theta_1|$. In other words, \tilde{V} is the polar map of the hypersurface $\alpha(\text{Cob}_X)$.

In particular, the base locus (equivalently, the locus of Frobenius destabilized bundle) is the intersection $\alpha(\text{Kum}_X) \cap \text{Cob}_{X_1}$.

All the computations have been carried out using MAGMA Computational Algebra System, on the servers MEDICIS hosted at the Ecole Polytechnique.

I would like to thank Y. Laszlo for having introduced me to this question, for his help and encouragements.

2. Vector bundles and Theta group representations

2.1. Action of $J[2]$ on the moduli space S_X

Following [21, 1.8], there is a morphism $D : S_X \rightarrow |2\Theta|$ mapping a (S-equivalence class of) semi-stable rank 2 vector bundle with trivial determinant E to the unique effective divisor in $|2\Theta|$ with support the set

$$(2.1) \quad \text{Supp } D[E] = \{j \in J \mid H^0(X, E \otimes j \otimes \kappa_0) \neq 0\}.$$

We will consider the morphism $b : J \rightarrow S_X$ defined by $j \mapsto [j \oplus j^{-1}]$, the Kummer morphism $K_X : J \rightarrow |2\Theta|$ which maps J onto the Kummer variety $\text{Kum}_X \cong J/\{\pm 1\}$ and the morphism $\varphi_{2\Theta} : J \rightarrow |2\Theta|^*$ associated to $\mathcal{O}(2\Theta)$. Also, following [1, Sect. 2], introduce the subvariety Δ of S_X with support the set

$$\{[E] \in S_X \mid H^0(E \otimes \kappa_0) \neq 0\}.$$

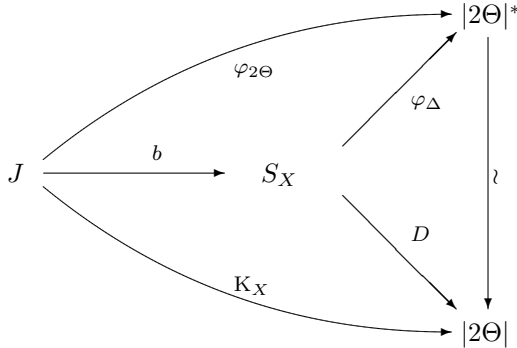
PROPOSITION 2.1.

(1) One has $b^*(\mathcal{O}(\Delta)) \cong \mathcal{O}(2\Theta)$ and the map

$$b^* : H^0(S_X, \mathcal{O}(\Delta)) \rightarrow H^0(X, \mathcal{O}(2\Theta))$$

is an isomorphism.

Identifying $|2\Theta|^*$ and $|\Delta|^*$ via this isomorphism, the morphism $\varphi_\Delta : S_X \rightarrow |2\Theta|^*$ associated to the linear system $|\Delta|$ gives a commutative diagram



(2.2)

(where the vertical arrow is Wirtinger’s isomorphism) [1, Proposition 2.5].

- (2) If $g = 2$, the morphism D is an isomorphism [16].
- (3) If $g = 3$ and X is not hyperelliptic, D is a closed immersion whose image is the Coble quartic surface Cob_X [17].

Define actions of $J[2]$ on S_X and $|2\Theta|$ (hence on $|2\Theta|^*$ by duality) respectively by $(\tau, [E]) \mapsto [E \otimes \tau]$, $(\tau, D) \mapsto T_\tau^* D$ (where T_τ is the translation by τ on the Jacobian J) All the maps in the diagram above are $J[2]$ -equivariant (see [1, Remark 2.6]).

2.2. Theta groups and representations

Let A be any abelian variety over k and L an ample line bundle A . Following [13, Sect. 1], let $\mathcal{G}(L)$ (resp. $K(L)$) be the group scheme (resp. the finite group scheme) such that, for any k -scheme S ,

$$\mathcal{G}(L)(S) = \{(x, \gamma) \mid x \in A(S), \gamma : L \xrightarrow{\sim} T_x^* L\}$$

$$\text{(resp. } K(L)(S) = \{x \in A(S) \mid T_x^* L \cong L\} \text{)}.$$

The commutator in the theta group $\mathcal{G}(L)$ induces a non degenerate skew-symmetric bilinear form denoted by $e^L : K(L) \times K(L) \rightarrow \mathbb{G}_m$. Suppose that $K(L)$ is reduced-reduced, i.e., $K(L)$ is reduced and its Cartier dual is also reduced, then L is said to be of separable type.

The ample line bundle $\mathcal{O}(2\Theta)$ over J is of that kind and we introduce the following notation:

$$W := H^0(J, \mathcal{O}(2\Theta)), \mathcal{G}(2) := \mathcal{G}(\mathcal{O}(2\Theta))$$

$$e_{2,J} := e^{\mathcal{O}(2\Theta)} : J[2] \times J[2] \rightarrow \mu_2 \subset \mathbb{G}_m.$$

The vector space W is the unique (up to isomorphism) irreducible representation of weight 1 of $\mathcal{G}(2)$ [13, Sect. 1]. By duality, there is an action (of weight -1) of $\mathcal{G}(2)$ on W^* .

We write $H = (\mathbb{Z}/2\mathbb{Z})^g$ and $\hat{H} = \text{Hom}((\mathbb{Z}/2\mathbb{Z})^g, k^*)$ (we identify H and \hat{H} by means of the bilinear form $(\alpha, \beta) \mapsto {}^t\alpha.\beta$ with values in \mathbb{F}_2) and we consider the associated Heisenberg group \mathcal{H} with underlying set $k^* \times H \times \hat{H}$. We let E denote the non-degenerate bilinear form on $H \times \hat{H}$ defined by the commutator in \mathcal{H} . Recall that a theta structure $\tilde{\phi} : \mathcal{H} \xrightarrow{\sim} \mathcal{G}(2)$ on $\mathcal{G}(2)$ is entirely determined by the images of H and \hat{H} and a theta basis $\{X_\alpha | \alpha \in H\}$ of W is canonically (up to multiplicative scalar) associated to it. It satisfies the following properties :

$$\beta.X_\alpha = X_{\alpha+\beta} \text{ for any } \alpha, \beta \in H, \alpha^*.X_\alpha = \alpha^*(\alpha)X_\alpha \text{ for any } \alpha, \alpha^* \in H \times \hat{H}.$$

2.3. τ -invariant vector bundles and étale double covers

Choose a non-zero element τ of $J[2]$ and consider the associated double étale cover $\pi : \tilde{X} := \mathbf{Spec}(\mathcal{O}_X \oplus \tau) \rightarrow X$ with genus $2g - 1$ (Hurwitz formula). Letting \tilde{J} denote the Jacobian of \tilde{X} , denote by

$$(2.3) \quad P_\tau := \ker(\text{Nm} : \tilde{J} \rightarrow J)^0$$

the Prym variety associated to π , defined as the neutral component of $\ker \text{Nm}$ (see [15] for general properties of Prym varieties). The homomorphism $\sigma : J \times P_\tau \rightarrow \tilde{J}$ defined set-theoretically by $(j, L) \mapsto \sigma(j, L) := \pi^*(j) \otimes L$ has reduced-reduced kernel K_σ . As π is étale and $\pi^*(z)^2 \cong \mathcal{O}_{\tilde{X}}$, $\pi^*(\kappa_0 \otimes z)$ is a theta characteristic for \tilde{X} . Denote by $\tilde{\Theta}_z$ the corresponding symmetric principal divisor on \tilde{J} . One has the set-theoretical equality

$$\text{Supp } D[\pi_*L \otimes z] = (\pi^*)^{-1} \text{Supp}(T_L^* \tilde{\Theta}_z).$$

PROPOSITION 2.2.

- (1) Choose an element z in $S_\tau = \{z \in J \mid z^2 = \tau\}$. Then, there is a well-defined morphism

$$d_{\tau,z} : P_\tau \rightarrow S_X$$

mapping L to $[\pi_*L \otimes z]$. Furthermore, if $\pi_*L \otimes z$ is strictly semi-stable, then L is in $P_\tau[2]$.

A vector bundle of this form is τ -invariant, i.e., it is equipped with an isomorphism $(\pi_*L \otimes z) \otimes \tau \xrightarrow{\sim} \pi_*L \otimes z$.

Conversely, any τ -invariant semi-stable rank 2 vector bundle with trivial determinant is S -équivalent to $\pi_*L \otimes z'$ for some L in P_τ and some z' in S_τ .

- (2) There is a well-defined morphism

$$\delta_{\tau,z} : P_\tau \rightarrow |2\Theta|$$

mapping L to the divisor $(\pi^*)^{-1}(T_L^*\tilde{\Theta}_z)$.

- (3) The morphism $\delta_{\tau,z}$ agrees with the composite $D \circ d_{\tau,z}$.

Proof.

- (1) Using projection formula, one finds that the degree zero rank 2 vector bundle π_*L is semi-stable and non-stable if and only if (M, L^{-1}) lie in K_σ , i.e., if $L \cong \pi^*M$ is in $P_\tau[2]$. Those statements hold after further imposing that L lies in P_τ and tensoring by z . In this case, $\det(\pi_*L \otimes z) \cong \text{Nm}(L) \otimes \tau \otimes z^2$ is trivial. The Poincaré line bundle over $\tilde{X} \times \tilde{J}$ provides a family of rank 2 and trivial determinant vector bundles over X parameterized by P_τ and the coarse moduli property induces the morphism $d_{\tau,z}$. Because $\pi^*\tau$ is trivial, the projection formula ensures that $\pi_*L \otimes z$ is τ -invariant. Conversely, a τ -invariant vector bundle with rank 2 and trivial determinant is isomorphic to π_*L for some line bundle L on \tilde{X} and the statement follows for determinant reasons.
- (2) follows from [15] and (3) is clear since a divisor in $|2\Theta|$ is entirely determined by its support.

□

2.4. Theta groups and Prym varieties

The Prym variety P_τ is actually a principally polarized abelian variety (see [15], Sections 2 and 3). Choosing any symmetric principal divisor Ξ on

P_τ , the line bundle $\mathcal{O}(2\Xi)$ is canonical and one has

$$\sigma^*(\mathcal{O}(\tilde{\Theta}_z)) \cong \mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi).$$

Therefore, [14, Sect. 23, Thm. 2] ensures that there is a unique level subgroup $K_\sigma \cong \tilde{K}_\sigma$ in the Heisenberg group $\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi))$ such that $\sigma_*(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi))^{\tilde{K}_\sigma} \cong \mathcal{O}(\tilde{\Theta}_z)$. Denote by $\tilde{\tau} := \tilde{\tau}(z)$ the image of τ via the lifting $K_\sigma \xrightarrow{\sim} \tilde{K}_\sigma$. It follows from [15, Sections 4 and 5] that there is a unique (up to multiplicative constant) isomorphism

$$(2.4) \quad \chi : H^0(J, \mathcal{O}(2\Theta))^{\langle \tilde{\tau} \rangle} \xrightarrow{\sim} H^0(P_\tau, \mathcal{O}(2\Xi))^*$$

of Heisenberg representations and that the morphism $\delta_{\tau,z} : P_\tau \rightarrow |2\Theta|$ factors as the composite

$$P_\tau \xrightarrow{\varphi_{2\Xi}} \mathbb{P}H^0(P_\tau, \mathcal{O}(2\Xi))^* \xrightarrow{\sim} \mathbb{P}H^0(J, \mathcal{O}(2\Theta))^{\tilde{\tau}} \subset |2\Theta|$$

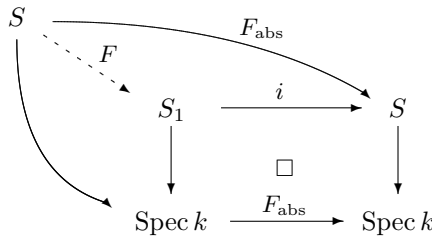
where the (canonical) isomorphism is deduced from χ .

Remark 2.3. — Notice that the set S_τ is principal homogeneous under the action $J[2]$ and check that $\tilde{\tau}(z + \alpha) = e_{2,J}(\alpha, \tau)\tilde{\tau}(z)$. In particular, $\tilde{\tau}(z) = \tilde{\tau}(-z)$.

3. The action of Frobenius

3.1. Theta groups in characteristic p

For any scheme S over k , we introduce the p -twist S_1 of S and the (relative) Frobenius $F : S \rightarrow S_1$ (which is k -linear by contrast with F_{abs}), both defined by the following commutative diagram



where the square is cartesian. Apart from J_1 and $S_X(r)_1$, define Θ_1 (resp. Δ_1) as the p -twists $i^*\Theta$ (resp. $i^*\Delta$). As before, define

$$W_1 := H^0(J_1, \mathcal{O}(2\Theta_1)), \mathcal{G}_1(2) := \mathcal{G}(\mathcal{O}(2\Theta_1)), e_{2,J_1} := e^{\mathcal{O}(2\Theta_1)}.$$

Assume from now on that X is an ordinary curve. The line bundle $\mathcal{O}(p\Theta_1)$ is no longer of separable type but Sekiguchi proved in [22] that the main results about theta groups proved by Mumford in case of line bundles of separable type can be extended in that case (we refer to [10, Section 2]). Because p is odd, $[p]$ induces identity on $J[2]$ and the restrictions $F : J[2] \rightarrow J_1[2]$ and $V : J_1[2] \rightarrow J[2]$ are isomorphisms, inverse one to each other, whence a natural action of $J[2]$ on the spaces $S_{X_1}, |2\Theta_1|...$

3.2. Finding equations for the Frobenius action on vector bundles

In the case of vector bundles, denote by $V_r : S_{X_1}(r) \dashrightarrow S_X(r)$ and call (generalized) Verschiebung the pull-back by $F : X \rightarrow X_1$. The diagram

$$(3.1) \quad \begin{array}{ccc} J_1 & \xrightarrow{V} & J \\ b_1 \downarrow & & \downarrow b \\ S_{X_1} & \overset{V_2}{\dashrightarrow} & S_X \end{array}$$

commutes and that means that $V_2 : S_{X_1} \rightarrow S_X$ extends $V : J_1 \rightarrow J$. In [11, Prop. 7.2], one finds the isomorphism

$$(3.2) \quad V_2^*(\mathcal{O}(\Delta)) \cong (\mathcal{O}(p\Delta_1))|_U$$

where U is the complementary open subset of the base locus \mathcal{B} in S_{X_1} .

PROPOSITION 3.1. — *If X is either a genus 2 curve or a non hyperelliptic genus 3 curve, there is a $J[2]$ -equivariant lifting $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ of the rational map $V_2 : S_{X_1} \dashrightarrow S_X$.*

Proof. — The proposition is equivalent to the existence of a $J[2]$ -equivariant rational map $\tilde{V} : |2\Theta_1|^* \dashrightarrow |2\Theta|^*$ such that the following diagram commutes

$$\begin{array}{ccc} S_{X_1} & \overset{V_2}{\dashrightarrow} & S_X \\ \varphi_{\Delta_1} \downarrow & & \downarrow \varphi_{\Delta} \\ |2\Theta_1|^* & \overset{\tilde{V}}{\dashrightarrow} & |2\Theta|^* \end{array}$$

In other words, one has to find a factorization

$$(3.3) \quad W \xrightarrow{\tilde{V}^*} \text{Sym}^p W_1 \xrightarrow{\varphi_{\Delta_1}^*} H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$$

of $V_2^* : W \rightarrow H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$, where the first arrow defines a $J[2]$ -equivariant map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|^*$, and where the last coincides with the canonical evaluation map.

Let us first prove the existence of a not necessarily $J[2]$ -equivariant map \tilde{V} .

LEMMA 3.2.

- (1) *If X has genus 2, there is a rational map \tilde{V} as in diagram (3.2). It is defined by 4 homogeneous degree p polynomials uniquely determined.*
- (2) *If X is a general genus 3 curve (it is in particular non hyperelliptic), the moduli space S_X is projectively normal in $|2\Theta|$. Therefore, there is a rational map \tilde{V} as in diagram (3.2). It is defined by 8 homogeneous degree p polynomials, uniquely determined modulo the Coble’s quartic.*

Proof.

- (1) The base locus \mathcal{B} is known to be finite (see [8]) and [18]) hence has codimension ≥ 2 and the isomorphism (3.2) extends to an isomorphism $V_2^*(\mathcal{O}(\Delta)) \cong \mathcal{O}(p\Delta_1)$. Because the morphism $\varphi_\Delta : S_X \rightarrow |2\Theta|^*$ is an isomorphism in that case, there is a map \tilde{V} such that the diagram (3.2) commute and it is uniquely determined.
- (2) The work of Mochizuki (see [12] and [19, Thm 4.4]) ensures that the base locus \mathcal{B} has dimension 2. Because S_X has dimension 6 in that case, the isomorphism (3.2) extends again to an isomorphism $V_2^*(\mathcal{O}(\Delta)) \cong \mathcal{O}(p\Delta_1)$. Because S_X (resp. S_{X_1}) is normal and complete intersection in $|2\Theta|^*$ (resp. $|2\Theta_1|^*$), it is projectively normal in $|2\Theta|^*$ (resp. $|2\Theta_1|^*$) (see [7, II, Exercise 8.4.(b), p. 188]). Letting $\mathcal{O}_{|2\Theta_1|^*}(n)$ denote the n -th power of the canonical twisting sheaf on $|2\Theta_1|^*$, the canonical evaluation map, which coincides with

$$H^0(|2\Theta_1|^*, \mathcal{O}_{|2\Theta_1|^*}(p)) \cong \text{Sym}^p W_1 \xrightarrow{\varphi_{\Delta_1}^*} H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$$

is therefore surjective with kernel isomorphic to the image of

$$\begin{aligned} H^0(|2\Theta_1|^*, \mathcal{O}_{|2\Theta_1|^*}(p-4)) &\cong \text{Sym}^{p-4} W_1 \xrightarrow{C_{X_1}} H^0(|2\Theta_1|^*, \mathcal{O}_{|2\Theta_1|^*}(p)) \\ &\cong \text{Sym}^p W_1. \end{aligned}$$

□

Remark 3.3. — In the case of a (general) genus 3 curve, and in characteristic 3 (the only case in which we will perform the computations), the rational map \tilde{V} is uniquely determined.

Because $J[2]$ does not act on sections but on multiplicative classes, we need to define actions of the corresponding Heisenberg group $\mathcal{G}(2)$.

LEMMA 3.4. — *There is a Theta group homomorphism $\mathcal{G}(2) \rightarrow \mathcal{G}_1(2)$ of weight p . In the cases of the previous lemma, it induces a weight p^2 action of $\mathcal{G}(2)$ on both $\text{Sym}^p W_1$ and $H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$, compatible with the evaluation map.*

Proof. — We check that the definition of [13, p. 310] can be generalized in our case. Namely, there is a homomorphism of Heisenberg groups $\eta_p : \mathcal{G}(\mathcal{O}(2p\Theta_1)) \rightarrow \mathcal{G}_1(2)$ mapping any $\gamma : \mathcal{O}(2p\Theta_1) \xrightarrow{\sim} T_x^* \mathcal{O}(2p\Theta_1)$ in $\mathcal{G}(\mathcal{O}(2p\Theta_1))$, $\eta_p(\gamma)$ to the unique isomorphism $\rho : \mathcal{O}(2\Theta_1) \xrightarrow{\sim} T_{p x}^* \mathcal{O}(2\Theta_1)$ such that the composite

$$[p]^* \mathcal{O}(2\Theta_1) \xrightarrow{\sim} \mathcal{O}(2p^2\Theta_1) \xrightarrow{\gamma^{\otimes p}} T_x^* \mathcal{O}(2p^2\Theta_1) \xrightarrow{\sim} [p]^* T_{p x}^* \mathcal{O}(2\Theta_1)$$

coincides with $[p]^* \rho$. It raises elements of the center to the p -th power and the composite homomorphism $\eta_p \circ V^* : \mathcal{G}(2) \rightarrow \mathcal{G}_1(2)$ has therefore weight p .

Using p -symmetric power, $\mathcal{G}_1(2)$ has a natural weight p action on $\text{Sym}^p W_1$ and composing, it gives a weight p^2 action of $\mathcal{G}(2)$ on $\text{Sym}^p W_1$. Because the evaluation map $\text{Sym}^p W_1 \rightarrow H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$ is an isomorphism in genus 2 and has kernel invariant under the action of $\mathcal{G}(2)$ in case of a general genus 3 curve, it induces a weight p^2 action of $\mathcal{G}(2)$ on $H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$. □

Because \tilde{V}^* is linear, there is no chance that it can be $\mathcal{G}(2)$ -equivariant. We therefore define the subgroup $\mathcal{G}(2)[2]$ of order 2 elements in $\mathcal{G}(2)$ and consider the induced actions on W , $\text{Sym}^p W_1$ and $H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$. Because $V_2^* : W \rightarrow H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$ is linear and comes from the $J[2]$ -equivariant map $V_2 : S_{X_1} \dashrightarrow S_X$, it is $\mathcal{G}(2)[2]$ -equivariant.

Choose a theta structure on $\mathcal{G}(2)$ and let $\{X_h, h \in H\}$ denote the associated theta basis. Assume that the map \tilde{V} provided by the lemma above is not $J[2]$ -equivariant. Despite $\tilde{V}^*(X_0)$ might not be \hat{H} -invariant, its class in $H^0(S_{X_1}, \mathcal{O}(p\Delta_1))$ is for V_2^* is $\mathcal{G}(2)[2]$ -equivariant. The element

$$V'_0 := \frac{1}{2^g} \sum_{h^* \in \hat{H}} h^*(\tilde{V}^*(X_0))$$

in $(\text{Sym}^p W_1)^{\hat{H}}$ is \hat{H} -invariant and maps onto $V_2^*(X_0)$. Defining $V'_h := h.V'_0$, the map $W \rightarrow \text{Sym}^p W_1$ defined by $X_h \mapsto V'_h$ is $\mathcal{G}(2)[2]$ -equivariant and induces V_2^* . The associated map $V' : |2\Theta_1|^* \dashrightarrow |2\Theta|^*$ is $J[2]$ -equivariant and we have proved the proposition. □

Remark 3.5. — One checks that $[p]^*i^*\gamma = V^*(F^*i^*)\gamma = V^*(\gamma^{\otimes p}) = (V^*\gamma)^{\otimes p}$ for any γ in $\mathcal{G}(\mathcal{O}(2p\Theta_1))$. Therefore, the homomorphism $\eta_p \circ V^*$ coincides with the homomorphism $i^* : \mathcal{G}(2) \rightarrow \mathcal{G}_1(2)$ induced by the pull-back by the quasi-isomorphism i . In particular, for any two elements $\bar{\alpha}$ and $\bar{\beta}$ in $J[2]$,

$$e_{2, J_1}(F(\bar{\alpha}), F(\bar{\beta})) = e_{2, J}(\bar{\alpha}, \bar{\beta})^p.$$

Because $J[2]$ is reduced and because e_2 takes its values in μ_2 , we find that F (hence V) is a symplectic isomorphism. This implies that the choice of a Göpel system for $J[2]$ (resp. a theta structure on $\mathcal{G}(2)$) determines a Göpel system for $J_1[2]$ (resp. a theta structure $\mathcal{H}_1 \xrightarrow{\sim} \mathcal{G}_1(2)$ (where $\mathcal{H}_1 := \mathcal{H} \otimes_{F_{\text{abs}, k}} k$)), and that the associated theta bases $\{X_\alpha\}_{\alpha \in H}$ and $\{Y_{\alpha_1}\}_{\alpha_1 \in H_1}$ are compatible in the sense that $Y_{\alpha_1} = i^*X_{V(\alpha_1)}$.

3.3. Frobenius action and Prym varieties

The functoriality of Frobenius is also compatible with the correspondence between order 2 line bundles over X and double étale covers of X . If F is the relative Frobenius, [3, I.11] says that the diagram

$$(3.4) \quad \begin{array}{ccc} \tilde{X} & \xrightarrow{F} & \tilde{X}_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ X & \xrightarrow{F} & X_1 \end{array}$$

is cartesian. As a consequence, the morphisms $\pi^* : J \rightarrow \tilde{J}$ and $\text{Nm} : \tilde{J} \rightarrow J$ commute with V .

PROPOSITION 3.6. — *The following diagram*

$$\begin{array}{ccc} J \times P_\tau & \xrightarrow{\sigma} & \tilde{J} \\ V \times V_\tau \uparrow & & \uparrow V \\ J_1 \times P_{\tau_1} & \xrightarrow{\sigma_1} & \tilde{J}_1 \end{array}$$

is commutative.

Furthermore, σ induces an isomorphism $J[p] \times P_\tau[p] \xrightarrow{\sim} \tilde{J}[p]$. In particular, if J is an ordinary abelian variety, then \tilde{J} is ordinary if and only if P_τ is ordinary.

Proof. — The Prym variety

$$P_{\tau_1} := \ker(\text{Nm})^0 \subseteq \tilde{J}_1$$

coincides with the p -twist of $P_\tau := \ker(\text{Nm})^0 \subseteq \tilde{J}$ and it is mapped by V to P_τ . The restriction $V|_{P_{\tau_1}} : P_{\tau_1} \rightarrow P_\tau$ coincides with the Verschiebung $V_\tau : P_{\tau_1} \rightarrow P_\tau$ for P_τ and the commutation of the diagram follows from the fact that V is a homomorphism and commutes with π^* .

The isomorphism $J[p] \times P_\tau[p] \xrightarrow{\sim} \tilde{J}[p]$ follows from the inclusion $\ker \sigma \subseteq J[2] \times P_\tau[2]$ and the hypothesis $p \geq 3$. Because X was supposed to be ordinary, the last assertion comes from the fact that the ordinariness of any abelian variety can be read on the reduced part of its p -torsion. \square

For later use, let us mention a result due to Nakajima, proving that for a sufficiently general curve X , all the abelian varieties appearing in the Proposition above are ordinary (one can find a proof in [24]).

PROPOSITION 3.7. — *Let X be a general, proper and smooth connected curve over an algebraically closed field of characteristic p and let $f : Y \rightarrow X$ be an étale cover with abelian Galois group G . Then Y is ordinary.*

Choose an element z in $S_\tau = \{z \in J \mid z^2 = \tau\} \subset J[4]$ and let z_1 be $F(z)$ (equivalently i^*z). Let ν_p be 0 if $p \equiv 1 \pmod 4$ and be 1 if $p \equiv 3 \pmod 4$, then

$$F^*z_1 = V(z_1) = (-1)^{\frac{p-1}{2}} z = z \otimes \tau^{\nu_p}.$$

Therefore, for any L_1 of P_{τ_1} , the cartesian square (3.4) gives

$$\begin{aligned} F^*((\pi_1)_*(L_1) \otimes z_1) &\cong \pi_*(F^*(L_1)) \otimes F^*z_1 \cong \pi_*(F^*(L_1)) \otimes z \otimes \tau^{\nu_p} \\ &\cong \pi_*(F^*(L_1)) \otimes z. \end{aligned}$$

Assuming that the curve X is sufficiently general, the Proposition 3.7 ensures that all the Prym varieties are ordinary and one can choose symmetric principal divisors Ξ and Ξ_1 on P_τ and P_{τ_1} respectively such that $\mathcal{O}(p\Xi_1) \cong V_\tau^*\mathcal{O}(\Xi)$ [10, Lemma 2.2].

Let $\varphi_{2\Xi}$ be the canonical map $P_\tau \rightarrow P_\tau/\pm \subseteq |2\Xi|^*$ and define $\varphi_{2\Xi_1}$ analogously. Let $V_\tau^\pm : P_{\tau_1}/\pm \rightarrow P_\tau/\pm$ be the morphism induced by $V_\tau : P_{\tau_1} \rightarrow P_\tau$ (which commutes with $[-1]$). Assume that, as in the diagram (3.2), there is a $J[2]$ -equivariant rational map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ lifting $V_2 : S_{X_1} \dashrightarrow S_X$. Restricting the map \tilde{V} to $|2\Xi_1|^*$ (which identifies with one of the two connected components of $|2\Theta_1|^{\tau_1}$) yields a rational map $\tilde{V}|_{|2\Xi_1|^*} : |2\Xi_1|^* \dashrightarrow |2\Xi|^*$ that lifts the morphism $V_\tau^\pm : P_{\tau_1}/\pm \rightarrow P_\tau/\pm$.

Therefore, one obtains the following commutative diagram

$$\begin{array}{ccccccc}
 P_\tau & \xrightarrow{\varphi_{2\Xi}} & P_\tau/\pm \subseteq & |2\Xi|^* \subseteq & |2\Theta|^\tau \subseteq & |2\Theta| \\
 \uparrow V_\tau & & \uparrow V_\tau^\pm & \uparrow \tilde{V}_{|2\Theta_1|^*} & & \uparrow \tilde{V} \\
 (3.5) \quad P_{\tau_1} & \xrightarrow{\varphi_{2\Xi_1}} & P_{\tau_1}/\pm \subseteq & |2\Xi_1|^* \subseteq & |2\Theta_1|^{\tau_1} \subseteq & |2\Theta_1|
 \end{array}$$

In terms of coordinates functions, we have therefore proven the following :

PROPOSITION 3.8. — Assume that X is proper and smooth genus $g = 2$ or 3 curve over k , sufficiently general. Then, for any non zero τ in $J[2]$, there is a $P_\tau[2]$ -equivariant map $\tilde{V}_\tau : |2\Xi_1|^* \dashrightarrow |2\Xi|^*$ such that the pull-back $V_\tau^* : H^0(P_\tau, \mathcal{O}(2\Xi)) \rightarrow H^0(P_{\tau_1}, \mathcal{O}(2p\Xi_1))$ factors as the composite

$$(3.6) \quad H^0(P_\tau, \mathcal{O}(2\Xi)) \xrightarrow{\tilde{V}_\tau^*} \text{Sym}^p H^0(P_{\tau_1}, \mathcal{O}(2\Xi_1)) \rightarrow H^0(P_{\tau_1}, \mathcal{O}(2p\Xi_1))$$

where the last arrow is the evaluation map.

Letting $\tilde{\tau}$ (resp. $\tilde{\tau}_1$) be the order 2 lift of τ in $\mathcal{G}(2)$ (resp. τ_1 in $\mathcal{G}_1(2)$) associated to z (resp. z_1), \tilde{V}_τ agrees (up to a multiplicative scalar) with the restriction

$$\tilde{V}_{|(W^{\tilde{\tau}})^*} : (W^{\tilde{\tau}})^* \longrightarrow \text{Sym}^p (W_1^{\tilde{\tau}_1})^*$$

4. The proof of the Theorem 1.1

From now on, we choose once for all a theta structure $\tilde{\phi}_0 : \mathcal{H} \xrightarrow{\sim} \mathcal{G}(2)$ and we let $\{X_\alpha, \alpha \in H\}$ be the associated theta basis. It gives a system of homogeneous coordinates $\{x_\alpha, \alpha \in H\}$ for $|2\Theta|$. Recall from Remark 3.5 that this choice also determines a theta structure $\tilde{\phi}_1 : \mathcal{H}_1 \xrightarrow{\sim} \mathcal{G}_1(2)$, hence a theta basis for W_1 and coordinate functions for $|2\Theta_1|$ adapted to the action of $J[2]$ that we denote $\{Y_\alpha | \alpha \in H\}$ and $\{y_\alpha | \alpha \in H\}$ respectively. Also, given $\tau = (\alpha_0, \alpha_0^*)$ in $H \times \hat{H}$, recall that the choice of a lift $\tilde{\tau} = (\mu, \alpha_0, \alpha_0^*)$ (with $\mu^2 = \alpha_0^*(\alpha_0)$) determines a lift $\tilde{\tau}_1 = (\mu^p, \alpha_0, \alpha_0^*)$ of τ_1 in \mathcal{H}_1 . Fix once for all a square root μ_0 of -1 in k . We will take $\mu = 1$ if $\alpha_0^*(\alpha_0) = 1$ and $\mu = \mu_0$ otherwise.

4.1. From geometry to linear algebra

The direct sum $\text{Sym}^p (W_1^{\tilde{\tau}_1})^* \oplus \text{Sym}^p (W_1^{-\tilde{\tau}_1})^*$ (which depends only on the choice of τ) is endowed with an action of \mathcal{H} of weight p^2 and the quotient

map

$$\text{Sym}^p W_1^* \rightarrow \text{Sym}^p (W_1^{\tilde{\tau}_1})^* \oplus \text{Sym}^p (W_1^{-\tilde{\tau}_1})^*$$

is equivariant for the action of \mathcal{H} on both spaces. Taking all order 2 elements of $J[2]$ together, we find a morphism of \mathcal{H} -representations

$$R : \text{Sym}^p W_1^* \rightarrow S_p := \bigoplus_{\tau \in J[2] \setminus \{0\}} \text{Sym}^p (W_1^{\tilde{\tau}_1})^* \oplus \text{Sym}^p (W_1^{-\tilde{\tau}_1})^*.$$

Because \tilde{V} is given by a linear sub-space of $\text{Sym}^p W_1^*$, isomorphic to W^* and endowed with an (irreducible) action of $\mathcal{G}(2)$ of weight p^2 , it is determined by its \bar{H} -invariant part and Theorem 1.1 follows from the Proposition

PROPOSITION 4.1. — *When $g = 2$ or 3 and $p = 3, 5$ or 7 , the restriction map*

$$\hat{R} : (\text{Sym}^p W_1^*)^{\hat{H}} \rightarrow S_p$$

is injective.

Remark 4.2. — Certainly, a necessary condition is that

$$\dim \text{Sym}^p W_1^* = \binom{2^g + p - 1}{2^g - 1} \leq \dim S_p = 2(2^{2g} - 1) \binom{2^{g-1} + p - 1}{2^{g-1} - 1}.$$

This cannot be the case for large p . More precisely, when $g = 2$, $\dim \text{Sym}^p W_1^* > \dim S_p$ for $p > 7$, and when $g = 3$, $\dim \text{Sym}^p W_1^* > \dim S_p$ for $p > 11$.

4.2. Preparation in small genus

Since $\text{Sym}^p W_1^*$ is generated by the free family of monomials $\prod_{\alpha \in H} y_\alpha^{e_\alpha}$ with $\sum e_\alpha = p$, $(\text{Sym}^p W_1^*)^{\hat{H}}$ is generated by the subfamily consisting of elements whose set of exponents satisfies

$$\sum_{\alpha \in H \mid \alpha_0^*(\alpha) = -1} e_\alpha \equiv 0 \pmod{2} \text{ for all } \alpha_0^* \text{ in } \hat{H}.$$

Writing such an element under the form $\prod_{\alpha \in H} y_\alpha^{\bar{e}_\alpha} \left[\prod_{\alpha \in H} y_\alpha^{f_\alpha} \right]^2$ with $\bar{e}_\alpha = 0$ or 1 and $\sum \bar{e}_\alpha + 2 \sum f_\alpha = p$, we find that

$$\sum_{\alpha \in H \mid \bar{e}_\alpha = 1} \alpha = 0.$$

It is easily seen that E can be $\{0\}$ and $H - \{0\}$. In genus 2, these are the only possibilities. In genus 3, one also has to consider the case where E has cardinal 3 and $E' = \{0\} \cup E$ is a cardinal 4 subgroup of H (the set of

such subgroups is in 1-1 correspondance with the set of non zero elements of \hat{H}). Consider therefore the sets

$$A(p) = \left\{ A_{\underline{f}} = y_0 \left[\prod_{\alpha \in H} y_{\alpha}^{f_{\alpha}} \right]^2, \text{ with } |\underline{f}| = \frac{p-1}{2} \right\}$$

$$B(p) = \left\{ B_{\underline{f}} = \prod_{\alpha \in H - \{0\}} y_{\alpha} \left[\prod_{\alpha \in H} y_{\alpha}^{f_{\alpha}} \right]^2, \text{ with } |\underline{f}| = \frac{p-2^g+1}{2} \right\}$$

and

$$C(p) = \left\{ C_{\alpha^*, \underline{f}} = \prod_{\alpha \in H - \{0\}, \alpha^*(\alpha)=1} y_{\alpha} \left[\prod_{\alpha \in H} y_{\alpha}^{f_{\alpha}} \right]^2, \right. \\ \left. \text{with } \alpha^* \text{ in } \hat{H} - \{0\} \text{ and } |\underline{f}| = \frac{p-3}{2} \right\}$$

(where \underline{f} is, in each case, the multi-index $(f_{\alpha}, \alpha \in H)$ with $|\underline{f}| = \sum f_{\alpha}$).

To compute the image of these monomials in the various $\text{Sym}^p(W_1^{\tilde{\tau}_1})^* \oplus \text{Sym}^p(W_1^{-\tilde{\tau}_1})^*$, it is convenient to distinguish whether τ is an element of \hat{H} or not. In the first case, the following lemma is straightforward

LEMMA 4.3. — *For a given $\tau = \tilde{\tau} = \alpha_0^*$, all the monomials in $A(p)$, $B(p)$ or $C(p)$ map to 0 in $\text{Sym}^p(W_1^{-\tilde{\tau}_1})^*$. The only monomials in $A(p) \sqcup B(p)$ (resp. $A(p) \sqcup B(p) \sqcup C(p)$) not mapping to 0 in $\text{Sym}^p(W_1^{\tilde{\tau}_1})^*$ are those that can be written under the form*

$$A_{\underline{f}} = y_0 \left[\prod_{\alpha \in H, \alpha_0^*(\alpha)=1} y_{\alpha}^{f_{\alpha}} \right]^2$$

(resp. as well as

$$C_{\alpha_0^*, \underline{f}} = \left(\prod_{\alpha \in H - \{0\}, \alpha_0^*(\alpha)=1} y_{\alpha} \left[\prod_{\alpha \in H} y_{\alpha}^{f_{\alpha}} \right]^2 \right)$$

and they all map to a different monomial in $\text{Sym}^p(W_1^{\tilde{\tau}_1})^*$.

4.3. Proof of Proposition 4.1 in the genus 2 case

As indicated above, a basis of $(\text{Sym}^p W_1^*)^{\hat{H}}$ in genus 2 is the (disjoint) union of the two sets $A(p)$ and $B(p)$.

LEMMA 4.4. — Assume that

$$(4.1) \quad V_0 = \sum_{A(p)} a_f A_f + \sum_{B(p)} b_f B_f$$

maps to zero in S_p . Then, for any integer $0 \leq k \leq \frac{p-1}{2}$

$$f_{00}+f_{01}=k \text{ and } f_{01}+f_{11} \text{ even} \quad a_f = 0; \quad f_{00}+f_{01}=k \text{ and } f_{01}+f_{11} \text{ odd} \quad a_f = 0$$

$$f_{00}+f_{10}=k \text{ and } f_{10}+f_{11} \text{ even} \quad a_f = 0; \quad f_{00}+f_{10}=k \text{ and } f_{10}+f_{11} \text{ odd} \quad a_f = 0$$

$$f_{00}+f_{11}=k \text{ and } f_{10}+f_{11} \text{ even} \quad a_f = 0; \quad f_{00}+f_{11}=k \text{ and } f_{10}+f_{11} \text{ odd} \quad a_f = 0$$

and we have analogous equalities for the b_f (with $0 \leq k \leq \frac{p-3}{2}$).

Proof. — Choose a non zero α_0 in H and for $\tau = (\alpha_0, \alpha_0^*)$, let $\tilde{\tau}$ be defined as above. In particular, $\tilde{\tau}_1 = (\mu^p, \alpha_0, \alpha_0^*)$. A generating system for $W_1^{\tilde{\tau}_1}$ (resp. $W_1^{-\tilde{\tau}_1}$) is

$$\{(Y_\alpha + \tilde{\tau}_1.Y_\alpha), (\alpha \in H)\} \text{ (resp. } \{(Y_\alpha - \tilde{\tau}_1.Y_\alpha), (\alpha \in H)\} \}.$$

Since one has $\tilde{\tau}_1.Y_\alpha = \mu^p \alpha_0^*(\alpha) Y_{\alpha+\alpha_0}$, one finds, letting \bar{y}_α denote the class of y_α in $(W_1^{\tilde{\tau}_1})^*$, that $\bar{y}_{\alpha+\alpha_0} = \mu^p \alpha_0^*(\alpha) \bar{y}_\alpha$. Similarly, letting \tilde{y}_α denote also the class of y_α in $(W_1^{-\tilde{\tau}_1})^*$ (there will be no risk of confusion), one finds that $\tilde{y}_\alpha = -\mu^p \alpha_0^*(\alpha) \tilde{y}_\alpha$.

Choose a non zero α in H such that $H = \langle \alpha_0, \alpha \rangle$. Then A_f maps to

$$(4.2) \quad \alpha_0^*(\alpha_0) f_{\alpha_0+f_{\alpha+\alpha_0}} \left(\bar{y}_0^{1+2(f_0+f_{\alpha_0})} \bar{y}_\alpha^{-2(f_\alpha+f_{\alpha+\alpha_0})} \right)$$

in both $\text{Sym}^p(W_1^{\tilde{\tau}_1})^*$ and $\text{Sym}^p(W_1^{-\tilde{\tau}_1})^*$. Similarly, the monomial B_f maps to

$$(4.3) \quad \alpha_0^*(\alpha) . \alpha_0^*(\alpha_0) f_{\alpha_0+f_{\alpha+\alpha_0}} \left(\tilde{y}_0^{1+2(f_0+f_{\alpha_0})} \tilde{y}_\alpha^{-2(f_\alpha+f_{\alpha+\alpha_0})} \right)$$

in both $\text{Sym}^p(W_1^{\tilde{\tau}_1})^*$ and $\text{Sym}^p(W_1^{-\tilde{\tau}_1})^*$.

Specifically, fix $\alpha_0 = 01$, choose $\alpha = 10$ and a positive integer k , and look at the coefficient of the monomial $\bar{y}_{00}^{1+2k} \bar{y}_{10}^{p-1-2k}$ in the image of V_0 in $\text{Sym}^p(W_1^{\tilde{\tau}_1})^*$ for all the elements $\tau = (01, \alpha_0^*)$ in $J[2]$. Using (4.2), (4.3)

above, we find the following expressions

$$\begin{aligned}
 (\alpha_0^* = 00) : & \quad \sum_{f_{00}+f_{01}=k} a_{\underline{f}} + \sum_{f_{00}+f_{01}=k} b_{\underline{f}} \\
 (\alpha_0^* = 01) : & \quad \sum_{f_{00}+f_{01}=k} (-1)^{f_{01}+f_{11}} a_{\underline{f}} - \sum_{f_{00}+f_{01}=k} (-1)^{f_{01}+f_{11}} b_{\underline{f}} \\
 (\alpha_0^* = 10) : & \quad \sum_{f_{00}+f_{01}=k} a_{\underline{f}} - \sum_{f_{00}+f_{01}=k} b_{\underline{f}} \\
 (\alpha_0^* = 11) : & \quad \sum_{f_{00}+f_{01}=k} (-1)^{f_{01}+f_{11}} a_{\underline{f}} + \sum_{f_{00}+f_{01}=k} (-1)^{f_{01}+f_{11}} b_{\underline{f}}.
 \end{aligned}$$

If $\hat{R}(V_0) = 0$, we easily derive from the expressions above the first line in the equations of the statement for both the $a_{\underline{f}}$'s and the $b_{\underline{f}}$'s. Repeating the process with $\alpha_0 = 10$ or 11 , $\alpha = 01$ in both cases, and letting k vary through the relevant sets proves the lemma. □

The proof of proposition 4.1 in genus 2 now boils down, using Lemmas 4.3 and 4.4, to an easy exercise of linear algebra left to the reader.

4.4. Proof of Proposition 4.1 in the genus 3 case

In genus 3, a basis of $(\text{Sym}^p W_1^*)^{\hat{H}}$ in the (disjoint) union of the three sets $A(p)$, $B(p)$ and $C(p)$. The following lemma is very similar to the analogous Lemma in genus 2 case, though more complicated and more inconvenient to write down. We leave the proof to the reader and we simply indicate the tricks we found useful to do the computations. For any non zero α_0 in H , we will choose a subgroup $H(\alpha_0)$ in H , not containing α_0 , such that $H(\alpha_0)$ and α_0 together generate H (it is analogous to the choice of an element α in the proof of Lemma 4.4). Notice that $H(\alpha_0)$ inherits an ordering from the lexicographic order on H . More specifically, we will choose $H(001) = \{000, 010, 100, 110\}$, $H(010) = H(011) = \{000, 001, 101, 101\}$, and $H(\alpha_0) = \{000, 001, 010, , 011\}$ otherwise. For any multi-index \underline{f} , let $[f]_{\alpha_0}$ denote the 4-tuple $(f_{\alpha} + f_{\alpha_0+\alpha})_{\alpha \in H(\alpha_0)}$ (with lexicographic order). Also, let $\Sigma_{\alpha_0} \underline{f}$ stand for the sum $\sum_{\alpha \in H(\alpha_0)} f_{\alpha_0+\alpha}$. We will keep on denoting by $|\underline{f}|$ the sum $\sum f_{\alpha}$ for any n -tuple.

LEMMA 4.5. — Assume that

$$V_0 = \sum_{A(p)} a_{\underline{f}} A_{\underline{f}} + \sum_{B(p)} b_{\underline{f}} B_{\underline{f}} + \sum_{C(p)} c_{\alpha^*, \underline{f}} C_{\alpha^*, \underline{f}}$$

maps to zero in S_p . Then, for all 4-tuples $\underline{k} = (k_{00}, k_{01}, k_{10}, k_{11})$ of positive integers such that $|\underline{k}| = \frac{p-1}{2}$ and for all non-zero α_0 in H , we have the

equalities

$$\sum_{[\underline{f}]_{\alpha_0}=\underline{k} \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ even}} a_{\underline{f}} + \sum_{[\underline{f}]_{\alpha_0}=(k_{00}, k_{01}-1, k_{10}-1, k_{11}-1) \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ even}} b_{\underline{f}} = 0$$

and

$$\sum_{[\underline{f}]_{\alpha_0}=\underline{k} \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ odd}} a_{\underline{f}} + \sum_{[\underline{f}]_{\alpha_0}=(k_{00}, k_{01}-1, k_{10}-1, k_{11}-1) \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ odd}} b_{\underline{f}} = 0$$

Also, for all 4-tuples $\underline{k} = (k_{00}, k_{01}, k_{10}, k_{11})$ of positive integers such that $|\underline{k}| = \frac{p-3}{2}$ and for all non-zero α_0 in H , we have the equalities

$$\sum_{[\underline{f}]_{\alpha_0}=\underline{k} \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ even}} c_{\alpha^*, \underline{f}} = 0 \text{ and } \sum_{[\underline{f}]_{\alpha_0}=\underline{k} \text{ and } \Sigma_{\alpha_0}(\underline{f}) \text{ odd}} c_{\alpha^*, \underline{f}} = 0$$

for all α^* in $\hat{H} - \{0\}$.

Combining the data given by the Lemmas 4.3 and 4.5, we reduce once again our problem to an easy question of linear algebra. Nonetheless, it involves a large number of unknowns and equations and those equations should not be written bluntly since a few (easy) tricks simplify the matter considerably.

5. Elliptic curves, Kummer’s quartic surface and Coble’s quartic hypersurface

In this section, we give some preparation for the computations to come in the next section.

5.1. Kummer’s quartic surface and associated elliptic curves

Let us begin with recalling some well-known results dealing with the geometry of the Kummer’s quartic surface. Proofs can be found in [5] (PhD thesis) or in [6] in the complex case but they can be carried over to any algebraically closed base field k of characteristic different from 2.

LEMMA 5.1.

- (1) *Let X be a smooth and projective genus 2 curve over k and let J be its Jacobian. The scheme-theoretic image of the morphism $K_X : J \rightarrow |2\Theta|$ identifies with the quotient of J under the action of $\{\pm 1\}$. It is a reduced, irreducible, $J[2]$ -invariant quartic in $|2\Theta|$ with 16 nodes and no other singularities, i.e., a Kummer surface.*

(2) In the coordinate system $\{x_\bullet\}$ defined above, there are scalars k_{00}, k_{01}, k_{10} and k_{11} such that the equation defining the Kummer quartic surface Kum_X is

$$(5.1) \quad K_X = S^K + 2k_{00}P^K + k_{01}Q_{01}^K + k_{10}Q_{10}^K + k_{11}Q_{11}^K$$

where

$$S^K = x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4, \quad P^K = x_{00}x_{01}x_{10}x_{11},$$

$$Q_{01}^K = x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2, \quad Q_{10}^K = x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2, \quad Q_{11}^K = x_{00}^2x_{11}^2 + x_{01}^2x_{10}^2.$$

(3) These scalars k_{00}, k_{01}, k_{10} and k_{11} satisfy the cubic relationship

$$(5.2) \quad 4 + k_{01}k_{10}k_{11} - k_{01}^2 - k_{10}^2 - k_{11}^2 + k_{00}^2 = 0$$

and one has

$$(5.3) \quad \begin{cases} k_{01} \neq \pm 2, \quad k_{10} \neq \pm 2, \quad k_{11} \neq \pm 2, \\ k_{01} + k_{10} + k_{11} + 2 \pm k_{00} \neq 0, \\ k_{01} + k_{10} - k_{11} - 2 \pm k_{00} \neq 0, \\ k_{01} - k_{10} + k_{11} - 2 \pm k_{00} \neq 0, \\ -k_{01} + k_{10} + k_{11} - 2 \pm k_{00} \neq 0. \end{cases}$$

Let $\tau = (\alpha_0, \alpha_0^*)$ be a non zero element of $J[2] = H \times \hat{H}$. Fix an order 2 lift $\tilde{\tau}$ of τ in \mathcal{H} as in the previous section. The space W splits in the direct sum $W = W^{\tilde{\tau}} \oplus W^{-\tilde{\tau}}$ of the two 2-dimensional spaces of eigenvectors of $\tilde{\tau}$. Denote by $\Delta^+(\tilde{\tau})$ (resp. $\Delta^-(\tilde{\tau})$) the corresponding projective lines in $|\mathcal{O}(2)|$. Again, one can find a non zero $\alpha = \alpha(\tau)$ in H such that the images \bar{x}_0 and \bar{x}_α (of x_0 and x_α respectively) via the canonical map $W^* \rightarrow (W^{\tilde{\tau}})^*$ give a set of homogeneous coordinates for $\Delta^+(\tilde{\tau})$. We will write λ_0 for \bar{x}_0 and λ_1 for \bar{x}_α .

Remark 5.2. — We can similarly construct a system of coordinates $\{\bar{\lambda}_0, \bar{\lambda}_1\}$ for $\Delta^-(\tilde{\tau})$. Via Wirtinger’s isomorphism, it gives dually a basis $\{\Lambda_0, \Lambda_1, \bar{\Lambda}_0, \bar{\Lambda}_1\}$ of W that splits into bases $\{\Lambda_0, \Lambda_1\}$ and $\{\bar{\Lambda}_0, \bar{\Lambda}_1\}$ of $W^{\tilde{\tau}}$ and $W^{-\tilde{\tau}}$ respectively. One then notices that this is the theta basis associated to a suitable theta structure on $\mathcal{G}(2)$.

Restricted to $\Delta^+(\tilde{\tau})$, the equation of the Kummer surface reduces, up to a multiplicative scalar, to

$$(5.4) \quad \lambda_0^4 + \lambda_1^4 + \omega \lambda_0^2 \lambda_1^2 = 0$$

where $\omega := \omega(\tau)$ depends on τ but not on our particular choice of a lifting $\tilde{\tau}$ of τ . In particular, the equation of the Kummer surface restricts in the same way to $\Delta^-(\tilde{\tau})$. The points of $\Delta^+(\tilde{\tau}) \cap \text{Kum}_X$ correspond to the four points of $P_\tau[2]$ (in particular, $\omega \neq \pm 2$) and they must have homogeneous

coordinates $(a : b)$, $(a : -b)$, $(b : a)$ and $(b : -a)$, whence $\omega = -\frac{a^4 + b^4}{a^2b^2}$. It is easy to compute the various $\omega(\tau)$ in terms of the k_\bullet . Namely,

(5.5)

α_0	$\alpha_0^* = 00$	$\alpha_0^* = 01$	$\alpha_0^* = 10$	$\alpha_0^* = 11$
00	*	k_{10}	k_{01}	k_{11}
01	$\frac{2(k_{00} + k_{10} + k_{11})}{2 + k_{01}}$	$\frac{2(-k_{00} + k_{10} - k_{11})}{2 - k_{01}}$	$\frac{2(-k_{00} + k_{10} + k_{11})}{2 + k_{01}}$	$\frac{2(k_{00} + k_{10} - k_{11})}{2 - k_{01}}$
10	$\frac{2(k_{00} + k_{01} + k_{11})}{2 + k_{10}}$	$\frac{2(-k_{00} + k_{01} + k_{11})}{2 + k_{10}}$	$\frac{2(-k_{00} + k_{01} - k_{11})}{2 - k_{10}}$	$\frac{2(k_{00} + k_{01} - k_{11})}{2 - k_{10}}$
11	$\frac{2(k_{00} + k_{01} + k_{10})}{2 + k_{11}}$	$\frac{2(k_{00} + k_{01} - k_{10})}{2 - k_{11}}$	$\frac{2(-k_{00} + k_{01} - k_{10})}{2 - k_{11}}$	$\frac{2(-k_{00} + k_{01} + k_{10})}{2 + k_{11}}$

Notice that the inequations (5.3) ensure that those coefficients are well-defined scalars, and they give another reason why $\omega(\tau)$ cannot equal ± 2 . Because an elliptic curve E is completely determined by the branch locus of its Kummer map $E \rightarrow E/\{\pm\} \cong \mathbb{P}^1$, these data allow one to determine the elliptic curve P_τ arising as the Prym variety associated to the double cover corresponding to τ .

5.2. Coble’s quartic and associated Kummer surfaces

Now, X is a non hyperelliptic curve of genus 3 over k . Take $\tau = (\alpha_0, \alpha_0^*)$ to be non-zero and fix a lift $\tilde{\tau}$ in \mathcal{H} . The space W splits again in the direct sum $W = W^{\tilde{\tau}} \oplus W^{-\tilde{\tau}}$ of the two 4-dimensional spaces of eigenvectors $\tilde{\tau}$ and we let $\Delta^+(\tilde{\tau})$ and $\Delta^-(\tilde{\tau})$ denote the corresponding projective space \mathbb{P}^3 in $|2\Theta|$. Again, one can find a cardinal 4 subgroup $H(\tau)$ of H such that the images \bar{x}_α of x_α (α in $H(\tau)$) via the canonical map $W^* \rightarrow (W^{\tilde{\tau}})^*$ give a set of homogeneous coordinates for $\Delta^+(\tilde{\tau})$. This set of coordinates will be denoted $\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}$ in such a way that the lexicographical order is respected and if α is in $H(\tau)$, we will write $\bar{x}_\alpha = \lambda_{\bar{\alpha}}$ (with our conventions, $\alpha \mapsto \bar{\alpha}$ is an group isomorphism). In particular, $\bar{x}_0 = \lambda_{\bar{0}} = \lambda_{00}$.

The following results are also well-known (the reader may refer to [2], [6] and [20]).

LEMMA 5.3. — *There is a unique $J[2]$ -invariant quartic Cob_X in $|2\Theta|$ whose singular locus is the Kummer variety Kum_X associated to X .*

If τ is a non zero element of $J[2]$ and if $\Delta \cong \mathbb{P}^3$ is one of the two connected components of $|2\Theta|^\tau$, then the intersection $\Delta \cap \text{Cob}_X$ is isomorphic to the Kummer surface Kum_τ associated to the genus 2 curve Y_τ whose Jacobian variety is isomorphic to the Prym variety P_τ corresponding to the double étale covering $\tilde{X} \rightarrow X$ associated to τ .

The hypersurface Cob_X is completely determined by the data of the 63 Kummer surface Kum_τ defined above.

For later use, we give some details on the last point, involving some calculations. Using the theta structure we have chosen, the equation of Cob_X goes under the form

$$(5.6) \quad C_X = S^C + \sum_{\alpha \in H - \{0\}} \gamma_\alpha Q_\alpha^C + \sum_{\alpha^* \in \hat{H} - \{0\}} \delta_{\alpha^*} P_{\alpha^*}^C$$

where, letting $\hat{\alpha}$ denote the dual of α in \hat{H} ,

$$S^C = \sum_{\beta \in H} x_\beta^4; \quad Q_\alpha^C = \sum_{\beta \in H \mid \hat{\alpha}(\beta)=1} x_\beta^2 x_{\beta+\alpha}^2;$$

$$P_{\alpha^*}^C = \prod_{\beta \in H \mid \alpha^*(\beta)=1} x_\beta + \prod_{\beta \in H \mid \alpha^*(\beta)=-1} x_\beta.$$

Once again, the application $\alpha^* \mapsto \{\beta \in H \mid \alpha^*(\beta) = 1\}$ gives a one-to-one correspondence between the set $\hat{H} - \{0\}$ and the set H_4 of cardinal 4 subgroups of H . If G is element of H_4 , we denote by α_G^* the corresponding element of $\hat{H} - \{0\}$ (in such a way that $G = \{\beta \in H \mid \alpha_G^*(\beta) = 1\}$) and we define $\delta_G P_G^C := \delta_{\alpha_G^*} P_{\alpha_G^*}^C$.

Restricted to $\Delta^+(\tilde{\tau})$, the equation C_X of the Coble’s quartic hypersurface reduces, up to a multiplicative scalar, to the equation of a Kummer surface whose coefficients depend only on τ (and not on $\tilde{\tau}$). Those coefficients $k_{00}(\tau)$, $k_{01}(\tau)$, $k_{10}(\tau)$, $k_{11}(\tau)$ are determined in terms of the γ_α ’s and the δ_α ’s. Namely, one finds that either

$$(5.7) \quad 2k_{00}(\alpha_0^*) = \delta_{\alpha_0^*} \text{ and } k_{\bar{\alpha}}(\alpha_0^*) = \gamma_\alpha \text{ for all non zero } \alpha \text{ in } H(\tau)$$

if $\alpha_0 = 0$ or

$$(5.8) \quad k_{00}(\tau) = 2 \frac{\delta_{H(\tau)} + \sum_{G \in H_4 \mid G \cap H(\tau) = \langle \alpha_1 \rangle, \alpha_G^*(\alpha_0) = -1} \alpha_0^*(\alpha + \alpha_0) \delta_G}{2 + \alpha_0^*(\alpha_0) \gamma_{\alpha_0}}$$

and $k_{\bar{\alpha}}(\tau) = \frac{2(\gamma_\alpha + \alpha_0^*(\alpha_0) \gamma_{\alpha+\alpha_0}) + \delta_{\langle \alpha, \alpha_0 \rangle}}{2 + \alpha_0^*(\alpha_0) \gamma_{\alpha_0}}$ for all non zero α in $H(\tau)$.

6. Performing the computations

6.1. Multiplication by p on an elliptic curve

We refer the reader to [23] where it is explained how one can recover the group law on an elliptic curve E via its geometry. More specifically, there

are duplication and addition formulae (see [23, III,2]) given for an affine model $y^2 = x(x-1)(x-\mu)$ of E as well as the division polynomials in characteristic $p \geq 5$ (see [23, Exercise 3.7]) which are much more convenient when implemented with a computer. As the action of $\{\pm\}$ commutes with multiplication by p , the latter induces a map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by two homogeneous polynomials of degree p (say D and N) with $x \mapsto N(x^p)/D(x^p)$ and the map induced by Verschiebung is defined by $x \mapsto N(u)/D(u)$. The computations yield

LEMMA 6.1. — *In characteristic $p = 3$,*

$$N(u) = u(u + \mu(\mu + 1))^2 \text{ and } D(u) = ((\mu + 1)u + \mu^2)^2.$$

In characteristic $p = 5$,

$$N(u) = u [u^2 - \mu(\mu + 1)(\mu^2 - \mu + 1)u + \mu^4(\mu^2 - \mu + 1)]^2$$

and

$$D(u) = [(\mu^2 - \mu + 1) [u^2 - \mu^2(\mu + 1)u] + \mu^6]^2.$$

In characteristic $p = 7$,

$$N(u) = u [u^3 + 2\mu(\mu + 1)(\mu - 2)(\mu - 4)(\mu^2 + 3\mu + 1)u^2 + \mu^4(\mu + 1)^2\mu - 2)(\mu - 4)(\mu^2 + 1)u + \mu^9(\mu + 1)(\mu - 2)(\mu - 4)]^2$$

and

$$D(u) = [(\mu + 1)(\mu - 2)(\mu - 4) [u^3 + \mu^2(\mu + 1)(\mu^2 + 1)u^2 + \mu^6(\mu^2 + 3\mu + 1)u] + \mu^{12}]^2.$$

Remark 6.2. — These results are consistent with the classification of supersingular elliptic curves in small characteristics (see [7, Chapter IV, Examples 4.23.1, 4.23.2 and 4.23.3]).

Choose non-zero scalars a and b such that $\omega = -\frac{a^4 + b^4}{a^2b^2}$ is different from ± 2 (in particular, $a \neq \pm b$). There is a unique linear automorphism of \mathbb{P}^1 mapping $(a : b)$ to 0, $(a : -b)$ to 1 and $(b : a)$ to ∞ . It maps $(b : -a)$ to the point $(\mu : 1)$ with $\mu = \left(\frac{b^2 + a^2}{2ab}\right)^2 = \frac{2 - \omega}{4}$. Letting λ_0, λ_1 denote the corresponding pair of homogeneous of \mathbb{P}^1 , the homogeneous polynomials Q_0 and Q_1 corresponding to N and D under this linear transformation can be exchanged by the action of a suitable element of $E[2]$ and the computations yield expressions that depend only on ω and not on the choice of a and b as expected.

LEMMA 6.3. — *With the notations given above, one has*

- $p = 3$. $Q_0(\lambda_0, \lambda_1) = \lambda_0^3 - \omega\lambda_0\lambda_1^2$.
- $p = 5$. $Q_0 = \lambda_0^5 + \omega(\omega^2 + 2)\lambda_0^3\lambda_1^2 + (\omega^2 + 2)\lambda_0\lambda_1^4$.
- $p = 7$. $Q_0 = \lambda_0^7 - 2\omega(\omega^4 - 1)\lambda_0^5\lambda_1^2 + \omega^2(\omega^2 - 1)(\omega^2 - 2)\lambda_0^3\lambda_1^4 - \omega(\omega^2 - 1)\lambda_0\lambda_1^6$.

Remark 6.4. — Using the expression of ω in terms of μ , the chart (5.5) as well as the remark 6.2 allow one to give a more precise description of the locus of the Kummer surfaces such that the corresponding genus 2 curves only have ordinary Prym varieties. Namely, viewing the set of Kummer surfaces as an open subset of the double covering of the affine 3-space \mathbb{A}^3 given by equation (5.2), one has to exclude the inverse image of a finite set of affine planes in \mathbb{A}^3 . Therefore, one can easily say when a Coble quartic has associated Kummer surfaces such that the corresponding genus 2 curves only have ordinary Prym varieties (there is a finite set of linear relations that the coefficients of the Coble quartic should not satisfy) and these form a dense subset in the set of all Coble quartics. Together with the Proposition 3.7, this ensures that a general genus 3 curve is ordinary, that all of its Prym varieties P_τ are ordinary, and that if Y_τ is the genus 2 curve associated to P_τ , then all the Prym varieties of Y_τ also are ordinary.

6.2. Equations of \tilde{V} for $p = 3$

Let us start with the genus 2 case and provide an alternative proof to the following result of Laszlo and Pauly ([11, Thm 6.1] where the result is proven for any curve, not only a general one).

THEOREM 6.5. — *Let X be a general smooth and projective curve of genus 2 over an algebraically closed field of characteristic 3.*

- (1) *There is an embedding $\alpha : \text{Kum}_X \hookrightarrow |2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$*

$$\tilde{V}^{-1}(\text{Kum}_X) = \text{Kum}_{X_1} + 2\alpha(\text{Kum}_X)$$

holds scheme-theoretically.

- (2) *The cubic equations of \tilde{V} are given by the 4 partial derivatives of the quartic equation of the Kummer surface $\alpha(\text{Kum}_X) \subseteq |2\Theta_1|$. In other words, \tilde{V} is the polar map of the surface $\alpha(\text{Kum}_X)$.*

Proof. — Because X is general, all its Prym varieties are ordinary (Proposition 3.7). Let V_0 be a generator for the \hat{H} -invariant part of $\tilde{V}^*(W)$ in $\text{Sym}^p W_1$. For $g = 2$ and $p = 3$, it comes under the form

$$V_0 = y_{00}^3 + a_{01}y_{00}y_{01}^2 + a_{10}y_{00}y_{10}^2 + a_{11}y_{00}y_{11}^2 + by_{01}y_{10}y_{11}.$$

We use the Lemmas 4.3 and 4.4, the chart (5.5) and the Lemma 6.3 to obtain

$$(6.1) \quad V_0 = y_{00}^3 + 2k_{01}y_{00}y_{01}^2 + 2k_{10}y_{00}y_{10}^2 + 2k_{11}y_{00}y_{11}^2 + 2k_{00}y_{01}y_{10}y_{11}.$$

Then, one can deduce the $V_\alpha := \tilde{V}(x_\alpha)$ ($\alpha = 01, 10, 11$) by permuting suitably the coordinate functions y_\bullet in V_0 . Notice that V_α is the partial (with respect to y_α) of a quartic surface with equation

$$S + 2k_{00}P + k_{10}Q_{01} + k_{01}Q_{10} + k_{11}Q_{11}$$

(with S, P, Q_{01}, Q_{10} and Q_{11} as in Lemma 5.1) hence isomorphic to Kum_X . Thus, the second point above is proven.

The inverse image $\tilde{V}^{-1}(\text{Kum}_X)$ can be computed explicitly as it is defined by the pull-back $\tilde{V}^*(K_X)$ of the equation (5.1). In other words, a few more computations enable us to recover the first assertion of the Theorem. Namely, one knows (see Diagram (3.1)) that the equation K_{X_1} of Kum_{X_1} divides $\tilde{V}^*(K_X)$. The exact quotient $\tilde{V}^*(K_X)/K_{X_1}$ coincide with the square of K_X . □

This geometric interpretation of \tilde{V} in genus 2 and characteristic 3 has an analogous interpretation of the unique \tilde{V} lifting the generalized Verschiebung in genus 3 and characteristic 3, namely the theorem 1.3.

Proof of the Theorem 1.3. — The proposition 3.1 and the Remark 3.3 ensure the existence and the uniqueness of $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ lifting $V_2 : S_{X_1} \dashrightarrow S_X$ which is given by a system of 8 cubics. Because X is general, the Remark 6.4 tells us that the associated Prym varieties (that are ordinary by Proposition 3.7) are general enough to have Kummer surfaces for which the previous theorem is valid. Let V_0 be the only \tilde{H} -invariant cubic of this system of 8 cubics. In the same way as in genus 2, we use the Lemmas 4.3 and 4.5, the equations (5.7) and (5.8) (instead of the chart (5.5)), and the Theorem 6.5 (instead of the Lemma 6.3) to obtain

$$\begin{aligned} V_0 = & y_0^3 + 2\gamma_{001}y_0y_{001}^2 + 2\gamma_{010}y_0y_{010}^2 + 2\gamma_{011}y_0y_{011}^2 \\ & + 2\gamma_{100}y_0y_{100}^2 + 2\gamma_{101}y_0y_{101}^2 + 2\gamma_{110}y_0y_{110}^2 + 2\gamma_{111}y_0y_{111}^2 \\ & + \delta_{010}y_{010}y_{100}y_{110} + \delta_{010}y_{001}y_{100}y_{101} + \delta_{011}y_{011}y_{100}y_{111} \\ & + \delta_{100}y_{001}y_{010}y_{011} + \delta_{101}y_{010}y_{101}y_{111} + \delta_{110}y_{001}y_{110}y_{111} \\ & + \delta_{111}y_{011}y_{101}y_{110}. \end{aligned}$$

Then, one deduces the $V_\alpha := \tilde{V}(x_\alpha)$ by permuting suitably the coordinate functions y_\bullet in V_0 and checks that V_α is the partial (with respect to y_α) of

a quartic hypersurface with equation

$$S^C + \sum_{\alpha \in H - \{0\}} \gamma_\alpha Q_\alpha^C + \sum_{\alpha^* \in \hat{H} - \{0\}} \delta_{\alpha^*} P_{\alpha^*}^C$$

as in the Subsection 5.2, hence isomorphic to Cob_X . □

6.3. Equations of \tilde{V} for $p = 5$ and 7 and geometric consequences

For these characteristics, we have only performed the calculations in the genus 2 case. The equations obtained are already huge and interpretation requires (much time to) computational softwares. In much the same way as we proved the Theorem 6.5, we prove the following

PROPOSITION 6.6. — *Let X be a general proper and smooth curve of genus 2 over an algebraically closed field of characteristic 5. There are coordinate functions $\{x_\alpha\}$ and $\{y_\alpha\}$ for $|2\Theta|$ and $|2\Theta_1|$ respectively such that the Kummer surface Kum_X in $|2\Theta|$ has an equation of the form (5.1) and such that, if $V_\alpha(y) = \tilde{V}^*(x_\alpha)$, then*

$$\begin{aligned} V_{00} = & y_{00}^5 + a_{1100}y_{00}^3y_{01}^2 + a_{1010}y_{00}^3y_{10}^2 + a_{1001}y_{00}^3y_{11}^2 \\ & + a_{0200}y_{00}y_{01}^4 + a_{0110}y_{00}y_{01}^2y_{10}^2 + a_{0101}y_{00}y_{01}^2y_{11}^2 + a_{0020}y_{00}y_{10}^4 \\ & + a_{0011}y_{00}y_{10}^2y_{11}^2 + a_{0002}y_{00}y_{11}^4 + b_{00}y_{00}^2y_{01}y_{10}y_{11} + b_{01}y_{01}^3y_{10}y_{11} \\ & + b_{10}y_{01}y_{10}^3y_{11} + b_{11}y_{01}y_{10}y_{11}^3 \end{aligned}$$

with

$$\begin{aligned} a_{1100} = k_{01}(k_{01}^2 + 2), & \quad a_{1010} = k_{10}(k_{10}^2 + 2), & \quad a_{1001} = k_{11}(k_{11}^2 + 2), \\ a_{0200} = (k_{01}^2 + 2), & \quad a_{0020} = (k_{10}^2 + 2), & \quad a_{0002} = (k_{11}^2 + 2), \end{aligned}$$

$$\begin{aligned} a_{0110} &= 3k_{11}(k_{00}^2 + k_{11}^2) + k_{01}k_{10}(1 - k_{11}^2), \\ a_{0101} &= 3k_{10}(k_{00}^2 + k_{10}^2) + k_{01}k_{11}(1 - k_{10}^2), \\ a_{0011} &= 3k_{01}(k_{00}^2 + k_{01}^2) + k_{10}k_{11}(1 - k_{01}^2), \\ b_{00} &= 2k_{00}(k_{00}^2 + 1) - k_{00}k_{01}k_{10}k_{11}, \end{aligned}$$

$b_{01} = k_{00}(k_{01} + 3k_{10}k_{11})$, $b_{10} = k_{00}(k_{10} + 3k_{01}k_{11})$, $b_{11} = k_{00}(k_{11} + 3k_{01}k_{10})$ where the k_i are the coefficients of the equation (5.1) of Kum_X . The V_α ($\alpha = 01, 10, 11$) can be deduced from V_{00} by a suitable permutation of the coordinate functions y_i , namely the unique pairwise permutation that exchanges y_{00} and y_α .

Remark 6.7. — In characteristic 7, one has the same kind of statement except that V_0 is now the sum of 30 monomials, whose coefficients are as above polynomials (of degree at most 7) in the parameters k_\bullet .

Let us recall some features of the geometry of the map $\tilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ when $g = 2$ and $p = 3$ exhibited in [11]. First, there is an irreducible reduced hypersurface $H = \alpha(\text{Kum}_X)$ of degree $2p - 2 = 4$ in S_{X_1} such that the equality of divisors

$$\tilde{V}^{-1}(\text{Kum}_X) = \text{Kum}_{X_1} + 2H$$

holds in S_{X_1} and such that base locus of \tilde{V} coincides with the singular locus of H . Second, H contains 16 curves (namely, conics containing 6 of the 16 singular points of H) each of which was contracted by \tilde{V} on a singular point of Kum_X . Therefore, the inverse image of the singular locus of Kum_X by \tilde{V} is 1-dimensional and contains all the singular points of H .

We tried to check these properties in characteristic 5 and 7 in exploiting the equations we could compute. Unfortunately, most of these assertions require a Groebner basis computation which seems beyond the capacities of the machines (or the software) used when working with the generic curve (that is to say over the extension

$$L := \mathbb{F}_p(k_{01}, k_{10}, k_{11})[k_{00}]/(k_{00}^2 - k_{01}^2 - k_{10}^2 - k_{11}^2 + k_{01}k_{10}k_{11} + 4)$$

of the prime field \mathbb{F}_p as field of coefficients). Still, we could prove Proposition 1.2.

Proof of the proposition 1.2. — Define the field L as above and R as the L -graded algebra generated by y_{00}, y_{01}, y_{10} and y_{11} . The homogeneous polynomials V_{00}, V_{01}, V_{10} and V_{11} define an endomorphism of L -graded algebras $\tilde{V}^* : R \rightarrow R$. Letting K (resp. K_1) be the equation (5.1) of the Kummer surface Kum_X in $|2\Theta|$ (resp. Kum_{X_1} in $|2\Theta_1|$), which is obtained by raising the coefficients of (5.1) to the power p , one checks that K_1 divides $\tilde{V}^*(K)$. Letting Q be the exact quotient $\tilde{V}^*(K)/K_1$, one checks that it is a square S^2 and that S is irreducible. □

Although the computation does not end for the generic curve, we checked the other assertions for a one hundred plus particular curves in each characteristic ($p = 5$ and 7), randomly choosing the coefficients k_{01}, k_{10}, k_{11} in $\mathbb{F}_{p^{10}}$. The following pattern arises, with no exception once put aside the (non-relevant) cases contradicting the inequalities (5.3). The base locus of \tilde{V} is contained, scheme-theoretically, in the singular locus of H , which is 0-dimensional and has total length 96 (resp. 304) in characteristic 5 (resp. 7). This singular locus is itself in the stable locus of S_{X_1} as well as in the inverse image of the singular points of Kum_X which again defines a 1-dimensional subset of H . The inclusion of the base locus of \tilde{V} in the singular locus of H is strict and, unfortunately, its reducedness is too expensive to be checked out by the computation.

Of course, one would like to find a geometric (and characteristic free) proof of these facts that we believe are true for any general curve and any odd characteristic.

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Manuscrit reçu le 6 mars 2008,
révisé le 19 septembre 2008,
accepté le 7 novembre 2008.

Laurent DUCROHET
École Polytechnique
CMLS
91128 Palaiseau Cedex (France)
ducrohet@math.polytechnique.fr